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Bernard Derrida, Véronique Gayrard, Peter Mottishaw

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## **RANDOM ENERGY MODELS: BROKEN REPLICA SYMMETRY AND ACTIVATED DYNAMICS**

BERNARD DERRIDA, VÉRONIQUE GAYRARD, AND PETER MOTTISHAW

**ABSTRACT.** The random energy model (REM) and the generalized random energy model (GREM) are simple spin glass models which play an important role in the theory of spin glasses. The connection with more complex spin glass models can be made using the  $p$ -spin generalisation of the SK model, which, in the large  $p$  limit, gives precisely the REM. The REM and GREM allow us to illustrate and to test in a simple framework several central ideas of the theory both from an equilibrium and from a dynamical point of view. They were also used in several contexts such as the problem of protein folding or error-correcting codes. This chapter presents the basic ideas and some recent developments for two aspects:

- (1) The random energy models and replica symmetry breaking by B. Derrida and P. Mottishaw
- (2) Aging in the activated dynamics of the REM and the  $p$ -spin SK models by V. Gayrard

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## 1. THE RANDOM ENERGY MODELS AND REPLICA SYMMETRY BREAKING

1.1. **Introduction.** At the end of the 70's there were considerable efforts among theoretical physicists to try to solve the Sherrington Kirkpatrick (SK) model [85]. This led to the invention by Parisi of his replica symmetry breaking (RSB) scheme in 1979 (see [69] and references therein).

Trying to develop possible ways of understanding the SK model, a generalization of the SK model, the  $p$ -spin model [30], was introduced in 1980 which reduces to the SK model when  $p = 2$  and has two simple limits, the case  $p = 1$  where the spins are independent and the case  $p = \infty$  which is precisely the random energy model (REM). Initially the goal was to try to approach the SK model through perturbative expansions around these two exactly soluble cases.

The  $p$ -spin model describes a system of  $N$  Ising spins  $\sigma_i = \hat{A} \pm 1$  with infinite-ranged random  $p$ -spin interactions. The energy of a configuration  $\mathcal{C} = \{\sigma_1 \cdots \sigma_N\}$  can be written as

$$E(\mathcal{C}) = - \sum_{i_1 \leq i_2 \cdots i_p} A_{i_1 \cdots i_p} \sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_p} \quad (1.1)$$

where the  $p$ -spin interactions  $A_{i_1 \cdots i_p}$  are quenched random variables distributed according to

$$\rho(A_{i_1 \cdots i_p}) = \sqrt{\frac{N^{p-1}}{\pi J^2 p!}} \exp \left[ - \frac{(A_{i_1 \cdots i_p})^2 N^{p-1}}{J^2 p!} \right]. \quad (1.2)$$

For the  $p$ -spin model, (as well as for many other disordered systems) the partition function  $Z(\beta)$

$$Z(\beta) = \sum_{\mathcal{C}=1}^{2^N} e^{-\beta E(\mathcal{C})} \quad (1.3)$$

is nothing but a sum of exponentials of correlated Gaussian random variables,

$$P(E(\mathcal{C})) \simeq \frac{1}{\sqrt{N\pi} J} \exp \left[ - \frac{E(\mathcal{C})^2}{N J^2} \right] \quad (1.4)$$

with known correlations and

$$\langle E(\mathcal{C}) E(\mathcal{C}') \rangle = \frac{N J^2}{2} q(\mathcal{C}, \mathcal{C}')^p \quad (1.5)$$

where the overlap between the pair of configurations  $\mathcal{C}, \mathcal{C}'$  is defined by

$$q(\mathcal{C}, \mathcal{C}') = \frac{1}{N} \sum_{i=1}^N \sigma_i \sigma'_i. \quad (1.6)$$

Here, and elsewhere in this chapter,  $\langle \cdot \rangle$  denotes an average over disorder, i.e. over the  $2^N$  energies  $E(\mathcal{C})$ . Clearly in the large  $p\hat{A}$  limit the energies of different configurations become uncorrelated. Therefore in this limit, the  $p$ -spin model becomes the random energy model with a partition function  $Z(\beta)$  given by (1.3), the energies  $E(\mathcal{C})$  being  $2^N$  i.i.d. random variables distributed according to (1.4). That the partition function can be seen as a sum of independent random variables greatly simplifies the analysis of the REM [50, 76, 79].

