



HAL
open science

A UNIQUENESS RESULT FOR A TWO DIMENSIONAL VARIATIONAL PROBLEM

Benjamin Lledos

► **To cite this version:**

Benjamin Lledos. A UNIQUENESS RESULT FOR A TWO DIMENSIONAL VARIATIONAL PROBLEM. 2022. hal-03845599

HAL Id: hal-03845599

<https://hal.science/hal-03845599v1>

Preprint submitted on 9 Nov 2022

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

A UNIQUENESS RESULT FOR A TWO DIMENSIONAL VARIATIONAL PROBLEM

LLEDOS Benjamin¹

ABSTRACT. We investigate the uniqueness of the solutions for a non-strictly convex problem in the Calculus of Variations of the form $\int \varphi(\nabla v) - \lambda v$. Here, $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}$ is a convex function and λ is Lipschitz continuous. We prove the uniqueness when $\nabla \lambda$ is small and give some counterexamples when that is not the case. The proof is based on the global Lipschitz regularity of the minimizers and on the study of their level sets.

Mathematics Subject Classification: 35A02, 49J45, 49N99.

1. INTRODUCTION

1.1. A model case. The motivation of this article is to study non strictly convex problems in the Calculus of Variations in dimension two as in the following model case:

$$(1.1) \quad \tilde{\mathcal{I}}_\lambda : u \rightarrow \int_{\Omega} F(\nabla u(x)) - \lambda(x)u(x)dx$$

where Ω is an open bounded set in \mathbb{R}^2 , $\lambda \in L^\infty(\Omega)$ and $F(y) = f(|y|)$ with

$$(1.2) \quad f(t) = \begin{cases} \frac{1}{2}|t|^2 & \text{if } |t| \leq 1, \\ |t| - \frac{1}{2} & \text{if } 1 < |t| < 2, \\ \frac{1}{4}|t|^2 + \frac{1}{2} & \text{if } 2 \leq |t|. \end{cases}$$

For this functional, the admissible functions u belong to the Sobolev space $W^{1,2}(\Omega)$ with a prescribed trace $\psi : \mathbb{R}^2 \mapsto \mathbb{R}$ on the boundary $\partial\Omega$ of Ω . Our goal is to prove the uniqueness of solutions to the following minimization problem:

$$\tilde{\mathcal{P}}_\lambda : \min_{u \in W_\psi^{1,2}(\Omega)} \tilde{\mathcal{I}}_\lambda(u).$$

When $\lambda \equiv \lambda_0 \in \mathbb{R}_+$, this problem studied by Kawohl, Stara and Wittum in [15] arises as the convexification a non convex problem of shape optimization in the theory of elasticity. In this example, f is the convexification of the minimum of two parabolas: $t \mapsto \frac{1}{2}|t|^2$ and $t \mapsto \frac{1}{4}|t|^2 + \frac{1}{2}$. Observe in particular that f is affine on the interval $(1, 2)$. Since f is convex but not strictly convex, there is no obvious reason for $\tilde{\mathcal{P}}_\lambda$ to have a unique solution.

In fact, the authors of [15] rely on the assumption that the level sets of one minimizer u are star-shaped. Furthermore, they suppose that the boundary of the set in Ω where $f'(|\nabla u|) = 1$ is piecewise \mathcal{C}^1 . In this paper, we do not require such additional assumptions.

¹Institut de Mathématiques de Toulouse, CNRS UMR 5219
 Université de Toulouse, F-31062 Toulouse Cedex 9, France
 E-mail address: benjamin.lledos@math.univ-toulouse.fr
 ORCID: 0000-0002-2339-778X

There is no general answer to the question of uniqueness for non strictly convex problems in the Calculus of Variations, especially when λ is not constant. Hence we restrict our attention to the framework (1.1), where f can be replaced by more general convex functions provided that they are strictly convex around the origin and at infinity.

1.2. Main results. More precisely, let $\varphi : \mathbb{R} \mapsto \mathbb{R}$ be an even \mathcal{C}^1 convex function with $\varphi(t) > \varphi(0) = 0$ for all $t \neq 0$. Moreover, we assume that $\varphi \in \mathcal{C}_{loc}^{1,1}(\mathbb{R} \setminus \{0\})$

We suppose that φ has p -growth for $p > 1$, namely, there exist $C_1 > 0$ and $C_2 > 0$ such that:

$$(1.3) \quad C_1|t|^p \leq \varphi(t) \leq C_2(1 + |t|^p) \text{ for all } t \in \mathbb{R}.$$

We introduce the following set of strict convexity of φ :

$$\mathcal{SC} = \{x \in \mathbb{R}, \forall y \in \mathbb{R} \setminus \{x\}, \forall t \in]0, 1[, \varphi(tx + (1-t)y) < t\varphi(x) + (1-t)\varphi(y)\}.$$

For instance, $\mathcal{SC} = (-\infty, -2) \cup (-1, 1) \cup (2, +\infty)$ when φ is equal to the function f in (1.2).

We make some *structural assumptions* on \mathcal{SC} :

- The set \mathcal{SC} has finitely many connected components, in particular \mathcal{SC} is open and

$$\mathcal{SC} \cap \mathbb{R}_+ = \bigcup_{n=0}^N \mathcal{SC}_n$$

with $\mathcal{SC}_0 := [0, b_0)$, $\mathcal{SC}_n := (a_n, b_n)$ for every $n \in \mathbb{N}^*$, $n < N$ and $\mathcal{SC}_N := (a_N, +\infty)$. For every $n \in \mathbb{N}$, $n < N$ we introduce $d_n := \varphi'(b_n) = \varphi'(a_{n+1})$.

- We assume that φ is \mathcal{C}^2 and $\varphi'' > 0$ on $\mathcal{SC} \setminus \{0\}$ and that φ is strongly convex at $+\infty$ in the following sense:

$$\liminf_{t \rightarrow +\infty} \frac{t\varphi''(t)}{\varphi'(t)} > 0.$$

We define $\Phi(\cdot) := \varphi(|\cdot|)$ and for $\lambda \in L^\infty(\Omega)$ we introduce the following functional:

$$\mathcal{I}_\lambda : u \mapsto \int_{\Omega} \Phi(\nabla u(x)) - \lambda(x)u(x) dx$$

on $W_\psi^{1,p}(\Omega)$, where Ω is an open simply connected bounded set of \mathbb{R}^2 with a Lipschitz continuous boundary $\partial\Omega$ and ψ is a Lipschitz-continuous function on $\partial\Omega$. Here, $W_\psi^{1,p}(\Omega)$ is the subset of those functions in $W^{1,p}(\Omega)$ that are equal to ψ on $\partial\Omega$.

We introduce the minimization problem:

$$\mathcal{P}_\lambda : \min_{u \in W_\psi^{1,p}(\Omega)} \mathcal{I}_\lambda(u).$$

The main result of the paper is the following:

Theorem 1.1. *Let Ω be a simply connected bounded open set of \mathbb{R}^2 . We assume that Ω has a $\mathcal{C}^{1,1}$ boundary, $\psi \in \mathcal{C}^{1,1}(\mathbb{R}^2)$, λ is Lipschitz continuous on $\overline{\Omega}$, $\min_{x \in \overline{\Omega}} \lambda(x) > 0$. There exists a positive constant*

$$C := C(N, |\Omega|, \max_{\overline{\Omega}} \lambda, \min_{\overline{\Omega}} \lambda, \|\psi\|_{\mathcal{C}^{1,1}(\Omega)}, \kappa)$$

where κ is the maximum curvature of Ω such that if $\|\nabla \lambda\|_{L^\infty(\Omega)} \leq C$ then \mathcal{P}_λ admits a unique solution on $W_\psi^{1,p}(\Omega)$.

Remark 1.2. When λ is constant, a more general result, true in any dimension, can be found in [16].

Moreover, the boundedness condition on $\nabla\lambda$ is optimal since:

Proposition 1.3. *There exists $\lambda \in C^\infty(\overline{B_1(0)})$ with $\min_{B_1(0)} \lambda > 0$ such that \mathcal{P}_λ has more than one solution on $W_0^{1,p}(B_1(0))$.*

1.3. Ideas of the proof. We want to prove the uniqueness of the solution u for the variational problem \mathcal{P}_λ . We know by classical theory (see [9] and [13]) that the problem \mathcal{P}_λ admits at least one minimizer u , this function u is bounded, globally Hölder continuous by [13, Theorem 7.8] and locally Lipschitz continuous by [7, Theorem 1.1].

When $\lambda = 0$, the proof is substantially simplified. In this case, the strategy has been developed by Marcellini in [19] under additional assumptions and the proof itself in a general framework is due to Lussardi and Mascolo in [18]. In those two papers the proof is divided into two parts:

- **Part 1** If u and v are two solutions of the same problem, then v is constant on the level sets of u .
- **Part 2** The level sets of u intersect the boundary $\partial\Omega$ of Ω . Since u and v are equal on $\partial\Omega$ they are equal on Ω .

As observed in Remark 1.2, when $\lambda \equiv \lambda_0 \in \mathbb{R}_+$, a shorter proof can be found in [16] but when $\lambda \in W^{1,\infty}(\Omega)$ the proof requires new ideas and turns out to be fairly intricate. **Part 1** remains true but **Part 2** fails to be true. In fact, the term $u \mapsto \int_\Omega \lambda u$ changes the geometry of the level lines: they do not necessarily intersect the boundary $\partial\Omega$ of Ω . It is even possible that only one level set intersects the boundary, see Proposition 2.10.

A very important subset of Ω is the following:

Proposition 1.4. *There exists an open set U such that for every minimizer u , one has $u \in C^1(U)$ and for every $x \in U$, $|\nabla u(x)| \in \mathcal{SC} \setminus \{0\}$ while for a.e. $x \notin U$, $|\nabla u(x)| \notin \mathcal{SC} \setminus \{0\}$.*

When $\lambda \neq 0$, the set $U \cup \partial\Omega$ plays the same role as the one played by $\partial\Omega$ in **Part 2** when there is no lower order term. However, the fact that $u = v$ on U is far from being obvious. Nevertheless, if u and v are two minimizers of the same problem, we can easily see that ∇u and ∇v are equal on U and we can even prove that this is also the case on the level sets that intersect U . The aim of the proof is to show that $u = v$ or $\nabla u = \nabla v$ on the level sets of u and v . Hence, for a.e. $x \in \Omega$, the Lipschitz map $w(x) := u(x) - v(x)$ is equal to 0 or $\nabla w(x) = 0$, thus $u - v = w = 0$.

This idea of using $U \cup \partial\Omega$ comes from a paper by Bouchitté and Bousquet [5] but in their case the fact that \mathcal{SC} is of the form $(1, +\infty)$ implies that the boundary of every connected component of U intersects $\partial\Omega$. Since $\nabla u = \nabla v$ on U and $u = v$ on $\partial\Omega$ we readily obtain that $u = v$ on U and this part of the proof is easier. We warn the reader that this paper is not a generalization of [5] because φ has no singularity at the origin unlike in [5]. This singularity of φ in [5] and in its generalization [17] creates some regularity issues that will not appear here.

For instance in our situation, we exploit the fact that thanks to [10], $\max(\alpha, \varphi'(|\nabla u|)) \in W_{loc}^{1,2}(\Omega)$ for any $\alpha > 0$. Then we prove that $\max(d_0, \varphi'(|\nabla u|))$ has a representative that is absolutely continuous on almost every level sets.

The other major difference between this paper and [5, 17] is that in these references, $\max(1, \varphi'(|\nabla u|))$ is continuous on Ω which allows to prove their results for any dimension larger or equal to two. Here, we heavily rely on two results which are only valid in dimension two: a general regularity result for

Lipschitz continuous functions, see Theorem 2.8 below and the Jordan curve theorem. The latter is the reason why we assume that Ω simply connected: this prevents the existence of holes inside the connected components of the upper level set $E_s := [u > s]$ for $s \in \mathbb{R}$.

When λ is small we can prove that almost every level set intersects a connected component of U where $|\nabla u| < b_0$. This allows us to prove the following theorem:

Theorem 1.5. *When $0 \leq \lambda(x) \leq d_0 h_\Omega$ for a.e. $x \in \Omega$, the problem \mathcal{P}_λ admits a unique minimizer.*

Here, $d_0 = \varphi'(b_0)$ and h_Ω is the Cheeger constant of Ω :

Definition 1.6. The Cheeger constant of Ω is defined as:

$$h_\Omega = \inf_{D \subset \bar{\Omega}} \frac{\text{Per}(D, \mathbb{R}^2)}{|D|}$$

where

$$\text{Per}(D, \mathbb{R}^2) = \sup \left\{ \int_D \text{div } g \mid g \in C_c^1(\mathbb{R}^2; \mathbb{R}^2), |g(x)| \leq 1, \forall x \in \mathbb{R}^2 \right\}$$

is called the *Perimeter* of the set D . A set $D \subset \bar{\Omega}$ of finite perimeter is said to be a *Cheeger set* if $\text{Per}(D, \mathbb{R}^2) = h_\Omega |D|$.

The proof of the main theorem will be based on an induction argument related to the family $\{d_n, n \in \mathbb{N}, 0 \leq n < N\}$ with the previous theorem as the initialization step.

