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Lévy measures of infinitely divisible positive processes - examples and distributional identities

Nathalie Eisenbaum and Jan Rosiński

Abstract

The law of a positive infinitely divisible process with no drift is characterized by its Lévy measure on the paths space. Based on recent results of the two authors, it is shown that even for simple examples of such process, the knowledge of their Lévy measures allows to obtain remarkable distributional identities.

1 Introduction

A random process is infinitely divisible if all its finite dimensional marginals are infinitely divisible. Let $\psi = (\psi(x), x \in E)$ be a nonnegative infinitely divisible process with no drift. The infinite divisibility of ψ is characterized by the existence of a unique measure ν on \mathbb{R}_+^E , the space of all functions from E into \mathbb{R}_+ , such that for every $n > 0$, every $\alpha_1, \dots, \alpha_n$ in \mathbb{R}_+ and every x_1, \dots, x_n in E :

$$\mathbb{E}[\exp\{-\sum_{i=1}^n \alpha_i \psi(x_i)\}] = \exp\{-\int_{\mathbb{R}_+^E} (1 - e^{-\sum_{i=1}^n \alpha_i y(x_i)}) \nu(dy)\}. \quad (1.1)$$

The measure ν is called the Lévy measure of ψ . The existence and uniqueness of such measures was established in complete generality in [16]. In section 2, we recall some definitions and facts about Lévy measures.

It might be difficult to obtain an expression for the Lévy measure ν directly from (1.1). In [3], a general expression for ν has been established. Its proof is based on several identities involving ψ . Among them:

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For every $a \in E$ with $0 < \mathbb{E}[\psi(a)] < \infty$, there exists a nonnegative process $(r^{(a)}(x), x \in E)$ independent of ψ such that

$$\psi + r^{(a)} \text{ has the law of } \psi \text{ under } \mathbb{E} \left[\frac{\psi(a)}{\mathbb{E}[\psi(a)]}, \cdot \right] \quad (1.2)$$

Actually, the existence of $(r^{(a)}, a \in E)$ characterizes the infinite divisibility of ψ . This characterization has been established in [2], see also [16, Proposition 4.7].

Under an assumption of stochastic continuity for ψ , the general expression for ν obtained in [3], is the following:

$$\nu(F) = \int_E \mathbb{E} \left[\frac{F(r^{(a)})}{\int_E r^{(a)}(x) m(dx)} \right] \mathbb{E}[\psi(a)] m(da), \quad (1.3)$$

for any measurable functional F on \mathbb{R}_+^E , where m is any σ -finite measure with support equal to E such that $\int_E \mathbb{E}[\psi(x)] m(dx) < \infty$.

Moreover the law of $r^{(a)}$ is connected to ν as follows (see [3], [16]):

$$\mathbb{E}[F(r^{(a)})] = \frac{1}{\mathbb{E}[\psi(a)]} \int_{\mathbb{R}_+^E} y(a) F(y) \nu(dy). \quad (1.4)$$

The problem of determining ν is hence equivalent to the one of the law of $r^{(a)}$ for every a in E . But knowing ν , one can not only write (1.2) but many other identities of the same type. In each one, the process $r^{(a)}$ is replaced by a process with an absolutely continuous law with respect to ν (see [16, Theorem 4.3(a)]).

Some conditionings on ψ lead to a splitting of ν . This allows to obtain decompositions of ψ into independent infinitely divisible components (see [3, Theorems 1.1, 1.2 and 1.3]). As an example:

For every $a \in E$, there exists a nonnegative infinitely divisible process $(\mathcal{L}^{(a)}(x), x \in E)$ independent of an infinitely divisible process $((\psi(x), x \in E) | \psi(a) = 0)$ such that

$$\psi \stackrel{(\text{law})}{=} (\psi | \psi(a) = 0) + \mathcal{L}^{(a)}. \quad (1.5)$$

By [3, Theorem 1.2], the processes $(\psi | \psi(a) = 0)$ and $\mathcal{L}^{(a)}$ have the respective Lévy measures ν_a and $\tilde{\nu}_a$, where

$$\nu_a(dy) = \mathbb{1}_{\{y(a)=0\}} \nu(dy) \text{ and } \tilde{\nu}_a(dy) = \mathbb{1}_{\{y(a)>0\}} \nu(dy). \quad (1.6)$$

In section 3, to illustrate the relations and identities (1.1)–(1.5) we choose to consider simple examples of nonnegative infinitely divisible processes. In each case the

Lévy measure is directly computable from (1.1) or from the stochastic integral representation of ψ (see [12]). Thanks to (1.2) and its extensions, and (1.5), we present remarkable identities satisfied by the considered nonnegative infinitely divisible processes. Moreover the general expression (1.3) provides alternative formulas for the Lévy measure, which are also remarkable. We treat the cases of Poisson processes, Sato processes, stochastic convolutions and tempered stable subordinators. We also point out a connection with infinitely divisible random measures. We end section 3 by reminding the case of infinitely divisible permanental processes which is the first case for which identities in law of the same type as (1.2) have been established. In this case, such identities in law are called "isomorphism theorems" in reference to the very first one established by Dynkin [1] the so-called "Dynkin isomorphism Theorem".

When ψ is an infinitely divisible permanental process, the two processes $r^{(a)}$ and $\mathcal{L}^{(a)}$ have the same law. If moreover ψ is a squared Gaussian process, Marcus and Rosen [9] have established correspondences between path properties of ψ and the ones of $\mathcal{L}^{(a)}$. The extension of these correspondences to general infinitely divisible permanental processes has been undertaken by several authors (see [4], [3], [10] or [11]). Similarly, in section 4, we consider a general infinitely divisible nonnegative process ψ and state some trajectories correspondences between ψ and $\mathcal{L}^{(a)}$ resulting from an iteration of (1.5) (see also [16]).

Finally, observing that given an infinitely divisible positive process ψ , $r^{(a)}$ is not a priori "naturally" connected to ψ , we present, in section 5, $r^{(a)}$ as the limit of a sequence of processes naturally connected to ψ .

2 Preliminaries on Lévy measures

In this section we recall some definitions and facts about general Lévy measures given in [16, Section 2]. Some additional material can be found in [15]. Let $(\xi(x), x \in E)$ be a real-valued infinitely divisible process, where E is an arbitrary nonempty set. A measure ν defined on the cylindrical σ -algebra \mathcal{R}^E of \mathbb{R}^E is called the Lévy measure of ξ if the following two conditions hold:

- (i) for every $x_1, \dots, x_n \in E$, the Lévy measure of the random vector $(\xi(x_1), \dots, \xi(x_n))$ coincides with the projection of ν onto $\mathbb{R}^{\{x_1, \dots, x_n\}}$, modulo the mass at the origin;
- (ii) $\nu(A) = \nu_*(A \setminus 0_E)$ for all $A \in \mathcal{R}^E$, where ν_* denotes the inner measure and 0_E is the origin of \mathbb{R}^E .

The Lévy measure of an infinitely divisible process always exists and (ii) guarantees its uniqueness. Condition (i) implies that $\int_{\mathbb{R}^E} (f(x)^2 \wedge 1) \nu(df) < \infty$ for every $x \in E$.

A Lévy measure ν is σ -finite if and only if then there exists a countable set $E_0 \subset E$ such that

$$\nu\{f \in \mathbb{R}^E : f|_{E_0} = 0\} = 0. \quad (2.1)$$

Actually, if (i) and (2.1) hold, then does so (ii) and ν is a σ -finite Lévy measure.

Condition (2.1) is usually easy to verify. For instance, if an infinitely divisible process $(\xi(x), x \in E)$ is separable in probability, then its Lévy measure satisfies (2.1), so is σ -finite. The separability in probability is a weak assumption. It says that there is a countable set $E_1 \subset E$ such that for every $x \in E$ there is a sequence $(x_n) \subset E_1$ such that $\xi(x_n) \rightarrow \xi(x)$ in \mathbb{P} . Infinitely divisible processes whose Lévy measures do not satisfy (2.1) include such pathological cases as an uncountable family of independent Poisson random variables with mean 1.

If the process ξ has paths in some “nice” subspace of \mathbb{R}^E , then due to the transfer of regularity [16, Theorem 3.4], its Lévy measure ν is carried by the same subspace of \mathbb{R}^E . Thus, one can investigate the canonical process on $(\mathbb{R}^E, \mathcal{R}^E)$ under the law of ξ and also under the measure ν , and relate their properties. This approach was successful in the study of distributional properties of subadditive functionals of paths of infinitely divisible processes [17] and the decomposition and classification of stationary stable processes [13], among others.

If an infinitely divisible process ξ without Gaussian component has the Lévy measure ν , then it can be represented as

$$(\xi(x), x \in E) \stackrel{(\text{law})}{=} \left(\int_{\mathbb{R}^E} f(x) [N(df) - \chi(f(t))\nu(df)] + b(x), x \in E \right) \quad (2.2)$$

where N is a Poisson random measure on $(\mathbb{R}^E, \mathcal{R}^E)$ with intensity measure ν , $\chi(u) = \mathbb{1}_{[-1,1]}(u)$, and $b \in \mathbb{R}^E$ is deterministic. Relation (2.2) can be strengthened to the equality almost surely under some minimal regularity conditions on the process ξ , provided the probability space is rich enough (see [16, Theorem 3.2]). This is an extension to general infinitely divisible processes of the celebrated Lévy-Itô representation.