**1.2. The phase diagram of the REM.** A simple way of obtaining the phase diagram of the REM is to use the microcanonical ensemble. For a given sample, i.e. for a given realization of the  $2^N$  energies  $E(\mathcal{C})$ , let  $\mathcal{N}(E)$  be the number of configurations with an energy in the interval  $(E, E + \delta E)$ , (we choose  $\delta E$  to be small compared to  $N$ , but not exponentially small in  $N$ ). Obviously the average of  $\mathcal{N}(E)$  over the samples is  $\langle \mathcal{N}(E) \rangle = 2^N P(E) \delta E$ . Then because the energies are independent, one has that for a typical sample  $\mathcal{N}(E) \simeq \langle \mathcal{N}(E) \rangle$  in the range of energies where  $\langle \mathcal{N}(E) \rangle \gg 1$  (i.e. when  $|E/N| < J\sqrt{\log 2}$ ) and  $\mathcal{N}(E) = 0$  in the range where  $\langle \mathcal{N}(E) \rangle \ll 1$ . This immediately tells us that the ground state energy is such  $E_{\text{GS}}/N = -J\sqrt{\log 2}$  and that the entropy in the range  $|E|/N < J\sqrt{\log 2}$  is given by  $S(E) = N \log 2 - E^2/(NJ^2)$ . Outside this range, there is no energy level (for a typical sample) and thus  $S(E) = -\infty$ . In summary,

$$\frac{S(E)}{N} = \begin{cases} \log 2 - \frac{E^2}{N^2 J^2} & \text{for } \frac{|E|}{NJ} < \sqrt{\log 2} \\ -\infty & \text{otherwise.} \end{cases} \quad (1.7)$$

Then using the fact that  $\beta = \frac{dS(E)}{dE}$  one gets the following expression for the free energy of a typical sample:

$$\frac{\log Z(\beta)}{N} = \begin{cases} \log 2 + \frac{\beta^2 J^2}{4} & \text{for } \beta < \beta_c \\ \beta J \sqrt{\log 2} & \text{for } \beta > \beta_c \end{cases} \quad \text{where } \beta_c = 2\sqrt{\log 2} J^{-1}. \quad (1.8)$$

This shows that the REM exhibits a phase transition at  $\beta = \beta_c$ . In the low temperature phase the energy becomes independent of  $\beta$  which means that the system is frozen in its lowest energy levels (see Section 1.3).

To investigate the magnetic properties of the REM [30, 31], one can introduce a magnetic field and a magnetization in the REM by considering that among the  $2^N$  configurations,  $\binom{N}{\frac{N+M}{2}}$  have a total magnetization  $M$  (with  $M = -N, -N+2, -N+4, \dots, N$ ). By repeating the above reasoning which led to (1.8) for the typical number  $\mathcal{N}(E, M)$  of configurations with energy  $E$  and magnetization  $M$ , one can calculate the entropy and then the free energy as a function of the temperature and of the magnetic field. The outcome is the following expression of the zero field magnetic susceptibility

$$\chi = \begin{cases} \beta & \text{for } \beta < \beta_c \\ \beta_c & \text{for } \beta > \beta_c \end{cases} \quad (1.9)$$

with a cusp at the transition very reminiscent of the susceptibility of spin glasses. Moreover the REM exhibits a phase transition in a non-zero magnetic field [30] as the SK model does along the de Almeida-Thouless line [69].

