On the set of points of zero torsion for negative-torsion maps of the annulus

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Abstract

For negative-torsion maps on the annulus we show that on every $\mathcal{C}^1$ essential curve there is at least one point of zero torsion. As an outcome we deduce that the Hausdorff dimension of the set of points of zero torsion is greater or equal 1. As a byproduct we obtain a Birkhoff's Theorem-Like result for $\mathcal{C}^1$ essential curves in the framework of negative-torsion maps.

1 Introduction

Let $\mathbb{A} = \mathbb{T} \times \mathbb{R}$ where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. Endow $\mathbb{A}$ with the standard Riemannian metric and the standard trivialization. Let $f : \mathbb{A} \to \mathbb{A}$ be a $\mathcal{C}^1$ diffeomorphism isotopic to the identity. Let $(f_t)_{t\in[0,1]}$ be an isotopy in Diff $^1(\mathbb{A})$ joining the identity to $f_1 = f$. We are interested in the linearized dynamics: more precisely we look at the *torsion*, a dynamical invariant first introduced by D. Ruelle in 1985 (see [Rue85]), who called it *rotation number*.
Roughly speaking, the torsion of the orbit of a point x, denoted as Torsion(f, x), describes the average asymptotic velocity at which the differential of the diffeomorphism makes the tangent vectors turn along the considered orbit. The torsion at finite time T of a point (x, ξ) of the tangent bundle is the variation between 0 and T of a continuous determination of the angle function associated to $Df_t(x)\xi, t \in [0, T]$, divided by T. The torsion of the orbit of x is the limit for $T \to +\infty$ of the torsion at finite time T, whenever the limit exists. If μ is a compactly supported f-invariant Borel probability measure, then the torsion of the orbit of x exists for μ-almost every point $x \in \mathbb{A}$, see [Rue85].
Besides Ruelle's work, in the framework of conservative twist maps, the structure of some null torsion sets, called Aubry-Mather sets, has been studied by Mather (in [Mat82] and [Mat91]) and Angenent (in [Ang88]) through a variational approach. Using topological tools, Crovisier has generalised some results in the non conservative twist framework. He has shown that for every rotation number there exists an Aubry-Mather sets of zero torsion, see Theorem 1.2 in [Cro03].

A negative-torsion map is a $\mathcal{C}^1$ diffeomorphism isotopic to the identity such that at every point the vertical vector has a negative average rotational velocity: that is, for every point $x \in \mathbb{A}$ the torsion at finite time 1 at x with respect to the vertical vector is negative. Positive twist maps and Tonelli Hamiltonian flows at finite time are examples

of negative-torsion maps. The notion of negative-torsion map is equivalent to the notion of positive tilt maps as presented in [Hu98] and [GR13]. A notion of positive-torsion maps can be given similarly.
In this paper we consider the set of points of zero torsion for negative-torsion maps. The main result of the work is the following.

Theorem 1.1. *Let $f : \mathbb{A} \to \mathbb{A}$ be a negative-torsion map. Then for any $\mathcal{C}^1$ essential curve $\gamma : \mathbb{T} \to \mathbb{A}$ there exists a point $z \in \gamma(\mathbb{T})$ such that Torsion$(z) = 0$.*

Remark that we do not ask any conservative hypothesis on f. Applying Theorem 1.1 to every simple circle curves, i.e. any curve $\mathbb{T} \times \{r\}$ for $r \in \mathbb{R}$, we can deduce the following

Corollary 1.1. *Let $f : \mathbb{A} \to \mathbb{A}$ be a negative-torsion map. Then*

$$dim_H\left(\{z \in \mathbb{A} : \ \ Torsion(z) = 0\}\right) \geq 1,$$

where $dim_H(\cdot)$ denotes the Hausdorff dimension of the set.

The idea of the proof of Theorem 1.1 is to consider, for every $N \in \mathbb{N}^*$, the preimage on γ of points of maximal height of $f^N \circ \gamma(\mathbb{T})$. We then show that the angle variation of the vector tangent to γ at these points between 0 and N is bounded uniformly in N. By the negative-torsion property, we control the torsion at finite time m of these points for every $m \in [\![1, N]\!]$. Consider then the sequence of preimages on γ of such points of maximal height. Every limit point of the sequence is a point of zero torsion.
The idea of looking at points of maximal height can be adapted to extend the result to $\mathcal{C}^0$ essential curves which are graphs of functions (see Theorem 3.1). We do not know if the result can be extended also to $\mathcal{C}^0$ essential curves which are not graphs.

As a by-product of the proof of Theorem 1.1, we deduce that the torsion of an orbit of a point belonging to a $\mathcal{C}^1$ f-invariant essential curve can be calculated through the angle variation of the vector tangent to the curve along the curve itself. We obtain so as a by-product of the proof of Theorem 1.1 a version of Birkhoff's theorem for negative-torsion maps.

Theorem 1.2. *Let $f : \mathbb{A} \to \mathbb{A}$ be a negative-torsion map. Let $\gamma : \mathbb{T} \to \mathbb{A}$ be a $\mathcal{C}^1$ f-invariant essential curve such that $f_{|\gamma}$ is non wandering. Then γ is the graph of a $\mathcal{C}^1$ function.*

All the above results still hold also in the case of positive-torsion maps. The paper is organised as follows. In Section 2 we fix the notation and we provide the main definitions of torsion and negative-torsion maps. Useful properties of torsion are recalled. Section 3 is devoted to the discussion of the proof of Theorem 1.1. Its demonstration relies on two main propositions, see Propositions 3.1 and 3.2. In particular, Proposition 3.1 is discussed first in a simpler case in Subsection 3.1 and then in the general framework in Subsection 3.2. Subsection 3.3 concerns the partial generalisation of Theorem 1.1 to continuous essential curves. Finally, we present the Birkhoff's tTheorem-Like result (see Theorem 1.2) for negative-torsion maps in Section 4.

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2 Notation, main definitions and first properties of torsion

Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ and let $\mathbb{A} = \mathbb{T} \times \mathbb{R}$. Denote as $p : \mathbb{R} \to \mathbb{T}$ and as $p \times \mathrm{Id} : \mathbb{R}^2 \to \mathbb{A}$ the universal covering maps of $\mathbb{T}$ and of $\mathbb{A}$ respectively. The functions p_1, p_2 denote the projections over the first and the second coordinate respectively over $\mathbb{A}$ and, with an abuse of notation, also over $\mathbb{R}^2$.
Endow $\mathbb{A}$ with the standard Riemannian metric and the standard trivialization. We fix the counterclockwise orientation of $\mathbb{R}^2$. Thus the notion of oriented angle between two non zero vectors is well-defined.

Definition 2.1 (see [Hir76])**.** Let M, N be differential manifolds and let $f, g : M \to N$ be in Diff $^1(M, N)$. An isotopy $(\phi_t)_{t\in[0,1]}$ joining f to g is an arc in Diff $^1(M, N)$ such that $\phi_0 = f, \phi_1 = g$ and which is continuous with respect to the weak or compact-open $\mathcal{C}^1$ topology on Diff $^1(M, N)$.

Usually, isotopies are paths in the space of homeomorphisms, rather than diffeomorphisms. Nevertheless, in our case Definition 2.1 is not restrictive: indeed, any diffeomorphism of the annulus $\mathbb{A}$ that is isotopic to the identity through a path in the space of homeomorphisms, it is also isotopic to the identity by a path in the space of diffeomorphisms, see for example [Sma59] or [Gra73].

Definition 2.2. Let $I \subset \mathbb{R}$ be an interval. A continuous determination of an angle function $\theta : I \to \mathbb{T}$ is a continuous lift of θ, that is a continuous function $\tilde{\theta} : I \to \mathbb{R}$ such that $\tilde{\theta}(s)$ is a preimage (by $p : \mathbb{R} \to \mathbb{T}$) of the oriented angle $\theta(s)$ for any $s \in I$.

Let $f : \mathbb{A} \to \mathbb{A}$ be a $\mathcal{C}^1$ diffeomorphism isotopic to the identity. Let $(f_t)_{t\in[0,1]}$ be an isotopy joining the identity to f. Extend the isotopy for any positive time so that for any $t \in \mathbb{R}_+$ the $\mathcal{C}^1$ diffeomorphism $f_t : \mathbb{A} \to \mathbb{A}$ is defined as $f_t := f_{\{t\}} \circ f^{\lfloor t \rfloor}$, where $\{\cdot\}, \lfloor\cdot\rfloor$ are the fractionary and integer part respectively. We denote $I = (f_t)_{t\in\mathbb{R}_+}$ the extended isotopy. For any $x \in \mathbb{A}$ denote as $\chi = (0, 1) \in T_x\mathbb{A}$ the unitary positive vertical vector.
The definition of torsion we adopt is the one given by Béguin and Boubaker in [BB13]. The asymptotic torsion is actually Ruelle's rotation number, see [Rue85].