Obviously, all the above apply to processes presented in the introduction but in more transparent form. Namely, if $(\psi(x), x \in E)$ is an infinitely divisible nonnegative process, then its Lévy measure ν is concentrated on \mathbb{R}_+^E and (2.2) becomes

$$(\psi(x), x \in E) \stackrel{(\text{law})}{=} \left(f_0(x) + \int_{\mathbb{R}_+^E} f(x) N(df), x \in E \right), \quad (2.3)$$

where N is a Poisson random measure on \mathbb{R}_+^E with intensity measure ν such that $\int_{\mathbb{R}_+^E} (f(x) \wedge 1) \nu(df) < \infty$ for every $x \in E$. Moreover, $\mathbb{E}[\psi(x)] < \infty$ if and only if $\int_{\mathbb{R}_+^E} f(x) \nu(df) < \infty$ and $f_0 \geq 0$ is a deterministic drift.

Since N can be seen as a countable random subset of \mathbb{R}_+^E , one can write (2.3) as

$$(\psi(x), x \in E) \stackrel{(\text{law})}{=} \left(f_0(x) + \sum_{f \in N} f(x), x \in E \right). \quad (2.4)$$

We end this section with a necessary and sufficient condition for a measure ν to be the Lévy measure of a nonnegative infinitely divisible process. It is a direct consequence of [16] section 2. From now on will assume that ψ has no drift, in which case $f_0 = 0$ in (2.3)-(2.4).

Let ν be a measure on $(\mathbb{R}_+^E, \mathcal{B}^E)$, where \mathcal{B}^E denotes the cylindrical σ -algebra associated to \mathbb{R}_+^E the space of all functions from E into \mathbb{R}_+ . There exists an infinitely divisible nonnegative process $(\psi(x), x \in E)$ such that for every $n > 0$, every x_1, \dots, x_n in E :

$$\mathbb{E}[\exp\{-\sum_{i=1}^n \alpha_i \psi(x_i)\}] = \exp\{-\int_{\mathbb{R}_+^E} (1 - e^{-\sum_{i=1}^n \alpha_i y(x_i)}) \nu(dy)\},$$

if and only if ν satisfies the two following conditions:

(L1) for every $x \in E$ $\nu(y(x) \wedge 1) < \infty$,

(L2) for every $A \in \mathcal{B}^E$, $\nu(A) = \nu_*(A \setminus 0_E)$, where ν_* is the inner measure.

3 Illustrations

By a standard uniform random variable we mean a random variable with the uniform law on $[0, 1]$. A random variable with exponential law and mean 1 will be called standard exponential.

3.1 Poisson process

A Poisson process $(N_t, t \geq 0)$ with intensity λm , where $\lambda > 0$ and m is the Lebesgue measure on \mathbb{R}_+ , is the simplest Lévy process but its Lévy measure ν is even simpler. It is a σ -finite measure given by

$$\nu(F) = \lambda \int_0^\infty F(\mathbb{1}_{[s, \infty)}) ds, \quad (3.1)$$

for every measurable functional $F : \mathbb{R}_+^{[0, \infty)} \mapsto \mathbb{R}_+$. Thus (3.1) says that ν is the image of λm by the mapping $s \mapsto \mathbb{1}_{[s, \infty)}$ from \mathbb{R}_+ into $\mathbb{R}_+^{[0, \infty)}$.

Formula (3.1) is a special case of [16, Example 2.23]. We will derive it here for the sake of illustration and completeness.

Let $(N_t, t \geq 0)$ be a Poisson process as above. By a routine computation of the Laplace transform, we obtain that for every $0 \leq t_1 < \dots < t_n$ the Lévy measure ν_{t_1, \dots, t_n} of $(N_{t_1}, \dots, N_{t_n})$ is of the form

$$\nu_{t_1, \dots, t_n} = \sum_{i=1}^n \lambda \Delta t_i \delta_{\mathbf{u}_i},$$

where $\Delta t_i = t_i - t_{i-1}$, $t_0 = 0$, and $\mathbf{u}_i = \underbrace{(0, \dots, 0)}_{i-1 \text{ times}}, 1, \dots, 1) \in \mathbb{R}^n$, $i = 1, \dots, n$.

To verify that (3.1) satisfies (i) of Section 1, consider a finite dimensional functional F , that is $F(f) = F_0(f(t_1), \dots, f(t_n))$, where $F_0 : \mathbb{R}_+^n \mapsto \mathbb{R}_+$ is a Borel function with $F_0(0, \dots, 0) = 0$ and $0 \leq t_1 < \dots < t_n$. From (3.1) we have

$$\begin{aligned} \nu(F) &= \lambda \int_0^\infty F(\mathbb{1}_{[s, \infty)}) ds = \lambda \int_0^\infty F_0(\mathbb{1}_{[s, \infty)}(t_1), \dots, \mathbb{1}_{[s, \infty)}(t_n)) ds \\ &= \lambda \sum_{i=1}^n \int_{t_{i-1}}^{t_i} F_0(\mathbf{u}_i) ds = \int_{\mathbb{R}_+^n} F_0(x) \nu_{t_1, \dots, t_n}(dx) \end{aligned}$$

which proves (i). Condition (2.1) holds for any unbounded set, for instance $E_0 = \mathbb{N}$. Indeed,

$$\nu\{f \in \mathbb{R}_+^{[0, \infty)} : f|_{\mathbb{N}} = 0\} = \lambda \int_0^\infty \mathbb{1}\{s : \mathbb{1}_{[s, \infty)}(n) = 0 \forall n \in \mathbb{N}\} ds = 0,$$

so that ν is the Lévy measure of $(N_t, t \geq 0)$.

The next proposition exemplifies remarkable identities resulting from (1.5) and (1.2). It also gives an alternative “probabilistic” form of the Lévy measure ν .

Proposition 3.1 *Let $N = (N_t, t \geq 0)$ be a Poisson process with intensity λm , where m is the Lebesgue measure on \mathbb{R}_+ and $\lambda > 0$.*

- (a1) *Given $a > 0$, let $r^{(a)}$ be the process defined by: $r^{(a)}(t) := \mathbb{1}_{[aU, \infty)}(t)$, $t \geq 0$, where U is a standard uniform random variable independent of $(N_t, t \geq 0)$. Then $(r^{(a)}(t), t \geq 0)$ satisfies (1.2), that is,*

$$(N_t + \mathbb{1}_{[aU, \infty)}(t), t \geq 0) \stackrel{(\text{law})}{=} (N_t, t \geq 0) \text{ under } \mathbb{E} \left[\frac{N_a}{\lambda a}; \cdot \right].$$

- (b1) *For any nonnegative random variable Y whose support equals \mathbb{R}_+ and $\mathbb{E}Y < \infty$, the Lévy measure ν of $(N_t, t \geq 0)$ can be represented as*

$$\nu(F) = \lambda \mathbb{E} [Y h(UY) F(\mathbb{1}_{[UY, \infty)})]$$

for every measurable functional $F : \mathbb{R}_+^{[0,\infty)} \mapsto \mathbb{R}_+$, where U is a standard uniform random variable independent of Y and $h(x) = 1/\mathbb{P}[Y \geq x]$.

In particular, if Y is a standard exponential random variable independent of U , then

$$\nu(F) = \lambda \mathbb{E}[Y e^{UY} F(\mathbb{1}_{[UY,\infty)})].$$

(c1) The components of the decomposition (1.5): $N \stackrel{(\text{law})}{=} (N | N_a = 0) + \mathcal{L}^{(a)}$, can be identified as

$$(N_t, t \geq 0 | N_a = 0) \stackrel{(\text{law})}{=} (N_{t \vee a} - N_a, t \geq 0).$$

and

$$(\mathcal{L}_t^{(a)}, t \geq 0) \stackrel{(\text{law})}{=} (N_{t \wedge a}, t \geq 0).$$

The Lévy measures ν_a and $\tilde{\nu}_a$ of $(N_t, t \geq 0 | N_a = 0)$ and of $(\mathcal{L}_t^{(a)}, t \geq 0)$, respectively, are given by

$$\nu_a(F) = \lambda \int_a^\infty F(\mathbb{1}_{[s,\infty)}) ds,$$

and

$$\tilde{\nu}_a(F) = \lambda \int_0^a F(\mathbb{1}_{[s,\infty)}) ds,$$

for every measurable functional $F : \mathbb{R}_+^{[0,\infty)} \mapsto \mathbb{R}_+$.

Proof (a1): By (1.4) we have for any measurable functional $F : \mathbb{R}^{[0,\infty)} \mapsto \mathbb{R}_+$

$$\begin{aligned} \mathbb{E}F(r_t^{(a)}, t \geq 0) &= \frac{1}{\mathbb{E}N_a} \int F(y) y(a) \nu(dy) \\ &= \frac{1}{a} \int_0^\infty F(\mathbb{1}_{[s,\infty)}) \mathbb{1}_{[s,\infty)}(a) ds \\ &= \frac{1}{a} \int_0^a F(\mathbb{1}_{[s,\infty)}) ds = \mathbb{E}F(\mathbb{1}_{[aU,\infty)}). \end{aligned}$$

Thus $(r_t^{(a)}, t \geq 0) \stackrel{(\text{law})}{=} (\mathbb{1}_{[aU,\infty)}(t), t \geq 0)$. Choosing U independent of N , we have (1.2) for $r_t^{(a)} = \mathbb{1}_{[aU,\infty)}(t)$, $t \geq 0$, which completes the proof of (a1).

