

CENTRAL WEYL INVOLUTIONS ON FANO-MUKAI FOURFOLDS OF GENUS 10

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ABSTRACT. It is known that every Fano-Mukai fourfold X of genus 10 is acted upon by an involution τ which comes from the center of the Weyl group of the simple algebraic group of type G_2 , see [PZ18, PZ22]. This involution is uniquely defined up to conjugation in the group $\text{Aut}(X)$. In this note we describe the set of fixed points of τ and the surface scroll swept out by the τ -invariant lines.

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INTRODUCTION

Let V be a Fano-Mukai fourfold of genus 10 over \mathbb{C} . A classification of the automorphism groups of these fourfolds was started in [PZ18] and was completed in [PZ22, Theorem A]. In particular, every V is acted upon by an involution $\tau \in \text{Aut}(V) \setminus \text{Aut}^0(V)$ defined uniquely up to conjugation in $\text{Aut}(V)$. It interchanges any pair of disjoint $\text{Aut}^0(V)$ -invariant cubic surface cones on V , see [PZ22, Proposition 2.11] and Corollary 1.7 below.

This note is devoted to the following problem, see [PZ18, Problem 15.4].

Given a Fano-Mukai fourfold V of genus 10, describe the involutions acting on V and interchanging the pairs of $\text{Aut}^0(V)$ -invariant cubic cones.

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Let \mathfrak{g} be the Lie algebra of the simple algebraic group G of type G_2 . The adjoint variety Ω of G is the unique closed orbit in the projectivized adjoint representation of G on $\mathbb{P}\mathfrak{g} \cong \mathbb{P}^{13}$. We have $\dim \Omega = 5$. Every Fano-Mukai fourfold V of genus 10 admits a realization as a hyperplane section $\Omega \cap \mathbb{P}^{12}$, see [Muk89, Theorem 2]. Such a hyperplane section is unique up to the G -action on Ω . Under this realization, $\text{Aut}(V)$ coincides with the stabilizer of V in G and τ extends to an element of order 2 of G .

Let T be a maximal torus of G . The Weyl group $W = N_G(T)/T$ is isomorphic to the dihedral group \mathfrak{D}_6 ; there exists a splitting $N_G(T) \cong T \rtimes W$ [AH17, Theorem A]. Up to a choice of such a splitting and up to conjugation in G , τ can be identified as the unique element of order 2 from the center of W . This is why we call τ a *central Weyl involution*.

The main result of the present note is the following theorem.

Theorem 0.1. *Let V be a Fano-Mukai fourfold of genus 10 half-anticanonically embedded in \mathbb{P}^{12} , and let $\tau \in \text{Aut}(V) \setminus \text{Aut}^0(V)$ be an involution. Then $\tau \in G$ is a central Weyl involution. The fixed point set V^τ is a union of two disjoint smooth rational sextic curves E^+ and E^- such that*

$$\langle E^+ \rangle = \mathbb{P}^5, \quad N_{E^+/\mathbb{P}^5} = \mathcal{O}_{\mathbb{P}^1}(8)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^1}(9)^{\oplus 2} \quad \text{and} \quad \langle E^- \rangle = \mathbb{P}^6, \quad N_{E^-/\mathbb{P}^6} = \mathcal{O}_{\mathbb{P}^1}(8)^{\oplus 5}.$$

Furthermore, there is a surface scroll $\Pi = \Pi(V, \tau)$ in V verifying the following.

- (i) Each ruling of Π is τ -invariant and each τ -invariant line on V is a ruling of Π .
- (ii) Π has degree 12, is linearly nondegenerate in \mathbb{P}^{12} and is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ embedded in \mathbb{P}^{12} by a linear system of type $(1, 6)$.
- (iii) An isomorphism $\Pi \cong \mathbb{P}^1 \times \mathbb{P}^1$ sends the curves E^\pm into constant sections of the natural projection $\text{pr}: \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$.

Due to (i) we call Π the *scroll in τ -invariant lines*. Notice that neither τ nor Π are $\text{Aut}^0(V)$ -invariant.

Results of [PZ22] were extended in [BM22] to the automorphism groups of smooth hyperplane sections of other generalized flag varieties G/P ; most of these hyperplane sections also possess Weyl involutions. It is worth to obtain a geometric description similar to that of Theorem 0.1 in this more general setting; see e.g. [DM22, Proposition 24] for some results in this direction.

We end this Introduction with the following open question. Recall that the Fano-Mukai fourfolds of genus 10 are rational.

Question. *Is any central Weyl involution acting on a Fano-Mukai fourfold V of genus 10 linearizable, that is, conjugate to a linear involution of \mathbb{P}^4 via a birational map $V \dashrightarrow \mathbb{P}^4$?*

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1. PRELIMINARIES

1.1. Lines and cubic scrolls on Fano-Mukai fourfolds. We gather here various results from [KR13], [PZ18] and [PZ22] that will be used in the sequel.

Fix a Fano-Mukai fourfold V of genus 10 together with an embedding $V \hookrightarrow \mathbb{P}^{12}$ by the half-anticanonical system. Let $\Sigma = \Sigma(V)$ be the Hilbert scheme of lines on V and $\mathcal{S} = \mathcal{S}(V)$ be the Hilbert scheme of cubic surface scrolls on V . By a *cubic cone* we mean the projective cone over a rational twisted cubic curve. We fix an involution

$\tau \in \text{Aut}(V) \setminus \text{Aut}^0(V)$ and we denote by the same letter the induced involutions acting on Σ and on \mathcal{S} . The geometry of a Fano-Mukai fourfold can be described as follows.

Proposition 1.1.

- (i) Σ is a smooth hyperplane section of $\mathbb{P}^2 \times \mathbb{P}^2$ embedded in \mathbb{P}^7 via the Segre embedding. Up to an automorphism of $\mathbb{P}^2 \times \mathbb{P}^2$ one can choose this section in such a way that the action of τ on Σ is induced by the involution on $\mathbb{P}^2 \times \mathbb{P}^2$ interchanging the factors.
- (ii) There is the diagram

$$(1) \quad \begin{array}{ccc} & \mathcal{L}(V) & \\ \rho \swarrow & & \searrow s \\ \Sigma(V) & & V \end{array}$$

where $\mathcal{L}(V) \subset \Sigma \times V$ is the incidence relation between lines and points on V . The \mathbb{P}^1 -bundle $\rho: \mathcal{L}(V) \rightarrow \Sigma(V)$ is the universal family of lines on V . The action of τ on $\Sigma \times V$ leaves $\mathcal{L}(V)$ invariant and respects the \mathbb{P}^1 -bundle structure.

- (iii) The map $s: \mathcal{L}(V) \rightarrow V$ in (1) is a generically finite morphism of degree 3. So, V is covered by lines and through a general point of V pass precisely 3 lines.
- (iv) Any cubic cone S on V is contained in the branching divisor $\mathcal{B} \subset V$ of s .
- (v) Any line on V passing through the vertex of a cubic cone S is a ruling of S . If a point $v \in \mathcal{B}$ is different from the vertices of cubic cones on V then the number of lines passing through v equals either 1 or 2. Through any point on $V \setminus \mathcal{B}$ passes exactly 3 distinct lines on V .
- (vi) The Hilbert scheme $\mathcal{S} = \mathcal{S}(V)$ of cubic scrolls on V has exactly two irreducible components $\mathcal{S}_i \cong \mathbb{P}^2$, $i = 1, 2$. These components are disjoint and interchanged by τ .
- (vii) $H^4(V, \mathbb{Z}) = \mathbb{Z}[S_1] \oplus \mathbb{Z}[S_2]$, where $S_i \in \mathcal{S}_i$ satisfy the relations $[S_1]^2 = [S_2]^2 = 1$ and $[S_1] \cdot [S_2] = 0$.
- (viii) Any cubic scroll S on V coincides with the singular locus of a unique hyperplane section A_S of V , where A_S is the union of lines on V meeting S . Any line contained in A_S meets S . Through any point of $A_S \setminus S$ passes a unique line on V which meets S .
- (ix) Two scrolls $S_1 \in \mathcal{S}_1$ and $S_2 \in \mathcal{S}_2$ from different components of the Hilbert scheme \mathcal{S} either are disjoint or contain a unique common ruling. Any line l on V is the unique common ruling of exactly two cubic scrolls $S_i(l) \in \mathcal{S}_i$, $i = 1, 2$. The morphisms

$$\text{pr}_i: \Sigma \rightarrow \mathcal{S}_i = \mathbb{P}^2, \quad l \mapsto S_i(l), \quad i = 1, 2$$

coincide with the standard projections in (i):

$$\text{pr}_i: \Sigma \subset \mathbb{P}^2 \times \mathbb{P}^2 \rightarrow \mathbb{P}^2.$$

The fiber of pr_i over $S \in \mathcal{S}_i$ is the line $\Lambda(S)$ on Σ which is the Hilbert scheme of the rulings of S .

- (x) For a cubic scroll S on V the Hilbert scheme $\Sigma(S) \subset \Sigma$ of lines on V meeting S is the pull-back of a line in $\mathbb{P}^2 = \mathcal{S}_j$ under the second projection pr_j , $j \neq i$. One has $\Sigma(S) \cong \mathbb{F}_1$ and $\Lambda(S)$ is the exceptional section of $\Sigma(S)$.
- (xi) The Hilbert scheme of cubic cones on V is either finite or one-dimensional. The number of $\text{Aut}^0(V)$ -invariant cubic cones on V is finite and V contains a pair (S_1, S_2) of disjoint $\text{Aut}^0(V)$ -invariant cubic cones.
- (xii) $\bigcap_{S \in \mathcal{S}_1} A_S$ is the union of vertices of cubic cones in \mathcal{S}_1 and $\bigcap_{S \in \mathcal{S}_1 \cup \mathcal{S}_2} A_S = \emptyset$.

Proof. See

- [KR13, Proposition 2], [PZ18, Theorem 9.1(a), Lemma 9.5.1 and its proof] for (i),
- [PZ18, Proposition 8.2(d), (8.2.2)] for (ii),
- [PZ18, Proposition 8.2(d)] for (iii),
- [PZ18, Lemma 9.4] for (iv),
- [PZ18, Proposition 8.2(e)] for (v),
- [KR13, Proposition 1] [PZ18, Theorem 9.1(b)] for (vi),
- [PZ18, Proposition 9.6] for (vii),
- [PZ18, Lemma 9.2] for (viii),
- [PZ18, Corollaries 9.7.3, 9.7.4, and 9.10.1] for (ix),
- [PZ18, Theorem 9.1(a), Lemma 9.7.2] for (x),
- [PZ20, Lemma 1.8, Proposition 2.11] for (xi),
- [PZ20, Lemma 1.9] for (xii).

□

1.2. Mukai G_2 -presentation. Let $\Omega \subset \mathbb{P}^{13}$ be the adjoint variety of the simple algebraic group G of type G_2 , that is, the minimal nilpotent orbit $\Omega = G/P$ of the adjoint action of G on $\mathbb{P}\mathfrak{g} = \mathbb{P}^{13}$, where $\mathfrak{g} = \text{Lie}(G)$ and $P \subset G$ is a parabolic subgroup of dimension 9. Recall that Ω is a Fano-Mukai fivefold embedded in \mathbb{P}^{13} by the linear system $|-\frac{1}{3}K_\Omega|$. Using the duality given by the Killing form on \mathfrak{g} we identify $\mathbb{P}\mathfrak{g}$ with its dual projective space $\mathbb{P}\mathfrak{g}^\vee$. Abusing notation we let h^\perp denote the hyperplane in $\mathbb{P}\mathfrak{g}$ orthogonal to a vector $h \in \mathfrak{g} \setminus \{0\}$. Letting $\mathbb{P}h$ stand for the vector line $\mathbb{C}h \in \mathbb{P}\mathfrak{g}$ we let $V(h) = h^\perp \cap \Omega$ denote the corresponding hyperplane section of Ω . According to [Muk89, Theorem 2] any Fano-Mukai fourfolds V of genus 10 admits a *Mukai realization* as a smooth hyperplane section $V = V(h)$ of Ω .

The dual variety $D_\ell = \Omega^*$ is a sextic hypersurface in \mathbb{P}^{13} whose points $\mathbb{P}h \in D_\ell$ correspond to the singular hyperplane sections $V(h)$ of Ω . Thus, the smooth hyperplane sections $V(h)$ are parameterized by the affine variety $\mathbb{P}^{13} \setminus D_\ell$. The group G acts on \mathbb{P}^{13} via the projective adjoint representation. Each G -orbit in $\mathbb{P}^{13} \setminus D_\ell$ corresponds to a class of isomorphic smooth hyperplane sections of Ω , that is, to a class of isomorphic Fano-Mukai fourfolds of genus 10. Indeed, according to the Mukai theorem [Muk89, Theorem 0.9], any isomorphism between two smooth hyperplane sections of Ω extends to an automorphism of Ω and the group $\text{Aut}(\Omega)$ coincides with $G|_\Omega$. For a smooth hyperplane section $V = V(h)$ of Ω one has (see [PZ22, Corollary 2.3.2])

$$(2) \quad \text{Aut}(V) = \text{Stab}_G(\mathbb{P}h).$$

Given a maximal torus T of G we let $\mathfrak{h} = \text{Lie}(T)$ be the corresponding Cartan subalgebra in \mathfrak{g} and $\Delta \subset \mathfrak{h}^*$ be the corresponding root system. For a root $\alpha \in \Delta$ we let \mathfrak{g}_α be the corresponding one-dimensional root subspace of \mathfrak{g} . The points $\mathbb{P}\mathfrak{g}_{\alpha_i} \in \mathbb{P}\mathfrak{g}$, $i = 1, \dots, 6$ that correspond to the 6 long roots $\alpha_1, \dots, \alpha_6$ lie on Ω , while the points $\mathbb{P}\mathfrak{g}_{\alpha_i}$, $i = 7, \dots, 12$ that correspond to the 6 short roots $\alpha_7, \dots, \alpha_{12}$ lie on another nilpotent orbit Ω_s of G . The dual hypersurface $D_s = (\overline{\Omega_s})^*$ of $\overline{\Omega_s}$ is also a sextic. Let $(D_t)_{t \in \mathbb{P}^1}$ be the pencil of sextic hypersurfaces in $\mathbb{P}^{13} = \mathbb{P}\mathfrak{g}$ generated by D_ℓ and D_s with base locus $D_\ell \cap D_s$. Any sextic hypersurface D_t is invariant under the adjoint G -action on $\mathbb{P}\mathfrak{g}$. Thus, any G -orbit in $\mathbb{P}\mathfrak{g}$ is contained in D_t for a suitable $t \in \mathbb{P}^1$. There is the following description of G -orbits in $\mathbb{P}\mathfrak{g}$.