We study the connected components $l_s(u)$ of $L_s(u) := u^{-1}(s) \subset \mathbb{R}^2$ such that $l_s(u)$ is a closed simple curve. The case $l_s(u) \cap \partial\Omega \neq \emptyset$ is easy because $u - v$ is constant on $l_s(u)$ and $u - v = 0$ on $\partial\Omega$ so that $u - v$ on $l_s(u)$. Hence, we can assume that $l_s(u) \Subset \Omega$ and by the Jordan curve theorem, we can define F_s as the bounded connected component of $\mathbb{R}^2 \setminus l_s(u)$. If $l_s(u) \cap U = \emptyset$ then we use the following proposition:

Proposition 1.7. *There exists a representative f_0 of $\max(d_0, \varphi'(|\nabla u|))$ such that for a.e. $s \in \mathbb{R}$, if $l_s(u) \cap U = \emptyset$ then f_0 is equal to a constant $C(l_s(u)) \in \{d_i, 0 \leq i < N\}$ on $l_s(u)$.*

Another important result is a maximum principle proved in section 5 for smooth approximations of our problem \mathcal{P}_λ . We first regularize the problem to obtain a sequence $(u_n)_{n \in \mathbb{N}}$ of smooth minimizers of smooth problems \mathcal{P}_{λ_n} , with $(\varphi_n)_{n \in \mathbb{N}}$ and $(\lambda_n)_{n \in \mathbb{N}}$ smooth approximations of φ and λ . In section 4, we use the fact that the sequence $(\nabla u_n)_{n \in \mathbb{N}}$ generates Young measures $(\nu_x)_{x \in \Omega}$ to prove that $\varphi'_n(\nabla u_n) \rightarrow \varphi'(\nabla u)$ a.e. in Ω when $n \rightarrow +\infty$. For such approximations we have:

Proposition 1.8. *For a.e. $s \in \mathbb{R}$, if $l_s(u)$ is a connected component of $L_s(u)$ which is a closed simple curve and such that $l_s(u) \Subset \Omega$ then*

$$\sup_{l_s(u)} \max(d_0, \varphi'_n(|\nabla u_n|)) = \sup_{F_s} \varphi'_n(|\nabla u_n|).$$

1.4. Plan of the paper. In the next section, we recall some classical results and we introduce the notations and notions that are useful throughout the article. In Section 3, we study the regularity properties of the level sets of the minimizers. In the subsequent Section 4, we prove that $\max(\alpha, \varphi'(|\nabla u|)) \in W_{loc}^{1,2}(\Omega)$. The maximum principle for $\max(d_0, |\sigma_n|)$ is proved in Section 5. Section 6 is dedicated to the proof of Theorem 1.1 and Theorem 1.5. In the last section, we state a possible extension to the main theorem.

2. PRELIMINARY RESULTS

In this section we introduce some known results related to this problem.

2.1. Direct methods. We know by the direct method in the calculus of variations (see [9] and [13]) that the problem \mathcal{P}_λ admits at least one minimizer. We recall that every minimizer u is bounded, globally Hölder continuous by [13, Theorem 7.8] and locally Lipschitz continuous by [7, Theorem 1.1].

We begin this subsection by observing that the minimum of a minimizer is attained on the boundary of Ω .

Proposition 2.1. *Let u be a minimizer of \mathcal{P}_λ on $W_\psi^{1,p}(\Omega)$ with $\lambda \in L^\infty(\Omega)$ and $\lambda(x) \geq 0$ for a.e. $x \in \Omega$. Then $\min_{\bar{\Omega}} u = \min_{\partial\Omega} \psi$.*

Proof. Since $\min_{\bar{\Omega}} u \leq c := \min_{\partial\Omega} \psi$, we have to prove that $\min_{\bar{\Omega}} u \geq \min_{\partial\Omega} \psi$. We introduce $w := \max(u, c)$. If there exists a point $x \in \Omega$ such that $u(x) < c$, then by continuity of u the set $[u < c]$ has a positive measure. We have $w = u$ and $\nabla w = \nabla u$ on $[u > c]$. Moreover, since $[u < c]$ has positive measure we have:

$$0 = \int_{[u \leq c]} \varphi(|\nabla w|) < \int_{[u \leq c]} \varphi(|\nabla u|) \quad \text{and} \quad - \int_{[u \leq c]} \lambda w = - \int_{[u \leq c]} \lambda c \leq - \int_{[u \leq c]} \lambda u.$$

Hence, $I_\lambda(w) < I_\lambda(u)$ on Ω , which contradicts the fact that u is a minimizer. Thus, $u \geq c$ on Ω . \square

We now prove that the gradients of two minimizers of the same problem are collinear. This property is used in many subsequent proofs.

Lemma 2.2. *Let u and v be two minimizers of \mathcal{P}_λ with $\lambda \in L^\infty(\Omega)$. Then $\nabla u(x)$ and $\nabla v(x)$ are collinear and φ is affine on the interval $[|\nabla u(x)|, |\nabla v(x)|]$ for a.e. $x \in \Omega$.*

Proof. Since u is a solution of \mathcal{P}_λ ,

$$\mathcal{I}_\lambda(u) \leq \mathcal{I}_\lambda\left(\frac{u+v}{2}\right).$$

By the fact that φ is non decreasing and the convexity of φ and of the Euclidean norm,

$$\begin{aligned} \mathcal{I}_\lambda\left(\frac{u+v}{2}\right) &= \int_{\Omega} \varphi\left(\left|\frac{\nabla u + \nabla v}{2}\right|\right) - \lambda \frac{u+v}{2} \\ &\leq \frac{1}{2} \int_{\Omega} (\varphi(|\nabla u|) - \lambda u) + \frac{1}{2} \int_{\Omega} (\varphi(|\nabla v|) - \lambda v) \\ &= \frac{1}{2} I_\lambda(u) + \frac{1}{2} I_\lambda(v). \end{aligned}$$

Since v is another solution,

$$I_\lambda(u) = \frac{1}{2} I_\lambda(u) + \frac{1}{2} I_\lambda(v).$$

This implies that

$$\int_{\Omega} \varphi\left(\left|\frac{\nabla u + \nabla v}{2}\right|\right) = \int_{\Omega} \frac{1}{2} (\varphi(|\nabla u|) + \varphi(|\nabla v|)).$$

Hence for a.e. $x \in \Omega$,

$$(2.1) \quad \varphi\left(\left|\frac{\nabla u(x) + \nabla v(x)}{2}\right|\right) = \frac{1}{2} (\varphi(|\nabla u(x)|) + \varphi(|\nabla v(x)|)).$$

Hence for a.e. $x \in \Omega$, $\varphi \circ |\cdot|$ is affine on the segment $[\nabla u(x), \nabla v(x)]$. In view of the definition of φ and the strict convexity of the lower level sets of $|\cdot|$, this means that $\nabla u(x)$ and $\nabla v(x)$ are collinear for a.e. $x \in \Omega$ but also that φ is affine on $[|\nabla u(x)|, |\nabla v(x)|]$ for a.e. $x \in \Omega$. \square

We use a result of [3] to introduce the following set where ∇u is continuous and $|\nabla u|$ takes its values in $\mathcal{SC} \setminus \{0\}$, which is defined in Section 1.2.

Proposition 2.3. *When $\lambda \in \mathcal{C}^0(\overline{\Omega})$, there exists an open set U such that $u \in \mathcal{C}^1(U)$ and for every $x \in U$, $|\nabla u(x)| \in \mathcal{SC} \setminus \{0\}$ while for a.e. $x \notin U$, $|\nabla u(x)| \notin \mathcal{SC} \setminus \{0\}$.*

Proof. By [3, Theorem 6.1], for a.e. $x \in \Omega$ such that $|\nabla u(x)| \in \mathcal{SC} \setminus \{0\}$ there exists a neighborhood \mathcal{V} of x such that $u \in \mathcal{C}^{1,\alpha}(\mathcal{V})$. Since $\mathcal{SC} \setminus \{0\}$ is open there exists $\epsilon > 0$ such that for every $x' \in B_\epsilon(x)$, $|\nabla u(x')| \in \mathcal{SC} \setminus \{0\}$. Let U be the set of such x , then U is open and for a.e. $x \notin U$, $|\nabla u(x)| \notin \mathcal{SC} \setminus \{0\}$. \square

One of the interests of this set is the following:

Proposition 2.4. *The set U does not depend on the choice of a minimizer. Moreover, let u and v be two minimizers of \mathcal{P}_λ , then $\nabla u = \nabla v$ on U .*

Proof. Let us consider two minimizers u and v of the same problem. We define respectively, U_u and U_v as the open sets of the previous proposition for u and v . By Lemma 2.2 and strict convexity of φ on \mathcal{SC} , we have that $\nabla u = \nabla v$ a.e. on U_u . Hence, $v \in \mathcal{C}^1(U_u)$ and for every $x \in U_u$, $|\nabla v(x)| \in \mathcal{SC} \setminus \{0\}$. Thus, by definition of U_v , we have that $U_u \subset U_v$. To prove the other inclusion we just have to exchange u and v . Hence, $U_u = U_v = U$ and $\nabla u = \nabla v$ on U . \square

A direct consequence of this result is that:

Remark 2.5. For every connected component U_i of U , $u - v$ is constant on U_i .

To conclude this section, we introduce the weak Euler-Lagrange equation associated to \mathcal{P}_λ :

$$(2.2) \quad \operatorname{div}(\nabla \Phi(\nabla u)) = -\lambda \text{ on } \Omega.$$

Remark 2.6. By Lemma 2.2 the function $\nabla \Phi(\nabla u) = \varphi'(|\nabla u|) \frac{\nabla u}{|\nabla u|}$ is independent of the choice of the minimizer of \mathcal{P}_λ and will be denoted by σ .

2.2. Lipschitz regularity of a minimizer u and its level lines. In this subsection, we recall some Lipschitz regularity results for u and its level lines.

We use the following result from [17, Theorem 1.6] to show that our minimizers are globally Lipschitz-continuous on Ω :

Proposition 2.7. *We assume that Ω has a $\mathcal{C}^{1,1}$ boundary and $\psi \in \mathcal{C}^{1,1}(\mathbb{R}^2)$. Then any minimizer u of \mathcal{P}_λ is globally Lipschitz-continuous on Ω . Moreover, there exists $L > 0$ such that*

$$\|\nabla u\|_{L^\infty(\Omega)} \leq L(p, C_1, |\Omega|, \|\lambda\|_{L^\infty(\Omega)}, \|\psi\|_{\mathcal{C}^{1,1}(\mathbb{R}^2)}, \kappa)$$

where κ is the maximum of the curvatures of Ω and C_1 is introduced in (1.3).

For a function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ and for every $s \in \mathbb{R}$, we introduce the following notation : $L_s^*(f)$ is the union of all connected components $l_s(f)$ of $L_s(f) = f^{-1}(s) \subset \mathbb{R}^2$ such that $\mathcal{H}^1(l_s(f)) > 0$. Here, \mathcal{H}^1 is the one-dimensional Hausdorff measure.

We will apply the following theorem from [1, Theorem 2.5] to prove Proposition 2.9.

Theorem 2.8. *Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a Lipschitz continuous function with compact support. For a.e. $s \in \mathbb{R}$, we have:*

- $\mathcal{H}^1(L_s(f) \setminus L_s^*(f)) = 0$.
- Every connected component $l_s(f)$ of $L_s(f)$ that is not a point is a closed simple curve with a Lipschitz parametrization γ_s .
- $L_s^*(f)$ has a countable number of connected components.

It follows that the level lines of a minimizer u have a Lipschitz-continuous parametrization:

Remark 2.9. Let u be a globally Lipschitz-continuous minimizer of \mathcal{P}_λ with $\lambda \in L^\infty(\Omega)$. We extend it outside Ω by ψ that can be assumed compactly supported. For a.e. $s \in \mathbb{R}$ every connected component of $L_s^*(u) \subset \mathbb{R}^2$ has a Lipschitz-continuous parametrization.

2.3. Explicit solution on the ball and counter-example. The application of [6, Theorem 1] to our problem \mathcal{P}_λ gives an explicit form for the unique solution on $W_0^{1,p}(B_r(x_0))$ in dimension two when $\lambda \equiv \lambda_0 \in \mathbb{R}_+$:

Proposition 2.10. *When $\Omega = B_r(x_0)$ and λ is constant, the problem \mathcal{P}_λ admits a unique minimizer on $W_0^{1,p}(B_r(x_0))$. We can compute it explicitly:*

$$u(x) := C - \frac{2}{\lambda} \varphi^* \left(\frac{\lambda}{2} |x - x_0| \right)$$

with $\varphi^*(x) := \sup_{y \in \mathbb{R}} \langle x, y \rangle - \varphi(y)$, and C the constant such that $u \in W_0^{1,p}(B_r(x_0))$.

The following proposition uses the Euler-Lagrange equation (2.2) to prove that a function is a minimizer.

Proposition 2.11. *Let u be in $W^{1,p}(\Omega)$ and Φ a convex function. If there exist $\sigma \in \partial\Phi(\nabla u) \in L^p(\Omega; \mathbb{R}^2)$ and $\lambda \in L^\infty(\Omega)$ such that $\operatorname{div} \sigma = -\lambda$ then u is a minimizer of \mathcal{P}_λ on $W_u^{1,p}(\Omega)$.*

Here $\partial\Phi$ is the convex subdifferential of Φ .