(b1): This point is an illustration of the invariance property in m of (1.3). Indeed, since the process $(N_t, t \geq 0)$ is stochastically continuous we have for every σ -finite measure \tilde{m} whose support is $[0, \infty)$ and $\int_0^\infty t \tilde{m}(dt) < \infty$

$$\begin{aligned} \nu(F) &= \int_0^\infty \mathbb{E} \left[\frac{F(r^{(a)})}{\int_0^\infty r_s^{(a)} \tilde{m}(ds)} \right] \mathbb{E}[N_a] \tilde{m}(da) \\ &= \lambda \int_0^\infty \mathbb{E} \left[\frac{F(\mathbb{1}_{[aU,\infty)})}{\tilde{m}([aU, \infty))} \right] a \tilde{m}(da). \end{aligned}$$

If \tilde{m} is the law of a nonnegative random variable Y , then

$$\begin{aligned}\nu(F) &= \lambda \int_0^\infty \mathbb{E} [a h(aU) F(\mathbb{1}_{[aU, \infty)})] \tilde{m}(da) \\ &= \lambda \mathbb{E} [Y h(UY) F(\mathbb{1}_{[UY, \infty)})],\end{aligned}$$

which is the formula in (b1).

(c1): Since $(N_t, t \geq 0 \mid N_a = 0)$ has the Lévy measure $\nu_a(dy) = \mathbb{1}_{\{y(a)=0\}}\nu(dy)$ (see [3]), by (3.1) we get

$$\begin{aligned}\nu_a(F) &= \int F(y) \mathbb{1}_{\{y(a)=0\}}\nu(dy) \\ &= \lambda \int_0^\infty F(\mathbb{1}_{[s, \infty)}) \mathbb{1}_{\{\mathbb{1}_{[s, \infty)}(a)=0\}} ds \\ &= \lambda \int_a^\infty F(\mathbb{1}_{[s, \infty)}) ds.\end{aligned}$$

Since $\tilde{\nu}_a = \nu - \nu_a$, by (3.1) we have

$$\tilde{\nu}_a(F) = \lambda \int_0^a F(\mathbb{1}_{[s, \infty)}) ds.$$

Let $0 = t_0 < t_1 < \dots < t_n$ be such that $t_m = a$ for some $m \leq n$. For $\alpha_i > 0$ we obtain

$$\begin{aligned}\mathbb{E} \exp \left\{ - \sum_{i=1}^n \alpha_i (\mathcal{L}_{t_i}^{(a)} - \mathcal{L}_{t_{i-1}}^{(a)}) \right\} &= \exp \left\{ -\tilde{\nu}_a(1 - e^{-\sum_{i=1}^n \alpha_i (y(t_i) - y(t_{i-1}))}) \right\} \\ &= \exp \left\{ -\lambda \int_0^a (1 - e^{-\sum_{i=1}^n \alpha_i (\mathbb{1}_{[s, \infty)}(t_i) - \mathbb{1}_{[s, \infty)}(t_{i-1}))}) ds \right\} \\ &= \exp \left\{ -\lambda \sum_{i=1}^m \int_{t_{i-1}}^{t_i} (1 - e^{-\sum_{i=1}^n \alpha_i (\mathbb{1}_{[s, \infty)}(t_i) - \mathbb{1}_{[s, \infty)}(t_{i-1}))}) ds \right\} \\ &= \exp \left\{ -\lambda \sum_{i=1}^m (t_i - t_{i-1})(1 - e^{-\alpha_i}) \right\} = \mathbb{E} \exp \left\{ - \sum_{i=1}^n \alpha_i (N_{t_i \wedge a} - N_{t_{i-1} \wedge a}) \right\}\end{aligned}$$

which shows that $(\mathcal{L}_t^{(a)}, t \geq 0) \stackrel{(\text{law})}{=} (N_{t \wedge a}, t \geq 0)$.

Since $(N_{t \wedge a}, t \geq 0)$ and $(N_{t \vee a} - N_a, t \geq 0)$ are independent and they add to $(N_t, t \geq 0)$, $(N_t, t \geq 0 \mid N_a = 0) \stackrel{(\text{law})}{=} (N_{t \vee a} - N_a, t \geq 0)$. \square

Remarks 3.2

(1) By Proposition 3.1(b1) the Lévy measure ν of N can be viewed as the law of the stochastic process

$$(\mathbb{1}_{[UY, \infty)}(t), t \geq 0)$$

under the infinite measure $\lambda Y h(UY) d\mathbb{P}$. This point of view provides some intuition about the support of a Lévy measure and better understanding how its mass is distributed on the path space.

(2) The process $(r_t^{(a)}, t \geq 0)$ of Proposition 3.1(a1) is not infinitely divisible. Indeed, for each $t > 0$, $r_t^{(a)}$ is a Bernoulli random variable.

(3) While the decomposition (1.5) is quite intuitive in case (c1), it is not so for general infinitely divisible random fields (cf. [3]).

3.2 Sato processes

Recall that a process $X = (X_t, t \geq 0)$ is H -self-similar, $H > 0$, if for every $c > 0$

$$(X_{ct}, t \geq 0) \stackrel{(\text{law})}{=} (c^H X_t, t \geq 0).$$

It is well-known that a Lévy process is H -self-similar if and only if it is strictly α -stable with $\alpha = 1/H \in (0, 2]$, see [19, Proposition 13.5]. In short, there are only obvious examples of self-similar Lévy processes.

Sato [18] showed that within a larger class of additive processes there is a rich family of self-similar processes which is generated by selfdecomposable laws. These processes are known as *Sato processes* and will be precisely defined below.

Recall that the law of a random variable S is said to be selfdecomposable if for every $b > 1$ there exists an independent of S random variable R_b such that

$$S \stackrel{(\text{law})}{=} b^{-1}S + R_b.$$

Wolfe [21] and Jurek and Vervaat [7], showed that a random variable S is selfdecomposable if and only if

$$S \stackrel{(\text{law})}{=} \int_0^\infty e^{-s} dY_s \tag{3.2}$$

for some Lévy process $Y = (Y_s, s \geq 0)$ with $\mathbb{E}(\ln^+ |Y_1|) < \infty$. Moreover, there is a 1-1 correspondence between the laws of S and Y_1 . The process Y is called the background driving Lévy process (BDLP) of S .

Sato [18] proved that a random variable S has the selfdecomposable law if and only if for each $H > 0$ there exists a unique additive H -self-similar process $(X_t, t \geq 0)$ such that $X_1 \stackrel{(\text{law})}{=} S$. An additive self-similar processes, whose law at time 1 is selfdecomposable, will be called a Sato processes.

Jeanblanc, Pitman, and Yor [6, Theorem 1] gave the following representation of Sato processes. Let Y be the BDLP specified in (3.2) and let $\hat{Y} = (\hat{Y}_s, s \geq 0)$ be an independent copy of Y . Then, for each $H > 0$, the process

$$X_r := \begin{cases} \int_{\ln(r^{-1})}^{\infty} e^{-Ht} d_t(Y_{Ht}) & \text{if } 0 \leq r \leq 1 \\ X_1 + \int_0^{\ln r} e^{Ht} d_t(\hat{Y}_{Ht}) & \text{if } r \geq 1. \end{cases} \quad (3.3)$$

is the Sato process with selfsimilarity exponent H . Stochastic integrals in (3.2) and (3.3) can be evaluated pathwise by parts due to the smoothness of the integrands. We will give another form of this representation that is easier to use for our purposes.

Theorem 3.3 *Let $\bar{Y} = (\bar{Y}_s, s \in \mathbb{R})$ be a double sided Lévy process such that $\bar{Y}_0 = 0$ and $\mathbb{E}(\ln^+ |\bar{Y}_1|) < \infty$. Then, for each $H > 0$, the process*

$$X_t := \int_{\ln(t^{-H})}^{\infty} e^{-s} d\bar{Y}_s, \quad t \geq 0, \quad (3.4)$$

is a Sato process with selfsimilarity exponent H . Conversely, any Sato process with selfsimilarity exponent H has a version given by (3.4).

Proof By definition, a double sided Lévy process \bar{Y} is indexed by \mathbb{R} , has stationary and independent increments, càdlàg paths, and $\bar{Y}_0 = 0$ a.s. Since (3.4) coincides with (3.2) when $t = 1$, the improper integral $X_1 = \int_0^{\infty} e^{-s} d\bar{Y}_s$ converges a.s. and it has a selfdecomposable distribution. Moreover,

$$X_{0+} = \lim_{t \downarrow 0} \int_{\ln(t^{-H})}^{\infty} e^{-s} d\bar{Y}_s = 0 \quad a.s.$$

For every $0 < t_1 < \dots < t_n$ and $u_k = \ln(t_k^{-H})$ the increments

$$X_{t_k} - X_{t_{k-1}} = \int_{u_k}^{\infty} e^{-s} d\bar{Y}_s - \int_{u_{k-1}}^{\infty} e^{-s} d\bar{Y}_s = \int_{u_k}^{u_{k-1}} e^{-s} d\bar{Y}_s, \quad k = 2, \dots, n$$

are independent as \bar{Y} has independent increments. Thus X is an additive process.

To prove the H -selfsimilarity of X , notice that since X is an additive process, it is enough to show that for every $c > 0$ and $0 < t < u$

$$X_{cu} - X_{ct} \stackrel{\text{(law)}}{=} c^H (X_u - X_t). \quad (3.5)$$

Since \bar{Y} has stationary increments, we get

$$\begin{aligned} X_{cu} - X_{ct} &= \int_{\ln((cu)^{-H})}^{\ln((ct)^{-H})} e^{-s} d\bar{Y}_s = \int_{\ln(u^{-H}) + \ln(c^{-H})}^{\ln(t^{-H}) + \ln(c^{-H})} e^{-s} d\bar{Y}_s \\ &\stackrel{\text{(law)}}{=} \int_{\ln(u^{-H})}^{\ln(t^{-H})} e^{-s - \ln(c^{-H})} d\bar{Y}_s = c^H (X_u - X_t), \end{aligned}$$

which proves (3.5).