Theorem 1.2 ([KR13, Lemma 1], [PZ22, Proposition 1.4]).

- (i) *The base locus $D_\ell \cap D_s$ coincides with the projectivized nilpotent cone of \mathfrak{g} . Thus, all the nilpotent G -orbits in $\mathbb{P}\mathfrak{g}$, including Ω and Ω_s , are contained in $D_\ell \cap D_s$.*

- (ii) For any $t \notin \{\ell, s\}$ the complement $D_t \setminus D_\ell$ coincides with the orbit of a point $\mathbb{P}s \in \mathbb{P}\mathfrak{g}$ where $s \in \mathfrak{g}$ is a regular semisimple element.
- (iii) $D_s \setminus D_\ell$ is the union of two G -orbits Ω^a and Ω^{sr} , where Ω^a of dimension 12 is open and dense in $D_s \setminus D_\ell$ and Ω^{sr} of dimension 10 is closed in $D_s \setminus D_\ell$. There exists a pair of commuting elements $s, n \in \mathfrak{g}$, where s is subregular semisimple, n is nilpotent and $s + n$ is regular such that Ω^a is the G -orbit of $\mathbb{P}(s + n)$ and Ω^{sr} is the G -orbit of $\mathbb{P}s$.

1.3. Automorphism groups and involutions of Fano-Mukai fourfolds. According to (2) the classification of the automorphism groups of Fano-Mukai fourfolds of genus 10 is ultimately related to the classification of non-nilpotent G -orbits in $\mathbb{P}\mathfrak{g}$. Indeed, due to the existence of the Mukai presentation, the latter groups are stabilizers of the G -orbits in $\mathbb{P}\mathfrak{g} \setminus D_\ell$. These orbits are $D_t \setminus D_\ell$, $t \in \mathbb{P}^1 \setminus \{\ell, s\}$, Ω^a and Ω^{sr} , see Theorem 1.2. Accordingly, we have the following description.

Theorem 1.3 ([PZ18, Theorem 1.3], [PZ22, Proposition 1.5]). *Let $V = V(h)$ be a Fano-Mukai fourfold of genus 10 under a Mukai realization, where $\mathbb{P}h \in \mathbb{P}\mathfrak{g} \setminus D_\ell$. Then the identity component $\text{Aut}^0(V)$ is as follows:*

$$\text{Aut}^0(V(h)) = \text{Stab}_G(\mathbb{P}h) \cong \begin{cases} \text{GL}(2, \mathbb{C}), & \mathbb{P}h \in \Omega^{sr} \\ \mathbb{G}_a \times \mathbb{G}_m, & \mathbb{P}h \in \Omega^a \\ \mathbb{G}_m^2, & \mathbb{P}h \in \mathbb{P}\mathfrak{g} \setminus (D_\ell \cup D_s) = \bigcup_{t \neq s, \ell} (D_t \setminus D_\ell) \end{cases}$$

Consequently, there is a unique up to isomorphism Fano-Mukai fourfold V of genus 10 with $\text{Aut}^0(V) \cong \text{GL}(2, \mathbb{C})$ (resp. $\text{Aut}^0(V) \cong \mathbb{G}_a \times \mathbb{G}_m$), while the isomorphism classes of Fano-Mukai fourfolds with $\text{Aut}^0(V) \cong \mathbb{G}_m^2$ form a one-parameter family.

The geometry of a Fano-Mukai fourfold V and its automorphism group are also ultimately related.

Proposition 1.4 ([PZ18,]). *A Fano-Mukai fourfold V carries*

- exactly 6 cubic cones if $\text{Aut}^0(V) \cong \mathbb{G}_m^2$,
- exactly 4 cubic cones if $\text{Aut}^0(V) \cong \mathbb{G}_a \times \mathbb{G}_m$,
- a one-parameter family of cubic cones if $\text{Aut}^0(V) \cong \text{GL}(2, \mathbb{C})$.

In the first two cases every cubic cone on V is $\text{Aut}^0(V)$ -invariant. In the last case there are exactly two $\text{Aut}^0(V)$ -invariant cubic cones on V , they are disjoint and belong to different connected components \mathcal{S}_1 and \mathcal{S}_2 of \mathcal{S} .

Concerning the discrete part of the automorphism group $\text{Aut}(V)$ and its action on V we have the following results.

Theorem 1.5 ([PZ22, Theorem A, Proposition 3.11]).

- (i) If $\text{Aut}^0(V)$ is one of the groups $\text{GL}(2, \mathbb{C})$ and $\mathbb{G}_a \times \mathbb{G}_m$ then $\text{Aut}(V) = \text{Aut}^0(V) \rtimes \mathbb{Z}/2\mathbb{Z}$. The same holds in the case $\text{Aut}^0(V) \cong \mathbb{G}_m^2$ except for $V_r = V(h_r)$ with $\mathbb{P}h_r \in D_r = 3Q$ where $Q \subset \mathbb{P}\mathfrak{g}$ is the quadric defined by the Killing form. In the latter case $\text{Aut}(V_r) \cong \mathbb{G}_m^2 \rtimes \mathbb{Z}/6\mathbb{Z}$.
- (ii) Let τ be the generator of the factor $\mathbb{Z}/2\mathbb{Z}$ in the former cases and the unique order 2 element $\tau = \zeta^3$ in the latter case, where ζ is a generator of the factor $\mathbb{Z}/6\mathbb{Z}$. Then τ is an involution on V which interchanges any pair of disjoint $\text{Aut}^0(V)$ -invariant cubic cones on V . There is precisely one such pair of cubic cones in the cases where $\text{Aut}^0(V)$ is one of the groups $\text{GL}(2, \mathbb{C})$ and $\mathbb{G}_a \times \mathbb{G}_m$, and precisely 3 such pairs in the general case where $\text{Aut}^0(V) \cong \mathbb{G}_m^2$.

- (iii) If $\text{Aut}^0(V) \cong \mathbb{G}_m^2$ then the 6 cubic cones on V are naturally arranged in a cycle $\sigma = (S_1, \dots, S_6)$ such that S_i and S_j are neighbors if and only if they share a common ruling. For $V = V_r$ any element of the factor $\mathbb{Z}/6\mathbb{Z}$ of $\text{Aut}(V_r)$ acts on σ via a cyclic shift.
- (iv) The involution $\tau \in \text{Aut}(V) \setminus \text{Aut}^0(V)$ can be chosen in such a way that the action of τ by conjugation on $\text{Aut}^0(V)$ is the inversion $g \mapsto g^{-1}$ if $\text{Aut}^0(V)$ is one of the groups \mathbb{G}_m^2 and $\mathbb{G}_a \times \mathbb{G}_m$ and the Cartan involution $c: g \mapsto (g^t)^{-1}$ if $\text{Aut}^0(V) = \text{GL}(2, \mathbb{C})$, where $g \mapsto g^t$ stands for the matrix transposition.

We need in the sequel the following elementary facts.

Proposition 1.6.

- (i) Any involution $\tau \in \text{Aut}(\text{GL}(2, \mathbb{C})) \setminus \text{Inn}(\text{GL}(2, \mathbb{C}))$ is conjugate in $\text{Aut}(\text{GL}(2, \mathbb{C}))$ to the Cartan involution c .
- (ii) Let H be a group admitting square roots, that is, for any $h \in H$ there exists $g \in H$ such that $h = g^2$. Consider the semidirect product $\tilde{H} = H \rtimes \mathbb{Z}/2\mathbb{Z}$ where the generator c of the factor $\mathbb{Z}/2\mathbb{Z}$ acts on H via inversion. Then for any $h \in H$ the element $ch \in \tilde{H} \setminus H$ has order 2 and is conjugate to c .
- (iii) Let $\tilde{H} = H \rtimes \mathbb{Z}/2\mathbb{Z}$ where $H = \mathbb{G}_a \times \mathbb{G}_m$ and the generator c of the factor $\mathbb{Z}/2\mathbb{Z}$ acts on H via inversion. Then any element $\tau \in \tilde{H} \setminus H$ of order 2 is conjugate to c .
- (iv) Let $\tilde{T} = T \rtimes \mathbb{Z}/2\mathbb{Z}$ where $T = \mathbb{G}_m^2$ and the generator c of the factor $\mathbb{Z}/2\mathbb{Z}$ acts on T via inversion. Then any element $\tau \in \tilde{T} \setminus T$ of order 2 is conjugate to c .
- (v) There exist exactly two different (conjugated) subgroups H^\pm of $\text{GL}(2, \mathbb{C})$ such that
 - (a) $H^\pm \cong \mathbb{G}_a \times \mathbb{G}_m$;
 - (b) H^\pm is invariant under the Cartan involution c ;
 - (c) $c|_{H^\pm}$ is the inversion $g \mapsto g^{-1}$.

Proof. (i) Let us first show the assertion of (i) for the derived subgroup $\text{SL}(2, \mathbb{C})$ of $\text{GL}(2, \mathbb{C})$. This group is invariant under the $\text{Aut}(\text{GL}(2, \mathbb{C}))$ -action and any automorphism of $\text{SL}(2, \mathbb{C})$ extends to an automorphism of $\text{GL}(2, \mathbb{C})$.

It is known that $\text{Aut}(\text{SL}(2, \mathbb{C})) \cong \text{Inn}(\text{SL}(2, \mathbb{C})) \rtimes \mathbb{Z}/2\mathbb{Z}$, see [Pro06, Theorem 10.6.10]. Since $c|_{\text{SL}(2, \mathbb{C})} \in \text{Aut}(\text{SL}(2, \mathbb{C})) \setminus \text{Inn}(\text{SL}(2, \mathbb{C}))$ we may suppose that the factor $\mathbb{Z}/2\mathbb{Z}$ is generated by c . Then $\tau = \text{Ad}(a) \cdot c$ for some $a \in \text{SL}(2, \mathbb{C})$ where $\text{Ad}(a): f \mapsto afa^{-1}$ for $f \in \text{SL}(2, \mathbb{C})$. Suppose that τ is a conjugate of c in $\text{Aut}(\text{SL}(2, \mathbb{C}))$. Then for some $b \in \text{SL}(2, \mathbb{C})$ we have

$$\tau = \text{Ad}(b) \cdot c \cdot \text{Ad}(b^{-1}) = \text{Ad}(bb^t) \cdot c.$$

Now, from $\tau^2 = 1$ and $\tau = \text{Ad}(a) \cdot c$ we deduce that $a = a^t$, that is, a is a symmetric matrix. Then a can be written as $a = bb^t$ for some $b \in \text{SL}(2, \mathbb{C})$. It follows by the preceding formula that $\tau = \text{Ad}(\text{Ad}(b))(c)$, that is, τ is conjugate to c in $\text{Aut}(\text{SL}(2, \mathbb{C}))$, as stated.

Any automorphism $\alpha \in \text{Aut}(\text{GL}(2, \mathbb{C}))$ is a composition of an inner automorphism and the multiplication by a character \det^k of $\text{GL}(2, \mathbb{C})$ for some $k \in \mathbb{Z}$ [Die55, Ch. 4, § 1]. However, if α has a finite order then $k = 0$. Now the preceding argument applied mutatis mutandis shows that any element $\alpha \in \text{Aut}(\text{GL}(2, \mathbb{C})) \setminus \text{Inn}(\text{GL}(2, \mathbb{C}))$ of order 2 is conjugate to the Cartan involution c in $\text{Aut}(\text{GL}(2, \mathbb{C}))$.

(ii) For $h \in H$ the element $ch \in \tilde{H} \setminus H$ satisfies $ch = h^{-1}c$. Therefore, for $g, h \in H$ the equality $ch = gcg^{-1}$ holds if and only if $g^2 = h^{-1}$. Since H admits square roots it follows that for any $h \in H$ there exists $g \in H$ such that $ch = gcg^{-1}$ in \tilde{H} , that is, ch is a conjugate of c in \tilde{H} .

Now (iii)–(iv) follow since the both groups H in (iii) and T in (iv) admit square roots.

(v) Let H be a subgroup of $\mathrm{GL}(2, \mathbb{C})$ isomorphic to $\mathbb{G}_a \times \mathbb{G}_m$. Then H is contained in a Borel subgroup B of $\mathrm{GL}(2, \mathbb{C})$. The unipotent \mathbb{G}_a -factor of H coincides with the unipotent radical U of B and the \mathbb{G}_m -factor of H coincides with the center $z(B) = z(\mathrm{GL}(2, \mathbb{C}))$. Hence $B = \mathrm{Norm}_{\mathrm{GL}(2, \mathbb{C})}(U)$ is the unique Borel subgroup containing H .

The induced action of the outer automorphism c of $\mathrm{GL}(2, \mathbb{C})$ by an involution on the flag variety $\mathrm{GL}(2, \mathbb{C})/B = \mathbb{P}^1$ is effective and has exactly two fixed points, say B^\pm . It follows by the preceding observations that there are exactly two c -invariant subgroups $H^\pm \subset B^\pm$ with desired properties. Let us find these subgroups explicitly.