Proof. Since $\sigma \in \partial\Phi(\nabla u)$, for every $w \in W_u^{1,p}(\Omega)$ we have:

$$\int_{\Omega} \Phi(\nabla w) \geq \int_{\Omega} \Phi(\nabla u) + \langle \sigma, \nabla w - \nabla u \rangle.$$

Since $\operatorname{div} \sigma = -\lambda$ we get:

$$\int_{\Omega} \langle \sigma, \nabla w - \nabla u \rangle = \int_{\Omega} \lambda(w - u).$$

Hence, $\mathcal{I}_\lambda(w) \geq \mathcal{I}_\lambda(u)$ for every $w \in W_u^{1,p}(\Omega)$. Thus, u is a minimizer of \mathcal{P}_λ on $W_u^{1,p}(\Omega)$. □

We apply this result to show that when λ is not constant, we can have more than one solution.

Proposition 2.12. *Let $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}$ be a non-strictly convex function such that $\varphi(0) < \varphi(t)$ for every $t > 0$. We assume that $\varphi \in \mathcal{C}^1(\mathbb{R}_+)$. Then, there exists $\lambda_\infty \in \mathcal{C}^\infty(\overline{B_1(0)})$, $\lambda_\infty > 0$ such that $\mathcal{P}_{\lambda_\infty}$ has an infinite number of solutions on $W_0^{1,p}(B_1)$ with $\Phi(\cdot) = \varphi(|\cdot|)$ and*

$$\mathcal{I}_{\lambda_\infty}(u) := \int_{B_1} \Phi(\nabla u(x)) - \lambda_\infty(x)u(x)dx.$$

Proof. We construct two different radial solutions u and v of the same problem. For every $x \in B_1(0)$, we set $u(x) := \tilde{u}(|x|)$ and $v(x) := \tilde{v}(|x|)$. Our goal is to define \tilde{u}' and \tilde{v}' on $(0, 1)$.

Since φ is not strictly convex on \mathbb{R}_+ , there exist $a, b \in \mathbb{R}_+$ such that φ' is constant on (a, b) and $\varphi'(t) \neq \varphi'(a)$ for every $t \notin [a, b]$.

a) We assume that the smallest possible a is strictly positive. Let us introduce a smooth increasing function $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $f(t) = t$ for every $t \geq 0$ small and $f(t) = \varphi'(a)$ for every $t \geq \frac{1}{2}$.

We use the fact that for every $x > 0$ if $x \in \partial\varphi^*(y)$ then $\varphi'(x) = y$. Hence for every $t > 0$, $\varphi'(\partial\varphi^*(f(t))) = \{f(t)\}$. For every $t \in (0, 1)$, we set $\tilde{u}'(t) = -x_t$ with $x_t \in \partial\varphi^*(f(t))$ such that \tilde{u}' is measurable. Such a choice is possible by [8, Theorem 5.3, page 151]. In order to define \tilde{v}' , we set $\tilde{v}'(t) = \tilde{u}'(t)$ on $(0, \frac{1}{2})$ and $\tilde{v}'(t) = -b$ for every $t > \frac{1}{2}$.

Hence, $\varphi'(|\tilde{v}'|) = \varphi'(|\tilde{u}'|) = f$ is a smooth function.

b) We assume that $a = 0$. We set $\tilde{u}' = 0$ on $(0, 1)$, $\tilde{v}'(t) = 0$ on $(0, \frac{1}{2})$ and $\tilde{v}'(t) = -b$ on $(\frac{1}{2}, 1)$.

Now, we can set $u(x) := \int_{|x|}^1 -\tilde{u}'(t)dt$ and $v(x) := \int_{|x|}^1 -\tilde{v}'(t)dt$ that are Lipschitz-continuous on $B_1(0)$ and vanish at the boundary. It remains to find σ and λ_∞ .

In the case **a)**, we set for every $x \in B_1(0)$, $\sigma(x) = -f(|x|)\frac{x}{|x|}$ that is smooth by assumptions on f .

In the case **b)**, we set $\sigma(x) = -\varphi'(0)\frac{x}{|x|}$ for every $x \in B_1(0)$ such that $|x| > \frac{1}{2}$. On $B_{\frac{1}{2}}(0)$, we set $\sigma(x) = -\tilde{f}(x)\frac{x}{|x|}$ with \tilde{f} smooth satisfying the same assumptions as f .

Finally, we set

$$\lambda_\infty(x) = -\operatorname{div} \sigma \in C^\infty(B_1(0)).$$

Hence, by Proposition 2.11, u and v are solutions of the same problem $\mathcal{P}_{\lambda_\infty}$. Moreover, a direct computation shows that $\lambda_\infty(x) > 0$ on $\overline{B_1(0)}$. \square

Now, we give an explicit counter-example where \mathcal{P}_λ has more than one solution with $\lambda > 0$ and $U \neq \emptyset$ with φ is as in (1.2), namely:

$$\varphi(t) = \begin{cases} \frac{1}{2}|t|^2 & \text{if } |t| \leq 1, \\ |t| - \frac{1}{2} & \text{if } 1 < |t| < 2, \\ \frac{1}{4}|t|^2 + \frac{1}{2} & \text{if } 2 \leq |t|. \end{cases}$$

Proposition 2.13. *There exists $\lambda \in C^\infty(\overline{B_1(0)})$, $\min_{B_1(0)} \lambda > 0$ such that \mathcal{P}_λ has more than one solution on $W_0^{1,2}(B_1(0))$ and $U \neq \emptyset$.*

Proof. We take the same notations as in the previous proof. In this case, we have

$$\varphi^*(t) = \begin{cases} \frac{1}{2}|t|^2 & \text{if } |t| \leq 1, \\ |t|^2 - \frac{1}{2} & \text{if } 1 < |t|. \end{cases}$$

For $t \in \mathbb{R}$, we define:

$$g(t) = \begin{cases} t & \text{if } t \leq \frac{1}{4}, \\ 3t - \frac{1}{2} & \text{if } \frac{1}{4} < t < \frac{1}{2}, \\ 1 & \text{if } \frac{1}{2} \leq t. \end{cases}$$

Hence, we can take f as the convolution of g with a smooth standard mollifier. For instance,

$$f(t) := C \int_{t-\frac{1}{8}}^{t+\frac{1}{8}} g(s) \exp\left(-\frac{1}{\frac{1}{64} - |t-s|^2}\right) ds$$

with

$$C^{-1} = \int_{-\frac{1}{8}}^{\frac{1}{8}} \exp\left(-\frac{1}{\frac{1}{64} - |s|^2}\right) ds.$$

Then $\tilde{u}' = -f$ on $[0, 1]$, $\tilde{v}' = -f$ on $[0, \frac{5}{8}]$ and $\tilde{v}' = -2$ on $(\frac{5}{8}, 1]$. Thus, $u(x) := \int_{|x|}^1 -\tilde{u}'(t) dt$ and $v(x) := \int_{|x|}^1 -\tilde{v}'(t) dt$ are two solutions on $W_0^{1,2}(B_1(0))$ with $\lambda(x) := \frac{N-1}{|x|} f(|x|) + f'(|x|)$.

Moreover, $U \neq \emptyset$ since $|\nabla u| < 1$ on $B_{\frac{1}{2}}(0)$. \square

2.4. BV functions. We start by giving the definitions of functions of bounded variations and sets of finite perimeter:

Definition 2.14. A function $f \in L^1(\Omega)$ has bounded variations in Ω if

$$\int_{\Omega} |Df| := \sup \left\{ \int_{\Omega} f \operatorname{div} g \, dx \mid g \in C_c^1(\Omega; \mathbb{R}^2), |g(x)| \leq 1, \forall x \in \Omega \right\} < \infty.$$

We denote by $BV(\Omega)$ the set of functions in $L^1(\Omega)$ having bounded variations in Ω .

If $f \in BV(\Omega)$, the distributional gradient of f is a vector valued Radon measure that we denote by Df and $|Df|$ is the total variation of Df .

Definition 2.15. Let E be a Borel set. We say that E has finite perimeter in Ω if the characteristic function χ_E of E belongs to $BV(\Omega)$. The perimeter $\operatorname{Per}(E, \Omega)$ is defined as:

$$\operatorname{Per}(E, \Omega) = \int_{\Omega} |D\chi_E| = \sup \left\{ \int_E \operatorname{div} g \mid g \in C_c^1(\Omega; \mathbb{R}^2), |g(x)| \leq 1, \forall x \in \Omega \right\}.$$

Definition 2.16. For a set E of finite perimeter in \mathbb{R}^2 , we define the reduced boundary $\partial^* E$ of E as the subset of $\operatorname{supp} |D\chi_E|$ such that for every $x \in \partial^* E$,

$$\nu_E(x) := \lim_{r \rightarrow 0} \frac{\int_{B_r(x)} D\chi_E}{\int_{B_r(x)} |D\chi_E|}$$

exists and $|\nu_E(x)| = 1$.

Remark 2.17. The reduced boundary $\partial^* E$ is a subset of ∂E .

We recall the coarea formula for Lipschitz continuous functions from [11, Theorem 3.4.2.1, page 112] that will be useful throughout the article.

Proposition 2.18. (Coarea formula) *Let u be a Lipschitz continuous function with compact support and f be a nonnegative measurable function. Then*

$$\int_{\mathbb{R}^2} f|\nabla u|dx = \int_{\mathbb{R}} \int_{L_s(u)} f(x)d\mathcal{H}^1(x)ds$$

where $L_s(u) := u^{-1}(s) \subset \mathbb{R}^2$.

Remark 2.19. By replacing f by the indicator function

$$\mathbf{1}_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A, \end{cases}$$

we observe that if $|A| = 0$ then for a.e. $s \in \mathbb{R}$, $\mathcal{H}^1(L_s(u) \cap A) = 0$.

Proposition 2.20. *Let v be a Lipschitz continuous function with compact support in \mathbb{R}^2 . For a.e. $s \in \mathbb{R}$ we have*

$$\frac{\nabla v(x)}{|\nabla v(x)|} = \frac{D\mathbf{1}_{[v>s]}(x)}{|D\mathbf{1}_{[v>s]}(x)|} \text{ for } \mathcal{H}^1 \text{ a.e. } x \in L_s(v).$$

Proof. By the vector valued coarea formula [2, Theorem 3.40] we have that

$$\int_A \nabla v = \int_{\mathbb{R}} \int_A D\mathbf{1}_{[v>s]} ds$$

for every Borel set A .

By linearity, for every linear combination of indicator functions χ , we have

$$\int_{\text{supp } v} \langle \chi, \nabla v \rangle = \int_{\mathbb{R}} \int_{\text{supp } v} \langle \chi, D\mathbf{1}_{[v>s]} \rangle ds.$$

By density, for every $g \in L^\infty(\text{supp } v; \mathbb{R}^N)$, we get

$$\int_{\mathbb{R}^2} \langle g, \nabla v \rangle = \int_{\mathbb{R}} \int_{\mathbb{R}^2} \langle g, D\mathbf{1}_{[v>s]} \rangle ds.$$

We fix $g := \frac{\nabla v}{|\nabla v|}$ when $\nabla v \neq 0$, $g := 0$ when $\nabla v = 0$ and obtain

$$(2.3) \quad \int_{\mathbb{R}^2} |\nabla v| = \int_{\mathbb{R}} \int_{\mathbb{R}^2} \left\langle \frac{\nabla v}{|\nabla v|}, D\mathbf{1}_{[v>s]} \right\rangle ds.$$

But, by [12, Theorem 4.4] we have for a.e. $s \in \mathbb{R}$ that $|D\mathbf{1}_{[v>s]}| = \mathcal{H}^1 \llcorner \partial^*[v > s]$. Hence,

$$(2.4) \quad \int_{\mathbb{R}^2} \langle g, D\mathbf{1}_{[v>s]} \rangle = \int_{\mathbb{R}^2} \left\langle g, \frac{D\mathbf{1}_{[v>s]}}{|D\mathbf{1}_{[v>s]|}} \right\rangle d|D\mathbf{1}_{[v>s]}| = \int_{\partial^*[v>s]} \left\langle g, \frac{D\mathbf{1}_{[v>s]}}{|D\mathbf{1}_{[v>s]|}} \right\rangle d\mathcal{H}^1.$$

Since $\left\langle g, \frac{D\mathbf{1}_{[v>s]}}{|D\mathbf{1}_{[v>s]|}} \right\rangle \leq 1$ for a.e. $s \in \mathbb{R}$, with (2.3) and (2.4) we get

$$\int_{\mathbb{R}^2} |\nabla v| \leq \int_{\mathbb{R}} \mathcal{H}^1(\partial^*[v > s]) ds.$$

By remark 2.17, we have

$$\int_{\mathbb{R}^2} |\nabla v| \leq \int_{\mathbb{R}} \mathcal{H}^1(\partial^*[v > s]) ds \leq \int_{\mathbb{R}} \mathcal{H}^1(L_s(v)) ds.$$

By the coarea formula given in Proposition 2.18, the following equalities hold true:

$$\int_{\mathbb{R}^2} |\nabla v| = \int_{\mathbb{R}} \mathcal{H}^1(\partial^*[v > s]) ds = \int_{\mathbb{R}} \mathcal{H}^1(L_s(v)) ds.$$

Hence, we get $\langle g, \frac{D\mathbf{1}_{[v>s]}}{|D\mathbf{1}_{[v>s]|}} \rangle = 1$ for \mathcal{H}^1 a.e. $x \in \partial^*[v > s]$ and $\mathcal{H}^1(L_s(v) \setminus \partial^*[v > s]) = 0$ for a.e. $s \in \mathbb{R}$. Thus, $g = \frac{D\mathbf{1}_{[v>s]}}{|D\mathbf{1}_{[v>s]|}} \mathcal{H}^1$ a.e. on $L_s(v)$ for a.e. $s \in \mathbb{R}$, as desired. \square

3. RELATION BETWEEN THE LEVEL LINES AND U .

In this section we use the Lipschitz regularity of the level lines of a minimizer u to prove that they are, in a generic sense, level sets for the other minimizers. We then study the case when a level line intersects the set U , which implies that the gradient of another solution is equal to ∇u on that particular level line.