Conversely, let $X = (X_t : t \geq 0)$ be a H -selfsimilar Sato process. By (3.2) there exists a unique in law Lévy process $Y = (Y_t : t \geq 0)$ such that $\mathbb{E}(\ln^+ |Y_1|) < \infty$ and

$$X_1 \stackrel{\text{(law)}}{=} \int_0^\infty e^{-s} dY_s.$$

Let $Y^{(1)}$ and $Y^{(2)}$ be independent copies of the Lévy process Y . Define $\bar{Y}_s = Y_s^{(1)}$ for $s \geq 0$ and $\bar{Y}_s = Y_{(-s)-}^{(2)}$ for $s < 0$. Then \bar{Y} is a double sided Lévy process with $\bar{Y}_1 \stackrel{\text{(law)}}{=} Y_1$. Then

$$\tilde{X}_t := \int_{\ln(t^{-H})}^\infty e^{-s} d\bar{Y}_s, \quad t \geq 0,$$

is a version of X . \square

Corollary 3.4 *Let $X = (X_t : t \geq 0)$ be a H -selfsimilar Sato process given by (3.4). Let ρ be the Lévy measure of \bar{Y}_1 . Then the Lévy measure ν of X is given by*

$$\nu(F) = \int_{\mathbb{R}} \int_{\mathbb{R}} F(xe^{-s} \mathbb{1}_{[e^{-s}/H, \infty)}) \rho(dx) ds. \quad (3.6)$$

Proof We can write (3.4) as $X_t = \int_{\mathbb{R}} f_t(s) d\bar{Y}_s$, where $f_t(s) = e^{-s} \mathbb{1}_{[e^{-s}/H, \infty)}(t)$. It follows from [12, Theorem 2.7(iv)] that the Lévy measure ν of X is the image of $m \otimes \rho$ by the map $(s, x) \mapsto xf_{(\cdot)}(s)$ from \mathbb{R}^2 into $\mathbb{R}^{[0, \infty)}$. \square

From now on we will consider a H -selfsimilar nonnegative Sato process with finite mean and no drift. By Theorem 3.3 we have

$$\psi(t) = \int_{\ln(t^{-H})}^\infty e^{-s} d\bar{Y}_s, \quad t \geq 0, \quad (3.7)$$

where $\bar{Y} = (\bar{Y}_t, t \in \mathbb{R})$ is a double sided subordinator without drift such that $\bar{Y}_0 = 0$ and $\mathbb{E}\bar{Y}_1 < \infty$. Consequently, $\mathbb{E}\psi(t) = \kappa t^H$, $t \geq 0$, where $\kappa := \mathbb{E}\psi(1) = \mathbb{E}\bar{Y}_1$.

Proposition 3.5 *Let $(\psi(t), t \geq 0)$ be a nonnegative H -selfsimilar Sato process given by (3.7). Therefore, the Lévy measure ρ of \bar{Y}_1 is concentrated on \mathbb{R}_+ .*

(a2) *Given $a > 0$, let $(r^{(a)}(t), t \geq 0)$ be the process defined by:*

$$r^{(a)}(t) := a^H UV \mathbb{1}_{[aU^{1/H}, \infty)}(t), t \geq 0,$$

where U is a standard uniform random variable and V has the distribution $\kappa^{-1}x\rho(dx)$, with U, V and $(\psi(t), t \geq 0)$ independent. Then $r^{(a)}$ satisfies (1.2), that is,

$$\{\psi(t) + a^H UV \mathbb{1}_{[aU^{1/H}, \infty)}(t), t \geq 0\} \stackrel{\text{(law)}}{=} \{\psi(t), t \geq 0\} \text{ under } \mathbb{E} \left[\frac{\psi(a)}{\kappa a^H}; \cdot \right].$$

(b2) Let G be a standard exponential random variable, U and V be as above, and assume that G , U , and V are independent. Then the Lévy measure ν of the process $(\psi(t), t \geq 0)$ can be represented as

$$\nu(F) = \kappa \mathbb{E} \left[(UV)^{-1} e^{GU^{1/H}} F(G^H UV \mathbb{1}_{[GU^{1/H}, \infty)}) \right]$$

for every measurable functional $F : \mathbb{R}_+^{[0, \infty)} \mapsto \mathbb{R}_+$. Therefore, ν is the law of the process $(G^H UV \mathbb{1}_{[GU^{1/H}, \infty)}(t), t \geq 0)$ under the measure $\kappa(UV)^{-1} e^{GU^{1/H}} d\mathbb{P}$.

(c2) The components of the decomposition (1.5): $\psi \stackrel{(\text{law})}{=} (\psi | \psi(a) = 0) + \mathcal{L}^{(a)}$, can be identified as

$$(\psi(t), t \geq 0 | \psi(a) = 0) \stackrel{(\text{law})}{=} (\psi(t \vee a) - \psi(a), t \geq 0).$$

and

$$(\mathcal{L}_t^{(a)}, t \geq 0) \stackrel{(\text{law})}{=} (\psi(t \wedge a), t \geq 0).$$

The Lévy measures ν_a and $\tilde{\nu}_a$ of $(\psi(t), t \geq 0 | \psi(a) = 0)$ and of $(\mathcal{L}_t^{(a)}, t \geq 0)$, respectively, are given by

$$\nu_a(F) = \int_{-\infty}^{\ln(a^{-H})} \int_0^{\infty} F(xe^{-s} \mathbb{1}_{[e^{-s/H}, \infty)}) \rho(dx) ds$$

and

$$\tilde{\nu}_a(F) = \int_{\ln(a^{-H})}^{\infty} \int_0^{\infty} F(xe^{-s} \mathbb{1}_{[e^{-s/H}, \infty)}) \rho(dx) ds,$$

for every measurable functional $F : \mathbb{R}_+^{[0, \infty)} \mapsto \mathbb{R}_+$.

Proof (a2): By (1.4) we have for any measurable functional $F : \mathbb{R}^{[0, \infty)} \mapsto \mathbb{R}_+$

$$\begin{aligned} \mathbb{E}F(r_t^a, t \geq 0) &= \frac{1}{\mathbb{E}\psi(a)} \int_{\mathbb{R}_+^E} F(y)y(a) \nu(dy) \\ &= \frac{1}{a^H \mathbb{E}\psi(1)} \int_{\mathbb{R}} \int_{\mathbb{R}_+} F(xe^{-s} \mathbb{1}_{[e^{-s/H}, \infty)}) xe^{-s} \mathbb{1}_{[e^{-s/H}, \infty)}(a) \rho(dx) ds \\ &= \frac{a^{-H}}{\mathbb{E}\psi(1)} \int_{\ln(a^{-H})}^{\infty} \int_{\mathbb{R}_+} F(xe^{-s} \mathbb{1}_{[e^{-s/H}, \infty)}) x \rho(dx) e^{-s} ds \\ &= a^{-H} \int_{\ln(a^{-H})}^{\infty} \mathbb{E}F(Ve^{-s} \mathbb{1}_{[e^{-s/H}, \infty)}) e^{-s} ds \\ &= \mathbb{E} \left[F(a^H UV \mathbb{1}_{[aU^{1/H}, \infty)}) \right]. \end{aligned}$$

Thus $(r_t^a, t \geq 0) \stackrel{(\text{law})}{=} (a^H UV \mathbb{1}_{[aU^{1/H}, \infty)}(t), t \geq 0)$. Since U, V and ψ are independent, (1.2) completes the proof of (a2).

(b2): Since the process $(\psi(t), t \geq 0)$ is stochastically continuous we have for every σ -finite measure \tilde{m} whose support is $[0, \infty)$ and $\int_0^\infty t^H \tilde{m}(dt) < \infty$

$$\begin{aligned} \nu(F) &= \int_0^\infty \mathbb{E} \left[\frac{F(r^{(a)})}{\int_0^\infty r_s^{(a)} \tilde{m}(ds)} \right] \mathbb{E}[\psi(a)] \tilde{m}(da) \\ &= \mathbb{E}[\psi(1)] \int_0^\infty \mathbb{E} \left[\frac{F(a^H UV \mathbb{1}_{[aU^{1/H}, \infty)})}{UV \tilde{m}([aU^{1/H}, \infty))} \right] \tilde{m}(da). \end{aligned}$$

If \tilde{m} is the law of a nonnegative random variable W , then

$$\begin{aligned} \nu(F) &= \mathbb{E}[\psi(1)] \int_0^\infty \mathbb{E} \left[\frac{h(aU^{1/H})}{UV} F(a^H UV \mathbb{1}_{[aU^{1/H}, \infty)}) \right] \tilde{m}(da) \\ &= \mathbb{E}[\psi(1)] \mathbb{E} \left[\frac{h(U^{1/H}W)}{UV} F(UVW^H \mathbb{1}_{[U^{1/H}W, \infty)}) \right] \end{aligned}$$

which is the formula in (b2).