Many other properties of the REM can be determined like the finite size corrections to the average free energy [30, 31]. For example, when  $\beta > \beta_c$ ,

$$\langle \log Z(\beta) \rangle = N \frac{\beta \beta_c J^2}{2} - \frac{\beta}{2\beta_c} \log N + R(\beta) \quad (1.10)$$

including the  $O(1)$  correction

$$R(\beta) = \frac{\beta}{\beta_c} \log \left( \frac{\Gamma \left( 1 - \frac{\beta_c}{\beta} \right)}{J \beta_c \sqrt{\pi}} \right) + \left( 1 - \frac{\beta}{\beta_c} \right) \Gamma'(1) + o(1) \quad (1.11)$$

or the location of the zeroes in the complex temperature plane [72, 33, 81].

Expressions like (1.8-1.10) or other properties can also be generalized to cases where the energies  $E(\mathcal{C})$  are still independent but are no longer Gaussian random variables [50, 63, 74, 75, 42].

**1.3. The low temperature phase of the REM.** In the low temperature phase ( $\beta > \beta_c$ ), the extensive part of the free energy is frozen and the partition function is dominated by the ground state and the energy levels at a distance of order 1 from this ground state. Close to the ground state energy, (in fact for all energies  $E$  such that  $|E + NJ\sqrt{\log 2}| \ll \sqrt{N}$ ) the distribution (1.4) can be approximated by an exponential distribution

$$P(E(\mathcal{C})) \sim \frac{1}{\sqrt{N\pi} J} \frac{1}{2^N} e^{\beta_c (E(\mathcal{C}) + NJ\sqrt{\log 2})}. \quad (1.12)$$

One can then rewrite the partition function (1.3) as a sum

$$Z(\beta) = Z_0 \sum_{\mathcal{C}=1}^{2^N} x(\mathcal{C}) \quad \text{where} \quad Z_0 = e^{N\beta J\sqrt{\log 2}} \quad (1.13)$$

of  $2^N$  i.i.d. random variables  $x(\mathcal{C}) = e^{-\beta(E(\mathcal{C}) + NJ\sqrt{\log 2})}$  which, using (1.12), have a distribution  $\mathcal{P}(x)$  with a heavy tail in the range of energies close to the ground state energy

$$\mathcal{P}(x(\mathcal{C})) \simeq \frac{1}{\sqrt{N\pi} \beta J} \frac{1}{2^N} x(\mathcal{C})^{-1 - \frac{\beta_c}{\beta}} \quad \text{in the range} \quad |\log x| \ll \sqrt{N}. \quad (1.14)$$

Thinking of the partition function (1.13) as the sum of  $2^N$  random variables distributed according to a heavy tail distribution like (1.14) allows to determine most of the properties of the low temperature phase in the large  $N$  limit [35, 34]. For example one can recover the low temperature expression (1.10) including the  $O(1)$  correction (1.11) of the free energy using the identity

$$\langle \log Z(\beta) \rangle = \int_0^\infty \frac{dt}{t} \left( 1 - \langle e^{-tZ(\beta)} \rangle \right)$$

and that

$$\begin{aligned} \langle e^{-tZ(\beta)} \rangle &= \left[ 1 + \int_0^\infty dx \mathcal{P}(x) (e^{-Z_0 t x} - 1) \right]^{2^N} \\ &\simeq \exp \left[ \frac{\Gamma(-\frac{\beta_c}{\beta})}{\sqrt{N\pi} \beta J} (Z_0 t)^{\frac{\beta_c}{\beta}} \right]. \end{aligned} \quad (1.15)$$

This also allows to obtain the distribution of the fluctuations of order 1 of the free-energy [53] in particular that [31]

$$\langle (\log Z(\beta))^2 \rangle - \langle \log Z(\beta) \rangle^2 = \frac{\pi^2}{6} \frac{\beta^2 - \beta_c^2}{\beta_c^2}$$

or to predict that the ground state energy has a Gumbel distribution [14].

**1.4. The overlaps in the low temperature phase of the REM.** Overlaps (1.6) which measure the distance between configurations play a central role in the theory of spin glasses. One striking outcome of the RSB theory [69] is that, even in the large  $N$  limit, the distribution  $P(q)$  defined by

$$P(q) = \frac{1}{Z(\beta)^2} \sum_{\mathcal{C}} \sum_{\mathcal{C}'} e^{-\beta(E(\mathcal{C}+E(\mathcal{C}')))} \delta(q(\mathcal{C}, \mathcal{C}') - q) \quad (1.16)$$

remains broad and non-self averaging, (i.e. it remains sample dependent). As discussed below, this is indeed what also happens for the REM.

