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Convergence towards equilibrium for a model with partial diffusion

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Abstract

We study the asymptotic behavior of a two dimensional linear PDE with a degenerate diffusion and a drift term. The structure of this equation typically arises in some mathematical mean fields models of neural network, and the investigation of the qualitative properties of this equation is still open, and a challenging question. We prove, via a Doeblin-Harris type method, that the solutions converge exponentially fast to the unique stationary state in a L^1 -weighted norm.

1 Introduction

The study of partial differential equations (PDE) associated to neurosciences has seen a growing interest in recent years. These models often involve a mixture of effects leading to singular terms, heterogeneities, nonlocal terms, jumps, partial diffusion mechanisms and long term memory effects, which overall make their qualitative study somewhat challenging (see for instance [25, 17, 2, 24, 16, 5, 13, 18, 22]). In this paper, we propose to study the following toy model, which is directly derived from a model in [21]. It reads

$$\partial_t u(t, x, y) - \partial_x(xu) - \partial_y(yu) - \partial_{xx}u = \delta_{x=1}N(t, y), \quad (1.1)$$

$$N(t, y) = \partial_x u(t, 0, y - 1), \quad t \geq 0, \quad x \geq 0, \quad y \in \mathbb{R}, \quad (1.2)$$

$$u(t, 0, y) = 0, \quad u(t, x, y) = 0 \text{ if } y \leq 0, \quad u(0, x, y) \geq 0, \quad \int_{x=0}^{+\infty} \int_{y=0}^{+\infty} u(0, x, y) dx dy = 1. \quad (1.3)$$

Modulo a change of sign in the variable x , the solution $u(t, x, y)$ can be interpreted as the probability density of a neuron to be at the membrane potential x with an adaptation current y at time t . The point $x = 0$ models the activation potential, and $x = 1$ the reset value. The function $N(t, y)$ models the flux of neurons which discharge at time t with an adaptation current $y - 1$. If we integrate the equation in the y variable, we find that $w(t, x) := \int_{y=0}^{+\infty} u(t, x, y) dy$ is a solution to

$$\partial_t w(t, x) - \partial_x(xw) - \partial_{xx}w = \delta_{x=1}\mathcal{N}(t),$$

$$\mathcal{N}(t) = \partial_x w(t, 0), \quad w(t, 0) = 0, \quad w(0, x) \geq 0, \quad \int_{x=0}^{+\infty} w(0, x) dx = 1.$$

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In other words w is solution to the classical LIF Fokker Planck equation, which was widely studied in the linear and non linear case in several articles (see for instance [14, 8, 9, 11, 12, 5, 23, 6, 26]). In particular, it was proven in [5] that, in the linear case, the solution converges exponentially fast to its stationary state, in an exponential weighted norm in L^2 . The proof consists of the use of an entropy inequality combined with Poincaré type estimates. The adaptation of this strategy fails in the context of Equation (1.1), due to the degenerate diffusion. The so-called hypocoercivity method [1, 28, 15, 20] are other possible methods which do not apply immediately here. For this reason, we propose here an entirely different approach based on an adaptation of the so-called Doeblin-Harris method. The latter has proved very performant in a number of contexts recently, as a growing number of papers have focused on the use of this method for many equations from neuroscience or physics [4, 3, 10, 16, 17, 27, 19]. Following these ideas, we eventually prove exponential convergence to the stationary state for Equation (1.1). More precisely, the next theorem holds where, here and in the sequel, we denote by $S(t)u_0(x, y) := u(t, x, y)$ the solution operator¹:

Theorem 1.1 *There exist two constants $C > 0$, $\mu > 0$, such that for all initial data u_1, u_2 satisfying (1.3), the following estimate holds*

$$\iint (1 + |y| + |x|^2) |S(t)u_1 - S(t)u_2|(x, y) dx dy \leq C e^{-\mu t} \iint (1 + |y| + |x|^2) |u_1 - u_2|(x, y) dx dy. \quad (1.4)$$

As a consequence, there exists a unique stationary state u^* of (1.1), (1.2), (1.3), with

$$\iint (1 + |y| + |x|^2) u^*(x, y) dx dy < +\infty,$$

and for all initial data u_0 satisfying (1.3),

$$\iint (1 + |y| + |x|^2) |u(t, x, y) - u^*(x, y)| dx dy \leq C e^{-\mu t} \iint (1 + |y| + |x|^2) |u(0, x, y) - u^*(x, y)| dx dy,$$

for all $t \geq 0$.

The Doeblin method (see e.g. [4]) relies on the following simple and efficient result. Assume that $\Omega \subset \mathbb{R}^d$, $d \geq 1$, and that $S(t) : L^1(\Omega) \rightarrow L^1(\Omega)$ is a linear semi-group for $t \geq 0$ which preserves mass and positivity: for all $f \in L^1(\Omega)$

$$\int_{\Omega} S(t)f(x) dx = \int_{\Omega} f(x) dx, \quad \text{and} \quad f \geq 0 \Rightarrow S(t)f \geq 0.$$

Assume also that for some time $t_0 > 0$, one is able to prove a uniform non trivial lower bound on the solution, relative to its initial mass. More precisely, assume that there exists $t_0 > 0$, $\alpha \in]0, 1[$, and a non negative $\nu \in L^1(\Omega)$ with total mass 1 such that for all non negative $f \in L^1(\Omega)$

$$S(t_0)f \geq \alpha \nu \int_{\Omega} f. \quad (1.5)$$

Then $S(t)$ possesses a unique non negative stationary state $f^* \in L^1(\Omega)$ with unit mass, and moreover there exists $C > 0$ and $\mu > 0$ such that for all $f \in L^1(\Omega)$, and for all $t \geq 0$

$$\|S(t)f - f^*\|_{L^1(\Omega)} \leq C e^{-\mu t} \|f - f^*\|_{L^1(\Omega)}.$$

¹Although we shall not study the existence part for the Cauchy problem here, our estimates below could actually be used to do so.