(c2): Since the conditional process $(\psi(t), t \geq 0 | \psi(a) = 0)$ has the Lévy measure $\nu_a(dy) = \mathbb{1}_{\{y(a)=0\}} \nu(dy)$ (see [3]), by (3.6) we obtain for any measurable functional $F : \mathbb{R}_+^{[0, \infty)} \mapsto \mathbb{R}_+$ and $a > 0$

$$\begin{aligned} \nu_a(F) &= \int F(y) \mathbb{1}_{\{y(a)=0\}} \nu(dy) \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}_+} F(xe^{-s} \mathbb{1}_{[e^{-s/H}, \infty)}) \mathbb{1}_{\{xe^{-s} \mathbb{1}_{[e^{-s/H}, \infty)}(a)=0\}} \rho(dx) ds \\ &= \int_{-\infty}^{\ln(a^{-H})} \int_0^\infty F(xe^{-s} \mathbb{1}_{[e^{-s/H}, \infty)}) \rho(dx) ds. \end{aligned}$$

Since $\tilde{\nu}_a = \nu - \nu_a$,

$$\tilde{\nu}_a(F) = \int_{\ln(a^{-H})}^\infty \int_0^\infty F(xe^{-s} \mathbb{1}_{[e^{-s/H}, \infty)}) \rho(dx) ds$$

Let $0 = t_0 < t_1 < \dots < t_n$ be such that $t_m = a$ for some $m \leq n$. For $\alpha_i > 0$ we obtain

$$\begin{aligned}
\mathbb{E} \exp \left\{ - \sum_{i=1}^n \alpha_i (\mathcal{L}_{t_i}^{(a)} - \mathcal{L}_{t_{i-1}}^{(a)}) \right\} &= \exp \{ -\tilde{\nu}_a (1 - e^{-\sum_{i=1}^n \alpha_i (y(t_i) - y(t_{i-1}))}) \} \\
&= \exp \left\{ - \int_{\ln(a^{-H})}^{\infty} \int_0^{\infty} (1 - e^{-\sum_{i=1}^n \alpha_i x e^{-s} (\mathbb{1}_{[e^{-s/H}, \infty)}(t_i) - \mathbb{1}_{[e^{-s/H}, \infty)}(t_{i-1})]}) \rho(dx) ds \right\} \\
&= \exp \left\{ - \int_{\ln(a^{-H})}^{\infty} \int_0^{\infty} (1 - e^{-\sum_{i=1}^n \alpha_i x e^{-s} \mathbb{1}_{(t_{i-1}, t_i]}(e^{-s/H})}) \rho(dx) ds \right\} \\
&= \exp \left\{ - \sum_{i=1}^m \int_{\ln(t_i^{-H})}^{\ln(t_{i-1}^{-H})} \int_0^{\infty} (1 - e^{-\alpha_i x e^{-s}}) \rho(dx) ds \right\} = \prod_{i=1}^m \mathbb{E} \exp \{ - \alpha_i (\psi(t_i) - \psi(t_{i-1})) \} \\
&= \mathbb{E} \exp \left\{ - \sum_{i=1}^n \alpha_i (\psi(t_i \wedge a) - \psi(t_{i-1} \wedge a)) \right\},
\end{aligned}$$

which shows that $(\mathcal{L}_t^{(a)}, t \geq 0) \stackrel{(\text{law})}{=} (\psi(t \wedge a), t \geq 0)$.

Since $(\psi(t \wedge a), t \geq 0)$ and $(\psi(t \vee a) - \psi(a), t \geq 0)$ are independent and they add to $(\psi(t), t \geq 0)$, we get $(\psi(t), t \geq 0 | \psi(a) = 0) \stackrel{(\text{law})}{=} (\psi(t \vee a) - \psi(a), t \geq 0)$. \square

3.3 Stochastic convolution

Let $Z = (Z_t, t \geq 0)$ be a subordinator with no drift. For a fixed function $f : \mathbb{R}_+ \mapsto \mathbb{R}_+$ and $t \geq 0$, the stochastic convolution $f * Z$ is given by

$$(f * Z)(t) = \int_0^t f(t-s) dZ_s.$$

Assume that $\kappa := \mathbb{E}Z_1 \in (0, \infty)$ and $\int_0^t f(s) ds < \infty$ for every $t > 0$. Therefore, $\mathbb{E}[(f * Z)(t)] = \kappa \int_0^t f(s) ds < \infty$. Set $f(u) = 0$ when $u < 0$.

We will consider the stochastic convolution process

$$\psi(t) := \int_0^t f(t-s) dZ_s, \quad t \geq 0. \quad (3.8)$$

Clearly, $(\psi(t), t \geq 0)$ is an infinitely divisible process. To determine its Lévy measure we write $\psi(t) = \int_0^\infty f_t(s) dZ_s$, where $f_t(s) = f(t-s)$. It follows from [12, Theorem 2.7(iv)] that the Lévy measure ν of the process ψ is the image of $m \otimes \rho$ by the map $(s, x) \mapsto x f_{\cdot}(s)$ acting from \mathbb{R}_+^2 into $\mathbb{R}^{[0, \infty)}$. That is,

$$\nu(F) = \int_0^\infty \int_0^\infty F(x f(t-s), t \geq 0) \rho(dx) ds \quad (3.9)$$

for every measurable functional $F : \mathbb{R}_+^{[0, \infty)} \mapsto \mathbb{R}_+$.

Proposition 3.6 *Let $(\psi(t), t \geq 0)$ be a stochastic convolution process as in (3.8). Let ρ be the Lévy measure of Z_1 and $I(a) := \int_0^a f(s) ds$.*

(a3) *Given $a > 0$ such that $I(a) > 0$, let $r^{(a)}$ be the process defined by:*

$$r^{(a)}(t) := Vf(t - U_a), \quad t \geq 0$$

where the random variable U_a has density $\frac{f(a-s)}{I(a)}$ on $[0, a]$, V has the law $\kappa^{-1}x\rho(dx)$ on \mathbb{R}_+ , and U_a, V , and $(\psi(t) : t \geq 0)$ are independent. Then $r^{(a)}$ satisfies (1.2), that is,

$$(\psi(t) + Vf(t - U_a), t \geq 0) \stackrel{(\text{law})}{=} (\psi(t), t \geq 0) \text{ under } \mathbb{E} \left[\frac{\psi(a)}{\kappa I(a)}; \cdot \right]$$

(b3) *Suppose that $\int_0^\infty e^{-\theta s} f(s) ds < \infty$ for some $\theta > 0$. Let Y be a random variable with the exponential law of mean θ^{-1} and independent of V specified in (a3). Then the Lévy measure ν of $(\psi(t), t \geq 0)$ can be represented as*

$$\nu(F) = \frac{\kappa}{\theta} \mathbb{E} [V^{-1} e^{\theta Y} F(Vf(t - Y), t \geq 0)].$$

for every measurable functional $F : \mathbb{R}_+^{[0, \infty)} \mapsto \mathbb{R}_+$. Therefore, ν is the law of the process $(Vf(t - Y), t \geq 0)$ under the measure $\kappa\theta^{-1}V^{-1}e^{\theta Y} d\mathbb{P}$.

(c3) *The components of the decomposition (1.5): $\psi \stackrel{(\text{law})}{=} (\psi | \psi(a) = 0) + \mathcal{L}^{(a)}$, can be identified as*

$$(\psi(t), t \geq 0 | \psi(a) = 0) \stackrel{(\text{law})}{=} \left(\int_0^t f(t-s) \mathbb{1}_{D_a}(s) dZ_s, t \geq 0 \right)$$

and

$$(\mathcal{L}_t^{(a)}, t \geq 0) \stackrel{(\text{law})}{=} \left(\int_0^t f(t-s) \mathbb{1}_{D_a^c}(s) dZ_s, t \geq 0 \right)$$

where $D_a = \{s \geq 0 : f(a-s) = 0\}$ and $D_a^c = \mathbb{R}_+ \setminus D_a$.

The Lévy measures ν_a and $\tilde{\nu}_a$ of $(\psi(t), t \geq 0 | \psi(a) = 0)$ and of $(\mathcal{L}_t^{(a)}, t \geq 0)$, respectively, are given by

$$\nu_a(F) = \int_{D_a} \int_0^\infty F(xf(t-s), t \geq 0) \rho(dx) ds$$

and

$$\tilde{\nu}_a(F) = \int_{D_a^c} \int_0^\infty F(xf(t-s), t \geq 0) \rho(dx) ds,$$

for every measurable functional $F : \mathbb{R}_+^{[0, \infty)} \mapsto \mathbb{R}_+$.

Proof (a3): From (1.4) and (3.9) we get

$$\begin{aligned}
\mathbb{E}F(r_t^{(a)}, t \geq 0) &= \frac{1}{\mathbb{E}\psi(a)} \int F(y) y(a) \nu(dy) \\
&= \frac{1}{\kappa I(a)} \int_0^\infty \int_0^\infty F(xf(t-s), t \geq 0) xf(a-s) \rho(dx) ds \\
&= \int_0^a \int_0^\infty F(xf(t-s), t \geq 0) \frac{x\rho(dx)}{\kappa} \frac{f(a-s) ds}{I(a)} \\
&= \mathbb{E}[F(Vf(t-U_a), t \geq 0)].
\end{aligned}$$

(b3): Since ψ is stochastically continuous, using (1.3) and (a3), we have for every σ -finite measure \tilde{m} whose support is $[0, \infty)$ and $\int_0^\infty I(a) \tilde{m}(da) < \infty$

$$\begin{aligned}
\nu(F) &= \int_0^\infty \mathbb{E} \left[\frac{F(r^{(a)})}{\int_0^\infty r_s^{(a)} \tilde{m}(ds)} \right] \mathbb{E}[\psi(a)] \tilde{m}(da) \\
&= \kappa \int_0^\infty \mathbb{E} \left[\frac{F(Vf(t-U_a), t \geq 0)}{V \int_0^\infty f(s-U_a) \tilde{m}(ds)} \right] I(a) \tilde{m}(da).
\end{aligned}$$