The c -action on the Lie algebra $\mathfrak{gl}(2, \mathbb{C})$ is given by $h \mapsto -h^t$. Up to proportionality, there are exactly two nonzero elements of $\mathfrak{gl}(2, \mathbb{C})$ satisfying $n^2 = 0$ and $c(n) = -n^t = \lambda n$, namely,

$$n^\pm = \begin{pmatrix} 1 & \pm i \\ \pm i & 1 \end{pmatrix}$$

where in fact $\lambda = -1$. Therefore, the unipotent radicals of the subgroups H^\pm are, respectively,

$$R_u(H^\pm) = \{\exp(tn^\pm), \quad t \in \mathbb{C}\} = \left\{ \begin{pmatrix} 1+t & \pm it \\ \pm it & 1-t \end{pmatrix}, \quad t \in \mathbb{C} \right\}$$

and their \mathbb{G}_m -factors coincide with the center of $\mathrm{GL}(2, \mathbb{C})$. \square

Due to the next corollary the choice of a concrete involution $\tau \in \mathrm{Aut}(V) \setminus \mathrm{Aut}^0(V)$ in Theorem 0.1 is irrelevant. Let $\Pi(\tau)$ be the scroll swept out by the τ -invariant lines on V , cf. Proposition 2.1 below.

Corollary 1.7. *Let V be a Fano-Mukai fourfold V of genus 10. Then any two involutions $\tau_1, \tau_2 \in \mathrm{Aut}(V) \setminus \mathrm{Aut}^0(V)$ are conjugate in the group $\mathrm{Aut}(V)$ via an element of $\mathrm{Aut}^0(V)$ which sends the fixed point set V^{τ_1} and the scroll in invariant lines $\Pi(\tau_1)$ to V^{τ_2} and $\Pi(\tau_2)$, respectively. Every involution $\tau \in \mathrm{Aut}(V) \setminus \mathrm{Aut}^0(V)$ interchanges any pair of disjoint $\mathrm{Aut}^0(V)$ -invariant cubic cones on V .*

Proof. According to Theorem 1.5(i) $\mathrm{Aut}(V) = \mathrm{Aut}^0(V) \rtimes \mathbb{Z}/2\mathbb{Z}$ except for $V = V(h)$ with $\mathbb{P}h \in D_t = 3Q$. In the latter case $\mathrm{Aut}(V) \cong \mathbb{G}_m^2 \rtimes \mathbb{Z}/6\mathbb{Z}$ and the factor $\mathbb{Z}/6\mathbb{Z}$ contains a unique element $c = \zeta^3$ of order 2. Due to Theorem 1.5 (iv) one can choose the generator c of the factor $\mathbb{Z}/2\mathbb{Z}$ and the order 2 element $c \in \mathbb{Z}/6\mathbb{Z}$ in the exceptional case so that the conjugation with c acts via the inversion on $\mathrm{Aut}^0(V)$ if $\mathrm{Aut}^0(V)$ is an abelian group and acts on $\mathrm{Aut}^0(V)$ via the Cartan involution if $\mathrm{Aut}^0(V) \cong \mathrm{GL}(2, \mathbb{C})$. Furthermore, c interchanges any pair of disjoint $\mathrm{Aut}^0(V)$ -invariant cubic cones on V , see Theorem 1.5(ii). However, by Proposition 1.6(i), (iii) and (iv) every involution $\tau \in \mathrm{Aut}(V) \setminus \mathrm{Aut}^0(V)$ is conjugate to c . Now the assertion follows. \square

2. SCROLL Π IN τ -INVARIANT LINES

2.1. The first properties. We fix as before a Fano-Mukai fourfold V of genus 10 and an involution $\tau \in \mathrm{Aut}(V) \setminus \mathrm{Aut}^0(V)$. Many objects considered in this section, including the scroll Π in τ -invariant lines, depend on the choice of τ . To simplify the notation we don't mention this dependence explicitly.

Let $\Sigma = \Sigma(V)$ stand as before for the Hilbert scheme of lines on $V \subset \mathbb{P}^{12}$ regarded as a subscheme of the Grassmannian of lines $\mathbb{G}(1, 12)$. Then the fixed point subscheme $C = \Sigma^\tau$ parameterizes the τ -invariant lines on V . Let $\mathcal{L} \subset \Sigma \times V$ be the line-point incidence relation on V , and let $\rho: \mathcal{L} \rightarrow \Sigma$ be the natural projection, see (1); it comes from the projection of the universal \mathbb{P}^1 -bundle over $G(1, 12)$ restricted to Σ . Letting

$\tilde{\Pi} = \rho^{-1}(C) \subset \mathcal{L}$, the projection $\rho|_{\tilde{\Pi}}: \tilde{\Pi} \rightarrow C$ defines a τ -invariant ruling on $\tilde{\Pi}$ over C . The image $\Pi = \Pi(V, \tau) := s(\tilde{\Pi})$ under the second projection $s: \mathcal{L} \rightarrow V$ is the union of all the τ -invariant lines on V .

Proposition 2.1. *Let $\Sigma \subset \mathbb{P}^2 \times \mathbb{P}^2$ be equipped with the polarization induced by the Segre embedding $\mathbb{P}^2 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^8$. Then the following hold.*

- (i) C is a smooth rational quartic curve on Σ ;
- (ii) $\tilde{\Pi}$ is a smooth rational ruled surface;
- (iii) Π is a rational surface scroll of degree 12 on V ;
- (iv) each τ -invariant line on V is a ruling of Π , and vice versa.

Proof. (i) The threefold Σ admits a realization as the projectivization of the variety of square matrices of order 3 and of rank 1 with zero trace [PZ18, 11.2.2]. Such a matrix A can be written as the tensor product $a \otimes b$ of two nonzero 3-vectors $a = (x_1, x_2, x_3)$ and $b = (y_1, y_2, y_3)$ with zero trace

$$\operatorname{tr}(A) = x_1y_1 + x_2y_2 + x_3y_3 = 0.$$

The involution τ interchanges each pair of disjoint $\operatorname{Aut}^0(V)$ -invariant cubic cones on V . Hence, τ also interchanges the connected components \mathcal{S}_1 and \mathcal{S}_2 of the Hilbert scheme \mathcal{S} of cubic scrolls on V , that is, the factors of $\mathbb{P}^2 \times \mathbb{P}^2 = \mathcal{S}_1 \times \mathcal{S}_2$, and so, each pair of disjoint cubic cones, see Proposition 1.1(ix). Up to a choice of coordinates, the induced action of τ on Σ results in the twist $a \leftrightarrow b$. The fixed point set $C = \Sigma^\tau$ is the intersection $\Sigma \cap \Delta$ where $\Delta \subset \mathbb{P}^2 \times \mathbb{P}^2$ is the diagonal. The image of $A = a \otimes b$ in Σ is fixed by τ if and only if a and b are proportional. The condition $\operatorname{tr}(A) = 0$ leads in the latter case to the relations

$$x_1^2 + x_2^2 + x_3^2 = 0 \quad \text{and} \quad y_1^2 + y_2^2 + y_3^2 = 0.$$

It follows that $C = \Sigma^\tau$ is a smooth rational curve on $\mathbb{P}^2 \times \mathbb{P}^2 = \mathcal{S}_1 \times \mathcal{S}_2$ of bidegree $(2, 2)$ which projects isomorphically onto a smooth conic $C_i \subset \mathcal{S}_i$. Thus, C is a conic in $\Delta \cong \mathbb{P}^2$ and a quartic curve in $\mathbb{P}^2 \times \mathbb{P}^2$.

(ii) The smooth morphism $\rho: \mathcal{L} \rightarrow \Sigma$ is the projection of a \mathbb{P}^1 -bundle. Hence (ii) follows.

(iii) By virtue of (ii), Π is an irreducible surface scroll on V whose rulings are parameterized by C . The scroll Π being τ -invariant, one has $[\Pi] = \alpha([S_1] + [S_2])$ in $H^4(V, \mathbb{Z})$, see Proposition 1.1(vii). Therefore, $\deg(\Pi) = 6\alpha$, where $\alpha = \Pi \cdot S_i$, $i = 1, 2$. For a general cubic scroll S on V the associated divisor $\Sigma(S)$ on Σ (see Proposition 1.1(x)) is the pull-back of a general line in one of the factors \mathbb{P}^2 under the corresponding projection. It follows that C meets $\Sigma(S)$ transversally in two distinct points. The latter means that among the rulings of Π exactly two meet S . Each of them meets S transversally in a single point, see Proposition 1.1(viii). Consequently, $\alpha = \Pi \cdot S = 2$, and so, $\deg(\Pi) = 12$.

(iv) By construction, each ruling of Π is τ -invariant and each τ -invariant line on V is a ruling of Π . \square

Lemma 2.2. *The fixed point set V^τ is contained in Π . No point of V^τ is a vertex of a cubic cone on V .*

Proof. Assume on the contrary that $v \in V^\tau$ is a vertex of a cubic cone S on V . Then $\tau(S)$ is a cubic cone with the same vertex as S , hence $\tau(S) = S$ by virtue of Proposition 1.1(iv). However, this contradicts Proposition 1.1(vi) which says that S and $\tau(S)$ belong to different components \mathcal{S}_i of the Hilbert scheme \mathcal{S} of cubic scrolls on V .

Thus, any $v \in V^\tau$ is different from the vertices of the cubic cones on V . By Proposition 1.1(v) there is at least one and at most three lines on V passing through v . At least

one of these lines is invariant under τ . Indeed, this is clear in the case where the number of lines passing through v is odd. Recall that V is a hyperplane section of the adjoint fivefold variety Ω , and the lines on Ω passing through P form a cubic cone S_P on Ω . If there are exactly two lines, say l and l' on V passing through v , then exactly one of these lines, say l is multiple, meaning that $V = h^\perp \cap \Omega$ is tangent along l to S_P . So l and l' are τ -invariant, and hence $v \in l \subset \Pi$. \square

Lemma 2.3. *No line on V is contained in V^τ .*

Proof. Suppose a line l on V is pointwise fixed by τ . By Proposition 2.1(iv) l is a ruling of Π . Then any ruling of Π is pointwise fixed by τ because of the rigidity of the τ -action, and so $\tau|_\Pi = \text{id}_\Pi$.

By Propositions 1.1(xi) and Theorem 1.5(ii) there exists on V a pair $(S_0, \tau(S_0))$ of disjoint Aut^0 -invariant cubic cones. Then a general cubic scroll $S \in \mathcal{S}_1$ and its τ -dual scroll $\tau(S) \in \mathcal{S}_2$ are disjoint as well. Each of them intersects Π , see the last paragraph in the proof of Proposition 2.1. Since $\tau|_\Pi = \text{id}_\Pi$ any point $x \in S \cap \Pi$ is fixed by τ , hence it is a common point of S and $\tau(S)$. The latter contradicts the fact that S and $\tau(S)$ are disjoint. \square

Corollary 2.4. *$s|_{\tilde{\Pi}}: \tilde{\Pi} \rightarrow \Pi$ is a birational morphism.*

Proof. We have to show that $s|_{\tilde{\Pi}}: \tilde{\Pi} \rightarrow \Pi$ has degree 1. Notice that two distinct points of the curve $C \subset \Sigma$ correspond to distinct lines on V . Hence, two distinct rulings of $\tilde{\Pi}$ project under s into two distinct rulings of Π . Assume on the contrary that $\deg(s|_{\tilde{\Pi}}) = m \geq 2$, that is, through a general point P of Π pass $m \geq 2$ distinct rulings of Π . Then P is fixed by τ and therefore, $\tau|_\Pi = \text{id}_\Pi$. The latter contradicts Lemma 2.3. \square

Lemma 2.5. *The Euler characteristic $e(V^\tau)$ equals 4.*

Proof. Given a smooth compact manifold M and a periodic diffeomorphism f of M , due to a version of the Lefschetz index theorem, see e.g. [tD79, Proposition 5.3.11], the Euler characteristic of the fixed point set M^f equals the Lefschetz number $L(f, M)$. Since $H^2(V, \mathbb{Z}) = \text{Pic}(V) = \mathbb{Z}$ and the trace of τ_* acting on $H^4(V, \mathbb{Z}) = \mathbb{Z}[S] \oplus \mathbb{Z}[\tau(S)]$ vanishes we have $e(V^\tau) = L(\tau, V) = 4$. \square

The next proposition is the main result of this subsection.

Proposition 2.6.

- (i) *The fixed point set $V^\tau = \Pi^\tau$ is the union of two disjoint smooth rational curves E_1, E_2 on Π . Each of them meets any ruling of Π in a unique point. Through any point of $\Pi \setminus (E_1 \cup E_2)$ passes a unique ruling of Π .*
- (ii) *The morphism $s|_{\tilde{\Pi}}: \tilde{\Pi} \rightarrow \Pi$ is bijective over $\Pi \setminus (E_1 \cup E_2)$.*

Proof. (i) Recall that $\rho|_{\tilde{\Pi}}: \tilde{\Pi} \rightarrow C$ is the projection of a smooth rational surface scroll. By Lemma 2.3 any ruling \tilde{r} of $\tilde{\Pi}$ is τ -invariant and $\tau|_{\tilde{r}}$ is not identical. Hence $\tau|_{\tilde{r}}$ has exactly two fixed points. So, the projection $\tilde{\Pi}^\tau \rightarrow C \cong \mathbb{P}^1$ is two-sheeted and unramified. It follows that $\tilde{\Pi}^\tau$ is a disjoint union of two sections, say \tilde{E}_1 and \tilde{E}_2 of the projection $\rho|_{\tilde{\Pi}}: \tilde{\Pi} \rightarrow C$, where \tilde{E}_1 and \tilde{E}_2 are smooth rational curves.

We have $V^\tau = \Pi^\tau = s(\tilde{\Pi}^\tau) = E_1 \cup E_2$ where $E_i = s(\tilde{E}_i)$, see Lemma 2.2. The fixed point set of a reductive group action on a smooth variety is smooth, so $V^\tau = E_1 \cup E_2$ is. By Lemma 2.5 $e(\Pi^\tau) = 4 = e(\tilde{\Pi}^\tau)$, hence $E_1 \neq E_2$, $E_1 \cap E_2 = \emptyset$ and E_i is a smooth rational curve for $i = 1, 2$. Since s maps any ruling in $\tilde{\Pi}$ isomorphically to a ruling in Π

and maps the section \tilde{E}_i of $\tilde{\Pi} \rightarrow C$ onto E_i for $i = 1, 2$ then E_i intersects any ruling of Π at a single point.