In the low temperature phase the configurations  $\mathcal{C} = \{\sigma_1 \cdots \sigma_N\}$  which dominate the partition function, i.e. whose energies are close to the ground state, are likely to be very scattered in phase space and have zero overlaps between themselves. Therefore, in the large  $N$  limit,  $P(q)$  has the form

$$P(q) = (1 - Y_2) \delta(q) + Y_2 \delta(q - 1) \quad (1.17)$$

where  $Y_2$  is the probability of finding at equilibrium two copies of the system in the same configuration

$$Y_2 = \frac{\sum_{\mathcal{C}} e^{-2\beta E(\mathcal{C})}}{(\sum_{\mathcal{C}} e^{-\beta E(\mathcal{C})})^2} = \frac{\sum_{\mathcal{C}} x(\mathcal{C})^2}{(\sum_{\mathcal{C}} x(\mathcal{C}))^2}.$$

Averaging over the  $x(\mathcal{C})$  distributed according to the heavy tail distribution (1.14) one gets [44, 38, 42] in the large  $N$  limit

$$\langle Y_2 \rangle = 1 - \frac{\beta_c}{\beta} \quad ; \quad \langle Y_2^2 \rangle - \langle Y_2 \rangle^2 = \frac{1}{3} \frac{\beta_c}{\beta} \left( 1 - \frac{\beta_c}{\beta} \right). \quad (1.18)$$

This shows that, in the whole low temperature phase,  $P(q)$  in (1.17) is a sum of two delta functions whose relative weights fluctuate even in the large  $N$  limit. In fact the whole distribution of  $Y_2$ , in the large  $N$  limit, can be determined [69, 35].

The expressions (1.18) can be generalized to calculate various correlations between the probabilities  $Y_k$  of finding  $k$  copies of the same system in the same configuration. For example

$$\langle Y_k \rangle = \left\langle \frac{\sum_{\mathcal{C}} e^{-k\beta E(\mathcal{C})}}{(\sum_{\mathcal{C}} e^{-\beta E(\mathcal{C})})^k} \right\rangle = \frac{\Gamma(k - \frac{\beta_c}{\beta})}{\Gamma(k) \Gamma(1 - \frac{\beta_c}{\beta})}. \quad (1.19)$$

An equivalent but slightly different way of thinking of the statistical properties of the energy levels which contribute in the low temperature phase and to recover all the statistical properties of the overlaps (such as (1.18) or (1.19)) is to say that the energies  $E(C)$  are generated by a Poisson point process with a density [38]

$$\rho(E) = 2^N P(E) \sim \frac{1}{\sqrt{N\pi} J} e^{\beta_c(E + NJ\sqrt{\log 2})}. \quad (1.20)$$

As we will see in section 1.6 this Poisson point process is at the basis of the Ruelle cascade which allows to calculate the overlaps in the GREM [80, 9].

More recently it has also been possible to calculate the finite size corrections to (1.18, 1.19), the effect of a discrete distribution of energies  $P(E(C))$  or the overlap between two copies of the same system at different temperatures [38, 42, 77, 41].

**1.5. The replica approach and the REM.** In the theory of spin glasses the replica approach consists in trying to determine the average free energy  $\langle \log Z(\beta) \rangle$  from the knowledge of the moments of the partition function  $\langle Z(\beta)^n \rangle$  via

$$\langle \log Z(\beta) \rangle = \lim_{n \rightarrow 0} \frac{\log \langle Z(\beta)^n \rangle}{n}. \quad (1.21)$$

The well known difficulty with (1.21) is that it requires the knowledge of non integer moments of the partition function and it is based on the wishful idea that these non integer moments can be determined by some kind of continuation to non-integer  $n$  of expressions for the integer moments which are usually easier to obtain. The breakthrough of the RSB theory was to discover the correct continuation for the SK model [78].