Since \tilde{m} is the law of Y in our case, it is easy to check that $\beta := \int_0^\infty I(a) \tilde{m}(da) < \infty$. Also,

$$\int_0^\infty f(s-U_a) \tilde{m}(ds) = \beta \theta e^{-\theta U_a}.$$

Then we get

$$\begin{aligned}
\nu(F) &= \frac{\kappa}{\beta \theta} \int_0^\infty \mathbb{E} [V^{-1} e^{\theta U_a} F(Vf(t-U_a), t \geq 0)] I(a) \theta e^{-\theta a} da \\
&= \frac{\kappa}{\beta \theta} \int_0^\infty \int_0^a \mathbb{E} [V^{-1} e^{\theta s} F(Vf(t-s), t \geq 0)] f(a-s) ds \theta e^{-\theta a} da \\
&= \frac{\kappa}{\theta} \int_0^\infty \mathbb{E} [V^{-1} e^{\theta s} F(Vf(t-s), t \geq 0)] \theta e^{-\theta s} ds \\
&= \frac{\kappa}{\theta} \mathbb{E} [V^{-1} e^{\theta Y} F(Vf(t-Y), t \geq 0)].
\end{aligned}$$

(c3): Since the conditional process $(\psi(t), t \geq 0 | \psi(a) = 0)$ has the Lévy measure $\nu_a(dy) = \mathbb{1}_{\{y(a)=0\}} \nu(dy)$ (see [3]), by (3.6) we obtain for any measurable functional $F : \mathbb{R}_+^{[0, \infty)} \mapsto \mathbb{R}_+$ and $a > 0$

$$\begin{aligned}
\nu_a(F) &= \int F(y) \mathbb{1}_{\{y(a)=0\}} \nu(dy) \\
&= \int_0^\infty \int_0^\infty F(xf(t-s), t \geq 0) \mathbb{1}_{\{(x,s): xf(a-s)=0\}} \rho(dx) ds \\
&= \int_0^\infty \int_0^\infty F(xf(t-s), t \geq 0) \mathbb{1}_{D_a}(s) \rho(dx) ds.
\end{aligned}$$

Using again [12, Theorem 2.7(iv)] we see that ν_a is the Lévy measure of the process

$$\left(\int_0^t f(t-s) \mathbb{1}_{D_a}(s) dZ_s, t \geq 0 \right)$$

which is a nonnegative infinitely divisible process without drift. Since the law of such process is completely characterized by its Lévy measure, we infer that

$$(\psi(t), t \geq 0 \mid \psi(a) = 0) \stackrel{(\text{law})}{=} \left(\int_0^t f(t-s) \mathbb{1}_{D_a}(s) dZ_s, t \geq 0 \right).$$

Since $\tilde{\nu}_a = \nu - \nu_a$ and $\psi \stackrel{(\text{law})}{=} (\psi \mid \psi(a) = 0) + \mathcal{L}^{(a)}$, we can apply the same argument as above to get

$$(\mathcal{L}_t^{(a)}, t \geq 0) \stackrel{(\text{law})}{=} \left(\int_0^t f(t-s) \mathbb{1}_{D_a}(s) dZ_s, t \geq 0 \right).$$

□

3.4 Tempered stable subordinator

Tempered α -stable subordinators behave at short time like α -stable subordinators and may have all moments finite, while the latter have the first moment infinite. Therefore, we can make use of tempered stable subordinators to illustrate identities (1.2)–(1.5). For concreteness, consider a tempered α -stable subordinator $(\psi(t), t \geq 0)$ determined by the Laplace transform

$$\mathbb{E}e^{-u\psi(1)} = \exp\{1 - (1+u)^\alpha\} \quad (3.10)$$

where $\alpha \in (0, 1)$. When $\alpha = 1/2$, ψ is also known as the inverse Gaussian subordinator. A systematic treatment of tempered α -stable laws and processes can be found in [14]. In particular, the Lévy measure of $\psi(1)$ is given by

$$\rho(dx) = \frac{1}{|\Gamma(-\alpha)|} x^{-\alpha-1} e^{-x} dx, \quad x > 0,$$

[14, Theorems 2.3 and 2.9(2.17)]. Therefore, the Lévy measure ν of the process ψ is given by

$$\begin{aligned} \nu(F) &= \int_0^\infty \int_0^\infty F(x \mathbb{1}_{[s, \infty)}) \rho(dx) ds \\ &= \frac{1}{|\Gamma(-\alpha)|} \int_0^\infty \int_0^\infty F(x \mathbb{1}_{[s, \infty)}) x^{-\alpha-1} e^{-x} dx ds, \end{aligned} \quad (3.11)$$

for every measurable functional $F : \mathbb{R}_+^{[0, \infty)} \mapsto \mathbb{R}_+$.

Proposition 3.7 *Let $(\psi(t), t \geq 0)$ be a tempered α -stable subordinator as above.*

(a4) *Given $a > 0$, let $r^{(a)}$ be the process defined by:*

$$r^{(a)}(t) := G\mathbb{1}_{[aU, \infty)}(t), \quad t \geq 0$$

where G has a Gamma($1-\alpha, 1$) law and U is a standard uniform random variable independent of G . Then $r^{(a)}$ satisfies (1.2), that is,

$$(\psi(t) + G\mathbb{1}_{[aU, \infty)}(t), t \geq 0) \stackrel{(\text{law})}{=} (\psi(t), t \geq 0) \text{ under } \mathbb{E} \left[\frac{\psi(a)}{\alpha a}; \cdot \right]$$

(b4) *The Lévy measure ν of $(\psi(t), t \geq 0)$ can be represented as*

$$\nu(F) = \alpha^{-1} \mathbb{E} [G^{-1} Y e^{UY} F(G\mathbb{1}_{[UY, \infty)})]$$

for every measurable functional $F : \mathbb{R}_+^{[0, \infty)} \mapsto \mathbb{R}_+$. Here G, U are as in (a4), Y is a standard exponential variable, and G, U and Y are independent. Consequently, ν is the law of the process $(G\mathbb{1}_{[UY, \infty)}, t \geq 0)$ under the measure $\alpha^{-1} G^{-1} Y e^{UY} d\mathbb{P}$.

(c4) *The components of the decomposition (1.5): $\psi \stackrel{(\text{law})}{=} (\psi | \psi(a) = 0) + \mathcal{L}^{(a)}$, can be identified as*

$$(\psi(t), t \geq 0 | \psi(a) = 0) \stackrel{(\text{law})}{=} (\psi(t \vee a) - \psi(a), t \geq 0).$$

and

$$(\mathcal{L}_t^{(a)}, t \geq 0) \stackrel{(\text{law})}{=} (\psi(t \wedge a), t \geq 0).$$

The Lévy measures ν_a and $\tilde{\nu}_a$ of $(\psi(t), t \geq 0 | \psi(a) = 0)$ and of $(\mathcal{L}_t^{(a)}, t \geq 0)$, respectively, are given by

$$\nu_a(F) = \frac{1}{|\Gamma(-\alpha)|} \int_a^\infty \int_0^\infty F(x\mathbb{1}_{[s, \infty)}) x^{-\alpha-1} e^{-x} dx ds$$

and

$$\tilde{\nu}_a(F) = \frac{1}{|\Gamma(-\alpha)|} \int_0^a \int_0^\infty F(x\mathbb{1}_{[s, \infty)}) x^{-\alpha-1} e^{-x} dx ds,$$

for every measurable functional $F : \mathbb{R}_+^{[0, \infty)} \mapsto \mathbb{R}_+$.

Proof (a4): From (3.10) we get $\mathbb{E}\psi(a) = \alpha a$. Using (3.11). and (1.4), we get

$$\begin{aligned} \mathbb{E}F(r_t^{(a)}, t \geq 0) &= \frac{1}{\mathbb{E}\psi(a)} \int F(y) y(a) \nu(dy) \\ &= \frac{1}{\alpha a} \int_0^\infty \int_0^\infty F(x\mathbb{1}_{[s, \infty)}) x\mathbb{1}_{[s, \infty)}(a) \rho(dx) ds \\ &= \frac{1}{\Gamma(1-\alpha)a} \int_0^a \int_0^\infty F(x\mathbb{1}_{[s, \infty)}) x^{-\alpha} e^{-x} dx ds \\ &= \mathbb{E}[F(G\mathbb{1}_{[aU, \infty)})]. \end{aligned}$$

(b4): We apply [3, Theorem 1.2] to $(\psi(t), t \geq 0)$ and $(r_t^{(a)}, t \geq 0)$ specified in (a4). Proceeding analogously to the previous examples we get for any σ -finite measure \tilde{m} whose support equals \mathbb{R}_+ and $\int_{\mathbb{R}_+} a \tilde{m}(da) < \infty$

$$\begin{aligned} \nu(F) &= \int_0^\infty \mathbb{E} \left[\frac{F(r^{(a)})}{\int_0^\infty r_s^{(a)} \tilde{m}(ds)} \right] \mathbb{E}[\psi(a)] \tilde{m}(da) \\ &= \frac{1}{\alpha} \int_0^\infty \mathbb{E} \left[\frac{F(G\mathbb{1}_{[Ua, \infty)})}{G\tilde{m}([aU, \infty))} \right] a \tilde{m}(da). \end{aligned}$$

When \tilde{m} is the law of a standard exponential random variable we obtain

$$\begin{aligned} \nu(F) &= \frac{1}{\alpha} \int_0^\infty \mathbb{E} [e^{aU} G^{-1} F(G\mathbb{1}_{[aU, \infty)})] a e^{-a} da \\ &= \alpha^{-1} \mathbb{E} [G^{-1} Y e^{UY} F(G\mathbb{1}_{[UY, \infty)})] . \end{aligned}$$

(c4): We will omit this proof as it is similar to the proof of (c1) in the Poisson case. \square

3.5 Connection with infinitely divisible random measures

Let $\mathcal{M}(S)$ denote the space of finite measures on a Borel space (S, \mathcal{S}) . $\mathcal{M}(S)$ is a Borel space under the topology of weak convergence of finite measures. A measurable map $\xi : \Omega \mapsto \mathcal{M}(S)$ is called a random measure on S . Any random measure ξ can also be viewed as a stochastic process indexed by \mathcal{S} and having paths in $\mathcal{M}(S) \subset \mathbb{R}^{\mathcal{S}}$, $\xi = \{\xi(A), A \in \mathcal{S}\}$. A random measure is called infinitely divisible if the corresponding stochastic process is infinitely divisible.