Suppose that through a point $p \in \Pi \setminus (E_1 \cup E_2) = \Pi \setminus \Pi^\tau$ pass two rulings of Π . These rulings are τ -invariant, and so, they also pass through the point $\tau(p) \neq p$, a contradiction. Thus, any multi-branched point of Π is contained in $E_1 \cup E_2$. This proves (i).

(ii) Since $s|_{\tilde{\Pi}}: \tilde{\Pi} \rightarrow \Pi$ restricted to any ruling of $\tilde{\Pi}$ is bijective and by (i) no two rulings of Π intersect outside $E_1 \cup E_2$, $s|_{\tilde{\Pi}}: \tilde{\Pi} \rightarrow \Pi$ is bijective over $\Pi \setminus (E_1 \cup E_2)$. \square

2.2. Winding families of scrolls. Any line l on V is a common ruling of a unique pair of cubic scrolls, see Proposition 1.1(ix). For a ruling l of Π the corresponding pair has the form $(S_l, \tau(S_l))$ where $S_l \in \mathcal{S}_1$ and $\tau(S_l) \in \mathcal{S}_2$. The cubic scrolls $(S_l)_{l \in C}$ roll around Π and $(\tau(S_l))_{l \in C}$ is a second such family. There is also a two-parameter family of τ -invariant sextic scrolls $(D_S)_{S \in \mathcal{S}_1 \setminus C_2}$ wrapping around Π ; we'll use this one to show that Π is smooth.

By Proposition 1.1(xi) there exists on V a pair $(S, \tau(S))$ of disjoint Aut^0 -invariant cubic cones. To every such pair there corresponds a unique Aut^0 -invariant rational sextic scroll, see [PZ20, Lemma 2.2]. More generally, we have the following facts.

Proposition 2.7. *Assume that a cubic scroll S on V contains no ruling of Π ¹. Then the following hold.*

- (i) S and $\tau(S)$ are disjoint.
- (ii) Let A_S be as in Proposition 1.1(viii). Then $\Gamma_S := S \cap \tau(A_S)$ is a rational twisted cubic curve on S .
- (iii) The twisted cubic curves Γ_S and $\tau(\Gamma_S)$ are disjoint sections of a unique smooth rational sextic scroll D_S on V . Any line on V meeting both Γ_S and $\tau(\Gamma_S)$ is a ruling of D_S , and vice versa. The scroll D_S is the image of $\mathbb{P}^1 \times \mathbb{P}^1$ embedded in \mathbb{P}^{12} by a linear system of bidegree $(1, 3)$.
- (iv) The scroll D_S is τ -invariant and carries exactly 2 distinct τ -invariant rulings r_1, r_2 and exactly 4 fixed points of τ . Namely, the intersection

$$V^\tau \cap D_S = \Pi^\tau \cap D_S = (E_1 \cup E_2) \cap (r_1 \cup r_2)$$

consists of the τ -fixed points $p_{i,j} = E_i \cdot r_j$, $i, j = 1, 2$.

- (v) Every line on $A_S \cap \tau(A_S)$ is a ruling of D_S .

Proof. (i) Suppose $S \cap \tau(S) \neq \emptyset$. Since S and $\tau(S)$ belong to different components of the Hilbert scheme \mathcal{S} they share a unique common ruling r , see Proposition 1.1(ix). Being unique, r is τ -invariant, hence r is a common ruling of Π and S contrary to our assumption. Thus, $S \cap \tau(S) = \emptyset$, as stated in (i).

(ii)–(iii) The proof of Lemma 2.2(i)–(iv) in [PZ20] goes mutatis in our more general setup. This proves (ii) and the first two statements in (iii).

To show the remaining statement of (iii) suppose on the contrary that D_S is isomorphic to a Hirzebruch surface \mathbb{F}_e where $e > 0$. Let E_0 be the exceptional section of D_S and f be a ruling of D_S . Then on $D_S \cong \mathbb{F}_e$ one has

$$\Gamma_S \sim E_0 + \alpha f \sim \tau(\Gamma_S) \quad \text{for some } \alpha \geq e > 0.$$

It follows that

$$0 = \Gamma_S \cdot \tau(\Gamma_S) = \Gamma_S^2 = 2\alpha - e \quad \text{and so } e = 2\alpha \leq \alpha,$$

a contradiction. Therefore, $e = 0$, that is, $D_S \cong \mathbb{P}^1 \times \mathbb{P}^1$. Now (iii) follows.

¹Since any cubic cone S on V verifies this assumption, see Lemma 2.2, so does the general cubic scroll.

(iv) The τ -invariance of D_S in (iv) follows from the facts that the pair $(S, \tau(S))$ is τ -invariant and D_S is uniquely defined by S due to (iii). The rulings of D_S form a unique pencil of lines on D_S , hence this pencil is τ -invariant. The induced action of τ on this pencil is not identical, since otherwise any ruling of D_S is τ -invariant, and so, $D_S = \Pi$. However, the latter is impossible because $\deg(D_S) = 6$ and $\deg(\Pi) = 12$, see Proposition 2.1.

Thus, the action of τ on \mathbb{P}^1 parameterizing the pencil has exactly 2 fixed points, that is, there are exactly 2 τ -invariant rulings of D_S , say r_1 and r_2 . By Lemma 2.3 the τ -action on r_i is not identical. So, r_i contains exactly 2 fixed points of τ for $i = 1, 2$. If a ruling r of D_S different from r_1 and r_2 carries a fixed point x of τ then $\tau(r)$ is as well a ruling of D_S passing through x . Since the scroll D_S is smooth we have $r = \tau(r)$. Hence r is a third τ -invariant ruling of D_S , a contradiction. Therefore, the points in $(r_1 \cup r_2) \cap (E_1 \cup E_2)$ are the only τ -fixed points on D_S .

(v) Any line l on $A_S \cdot \tau(A_S)$ meets S and $\tau(S)$ in points of Γ_S and $\tau(\Gamma_S)$, respectively, see Proposition 1.1(viii). Due to (iii) l is a ruling of D_S . \square

Remark 2.8. Assume that S and $\tau(S)$ are disjoint. One can show that the codimension 2 linear section $A_S \cap \tau(A_S)$ of V is a union of 3 sextic surfaces D_S , R_S and $\tau(R_S)$. In the case that $\text{Aut}^0(V) \cong \text{GL}(2, \mathbb{C})$, and only in this case, one has $R_S = D_S$, so that $A_S \cdot \tau(A_S) = 3D_S$, cf. [PZ18, Theorem 13.5(c)].

Recall that the Hilbert scheme C of rulings of Π is a smooth rational quartic curve in Σ contained in the diagonal of $\mathcal{S}_1 \times \mathcal{S}_2 \cong \mathbb{P}^2 \times (\mathbb{P}^2)^\vee$, see Proposition 2.1. The projection $C_i = \text{pr}_i(C) \subset \mathcal{S}_i$ is a smooth conic on $\mathcal{S}_i \cong \mathbb{P}^2$, $i = 1, 2$.

Corollary 2.9.

- (i) A pair of cubic scrolls $(S, \tau(S)) \in \mathcal{S}_1 \times \mathcal{S}_2$ shares a unique common ruling $r = r(S)$ if and only if $S \in C_1$. In the latter case r is a ruling of Π .
- (ii) Let $S \in \mathcal{S}_1 \setminus C_1$ and let D_S be the smooth rational sextic scroll as in Proposition 2.7(iii). Then $D_S \cap \Pi = r_1(S) \cup r_2(S)$ is a union of two distinct rulings.
- (iii) Let $C^{(2)}$ stand for the symmetric square of C and δ stand for the diagonal of $C^{(2)}$. The correspondence

$$(3) \quad \mathcal{S}_1 \setminus C_1 \ni S \mapsto \{r_1(S), r_2(S)\} \in C^{(2)} \setminus \delta \cong \mathbb{P}^2 \setminus \{a \text{ smooth conic}\}$$

is an isomorphism.

Proof. (i) By virtue of Proposition 1.1(ix) S and $\tau(S)$ intersect if and only if they share a unique common ruling. This ruling, say r , is τ -invariant, hence is a ruling of Π , see Proposition 2.1. Thus, $r \in C$, and so $S = \text{pr}_1(r) \in C_1$, see again Proposition 1.1(ix). The converse is immediate from Proposition 2.7(i).

Statement (ii) follows from (i) due to Proposition 2.7(iv) and its proof.

To show (iii) recall that the lines on V meeting S sweep out a hyperplane section A_S of V singular along S , see Proposition 1.1(viii). The Hilbert scheme $\Sigma(S)$ of these lines is the preimage $\text{pr}_2^{-1}(h_S)$ where $h_S \subset \mathcal{S}_2 = (\mathcal{S}_1)^\vee$ is the dual line of the point $S \in \mathcal{S}_1$, see Proposition 1.1(x). The smooth conic

$$\sigma_S := \Sigma(S) \cap \Sigma(\tau(S)) = \text{pr}_2^{-1}(h_S) \cap \text{pr}_1^{-1}(h_{\tau(S)})$$

parameterizes the lines on V which meet both S and $\tau(S)$, that is, the lines on $A_S \cap A_{\tau(S)}$. By Proposition 2.7(v) these are the rulings of D_S .

The line $h_S \subset \mathcal{S}_2$ meets the smooth conic $C_2 = \text{pr}_2(C)$ in an unordered pair of points. Hence $\Sigma(S) = \text{pr}_2^{-1}(h_S)$ meets C in an unordered pair of points $\{r_1(S), r_2(S)\}$. The corresponding rulings $r_1(S), r_2(S)$ of Π intersect S . Since these rulings are τ -invariant

and so is the scroll D_S , these are the common rulings of Π and D_S . By (i) $r_1(S) = r_2(S)$ if and only if $S \in C_1$. This yields the morphism in (3).

To show that the correspondence in (3) is one-to-one, choose $\{r_1, r_2\} \in C^{(2)} \setminus \delta = \mathbb{P}^2 \setminus \{\text{a smooth conic}\}$. Let l be the line on \mathcal{S}_2 such that $l \cdot C_2 = \text{pr}_2(r_1) + \text{pr}_2(r_2)$. The point $S = l^\vee$ in the dual projective plane \mathcal{S}_1 corresponds to a unique cubic scroll such that D_S and Π share the common rulings r_1 and r_2 . Thus, the morphism in (3) is a bijection, so an isomorphism. \square

Remark 2.10. Notice that $r_1 = r_2$ if and only if D_S and Π are tangent along the unique common ruling $r_1 = r_2$, if and only if l is tangent to C_2 at the corresponding point, that is, S and $\tau(S)$ share a common ruling.

Proposition 2.11.

- (i) No two distinct rulings of Π meet.
- (ii) $s|_{\tilde{\Pi}}: \tilde{\Pi} \rightarrow \Pi$ is a bijective normalization morphism.
- (iii) $\deg(E_1) + \deg(E_2) = 12$.
- (iv) Any line on V that intersects both E_1 and E_2 is a ruling of Π .

Proof. (i) The unordered pairs of distinct rulings of Π are in bijection with the points of $C^{(2)} \setminus \delta$, see (3). Every such pair corresponds to a pair of rulings $\{r_1, r_2\}$ of a smooth sextic scroll D_S for some $S \in \mathcal{S}_1 \setminus C_1$, see Proposition 2.9(iii). However, D_S is smooth and therefore, no two distinct rulings of D_S meet, see Proposition 2.7(iii).

(ii) Since $\tilde{\Pi}$ is smooth and $s|_{\tilde{\Pi}}: \tilde{\Pi} \rightarrow \Pi$ is bijective due to (i), the latter is a normalization morphism.

(iii) Recall that $\deg(\Pi) = 12$ and that \tilde{E}_i and E_i are smooth rational curves, see Propositions 2.1(iii) and 2.6(i). Hence by (ii) $s|_{\tilde{E}_i}: \tilde{E}_i \rightarrow E_i$ is an isomorphism. Now (iii) follows from the classical Edge formula [Edge31, Section I, p. 17], see [Har92, Example 19.5] for a modern treatment.

(iv) Let l be a line on V which meets E_i in a point P_i , $i = 1, 2$. Since $\tau(P_i) = P_i$ we have $P_i \in \tau(l)$ for $i = 1, 2$, and so $\tau(l) = l$. Now the result follows from Proposition 2.1(iv). \square

2.3. Numerical data of Π .

Lemma 2.12. *Given a cubic scroll $S \in \mathcal{S}_1$ let A_S be the associated hyperplane section of V singular along S , see Proposition 1.1(viii). Then*

$$(4) \quad A_S \cdot \Pi = n_1 r_1 + n_2 r_2 + \gamma_S$$

where $n_i = n_i(S) \geq 1$, γ_S is a section of Π and the rulings r_1 and r_2 of Π are disjoint if $S \in \mathcal{S}_1 \setminus C_1$ and equal if $S \in C_1$.