Definition 2.3. Let $x \in \mathbb{A}, \xi \in T_x\mathbb{A}, \xi \neq 0$. Define the oriented angle function

$$\mathbb{R}_+ \ni t \mapsto v(I, x, \xi)(t) := \theta(\chi, Df_t(x)\xi) \in \mathbb{T}, \tag{1}$$

where $\theta(u, v)$ denotes the oriented angle between the two non zero vectors u, v.
Denote

$$\mathbb{R}_+ \ni t \mapsto \tilde{v}(I, x, \xi)(t) \in \mathbb{R} \tag{2}$$

a continuous determination of the continuous oriented angle function $v(I, x, \xi)(\cdot)$.
The torsion at finite time $n \in \mathbb{N}^*$ of $(x, \xi) \in T\mathbb{A}, \xi \neq 0$ is

$$\mathrm{Torsion}_n(I, x, \xi) := \frac{\tilde{v}(I, x, \xi)(n) - \tilde{v}(I, x, \xi)(0)}{n}. \tag{3}$$

The torsion at $x \in \mathbb{A}$ is, whenever it exists,

$$\mathrm{Torsion}(I, x) := \lim_{n\to+\infty} \mathrm{Torsion}_n(I, x, \xi). \tag{4}$$

Remark 2.1. The torsion at finite time does not depend on the choice of the continuous determination of the oriented angle function. Moreover, it is independent from the choice of the isotopy joining the identity to f, see Proposition 2.5 in [Flo19b]. The (asymptotic) torsion, whenever it exists, does not depend on the tangent vector used to calculate the finite time torsion. We refer to [BB13] for these properties.

Notation 2.1. Since the torsion does not depend on the chosen isotopy, we write $\text{Torsion}_n(f, x, \xi)$ and $\text{Torsion}(f, x)$. In addition, in order to lighten the notation, we omit the dependance on the dynamics f of the torsion, writing just $\text{Torsion}_n(x, \xi)$ and $\text{Torsion}(x)$, when it will be clear from the contest.

Example 2.1. Consider the dynamical system of the simple pendulum obtained by the Hamiltonian $H(\theta, r) = \frac{r^2}{2} - \frac{\cos(2\pi\theta)}{4\pi^2}$. Let $(\phi_t)_{t\in\mathbb{R}}$ be the associated flow and consider the time-one flow $f = \phi_1$. Let U denote the open region contained between the separatrices of the pendulum system, see Figure 1. We can calculate the torsion with respect to the Hamiltonian vector field X_H, which is always tangent to the trajectories. So, every point z not belonging to U has zero (asymptotic) torsion, since the corresponding vector $X_H(\phi_t(z))$ is always contained in the same open half-plane and thus its angle variation is uniformly bounded in time. The elliptic point $(0, 0)$ has torsion equal to -1, since each differential $D\phi_t(0, 0)$ is a clockwise rotation of angle t. Every point $z \in U \setminus \{(0, 0)\}$ is periodic and has torsion equal to $-\frac{1}{T(z)}$, where $T(z)$ is the period of z, since the vector $X_H(\phi_t(z))$ is turning clockwisely once over a time interval of length $T(z)$.

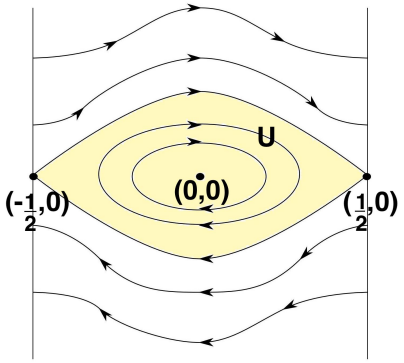

Figure 1 – Phase portrait of the pendulum system of Example 2.1.

Definition 2.4. A $\mathcal{C}^1$ diffeomorphism isotopic to the identity $f : \mathbb{A} \to \mathbb{A}$ is a negative-torsion map (respectively a positive-torsion map) if for any $z \in \mathbb{A}$ it holds

$$\text{Torsion}_1(z, \chi) < 0 \quad (\text{ respectively } > 0)\,. \tag{5}$$

Example 2.2. Every positive twist map is a negative-torsion map (see [Flo19b]). Actually the notion of negative-torsion map is equivalent to the notion of positive tilt map: see [Hu98], [GR13] and [Flo19a]. We remark that the notion of negative-torsion (positive-torsion) maps can be given also in terms of positive (negative) paths according to the definitions in [Her83] and [LC88].

Definition 2.5. An essential curve is a $\mathcal{C}^0$ embedded circle in $\mathbb{A}$ not homotopic to a point.

We recall here some properties of finite-time torsion. We refer to [BB13], [Flo19b] and [Flo19a] for the proofs.

Property 2.1. Fix $x \in \mathbb{A}$. If we calculate finite-time torsion at x with respect to different tangent vectors, we can control the error. Indeed, for $n \in \mathbb{N}^*$ and for $\xi, \delta \in T_x\mathbb{A} \setminus \{0\}$

$$|\text{Torsion}_n(x, \xi) - \text{Torsion}_n(x, \delta)| < \frac{1}{2n},$$

see Lemma 2.1 in [Flo19b].

Property 2.2. Let $f : \mathbb{A} \to \mathbb{A}$ be a $\mathcal{C}^1$ diffeomorphism isotopic to the identity. Let I be an isotopy in Diff $^1(\mathbb{A})$ joining the identity to f. Let $x \in \mathbb{A}$ and $\xi_1, \xi_2 \in T_x\mathbb{A} \setminus \{0\}$. Let $\tilde{v}(I, x, \xi_1)(\cdot), \tilde{v}(I, x, \xi_2)(\cdot)$ be continuous determinations of the angle functions $v(I, x, \xi_1)(\cdot), v(I, x, \xi_2)(\cdot)$ respectively.
If

$$\tilde{v}(I, x, \xi_1)(0) > \tilde{v}(I, x, \xi_2)(0),$$

then for any $t \in \mathbb{R}$ it holds

$$\tilde{v}(I, x, \xi_1)(t) > \tilde{v}(I, x, \xi_2)(t).$$

See Proposition 2.2 in [Flo19b].

Property 2.3. Let $f : \mathbb{A} \to \mathbb{A}$ be a $\mathcal{C}^1$ diffeomorphism isotopic to the identity. Let $a \in \mathbb{A}$. Let $N \in \mathbb{N}^*, (k_i)_{i \in [\![0, N-1]\!]} \in \mathbb{N}^N$ and $l_0 = 0 < l_1 < \cdots < l_N$ with $l_i \in \mathbb{N}$. Assume that for all $i \in [\![0, N-1]\!]$ it holds

$$(l_{i+1} - l_i)\text{Torsion}_{l_{i+1}-l_i}(f^{l_i}(a), \chi) < -\frac{k_i}{2}.$$

Then for any vector $\xi \in T_a\mathbb{A} \setminus \{0\}$ we have

$$l_N\text{Torsion}_{l_N}(a, \xi) < -\frac{\sum_{i=0}^{N-1} k_i}{2} + \frac{1}{2}.$$

In particular, when $\xi = \chi$, we have

$$l_N\text{Torsion}_{l_N}(a, \chi) < -\frac{\sum_{i=0}^{N-1} k_i}{2}.$$

We refer to [Flo19a, Lemma 2.2.3] (see Appendix 2.5).

Thus, from Property 2.3, we bound from above finite-time torsion for negative-torsion maps.

Property 2.4. Let $f : \mathbb{A} \to \mathbb{A}$ be a negative-torsion map. Let $m \in \mathbb{N}^*$ and let $z \in \mathbb{A}$ be such that $m\text{Torsion}_m(z, \chi) < -\frac{k}{2}$ for some $k \in \mathbb{N}^*$. Then for any $n \geq m, n \in \mathbb{N}^*$ it holds $n\text{Torsion}_n(z, \chi) < -\frac{k}{2}$.

3 Set of points of zero torsion

Theorem 1.1 is an outcome of the following two propositions.

Proposition 3.1. *Let $f : \mathbb{A} \to \mathbb{A}$ be a $\mathcal{C}^1$ diffeomorphism isotopic to the identity. Let $\gamma : \mathbb{T} \to \mathbb{A}$ be a $\mathcal{C}^1$ essential curve. There exists $C \in \mathbb{R}_+$ such that for any $n \in \mathbb{N}^*$ there exists $s_n \in \mathbb{T}$ so that*

$$|n\,Torsion_n(\gamma(s_n), \gamma'(s_n))| \leq C. \tag{6}$$

We postpone the proof of Proposition 3.1 to Subsections 3.1 and 3.2. In Subsection 3.1, we will first show it in the simpler case of simple circle curves, that is $\gamma(\mathbb{T}) = \mathbb{T} \times \{r\}$ for $r \in \mathbb{R}$, using the link between torsion and linking number. Then we will prove the proposition in the general case of $\mathcal{C}^1$ essential curves, see Subsection 3.2.