The key observation leading to this result is the following. Take $f \in L^1(\Omega)$ such that $\int_{\Omega} f = 0$ and decompose $f = f_+ - f_-$, where f_{\pm} are respectively the positive and the negative part of f , in particular

$$\int |f| = 2 \int f_+ = 2 \int f_-.$$

We write then

$$S(t_0)f = \left(S(t_0)f_+ - \frac{\alpha}{2} \int_{\Omega} |f|(x) dx \, \nu \right) - \left(S(t_0)f_- - \frac{\alpha}{2} \int_{\Omega} |f|(x) dx \, \nu \right) =: g_1 - g_2,$$

where ν and α are given in (1.5). Since the functions g_1 and g_2 are non negative by (1.5), it follows that

$$\int |S(t_0)f|(x) dx \leq \int g_1(x) dx + \int g_2(x) dx = (1 - \alpha) \int |f|(x) dx,$$

where we have used the fact that $S(\cdot)$ is mass preserving. The existence of a unique stationary state and exponential convergence towards it can then be proved following standard semi-group theory arguments, mainly through iteration, the details of which can be found e.g. in [4].

In order to prove Theorem 1.1, we will have to adapt the previous strategy because the equivalent of Estimate (1.5) clearly cannot hold for the semi-group associated to Equation (1.1), in particular due to the unboundedness of our domain. To overcome this difficulty, we shall first prove some uniform lower bounds in the spirit of (1.5), provided some control of the initial data in suitable weighted norms holds (see Theorem 3.1). For that purpose, and because the diffusion term only operates in the variable x , we need to understand the delicate mechanism that allows mass to spread along y as well. The next difficulty for the proof of Theorem 1.1, assuming Theorem 3.1 holds, is that we cannot iterate easily the contraction argument as one would do in a classical Doeblin method. Instead, in the spirit of Doeblin-Harris type method (see [7]), we need to understand the global dynamics of the solution with respect to the added weighted norm, combined with the information of the uniform non trivial lower-bound on the solution obtained in Theorem 3.1.

The organisation of the paper is as follows. In the next section we obtain uniform lower-bounds on the solutions on compact subsets $K \subset]0, +\infty[\times]0, +\infty]$, at the cost of integrating in the variables x and y and assuming that the second moment in the variable x and the first moment on the variable y are a priori bounded by a given constant $C > 0$. In the third section, still assuming these uniform bounds on the moments, we get rid of integration and obtain pointwise lower-bounds estimates. This part is complicated due to the degenerate diffusion in Equation (1.1). Finally in the last section, we finish the proof of Theorem 1.1, by combining the pointwise estimates with controls on the moments of the solution.

2 Uniform local mass bounds

Our main goal in this section is to prove that the solution can neither concentrate too much near the boundary in the x variable, nor have too much mass flowing to infinity. We rely on weighted norms, and define the quantities

$$M(t) := \int_{x=0}^{+\infty} \int_{y=0}^{+\infty} ((x-1)^2 + y) S(t) u_0(x, y) dx dy. \quad (2.1)$$

We shall prove

Proposition 2.1 *Let $C > 0$ be given. Then, there exists $\sigma > 0$, and for each $t \geq \sigma$ there exist $\varepsilon > 0$, $A > 0$, and $R > 0$ such that, for all initial data u_0 satisfying (1.3) with $M(0) \leq C$, the following estimate holds*

$$\int_{K_{\varepsilon,R}} S(t)u_0(x,y)dxdy \geq A,$$

where $K_{\varepsilon,R} = [\varepsilon, R] \times [0, R^2]$.

Proof of Proposition 2.1. We proceed in two steps. First, we establish a control on the growth $M(t)$, which allows us to ensure that the mass of the solution does not too much concentrate near infinity. Next, we prove on control on the possible mass concentration near the boundary at $x = 0$. More precisely, we shall prove

Lemma 2.2 *Let u_0 satisfying (1.3) with $M(0) < +\infty$. Then for $t \geq 0$,*

$$M(t) \leq M(0)e^{-t} + 3,$$

and

Lemma 2.3 *Let $C > 0$. Then, there exists $\sigma > 0$, and for each $t \geq \sigma$ there exists $B > 0$ such that for all u_0 satisfying (1.3) with $M(0) \leq C$,*

$$X_1(t) := \int_{x=0}^{+\infty} \int_{y=0}^{+\infty} xS(t)u_0(x,y)dxdy \geq B.$$