Proof. By Corollary 2.9 S and $\tau(S)$ are disjoint if $S \in \mathcal{S}_1 \setminus C_1$ and share a unique common ruling r otherwise, where r is a ruling of Π . In the former case the corresponding sextic scroll $D_S \subset A_S \cap \tau(A_S)$ shares with Π precisely two common rulings r_1 and r_2 , see Proposition 2.7(iv). These rulings r_1 and r_2 participate in the 1-cycle $\Upsilon_S = A_S \cdot \Pi$ with positive multiplicities n_1 and n_2 , respectively. No third ruling r of Π is contained in A_S . Indeed, otherwise $r = \tau(r) \subset A_S \cap \tau(A_S)$ (see Proposition 2.1) would be a third common ruling of D_S and Π , which contradicts Proposition 2.9(iv). Thus, the residual effective 1-cycle γ_S in (4) is reduced and a section of Π . \square

Lemma 2.13. *For the general cubic scroll $S \in \mathcal{S}_1$ one has $n_1 = n_2$ in (4).*

Proof. By Proposition 2.9 the rulings $r_1(S)$ and $r_2(S)$ are distinct for $S \in \mathcal{S}_1 \setminus C_1$. The unordered pair of coefficients $\{n_1(S), n_2(S)\}$ in (4) is constant for S from a suitable open subset $U \subset \mathcal{S}_1 \setminus C_1$. Identifying $\mathcal{S}_1 \setminus C_1$ with $C^{(2)} \setminus \delta = \mathbb{P}^2 \setminus \delta$ via (3) consider the 2-sheeted covering $(\mathbb{P}^1 \times \mathbb{P}^1) \setminus \tilde{\delta} \rightarrow \mathbb{P}^2 \setminus \delta$, where $\tilde{\delta}$ stands for the diagonal in $\mathbb{P}^1 \times \mathbb{P}^1$. The monodromy of the covering restricted to U interchanges the members of the ordered pair of points from $(\mathbb{P}^1 \times \mathbb{P}^1) \setminus \tilde{\delta}$. It follows that $n_1(S) = n_2(S)$ for $S \in U$. \square

According to Proposition 2.11 $\tilde{\Pi}$ is isomorphic to a Hirzebruch surface \mathbb{F}_e for some $e \geq 0$.

Lemma 2.14. *The following conditions are equivalent:*

- $e = 0$, that is, $\tilde{\Pi} \cong \mathbb{P}^1 \times \mathbb{P}^1$;
- $\deg(E_1) = \deg(E_2) = 6$.

These conditions are fulfilled provided $n_1 = n_2 = 3$ in (4) for some cubic scroll $S \in \mathcal{S}_1$ such that $\gamma_S \neq E_i$ for $i = 1, 2$.

Proof. Notice that $\tilde{\gamma}_S \neq \tilde{E}_i$ for a general $S \in \mathcal{S}_1$, $i = 1, 2$. Indeed, otherwise $E_i \subset A_S \cap \tau(A_S)$ for any $S \in \mathcal{S}_1$ contrary to Proposition 1.1(xii).

Assume that $e > 0$, and let \tilde{E}_0 be the exceptional section of $\tilde{\Pi} \cong \mathbb{F}_e$. Since τ preserves each ruling of $\tilde{\Pi}$ and $\tau(\tilde{E}_0) = \tilde{E}_0$ one has $\tau|_{\tilde{E}_0} = \text{id}_{\tilde{E}_0}$. So, \tilde{E}_0 is one of the components of the fixed point set $\tilde{\Pi}^\tau = \tilde{E}_1 \cup \tilde{E}_2$; we may assume that $\tilde{E}_0 = \tilde{E}_1$.

Let now $e \geq 0$. Letting F be a ruling of $\tilde{\Pi}$ we have $\tilde{E}_2 \sim \tilde{E}_1 + eF$. Since $(\tilde{A}_S|_{\tilde{\Pi}})^2 = 12$ and the linear system $|\tilde{A}_S|$ restricted to $\tilde{\Pi}$ is base point free, see Proposition 2.11(ii), one has $\tilde{A}_S|_{\tilde{\Pi}} \sim \tilde{E}_1 + aF$ where $a = 6 + e/2 \geq e$. It follows that

$$(5) \quad \deg(E_1) = \tilde{A}_S \cdot \tilde{E}_1 = 6 - e/2 \quad \text{and} \quad \deg(E_2) = \tilde{A}_S \cdot \tilde{E}_2 = 6 + e/2,$$

what is in line with Proposition 2.11(iii). Therefore, $e = 0$ if and only if $\deg(E_1) = \deg(E_2) = 6$. Thus, the two conditions of the lemma are equivalent.

Using Proposition 2.1(iii) and Lemmas 2.12–2.13 and letting $n = n_1 = n_2$ in (4) we obtain

$$12 = \deg(\Pi) = \deg(\Upsilon_S) = 2n + \deg(\gamma_S).$$

Let

$$\tilde{A}_S = s^*(A_S), \quad \tilde{\Upsilon}_S = \tilde{A}_S \cdot \tilde{\Pi}, \quad \tilde{\gamma}_S = s^*(\gamma_S) \cdot \tilde{\Pi} \quad \text{and} \quad \tilde{r}_i = s^*(r_i) \cdot \tilde{\Pi}.$$

Thus, $\tilde{\Upsilon}_S$, $\tilde{\gamma}_S$ and \tilde{r}_i are the proper transforms of Υ_S , γ_S and r_i under the bijective morphism $s|_{\tilde{\Pi}}: \tilde{\Pi} \rightarrow \Pi$. Using the Projection Formula, Proposition 2.11(iii) and (4) we deduce

$$\tilde{\Upsilon}_S = n(\tilde{r}_1 + \tilde{r}_2) + \tilde{\gamma}_S, \quad 12 = \tilde{A}_S \cdot \tilde{\Upsilon}_S = 2n + \deg(\gamma_S)$$

and

$$12 = \tilde{A}_S \cdot (\tilde{E}_1 + \tilde{E}_2) = (\tilde{\Upsilon}_S \cdot (\tilde{E}_1 + \tilde{E}_2))_{\tilde{\Pi}} = 4n + (\tilde{\gamma}_S \cdot (\tilde{E}_1 + \tilde{E}_2))_{\tilde{\Pi}}.$$

It follows that $n \leq 3$ for $\gamma_S \neq E_1$ and $n = 3$ if and only if $\deg \tilde{\gamma}_S = 6$.

Let now $S \in \mathcal{S}_1$ be such that $n_1 = n_2 = 3$ and γ_S in (4) is different from E_1 and E_2 . Then $(\tilde{\gamma}_S \cdot \tilde{E}_i)_{\tilde{\Pi}} = 0$, therefore \tilde{E}_1 , \tilde{E}_2 and $\tilde{\gamma}_S$ are 3 disjoint sections of $\tilde{\Pi} \rightarrow C$. Hence $e = 0$, these 3 sections are constant sections of the projection $\tilde{\Pi} \cong \mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{\text{pr}_2} \mathbb{P}^1$, while the other projection $\tilde{\Pi} \cong \mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{\text{pr}_1} \mathbb{P}^1$ defines the ruled surface structure on $\tilde{\Pi}$. Thus, the both conditions of the lemma are fulfilled. \square

3. STARTING THE PROOF OF THE MAIN THEOREM

The following is a weaker version of Theorem 0.1.

Theorem 3.1. *Let V be a Fano-Mukai fourfold of genus 10 half-anticanonically embedded in \mathbb{P}^{12} , and let $\tau \in \text{Aut}(V) \setminus \text{Aut}^0(V)$ be an involution. Then the fixed point set V^τ is a union of two disjoint smooth rational sextic curves E_1 and E_2 . Furthermore, there is a surface scroll $\Pi = \Pi(V, \tau)$ in V verifying the following.*

- (i) *Each ruling of Π is τ -invariant and each τ -invariant line on V is a ruling of Π .*
- (ii) *Π has degree 12 and its normalization $\tilde{\Pi}$ is a rational normal scroll isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ and embedded in \mathbb{P}^{13} by the complete linear system of type $(1, 6)$. The normalization morphism $\tilde{\Pi} \rightarrow \Pi$ is bijective.*
- (iii) *An isomorphism $\tilde{\Pi} \cong \mathbb{P}^1 \times \mathbb{P}^1$ sends the proper transforms \tilde{E}_i of E_i on $\tilde{\Pi}$, $i = 1, 2$ into constant sections of the projection $\text{pr}_1: \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$.*

It suffices to show that $\deg(E_i) = 6$ for $i = 1, 2$, the remaining assertions being already established in Propositions 2.6, 2.11 and Lemma 2.14 or follow immediately from these. We prove the equalities $\deg(E_i) = 6$, $i = 1, 2$ separately for each type of the group $\text{Aut}^0(V)$, see Propositions 3.2, 3.3 and 3.4. Once we know that these equalities hold provided $\text{Aut}^0(V) = \text{GL}(2, \mathbb{C})$, the other two cases where $\text{Aut}^0(V) = \mathbb{G}_m^2$ and $\text{Aut}^0(V) = \mathbb{G}_a \times \mathbb{G}_m$ are reduced to this one by a specialization argument.

3.1. The case $\text{Aut}^0(V) = \text{GL}(2, \mathbb{C})$.

Proposition 3.2. *Suppose $\text{Aut}^0(V) = \text{GL}(2, \mathbb{C})$. Then $\deg(E_1) = \deg(E_2) = 6$.*

Proof. Let $S \in \mathcal{S}_1$ be an $\text{Aut}^0(V)$ -invariant cubic cone on V . There is a unique such cone in \mathcal{S}_1 , and so $\tau(S)$ is a unique $\text{Aut}^0(V)$ -invariant cubic cone in \mathcal{S}_2 , see Proposition 1.4. According to [PZ18, Theorem 13.5(c)] one has $A_S \cdot A_{\tau(S)} = 3D_S$. Hence

$$(6) \quad A_S \cdot A_{\tau(S)} \cdot \Pi = 3D_S \cdot \Pi = 3(r_1 + r_2).$$

It follows by (4) that $A_S \cdot \Pi = 3(r_1 + r_2) + \gamma_S$, that is, $n_1 = n_2 = 3$. We claim that γ_S is different from E_1 and E_2 . Indeed, otherwise $\tau(\gamma_S) = \gamma_S$, and therefore $\gamma_S \subset A_S \cap A_{\tau(S)} \cap \Pi = D_S \cap \Pi$. The latter contradicts (6), which proves our claim. Now Lemma 2.14 applies and gives the result. \square

3.2. The case $\text{Aut}^0(V) = \mathbb{G}_m^2$. In this case we use the Mukai presentation of the Fano-Mukai fourfolds of genus 10, see subsection 1.2.

Proposition 3.3. *Assume $\text{Aut}^0(V) = \mathbb{G}_m^2$. Then $\deg(E_1) = \deg(E_2) = 6$.*

Proof. Let T be a maximal torus of G and $\mathfrak{h} = \text{Lie}(T)$ be the corresponding Cartan subalgebra of $\mathfrak{g} = \text{Lie}(G)$. The T -action on $\mathbb{P}\mathfrak{g} = \mathbb{P}^{13}$ fixes pointwise the projective line $\mathbb{P}\mathfrak{h}$. Hence $T \subset \text{Aut}(V(h)) = \text{Stab}_G(\mathbb{P}h)$ for any point $\mathbb{P}h \in \mathbb{P}\mathfrak{h}$. If $h \in \mathfrak{g} \setminus \{0\}$ is semisimple then $\text{Aut}(V(h))$ contains a maximal torus of G . Since the maximal tori in G are conjugate, the line $\mathbb{P}\mathfrak{h}$ intersects each semisimple G -orbit in $\mathbb{P}\mathfrak{g}$ including Ω^{sr} and does not intersect the orbit Ω^a ; cf. [KR13, Proof of Lemma 1] and [PZ22, Corollary 2.5.1].

Let $\mathcal{F} \rightarrow \mathbb{P}\mathfrak{g}$ be the universal family of hyperplane sections of Ω . The restriction

$$\mathcal{F}_T := \mathcal{F}|_{\mathbb{P}\mathfrak{h} \setminus D_\ell} \rightarrow \mathbb{P}\mathfrak{h} \setminus D_\ell \cong \mathbb{P}^1 \setminus \{3 \text{ points}\},$$

see [PZ22, Lemma 2.8], is a smooth one-parameter family of Fano-Mukai fourfolds of genus 10 consisting of the smooth hyperplane sections $V(h)$ of Ω which satisfy $T \subset \text{Aut}(V(h))$ where $\text{Aut}(V(h))$ coincides with the stabilizer $\text{Stab}_G(\mathbb{P}h)$.

Let $N_G(T)$ stand for the normalizer of T in G and let $W = N_G(T)/T$ be the Weyl group of G . One has $W \cong \mathfrak{D}_6$ where \mathfrak{D}_6 is the dihedral group of order 12. By [AH17] there is a splitting $N_G(T) = T \rtimes W$. Let $\tau \in N_G(T)$ be the central element of order 2 in $W \subset N_G(T)$. Then τ acts on the Cartan subalgebra \mathfrak{h} via the central symmetry $h \mapsto -h$, acts on T via the inversion $t \mapsto t^{-1}$ and acts identically on the line $\mathbb{P}\mathfrak{h} \subset \mathbb{P}\mathfrak{g}$. For any point $\mathbb{P}h \in \mathbb{P}\mathfrak{h}$ the torus T acts effectively on $V(h)$ and τ acts on Ω via an involution preserving $V(h)$. Thus,

$$\mathrm{Aut}(V(h)) = \mathrm{Stab}_G(\mathbb{P}h) \supset \langle T, \tau \rangle \cong T \rtimes \mathbb{Z}/2\mathbb{Z}.$$

Hence τ acts on the total space of the family $\mathcal{F}_T \rightarrow \mathbb{P}\mathfrak{h} \setminus D_\ell$ and acts effectively on the fiber $V(h)$ for any $\mathbb{P}h \in \mathbb{P}\mathfrak{h} \setminus D_\ell$.