Proposition 3.2. *Let $f : \mathbb{A} \to \mathbb{A}$ be a negative-torsion map. Let $C \in \mathbb{R}_+$ and let $z \in \mathbb{A}$ be such that $|n\,Torsion_n(z, \chi)| \leq C$ for some $n \in \mathbb{N}^*$. Let $K = \lfloor 2C \rfloor + 2$. Then for any $m \in [\![1, n]\!]$ it holds*

$$m\,Torsion_m(z, \chi) \in \Big[-\frac{K}{2}, 0 \Big). \tag{7}$$

Proof. Since f is a negative-torsion map and by Property 2.4, for any $z \in \mathbb{A}$ and for any $m \in \mathbb{N}^*$ it holds $m\text{Torsion}_m(z, \chi) < 0$. Let $z \in \mathbb{A}$ and $n \in \mathbb{N}$ be such that $|n\text{Torsion}_n(z, \chi)| \leq C$. In particular, by the negative-torsion property, it holds

$$n\text{Torsion}_n(z, \chi) \in [-C, 0).$$

Argue by contradiction and assume that there exists $m \in [\![1, n]\!]$ such that

$$m\text{Torsion}_m(z, \chi) < -\frac{K}{2} \leq -C - \frac{1}{2}.$$

If $m = n$ we contradict the hypothesis. Thus, we have $m < n$. Again because f is a negative-torsion map, it holds $(n - m)\text{Torsion}_{n-m}(f^m(z), \chi) < 0$.
Apply then Property 2.3 for f at z with respect to $N = 2, l_1 = m, l_2 = n, k_1 = K, k_2 = 0$. We so obtain

$$n\text{Torsion}_n(z, \chi) < -\frac{K}{2} < -C,$$

which is the required contradiction. □

Proof of Theorem 1.1. Let $(s_n)_{n \in \mathbb{N}^*} \subset \mathbb{T}^{\mathbb{N}^*}$ be the sequence of points built in Proposition 3.1. That is, for any $n \in \mathbb{N}^*$ it holds

$$|n\text{Torsion}_n(\gamma(s_n), \gamma'(s_n))| \leq C,$$

where C does not depend on n. By the properties of finite-time torsion, see Property 2.1, we have

$$|n\text{Torsion}_n(\gamma(s_n), \chi)| \leq C + \frac{1}{2}.$$

Denote as $s_\infty \in \mathbb{T}$ a limit point of the sequence $(s_n)_{n \in \mathbb{N}^*}$. This is our candidate point of zero torsion.
Fix $N \in \mathbb{N}^*$. Let $\varepsilon > 0$. Up to subsequences and by the continuity of finite time torsion with respect to the point, there exists $\bar{n} \in \mathbb{N}^*, \bar{n} > N$ such that

$$|N\text{Torsion}_N(\gamma(s_\infty), \chi) - N\text{Torsion}_N(\gamma(s_{\bar{n}}), \chi)| < \varepsilon.$$

Let $K = \lfloor 2C + 1 \rfloor + 2$. By Proposition 3.2 and since f is a negative-torsion map (see Property 2.4), it holds

$$0 > N\text{Torsion}_N(\gamma(s_\infty), \chi) =$$

$$= (N\text{Torsion}_N(\gamma(s_\infty), \chi) - N\text{Torsion}_N(\gamma(s_{\bar{n}}), \chi)) + N\text{Torsion}_N(\gamma(s_{\bar{n}}), \chi) > -\varepsilon - \frac{K}{2}.$$

Thus

$$\text{Torsion}_N(\gamma(s_\infty), \chi) \in \Big[-\frac{K}{2N}, 0 \Big).$$

In particular K is independent from N and, as N goes to $+\infty$, we conclude that

$$\text{Torsion}(\gamma(s_\infty)) = 0.$$

□

3.1 A first simpler case: simple circle curves and Corollary 1.1

Proposition 3.1 can be proved easily in the framework of simple circle curves, that is if $\gamma(\mathbb{T}) = \mathbb{T} \times \{r\}$ for $r \in \mathbb{R}$, by using the notion of linking number and the relation between linking number and torsion. The linking number for a diffeomorphism of the plane of two points measures the average rotational velocity at which the orbit of the first point turns around the orbit of the second one. We refer to [BB13] for a deeper discussion. Recall that χ is the unitary positive vertical vector $(0, 1)$.

Definition 3.1. Let $I = (F_t)_{t \in \mathbb{R}_+}$ be an isotopy in $\text{Diff}^1(\mathbb{R}^2)$ joining the identity to $F_1 = F$ and such that $F_{1+t} = F_t \circ F$. Fix $x, y \in \mathbb{R}^2, x \neq y$. Define the oriented angle function

$$\mathbb{R}_+ \ni t \mapsto u(I, x, y)(t) := \theta(\chi, F_t(y) - F_t(x)) \in \mathbb{T},$$

where $\theta(u, v)$ denotes the oriented angle between the two non zero vectors u, v.
Since $u(I, x, y)(\cdot)$ is continuous, consider a continuous determination $\mathbb{R}_+ \ni t \mapsto \tilde{u}(I, x, y)(t) \in \mathbb{R}$ of such oriented angle function.
For every $n \in \mathbb{N}^*$ the linking number at finite time n of x and y is

$$\text{Linking}_n(I, x, y) := \frac{\tilde{u}(I, x, y)(n) - \tilde{u}(I, x, y)(0)}{n}.$$

The linking number of x and y is, whenever the limit exists,

$$\text{Linking}(I, x, y) := \lim_{n \to +\infty} \text{Linking}_n(I, x, y).$$

Concerning the relation between torsion and linking number, we recall here Corollary 3.1 in [Flo19b]. The torsion is calculated with respect to the standard trivialization. We measure oriented angles with respect to χ.

Corollary 3.1 (Corollary 3.1 in [Flo19b])**.** *Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be a $\mathcal{C}^1$ diffeomorphism isotopic to the identity and let I be an isotopy joining the identity to $F_1 = F$. Assume there exist $n \in \mathbb{N}^*$ and $x, y \in \mathbb{R}^2, x \neq y$ such that $Linking_n(I, x, y) = l \in \mathbb{R}$. Then there exists $z \in [x, y]$, where $[x, y]$ denotes the segment joining the points x, y, such that $Torsion_n(I, z, y - x) = l$.*

We proceed now with the proof of Proposition 3.1 in the case of a simple circle curve.

Proposition 3.3. *Let $f : \mathbb{A} \to \mathbb{A}$ be a $\mathcal{C}^1$ diffeomorphism isotopic to the identity. Let $r \in \mathbb{R}$ and let $n \in \mathbb{N}^*$. Then there exists $z(r,n) \in \mathbb{T} \times \{r\}$ such that*

$$Torsion_n(z(r,n), \mathcal{H}) = 0, \tag{8}$$

where $\mathcal{H} = (1,0) \in T_{z(r,n)}\mathbb{A}$ is the horizontal positive unitary vector.

Proof. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be a lift of f. Let $I = (F_t)_{t\in\mathbb{R}_+} \in \text{Diff}^{\,1}(\mathbb{R}^2)$ be the isotopy joining the identity of $\mathbb{R}^2$ to F, obtained as lift of an isotopy on $\mathbb{A}$ joining $\text{Id}_{\mathbb{A}}$ to f.
Observe that for any $t \in \mathbb{R}_+$ the function F_t commutes with the translation by $(1,0)$. Consequently, for any fixed $r \in \mathbb{R}$ and for any $n \in \mathbb{N}^*$ it holds

$$\text{Linking}_n(I, (0,r), (1,r)) = 0.$$

By Corollary 3.1 in [Flo19b] (here Corollary 3.1), there exists $z(r,n) \in (p\times\text{Id})([0,1]\times\{r\})$ such that

$$\text{Torsion}_n(f, z(r,n), \mathcal{H}) = 0,$$

where $\mathcal{H}$ is the unitary positive horizontal vector. □

In particular, this proves Theorem 1.1 for an essential curve γ such that $\gamma(\mathbb{T}) = \mathbb{T}\times\{r\}$ for $r \in \mathbb{R}$. Consequently, we can deduce Corollary 1.1.