For the REM, it is in fact possible to calculate directly integer as well as non-integer moments of the partition function [55]. For example from (1.15) (and an identity of the form  $\Gamma(-\nu) Z^\nu = \int dt \exp[-tZ] t^{-\nu-1}$  for negative  $\nu$ ) one can show that, in the low temperature phase, for  $-\infty < \nu < \frac{\beta_c}{\beta} < 1$

$$\langle Z^\nu \rangle \simeq Z_0^\nu \left( \frac{\Gamma(1 - \frac{\beta_c}{\beta})}{J\beta_c \sqrt{\pi N}} \right)^{\nu \frac{\beta}{\beta_c}} \frac{\Gamma(1 - \nu \frac{\beta}{\beta_c})}{\Gamma(1 - \nu)}$$

from which one can recover (1.10, 1.11) in the  $\nu \rightarrow 0$  limit.

If however one tries to follow the standard approach, i.e. to obtain the free energy (1.3) from the knowledge of the integer moments the starting point is the following expression of these integer moments [31, 48, 41]

$$\langle Z(\beta)^n \rangle = \sum_{r \geq 1} \frac{n!}{r!} 2^{Nr} \sum_{\mu_1 \geq 1} \dots \sum_{\mu_r \geq 1} \frac{\exp \left[ N \frac{J^2 \beta^2}{4} (\mu_1^2 + \dots + \mu_r^2) \right]}{\mu_1! \times \dots \times \mu_r!} \delta_{\mu_1 + \dots + \mu_r = n} \quad (1.22)$$

which is exact in the case of the Poisson REM [38]. In (1.22) each term in this sum corresponds to a partition of the  $n$  replicas into  $r$  blocks of  $\mu_1, \dots, \mu_r$  replicas. For integer  $n$ , the sum in the large  $N$  limit is dominated by the largest term. This leads to [31, 55]

$$\langle Z(\beta)^n \rangle \sim \begin{cases} \exp \left[ Nn \left( \log 2 + \frac{J^2 \beta^2}{4} \right) \right] & \text{for } \beta < \frac{\beta_c}{\sqrt{n}} \\ \exp \left[ N \left( \log 2 + n^2 \frac{J^2 \beta^2}{4} \right) \right] & \text{for } \beta > \frac{\beta_c}{\sqrt{n}} \end{cases}. \quad (1.23)$$

Although, for small  $\beta$  the replica formula (1.21) leads to (1.8), there is no way to recover the low temperature expression (1.8) from (1.23).

If however, one follows Parisi's original approach [78] by allowing  $r$  and the  $\mu_i$ 's to become real numbers, by assuming that for non-integer  $n$ , the sum in (1.22) is dominated by a single term where all the  $\mu_i$ 's are equal to  $\mu$  so that  $r = \frac{n}{\mu}$ , by reversing the inequality  $1 \leq \mu \leq n$  to become  $0 \leq \mu \leq 1$  in the  $n \rightarrow 0$  limit)

$$\langle Z(\beta)^n \rangle \sim \exp \left[ Nn \left( \frac{\log 2}{\mu} + \frac{\mu J^2 \beta^2}{4} \right) \right] \quad (1.24)$$

and by looking for a minimum over  $\mu$  rather than a maximum, one gets

$$\mu = \min \left[ \frac{\beta_c}{\beta}, 1 \right]. \quad (1.25)$$

Doing so one recovers in the  $n \rightarrow 0$  limit [31] the free energy (1.8) by replacing  $\mu$  by this value in (1.24).