Since $\langle \tau \rangle \cong \mathbb{Z}/2\mathbb{Z}$ is a reductive group the fixed point set $(\mathcal{F}_T)^\tau$ is a smooth subvariety of the smooth variety \mathcal{F}_T . Any fiber $V(h)^\tau$ of $(\mathcal{F}_T)^\tau \rightarrow \mathbb{P}\mathfrak{h} \setminus D_\ell$ is a union of two disjoint smooth rational curves $E_1(h), E_2(h)$ where $\deg(E_1(h) + E_2(h)) = 12$. Hence $(\mathcal{F}_T)^\tau$ is a smooth rational surface fibered over $\mathbb{P}\mathfrak{h} \setminus D_\ell = \mathbb{P}^1 \setminus \{3 \text{ points}\}$ into a family of smooth reducible curves $E_1(h) \cup E_2(h)$ of degree 12. It follows that the lower semicontinuous function $\deg(E_i(h))$ on $\mathbb{P}\mathfrak{h} \setminus D_\ell$ is constant for $i = 1, 2$. For a point $h_0 \in \mathbb{P}\mathfrak{h} \cap D_s = \mathbb{P}\mathfrak{h} \cap \Omega^{\mathrm{sr}}$ with $\mathrm{Aut}^0(V(h_0)) \cong \mathrm{GL}(2, \mathbb{Z})$ one has

$$\mathrm{Aut}(V(h_0)) = \mathrm{Aut}^0(V(h_0)) \rtimes \langle \tau \rangle \cong \mathrm{GL}(2, \mathbb{Z}) \rtimes \mathbb{Z}/2\mathbb{Z},$$

see [PZ22, Theorem A]. By Proposition 3.2 one has $\deg(E_1(h_0)) = \deg(E_2(h_0)) = 6$. Therefore, $\deg(E_1(h)) = \deg(E_2(h)) = 6$ for any $\mathbb{P}h \in \mathbb{P}\mathfrak{h} \setminus D_\ell$.

The same equalities hold for any $\mathbb{P}h' \in \mathbb{P}\mathfrak{g} \setminus (D_\ell \cup D_s)$, that is, for any h' with $\mathrm{Aut}^0(V(h')) \cong \mathbb{G}_m^2$. Indeed, the projective line $\mathbb{P}\mathfrak{h}$ meets any G -orbit $D_t \setminus D_\ell$, $t \notin \{\ell, s\}$, see Theorem 1.2(ii) and [PZ22, Corollary 3.5.1]. Hence the G -orbit of $\mathbb{P}h'$ contains a point $\mathbb{P}h \in \mathbb{P}\mathfrak{h} \setminus (D_\ell \cup D_s)$. Thus, there is an element $g \in G \subset \mathrm{PGL}(14, \mathbb{C})$ which sends $\mathbb{P}h$ to $\mathbb{P}h'$, $V(h)$ isomorphically onto $V(h')$, τ to an involution $\tau' \in \mathrm{Aut}(V(h')) \setminus \mathrm{Aut}^0(V(h'))$, and $E_i(h)$ to $E_i(h')$, $i = 1, 2$, up to switching the order.

By Corollary 1.7 the choice of an involution $\tau \in \mathrm{Aut}(V) \setminus \mathrm{Aut}^0(V)$ is irrelevant for our purposes. Therefore, the above argument proves the proposition. \square

3.3. The case $\mathrm{Aut}^0(V) = \mathbb{G}_a \times \mathbb{G}_m$. The proof in this case is similar to the preceding one. We construct a one-parameter family of Fano-Mukai fourfolds $\{V_t\}$ with $\mathrm{Aut}^0(V_t) = \mathbb{G}_a \times \mathbb{G}_m$ which specializes to V_0 with $\mathrm{Aut}^0(V_0) = \mathrm{GL}(2, \mathbb{C})$ and then apply the same argument as before.

Proposition 3.4. *Assume $\mathrm{Aut}^0(V) = \mathbb{G}_a \times \mathbb{G}_m$. Then $\deg(E_1) = \deg(E_2) = 6$.*

Proof. According to [KR13, Lemma 1 and its proof], see also Theorem 1.2(iii) there exist commuting elements $s, n \in \mathfrak{g}$ such that s is subregular semisimple, n is nilpotent and $g = s + n$ is regular. We have $\mathbb{P}s \in \Omega^{\mathrm{sr}}$ and $\mathbb{P}(s + n) \in \Omega^{\mathrm{a}}$. It follows from the description of G -orbits in \mathfrak{g} , see Theorem 1.2, that the same inclusions hold if one replaces n by λn with $\lambda \in \mathbb{C} \setminus \{0\}$ and $s + n$ by $s + \lambda n \in \mathfrak{f} := \mathrm{span}(s, n)$.

Notice that \mathfrak{f} is a two-dimensional abelian Lie subalgebra of \mathfrak{g} . Consider the projective line $\mathbb{P}\mathfrak{f} \subset \mathbb{P}\mathfrak{g}$. Let $\omega = \mathbb{P}\mathfrak{f} \setminus \{\mathbb{P}n\}$ be the image in $\mathbb{P}\mathfrak{g}$ of the affine line $\{s + \lambda n \mid \lambda \in \mathbb{C}\}$ and let $\omega^{\mathrm{a}} = \omega \setminus \{\mathbb{P}s\}$. Since s is subregular, any regular element of $\mathfrak{f} \setminus \{\lambda n \mid \lambda \in \mathbb{C}\}$ is mixed. Since n is nilpotent one has $\mathbb{P}n \in D_\ell \cap D_s$, see Theorem 1.2(i). It follows that

$$\omega = \mathbb{P}\mathfrak{f} \setminus D_\ell \subset D_s \setminus D_\ell \quad \text{and} \quad \omega^{\mathrm{a}} = \mathbb{P}\mathfrak{f} \cap \Omega^{\mathrm{a}} = \omega \cap \Omega^{\mathrm{a}}.$$

Consider the smooth one-parameter family $\mathcal{F}_\omega := \mathcal{F}|_\omega \rightarrow \omega$ of hyperplane sections $V(h)$ of Ω with $\mathbb{P}h \in \omega$. For $\mathbb{P}h \in \omega$ we have by Theorem 1.3,

$$\mathrm{Aut}^0(V(h)) \cong \mathbb{G}_a \times \mathbb{G}_m \Leftrightarrow \mathbb{P}h \in \omega^a \quad \text{and} \quad \mathrm{Aut}^0(V(h)) \cong \mathrm{GL}(2, \mathbb{C}) \Leftrightarrow \mathbb{P}h = \mathbb{P}s \in \omega \setminus \omega^a.$$

Let

$$L = (\mathrm{Stab}_G(\mathbb{P}s))^0 = \mathrm{Aut}^0(V(s)) \cong \mathrm{GL}(2, \mathbb{C})$$

and

$$\tilde{L} = \mathrm{Stab}_G(\mathbb{P}s) = \mathrm{Aut}(V(s)) = L \rtimes \mathbb{Z}/2\mathbb{Z}.$$

For $\lambda \neq 0$ we let $g_\lambda = s + \lambda n$,

$$H_\lambda = (\mathrm{Stab}_G(\mathbb{P}g_\lambda))^0 = \mathrm{Aut}^0(V(g_\lambda)) \cong \mathbb{G}_a \times \mathbb{G}_m$$

and

$$\tilde{H}_\lambda = \mathrm{Stab}_G(\mathbb{P}g_\lambda) = \mathrm{Aut}(V(g_\lambda)) = H_\lambda \rtimes \mathbb{Z}/2\mathbb{Z}.$$

Thus, for $h \in \tilde{H}_\lambda$ one has $\mathrm{Ad}(h)(g_\lambda) = \mu g_\lambda$ for some $\mu \in \mathbb{C} \setminus \{0\}$. In view of the uniqueness of the Jordan decomposition we have $\mathrm{Ad}(h)(s) = \mu s$ and $\mathrm{Ad}(h)(n) = \mu n$. It follows that $\tilde{H}_\lambda \subset \tilde{L}$ and $H_\lambda \subset L$.

The centralizer $\mathrm{Cent}_{\mathfrak{g}}(g_\lambda)$ is contained in $\mathrm{Lie}(\mathrm{Stab}_G(\mathbb{P}g_\lambda))$. Hence also

$$\mathfrak{f} \subseteq \mathrm{Lie}(\mathrm{Stab}_G(\mathbb{P}g_\lambda)) = \mathrm{Lie}(H_\lambda).$$

Both \mathfrak{f} and $\mathrm{Lie}(H_\lambda)$ are Lie subalgebras of dimension 2, hence they coincide. Thus, $\mathrm{Lie}(H_\lambda) = \mathfrak{f}$ does not depend on λ , and therefore $H_\lambda = H \subset L$ for all $\lambda \in \mathbb{C} \setminus \{0\}$ where $H := H_1$.

The generator $\tau \in G$ of the factor $\mathbb{Z}/2\mathbb{Z}$ in the decomposition $\tilde{H}_\lambda = H \rtimes \mathbb{Z}/2\mathbb{Z}$ stabilizes the point $\mathbb{P}g_1$, that is, $\mathrm{Ad}(\tau)(s + n) = \pm(s + n)$. So $\mathrm{Ad}(\tau)(s) = \pm s$ and $\mathrm{Ad}(\tau)(n) = \pm n$ with the same signs in both cases. Therefore, $\mathrm{Ad}(\tau)(g_\lambda) = \pm g_\lambda$ for all $\lambda \in \mathbb{C}$. The latter means that τ fixes any point $\mathbb{P}g_\lambda \in D_s$ and acts via involution on the Fano-Mukai fourfold $V(g_\lambda)$ interchanging the pairs of disjoint $\mathrm{Aut}^0(V(g_\lambda))$ -invariant cubic cones on $V(g_\lambda)$. Furthermore, $\tau \in \tilde{L}$ serves as a generator of the factor $\mathbb{Z}/2\mathbb{Z}$ in a decomposition $\tilde{L} = L \rtimes \mathbb{Z}/2\mathbb{Z}$. Due to Theorem 1.5(iv) and Corollary 1.7, up to conjugation by an element of L , τ acts on $L = \mathrm{GL}(2, \mathbb{C})$ via the Cartan involution $g \mapsto (g^t)^{-1}$ and acts on H via the inversion. Thus, H coincides with one of the subgroups H^\pm from Proposition 1.6(v).

Finally, the involution τ acts on the total space of the smooth family $\mathcal{F}_\omega \rightarrow \omega$ preserving each fiber $V(g_\lambda)$ with $\mathbb{P}g_\lambda \in \omega$. The fixed point set $(\mathcal{F}_\omega)^\tau$ is a smooth surface fibered into a family of smooth reduced curves of degree 12, each curve being the union of two disjoint rational components $E_1(\lambda)$ and $E_2(\lambda)$. The pair of degrees of these curves is locally constant in the family and equals $(6, 6)$ for $\lambda = 0$, hence also for any $\lambda \in \mathbb{C}$. Since the fourfolds $V(h)$ with $\mathbb{P}h \in \Omega^a$ are pairwise isomorphic and $\omega^a \subset \Omega^a$, the same conclusion holds for any $\mathbb{P}h \in \Omega^a$. \square

4. SMOOTHNESS OF Π

Due to the following proposition Π is smooth, so the normalization morphism $\tilde{\Pi} \rightarrow \Pi$ is an isomorphism.

Proposition 4.1. *Given a Fano-Mukai fourfold V of genus 10 and an involution $\tau \in \mathrm{Aut}(V) \setminus \mathrm{Aut}^0(V)$, the scroll Π in τ -invariant lines on V is smooth.*

The proof of Proposition 4.1 is done after Lemmas 4.2 and 4.4.

Lemma 4.2.

$$(i) \quad (\mathrm{sing} \Pi) \cap (E_1 \cup E_2) = \emptyset.$$

(ii) $\dim(\text{sing } \Pi) = 0$.

Proof. (i) Pick a point $\tilde{P} \in \tilde{E}_i$ and let \tilde{r} be the ruling of $\tilde{\Pi}$ passing through \tilde{P} and $r = s(\tilde{r})$ be the ruling of Π passing through $P = s(\tilde{P})$. The restrictions $s|_{\tilde{E}_i}: \tilde{E}_i \rightarrow E_i$ and $s|_{\tilde{r}}: \tilde{r} \rightarrow r$ are isomorphisms. Hence the rank of the differential $d(s|_{\tilde{\Pi}})_{\tilde{P}}: T_{\tilde{P}}\tilde{\Pi} \rightarrow T_P\Pi$ equals 2, so $s|_{\tilde{\Pi}}$ is étale at \tilde{P} . Now (i) follows.

(ii) Assume that $\text{sing } \Pi$ contains an irreducible curve γ , and let $\tilde{\gamma} = (s|_{\tilde{\Pi}})^{-1}(\gamma)$. By virtue of (i), $\tilde{\gamma}$ does not meet $\tilde{E}_1 \cup \tilde{E}_2$. Then $\deg(\tilde{\gamma}) = 6$, so $\tilde{\gamma}$ is a constant section of the projection $\tilde{\Pi} \cong \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$. The morphism $s|_{\tilde{\gamma}}: \tilde{\gamma} \rightarrow \gamma$ is bijective, so étale at a general point $\tilde{Q} \in \tilde{\gamma}$. By the same argument as in the proof of (i), $s|_{\tilde{\Pi}}: \tilde{\Pi} \rightarrow \Pi$ is étale at \tilde{Q} . This contradicts our assumption $\gamma \subset \text{sing } \Pi$. \square

In the sequel we use the following notation.

Notation 4.3. The complete linear system of type $(1, 6)$ on $\mathbb{P}^1 \times \mathbb{P}^1$ embeds the quadric $\mathbb{P}^1 \times \mathbb{P}^1$ onto a rational normal scroll $\Pi_{\text{norm}} \subset \mathbb{P}^{13}$ of degree 12. The fibers of the projection $\text{pr}_1: \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$ are sent to rulings of Π_{norm} and the constant sections of pr_1 are sent to smooth rational normal sextic curves $\tilde{\gamma}_t$ on Π_{norm} . Any two different curves $\tilde{\gamma}_t$ and $\tilde{\gamma}_{t'}$ span two disjoint \mathbb{P}^6 's in \mathbb{P}^{13} . We identify Π_{norm} with the normalization $\tilde{\Pi}$ of Π . Thus, the linear projection $\pi: \mathbb{P}^{13} \rightarrow \mathbb{P}^k := \langle \Pi \rangle \subset \mathbb{P}^{12}$ with center $\mathbb{P}^{12-k} \subset \mathbb{P}^{13}$ defined by the linear system of hyperplane sections of $\Pi \subset \mathbb{P}^{12}$ is the normalization morphism. By Proposition 2.11(ii) π is a bijection. So, \mathbb{P}^{12-k} intersects no non-tangent secant line to Π_{norm} . Furthermore, it does not meet Π_{norm} because $\deg(\Pi_{\text{norm}}) = \deg(\Pi) = 12$.