Proof of Corollary 1.1. By Theorem 1.1 applied at all simple circle curves, for any $r \in \mathbb{R}$ there exists $z(r) \in \mathbb{T} \times \{r\}$ such that $\text{Torsion}(z(r)) = 0$. Thus

$$p_2\left(\{z \in \mathbb{A} : \ \text{Torsion}(z) = 0\}\right) = \mathbb{R}.$$

We are now interested in the Hausdorff dimension, denoted as $\dim_H$, of the set of points of zero torsion. Recall that if g is a Lipschitz function, then for any set U it holds $\dim_H(U) \geq \dim_H(g(U))$.
Since the projection over the second coordinate p_2 is Lipschitz and since $\dim_H(\mathbb{R}) = 1$, we conclude that $\dim_H(\{z \in \mathbb{A} : \ \text{Torsion}(z) = 0\}) \geq 1$.

□

3.2 The general case of $\mathcal{C}^1$ essential curves

We are now interested in the general case of γ being a $\mathcal{C}^1$ essential curve. In order to deal with this case we need to introduce the notion of angle variation along the curve γ.
Let $\gamma : \mathbb{T} \to \mathbb{A}$ be a $\mathcal{C}^1$ essential curve and let $x, y \in \gamma(\mathbb{T})$. Let $s_1, s_2 \in \mathbb{T}$ be such that $\gamma(s_1) = x, \gamma(s_2) = y$. Fix $S_1 \in \mathbb{R}$ a lift of s_1 and let $S_2 \in \mathbb{R}$ be the lift of s_2 such that $S_2 \in (S_1, S_1 + 1]$.
Define the oriented angle function

$$\mathbb{R}_+ \ni t \mapsto \Theta(\gamma, S_1)(t) := \theta\left(\chi, \frac{d\gamma(\tau)}{d\tau}\Big|_{\tau=p(S_1+t)}\right) \in \mathbb{T},$$

where $p : \mathbb{R} \to \mathbb{T}$ is the covering map of $\mathbb{T}$. Equivalently, $\Theta(\gamma, S_1)(t)$ is the oriented angle between χ and the vector tangent to γ at $\gamma(p(S_1 + t))$. Denote as $\tilde{\Theta}(\gamma, S_1) : \mathbb{R}_+ \to \mathbb{R}$ a continuous determination of such oriented angle function.

Definition 3.2. The angle variation along γ between $x = \gamma(s_1)$ and $y = \gamma(s_2)$ is

$$Var_\gamma(x, y) := \tilde{\Theta}(\gamma, S_1)(S_2 - S_1) - \tilde{\Theta}(\gamma, S_1)(0),$$

where S_1 is a lift of s_1 and S_2 is the lift of s_2 in $(S_1, S_+1]$.

Observe that the angle variation along γ does not depend on the chosen continuous determination $\tilde{\Theta}(\gamma, S_1)$.

Fact 3.1. We recall here some useful properties of the angle variation along a $\mathcal{C}^1$ essential curve γ. Let $x, y, z \in \gamma(\mathbb{T})$.

(1) $Var_\gamma(x, y)$ does not depend on the choice of the lift S_1 of $s_1 \in \mathbb{T}$ such that $\gamma(s_1) = x$;

(2) $Var_\gamma(x, x) = 0$;

(3) $Var_\gamma(x, y) + Var_\gamma(y, z) = Var_\gamma(x, z)$.

Remark 3.1. Fix $\gamma(s) \in \gamma(\mathbb{T})$. We observe that the function $\mathbb{R}_+ \ni t \mapsto Var_\gamma(\gamma(s), \gamma(s + p(t))) \in \mathbb{R}$ is 1-periodic.

Remark 3.2. An essential curve γ on the annulus is isotopic to either

$$\mathbb{T} \ni t \mapsto \mathbf{c}_1(t) = (t, 0) \qquad \text{or} \qquad \mathbb{T} \ni t \mapsto \mathbf{c}_{-1}(t) = (-t, 0).$$

Proposition 3.4. *Let $\gamma : \mathbb{T} \to \mathbb{A}$ be a $\mathcal{C}^1$ essential curve. Let $s_0, s_1 \in \mathbb{T}, s_0 \neq s_1$ correspond to points of maximal height on γ, that is*

$$p_2 \circ \gamma(s_0) = p_2 \circ \gamma(s_1) = \max_{s \in \mathbb{T}} p_2 \circ \gamma(s),$$

where $p_2 : \mathbb{A} \to \mathbb{R}$ is the projection over the second coordinate. Then

$$Var_\gamma(\gamma(s_0), \gamma(s_1)) = 0.$$

Proof. Let $s_0, s_1 \in \mathbb{T}, s_0 \neq s_1$ be such that $p_2 \circ \gamma(s_0) = p_2 \circ \gamma(s_1) = \max_{s \in \mathbb{T}} p_2 \circ \gamma(s)$. Let $S_0 \in \mathbb{R}$ be a lift of $s_0 \in \mathbb{T}$ and let $S_1 \in (S_0, S_0 + 1)$ be the lift of s_1.
Look now at the lifted framework in $\mathbb{R}^2$ and denote as $\Gamma : \mathbb{R} \to \mathbb{R}^2$ a lift of γ. Consider the points $\Gamma(S_0), \Gamma(S_1)$ and build the piecewise $\mathcal{C}^1$ closed curve $\mathscr{C}$ by concatenating the following ones (see Figure 2):

— $\{\Gamma(s) : \ s \in [S_0, S_1]\}$;

— the vertical segment $\{(p_1 \circ \Gamma(S_1), p_2 \circ \Gamma(S_1) + \xi) : \ \xi \in [0, 1]\}$.

— the horizontal segment $\{(\xi p_1 \circ \Gamma(S_0) + (1 - \xi) p_1 \circ \Gamma(S_1), p_2 \circ \Gamma(S_1) + 1) : \ \xi \in [0, 1]\}$;

— the vertical segment $\{(p_1 \circ \Gamma(S_0), p_2 \circ \Gamma(S_0) + 1 - \xi) : \ \xi \in [0, 1]\}$ [1];

Such a piecewise $\mathcal{C}^1$ closed curve does not have self-intersections because both $\Gamma(S_0)$ and $\Gamma(S_1)$ are points of maximal height. We are then interested in

$$Var_\gamma(\gamma(s_0), \gamma(s_1)) = Var_\Gamma(\Gamma(S_0), \Gamma(S_1)) = \tilde{\Theta}(\Gamma, S_0)(S_1 - S_0) - \tilde{\Theta}(\Gamma, S_0)(0).$$

1. Recall that $p_2 \circ \Gamma(S_0) = p_2 \circ \Gamma(S_1)$.

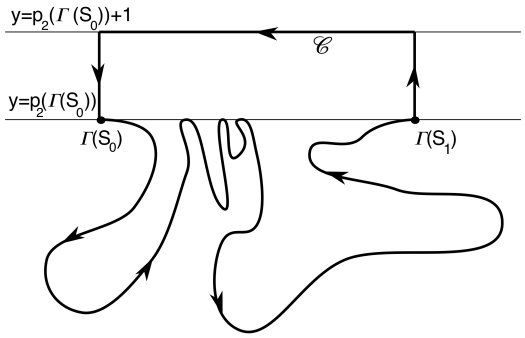

Figure 2 – The simple curve built in the proof of Proposition 3.4.

Claim 3.1. If γ is homotopic to $\mathbf{c}_1$ (respectively to $\mathbf{c}_{-1}$), then

$$p_1 \circ \Gamma(S_0) < p_1 \circ \Gamma(S_1) \qquad (\text{ resp. } p_1 \circ \Gamma(S_0) > p_1 \circ \Gamma(S_1)) \tag{9}$$

and

$$\Gamma'(S_0), \Gamma'(S_1) \in \mathbb{R}_+\mathcal{H} \qquad (\text{ resp. } \Gamma'(S_0), \Gamma'(S_1) \in \mathbb{R}_-\mathcal{H}.)\,. \tag{10}$$

Since both S_0 and S_1 are points of maximal height of Γ and since Γ is $\mathcal{C}^1$, both $\Gamma'(S_0)$ and $\Gamma'(S_1)$ are in $\mathbb{R}\mathcal{H}$. Let γ be homotopic to $\mathbf{c}_1$, the other case can be treated similarly. Assume, by contradiction, that $p_1 \circ \Gamma(S_1) < p_1 \circ \Gamma(S_0)$. Then, since $\Gamma(\mathbb{R} \setminus [S_0, S_1])$ cannot intersect the curve $\mathscr{C}$, it holds $\Gamma'(S_0) \in \mathbb{R}_-\mathcal{H}$, otherwise the unbounded set $\Gamma((-\infty, S_0])$ would be contained in a bounded region.