We postpone the proofs of these lemmas and proceed with the proof of Proposition 2.1. Let $C > 0$ be fixed as in the statement, and let $t \geq \sigma$ be given by Lemma 2.3 for that same value of C . Lemma 2.2 implies that $M(t) \leq C + 3$. Hence, comparing the weights and assuming $R \geq 2$, we deduce that

$$4\frac{C+3}{R} \geq \int_{x=R}^{+\infty} \int_{y=0}^{+\infty} xu(t,x,y)dxdy \quad \text{and} \quad \frac{C+3}{R^2} \geq \int_{x=0}^{+\infty} \int_{y=R^2}^{+\infty} u(t,x,y)dxdy. \quad (2.2)$$

Thanks to Lemma 2.3, we obtain that, for all $R > 0$ and $\varepsilon > 0$,

$$\begin{aligned} B &\leq \int_{x=0}^{+\infty} \int_{y=0}^{+\infty} x S(t)u_0(x,y)dxdy = \int_{x=0}^{\varepsilon} \int_{y=0}^{+\infty} x S(t)u_0(x,y)dxdy + \int_{x=\varepsilon}^R \int_{y=0}^{R^2} x S(t)u_0(x,y)dxdy \\ &\quad + \int_{x=\varepsilon}^R \int_{y=R^2}^{+\infty} x S(t)u_0(x,y)dxdy + \int_{x=R}^{+\infty} \int_{y=0}^{+\infty} x S(t)u_0(x,y)dxdy. \end{aligned}$$

From the latter and (2.2), we deduce that when $R \geq 2$,

$$R \int_{x=\varepsilon}^R \int_{y=0}^{R^2} S(t)u_0(x,y)dxdy \geq B - \varepsilon - 5\frac{C+3}{R}.$$

Taking ε sufficiently small (e.g. $\varepsilon = B/3$), and then and $R \geq 2$ sufficiently large (e.g. $R = \max(2, 15(C+3)/B)$), the conclusion follows. \square

We proceed now with the proofs of Lemma 2.2 and Lemma 2.3.

Proof of Lemma 2.2. The function M satisfies the identity

$$M'(t) = -M(t) + \int_{x=0}^{+\infty} \int_{y=0}^{+\infty} (x-1)(-x+3) S(t) u_0(x, y) dx dy + 2 \int_{x=0}^{+\infty} \int_{y=0}^{+\infty} S(t) u_0(x, y) dx dy.$$

Using that $(x-1)(-x+3) \leq 0$ for $x \geq 3$, and that $(x-1)(-x+3) \leq 1$ for $0 \leq x \leq 3$, we obtain the inequality

$$M'(t) \leq -M(t) + 3 \int_{x=0}^{+\infty} \int_{y=0}^{+\infty} S(t) u_0(x, y) dx dy = -M(t) + 3,$$

and therefore the announced

$$M(t) \leq M(0)e^{-t} + 3$$

which gives Lemma 2.2. □

Proof of Lemma 2.3. We have

$$X_1'(t) = -X_1(t) + \mathcal{N}_0(t),$$

where $\mathcal{N}_0(t) = \int_{y=0}^{+\infty} N(t, y) dy$, and therefore

$$X_1(t) = X_1(0)e^{-t} + e^{-t} \int_0^t e^s \mathcal{N}_0(s) ds. \quad (2.3)$$

If $X_1(0) \geq \frac{1}{2}$, we simply derive that

$$X_1(t) \geq \frac{1}{2}e^{-t}. \quad (2.4)$$

Assume now that $X_1(0) \leq \frac{1}{2}$, and set, for $\mu > 0$,

$$\Phi_\mu(t) := \iint e^{-\mu x} u(t, x, y) dx dy.$$

Multiplying Equation (1.1) by $e^{-\mu x}$, and integrating, we obtain that

$$\Phi_\mu'(t) = \mu^2(t) \Phi_\mu(t) + \mu \iint x e^{-\mu x} u(t, x, y) dx dy + (e^{-\mu} - 1) \mathcal{N}_0(t)$$

and therefore

$$\Phi_\mu'(t) \geq \mu^2(t) \Phi_\mu(t) + (e^{-\mu} - 1) \mathcal{N}_0(t).$$

Mass conservation implies that for all $t \geq 0$, $\Phi_\mu(t) \leq 1$, and so

$$\int_{s=0}^t e^{-\mu^2 s} \mathcal{N}_0(s) ds \geq \frac{1}{(1 - e^{-\mu})} \left(\Phi_\mu(0) - e^{-\mu^2 t} \right).$$

On the other hand, as $M(0) \leq C$, we deduce that there exists $C_1 > 0$ such that

$$\int_{x=0}^{C_1} \int_{y=0}^{C_1} u(0, x, y) dx dy \geq \frac{1}{2}$$

and so $\Phi_\mu(0) \geq \frac{1}{2}e^{-C_1\mu}$, which gives

$$\int_{s=0}^t \mathcal{N}_0(s) ds \geq \int_{s=0}^t e^{-\mu^2 s} \mathcal{N}_0(s) ds \geq \frac{1}{(1 - e^{-\mu})} \left(\frac{1}{2}e^{-C_1\mu} - e^{-\mu^2 t} \right).$$

Taking $\mu = 1$, we obtain that there exists a constant $\tilde{C}_1 > 0$ and $\sigma > 0$ such that for all $t \geq \sigma$,

$$\int_{s=0}^t \mathcal{N}_0(s) ds \geq \tilde{C}_1.$$

Coming back to Estimate (2.3), we obtain that, for $t \geq \sigma$,

$$X_1(t) \geq e^{-t} \tilde{C}_1,$$

which concludes the proof of Lemma 2.3. \square

3 Uniform pointwise lower bounds

The aim of this section is to prove the following theorem.