Following [Har92], for a point $\tilde{Q} \in \Pi_{\text{norm}}$ we let $T_{\tilde{Q}}\Pi_{\text{norm}}$ and $\mathbb{T}_{\tilde{Q}}\Pi_{\text{norm}} \subset \mathbb{P}^{13}$ be the tangent plane and the projective tangent plane of Π_{norm} at \tilde{Q} , respectively. Clearly, the image $Q = \pi(\tilde{Q})$ is a singular point of Π if and only if the center \mathbb{P}^{12-k} of π intersects the projective tangent plane $\mathbb{T}_{\tilde{Q}}(\Pi_{\text{norm}})$. Recall the following well known facts.

Lemma 4.4.

- (i) Π_{norm} is cut out by quadric hypersurfaces in \mathbb{P}^{13} .
- (ii) Every projective line tangent to Π_{norm} and different from its rulings intersects Π_{norm} in a single point.
- (iii) Every projective plane tangent to Π_{norm} intersects Π_{norm} in a single ruling.
- (iv) Given a ruling \tilde{r} of Π_{norm} , the projective planes tangent to Π_{norm} along \tilde{r} sweep out a \mathbb{P}^3 that contains \tilde{r} .
- (v) Projective planes tangent to Π_{norm} sweep a quadruple $\text{Tan}(\Pi_{\text{norm}})$ (also swept by the one-parameter family of \mathbb{P}^3 's from (iv)).
- (vi) The secant variety $\text{Sec}(\Pi_{\text{norm}})$ is a fivefold of degree 45.

Proof. Statement (i) follows from [Har92, 9.11–9.12] and implies (ii), which in turn implies (iii). See [Edge31, Section I.51] for (iv)-(vi). \square

Proof of Proposition 4.1. The action of τ on Π lifts to a τ -action on the normalization Π_{norm} , which in turn extends to the ambient space \mathbb{P}^{13} as a linear involution. Let \tilde{E}_i be the preimage of E_i in Π_{norm} and let \tilde{r} be the preimage of a ruling r of Π . The fixed point set of τ acting on \mathbb{P}^{13} is the union of two disjoint \mathbb{P}^6 's that are the linear spans $\langle \tilde{E}_1 \rangle$ and $\langle \tilde{E}_2 \rangle$. Any ruling \tilde{r} of Π_{norm} is τ -invariant. The projection $\pi: \mathbb{P}^{13} \dashrightarrow \mathbb{P}^k$ is τ -equivariant.

If $Q \in \Pi$ is a singular point, then $Q \notin E_1 \cup E_2$, see Lemma 4.2(i). So, $\tau(Q) \neq Q$ is a second singular point of Π . The ruling r of Π that contains Q is τ -invariant, therefore $r = (Q\tau(Q))$. Let $\tilde{Q} \in \Pi_{\text{norm}}$ be the preimage of Q , \tilde{r} be the ruling of Π_{norm} passing through \tilde{Q} and $\tau(\tilde{Q}) = \tau(\tilde{Q}) \in \tilde{r}$ be the preimage of $\tau(Q)$. The projective tangent planes

$\mathbb{T}_{\bar{Q}}^2 \Pi_{\text{norm}}$ and $\mathbb{T}_{\tau(\bar{Q})}^2 \Pi_{\text{norm}}$ intersect precisely along \bar{r} and span a $\mathbb{P}^3 \subset \mathbb{P}^{13}$ that contains each projective tangent plane $\mathbb{T}_{\bar{Q}'} \Pi_{\text{norm}}$ for $\bar{Q}' \in \bar{r}$, see Lemma 4.4(iv).

The center \mathbb{P}^{12-k} of π is τ -invariant. Since $Q \in \Pi$ is a singular point, \mathbb{P}^{12-k} meets $\mathbb{T}_{\bar{Q}}^2 \Pi_{\text{norm}}$ at a point, say $P \neq Q$. Since $\tau(\bar{Q})$ is singular on Π , \mathbb{P}^{12-k} also meets $\mathbb{T}_{\tau(\bar{Q})}^2 \Pi_{\text{norm}}$ at $\tau(P)$. The projective line

$$(7) \quad l_Q = (P, \tau(P)) = \mathbb{P}^{12-k} \cap \mathbb{P}^3$$

does not meet \bar{r} and intersects each projective tangent plane $\mathbb{T}_{\bar{Q}'} \Pi_{\text{norm}}$ from the pencil of planes in \mathbb{P}^3 with base locus \bar{r} . It follows that each point $Q' = \pi(\bar{Q}') \in r$ is a singular point of Π . The latter contradicts Lemma 4.2(ii). \square

5. LINEAR NONDEGENERACY OF Π IN \mathbb{P}^{12}

What is left to complete the proof of Theorem 0.1? We do not yet know the value of $k = \dim \langle \Pi \rangle$ and the splitting types of the normal bundles of the fixed curves E_1 and E_2 . The first is done in this section and the second in the next.

5.1. The first observations. Let as before \mathfrak{g} be the Lie algebra of G , T be a fixed maximal torus of G , $\mathfrak{h} = \text{Lie}(T)$ be the corresponding Cartan subalgebra of \mathfrak{g} , and Δ be the root system in \mathfrak{h}^\vee . For a root $\alpha \in \Delta$ let \mathfrak{g}_α be the corresponding root subspace of \mathfrak{g} . Consider the Cartan decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta^+} (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}).$$

Consider the Weyl group $W = N_G(T)/T$ of G . The central Weyl involution $\tau \in z(W) \subset G$ acts on \mathfrak{h} via $\tau|_{\mathfrak{h}} = -\text{id}_{\mathfrak{h}}$ and acts on every plane $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$ by interchanging \mathfrak{g}_α and $\mathfrak{g}_{-\alpha}$. So \mathfrak{g} can be decomposed into a direct sum $\mathfrak{g} = F^+ \oplus F^-$ of eigenspaces orthogonal with respect to the Killing form, where

$$(8) \quad \tau|_{F^+} = \text{id}_{F^+}, \quad \tau|_{F^-} = -\text{id}_{F^-}, \quad \dim F^+ = 6, \quad \dim F^- = 8, \quad \mathfrak{h} \subset F^-.$$

The fixed point set $(\mathbb{P}\mathfrak{g})^\tau \subset \mathbb{P}^{13}$ consists of two disjoint subspaces $\mathbb{P}F^+ \cong \mathbb{P}^5$ and $\mathbb{P}F^- \cong \mathbb{P}^7$, where $\mathbb{P}\mathfrak{h} \subset \mathbb{P}F^-$.

Since the torus T is an abelian group, $T|_{\mathfrak{h}} = \text{id}_{\mathfrak{h}}$, and so $T|_{\mathbb{P}\mathfrak{h}} = \text{id}_{\mathbb{P}\mathfrak{h}}$. The center $z(W) = \langle \tau \rangle$ of the Weyl group also acts identically on $\mathbb{P}\mathfrak{h}$. So, the action of the dihedral group $W = \mathfrak{D}_6$ on \mathfrak{h} descends to an effective action of the symmetric group $\mathfrak{S}_3 = W/z(W)$ on the projective line $\mathbb{P}\mathfrak{h} \subset \mathbb{P}\mathfrak{g} = \mathbb{P}^{13}$. This line meets the sextic hypersurfaces D_ℓ and D_s along a zero-cycles $2O_\ell$ and $2O_s$, respectively, where O_ℓ and O_s are \mathfrak{S}_3 -orbits of length 3, see [PZ22, Lemmas 3.7 and 3.8].

Choose a point $\mathbb{P}h \in \mathbb{P}\mathfrak{h} \setminus D_\ell$. Then the hyperplane section $V = V(h) = h^\perp \cap \Omega$ is a smooth Fano-Mukai fourfold of genus 10 with a reductive group $\text{Aut}(V)$ that contains the torus T and the involution τ normalizing T . Notice that $V = V(h)$ is singular exactly when $\mathbb{P}h \in O_\ell = \mathbb{P}h \cap D_\ell$.

Let now V be a smooth Fano-Mukai fourfold of genus 10 with a reductive group $\text{Aut}(V)$ that contains the torus T and the involution τ normalizing T . Then $V \cong V(h)$ for some $h \in \mathfrak{g}$ with $\mathbb{P}h \in \mathbb{P}\mathfrak{h} \setminus D_\ell$, see [PZ22, Corollary 3.5.1]. Besides, $\text{Aut}^0(V) \cong \text{GL}_2(\mathbb{C})$ if $\mathbb{P}h \in O_s$ and $\text{Aut}^0(V) = T \cong \mathbb{G}_m^2$ if $\mathbb{P}h \in \mathbb{P}\mathfrak{h} \setminus (O_s \cup O_\ell)$, see [PZ22, Corollary 3.8.1].

It is known that $V = V(h) \subset \mathbb{P}^{12}$ is linearly nondegenerate. Since the involution τ leaves $\langle V(h) \rangle = h^\perp$ invariant we have $h \in F^+ \cup F^-$. Letting

$$(9) \quad \Lambda^+ = \mathbb{P}F^+ \cap h^\perp \quad \text{and} \quad \Lambda^- = \mathbb{P}F^- \cap h^\perp$$

we have

$$(10) \quad \Lambda^+ \cap \Lambda^- = \emptyset, \quad \dim \Lambda^+ + \dim \Lambda^- = 11 \quad \text{and} \quad \langle \Lambda^+ \cup \Lambda^- \rangle = h^\perp \cong \mathbb{P}^{12}.$$

For $h \in \mathfrak{h} \subset F^-$ one has $h^\perp \supset F^+$, therefore $\Lambda^+ \cong \mathbb{P}^5$ and $\Lambda^- \cong \mathbb{P}^6$.

Let $\Pi \subset V = V(h)$ be the scroll in τ -invariant lines. Then $\Pi^\tau = E^+ \cup E^-$ where E^+ and E^- are disjoint smooth rational sextic curves, see Theorem 0.1. We may consider that $E^+ \subset \Lambda^+$ and $E^- \subset \Lambda^-$. Indeed, if both E^+ and E^- were contained in the same projectivized eigenspace, say $E^+ \cup E^- \subset \Lambda^-$, then $\Pi \subset \langle E^+ \cup E^- \rangle \subset \Lambda^-$, and so $\tau|_\Pi = \text{id}_\Pi$, a contradiction. Hence $E^\pm = \Lambda^\pm \cap V$. Since $\min\{\dim^+, \dim \Lambda^-\} = 5$, at least one of the sextics $E^\pm \subset \Lambda^\pm$ is not a rational normal curve, therefore $\Pi \subset \mathbb{P}^{12}$ is not linearly normal.

5.2. Linear nondegeneracy of Π .

Proposition 5.1.

- (i) *The scroll Π in τ -invariant lines is linearly nondegenerate in \mathbb{P}^{12} .*
- (ii) *Up to enumeration we have $\dim \langle E_1 \rangle = 5$ and $\dim \langle E_2 \rangle = 6$.*

We start with the following lemma.

Lemma 5.2. *Let $k = \dim \langle \Pi \rangle$ and let $\pi: \mathbb{P}^{13} \dashrightarrow \mathbb{P}^k = \langle \Pi \rangle$ be the linear projection with center $Z = \mathbb{P}^{12-k}$ which sends the rational normal scroll Π_{norm} onto Π and sends \bar{E}_i onto E_i , $i = 1, 2$. Let $H_i = \langle \bar{E}_i \rangle \subset \mathbb{P}^{13}$ and $Z_i = Z \cap H_i$, $i = 1, 2$. Then either*

- $Z = Z_i$ for some $i \in \{1, 2\}$, or
- Z coincides with the join $J(Z_1, Z_2) = \langle Z_1, Z_2 \rangle$. In the latter case $\dim(Z) = 12 - k = \dim(Z_1) + \dim(Z_2) + 1$.

Proof. We have $H_i \cong \mathbb{P}^6$, $H_1 \cap H_2 = \emptyset$ and $H_1 \cup H_2$ is the fixed point set $(\mathbb{P}^{13})^\tau$. Recall that through any point of $\mathbb{P}^{13} \setminus (H_1 \cup H_2)$ passes a unique line meeting both H_1 and H_2 .

Assume that $Z \neq Z_i$ for $i = 1, 2$ and let $P \in Z \setminus (H_1 \cup H_2)$. Since Z is τ -invariant, the τ -invariant line $l_P = (P\tau(P))$ is contained in Z and $P_i = l_P \cap H_i \in Z_i$, $i = 1, 2$, are the two fixed points of τ on l_P . Thus, $P \in l_P = (P_1, P_2) \subset J(Z_1, Z_2)$, and so $Z \subset J(Z_1, Z_2)$. The opposite inclusion is evidently true. \square

Corollary 5.3. *Π is linearly nondegenerate if and only if Z is a point in $H_i \setminus \text{Sec}(\bar{E}_i)$ for some $i \in \{1, 2\}$. In the latter case*

$$\dim \langle E_i \rangle = 5 \quad \text{and} \quad \dim \langle E_j \rangle = 6 \quad \text{for} \quad j \neq i.$$

Proof. This follows immediately from Lemma 5.2. \square

Let us recall the notation and some formulas from [Hir57].

Notation 5.4. Let C_1 and C_2 be curves in \mathbb{P}^n with no common component. For a point $P \in C_1 \cap C_2$ let $I(C_i)$ be the ideal of C_i in the local ring A of (\mathbb{P}^n, P) . Then the intersection index of C_1 and C_2 at P is defined as

$$i(C_1, C_2; P) = \text{length } A/(I(C_1), I(C_2))A.$$

It is known that $i(C_1, C_2; P) = 1$ if and only if the Zariski tangent spaces of C_1 and C_2 at P intersect only in P , see [Hir57, Proposition 3].

Let now $C_1 = C'_1 \cup C''_1$ where C'_1 , C''_1 and C_2 are smooth at P . If C_1 lies on a smooth surface S and C_2 is transversal to S at P then $i(C_1, C_2; P) = 1$. If C'_1 is a simple tangent line to C_2 at P and C''_1 is transversal to C_2 at P then $i(C_1, C_2; P) = 3$. If, finally, C'_1 is a simple tangent line to C_2 at P and $C''_1 = C'_1$ then $i(C_1, C_2; P) = 2i(C'_1, C_2; P) = 4$.