Since γ is homotopic to $\mathbf{c}_1$, there exists $n \in \mathbb{N}$ such that $p_1 \circ \Gamma(S_0 - n) < p_1 \circ \Gamma(S)$ for every $S \in [S_0, S_1]$. In particular, $\Gamma([S_0 - n, S_0))$ is contained in the unbounded region determined by $\mathscr{C}$. Since $p_2 \circ \Gamma(S_0 - n) = p_2 \circ \Gamma(S_0)$, we can build another curve $\mathscr{C}'$ as done with $\mathscr{C}$, but starting from $S_0 - n, S_0$. Thus, $\Gamma(S_1)$, as well as $\Gamma([S_1, +\infty))$, is contained in the bounded region determined by $\mathscr{C}'$, providing the required contradiction. This proves Claim 3.1.

In particular, if γ is homotopic to $\mathbf{c}_1$ (respectively to $\mathbf{c}_{-1}$) then the closed curve $\mathscr{C}$ is oriented counterclockwisely (respectively clockwisely).
Apply then the Turning Tangent Theorem to the simple piecewise $\mathcal{C}^1$ closed curve $\mathscr{C}$ described above (see Figure 2).
By (9) and (10), if γ is homotopic to $\mathbf{c}_1$, then we have

$$\tilde{\Theta}(\Gamma, S_0)(S_1 - S_0) - \tilde{\Theta}(\Gamma, S_0)(0) + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = 1.$$

A similar result can be obtained if γ is homotopic to $\mathbf{c}_{-1}$. Thus

$$\tilde{\Theta}(\Gamma, S_0)(S_1 - S_0) - \tilde{\Theta}(\Gamma, S_0)(0) = Var_\Gamma(\Gamma(S_0), \Gamma(S_1)) = Var_\gamma(\gamma(s_0), \gamma(s_1)) = 0.$$

This ends the proof of Proposition 3.4. □

Notation 3.1. Let $s_0 \in \mathbb{T}$ be a point of maximal height, that is such that $p_2 \circ \gamma(s_0) = \max_{t \in \mathbb{T}} p_2 \circ \gamma(t)$. Fix $S_0 \in \mathbb{R}$ a lift of s_0.

Definition 3.3 (Complexity of a $\mathcal{C}^1$ essential curve)**.** The complexity of the curve γ is

$$C(\gamma) := \sup_{t \in \mathbb{R}_+} |Var_\gamma(\gamma(p(S_0)), \gamma(p(S_0 + t))| = \max_{t \in [0,1]} |Var_\gamma(\gamma(p(S_0)), \gamma(p(S_0 + t))|,$$

where $p : \mathbb{R} \to \mathbb{T}$ is the covering map of $\mathbb{T}$.

Remark 3.3. By Proposition 3.4 and by the properties of the angle variation along γ (see Fact 3.1), we remark that the definition of $C(\gamma)$ is independent of the choice of the point $s_0 \in \mathbb{T}$ of maximal height.

We can now prove Proposition 3.1 for any $\mathcal{C}^1$ essential curve γ. More precisely, at the end of this section, we will show the following proposition.

Proposition 3.5. *Let $\gamma : \mathbb{T} \to \mathbb{A}$ be a $\mathcal{C}^1$ essential curve of complexity $C(\gamma)$. Let $n \in \mathbb{N}^*$. Then there exists $z(n) = \gamma(s_n) \in \gamma(\mathbb{T})$ such that*

$$|n\,Torsion_n(\gamma(s_n), \gamma'(s_n))| \leq C(\gamma).$$

Notation 3.2. For any $t \in \mathbb{R}_+$ denote as γ_t the curve

$$\mathbb{T} \ni s \mapsto \gamma_t(s) := f_t(\gamma(s)) \in \mathbb{A}.$$

Consider the maximal height function

$$M^h_\gamma : \mathbb{R}_+ \to \mathbb{R}$$

$$t \mapsto M^h_\gamma(t) := \max_{s\in\mathbb{T}} p_2 \circ \gamma_t(s).$$

For any $t \in \mathbb{R}_+$ denote

$$Argmax(p_2 \circ \gamma_t) = \{s \in \mathbb{T} :\ p_2 \circ \gamma_t(s) = M^h_\gamma(t)\}, \tag{11}$$

that is $Argmax(p_2 \circ \gamma_t)$ is the set of $s \in \mathbb{T}$ whose image through γ_t achieves the maximal height among $\gamma_t(\mathbb{T})$.
Observe that, since each γ_t is $\mathcal{C}^1$, for any $s \in Argmax(p_2 \circ \gamma_t)$ the tangent vector $\gamma_t'(s)$ belongs to $\mathbb{R}\mathcal{H}$. For any $t \in \mathbb{R}_+$ denote as s_t an element of $Argmax(p_2 \circ \gamma_t)$.

Notation 3.3. For any $t \in \mathbb{R}_+$ we denote as $t\,\mathrm{Torsion}_t(z, \xi)$ the angle variation $\tilde{v}(I, z, \xi)(t) - \tilde{v}(I, z, \xi)(0)$.

Notation 3.4. Define the function $\Phi : \mathbb{R}_+ \to \mathbb{Z}$

$$\mathbb{R}_+ \ni t \mapsto t\,\mathrm{Torsion}_t(\gamma(s_t), \gamma'(s_t)) + Var_\gamma(\gamma(s_0), \gamma(s_t)) \in \mathbb{R}. \tag{12}$$

The function Φ takes values in $\mathbb{Z}$ because if γ is homotopic to $\mathbf{c}_1$ (respectively to $\mathbf{c}_{-1}$) then both $Df_t(\gamma(s_t))\gamma'(s_t)$ and $\gamma'(s_0)$ belongs to $\mathbb{R}_+\mathcal{H}$ (respectively $\mathbb{R}_-\mathcal{H}$) (see (10)).

The idea of considering points of maximal (respectively minimal) height on a curve is due to P. Le Calvez (see Section 5 in [LC91]).

Lemma 3.1. *For any $t \in \mathbb{R}_+$, the value $\Phi(t)$ does not depend on the choice of $s_t \in Argmax(p_2 \circ \gamma_t)$.*

Proof. Let $s_t, \bar{s}_t \in Argmax(p_2 \circ \gamma_t), s_t \neq \bar{s}_t$. From Proposition 3.4 it holds that

$$Var_{\gamma_t}(s_t, \bar{s}_t) = 0. \tag{13}$$

Recall that the torsion at finite-time does not depend on the chosen continuous determination of the oriented angle function. So we calculate the torsion at $\gamma(\bar{s}_t)$ using the continuous lift

$$\mathbb{R}_+ \ni \tau \mapsto \tilde{v}(f, \gamma(s_t), \gamma'(s_t))(\tau) + Var_{\gamma_\tau}(\gamma_\tau(s_t), \gamma_\tau(\bar{s}_t)) \in \mathbb{R}, \tag{14}$$

where $\tilde{v}(f, \gamma(s_t), \gamma'(s_t))(\cdot)$ is a continuous lift of the angle function $\tau \mapsto \theta(\chi, Df_\tau(\gamma(s_t))\gamma'(s_t))$. In particular, the function in (14) is a continuous determination of the angle function $\tau \mapsto \theta(\chi, \gamma'_\tau(\bar{s}_t))$. The value $\Phi(t)$ calculated with respect to $\bar{s}_t \in Argmax(p_2 \circ \gamma_t)$ is

$$t \operatorname{Torsion}_t(\gamma(\bar{s}_t), \gamma'(\bar{s}_t)) + Var_\gamma(\gamma(s_0), \gamma(\bar{s}_t)).$$

Write then $t \operatorname{Torsion}_t(\gamma(\bar{s}_t), \gamma'(\bar{s}_t))$ using the continuous determination in (14). Using the properties of Var_{γ_t} (see Fact 3.4) and by (13), we can conclude that

$$t \operatorname{Torsion}_t(\gamma(\bar{s}_t), \gamma'(\bar{s}_t)) + Var_\gamma(\gamma(s_0), \gamma(\bar{s}_t)) = t \operatorname{Torsion}_t(\gamma(s_t), \gamma'(s_t)) + Var_\gamma(\gamma(s_0), \gamma(s_t)),$$

that is $\Phi(t)$ does not depend on the choice of $s_t \in Argmax(p_2 \circ \gamma_t)$.