3.1. Equality on level lines. We first prove that the difference between two minimizers is constant on every connected component of almost every level sets.

Proposition 3.1. *Let u and v be two minimizers of the same problem \mathcal{P}_λ . There exists a negligible subset N_0 of \mathbb{R} such that for every $s \in S_0 := \mathbb{R} \setminus N_0$, for every connected component $l_s(u)$ of $L_s(u)$, the map $u - v$ is constant on $l_s(u)$.*

Proof. We consider that u and v are extended by ψ outside of Ω . By Proposition 2.9 there is a negligible set N_∞ such that for every $s \in \mathbb{R} \setminus N_\infty$, every connected component $l_s(u)$ of $L_s(u) \subset \mathbb{R}^2$ that is not a point has a parametrization that is Lipschitz continuous.

Since ∇u and ∇v are defined and collinear a.e. on \mathbb{R}^2 , by the coarea formula we obtain that there exists a negligible set N'_∞ such that for every $s \in \mathbb{R} \setminus N'_\infty$, ∇u and ∇v are defined and collinear \mathcal{H}^1 a.e. on $L_s(u)$. We set $N_0 := N_\infty \cup N'_\infty$.

Hence, for every $s \in \mathbb{R} \setminus N_0$ we have that ∇v is orthogonal to each Lipschitz connected curve $l_s(u)$. We introduce $\gamma_s : [0, \text{length}(l_s(u))] \mapsto l_s(u)$ a Lipschitz-continuous parametrization of $l_s(u)$. Then, by the chain rule we have $(v \circ \gamma_s)' = \langle \nabla v(\gamma_s), \gamma_s' \rangle$ a.e. on $[0, \text{length}(l_s(u))]$. By orthogonality of ∇v to $l_s(u)$, we have that v is constant on $l_s(u)$. \square

The following proposition is the first step to prove the uniqueness result.

Proposition 3.2. *For $s \in S_0$, if $l_s(u) \cap (\mathbb{R}^2 \setminus \Omega) \neq \emptyset$ then $u = v$ on $l_s(u)$.*

Proof. Thanks to the previous proposition we know that $u - v$ is constant on $l_s(u)$. Since u and v are extended by ψ outside Ω , we have $u \equiv v$ on $\mathbb{R}^2 \setminus \Omega$. By assumption, we have $l_s(u) \cap \mathbb{R}^2 \setminus \Omega \neq \emptyset$ then $u = v$ on $l_s(u)$. \square

3.2. Relation between U and the level lines. In this section we consider two minimizers u and v of I_λ with the same boundary condition. We know by Proposition 2.4 that $\nabla u = \nabla v$ on U . We will extend this result to the level lines that intersect U .

Notation. We denote by $S \subset \mathbb{R}$ the set of these s that satisfy the following conditions:

- $s \in S_0$ with S_0 defined in Proposition 3.1,
- $\nabla u \neq 0$ \mathcal{H}^1 a.e. on $L_s(u)$.

We introduce the following set:

$$\Gamma := \{l_s^i(u), s \in S \text{ and } i \in I_s\}$$

where the index set I_s corresponds to the non-constant curves $l_s^i(u)$ among the connected components of $L_s(u)$ inside Ω which do not intersect $\partial\Omega$.

Proposition 3.3. *Given $s \in S$ and $i \in I_s$, let F_s^i be the bounded connected component of $\mathbb{R}^2 \setminus l_s^i(u)$ given by the Jordan curve theorem. Then for every $i \in I_s$, $u > s$ on F_s^i .*

Proof. We fix $s \in S$ and we call E_s the set $[u > s] \subset \mathbb{R}^2$. By continuity of u , E_s is an open set. Let $l_s^i(u)$ a connected component of $L_s(u)$ such that $l_s^i(u) \Subset \Omega$. Since Ω is simply connected, $F_s^i \subset \Omega$ and by Proposition 2.1, we have that $u \geq s$ on F_s^i . If $l'_s(u)$ is a connected component of $L_s^*(u)$ that is inside F_s^i then for \mathcal{H}^1 a.e. $y \in l'_s(u)$, $\nabla u(y)$ is defined and $\nabla u(y) \neq 0$. Since $u \geq s$ on F_s^i , every point y in $l'_s(u)$ is a local minimum on F_s^i and hence either $\nabla u(y) = 0$ or $\nabla u(y)$ is not defined. Thus there is no such $l'_s(u)$ in F_s^i . By assumptions on s , we have $\mathcal{H}^1(L_s \setminus L_s^*) = 0$. Hence, $\mathcal{H}^1([u = s] \cap F_s^i) = 0$.

Let us assume that there exists $x \in F_s^i$ such that $u(x) = s$. Then by Proposition 2.1 for every $\epsilon < \text{dist}(x, l_s^i(u))$ there exists $y_\epsilon \in \partial B_\epsilon(x)$ such that $u(y_\epsilon) = s$ and we define the following set $Y := \{y_\epsilon, 0 < \epsilon < \text{dist}(x, l_s^i(u))\}$. We have

$$\mathcal{H}^1(Y) := \lim_{\delta \rightarrow 0} \mathcal{H}_\delta^1(Y)$$

with

$$\mathcal{H}_\delta^1(Y) := \inf \left\{ \sum_{n \in \mathbb{N}} \text{diam}(V_n) \right\}$$

where the infimum is taken over the families of sets $(V_n)_{n \in \mathbb{N}}$ such that $Y \subset \bigcup_n V_n$ and $\text{diam}(V_n) < \delta$ for every $n \in \mathbb{N}$.

For every admissible $(V_n)_{n \in \mathbb{N}}$, we define E_n as the set of those ϵ such that $y_\epsilon \in V_n \cap Y$. We define $e_m := \inf\{\epsilon \in E_n\}$, $e^M := \sup\{\epsilon \in E_n\}$ and $\tilde{V}_n := [e_m, e^M]$. We have $\text{diam}(\tilde{V}_n) = e^M - e_m \leq \text{diam}(V_n) < \delta$ and $]0, \text{dist}(x, l_s^i(u))[\subset \bigcup \tilde{V}_n$.

Hence, $(\tilde{V}_n)_{n \in \mathbb{N}}$ is admissible for $\mathcal{H}_\delta^1((0, \text{dist}(x, l_s^i(u))))$ and

$$\mathcal{H}_\delta^1(Y) \geq \mathcal{H}_\delta^1((0, \text{dist}(x, l_s^i(u)))).$$

By taking the limit when δ goes to 0, we obtain:

$$\mathcal{H}^1(Y) \geq \mathcal{H}^1((0, \text{dist}(x, l_s^i(u))) = \text{dist}(x, l_s^i(u)).$$

Thus $\mathcal{H}^1([u = s] \cap F_s^i) \geq \text{dist}(x, l_s^i(u)) > 0$. That is a contradiction. Hence there is no such x , thus $u > s$ on F_s^i . \square

A direct consequence of that result is the following:

Proposition 3.4. *For every $s \in S$, I_s is countable. Moreover, for every $i \in I_s$, $l_s^i(u)$ is the boundary of a connected component F_s^i of $E_s = [u > s]$.*

Proof. By Theorem 2.8, I_s is countable. Let us consider $i \in I_s$. By the previous proposition, $l_s^i(u)$ is the boundary of a connected component of $E_s = [u > s]$. \square

We also have that:

Proposition 3.5. *For every $s \in S$, every connected component F_s of E_s , if $F_s \Subset \Omega$ then F_s is simply connected and its boundary is a closed simple curve $l_s(u)$ with Lipschitz parametrization.*

Proof. Since F_s is bounded, $\mathbb{R}^2 \setminus F_s$ has only one unbounded connected component. We call \tilde{F}_s the complement of this unbounded set. We claim that $\tilde{F}_s = F_s$. We have that $\partial\tilde{F}_s \subset \partial F_s$. Hence, $u \equiv s$ on $\partial\tilde{F}_s$. Since \tilde{F}_s is simply connected, $\partial\tilde{F}_s$ is a connected set in $L_s(u)$ with $\mathcal{H}^1(\partial\tilde{F}_s) > 0$. By Theorem 2.8, $\partial\tilde{F}_s$ is a closed subset of a closed simple curve with Lipschitz parametrization $l_s(u)$. Hence, \tilde{F}_s is a bounded set such that $\partial\tilde{F}_s \subset l_s(u)$. We have that \tilde{F}_s is an open set in $\mathbb{R}^2 \setminus l_s(u)$. Since, $\partial\tilde{F}_s \subset l_s(u)$, \tilde{F}_s is also closed in $\mathbb{R}^2 \setminus l_s(u)$. The fact that \tilde{F}_s is bounded and connected gives that \tilde{F}_s is the bounded connected component of $\mathbb{R}^2 \setminus l_s(u)$ and by the Jordan curve theorem we have $\partial\tilde{F}_s = l_s(u)$. By Proposition 3.3, $\tilde{F}_s \subset E_s$. Since \tilde{F}_s contains F_s , we get that $\tilde{F}_s = F_s$. Moreover, we proved that F_s is simply connected with $l_s(u)$ as boundary. \square

The main result of this subsection is the following:

Proposition 3.6. *For a.e. $s \in S$, for every $i \in I_s$, if $l_s^i(u) \cap U \neq \emptyset$ then $\nabla(u - v) = 0$ \mathcal{H}^1 a.e. on $l_s^i(u)$.*

In order to prove this result, we state two technical lemmata:

Lemma 3.7. *For a.e. $s \in S$, for every $i \in I_s$ there exists a decreasing sequence $(s_n)_{n \in \mathbb{N}}$ converging to s such that:*

- *There exists a simple connected curve $l_{s_n}(u)$ in L_{s_n} with Lipschitz parametrization that is inside F_s^i .*
- *$(F_{s_n})_{n \in \mathbb{N}}$ is an increasing sequence with $\bigcup_{n \in \mathbb{N}} F_{s_n} = F_s^i$.*

Here, F_t is the bounded connected component of $\mathbb{R}^2 \setminus l_t(u)$ given by the Jordan curve theorem and F_s^i is the bounded connected component of $\mathbb{R}^2 \setminus l_s^i(u)$.

Proof of Lemma 3.7. By Proposition 3.3 we have that $u > s$ on F_s^i . By the coarea formula 2.18 there exists $s_0 > s$, $s_0 \in S$ such that $\mathcal{H}^1(L_{s_0}(u) \cap F_s^i) > 0$. Moreover, by Theorem 2.8, $\mathcal{H}^1(L_{s_0}(u) \setminus L_{s_0}^*(u)) = 0$. Hence, there exists $l_{s_0}(u)$ in L_{s_0} satisfying the assumptions of Lemma 3.7. We next select $s < s_1 < s_0$ with $s_1 \in S$ such that $\mathcal{H}^1(L_{s_1}(u) \cap F_s^i) > 0$. We have that $F_{s_0} \subset E_{s_1}$. Hence, F_{s_0} is in one connected component of E_{s_1} , we call $l_{s_1}(u)$ the boundary of that connected component. By Proposition 3.5 we have that $l_{s_1}(u)$ is a simple connected curve with Lipschitz parametrization. We repeat this argument to find a sequence $(s_n)_{n \in \mathbb{N}}$ that satisfies the first part of the lemma. By construction $(F_{s_n})_{n \in \mathbb{N}}$ is an increasing sequence, it remains to prove that $\bigcup_{n \in \mathbb{N}} F_{s_n} = F_s^i$.

We introduce $F_\infty := \bigcup_{n \in \mathbb{N}} F_{s_n}$ a subset of F_s^i . If $y \in \partial F_\infty$ there exists a sequence $(y_n)_{n \in \mathbb{N}}$ such that $y_n \in \partial F_{s_n}$ and $y_n \rightarrow y$. By continuity of u and the fact that y_n converges to y we obtain that $u(y) = s$.