3.5.1 Cluster representation

The key result on infinitely divisible random measures is the cluster representation. It says that any infinitely divisible random measure ξ on (S, \mathcal{S}) is of the form

$$\xi = m + \int_{\mathcal{M}(S)} \mu \Lambda(d\mu) \quad a.s. \quad (3.12)$$

where Λ is a Poisson random measure on $\mathcal{M}(S)$ with intensity λ satisfying

$$\int_{\mathcal{M}(S)} (\mu(A) \wedge 1) \lambda(d\mu) < \infty, \quad A \in \mathcal{S} \quad (3.13)$$

and $m \in \mathcal{M}(S)$ is non-random, see [8, Theorem 3.20]. Notice that this result follows from (2.3) of Section 2 when $E = \mathcal{S}$ and $\psi = \xi$. We will sketch a proof to this claim.

Indeed, since (2.3) in this case states that

$$(\xi(A), A \in \mathcal{S}) \stackrel{(\text{law})}{=} \left(f_0(A) + \int_{\mathbb{R}_+^S} f(A) N(df), A \in \mathcal{S} \right),$$

pathwise additivity of ξ implies that ν , the Lévy measure of ξ , is concentrated on finite additive functions $f : \mathcal{S} \mapsto \mathbb{R}_+$. Since the σ -algebra \mathcal{S} is countably generated and ξ is pathwise σ -additive, ν is a σ -finite measure concentrated on $\mathcal{M}(S)$ with $\nu(\{0\}) = 0$. It follows that $f_0 \in \mathcal{M}(S)$. Hence

$$(\xi(A), A \in \mathcal{S}) \stackrel{(\text{law})}{=} \left(m(A) + \int_{\mathcal{M}(S)} \mu(A) N(d\mu), A \in \mathcal{S} \right).$$

This equality can be strengthened to the almost sure equality by the usual argument. Hence (3.12)-(3.13) hold with $\lambda = \nu$, $\Lambda = N$, and $m = f_0$.

3.5.2 A characterization of infinitely divisible random measures

One can make use of (1.2) for nonnegative processes indexed by \mathcal{S} to obtain the following characterization of infinitely divisible random measures on S .

A random measure ξ on S is infinitely divisible if and only if for every A in \mathcal{S} such that $0 < \mathbb{E}[\xi(A)] < \infty$, there exists a random measure $r^{(A)}$ on S , independent of ξ such that:

$$\xi + r^{(A)} \stackrel{(\text{law})}{=} \xi \text{ under } \mathbb{E}\left[\frac{\xi(A)}{\mathbb{E}[\xi(A)]}, \cdot\right] \quad (3.14)$$

The characterization (3.14) can be connected to another characterization given in [8, Theorem 6.17]. Namely, assume that ξ has a σ -finite intensity n , then ξ is infinitely divisible if and only if for every a in S there exists a random measure $R^{(a)}$ on S , independent of ξ such that

$$\xi + R^{(a)} \stackrel{(\text{law})}{=} \xi^a, \quad (3.15)$$

where ξ^a is the Palm measure of ξ at point a .

By definition, the Palm measures $\{\xi^a, a \in S\}$ of ξ satisfy for every A in \mathcal{S} and every measurable subset L of $\mathcal{M}(S)$

$$\mathbb{E}[\xi(A); \xi \in L] = \int_A n(da) \mathbb{P}[\xi^a \in L],$$

which leads to the following relation for A such that $0 < n(A) < \infty$

$$\mathbb{P}[\xi + r^{(A)} \in L] = \frac{1}{n(A)} \int_A n(da) \mathbb{P}[\xi + R^{(a)} \in L].$$

By computing the Laplace transforms one finally has:

$$r^{(A)} \stackrel{(\text{law})}{=} \frac{1}{n(A)} \int_A n(da) R^{(a)}. \quad (3.16)$$

In the special case of a point a of S such that $\mathbb{P}[\xi(\{a\}) > 0] > 0$ (e.g. S is discrete), one obtains: $R^{(a)} \stackrel{(\text{law})}{=} r^{\{a\}}$.

3.5.3 A decomposition formula

Given an infinitely divisible random measure ξ on S , one can take advantage of (1.6) to obtain for every A such that $0 < \mathbb{E}[\xi(A)] < \infty$, the existence of an infinitely divisible random measure $\mathcal{L}^{(A)}$ on S such that:

$$\xi \stackrel{(\text{law})}{=} (\xi \mid \xi(A) = 0) + \mathcal{L}^{(A)}, \quad (3.17)$$

with the two measures on the right hand side independent.

3.5.4 Some remarks

In this section we take $S = \mathbb{R}_+^E$. Let χ be a finite infinitely divisible random measure on \mathbb{R}_+^E with no drift and Lévy measure λ . Assume now that for every a in E :

$$\int_{\mathbb{R}_+^E} f(a) \int_{\mathcal{M}(\mathbb{R}_+^E)} \mu(df) \lambda(d\mu) < \infty.$$

Consider then the nonnegative process ψ on E defined by: $\psi(x) = \int_{\mathbb{R}_+^E} f(x) \chi(df)$. The process ψ is infinitely divisible and nonnegative. The following proposition gives its Lévy measure.

Proposition 3.8 *The infinitely divisible nonnegative process $(\int_{\mathbb{R}_+^E} f(x) \chi(df), x \in E)$ admits for Lévy measure ν given by:*

$$\nu = \int_{\mathcal{M}(\mathbb{R}_+^E)} \mu \lambda(d\mu).$$

Proof From (2.3), we know that there exists a Poisson point process \tilde{N} on \mathbb{R}_+^E with intensity the Lévy measure of ψ satisfying: $(\psi(x), x \in E) = (\int_{\mathbb{R}_+^E} f(x) \tilde{N}(df), x \in E)$. Besides, χ admits the following expression: $\chi = \int_{\mathcal{M}(\mathbb{R}_+^E)} \mu N(d\mu)$, with N Poisson point process on $\mathcal{M}(\mathbb{R}_+^E)$ with intensity λ . One obtains:

$$(\psi(x), x \in E) = \left(\int_{\mathbb{R}_+^E} f(x) \int_{\mathcal{M}(\mathbb{R}_+^E)} \mu(df) N(d\mu), x \in E \right).$$

Using then Campbell formula for every measurable subset A of \mathbb{R}_+^E , one computes the intensity of the Poisson point process $\int_{\mathcal{M}(\mathbb{R}_+^E)} \mu(df) N(d\mu)$

$$\mathbb{E}\left[\int_{\mathbb{R}_+^E} 1_A(f) \int_{\mathcal{M}(\mathbb{R}_+^E)} \mu(df) N(d\mu)\right] = \int_{\mathbb{R}_+^E} 1_A(f) \int_{\mathcal{M}(\mathbb{R}_+^E)} \mu(df) \lambda(d\mu) = \nu(A).$$

□

Proposition 3.8 allows to write every Lévy measure ν on \mathbb{R}_+^E in terms of a Lévy measure on $\mathcal{M}(\mathbb{R}_+^E)$. Indeed given a Lévy measure ν on \mathbb{R}_+^E , denote by $(\psi(x), x \in E)$ the corresponding infinitely divisible nonnegative process without drift. From (2.3), we know that ψ admits the representation $(\int_{\mathbb{R}_+^E} f(x)\chi(df), x \in E)$ with χ Poisson random measure on $\mathcal{M}(\mathbb{R}_+^E)$. The random measure χ is hence infinitely divisible. Proposition 3.8 gives us:

$$\nu = \int_{\mathcal{M}(\mathbb{R}_+^E)} \mu \lambda(d\mu) \quad (3.18)$$

where λ is the Lévy measure of χ . Proposition 3.8 allows to see that given ν , the Lévy measure λ satisfying (3.18) is not unique.

Proposition 3.9 *The intensity ν of a Poisson random measure on \mathbb{R}_+^E with Lévy measure λ satisfies:*

$$\nu = \int_{\mathcal{M}(\mathbb{R}_+^E)} \mu \lambda(d\mu).$$

3.6 Infinitely divisible permanental processes

A permanental process $(\psi(x), x \in E)$ with index $\beta > 0$ and kernel $k = (k(x, y), (x, y) \in E \times E)$ is a nonnegative process with finite dimensional Laplace transforms satisfying, for every $\alpha_1, \dots, \alpha_n \geq 0$ and every x_1, x_2, \dots, x_n in E :

$$\mathbb{E}[\exp\{-\frac{1}{2} \sum_{i=1}^n \alpha_i \psi(x_i)\}] = \det(I + \alpha K)^{-\beta} \quad (3.19)$$

where α is the diagonal matrix with diagonal entries $(\alpha_i)_{1 \leq i \leq n}$, I is the $n \times n$ -identity matrix and K is the matrix $(k(x_i, x_j))_{1 \leq i, j \leq n}$.

Note that the kernel of a permanental process is not unique.