This is the simplest form of RSB, the one step RSB: the  $n$  replicas have been divided into  $n/\mu$  blocks of  $\mu$  replicas each. It is then possible to recover the expressions (1.18,1.19) by simply writing that  $\langle Y_k \rangle$  is the probability of finding in the same block  $k$  replicas among  $n$  different replicas.

$$\langle Y_k \rangle = \lim_{n \rightarrow 0} \frac{n}{\mu} \times \frac{\mu(\mu-1) \cdots (\mu-k+1)}{n(n-1) \cdots (n-k+1)}.$$

For the  $p$ -spin model with Ising spins, it has been shown that for large enough  $p$ , the replica approach leads to such a one step RSB transition [62, 54, 89, 90] and to the free-energy (1.8) in the large  $p$  limit. This transition is followed at a lower temperature by a full RSB transition, the Gardner transition [54]. The  $p$ -spin model in its spherical version also exhibits a one step RSB [28, 91].

Trying to understand the finite size corrections to the free energy (1.10,1.11) or to the overlaps using the replica approach is not so easy [24]. To do so it seems that one should let the  $\mu_i$ 's fluctuate around (1.25) and even take complex values [38]. Fluctuating block sizes are also present when one looks at two-temperature overlaps of the REM or when the energies  $E(\mathcal{C})$  take only discrete energies [41, 42].

**1.6. The GREM.** The Generalized Random Energy Model (GREM) was invented as an elementary way of introducing correlations between the energies  $E(\mathcal{C})$  of the configurations. In its simplest version [32], the configurations are structured into  $\alpha_1^N$  groups  $\mathcal{G}$  of  $\alpha_2^N$  configurations each and the energy of each configuration  $\mathcal{C}$  is by definition the sum of two Gaussian random energies

$$E(\mathcal{C}) = E_1(\mathcal{G}) + E_2(\mathcal{C})$$

of variance  $NJ^2 a_1/2$  and  $NJ^2 a_2/2$ . All the energies  $E_1(\mathcal{G})$  and  $E_2(\mathcal{C})$  are independent, but configurations  $\mathcal{C}$  in the same group  $\mathcal{G}$  have all the same  $E_1(\mathcal{G})$ . Therefore the energies of two configurations  $\mathcal{C}, \mathcal{C}'$  are correlated if they are in the same group (we will say that  $q(\mathcal{C}, \mathcal{C}') = Q$ ) and are uncorrelated if they belong to two distinct groups (we will say that their overlap  $q(\mathcal{C}, \mathcal{C}') = 0$ ). The overlap can therefore take three possible values: 1,  $Q$  and 0 (taking by definition  $q(\mathcal{C}, \mathcal{C}) = 1$ ).

For the total number of configurations to be  $2^N$  and for the distribution of individual energies to still be given by (1.4) we impose the following normalization

$$\log \alpha_1 + \log \alpha_2 = \log 2 \quad ; \quad a_1 + a_2 = 1.$$



The GREM can be solved [32, 16, 17] by using the same reasoning as for the REM in section 1.2: the typical  $\mathcal{N}_1(E_1)$  number of groups with an energy  $E_1$  is equal to its average when this average is much larger than 1 and is 0 when this average is much smaller than 1.

$$\mathcal{N}_1(E_1) \sim \begin{cases} \exp \left[ N \log \alpha_1 - \frac{E_1^2}{NJ^2 a_1} \right] & \text{for } \frac{|E_1|}{N} < J\sqrt{a_1 \log \alpha_1} \\ 0 & \text{otherwise .} \end{cases}$$

Then the average of  $\mathcal{N}(E)$  given the function  $\mathcal{N}_1(E_1)$  is

$$\langle \mathcal{N}(E) \rangle \sim \max_{E_1} \left\{ \alpha_2^N \exp \left[ -\frac{(E - E_1)^2}{NJ^2 a_2} \right] \mathcal{N}_1(E_1) \right\} .$$

As  $E$  varies (in the range where  $\mathcal{N}(E) \gg 1$ ), the optimal  $E_1$  either sticks to one of its extremal values  $E_1/N = \pm J\sqrt{a_1 \log \alpha_1}$  or varies with  $E$ . This leads to distinguish two cases.