By Proposition 3.3 we have that $\partial F_\infty \subset \partial F_s^i$. We claim that $F_\infty = F_s^i$. Indeed, if those two sets are not equal there exists $x \in F_s^i \setminus F_\infty$. Since F_s^i is a connected open set, for every $y \in F_\infty$ there exists a continuous path from x to y included in F_s^i . By continuity this path must intersect $\partial F_\infty \subset \partial F_s^i$, contradicting the fact that the path is in F_s^i . Hence, $F_\infty = F_s^i$. \square

Lemma 3.8. *Let u be a minimizer. For a.e. $s \in S$ and every $i \in I_s$, we consider a sequence $(l_{s_n})_{n \in \mathbb{N}}$ as in the previous lemma. Then, we have that $\lim_{n \rightarrow +\infty} D(l_{s_n}, l_s^i(u)) = 0$ where $D(E, F) := \sup_{e \in E} \inf_{f \in F} |e - f|$. Moreover, let v be another minimizer, if there exists a constant C such that $u - v \equiv C$ on every l_{s_n} then $\nabla(u - v) = 0$ \mathcal{H}^1 a.e on $l_s^i(u)$.*

Proof of Lemma 3.8. For every $\epsilon > 0$, we claim that there exists $N \in \mathbb{N}$ such that for every $n \geq N$, $D(l_{s_n}, l_s^i(u)) < \epsilon$. Indeed, assume by contradiction that there exists $\epsilon > 0$ such that $\forall N$, there exist $n \geq N$ and $y_n \in l_{s_n}(u)$ such that $\forall x \in l_s^i(u)$, $d(y_n, x) \geq \epsilon$. Since all these y_n are in Ω , there exists a sequence $(y_{g(n)})_{n \in \mathbb{N}}$ converging towards some $y \in F_s^i$. We have that $d(y, x) \geq \epsilon$ for every $x \in l_s^i(u)$. By continuity of u , we have that $u(y) = s$. Thus, by Proposition 3.3, $y \in \partial F_s^i = l_s^i(u)$. That is a contradiction.

For \mathcal{H}^1 a.e. $x \in l_s^i(u)$ we have that $\nabla u(x) \neq 0$ and $\nabla v(x)$ exist. Moreover, for \mathcal{H}^1 a.e. $x \in l_s^i(u)$ we have that $\nabla u(x)$ and $\nabla v(x)$ are orthogonal to $l_s^i(u)$ at x in the sense that $\langle \nabla u(x), \gamma_s'(\gamma_s^{-1}(x)) \rangle = 0$ where γ_s is a Lipschitz parametrization of $l_s^i(u)$. We consider $d_x := x + \mathbb{R}\nabla u(x)$. Let us call H_- and H_+ the two half-planes of $\mathbb{R}^2 \setminus d_x$. For every $r > 0$, $H_\pm \cap l_s^i(u) \cap B_r(x) \neq \emptyset$, otherwise it would contradict the fact that $\nabla u(x)$ is orthogonal to $l_s^i(u)$. A direct consequence is the fact that $H_- \cap F_s^i \neq \emptyset$ and $H_+ \cap F_s^i \neq \emptyset$.

We assume that for every $N \in \mathbb{N}$, there exists $\tilde{n} \geq N$ such that $l_{s_{\tilde{n}}}(u) \cap d_x = \emptyset$. Thus, we can assume that $l_{s_{\tilde{n}}}(u) \cap H_- = \emptyset$. Hence, there exist $y \in l_s^i(u)$ and $\epsilon_0 > 0$ such that $d(y, F_{s_{\tilde{n}}}^i) > \epsilon_0$. Since $l_{s_{\tilde{n}}}(u) \rightarrow l_s^i(u)$ in the sense of D that is absurd. Then, there exists $N \in \mathbb{N}$ such that for every $n \geq N$, $l_{s_n}(u) \cap d_x \neq \emptyset$. For every such n we take x_n as a point that minimizes $d(x, y)$ on $l_{s_n}(u) \cap d_x$. Since this sequence $(x_n)_{n \in \mathbb{N}}$ is bounded, it converges, up to a subsequence, to a point $x' \in \overline{F_s^i}$ such that $u(x') = s$. By Proposition 3.3, $x' \in l_s^i(u)$. If $x' \neq x$ then there exists $r > 0$ such that $d_x \cap B_r(x) \cap l_{s_n}(u) = \emptyset$ for every $n \in \mathbb{N}$ large enough. Hence, $d_x \cap B_r(x) \cap F_{s_n}^i = \emptyset$ for every $n \in \mathbb{N}$ large enough which contradicts the fact that $\bigcup_{n \in \mathbb{N}} F_{s_n}^i = F_s^i$. Thus, $x = x'$ and we can find a sequence $(x_n)_{n \in \mathbb{N}}$ such that $x_n \in l_{s_n}(u) \cap d_x$ and $x_n \rightarrow x$. By assumption, $u - v \equiv C$ on $l_{s_n}(u)$ for n large enough. By continuity of $u - v$ we obtain that $u - v(x) = C$. Then, $\frac{(u-v)(x) - (u-v)(x_n)}{|x - x_n|} = \frac{C - C}{|x - x_n|} = 0$. Moreover, $\nabla(u - v)(x)$ is collinear to $\nabla u(x)$, hence, we obtain $\nabla(u - v)(x) = 0$. Since that is the case for \mathcal{H}^1 a.e. $x \in l_s^i(u)$, we have the desired conclusion. \square

Proof of Proposition 3.6. For every $s \in S$, $l_s^i(u)$ is a Lipschitz continuous closed curve such that ∇u and ∇v are defined and collinear \mathcal{H}^1 a.e. on $l_s^i(u)$ and $\nabla u \neq 0$ \mathcal{H}^1 a.e. on $l_s^i(u)$. If $l_s^i(u) \cap U \neq \emptyset$ then there exists U_i a connected component of U such that $l_s^i(u) \cap U_i \neq \emptyset$. By Proposition 2.5 and Proposition 3.1 we have $u - v \equiv C_i$ on $l_s^i(u)$. We consider the sequences $(s_n)_{n \in \mathbb{N}}$, $(l_{s_n}(u))_{n \in \mathbb{N}}$ and $(F_{s_n}^i)_{n \in \mathbb{N}}$ from Lemma 3.7.

Hence, by the first part of Lemma 3.8, there exists $N \in \mathbb{N}$ such that for every $n \geq N$, $l_{s_n}(u) \cap U_i \neq \emptyset$. Thus, by Proposition 2.5 and Proposition 3.1, $u - v \equiv C_i$ on $l_{s_n}(u)$ for every $n \geq N$. By the second part of Lemma 3.8, we have that $\nabla(u - v) = 0$ \mathcal{H}^1 a.e on $l_s^i(u)$. \square

4. $W^{1,2}$ REGULARITY OF $|\sigma|$.

In this section, we prove the following proposition:

Proposition 4.1. *For every $\alpha > 0$, the function $f := \max(\alpha, |\sigma|)$ is in $W^{1,2}(\Omega')$ for any $\Omega' \Subset \Omega$.*

We prove this result in four parts. In **Step 1**, we regularize our problem in order to work with smooth solutions $(u_n)_{n \in \mathbb{N}}$. Then in **Step 2**, we prove that $\|\max(\alpha, |\nabla \Phi_n(\nabla u_n)|)\|_{W^{1,2}(\Omega')}$ is uniformly bounded in $n \in \mathbb{N}$. In the subsequent **Step 3**, we show that $\max(\alpha, |\nabla \Phi_n(\nabla u_n)|) \rightarrow f$ a.e. on Ω . We conclude that f is in $W^{1,2}(\Omega')$ in **Step 4**.

Step 1 For every $n \in \mathbb{N}$, we introduce $(\rho_n)_{n \in \mathbb{N}}$ a standard mollifying sequence with $\text{supp } \rho_n \subset B_{\frac{1}{n}}(0)$. If we set $\varphi_n := \varphi * \rho_n$ and $\lambda_n := \lambda * \rho_n$ then $(\varphi_n)_{n \in \mathbb{N}}$ and $(\lambda_n)_{n \in \mathbb{N}}$ are sequences of smooth approximations of φ and λ . We consider $\Phi_n := \varphi_n(|\cdot|) + \frac{1}{n}g(|\cdot|)$. The function g is smooth quadratic around the origin such that $0 < g''(x)$ for every $x \in \mathbb{R}$ and

$$C_-|x|^p \leq g(z) \leq C_+(|x|^p + 1)$$

for all $|x| \geq 1$ with $0 < C_- < C_+$.

Let u_n be the minimizer of:

$$\mathcal{I}_n : v \rightarrow \int_{\Omega} \Phi_n(\nabla v(x)) - \lambda_n v(x) dx$$

on $W_{\psi}^{1,p}(\Omega)$.

Proposition 4.2. *The sequence $(u_n)_{n \in \mathbb{N}}$ is uniformly bounded in $W^{1,p}(\Omega)$. There exists a subsequence still denoted by $(u_n)_{n \in \mathbb{N}}$ that weakly converges in $W^{1,p}(\Omega)$ towards \tilde{u} . Moreover, \tilde{u} is a minimizer of \mathcal{P}_{λ} on $W_{\psi}^{1,p}(\Omega)$.*

Proof. By Proposition 2.7 we have that the sequence $(u_n)_{n \in \mathbb{N}}$ is uniformly bounded in $W^{1,\infty}(\Omega)$. Hence, we can extract a subsequence, still denoted by $(u_n)_{n \in \mathbb{N}}$, that converges strongly in $L^p(\Omega)$ and weakly in $W_{\psi}^{1,p}(\Omega)$ towards \tilde{u} . It remains to prove that \tilde{u} is a minimizer of \mathcal{P}_{λ} on $W_{\psi}^{1,p}(\Omega)$.

By Jensen's inequality, we have $\Phi_n \geq \Phi$. Hence

$$(4.1) \quad \liminf_{n \rightarrow +\infty} \int_{\Omega} \Phi_n(\nabla u_n) \geq \liminf_{n \rightarrow +\infty} \int_{\Omega} \Phi(\nabla u_n).$$

By weak lower semi-continuity of I_{λ} , (4.1) and the fact that u_n is the minimizer for \mathcal{I}_n we have

$$(4.2) \quad \int_{\Omega} \Phi(\nabla \tilde{u}) - \lambda \tilde{u} \leq \liminf_{n \rightarrow +\infty} \int_{\Omega} \Phi(\nabla u_n) - \lambda u_n \leq \liminf_{n \rightarrow +\infty} \int_{\Omega} \Phi_n(\nabla u_n) - \lambda_n u_n \leq \lim_{n \rightarrow +\infty} \int_{\Omega} \Phi_n(\nabla u) - \lambda_n u.$$

By the dominated convergence theorem applied to the last quantity we obtain

$$\int_{\Omega} \Phi(\nabla \tilde{u}) - \lambda \tilde{u} \leq \int_{\Omega} \Phi(\nabla u) - \lambda u.$$

Hence, \tilde{u} is a minimizer on $W_{\psi}^{1,p}(\Omega)$. □

Step 2 For every $n \in \mathbb{N}$, we introduce $\sigma_n := \nabla \Phi_n(\nabla u_n)$. In this part, we prove the following result on $f_n := \max(\alpha, |\sigma_n|)$:

Proposition 4.3. *For every $\alpha > 0$ and every $\Omega' \Subset \Omega$, the functions $f_n := \max(\alpha, |\sigma_n|)$ are uniformly bounded in $W^{1,2}(\Omega')$.*

Proof. By [10, Proposition 2.4], we have for every $b > 0$ and $k \in \{1, 2\}$ that:

$$\int_{\Omega' \cap \{\partial_k u_n \geq b\}} |\nabla \sigma_n|^2 \leq C_1 \left(b, \|\nabla u_n\|_{\mathcal{L}^{\infty}(\Omega)}, \sup_{b \leq t \leq \|\nabla u_n\|_{\mathcal{L}^{\infty}(\Omega)}} \varphi_n''(t) + \frac{g''(t)}{n} \right).$$

Thus,

$$\int_{\Omega' \cap \{|\nabla u_n| \geq b\}} |\nabla \sigma_n|^2 \leq C_2 \left(b, \|\nabla u_n\|_{\mathcal{L}^\infty(\Omega)}, \sup_{b \leq t \leq \|\nabla u_n\|_{\mathcal{L}^\infty(\Omega)}} \varphi_n''(t) + \frac{g''(t)}{n} \right).$$

Finally,

$$\int_{\Omega' \cap \{|\sigma_n| \geq \varphi'_n(b)\}} |\nabla \sigma_n|^2 \leq C_2 \left(b, \|\nabla u_n\|_{\mathcal{L}^\infty(\Omega)}, \sup_{b \leq t \leq \|\nabla u_n\|_{\mathcal{L}^\infty(\Omega)}} \varphi_n''(t) + \frac{g''(t)}{n} \right).$$

By Proposition 2.7 we have that $\|\nabla u_n\|_{\mathcal{L}^\infty(\Omega)}$ can be bounded by L uniformly in $n \in \mathbb{N}$. Moreover, $\varphi \in \mathcal{C}_{loc}^{1,1}(\mathbb{R} \setminus \{0\})$ and φ_n is a convolution of φ . Hence, $\sup_{b \leq t \leq \|\nabla u_n\|_{\mathcal{L}^\infty(\Omega)}} \varphi_n''(t) + \frac{g''(t)}{n}$ can be bounded by $\sup_{\frac{b}{2} \leq t \leq L} \varphi''(t) + 1$ for every $n \in \mathbb{N}$ such that φ_n'' is close enough to φ'' on $(\frac{b}{2}, +\infty)$ and larger than $\sup_{b \leq t \leq L} g''(t)$. Namely, every $n \in \mathbb{N}$ larger than $\max\{\frac{1}{2b}, \sup_{b \leq t \leq L} g''(t)\}$. Thus, we get

$$\int_{\Omega' \cap \{|\sigma_n| \geq \varphi'_n(b)\}} |\nabla \sigma_n|^2 \leq C_2(b, L, \sup_{\frac{b}{2} \leq t \leq L} \varphi''(t) + 1)$$

for every $n \in \mathbb{N}$ larger than $\max\{\frac{1}{2b}, \sup_{b \leq t \leq L} g''(t)\}$.