Theorem 3.1 *Let $C > 0$. Then, there exists $t_* > 0$, and for each $t_0 \geq t_*$ there exist $\alpha \in]0, 1[$ and an integrable function*

$$\nu : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}^+ \text{ with } \iint \nu(x, y) dx dy = 1,$$

such that for all initial data satisfying (1.3), with $M(0) \leq C$, where M is defined by (2.1), the following estimate holds

$$u(t_0, x, y) \geq \alpha \nu(x, y).$$

The main difficulty of the proof lies in the fact that we have a diffusion only in the variable x . It is therefore necessary to understand how the source term, coupled with the drift term in the variable y , allows to gain a sufficiently diffusive dynamics in y . The sketch of the proof goes as follows: First, by a suitable change of variable, we observe that Equation (1.1) can be written as infinitely many heat equations, parametrized by y , with a source term that couples all equations together. For the 1D linear heat equation, with a Dirichlet boundary term, it can be explicitly verified that the derivative in x of the solution at 0 can, for any time, be bounded from below using a weighted average of the initial data at time 0 (this is the object of Lemma 3.2 below). Using Duhamel's formula for our new set of equations, which involves a time integral of the source term, we obtain a control from below on the solution via the coupling term, which itself can be controlled using Lemma 3.2. One key point is that the integral in time obtained via Duhamel's formula also provides in the original variables some implicit integration in the y variable, because the derivative of the velocity of the drift term with respect to y is non trivial.

We begin with the explicit boundary derivative control for the heat equation, and then proceed with the actual proof of Theorem 3.1

Lemma 3.2 *Let $v : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be the solution of the equation*

$$\partial_t v - \partial_x^2 v = 0, \quad v(t, 0) = 0, \quad v(0, x) \geq 0.$$

Let $S(t) := \partial_x v(t, 0)$. Then,

$$S(t) = \frac{1}{t\sqrt{4\pi t}} \int_0^{+\infty} x e^{-\frac{x^2}{4t}} v(0, x) dx.$$

Proof of Lemma 3.2. We have

$$v(t, x) = \frac{1}{\sqrt{4\pi t}} \int_0^{+\infty} (e^{-\frac{(x-y)^2}{4t}} - e^{-\frac{(x+y)^2}{4t}}) v(0, y) dy.$$

Hence,

$$\partial_x v(t, x) = \frac{1}{t\sqrt{4\pi t}} \int_0^{+\infty} ((y-x)e^{-\frac{(x-y)^2}{4t}} + (x+y)e^{-\frac{(x+y)^2}{4t}}) v(0, y) dy,$$

which yields the result, taking $x = 0$. □

Proof of Theorem 3.1.

First observe that by direct computation, if u is a solution to (1.1), then

$$n(t, x, y) := \frac{1}{(2t+1)} u(\ln(\sqrt{2t+1}), \frac{x}{\sqrt{2t+1}}, \frac{y}{\sqrt{2t+1}}) \quad (3.1)$$

satisfies

$$\partial_t n - \partial_x^2 n = \delta_{x=\sqrt{2t+1}} G(t, y), \quad G(t, y) = \partial_x n(t, 0, y - \sqrt{2t+1}). \quad (3.2)$$

By Duhamel formula, it follows that

$$n(t, x, y) = e^{-t\Delta} n(0, x, y) + \int_0^t e^{-(t-s)\Delta} \delta_{x=\sqrt{2s+1}} G(s, y) ds,$$

and therefore when u and thus n is non negative,

$$n(t, x, y) \geq \int_0^t \frac{1}{\sqrt{4\pi(t-s)}} (e^{-\frac{(\sqrt{2s+1}-x)^2}{4(t-s)}} - e^{-\frac{(\sqrt{2s+1}+x)^2}{4(t-s)}}) G(s, y) ds.$$

From the latter, we deduce that for any compact set $K \subset]0, +\infty[$, and for any $t \geq 0$, there exists a positive constant $C(t, K)$ such that

$$n(t, x, y) \mathbb{I}_{x \in K} \geq C(t, K) \int_0^t G(s, y) ds. \quad (3.3)$$

We now estimate the term involving G . To this end, observe first that by the comparison principle $G(t, y) \geq S(t, y)$, where $S(t, y)$ is given by $S(t, y) := \partial_x v(t, 0, y)$, where

$$\partial_t v - \partial_x^2 v = 0, \quad v(t, 0, y) = 0, \quad v(0, x, y) = n(0, x, y) \geq 0.$$