□

Lemma 3.2. *The function $\Phi : \mathbb{R}_+ \to \mathbb{Z}$ is the constant zero function.*

Proof. If we show that Φ is continuous, then, since Φ takes values in $\mathbb{Z}$ and since $\Phi(0) = 0$, we conclude that Φ is the constant zero function.
Consider the function $\Phi_{|[0,1]} : [0, 1] \to \mathbb{R}$. If its graph is compact, then $\Phi_{|[0,1]}$ is continuous. Denote for any $t \in [0, 1]$

$$\mathbb{K}_t = \{s \in \mathbb{T} : \ s \in Argmax(p_2 \circ \gamma_t)\} \times \{t\}$$

and

$$\mathbb{K} = \bigcup_{t\in[0,1]} \mathbb{K}_t = \bigcup_{t\in[0,1]} \{(s, t) : \ s \in Argmax(p_2 \circ \gamma_t)\} \subset \mathbb{T} \times [0, 1].$$

The set $\mathbb{K}$ is bounded. Let $(s_n, t_n)_{n\in\mathbb{N}} \subset \mathbb{K}$ be a sequence converging to (s, t). The sequence $(t_n)_{n\in\mathbb{N}} \subset [0, 1]$ converges to $t \in [0, 1]$. Moreover, $s \in Argmax(p_2 \circ \gamma_t)$. That is, $\mathbb{K}$ is closed. Consequently, $\mathbb{K}$ is compact. Consider now the function

$$\mathbb{K} \ni (s, t) \mapsto (t, t \operatorname{Torsion}_t(\gamma(s), \gamma'(s)) + Var_\gamma(\gamma(s_0), \gamma(s))) \in [0, 1] \times \mathbb{R}.$$

It is continuous and, since $\mathbb{K}$ is compact, its image is compact too. Observe that its image is actually the graph of the function $\Phi_{|[0,1]}$. Thus $\Phi_{|[0,1]}$ is continuous.
Using the same argument, we deduce that the function Φ is continuous on every compact $[0, n]$ for $n \in \mathbb{N}^*$. Consequently, the function $\Phi : \mathbb{R}_+ \to \mathbb{Z}$ is continuous. This concludes the proof.

□

Proof of Proposition 3.5. Fix $n \in \mathbb{N}^*$ and let $s_n \in Argmax(p_2 \circ \gamma_n)$. By Lemma 3.1 the value $\Phi(n)$ does not depend on the element of $Argmax(p_2 \circ \gamma_n)$ and by Lemma 3.2 the function Φ is the constant zero function. Therefore

$$\Phi(n) = n \operatorname{Torsion}_n(\gamma(s_n), \gamma'(s_n)) + Var_\gamma(\gamma(s_0), \gamma(s_n)) = 0.$$

Thus

$$|n \operatorname{Torsion}_n(\gamma(s_n), \gamma'(s_n))| = |Var_\gamma(\gamma(s_0), \gamma(s_n))| \leq C(\gamma),$$

that is $z(n) := \gamma(s_n) \in \gamma(\mathbb{T})$ is the required point.

□

3.3 A partial result on $\mathcal{C}^0$ essential curves

It seems natural to wonder if Theorem 1.1 can be generalised to continuous essential curves. We provide here a partial result.

Theorem 3.1. *Let $f : \mathbb{A} \to \mathbb{A}$ be a negative-torsion map. Let $\gamma : \mathbb{T} \to \mathbb{A}$ be a $\mathcal{C}^0$ essential curve such that $\gamma(\mathbb{T})$ is the graph of a function. Then there exists a point $z \in \gamma(\mathbb{T})$ of zero torsion.*

The result follows from Proposition 3.2 and from the following

Proposition 3.6. *Let $f : \mathbb{A} \to \mathbb{A}$ be a $\mathcal{C}^1$ diffeomorphism isotopic to the identity. Let $\gamma : \mathbb{T} \to \mathbb{A}$ be a continuous essential curve such that $\gamma(\mathbb{T})$ is a graph. Let $n \in \mathbb{N}^*$. Then there exists $z(n) = \gamma(s_n) \in \gamma(\mathbb{T})$ such that*

$$|n\,Torsion_n(\gamma(s_n), \chi)| \leq \frac{1}{4}. \tag{15}$$

For the proof of Proposition 3.6 we need to introduce the notion of *tilt* angle variation.

Definition 3.4. Let $\psi : \mathbb{R} \to \mathbb{A}$ be a $\mathcal{C}^1$ embedded curve such that $\lim_{t\to\pm\infty} p_2 \circ \psi(t) = \pm\infty$.
The angle function $tilt(\psi)$ is defined by

$$\mathbb{R} \ni t \mapsto tilt(\psi)(t) := \theta(\chi, \psi'(t)) \in \mathbb{T},$$

where $\theta(u, v)$ denotes the oriented angle between the non-zero vectors u, v.
Let $\widetilde{tilt}(\psi) : \mathbb{R} \to \mathbb{R}$ be the continuous determination of the angle function $tilt(\psi)$ such that if $t \in \mathbb{R}$ is such that

$$p_2 \circ \psi(t) > p_2 \circ \psi(s) \quad \forall s < t,$$

then $\widetilde{tilt}(\psi)(t) \in [-\frac{1}{4}, \frac{1}{4}]$.

For more details about the well-definition of the lift $\widetilde{tilt}(\psi)$, we refer to [Hu98] and [Flo19a, Lemma 2.3.2].

Notation 3.5. For every $z \in \mathbb{A}$ denote as

$$\mathbb{R} \ni t \mapsto V_z(t) = (p_1(z), t) \in \mathbb{A}$$

the vertical line passing through z.

We can calculate the finite-time torsion looking at the continuous determination $\widetilde{tilt}$.

Proposition 3.7. *Let $f : \mathbb{A} \to \mathbb{A}$ be a $\mathcal{C}^1$ diffeomorphism isotopic to the identity. Then for any $z = (x, y) \in \mathbb{A}$ it holds*

$$Torsion_1(z, \chi) = \widetilde{tilt}(f \circ V_z)(y). \tag{16}$$

Proof. Recall that the time-one torsion at $z \in \mathbb{A}$ with respect to the vertical vector χ is

$$\tilde{v}(f, z, \chi)(1) - \tilde{v}(f, z, \chi)(0),$$

where $t \mapsto \tilde{v}(f, z, \chi)(t)$ is a lift of the oriented angle function $t \mapsto \theta(\chi, Df_t(z)\chi)$. Consider $\widetilde{tilt}(f \circ V_z)$. Observe that both $\tilde{v}(f, z, \chi)(1)$ and $\widetilde{tilt}(f \circ V_z)(y)$ are preimages of the same

oriented angle $\theta(\chi, Df(z)\chi)$.
The continuous function

$$\mathbb{A} \ni z = (x, y) \mapsto \Psi(z) := \text{Torsion}_1(z, \chi) - \widetilde{tilt}(f \circ V_z)(y) \in \mathbb{R}$$

takes value in $\mathbb{Z}$. Thus, it is constant.
We are going to exhibit a point $z \in \mathbb{A}$ such that $\Psi(z) = 0$. Thus we will conclude that for any $(x, y) \in \mathbb{A}$

$$\text{Torsion}_1((x, y), \chi) = \widetilde{tilt}(f \circ V_{(x,0)})(y). \tag{17}$$

Consider the $\mathcal{C}^1$ essential curve $\mathbb{T} \times \{0\}$ and its image $f(\mathbb{T} \times \{0\})$. The complexity of the curve $\mathbb{T} \times \{0\}$ is clearly zero, see Definition 3.3.
Let $\bar{z} = (x, 0) \in \mathbb{T} \times \{0\}$ correspond to a point of maximal height of $f(\mathbb{T} \times \{0\})$, that is

$$p_2 \circ f(\bar{z}) = \max_{\xi \in \mathbb{T} \times \{0\}} p_2 \circ f(\xi).$$

By Lemmas 3.1 and 3.2, it holds that $\text{Torsion}_1(\bar{z}, \mathcal{H}) = 0$. By Property 2.1 we have

$$|\text{Torsion}_1(\bar{z}, \chi)| = |\text{Torsion}_1(\bar{z}, \chi) - \text{Torsion}_1(\bar{z}, \mathcal{H})| < \frac{1}{2}. \tag{18}$$

We now show that the point $\bar{z} = (x, 0) \in \mathbb{T} \times \{0\}$ is such that for any $s < 0$ it holds

$$p_2 \circ f(x, 0) > p_2 \circ f(x, s).$$

Indeed if by contradiction there exists $\hat{s} < 0$ such that $p_2 \circ f(x, \hat{s}) \geq p_2 \circ f(x, 0)$, then, since $f(x, 0)$ is a point of maximal height of $f(\mathbb{T} \times \{0\})$ and since f preserves the boundaries, the curve $f(\mathbb{T} \times \{0\})$ should intersect the curve $\{f(x, \xi) : \ \xi \leq \hat{s}\}$. This contradicts the fact that $(\mathbb{T} \times \{0\}) \cap \{(x, \xi) : \ \xi < 0\}$ is empty and that f is a diffeomorphism.
Consequently, from the definition of $\widetilde{tilt}$ (see Definition 3.4) we have

$$\widetilde{tilt}(f \circ V_x)(0) \in \left[-\frac{1}{4}, \frac{1}{4}\right]. \tag{19}$$

Look now at $\text{Torsion}_1((x, 0), \chi)$. Choose the continuous determination such that $\tilde{v}(f, (x, 0), \chi)(0) = 0$. Both $\tilde{v}(f, (x, 0), \chi)(1)$ and $\widetilde{tilt}(f \circ V_x)(0)$ are preimages of the same angle. Thus

$$\tilde{v}(f, (x, 0), \chi)(1) - \widetilde{tilt}(f \circ V_x)(0) = \text{Torsion}_1((x, 0), \chi) - \widetilde{tilt}(f \circ V_x)(0) \in \mathbb{Z}.$$

From (18) and from (19), we have that

$$\text{Torsion}_1((x, 0), \chi) = \widetilde{tilt}(f \circ V_x)(0),$$

concluding so the proof.