By growing assumptions on φ , for every $\alpha > 0$ we can find $b > 0$ such that $\varphi'_n(b) \leq \alpha$ for $n \in \mathbb{N}$ large enough. Hence, for every $\alpha > 0$ and every $n \in \mathbb{N}$ large enough we have:

$$\int_{\Omega' \cap \{|\sigma_n| > \alpha\}} |\nabla \sigma_n|^2 \leq C(\alpha, \varphi'', L).$$

Thus, the sequence $(f_n)_{n \in \mathbb{N}}$ is uniformly bounded in $W^{1,2}(\Omega')$. \square

Since the functions $f_n := \max(\alpha, |\sigma_n|)$ are uniformly bounded in $W^{1,2}(\Omega')$, we can extract a subsequence which converges weakly.

Step 3 We prove that $\sigma_n \rightarrow \sigma$ a.e. Ω up to a subsequence. To do so we use the Young measures associated to $(\nabla u_n)_{n \in \mathbb{N}}$.

Proposition 4.4. *We have the following equality*

$$(4.3) \quad \liminf_{n \rightarrow +\infty} \int_{\Omega} \Phi(\nabla u_n) = \int_{\Omega} \Phi(\nabla \tilde{u}).$$

Proof. If we replace u by \tilde{u} in the last term of (4.2) we obtain that

$$(4.4) \quad \int_{\Omega} \Phi(\nabla \tilde{u}) - \lambda \tilde{u} \leq \liminf_{n \rightarrow +\infty} \int_{\Omega} \Phi(\nabla u_n) - \lambda u_n \leq \liminf_{n \rightarrow +\infty} \int_{\Omega} \Phi_n(\nabla u_n) - \lambda_n u_n \leq \liminf_{n \rightarrow +\infty} \int_{\Omega} \Phi_n(\nabla \tilde{u}) - \lambda_n \tilde{u}.$$

By Fatou's lemma with the limsup the last term is equal to the first term. Hence all those inequalities are equalities, in particular:

$$\liminf_{n \rightarrow +\infty} \int_{\Omega} \Phi(\nabla u_n) - \lambda u_n = \int_{\Omega} \Phi(\nabla \tilde{u}) - \lambda \tilde{u}.$$

Since $u_n \rightarrow \tilde{u}$ in $L^p(\Omega)$ we have that

$$\lim_{n \rightarrow +\infty} \int_{\Omega} \lambda u_n = \int_{\Omega} \lambda \tilde{u}.$$

Hence,

$$\liminf_{n \rightarrow +\infty} \int_{\Omega} \Phi(\nabla u_n) = \int_{\Omega} \Phi(\nabla \tilde{u}).$$

□

We consider $(u_{\psi(n)})_{n \in \mathbb{N}}$ a subsequence such that

$$(4.5) \quad \liminf_{n \rightarrow +\infty} \int_{\Omega} \Phi(\nabla u_n) = \lim_{n \rightarrow +\infty} \int_{\Omega} \Phi(\nabla u_{\psi(n)}).$$

In order to simplify the notations, we still denote $(u_{\psi(n)})_{n \in \mathbb{N}}$ by $(u_n)_{n \in \mathbb{N}}$.

Proposition 4.5. *For a.e. $x \in \Omega$ we have*

$$\Phi(\nabla \tilde{u}(x)) = \bar{\Phi}(x) := \int_{\mathbb{R}^2} \Phi(y) d\nu_x(y)$$

where ν_x is a probability measure that depends on x and on the weak convergence of $(\nabla u_n)_{n \in \mathbb{N}}$ towards $\nabla \tilde{u}$. Moreover, $\text{supp } \nu_x \subset \{y \in \mathbb{R}^2, \nabla \Phi(y) = \nabla \Phi(\nabla \tilde{u}(x))\}$ for a.e. $x \in \Omega$.

Proof. Let $(\nu_x)_{x \in \Omega}$ be the Young measures associated to a subsequence of $(\nabla u_n)_{n \in \mathbb{N}}$ given by [4, Theorem 2]. We have for every Carathéodory function F such that $\{F(\cdot, \nabla u_n(\cdot))\}_{n \in \mathbb{N}}$ is uniformly integrable:

$$(4.6) \quad \lim_{n \rightarrow +\infty} \int_{\Omega} F(x, \nabla u_n(x)) dx = \int_{\Omega} \bar{F}(x) dx$$

with $\bar{F}(x) = \int_{\mathbb{R}^2} F(x, y) d\nu_x(y)$. Moreover, for a.e. $x \in \Omega$,

$$(4.7) \quad \nabla \tilde{u}(x) = \int_{\mathbb{R}^2} y d\nu_x(y).$$

Since u_n is uniformly bounded in $W^{1, \infty}(\Omega)$,

$$(4.8) \quad \lim_{n \rightarrow +\infty} \int_{\Omega} \Phi(\nabla u_n(x)) dx = \int_{\Omega} \bar{\Phi}(x) dx \text{ where } \bar{\Phi}(x) = \int_{\mathbb{R}^2} \Phi(y) d\nu_x(y).$$

If we combine this last equation with (4.5) we get

$$\int_{\Omega} \Phi(\nabla \tilde{u}) = \lim_{n \rightarrow +\infty} \int_{\Omega} \Phi(\nabla u_n(x)) dx = \int_{\Omega} \bar{\Phi}(x) dx.$$

If we apply the triangle inequality and Jensen's inequality to (4.7) we obtain for a.e. $x \in \Omega$,

$$(4.9) \quad \Phi(\nabla \tilde{u}(x)) \leq \int_{\mathbb{R}^2} \Phi(y) d\nu_x(y) dx = \bar{\Phi}(x).$$

If we combine the two last equations we obtain for a.e. $x \in \Omega$

$$(4.10) \quad \Phi(\nabla \tilde{u}(x)) = \bar{\Phi}(x).$$

By Jensen's inequality, Φ is affine on $\text{supp } \nu_x$ and thus, for a.e. $x \in \Omega$ we have that $\text{supp } \nu_x \subset \{y \in \mathbb{R}^2, \nabla \Phi(y) = \nabla \Phi(\nabla \tilde{u}(x))\}$.

□

Now, we can prove the following convergence result:

Proposition 4.6. *We have that $\sigma_n \rightarrow \sigma$ in $L^1(\Omega)$ when $n \rightarrow +\infty$. Here, $\sigma_n = \nabla \Phi_n(\nabla u_n)$.*

Proof. If we set $F(x, y) = |\nabla\Phi(y) - \sigma(x)|$ in (4.6), we obtain that

$$\lim_{n \rightarrow +\infty} \int_{\Omega} |\nabla\Phi(\nabla u_n(x)) - \sigma(x)| dx = \int_{\Omega} \int_{\mathbb{R}^2} |\nabla\Phi(y) - \sigma(x)| d\nu_x(y) dx = 0.$$

Since ∇u_n is uniformly bounded in $L^\infty(\Omega)$, we have that

$$\lim_{n \rightarrow +\infty} \int_{\Omega} |\nabla\Phi_n(\nabla u_n) - \nabla\Phi(\nabla u_n)| = 0.$$

Hence, by the triangle inequality,

$$\lim_{n \rightarrow +\infty} \int_{\Omega} |\sigma_n(x) - \sigma(x)| dx = 0.$$

Hence, $\sigma_n \rightarrow \sigma$ in $L^1(\Omega)$ when $n \rightarrow +\infty$. \square

Thanks to the previous Proposition, we can extract a subsequence, we do not relabel, such that $\sigma_n \rightarrow \sigma$ a.e. on Ω when $n \rightarrow +\infty$.

Step 4 Since $\sigma_n \rightarrow \sigma$ a.e. on Ω when $n \rightarrow +\infty$, we have that $f_n \rightarrow \max(\alpha, |\sigma|)$ a.e. on Ω . By Proposition 4.3, we have that $\max(\alpha, |\sigma|) \in W^{1,2}(\Omega')$.

5. CONTINUITY OF $|\sigma|$ ON THE LEVEL LINES AND A MAXIMUM PRINCIPLE

In this section, we prove that generically, $\max(d_0, |\sigma|)$ is continuous on the level lines of u and satisfies a maximum principle.

For $\Omega' \Subset \Omega$, we introduce

$$\Gamma' := \{l_s^i(u), s \in S \text{ and } i \in I'_s\}$$

where $S \subset \mathbb{R}$ is the set of those s that satisfy the conclusion of Theorem 2.8 and such that ∇u , ∇v are defined, $\nabla u \neq 0$, ∇u and ∇v are collinear \mathcal{H}^1 a.e. on $L_s(u)$. The index set I'_s corresponds to the non-constant curves $l_s^i(u)$ among the connected components of $L_s(u)$ such that $l_s^i(u) \Subset \Omega'$.

Proposition 5.1. *There exists a representative f_0 of $\max(d_0, |\sigma|)$ that is absolutely continuous on $l_s^i(u)$ for a.e. $s \in S$ and every $i \in I_s$.*

Proof. We consider the sequence $(\sigma_n)_{n \in \mathbb{N}}$ introduced in the previous section. We have that $\sigma_n \rightarrow \sigma$ a.e. on Ω when $n \rightarrow +\infty$. By Proposition 4.1, we have $\|\nabla \max(d_0, |\sigma_n|)\|_{L^2(\Omega_1)} \leq C_1$ with $\Omega_1 \Subset \Omega$ and C_1 independent of $n \in \mathbb{N}$. Thus, there exists a constant C_2 independent of n such that

$$C_2 \geq \int_{\Omega_1} |\nabla \max(d_0, |\sigma_n|)|^2 |\nabla u| = \int_{\mathbb{R}} \int_{L_s(u) \cap \Omega_1} |\nabla \max(d_0, |\sigma_n|)|^2 d\mathcal{H}^1 ds$$

where the equality is given by Proposition 2.18. With Fatou's lemma we obtain that for a.e. $s \in S$,

$$(5.1) \quad \liminf_{n \rightarrow +\infty} \int_{L_s(u) \cap \Omega_1} |\nabla \max(d_0, |\sigma_n|)|^2 d\mathcal{H}^1 \leq C_3(s).$$

Here, the index I'_s corresponds to the non-constant curves $l_s^i(u)$ among the connected components of $L_s(u)$ such that $l_s^i(u) \Subset \Omega_1$. We define S_1 as the subset of S such that (5.1) holds. We have $|S \setminus S_1| = 0$.

Now, we fix $s \in S_1$ and $i \in I'_s$. We can extract a subsequence such that for every $n \in \mathbb{N}$:

$$\int_{l_s^i(u)} |\nabla \max(d_0, |\sigma_n|)|^2 d\mathcal{H}^1 \leq 2C_3(s).$$

Let us call $\gamma_s^i : [0, \text{length}(l_s^i(u))] \mapsto l_s^i(u)$ a Lipschitz continuous parametrization of $l_s^i(u)$. We have that $\max(d_0, |\sigma_n|) \circ \gamma_s^i$ is bounded in $W^{1,2}([0, \text{length}(l_s^i(u))])$. By the Arzelà-Ascoli theorem there exists a subsequence of $\max(d_0, |\sigma_n|) \circ \gamma_s^i$ converging uniformly to $v \in C^0([0, \text{length}(l_s^i(u))])$.

Since $\max(d_0, |\sigma_n|) \rightarrow \max(d_0, |\sigma|)$ \mathcal{H}^1 a.e. on $l_t^j(u)$ for a.e. $t \in \mathbb{R}$ and every $j \in I_t$, we choose f_0 as a representative of $\max(d_0, |\sigma|)$ such that $f_0 = v \circ (\gamma_s^i)^{-1}$ on $l_s^i(u)$.

Now, we introduce an increasing sequence of open sets $(\Omega_k)_{k \in \mathbb{N}}$ such that $\chi_{\Omega_k} \rightarrow \chi_\Omega$ in $L^1(\mathbb{R}^2)$. For a.e. $s \in S_1$ and for every $i \in I_s^2 \setminus I_s^1$, we can define f_0 as we did on Ω_1 . Hence, there exists $S_2 \subset S_1$ such that $|S \setminus S_2| = 0$ and for every $s \in S_2$ and every $i \in I_s^2$, we have f_0 absolutely continuous on $l_s^i(u)$.

Thus, we can select by induction a representative of $\max(d_0, |\sigma|)$ that is absolutely continuous on $l_s^i(u)$ for a.e. $s \in S$ and every $i \in I_s$. □

For a.e. $s \in S$, if $l_s^i(u) \cap U = \emptyset$ we have some additional information that will be useful in the final proof.