Hence, applying Lemma 3.2 in Estimate (3.3), we obtain that for all $t > \delta > 0$

$$n(t, x, y) \mathbb{I}_{x \in K} \geq C(t, K, \delta) \int_{s=\delta}^t \int_{x=0}^{+\infty} x e^{-\frac{x^2}{4s}} n(\delta, x, y - \sqrt{2s+1}) dx ds,$$

for some positive $C(t, K, \delta)$. In turn this implies that for any compact interval $K' \subset]0, +\infty[$, we have

$$n(t, x, y) \mathbb{I}_{x \in K} \geq C(t, K, K', \delta) \int_{z=y-\sqrt{2t+1}}^{y-\sqrt{2\delta+1}} \int_{x \in K'} n(\delta, x, z) dx dz,$$

for some positive $C(t, K, K', \delta)$. Using Relation (3.1) between u and n , and rescaling the variable z , we infer that for all compact interval K and K' of $]0, +\infty[$,

$$\begin{aligned} & u(\ln(\sqrt{2t+1}), \frac{x}{\sqrt{2t+1}}, \frac{y}{\sqrt{2t+1}}) \mathbb{I}_{x \in K} \\ & \geq C(t, K, \delta, K') \int_{z=\frac{y-\sqrt{2t+1}}{\sqrt{2\delta+1}}}^{\frac{y}{\sqrt{2\delta+1}}-1} \int_{x \in K'} u(\ln(\sqrt{2\delta+1}), x, z) dx dz. \end{aligned} \quad (3.4)$$

Let $A > 0$, $\varepsilon > 0$, $R > 0$ be given by Proposition 2.1, where in the statement of Proposition 2.1, we take the value of C identical to its value in the statement of Theorem 3.1 and with the choice $t = \sigma$. We next choose δ in such a way that $\ln(\sqrt{2\delta+1}) = \sigma$, and let $K' = [\varepsilon, R]$. The interval

$$K'' := \{y \geq 0 \text{ s.t. } y \leq \sqrt{2t+1} \text{ and } y \geq (R^2 + 1)e^\sigma\}$$

has positive length provided $t > t_* := \frac{1}{2}((R^2 + 1)^2 e^{2\sigma} - 1)$, and by construction we have

$$[0, R^2] \subset [\frac{y - \sqrt{2t+1}}{\sqrt{2\delta+1}}, \frac{y}{\sqrt{2\delta+1}} - 1], \quad \forall y \in K'',$$

so that by (3.4) and Proposition 2.1, for all $t > t_*$,

$$u(\ln(\sqrt{2t+1}), \frac{x}{\sqrt{2t+1}}, \frac{y}{\sqrt{2t+1}}) \mathbb{I}_{x \in K} \mathbb{I}_{y \in K''} \geq C(t, \sigma, K) \int_{K_{\varepsilon, R}} u(\sigma, x, z) dx dz \geq C(t, \sigma, K) A.$$

This implies that for all $t > t_*$, there exists a compact set of positive measure $K''' \subset \mathbb{R}^+ \times \mathbb{R}^+$, such that

$$u(\ln(\sqrt{2t+1}), x, y) \mathbb{I}_{(x, y) \in K'''} \geq C(t, \sigma, K) A.$$

As for all $t > 0$, $\ln(\sqrt{2t+1}) < t$, we deduce that for all $t_0 \geq t_*$, there exists a compact set of positive measure $K''' \subset \mathbb{R}^+ \times \mathbb{R}^+$, such that

$$u(t_0, x, y) \mathbb{I}_{(x, y) \in K'''} \geq C(t_0, \sigma, K) A,$$

which concludes the proof of Theorem 3.1 by taking $\nu = \frac{\mathbb{I}_{K'''}}{|K'''|}$, where $|K'''|$ denotes the Lebesgue measure of K''' . \square

4 Proof of Theorem 1.1

We first recall for completeness the classical argument showing that Estimate (1.4) implies that there exists a unique stationary state u^* of (1.1), (1.2), (1.3), such that

$$\int_{x=0}^{+\infty} \int_{y=0}^{+\infty} (1 + |x|^2 + |y|) u^*(x, y) dx dy < +\infty.$$

Let L_m^1 denote the Banach space of integrable functions $f : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}$, endowed with the norm $\|\cdot\|_{L_m^1}$, where

$$\|f\|_{L_m^1} = \int_{x=0}^{+\infty} \int_{y=0}^{+\infty} (1 + |x|^2 + |y|) |f|(x, y) dx dy.$$

For $t > 0$, let $\Psi_t : L_m^1 \rightarrow L_m^1$ defined by $\Psi_t(f) = S(t)f$. The application Ψ_t is well defined because of Lemma 2.2, and if t is sufficiently large, it follows from Estimate (1.4) that there exists $\kappa < 1$ such that for all $(f, g) \in L_m^1 \times L_m^1$,

$$\|\Psi_t(f) - \Psi_t(g)\|_{L_m^1} \leq \kappa \|f - g\|_{L_m^1},$$

and Picard fixed point theorem yields a unique $u^* \in L_m^1$ such that

$$S(t)u^* = u^*.$$

On the other hand, we have for all $t' \geq 0$,

$$S(t)(S(t')u^*) = S(t')(S(t)u^*) = S(t')u^*.$$

Hence, for all $t' \geq 0$, $S(t')u^*$ is a fixed point of Ψ_t which implies that for all $t' \geq 0$, $S(t')u^* = u^*$.

It remains to prove Estimate (1.4). For that purpose, we consider u_1 and u_2 two initial data satisfying (1.3), and set

$$m_0(t) := \int_{x=0}^{+\infty} \int_{y=0}^{+\infty} |S(t)(u_1 - u_2)|(x, y) dx dy$$

$$\overline{M}(t) := \int_{x=0}^{+\infty} \int_{y=0}^{+\infty} ((x-1)^2 + y) |S(t)(u_1 - u_2)|(x, y) dx dy.$$

We note that by conservation of mass and positivity, the quantity m_0 is a non increasing function.