□

Proof of Proposition 3.6. Fix $n \in \mathbb{N}^*$. Consider $f^n \circ \gamma(\mathbb{T})$ and let $s_n \in \mathbb{T}$ be such that

$$p_2 \circ f^n \circ \gamma(s_n) = \max_{s \in \mathbb{T}} p_2 \circ f^n \circ \gamma(s).$$

Denote as $V_{\gamma(s_n)}$ the vertical line passing through $\gamma(s_n)$. Denote $\gamma(s_n) = V_{\gamma(s_n)}(y_n)$. Since $\gamma(\mathbb{T})$ is a graph, $V_{\gamma(s_n)}$ intersects $\gamma(\mathbb{T})$ only once.
Consequently for every $y < y_n$ it holds

$$p_2 \circ f^n \circ V_{\gamma(s_n)}(y) < p_2 \circ f^n \circ V_{\gamma(s_n)}(y_n) = p_2 \circ f^n \circ \gamma(s_n)$$

because $f^n \circ \gamma(s_n)$ is a point of maximal height of $f^n \circ \gamma(\mathbb{T})$ and because f^n preserves the boundaries (otherwise we would have another point of intersection between $\gamma(\mathbb{T})$ and $V_{\gamma(s_n)}(\mathbb{R})$).
By the definition of the continuous determination $\widetilde{tilt}(V_{\gamma(s_n)})$, see Definition 3.4, we have $\widetilde{tilt}(V_{\gamma(s_n)})(y_n) \in [-\frac{1}{4}, \frac{1}{4}]$.
By Proposition 3.7 we conclude that [2]

$$\text{Torsion}_1(f^n, \gamma(s_n), \chi) = n\text{Torsion}_n(f, \gamma(s_n), \chi) \in \left[-\frac{1}{4}, \frac{1}{4}\right].$$

□

4 A Birkhoff's theorem through torsion

Using the tool of torsion, we can prove a Birkhoff's-theorem-like result (see [Bir22] and [Her83]) in a different hypothesis framework, see Theorem 1.2. The idea of using the torsion in order to prove a Birkhoff's-theorem-like result was already present in the work of M. Bialy and L. Polterovich (see [BP89], [Pol91] and [BP92]). This result arises from a question by V. Humiliére.
On one hand we do not require that f is either a twist map or a conservative map. On the other hand f has to be a negative-torsion (positive-torsion) map and we require that the dynamics restricted to the $\mathcal{C}^1$ curve is non-wandering (see [KH95]).

Lemma 4.1. *Let γ be a $\mathcal{C}^1$ essential curve. If γ is transversal to the vertical at every point, then γ is the graph of a function.*

Proof. Let $\Gamma : \mathbb{R} \to \mathbb{R}^2$ be a lift of γ. Consider the $\mathcal{C}^1$ function $p_1 \circ \Gamma : \mathbb{R} \to \mathbb{R}$. Since Γ is transversal to the vertical at every point, it holds $D(p_1 \circ \Gamma)(t) \neq 0$ for every $t \in \mathbb{R}$. Without loss of generality assume that it is positive at every $t \in \mathbb{R}$. Thus, $p_1 \circ \Gamma$ is an increasing diffeomorphism to its image.
Since Γ is a lift of an essential curve and since $p_1 \circ \Gamma$ is increasing, we have that for every $t \in \mathbb{R}$

$$p_1 \circ \Gamma(t+1) = p_1 \circ \Gamma(t) + 1.$$

In particular, we deduce that $p_1 \circ \Gamma(\mathbb{R}) = \mathbb{R}$. That is $p_1 \circ \Gamma$ is a $\mathcal{C}^1$ diffeomorphism. Denote $\phi = (p_1 \circ \Gamma)^{-1}$. Consequently, the $\mathcal{C}^1$ function $\mathbb{R} \ni s \mapsto p_2 \circ \Gamma \circ \phi(s) \in \mathbb{R}$ is such that its graph is $\Gamma(\mathbb{R})$.
The function $p_2 \circ \Gamma \circ \phi$ is 1-periodic. Thus, its projection on the annulus is well-defined and $\gamma(\mathbb{T})$ is the graph of the $\mathcal{C}^1$ function $\psi : \mathbb{T} \to \mathbb{R}$ such that $\psi \circ p = p_2 \circ \Gamma \circ \phi$, where $p : \mathbb{R} \to \mathbb{T}$ is the covering map of $\mathbb{T}$. □

We will now provide an upper bound of the N-finite time torsion along the curve γ. The bound is independent from N.

Notation 4.1. Let $x \in \mathbb{A}$ and let $\delta \in (0, \frac{1}{4})$. Denote

$$C(x, \chi, \delta) := \{v \in T_x\mathbb{A} : \ \theta(\chi, v) \text{ or } \theta(-\chi, v) \text{ admits a preimage in } (-\delta, \delta)\}.$$

2. Here we make explicit the dynamics with respect to which we are calculating the torsion, since it changes.

Lemma 4.2. *Let $f : \mathbb{A} \to \mathbb{A}$ be a negative-torsion map and let K be a compact f-invariant set. There exist $\varepsilon \in (0, \frac{1}{2})$ and $\delta \in (0, \frac{\varepsilon}{4})$ such that for any $x \in K$, for any $v \in C(x, \chi, \delta)$ and for any $N \in \mathbb{N}^*$ it holds*

$$N\,Torsion_N(x, v) < -\frac{\varepsilon}{2} < 0. \tag{20}$$

Proof. Let us argue by induction. Since f is a negative-torsion map, since K is compact and by the continuity of the torsion at time 1, there exist $\varepsilon \in (0, \frac{1}{2})$ and $\delta \in (0, \frac{\varepsilon}{4})$ such that for every $x \in K$ and for every $v \in C(x, \chi, \delta)$ it holds

$$\text{Torsion}_1(x, v) < -\varepsilon < 0. \tag{21}$$

Assume now that the result holds for $N - 1$. Let $x \in K$ and let $v \in C(x, \chi, \delta)$. Without loss of generality assume that the oriented angle $\theta(\chi, v)$ admits a preimage in $(-\delta, \delta)$. The case of $\theta(-\chi, v)$ admitting a preimage in $(-\delta, \delta)$ can be discussed similarly. Choose a continuous determination of the angle so that $\tilde{v}(I, x, v)(0) \in (-\delta, \delta)$. By inductive hypothesis it holds

$$\tilde{v}(I, x, v)(N - 1) < -\frac{\varepsilon}{2} + \delta < -\frac{\varepsilon}{4}.$$

Consider now the continuous determination such that $\tilde{v}(I, f^{N-1}(x), \chi)(0) = 0$. We point out the fact that we are considering a continuous determination with respect to a different point in $T_K\mathbb{A}$. In particular

$$\tilde{v}(I, x, v)(N - 1) < \tilde{v}(I, f^{N-1}(x), \chi)(0).$$

From Property 2.2, from the choice of the continuous determinations and by the base case, we have that

$$\tilde{v}(I, x, v)(N) < \tilde{v}(I, f^{N-1}(x), \chi)(1) = \tilde{v}(I, f^{N-1}(x), \chi)(1) - \tilde{v}(I, f^{N-1}(x), \chi)(0) < -\varepsilon.$$

Consequently, by the choice of $\tilde{v}(I, x, v)(\cdot)$, we conclude that

$$N\text{Torsion}_N(x, v) = \tilde{v}(I, x, v)(N) - \tilde{v}(I, x, v)(0) < -\varepsilon + \delta < -\frac{\varepsilon}{2}.$$

□

Lemma 4.3. *Let $f : \mathbb{A} \to \mathbb{A}$ be a negative-torsion map. Let $\gamma : \mathbb{T} \to \mathbb{A}$ be a $\mathcal{C}^1$ f-invariant essential curve. Then for every $s \in \mathbb{T}$ and for every $N \in \mathbb{N}^*$ it holds*

$$N\,Torsion_N(\gamma(s), \gamma'(s)) = Var_\gamma(\gamma(s), \gamma(s_N)), \tag{22}$$

where $f^N \circ \gamma(s) = \gamma(s_N)$.