Proposition 5.2. *For a.e. $s \in S$, for every $i \in I_s$, if $l_s^i(u) \cap U = \emptyset$ then $f_0 = C_s^i$ is constant on $l_s^i(u)$ with $C_s^i \in \{d_n, n \in \mathbb{N}, 0 \leq n < N\}$.*

Proof. For a.e. $x \in \Omega \setminus U$ we have $|\sigma(x)| \in \{0\} \cup \varphi'(\mathbb{R} \setminus \mathcal{SC})$. By the coarea formula for a.e. $s \in \mathbb{R}$, for \mathcal{H}^1 a.e. $x \in (\Omega \cap L_s(u)) \setminus U$ we have $|\sigma(x)| \in \varphi'(\mathbb{R} \setminus \mathcal{SC})$. Hence, for a.e. $s \in \mathbb{R}$ if $l_s^i(u) \cap U = \emptyset$ then $f_0(l_s^i(u)) \subset \varphi'(\mathbb{R} \setminus \mathcal{SC}) \cup f_0(X)$ for some $X \subset l_s^i(u)$ with $\mathcal{H}^1(X) = 0$.

Moreover for a.e. $s \in \mathbb{R}$ and every $i \in I_s$, $l_s^i(u)$ is a Lipschitz continuous curve such that f_0 is absolutely continuous on $l_s^i(u)$. Since $\varphi'(\mathbb{R} \setminus \mathcal{SC})$ is finite and $f_0(X)$ is the image of a negligible set by an absolutely continuous function we have $|\varphi'(\mathbb{R} \setminus \mathcal{SC}) \cup f_0(X)| = 0$. The continuity of f_0 on $l_s^i(u)$ implies that f_0 is constant on $l_s^i(u)$. Since $l_s^i(u) \cap U = \emptyset$ we obtain $f_0 = C_s^i \in \{d_n, n \in \mathbb{N}, 0 \leq n < N\}$. □

We use the notations of Section 4, where $(\sigma_n)_{n \in \mathbb{N}}$ is a smooth approximation that converges a.e. on Ω to σ . We prove the following maximum principle on $\max(d_0, |\sigma_n|)$:

Proposition 5.3. *We assume that Ω has a $C^{1,1}$ boundary, $\psi \in C^{1,1}(\mathbb{R}^2)$, λ is globally Lipschitz continuous on $\overline{\Omega}$ and $\lambda > 0$. There exists*

$$\Upsilon := \Upsilon(|\Omega|, \max_{\overline{\Omega}} \lambda, \min_{\overline{\Omega}} \lambda, \|\psi\|_{C^{1,1}(\mathbb{R}^2)}, \kappa) > 0$$

with κ the maximum of the principal curvatures of $\partial\Omega$ such that if $\|\nabla\lambda\|_{L^\infty(\Omega)} \leq \Upsilon$ then for $n \in \mathbb{N}$ large enough, for a.e. every $s \in S$, for every $i \in I_s$, if $l_s^i(u) \cap U = \emptyset$ we have

$$\sup_{F_s^i} |\sigma_n| \leq \sup_{l_s^i(u)} \max(d_0, |\sigma_n|)$$

where F_s^i is the bounded connected component of $\mathbb{R}^2 \setminus l_s^i(u)$.

Remark 5.4. When λ is constant this result is true even if Ω and ψ are only Lipschitz continuous.

Proof. By the coarea formula in Proposition 2.18, $\sigma_n \rightarrow \sigma$ \mathcal{H}^1 a.e. on $l_s^i(u)$ for a.e. $s \in S$ and every $i \in I_s$. We apply the maximum principle from [14, Theorem 15.1] to $|\nabla u_n|$ on F_s^i . To do so we

assume that

$$\|\nabla\lambda\|_{L^\infty(\Omega)} \leq \frac{\min \lambda^2}{2 \times L \times \sup_{x \in [\frac{b_0}{2}, L]} \varphi''(x) + \frac{\varphi'(x)}{x}}.$$

Here L is the Lipschitz constant introduced in Proposition 2.7. For every $n \in \mathbb{N}$, there exists b_n such that $\varphi'_n(b_n) = d_0$.

Hence, for $n \in \mathbb{N}$ large enough, $b_n \geq \frac{b_0}{2}$ and

$$\|\nabla\lambda_n\|_{L^\infty(\Omega)} \leq \frac{\min \lambda_n^2}{\|\nabla u_n\|_{L^\infty(\Omega)} \times \sup_{x \in [\frac{b_n}{2}, L]} \varphi''_n(x) + \frac{\varphi'_n(x)}{x}}.$$

Thus, thanks to [14, Theorem 15.1, Equation (15.15)] for a.e. $s \in \mathbb{R}$ we have

$$\sup_{F_s} |\nabla u_n| \leq \sup_{l_s^i(u)} \max\left(\frac{b_0}{2}, |\nabla u_n|\right) \leq \sup_{l_s^i(u)} \max(b_n, |\nabla u_n|)$$

for $n \in \mathbb{N}$ large enough. Since φ'_n is increasing we obtain $\sup_{F_s} |\sigma_n| \leq \sup_{l_s^i(u)} \max(d_0, |\sigma_n|)$. □

Proposition 5.5. *Let us consider $s \in S$ and $i \in I_s$ such that $l_s^i(u) \cap U = \emptyset$ and $f_0 = C_s^i$ on $l_s^i(u)$. Then for a.e. $t > s$, for every $j \in I_t$ such that $l_t^j(u) \Subset F_s^i$ and $l_t^j(u) \cap U = \emptyset$, we have $\max(d_0, |\sigma|) = C_t^j \mathcal{H}^1$ a.e. on $l_t^j(u)$ with $C_t^j \in \{d_n, n \in \mathbb{N}, 0 \leq n < N\}$ not larger than C_s^i .*

Proof. By construction of f_0 in the proof of Proposition 5.1 and Proposition 5.2, we can construct a subsequence $\max(d_0, |\sigma_{\psi(n)}|)$ converging uniformly to C_s^i on $l_s^i(u)$ that also converges uniformly to C_t^j on $l_t^j(u)$. Hence, with the previous proposition we get:

$$C_t^j \leq \limsup_{n \rightarrow +\infty} \sup_{l_t^j(u)} \max(d_0, |\sigma_{\psi(n)}|) \leq \limsup_{n \rightarrow +\infty} \sup_{F_s^i(u)} \max(d_0, |\sigma_{\psi(n)}|) \leq \lim_{n \rightarrow +\infty} \sup_{l_s^i(u)} \max(d_0, |\sigma_{\psi(n)}|) = C_s^i.$$

Thus, we have that $C_t^j \leq C_s^i$. □

6. PROOF OF THE MAIN THEOREM

6.1. Pseudo Cheeger problem. In this part we combine the maximum principle for $|\sigma|$ and the Euler-Lagrange equation to prove that the level sets are almost Cheeger sets.

We recall the definition of the Cheeger constant of a set:

Definition 6.1. The Cheeger constant of Ω is defined as:

$$h_\Omega = \inf_{D \subset \bar{\Omega}} \frac{\text{Per}(D, \mathbb{R}^2)}{|D|}$$

A set $D \subset \bar{\Omega}$ of finite perimeter is said to be a *Cheeger set* if $\text{Per}(D, \mathbb{R}^2) = h_\Omega |D|$.

Remark 6.2. There is no Cheeger set D of Ω such that $D \Subset \Omega$ because the function $t \mapsto \frac{\text{Per}(tD, \mathbb{R}^2)}{|tD|}$ is $\frac{1}{t}$ -homogeneous.

The following equality is a consequence of Proposition 5.2:

Proposition 6.3. *For a.e. $s \in S$, for every $i \in I_s$ if $l_s^i(u) \cap U = \emptyset$ we have*

$$\int_{F_s^i} \lambda = C_s^i \text{Per}(F_s^i)$$

where F_s^i is the bounded connected component of $\mathbb{R}^2 \setminus l_s^i(u)$ and C_s^i is the constant introduced Proposition in 5.2.

Proof. By [11, Section 5.11, Theorem 1], for a.e. $s \in S$ and for every $i \in I_s$, we have that $D\mathbf{1}_{F_s^i} \in BV(\Omega)$.

We consider the sequence $(\sigma_n)_{n \in \mathbb{N}}$ that converges a.e. on Ω towards σ introduced in Section 4. By [11, Section 5.8, Theorem 1] we obtain:

$$\int_{F_s^i} \text{div}(\sigma_n) dx = \int_{\Omega} \mathbf{1}_{F_s^i}(x) \text{div}(\sigma_n) dx = - \int_{\Omega} \langle \sigma_n, \frac{D\mathbf{1}_{F_s^i}}{|D\mathbf{1}_{F_s^i}|} \rangle d|D\mathbf{1}_{F_s^i}| = - \int_{\partial^* F_s^i} \langle \sigma_n, \frac{D\mathbf{1}_{F_s^i}}{|D\mathbf{1}_{F_s^i}|} \rangle d\mathcal{H}^1.$$

The set $\partial^* F_s^i$ is introduced in Definition 2.16. We can use Proposition 2.20 that gives:

$$- \int_{F_s^i} \text{div}(\sigma_n) dx = \int_{\partial^* F_s^i} \langle \sigma_n, \frac{\nabla u}{|\nabla u|} \rangle d\mathcal{H}^1.$$

But by the coarea formula $\sigma_n \rightarrow \sigma$ \mathcal{H}^1 a.e. on $\partial^* F_s^i \subset l_s^i(u)$ for a.e. $s \in \mathbb{R}$ and every $i \in I_s$. By Proposition 5.2 and since σ is collinear to $\frac{\nabla u}{|\nabla u|}$ \mathcal{H}^1 a.e. on $l_s^i(u)$, we get for such an s :

$$(6.1) \quad \lim_{n \rightarrow +\infty} - \int_{F_s^i} \text{div}(\sigma_n) dx = \int_{\partial^* F_s^i} |\sigma| d\mathcal{H}^1 = C_s^i \text{Per}(F_s^i).$$

Moreover,

$$(6.2) \quad - \int_{F_s^i} \text{div}(\sigma_n) = \int_{F_s^i} \lambda_n \rightarrow \int_{F_s^i} \lambda$$

when $n \rightarrow +\infty$, where $\lambda_n := \lambda * \rho_n$. Hence, with (6.1) and (6.2), we have the desired result:

$$\int_{F_s^i} \lambda = C_s^i \text{Per}(F_s^i)$$

for a.e. $s \in S$, for every $i \in I_s$ if $l_s^i(u) \cap U = \emptyset$. □

We also have:

Proposition 6.4. *For every set $F \subset \overline{F_s^i}$ of finite perimeter we have*

$$\int_F \lambda \leq C_s^i \text{Per}(F).$$

Proof. We follow the same ideas developed in the previous proof. We have:

$$- \int_F \text{div}(\sigma_n) = \int_{\partial^* F} \langle \sigma_n, \nu_F \rangle d\mathcal{H}^1.$$

The term in the left hand side tends to $\int_F \lambda$ when $n \rightarrow +\infty$. For the term in the right hand side we get:

$$\int_{\partial^* F} \langle \sigma_n, \nu_F \rangle d\mathcal{H}^1 \leq \int_{\partial^* F} |\sigma_n| d\mathcal{H}^1.$$

By Proposition 5.3, $\sup |\sigma_n| \leq \sup \max(d_0, |\sigma_n|)$. By Proposition 5.1 and Proposition 5.2 we have $\max(d_0, |\sigma_n|) \rightarrow C_s^i \mathcal{H}^1$ a.e. on $l_s^i(u)$ when $n \rightarrow +\infty$. Hence,

$$\int_F \lambda \leq C_s^i \text{Per}(F).$$

□

6.2. Main proof. We first prove Theorem 1.5:

Proof of Theorem 1.5. For a.e. $s \in \mathbb{R}$, for every $i \in I_s$ if $l_s^i(u) \cap (U \cup \partial\Omega) = \emptyset$ then by Proposition 6.3,

$$\int_{F_s^i} \lambda = C_s^i \text{Per}(F_s^i).$$

We assume that such a $l_s^i(u)$ exists. Since $F_s^i \Subset \Omega$ by Remark 6.2 and the previous equality, we have

$$h_\Omega < \frac{\text{Per}(F_s^i)}{|F_s^i|} = \frac{1}{C_s^i |F_s^i|} \int_{F_s^i} \lambda.$$

We have that $\|\lambda\|_{L^\infty(\Omega)} \leq d_0 h_\Omega$. Thus,

$$h_\Omega < \frac{d_0 h_\Omega}{C_s^i}.$$

Hence, $C_s^i < d_0$ which is a contradiction. Thus, for a.e. $s \in \mathbb{R}$ we have $l_s^i(u) \cap (U \cup \partial\Omega) \neq \emptyset$.

Let v be another minimizer. By Proposition 3.2 and Proposition 3.6, for a.e. $s \in \mathbb{R}$, on every connected component $l_s^i(u)$ of $L_s(u)$ that is not a point we have $u = v$ on $l_s^i(u)$ or $\nabla(u - v) = 0$ \mathcal{H}^1 a.e. on $l_s^i(u)$.