The next lemma, based on Theorem 3.1, provides a first contraction type result, *assuming* a uniform control on $\overline{M}(t)$ with respect to $m_0(t)$.

Lemma 4.1 *Let $\gamma > 0$. There exists $t_* > 0$, and for any $t_0 \geq t_*$ there exists $\alpha \in]0, 1[$, such that for any $t \geq 0$, if $\overline{M}(t) \leq \gamma m_0(t)$ then*

$$m_0(t + t_0) \leq \alpha m_0(t).$$

Proof of Lemma 4.1. We decompose $S(t)(u_1 - u_2) =: \mu_+ - \mu_-$, where μ_{\pm} denote respectively the positive and the negative parts of $S(t)(u_1 - u_2)$. We have

$$\int_{x=0}^{+\infty} \int_{y=0}^{+\infty} \mu_+(x, y) dx dy = \int_{x=0}^{+\infty} \int_{y=0}^{+\infty} \mu_-(x, y) dx dy = \frac{1}{2} m_0(t).$$

We also have, for all $(x, y) \in \mathbb{R}_+ \times \mathbb{R}_+$,

$$|S(t)(u_1 - u_2)(x, y)| \geq \mu_+(x, y) \quad \text{and} \quad |S(t)(u_1 - u_2)(x, y)| \geq \mu_-(x, y),$$

and therefore from the assumption $\overline{M}(t) \leq \gamma m_0(t)$ we deduce that

$$\int_{x=0}^{+\infty} \int_{y=0}^{+\infty} ((x-1)^2 + y) \mu_{\pm}(x, y) dx dy \leq 2\gamma \int_{x=0}^{+\infty} \int_{y=0}^{+\infty} \mu_{\pm}(x, y) dx dy.$$

Applying Theorem 3.1 with initial data μ_{\pm} , it follows that there exists a time $t_* > 0$ and for any $t_0 \geq t_*$ there exists $\alpha \in]0, 1[$ and a function ν as in Theorem 3.1, such that

$$S(t_0)(\mu_{\pm}) \geq \alpha \frac{m_0(t)\nu}{2}.$$

We write

$$S(t_0 + t)(u_1 - u_2) = S(t_0)(\mu_+) - \frac{m_0(t)}{2}\nu + \frac{m_0(t)}{2}\nu - S(t_0)(\mu_-),$$

and taking the absolute value and integrating yields

$$\int_{x=0}^{+\infty} \int_{y=0}^{+\infty} |S(t + t_0)(u_1 - u_2)|(x, y) dx dy \leq \alpha \int_{x=0}^{+\infty} \int_{y=0}^{+\infty} |S(t)(u_1 - u_2)|(x, y) dx dy$$

which ends the proof of Lemma 4.1. \square

Because it requires an a priori bound on $\overline{M}(t)$, it is not clear a priori that it can be iteratively used in order to derive an exponential decay. As a matter of fact, even assuming that there exists $t \geq 0$ such that $\frac{\overline{M}}{m_0}(t) \leq \gamma$, we do not know, for example, if for $k \geq 1$ we still have $\frac{\overline{M}}{m_0}(t + kt_0) \leq \gamma$. In order to overcome this difficulty, we shall split the time space into two regions determined by the ratio $\frac{\overline{M}}{m_0}$ with respect to a parameter γ : in one region we will directly make use of that ratio bound, while in the other will make use of Lemma 4.1.

We begin with the case where \overline{M} is large with respect to m_0 .

Lemma 4.2 *Let $\gamma > 0$. Assume that $\overline{M} \geq \gamma m_0$ on some interval $I = [T_1, T_2]$. Then*

$$\overline{M}(t) \leq e^{(-1 + \frac{3}{\gamma})t} \overline{M}(T_1),$$

for all $t \in I$.

Proof of Lemma 4.2. Set $p(t) := |S(t)(u_1 - u_2)|$. Then,

$$\partial_t p - \partial_x(xp) - \partial_y(y p) - \partial_{xx} p \leq \delta_{x=1} |N_1(t, y) - N_2(t, y)|$$

where $N_i(t, y) = \partial_x S(t) u_i(0, y - 1)$. Multiplying the above equation by $(x - 1)^2 + y$ and integrating, we derive that

$$\overline{M}'(t) \leq -\overline{M}(t) + 3m_0(t) \leq -\overline{M}(t) + \frac{3}{\gamma} \overline{M}(t), \quad (4.1)$$

and the conclusion follows from Gronwall's lemma. \square

We now proceed with the region where $\overline{M} \leq \gamma m_0$. To deal with that situation, we define a suitable area where one can iterate over Lemma 4.1, in order to control m_0 , and where, at the same time, a sufficient control on \overline{M} can be established. More precisely, we have

Lemma 4.3 *Let $\gamma > 0$, let t_* be given by Lemma 4.1 for this choice of γ , and let $t_0 > t_*$. Assume that for some $T \geq 0$, $\overline{M}(T) \leq \gamma m_0(T)$, and set*

$$N = N(T, \gamma, t_0) := \sup \left\{ k \geq 0 \text{ s.t. } \overline{M}(T + jt_0) \leq \gamma m_0(T + jt_0) \ \forall \ 0 \leq j \leq k \right\} \in \mathbb{N} \cup \{+\infty\}.$$

There exists $0 < \beta \leq \frac{1}{2}$, depending only on γ and t_0 , such that

$$m_0(t) \leq 2e^{-\beta(t-T)} m_0(T), \quad (4.2)$$

and

$$\overline{M}(t) \leq e^{-(t-T)} \overline{M}(T) + 12e^{-\beta(t-T)} m_0(T), \quad (4.3)$$

for all $t \in I_N$, where $I_N := [T, T + (N + 1)t_0]$ if N is finite, and $I_N := [T, +\infty)$ otherwise.