Proof. We start observing that for every $s \in \mathbb{T}$ both $\text{Torsion}_1(\gamma(s), \gamma'(s))$ and $Var_\gamma(\gamma(s), \gamma(s_1))$ are variations of the same oriented angle, where $\gamma(s_1) = f \circ \gamma(s)$. In particular there exists $k \in \mathbb{Z}$ such that for any $s \in \mathbb{T}$

$$\text{Torsion}_1(\gamma(s), \gamma'(s)) = Var_\gamma(\gamma(s), \gamma(s_1)) + k. \tag{23}$$

The integer $k \in \mathbb{Z}$ does not depend on $s \in \mathbb{T}$.
By Theorem 1.1 there exists a point $\gamma(s_\infty) \in \gamma(\mathbb{T})$ such that $\text{Torsion}(\gamma(s_\infty)) = 0$.
At the same time, since γ is f-invariant and from (23), we have that for any $N \in \mathbb{N}^*$

$$N\text{Torsion}_N(\gamma(s_\infty), \gamma'(s_\infty)) = Nk + \sum_{i=0}^{N-1} Var_\gamma(\gamma(s_i), \gamma(s_{i+1}) = Nk + Var_\gamma(\gamma(s_\infty), \gamma(s_N)),$$

where for every $i \in [\![0, N]\!]$ we denote as $s_i \in \mathbb{T}$ the point such that $\gamma(s_i) = f^i \circ \gamma(s_\infty)$.
Since $\gamma(s_\infty)$ has zero torsion and since we have $|Var_\gamma(\gamma(s_\infty), \gamma(s_N))| \leq C(\gamma) < +\infty$, we conclude that $k = 0$.
In particular for every $s \in \mathbb{T}$ and every $N \in \mathbb{N}^*$ it holds

$$N\text{Torsion}_N(\gamma(s), \gamma'(s)) = Var_\gamma(\gamma(s), \gamma(s_N)),$$

where $\gamma(s_N) = f^N \circ \gamma(s)$. □

An outcome of Lemma 4.3 is the following corollary, already proved by S. Crovisier for twist maps in [Cro03].

Corollary 4.1. *Let $f : \mathbb{A} \to \mathbb{A}$ be a negative-torsion map. Let $\gamma : \mathbb{T} \to \mathbb{A}$ be a $\mathcal{C}^1$ essential f-invariant curve on $\mathbb{A}$. Then, for any $s \in \mathbb{T}$ it holds Torsion$(\gamma(s)) = 0$.*

We can now finally prove Theorem 1.2.

Proof of Theorem 1.2. Argue by contradiction and assume that γ is not a graph. Then from Lemma 4.1 there exists a point $z = \gamma(s)$ such that $\gamma'(s) \in \mathbb{R}\chi$. Denote

$$\chi' = \begin{cases} \chi & \text{if } \gamma'(s) \in \mathbb{R}_+\chi, \\ \\ -\chi & \text{if } \gamma'(s) \in \mathbb{R}_-\chi. \end{cases}$$

Let $\varepsilon \in (0, \frac{1}{2})$ and $\delta \in (0, \frac{\varepsilon}{2})$ be the parameters of Lemma 4.2 applied at the f-invariant compact set $\gamma(\mathbb{T})$.
Let $U \subset \mathbb{T}$ be a neighborhood of s such that for any $t \in U$ the oriented angle $\theta(\chi', \gamma'(t))$ admits a preimage in $(-\delta, \delta)$. The dynamics $f_{|\gamma}$ is non-wandering and therefore there exists $N \in \mathbb{N}$ and $\tau \in U$ such that $\tau_N \in U$ where $f^N \circ \gamma(\tau) = \gamma(\tau_N)$.
From Lemma 4.3 it holds $N\text{Torsion}_N(f, \gamma(\tau), \gamma'(\tau)) = Var_\gamma(\gamma(\tau), \gamma(\tau_N))$. Observe that $Var_\gamma(\gamma(\tau), \gamma(\tau_N)) \in (-2\delta, 2\delta)$.
Consequently, since $\delta \in (0, \frac{\varepsilon}{4})$, we conclude that

$$N\text{Torsion}_N(\gamma(\tau), \gamma'(\tau)) \in \left(-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right).$$

This contradicts Lemma 4.2 and we conclude.

□

Remark 4.1. We have shown that the curve γ is the graph of a function and it is always transverse to the vertical. Thus, since γ is $\mathcal{C}^1$, we deduce that γ is the graph of a $\mathcal{C}^1$ function.

Remark 4.2. In order to obtain the result of Theorem 1.2 we need information over the dynamics on the curve. Indeed, there exist non conservative positive twist maps that admit $\mathcal{C}^1$ essential f-invariant curves which are not graphs of function. See Proposition 15.3 in [LC88].

References

[Ang88] S. B. Angenent. The periodic orbits of an area preserving twist map. *Communications in Mathematical Physics*, 115(3):353–374, 1988.

[BB13] François Béguin and Zouhour Rezig Boubaker. Existence of orbits with non-zero torsion for certain types of surface diffeomorphisms. *Journal of the Mathematical Society of Japan*, 65(1):137–168, 2013.

[Bir22] George D. Birkhoff. Surface transformations and their dynamical applications. *Acta Mathematica*, 43(1):1–119, 1922.

[BP89] Misha Bialy and Leonid Polterovich. Lagrangian singularities of invariant tori of Hamiltonian systems with two degrees of freedom. *Inventiones Mathematicae*, 97(2):291–303, 1989.

[BP92] Misha Bialy and Leonid Polterovich. Hamiltonian systems, Lagrangian tori and Birkhoff's theorem. *Mathematische Annalen*, 292(4):619–627, 1992.

[Cro03] Sylvain Crovisier. Ensembles de torsion nulle des applications déviant la verticale. *Bulletin de la Société mathématique de France*, 131(1):23–39, 2003.

[Flo19a] Anna Florio. *Asymptotic Maslov indices*. PhD thesis, Avignon Université, 2019.

[Flo19b] Anna Florio. Torsion and linking number for a surface diffeomorphism. *Mathematische Zeitschrift*, 292(1-2):231–265, 2019.

[GR13] Marian Gidea and Clark Robinson. Diffusion along transition chains of invariant tori and Aubry-Mather sets. *Ergodic Theory and Dynamical Systems*, 33(5):1401–1449, 2013.

[Gra73] André Gramain. Le type d'homotopie du groupe des difféomorphismes d'une surface compacte. *Ann. Sci. École Norm. Sup. (4)*, 6:53–66, 1973.

[Her83] Michael R. Herman. *Sur les courbes invariantes par les difféomorphismes de l'anneau. Vol. 1*, volume 103 of *Astérisque*. Société Mathématique de France, Paris, 1983. With an appendix by Albert Fathi, With an English summary.

[Hir76] Morris W. Hirsch. *Differential topology*. Springer-Verlag, new York-Heidelberg, 1976. Graduate Texts in Mathematics, No. 33.

[Hu98] Sen Hu. A variational principle associated to positive tilt maps. *Communications in Mathematical Physics*, 191(3):627–639, 1998.

[KH95] Anatole Katok and Boris Hasselblatt. *Introduction to the Modern Theory of Dynamical Systems*. Cambridge University Press, 1995.

[LC88] Patrice Le Calvez. Propriétés des attracteurs de Birkhoff. *Ergodic Theory and Dynamical Systems*, 8(2):241–310, 1988.

[LC91] Patrice Le Calvez. Propriétés dynamiques des difféomorphismes de l'anneau et du tore. *Astérisque*, (204):131, 1991.

[Mat82] John N. Mather. Existence of quasiperiodic orbits for twist homeomorphisms of the annulus. *Topology. An International Journal of Mathematics*, 21(4):457–467, 1982.

[Mat91] John N. Mather. Variational construction of orbits of twist diffeomorphisms. *Journal of the American Mathematical Society*, 4(2):207–263, 1991.

[Pol91] Leonid Polterovich. The second Birkhoff theorem for optical Hamiltonian systems. *Proceedings of the American Mathematical Society*, 113(2):513–516, 1991.

[Rue85] David Ruelle. Rotation numbers for diffeomorphisms and flows. *Annales de l'Institute Henri Poincaré. Physique théorique*, 42(1):109–115, 1985.

[Sma59] Stephen Smale. Diffeomorphisms of the 2-sphere. *Proc. Amer. Math. Soc.*, 10:621–626, 1959.