By the coarea formula:

$$\int_{\mathbb{R}^2 \cap [u \neq v]} |\nabla(u - v)| |\nabla u| = \int_{\mathbb{R}} \int_{L_s(u) \cap [u \neq v]} |\nabla(u - v)| d\mathcal{H}^1 ds.$$

By Theorem 2.8, for a.e. $s \in \mathbb{R}$, $\mathcal{H}^1(L_s \setminus L_s^*) = 0$ and L_s^* is composed by a countable number of curves $l_s^i(u)$. For every $i \in I_s$, we have:

$$\int_{l_s^i(u) \cap [u \neq v]} |\nabla(u - v)| d\mathcal{H}^1 = 0.$$

By Proposition 3.2, we get

$$\int_{L_s(u) \cap [u \neq v]} |\nabla(u - v)| d\mathcal{H}^1 = 0.$$

Hence,

$$\int_{\mathbb{R}^2 \cap [u \neq v]} |\nabla(u - v)| |\nabla u| dx = 0.$$

For the same reasons,

$$\int_{\mathbb{R}^2 \cap [u \neq v]} |\nabla(u - v)| |\nabla v| dx = 0.$$

Hence we have $\nabla(u - v) = 0$ a.e. on $[u \neq v]$. This implies that the map $u - v$ is constant on \mathbb{R}^2 . Since $u = v$ on $\partial\Omega$, we have that $u = v$ on Ω . □

Now, we are ready to prove the main theorem:

Proof of Theorem 1.1. Let u and v be two minimizers of \mathcal{P}_λ . We assume that $\|\nabla\lambda\|_{L^\infty(\Omega)} \leq \Upsilon < \frac{\min \lambda}{\text{diam}\Omega}$ up to decreasing the constant Υ from Proposition 5.3. For a.e. $s \in \mathbb{R}$, every $i \in I_s$, if $l_s^i(u) \cap (U \cup \partial\Omega) = \emptyset$ by Proposition 5.2, $|\sigma| = C_s^i \mathcal{H}^1$ a.e. on $l_s^i(u)$ with $C_s^i \in \{d_n, 0 \leq n < N\}$.

We prove by induction on $0 \leq n < N$ that if $|\sigma| = d_n$ on $l_s^i(u)$ then $\nabla(u - v) = 0 \mathcal{H}^1$ a.e. on l_s^i .

Step 1 As an initialisation step, we assume that $C_s^i = d_0$. By Proposition 3.3, $u > s$ on F_s^i . By the coarea formula, for a.e. $t > s$, t belongs to S . We assume that there exists $t > s$ and $j \in I_t$ such that $l_t^j(u) \cap U = \emptyset$ and $F_t \Subset F_s^i$. By Proposition 5.3, $|\sigma| = d_0$ a.e. on $l_t^j(u)$. Thus, by Proposition 6.3 we have that $\int_{F_t} \lambda = d_0 \text{Per}(F_t)$. For $r > 1$ close to 1 and $x_0 \in \Omega$, we introduce $F_t^r = r(F_t - x_0) + x_0 \Subset F_s^i$.

Hence, by Proposition 5.3 we have $|\sigma| \leq d_0$ on ∂F_t^r . Then, by Proposition 6.4,

$$r^2 \int_{F_t^r} \lambda(r(x - x_0) + x_0) dx = \int_{F_t^r} \lambda(y) dy \leq d_0 \text{Per}(F_t^r) = r d_0 \text{Per}(F_t) = r \int_{F_t} \lambda(x) dx.$$

Thus,

$$\int_{F_t^r} r\lambda(r(x - x_0) + x_0) - \lambda(x) \leq 0.$$

Since $\|\nabla\lambda\|_{L^\infty(\Omega)} < \frac{\min \lambda}{\text{diam}(\Omega)}$, we have that $r\lambda(r(x - x_0) + x_0) - \lambda(x) > 0$ for every $x \in F_s^i$. That is a contradiction. Hence, for a.e. $t > s$ and every $j \in I_t$ such that $l_t^j(u) \Subset F_s^i$ we have $l_t^j(u) \cap U \neq \emptyset$. By Proposition 3.6, $\nabla(u - v) = 0 \mathcal{H}^1$ a.e. on $l_t^j(u)$. By the coarea formula, $\nabla(u - v) = 0$ a.e. in F_s^i . By Lemma 3.8, we have that $\nabla(u - v) = 0 \mathcal{H}^1$ a.e. on $l_s^i(u)$.

Step 2 Now, we prove the induction part. We consider $1 \leq n < N$. Let us assume that for every $k < n$, for a.e. $t \in \mathbb{R}$ and every $j \in I_t$ if $l_t^j(u) \cap (U \cup \partial\Omega) = \emptyset$ and $C_t^j = d_k$ then $\nabla(u - v) = 0 \mathcal{H}^1$ a.e. on $l_t^j(u)$.

If $l_s^i(u)$ is such that $l_s^i(u) \cap (U \cup \partial\Omega) = \emptyset$ and $C_s^i = d_n$, we consider $t > s$ such that $l_t^j(u) \cap U = \emptyset$ and $F_t \Subset F_s^i$. Hence, by Proposition 5.5, either $C_t^j = d_n$ or $C_t^j < d_n$. If $C_t^j = d_n$, then as in **Step 1** we construct $F_t^r \Subset F_s^i$ and we prove the $\nabla(u - v) = 0 \mathcal{H}^1$ a.e. on $l_t^j(u)$. By induction we have $\nabla(u - v) = 0 \mathcal{H}^1$ a.e. on $l_t^j(u)$ in the second case. Hence, $\nabla(u - v) = 0 \mathcal{H}^1$ a.e. on $l_t^j(u)$. We can conclude as in **Step 1** that $\nabla(u - v) = 0 \mathcal{H}^1$ a.e. on $l_s^i(u)$.

Step 3 For a.e. $s \in S$, we consider $l_s(u)$ a connected component of $L^*(u)$. If $l_s(u) \cap (\mathbb{R}^2 \setminus \Omega) \neq \emptyset$ then by Proposition 3.2, $u = v$ on $l_s(u)$. If $l_s(u) \subset \Omega$ and $l_s(u) \cap U \neq \emptyset$ then by Proposition 3.6, $\nabla(u - v) = 0 \mathcal{H}^1$ a.e. on $l_s(u)$. Finally, thanks to **Step 2** if $l_s(u) \Subset \Omega$ and $l_s(u) \cap U = \emptyset$ then we have $\nabla(u - v) = 0 \mathcal{H}^1$ a.e. on $l_s(u)$. Hence, we can prove with the coarea formula, as in the proof of Theorem 1.5, that $u = v$. □

7. EXTENSIONS

In this section we present an extension of the main theorem where \mathcal{SC} has a countable number of connected components. We assume that φ is \mathcal{C}^2 and $\varphi'' > 0$ on $\text{int}(\mathcal{SC}) \setminus \{0\}$ and:

$$\mathcal{SC} \cap \mathbb{R}_+ = \mathcal{SC}_\infty \cup \left(\bigcup_{n \in \mathbb{N}} \mathcal{SC}_n \right)$$

with $\mathcal{SC}_0 := [0, b_0)$, $\mathcal{SC}_n := (a_n, b_n)$ for every $n \in \mathbb{N}^*$ and \mathcal{SC}_∞ is defined below. We assume that $(a_n)_{n \in \mathbb{N}^*}$ and $(b_n)_{n \in \mathbb{N}}$ are strictly increasing sequences. Moreover, the sequence $(a_n)_{n \in \mathbb{N}^*}$ is bounded and $\lim_{n \rightarrow +\infty} a_n = \alpha$. For every $n \in \mathbb{N}$, $d_n := \varphi'(b_n) = \varphi'(a_{n+1})$ is an increasing sequence.

The connected component \mathcal{SC}_∞ is exceptional because $\mathcal{SC}_\infty := (a_\infty, +\infty)$ if $\alpha < a_\infty$ and $\mathcal{SC}_\infty := [\alpha, +\infty)$ if $\alpha = a_\infty$.

Proposition 7.1. *In that case Theorem 1.1 is still valid.*

Proof. With this new *structural assumptions*, the minimizers are still globally Lipschitz-continuous on Ω . We can define U as previously with $\text{int}(\mathcal{SC})$ instead of \mathcal{SC} . The function $\max(d_0, \varphi'(\nabla u))$ is still in $H_{loc}^1(\Omega)$. Since $|\varphi'(\mathbb{R} \setminus \mathcal{SC})| = 0$, Proposition 5.2 remains valid. Hence, the last crucial point is the end of the induction argument in **Step 2** of the proof of Theorem 1.1. We assume that there exists $l_s^i(u) \Subset \Omega$ such that $l_s^i(u) \cap U = \emptyset$ and $C_s^i = \varphi'(a_\infty)$. Then for every $l_t \Subset F_s^i$, we either have that $C_t = \varphi'(a_\infty)$ or $\nabla(u - v) = 0$ \mathcal{H}^1 a.e. on l_t . Hence, we have that $\nabla(u - v) = 0$ \mathcal{H}^1 a.e. on $l_s^i(u)$. Thus, $u = v$ on Ω . \square

Remark 7.2. The sets

$$\left[0, \frac{1}{2}\right) \cup \left(\bigcup_{n \in \mathbb{N}^*} \left(\frac{2^{2n} - 1}{2^{2n}}, \frac{2^{2n+1} - 1}{2^{2n+1}} \right) \right) \cup [1, +\infty)$$

and

$$\left[0, \frac{1}{2}\right) \cup \left(\bigcup_{n \in \mathbb{N}^*} \left(\frac{2^{2n} - 1}{2^{2n}}, \frac{2^{2n+1} - 1}{2^{2n+1}} \right) \right) \cup (2, +\infty)$$

satisfy the new *structural assumptions* made on \mathcal{SC} .

REFERENCES

- [1] G. Alberti, S. Bianchini and G. Crippa, Structure of level sets and Sard-type properties of Lipschitz maps. *Annali della Scuola Normale Superiore di Pisa - Classe di Scienze, Serie 5, Volume 12*, (2013) no. 4, pp. 863-902.
- [2] L. Ambrosio, N. Fusco and D. Pallara, *Functions of bounded variation and free discontinuity problems*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, (2000).
- [3] G. Anzellotti and M. Giaquinta, Convex functionals and partial regularity. *Arch. Rational Mech. Anal.*, **102** (2013), pp. 243-272.
- [4] J. Ball, A version of the fundamental theorem for young measures, PDEs and continuum models of phase transitions, *Lecture Notes in Phys.* **344**, (1989), pp. 207-215.
- [5] G. Bouchitté and P. Bousquet, On a degenerate problem in the calculus of variations, *Trans. Amer. Math. Soc.*, **371**, (2019), pp. 777-807.
- [6] A. Cellina, Uniqueness and comparison results for functionals depending on ∇u and on u , *SIAM J. Optim.*, **18** (2007), 711-716.
- [7] P. Celada, G. Cupini, and M. Guidorzi, Existence and regularity of minimizers of nonconvex integrals with $p - q$ growth. *ESAIM: Control, Optimisation and Calculus of Variations*, **13** (2007), pp. 343-358.
- [8] F. H. Clarke, Yu. S. Ledyaev, R. J. Stern and R. R. Wolenski, *Nonsmooth Analysis and Control Theory*, Graduate Texts in Mathematics, **178**, Springer, (1998).
- [9] B. Dacorogna, *Direct Methods in the Calculus of Variations*, Springer New York, NY, 2nd ed. (2008).
- [10] D. De Silva, O. Savin, Minimizers of convex functionals arising in random surfaces. *Duke Mathematical Journal* **151**, (2010), pp. 487-532.
- [11] L. C. Evans and R. F. Gariepy, *Measure theory and fine properties of functions*, Advanced Mathematical Studies, CRC Press, Boca Raton, (1992).
- [12] E. Giusti, *Minimal surfaces and functions of bounded variation*, Monographs in Mathematics, Birkhäuser Verlag, Basel, (1984).

- [13] E. Giusti, *Direct methods in the calculus of variations*, World Scientific Publishing Co., Inc., River Edge, NJ, (2003).
- [14] D. Gilbarg and N. Trudinger, *Elliptic partial differential equations of second order*, Classics in Mathematics, Springer-Verlag, Berlin, (2001).
- [15] B. Kawohl, J. Stara and G. Wittum, Analysis and numerical studies of a problem of shape design. *Arch. Rational Mech. Anal.* **114**, (1991) pp. 349–363.
- [16] Lledos, B., A uniqueness result for a translation invariant problem in the calculus of variations, Submitted.
- [17] Lledos, B., A uniqueness result for a non-strictly convex problem in the calculus of variations. Submitted.
- [18] L. Lussardi and E. Mascolo, A uniqueness result for a class of non-strictly convex variational problems, *J. Math. Anal. Appl.*, **446** (2017), pp. 1687–1694.
- [19] P. Marcellini, A relation between existence of minima for nonconvex integrals and uniqueness for non strictly convex integrals of the calculus of variations, *Mathematical Theories of Optimization, Proceedings*, edited by J.P. Cecconi and T. Tozzi, Lecture Notes in Math., Springer, **979** (1983), pp. 216–231.