(1) First case: if  $\frac{\log \alpha_1}{a_1} < \frac{\log \alpha_2}{a_2}$  defining

$$\mathcal{E}_i = NJ\sqrt{a_i \log \alpha_i} \quad (1.26)$$

one gets for the entropy

$$\frac{S(E)}{N} = \begin{cases} \log 2 - \frac{E^2}{N^2 J^2} & \text{for } |E| < \left(1 + \frac{a_2}{a_1}\right) \mathcal{E}_1 \\ \log \alpha_2 - \frac{(E + \mathcal{E}_1)^2}{N^2 J^2 a_2} & \text{for } -\mathcal{E}_1 - \mathcal{E}_2 < E < -\left(1 + \frac{a_2}{a_1}\right) \mathcal{E}_1 \\ -\infty & \text{for } E < -\mathcal{E}_1 - \mathcal{E}_2 \end{cases} \quad (1.27)$$

which leads to the following expression of the free energy:

$$\frac{\langle \log Z(\beta) \rangle}{N} = \begin{cases} \log 2 + \frac{\beta^2 J^2}{4} & \text{for } \beta < \beta_c^{(1)} \\ \beta J\sqrt{a_1 \log \alpha_1} + \log \alpha_2 + \frac{\beta^2 J^2 a_2}{4} & \text{for } \beta_c^{(1)} < \beta < \beta_c^{(2)} \\ \beta J \left( \sqrt{a_1 \log \alpha_1} + (\sqrt{a_2 \log \alpha_2}) \right) & \text{for } \beta_c^{(2)} < \beta \end{cases} \quad (1.28)$$

where the inverse transition temperatures  $\beta_c^{(i)}$  are defined by

$$\beta_c^{(i)} = \frac{2}{J} \sqrt{\frac{\log \alpha_i}{a_i}} . \quad (1.29)$$

(Note that this first case is precisely the case where  $\beta_c^{(1)} < \beta_c^{(2)}$ ).

So the system undergoes two phase transitions. In the high temperature phase, both the energies  $E_1(\mathcal{G})$  and  $E_2(\mathcal{C})$  of typical configurations vary with temperature. In the intermediate phase, the energy  $E_1(\mathcal{G})$  is frozen at its minimal value ( $E_1(\mathcal{G})/N = -J\sqrt{a_1 \log \alpha_1}$ ) while the energy  $E_2(\mathcal{C})$  still varies with temperature. Lastly in the low temperature phase, both energies  $E_1$  and  $E_2$  are frozen at their minimal values.

In each of these phases, one can determine the distribution of overlaps

$$\langle P(q) \rangle = \begin{cases} \delta(q) & \text{for } \beta < \beta_c^{(1)} \\ \frac{\beta_c^{(1)}}{\beta} \delta(q) + \frac{\beta - \beta_c^{(1)}}{\beta} \delta(q - Q) & \beta_c^{(1)} < \beta < \beta_c^{(2)} \\ \frac{\beta_c^{(1)}}{\beta} \delta(q) + \frac{\beta_c^{(2)} - \beta_c^{(1)}}{\beta} \delta(q - Q) + \frac{\beta - \beta_c^{(2)}}{\beta} \delta(q - 1) & \beta_c^{(2)} < \beta. \end{cases} \quad (1.30)$$

This is one of the simplest examples of a 2-step RSB since in the lowest temperature phase the overlap can take two possible non-zero values. In this phase, only the groups with an energy  $E_1$  close to  $-NJ\sqrt{a_1 \log \alpha_1}$  and inside these groups, only the configurations with an energy  $E_2$  close to  $-NJ\sqrt{a_2 \log \alpha_2}$  contribute. The statistics of the energies  $E(\mathcal{C}) = E_1(\mathcal{G}) + E_2(\mathcal{C})$  of these configurations can be accurately described by a decorated Poisson process: the energies  $E_1(\mathcal{G})$  of the groups are generated by a Poisson process with an exponential density  $\rho_1(E_1) \sim \exp[\beta_c^{(1)}(E_1 + N\sqrt{a_1 \log \alpha_1})]$  and the energies  $E_2(\mathcal{C})$  of the configurations in each group are generated by a Poisson process of density  $\rho_2(E_2) \sim \exp[\beta_c^{(2)}(E_2 + N\sqrt{a_2 \log \alpha_2})]$ . This is an example of a Ruelle cascade where the levels are generated by a succession of nested Poisson processes [80, 9, 18]