In case $\beta = 1/2$ and k can be chosen symmetric positive semi-definite, $(\psi(x), x \in E)$ equals in law $(\eta_x^2, x \in E)$ where $(\eta_x, x \in E)$ is a centered Gaussian process with covariance k . The permanental processes hence represent an extension of the definition of squared Gaussian processes.

A necessary and sufficient condition on (β, k) for the existence of a permanental process $(\psi(x), x \in E)$ satisfying (3.19), has been established by Vere-Jones [20]. Since we are interested by the subclass of infinitely divisible permanental processes, we will only remind a necessary and sufficient condition for a permanental process to be infinitely divisible. Remark that if $(\psi(x), x \in E)$ is infinitely divisible then for every measurable nonnegative d , $(d(x)\psi(x), x \in E)$ is also infinitely divisible. Up to the product by a deterministic function, $(\psi(x), x \in E)$ is infinitely divisible if and only if it admits for kernel the 0-potential densities (the Green function) of a transient Markov process on E (see [4] and [5]).

Consider an infinitely divisible permanental process $(\psi(x), x \in E)$ admitting for kernel the Green function $(g(x, y), (x, y) \in E \times E)$ of a transient Markov process $(X_t, t \geq 0)$ on E . For simplicity assume that ψ has index $\beta = 1$. For $a \in E$ such that $g(a, a) > 0$, denote by $(L_\infty^{(a)}(x), x \in E)$ the total accumulated local times process of X conditioned to start at a and killed at its last visit to a . In [3], (1.5) has been explicitly written for ψ :

$$\psi \stackrel{(\text{law})}{=} (\psi | \psi(a) = 0) + \mathcal{L}^{(a)}$$

with $\mathcal{L}^{(a)}$ independent process of $(\psi | \psi(a) = 0)$, such that $\mathcal{L}^{(a)} \stackrel{(\text{law})}{=} (2L_\infty^{(a)}(x), x \in E)$. Moreover $(\psi | \psi(a) = 0)$ is a permanental process with index 1 and with kernel the Green function of X killed at its first visit to a .

One can also explicitly write (1.2) for ψ with $(r^{(a)}(x), x \in E) \stackrel{(\text{law})}{=} (2L_\infty^{(a)}(x), x \in E)$. Hence the case of infinitely divisible permanental processes is a special case since $r^{(a)}$ is infinitely divisible and $r^{(a)} \stackrel{(\text{law})}{=} \mathcal{L}^{(a)}$.

The easiest way to obtain the Lévy measure ν of ψ is to use (1.3) with m σ -measure with support equal to E such that: $\int_E g(x, x)m(dx) < \infty$, to obtain

$$\nu(F) = \int_E \mathbb{E}\left[\frac{F(2L_\infty^{(a)})}{\int_E L_\infty^{(a)}(x)m(dx)}\right]g(a, a)m(da),$$

for any measurable functional F on \mathbb{R}_+^E .

If moreover, the 0-potential densities $(g(x, y), (x, y) \in E \times E)$ were taken with respect to m then, for every a , $\int_E L_\infty^{(a)}(x)m(dx)$ represents the time of the last visit to a by X starting from a .

4 Transfer of continuity properties

Using (1.6), a nonnegative infinitely divisible process $\psi = (\psi(x), x \in E)$ with Lévy measure ν and no drift, is hence connected to a family of nonnegative infinitely divisible processes $\{\mathcal{L}^{(a)}, a \in E\}$. In case when ψ is an infinitely divisible squared Gaussian

process, Marcus and Rosen [9] have established correspondences between path properties of ψ and the ones of $\mathcal{L}^{(a)}$, $a \in E$. To initiate a similar study for a general ψ , we assume that (E, d) is a separable metric space with a dense set $D = \{a_k, k \in \mathbb{N}^*\}$.

One immediately notes that if ψ is a.s. continuous with respect to d , then for every a in E , $\mathcal{L}^{(a)}$ is a.s. continuous with respect to d and the measure ν is supported by the continuous functions from E into \mathbb{R}_+ i.e. $r^{(a)}$ is continuous with respect to d , for every a in E .

Conversely if $\mathcal{L}^{(a)}$ is continuous with respect to d for every a in E , what can be said about the continuity of ψ ?

As noticed in [16] (Proposition 4.7) the measure ν admits the following decomposition:

$$\nu = \sum_{k=1}^{\infty} 1_{A_k} \nu_k, \quad (4.1)$$

where $A_1 = \{y \in \mathbb{R}_+^E : y(a_1) > 0\}$ and for $k > 1$,

$$A_k = \{y \in \mathbb{R}_+^E : y(a_i) = 0, \forall i < k \text{ and } y(a_k) > 0\}$$

and ν_k is defined by

$$\nu_k(F) = \mathbb{E}\left[\frac{\mathbb{E}(\psi(a_k))}{r_{a_k}^{(a_k)}} 1_{A_k}(r^{(a_k)}) F(r^{(a_k)})\right]$$

for every measurable functional $F : \mathbb{R}_+^E \mapsto \mathbb{R}_+$.

For every k the measure ν_k is a Lévy measure. Since the supports of this measures are disjoint they correspond to independent nonnegative infinitely divisible processes that we denote by $L(k)$, $k \geq 1$. As a consequence of (4.1), ψ admits the following decomposition:

$$\psi \stackrel{(\text{law})}{=} \sum_{k=1}^{\infty} L(k). \quad (4.2)$$

Note that

$$L(1) \stackrel{(\text{law})}{=} \mathcal{L}^{(a_1)}$$

and similarly for every $k > 1$:

$$L(k) \stackrel{(\text{law})}{=} (\mathcal{L}^{(a_k)} | \mathcal{L}_{\{a_1, \dots, a_{k-1}\}}^{(a_k)} = 0).$$

Consequently, for every $k \geq 1$, $L(k)$ is continuous with respect to d .

From (4.2), one obtains all kind of 0 – 1 laws for ψ . For example:

- $\mathbb{P}[\psi \text{ is continuous on } E] = 0 \text{ or } 1$.

- ψ has a deterministic oscillation function w , such that for every a in E :

$$\liminf_{x \rightarrow a} \psi(x) = \psi(a) \text{ and } \limsup_{x \rightarrow a} \psi(x) = \psi(a) + w(a).$$

Exactly as in [3], one shows the following propositions.

Proposition 4.1 *If for every a in E , $\mathcal{L}^{(a)}$ is a.s. continuous, then there exists a dense subset Δ of E such that a.s. ψ is continuous at each point of Δ and $\psi|_{\Delta}$ is continuous.*

Proposition 4.2 *Assume that ψ is stationary. If for every a in E , $\mathcal{L}^{(a)}$ is a.s. continuous, then ψ is continuous.*

5 A limit theorem

Given a nonnegative infinitely divisible without drift process $(\psi_x, x \in E)$, the following result gives an intrinsic way to obtain $r^{(a)}$ for every a in E .

Theorem 5.1 *For a nonnegative infinitely divisible process $(\psi_x, x \in E)$ with Lévy measure ν , denote by $\psi^{(\delta)}$ an infinitely divisible process with Lévy measure $\delta\nu$. Then, for any a in E such that $\mathbb{E}[\psi_a] > 0$, $r^{(a)}$ is the limit in law of the processes $\psi^{(\delta)}$ under $\mathbb{E}\left[\frac{\psi_a^{(\delta)}}{\mathbb{E}[\psi_a^{(\delta)}]}; \cdot\right]$, as $\delta \rightarrow 0$.*

Proof We remind (1.4): $\mathbb{P}[r^{(a)} \in dy] = \frac{y^{(a)}}{\mathbb{E}[\psi_a]} \nu(dy)$. Since $\mathbb{E}[\psi_a^{(\delta)}] = \delta\mathbb{E}[\psi_a]$, one obtains immediately: $\mathbb{P}[r^{(a)} \in dy] = \frac{y^{(a)}}{\mathbb{E}[\psi_a^{(\delta)}]} \delta\nu(dy)$. Consequently $r^{(a)}$ satisfies:

$$\psi^{(\delta)} + r^{(a)} \stackrel{(\text{law})}{=} \psi^{(\delta)} \text{ under } \mathbb{E}\left[\frac{\psi_a^{(\delta)}}{\mathbb{E}[\psi_a^{(\delta)}]}; \cdot\right].$$

As $\delta \rightarrow 0$, $\psi^{(\delta)}$ converges to the 0-process in law, so $\psi^{(\delta)}$ under $\mathbb{E}\left[\frac{\psi_a^{(\delta)}}{\mathbb{E}[\psi_a^{(\delta)}]}; \cdot\right]$ must converge in law to $r^{(a)}$. \square

From (1.2) and (1.5), one obtains in particular:

$$\mathcal{L}^{(a)} + r^{(a)} \stackrel{(\text{law})}{=} \mathcal{L}^{(a)} \text{ under } \mathbb{E}\left[\frac{\mathcal{L}_a^{(a)}}{\mathbb{E}[\mathcal{L}_a^{(a)}]}; \cdot\right] \quad (5.1)$$

We know from [3], that the Lévy measure of $\mathcal{L}^{(a)}$ is $\nu(dy)1_{y(a)>0}$. Denote by $\ell^{(a,\delta)}$ a nonnegative process with Lévy measure $\delta\nu(dy)1_{y(a)>0}$. Using Theorem 5.1, one obtains that $r^{(a)}$ is also the limit in law of $\ell^{(a,\delta)}$ under $\mathbb{E}\left[\frac{\ell_a^{(a,\delta)}}{\mathbb{E}[\ell_a^{(a,\delta)}]}; \cdot\right]$.

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