Proof of Lemma 4.3. Inequality (4.2) follows from Lemma 4.1 and the fact that m_0 is a decreasing function. Indeed, for $0 \leq k \leq N$ we can apply Lemma 4.1 at the point $T + kt_0$ and therefore obtain that

$$m_0(T + (k + 1)t_0) \leq \alpha m_0(T + kt_0).$$

Using that m_0 is a decreasing function, it follows that

$$m_0(t) \leq \alpha^k m_0(T), \quad \forall t \in [T + kt_0, T + (k + 1)t_0], \quad \forall 0 \leq k \leq N,$$

and therefore (4.2) holds provided $e^{-\beta t_0} \geq \max(\frac{1}{2}, \alpha)$. We choose $\beta := \max(\frac{1}{2}, -\frac{1}{t_0} \log(\max(\frac{1}{2}, \alpha)))$. Estimate (4.3) is then a direct consequence of the first inequality in (4.1). Indeed, we have

$$\overline{M}(t) \leq e^{-(t-T)} \overline{M}(T) + 3e^{-t} \int_T^t e^s m_0(s) ds,$$

and using (4.2) as well as $\beta \leq \frac{1}{2}$, we bound

$$3e^{-t} \int_T^t e^s m_0(s) ds \leq 6e^{-\beta(t-T)} m_0(T) \int_T^t e^{-(1-\beta)(t-s)} ds \leq 12e^{-\beta(t-T)} m_0(T).$$

This concludes the proof of Lemma 4.3. □

Proof of Theorem 1.1 completed. Let $\gamma = 24$ and let t_* be given by Lemma 4.3 for this choice of γ . We then set $t_0 = \max(t_*, 2 \log(2))$, and denote by β the value provided by Lemma 4.3 for such values of γ and t_0 . We split the argument in three steps.

Step 1: Initialization. We distinguish two cases.

Case 1: $\overline{M}(0) \leq \gamma m_0(0)$. We may then apply Lemma 4.3 with $T = 0$, and we distinguish again two subcases depending on $N := N(T, \gamma, t_0)$.

Case 1a: $N = +\infty$. In that case, summation of (4.2) and (4.3) leads to

$$(\overline{M} + m_0)(t) \leq 14e^{-\beta t} (\overline{M} + m_0)(0), \quad \forall t \geq 0, \tag{4.4}$$

which immediately implies (1.4).

Case 1b: $N < +\infty$. In that case, we set $T_0 = (N + 1)t_0$ and similarly summation of (4.2) and (4.3) yields

$$(\overline{M} + m_0)(t) \leq 14e^{-\beta t} (\overline{M} + m_0)(0), \quad \forall t \in [0, T_0]. \tag{4.5}$$

By definition of N , we also have $\overline{M}(T_0) > \gamma m_0(T_0)$.

Case 2: $\overline{M}(0) > \gamma m_0(0)$. In that case, we simply set $T_0 = 0$. Obviously, (4.5) also holds.

Step 2: Induction. Assume that for some $k \in \mathbb{N}$ we have constructed $T_k \geq 0$ such that

$$\overline{M}(T_k) > \gamma m_0(T_k), \tag{4.6}$$

$$\overline{M}(T_k) \leq 14e^{-\beta T_k} (\overline{M} + m_0)(0), \tag{4.7}$$

$$(\overline{M} + m_0)(t) \leq 28e^{-\beta t} (\overline{M} + m_0)(0), \quad \forall 0 \leq t \leq T_k. \tag{4.8}$$

Let $T'_k = \sup\{t \geq T_k \text{ s.t. } \overline{M}(s) > \gamma m_0(s) \forall s \in [T_k, t]\}$. We distinguish two cases.

Case 1: $T'_k = +\infty$. Then by Lemma 4.2 and in view of the inequality $-1 + \frac{3}{\gamma} = -\frac{7}{8} \leq -\frac{1}{2} \leq -\beta$ we obtain that for all $t \geq T_k$

$$(\overline{M} + m_0)(t) \leq (1 + \frac{1}{\gamma}) \overline{M}(t) \leq (1 + \frac{1}{\gamma}) e^{-\beta(t-T_k)} \overline{M}(T_k) \leq 28e^{-\beta t} (\overline{M} + m_0)(0),$$

where we have used (4.7) for the last inequality. It follows that the equivalent of (4.8) holds for all $t \geq 0$, and there is nothing left to prove in that case.