Given r curves C_1, \dots, C_r with no common component, one consider the following zero-cycle supported at the singular points of $C = \bigcup_{i=1}^r C_i$:

$$\Lambda(C) = \sum_{P \in \text{sing}(C)} i(\Lambda(C), P)P \quad \text{where} \quad i(\Lambda(C), P) = \sum_{k=2}^r i \left(\left(\sum_{i=1}^{k-1} C_i \right), C_k; P \right).$$

In fact, this cycle does not depend on the choice of enumeration. The arithmetic genus $p_a(C)$ can be computed by the following formula, see [Hir57, Theorem 3]:

$$(11) \quad p_a(C) = \sum_{i=1}^r p_a(C_i) + \deg(\Lambda(C)) - (r - 1).$$

Lemma 5.5.

- (i) *The curve E_i is intersection of quadrics in $\langle E_i \rangle$ for $i = 1, 2$.*
- (ii) *We have $k \geq 11$.*

Proof. (i) It is known that $\Omega \subset \mathbb{P}^{13}$ and $V = \Omega \cap h^\perp$ in \mathbb{P}^{12} are intersections of quadrics, see [Isk77, Lemma 2.10], cf. also [PZ18, 4.5]. Then also $E_i = E^\pm = \Lambda^\pm \cap V = \langle E_i \rangle \cap V$ is intersection of quadrics in $\langle E_i \rangle$, $i = 1, 2$, see subsection 5.1.

(ii) Fix $i \in \{1, 2\}$ and let $E = E_i$, $\bar{E} = \bar{E}_i$ and $\mathcal{Z} = \mathcal{Z}_i \subset \langle \bar{E} \rangle = \mathbb{P}^6$. Since the curve $E = \pi(\bar{E})$ is smooth, the center \mathcal{Z} of the projection $\pi : \mathbb{P}^6 \rightarrow \langle E \rangle \subset \mathbb{P}^{12}$ does not intersect the threefold $\text{Sec}(\bar{E})$ in \mathbb{P}^6 . Hence $\dim \mathcal{Z} \leq 2$.

Suppose that $\dim \mathcal{Z} = 2$. Then $\langle E \rangle = \mathbb{P}^3$. By (i), E is intersection of quadrics in \mathbb{P}^3 . Thus, $E \subset Q_1 \cap Q_2$ for some distinct quadrics Q_1 and Q_2 in \mathbb{P}^3 . So, $6 = \deg(E) \leq 4$, a contradiction. It follows that $\dim \mathcal{Z} \leq 1$.

If $\dim \mathcal{Z} = 1$ then $\langle E \rangle = \pi(\langle \bar{E} \rangle) = \mathbb{P}^4$. By (i), E is contained in a proper intersection of quadrics Q_1, Q_2 and Q_3 in \mathbb{P}^4 . This intersection is a curve of degree 8 and of arithmetic genus 5. It consists of the smooth sextic E and a conic C . We claim that C is reducible. Indeed, let $L = \mathbb{P}^3$ be a hyperplane in \mathbb{P}^4 that contains C . Then the surface $L \cap \text{Sec}(E)$ in $L = \mathbb{P}^3$ meets C . Hence, there is a secant line l of E which intersects C , and so $l \cdot Q_i \geq 3$ for $i = 1, 2, 3$. Thus, $l \subset \bigcap_{i=1}^3 Q_i$ is a component of $C = l + l'$, where l' is the second component of C .

Letting $\Gamma = E + l + l'$ we obtain by (11)

$$5 = \pi_a(\Gamma) = \deg(\Lambda(\Gamma)) - 2,$$

and so $\deg(\Lambda(\Gamma)) = 7$. However, we claim that $\deg(\Lambda(\Gamma)) < 7$ whatever is the configuration (E, l, l') .

Indeed, E being a smooth intersection of quadrics, the intersection of E with any line is either transversal or a simple tangency at a single point. Therefore, $\sum_{P \in E} i(E, l_i; P) \leq 2$. Assume first that $l_1 \neq l_2$. If Γ has only nodes as singularities then the number of these points is ≤ 5 , and also $\deg(\Lambda(\Gamma)) \leq 5$, a contradiction.

Suppose now that Γ has a triple point P . Then either l_1, l_2 and E are transversal at P and then

$$i(\Lambda(\Gamma), P) = i(l_1, l_2; P) + i(l_1 + l_2, E; P) = 2,$$

or at most one of the l_i , say l_1 is a simple tangent to E at P and l_2 is transversal to $l_1 + E$ at P , hence

$$i(\Lambda(\Gamma), P) = i(E, l_1; P) + i(E + l_1, l_2; P) = 3.$$

It is easily seen that in any case $\deg(\Lambda(\Gamma)) \leq 5$. This leads again to a contradiction.

Suppose finally that $l_1 = l_2$ is a double line of the intersection $Q_1 \cdot Q_2 \cdot Q_3$. If this double line meets E transversally in two points P_1 and P_2 , then $\deg(\Lambda(\Gamma)) = 2(i(l_1, E; P_1) +$

$i(l_1, E; P_2) = 4$. If l_1 and E are tangent at a point P then again $\deg(\Lambda(\Gamma)) = 2i(l_1, E; P) = 4$. In all the other cases $\deg(\Lambda(\Gamma)) < 4$, which proves our claim.

Thus, $\dim Z_i \leq 0$ for $i = 1, 2$, hence $12 - k = \dim Z \leq 1$ and $k \geq 11$. \square

Proof of Proposition 5.1. Statement (ii) follows from (i) due to Corollary 5.3. To show (i) suppose that Π is linearly degenerate. By Lemma 5.5 one has $\langle \Pi \rangle = \mathbb{P}^{11}$, and so Π is contained in the hyperplane section $M = \langle \Pi \rangle \cap V$. By Lefschetz' theorem on hyperplane sections, $\text{Pic}(M) \cong \text{Pic}(V) = \mathbb{Z}$, see e.g. [Fuj80, Section 1]. Hence Π is a complete intersection in M . However, $\deg(\Pi) = 12$ does not divide $\deg(M) = 18$, a contradiction. \square

6. NORMAL BUNDLES OF THE FIXED CURVES

In this section we determine the splitting type of the normal bundles $N^- = N_{E^-/\mathbb{P}^6}$ and $N^+ = N_{E^+/\mathbb{P}^5}$ where E^\pm are the components of the fixed point set V^τ . Using the notation from subsection 5.1 we may consider that

$$E^+ \subset \Lambda^+ = \langle E^+ \rangle = \mathbb{P}^5 \quad \text{and} \quad E^- \subset \Lambda^- = \langle E^- \rangle = \mathbb{P}^6.$$

We will show that the first normal bundle is almost balanced and the second is balanced. More precisely, we have the following

Proposition 6.1. *The normal bundles N^\pm admit decompositions*

$$N^- = \mathcal{O}_{\mathbb{P}^1}(8)^5 \quad \text{and} \quad N^+ = \mathcal{O}_{\mathbb{P}^1}(8)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^1}(9)^{\oplus 2}.$$

Actually, the first equality is well known. Indeed, let $C \subset \mathbb{P}^n$ be a linearly nondegenerate smooth rational curve of degree $d \geq n$. Then

$$N_{C/\mathbb{P}^n} = \bigoplus_{i=1}^{n-1} \mathcal{O}_{\mathbb{P}^1}(d + b_i) \quad \text{where} \quad b_i \geq 2 \quad \text{and} \quad \sum_{i=1}^{n-1} b_i = 2d - 2,$$

see e.g. [Sac80], [CR18, Corollary 2.2]. In particular, for the rational normal curve $C = C_n$ of degree n in \mathbb{P}^n (i.e. for $d = n$) one has $b_1 = \dots = b_{n-1} = 2$, and so the normal bundle is balanced, that is,

$$N_{C_n/\mathbb{P}^n} = \mathcal{O}_{\mathbb{P}^1}(n + 2)^{n-1}.$$

For $n = 6$ this yields the first equality in Proposition 6.1.

As for the second, notice that the preceding formulas leaves just the following possibilities (a) and (b), depending on the choice of the center \mathcal{Z} of the linear projection $\pi: \mathbb{P}^6 \dashrightarrow \mathbb{P}^5$ which sends a rational normal curve C_6 to E^+ :

$$(12) \quad (a) \quad N^+ = \mathcal{O}_{\mathbb{P}^1}(8)^2 \oplus \mathcal{O}_{\mathbb{P}^1}(9)^2 \quad \text{and} \quad (b) \quad N^+ = \mathcal{O}_{\mathbb{P}^1}(8)^3 \oplus \mathcal{O}_{\mathbb{P}^1}(10).$$

Apriori, both of these splitting types could occur, due to the following facts.

Theorem 6.2.

- (i) ([Sac80]; see also [CR18, Theorem 2.7]) *For any sequence of integers $b_i \geq 2$, $1 \leq i \leq n - 1$ such that $\sum_{i=1}^{n-1} b_i = 2d - 2$ there exists an unramified map $f: \mathbb{P}^1 \rightarrow \mathbb{P}^n$ onto a linearly nondegenerate, immersed curve $C \subset \mathbb{P}^n$ of degree $d \geq n$ such that*

$$N_f = \bigoplus_{i=1}^{n-1} \mathcal{O}_{\mathbb{P}^1}(d + b_i).$$

(ii) ([Sac80]; see also [AR17, Theorem 2]) *For a generic rational curve C in \mathbb{P}^n of degree $d > n \geq 3$ one has*

$$N_f = \mathcal{O}_{\mathbb{P}^1}(d + q + 1)^{\oplus n-r-1} \oplus \mathcal{O}_{\mathbb{P}^1}(d + q + 2)^{\oplus r}$$

where $2d - n - 1 = q(n - 1) + r$ with $r < n - 1$.

From (ii) of Theorem 6.2 we deduce the following

Corollary 6.3. *The normal bundle of a generic rational sextic curve in \mathbb{P}^5 is of splitting type (a) in (12).*

Let us give concrete examples of rational sextic curves with normal bundles of splitting types (a) and (b) in (12).

Example 6.4. Consider the smooth monomial sextic curves C and C' in \mathbb{P}^5 with parametrizations

$$C = (u^6 : u^5v : u^4v^2 : u^2v^4 : uv^5 : v^6) \quad \text{resp.} \quad C' = (u^6 : u^5v : u^3v^3 : u^2v^4 : uv^5 : v^6).$$

Then

$$N_{C/\mathbb{P}^5} \cong N_{C'/\mathbb{P}^5} \cong \mathcal{O}_{\mathbb{P}^1}(8)^2 \oplus \mathcal{O}_{\mathbb{P}^1}(9)^2,$$

see [CR18, Theorem 3.2].

The other splitting type in (12) can be realized for immersed rational sextic curves. For example, let $p_i = a_iu + b_iv$, $i = 1, 2$ be two coprime linear forms with $a_i, b_i \neq 0$, and let $C \subset \mathbb{P}^5$ be the rational sextic curve with parameterization

$$f : (u : v) \mapsto (u^5p_1 : u^4vp_1 : u^3v^2p_1 : u^2v^3p_1 : uv^4p_1 : v^5p_2).$$

Then C is immersed with normal bundle

$$N_f = \mathcal{O}_{\mathbb{P}^1}(8)^3 \oplus \mathcal{O}_{\mathbb{P}^1}(10),$$

see [Sac80] and [CR18, Lemma 2.4]. Notice that C is the image of the rational normal curve $C_6 = h(\mathbb{P}^1) \subset \mathbb{P}^6$ parameterized via

$$(u : v) \mapsto (u^6 : u^5v : \dots : v^6)$$

under the projection $\pi_P : \mathbb{P}^6 \dashrightarrow \mathbb{P}^5$ with center

$$P = (1 : \lambda : \dots : \lambda^5 : \lambda^5\mu) \in \mathbb{P}^6 \quad \text{where} \quad \lambda = -a_1/b_1 \quad \text{and} \quad \mu = -a_2/b_2.$$

The center P is situated on the secant line $(Q_1(\lambda)Q_2)$ of C_6 where

$$Q_1(\lambda) = (1 : \lambda : \dots : \lambda^6) \in C_6 \quad \text{and} \quad Q_2 = (0 : 0 : \dots : 0 : 1) \in C_6.$$

Hence $\pi(Q_2) = (0 : \dots : 0 : 1) \in \mathbb{P}^5$ is a point of self-intersection of $C = \pi_P(C_6)$.

The above phenomenon is of general nature. Namely, the following theorem allows to replace the assumptions ‘‘generic’’ in Theorem 6.2(ii) and Corollary 6.3 by the assumption of smoothness.

Theorem 6.5 ([Ber11, Theorem 3.3.13], [Ber12, Theorem 2.12]). *Let $C_n \subset \mathbb{P}^n$ be the rational normal curve of degree n . Then the image $C = \pi_P(C_n) \subset \mathbb{P}^{n-1}$ has normal bundle*

$$N_{C/\mathbb{P}^{n-1}} = \mathcal{O}_{\mathbb{P}^1}(n + 2)^{\oplus n-4} \oplus \mathcal{O}_{\mathbb{P}^1}(n + 3)^{\oplus 2}$$

if and only if the center $P \in \mathbb{P}^n$ of the projection $\pi_P : \mathbb{P}^n \dashrightarrow \mathbb{P}^{n-1}$ does not lie on a secant or tangent line to C_n , if and only if C is smooth.

Since $E^+ \subset \mathbb{P}^5 = \langle E^+ \rangle$ is a smooth sextic curve, the following corollary is immediate.

Corollary 6.6. *One has*

$$N_{E^+/\mathbb{P}^5} = \mathcal{O}_{\mathbb{P}^1}(8)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^1}(9)^{\oplus 2}.$$

This ends the proofs of Proposition 6.1 and Theorem 0.1.

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