Case 2: $T'_k < +\infty$. By continuity it holds $\overline{M}(T'_k) = \gamma m_0(T'_k)$. By Lemma 4.2, and using once more that $-1 + \frac{3}{\gamma} = -\frac{7}{8} \leq -\frac{1}{2} \leq -\beta$, we have

$$\overline{M}(t) \leq e^{-\beta(t-T_k)} \overline{M}(T_k) \quad \forall t \in [T_k, T'_k], \quad (4.9)$$

and therefore by definition of T'_k

$$(\overline{M} + m_0)(t) \leq \left(1 + \frac{1}{\gamma}\right) \overline{M}(t) \leq \left(1 + \frac{1}{\gamma}\right) e^{-\beta(t-T_k)} \overline{M}(T_k) \quad \forall t \in [T_k, T'_k]. \quad (4.10)$$

We distinguish once more two subcases depending on $N := N(T'_k, \gamma, t_0)$.

Case 2a: $N = +\infty$. Summation of (4.2) and (4.3), together with (4.9) (evaluated at $t = T'_k$) yields

$$(\overline{M} + m_0)(t) \leq 14e^{-\beta(t-T'_k)} \overline{M}(T'_k) \leq 14e^{-\beta(t-T_k)} \overline{M}(T_k), \quad \forall t \geq T'_k. \quad (4.11)$$

Using (4.7) to control $\overline{M}(T_k)$ in both (4.11) and in (4.10), and combining the control so obtained for all $t \geq T_k$ with the existing control (4.8) for $t \leq T_k$ finishes the proof in that case.

Case 2b: $N < +\infty$. This is the only case that will potentially repeat itself, it is therefore important that it yields an exponential decay *without* a pre-multiplying factor. We set $T''_k = T'_k + t_0$ and $T_{k+1} = T'_k + (N+1)t_0$. Inequality (4.3) combined with $m_0(T'_k) = \overline{M}(T'_k)/\gamma$ yield

$$\overline{M}(t) \leq e^{-(t-T'_k)} \overline{M}(T'_k) + \frac{12}{\gamma} e^{-\beta(t-T'_k)} \overline{M}(T'_k), \quad \forall T'_k \leq t \leq T_{k+1}. \quad (4.12)$$

Since by construction $t_0 \geq 2 \log(2)$ and $\beta \leq \frac{1}{2}$, we have

$$e^{-(t-T'_k)} \leq e^{-\frac{1}{2}(t-T'_k)} e^{-\beta(t-T'_k)} \leq \frac{1}{2} e^{-\beta(t-T'_k)} \quad \forall t \geq T''_k = T'_k + t_0.$$

It follows therefore from (4.12) and the choice $\gamma = 24$ that

$$\overline{M}(t) \leq e^{-\beta(t-T'_k)} \overline{M}(T'_k) \quad \forall t \in [T''_k, T_{k+1}]. \quad (4.13)$$

Combining (4.13) and (4.9) (evaluated at $t = T'_k$) we deduce that

$$\overline{M}(t) \leq e^{-\beta(t-T_k)} \overline{M}(T_k) \quad \forall t \in [T_k, T_{k+1}] \setminus (T'_k, T''_k), \quad (4.14)$$

and therefore by the inductive assumption (4.7) we obtain

$$\overline{M}(t) \leq 14e^{-\beta t} (\overline{M} + m_0)(0) \quad \forall t \in [T_k, T_{k+1}] \setminus (T'_k, T''_k). \quad (4.15)$$

The latter evaluated at $t = T_{k+1}$ reproduces the equivalent of (4.7) at $t = T_{k+1}$. Note also that by definition of N we have $\overline{M}(T_{k+1}) > \gamma m_0(T_{k+1})$, i.e. (4.6) also holds at time T_{k+1} . It remains to prove the inequality in (4.8) holds for all $t \leq T_{k+1}$. In view of (4.15), this amounts to bound $\overline{M}(t)$ for $t \in (T'_k, T''_k)$ and $m_0(t)$ for $t \in [T_k, T_{k+1}]$. For the first one, we simply refer to (4.12) and combine it with (4.15) (evaluated at $t = T'_k$) to obtain

$$\overline{M}(t) \leq \frac{3}{2} 14e^{-\beta t} (\overline{M} + m_0)(0) \quad \forall t \in [T_k, T'_k]. \quad (4.16)$$

Regarding $m_0(t)$, first by definition of T'_k and (4.15)

$$m_0(t) \leq \frac{1}{\gamma} \overline{M}(t) \leq \frac{1}{\gamma} 14e^{-\beta t} (\overline{M} + m_0)(0) \quad \forall t \in [T_k, T'_k].$$

Second, by (4.2) we have

$$m_0(t) \leq 2e^{-\beta(t-T'_k)} m_0(T'_k) = \frac{2}{\gamma} e^{-\beta(t-T'_k)} \overline{M}(T'_k) \leq \frac{2}{\gamma} 14e^{-\beta t} (\overline{M} + m_0)(0) \quad \forall t \in [T'_k, T_{k+1}].$$

After summation with (4.16) or (depending on the value of t) (4.15), and using that $(\frac{3}{2} + \frac{2}{\gamma})14 \leq 28$, we finally derive (4.8) for all $t \leq T_{k+1}$.

Step 3: Conclusion. If the algorithm presented in Step 2 exits early (i.e. if either case 1 or case 2b holds at some point), then we already showed that there was nothing left to prove. If it doesn't then the sequence $(T_k)_{k \geq 0}$ is necessarily unbounded, since $T_{k+1} - T_k \geq t_0$ for all $k \geq 0$. Inequality (4.8) therefore eventually holds for all $t \geq 0$, and the proof of Theorem 1.1 is hereby completed. \square

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