- (2) Second case: if  $\frac{\log \alpha_1}{a_1} > \frac{\log \alpha_2}{a_2}$  i.e.  $\beta_c^{(1)} > \beta_c^{(2)}$  one can repeat the above reasoning, and one finds a single phase transition at  $\beta = \beta_c$  with a free energy and overlaps given by (1.8) and (1.18,1.19).

This is an example where the correlations between the energies  $E(\mathcal{C})$  are not strong enough to have any effect on the free energy.

The expressions of the free energy in all cases can be recovered from the following expression: [45]

$$\frac{\langle \log Z(\beta) \rangle}{N} = \min_{0 \leq \mu_1 \leq \mu_2 \leq 1} \left[ \frac{\log \alpha_1}{\mu_1} + \frac{\beta^2 J^2 a_1 \mu_1}{4} + \frac{\log \alpha_2}{\mu_2} + \frac{\beta^2 J^2 a_2 \mu_2}{4} \right]. \quad (1.31)$$

Depending on  $\beta$  and on the parameters  $a_i$  and  $\alpha_i$  one finds that the minimum is achieved for  $\mu_1 = \mu_2 = 1$  (high temperature phase),  $\mu_1 < \mu_2 = 1$  (intermediate phase in the first case),  $\mu_1 < \mu_2 < 1$  (low temperature phase in the first case),  $\mu_1 = \mu_2 < 1$  (low temperature phase in the second case).

The expression (1.31) can be recovered by a replica calculation [36, 73]: one writes the moments  $\langle Z^n \rangle$  as we did in (1.22,1.24) and then one assumes that the minimum is achieved when there are  $n/\mu_1$  groups of  $\mu_1$  replicas each and that in each group there are  $\mu_1/\mu_2$  blocks of  $\mu_2$  replicas.

The above two level structure of the GREM can be generalized to hierarchies with an arbitrary number  $k$  of levels [37, 26], the configurations being organized in groups inside groups inside groups etc. . . . : at each level  $i$  a group splits into  $\alpha_i^N$  subgroups with a Gaussian contribution to the energy  $E_i$ .

The solution leads to distinguish various cases depending on the relative values of the  $\beta_c^{(i)}$  but as for the above two level GREM, it can be written in a compact form valid at all temperatures and for arbitrary choices of the parameters  $a_i$  and  $\log \alpha_i$

$$\frac{\langle \log Z \rangle}{N} = \min_{0 \leq \mu_1 \leq \dots \leq \mu_k \leq 1} \left[ \sum_{1 \leq i \leq k} \left( \frac{\log \alpha_i}{\mu_i} + \frac{\mu_i a_i \beta^2 J^2}{4} \right) \right]. \quad (1.32)$$

This can also be interpreted as a scheme à la Parisi where at each level  $i$ , blocks of  $\mu_{i-1}$  replicas are divided into  $\mu_{i-1}/\mu_i$  blocks of  $\mu_i$  replicas each.

As for the REM, one can introduce a magnetic field in the GREM [36] and one finds a cusp in the magnetic susceptibility and a de Aledida Thouless line.

**1.7. The directed polymer on a tree.** For a GREM with an arbitrary number  $k$  of levels, if all the  $\alpha_i$  and all the  $a_i$  are equal, the extensive part of the free energy and the overlaps are still given by (1.8,1.18,1.19). The finite size corrections are however different [40, 27, 71]. The same is true [43] for the directed polymer on a binary tree, of height  $N$ , where on the  $2^{N+1} - 2$  edges  $b$  there are i.i.d. Gaussian energies  $\epsilon_b$  (with  $P(\epsilon_b) = (\pi J^2)^{-1/2} \exp[-\epsilon_b^2/J^2]$ ). The  $2^N$  configurations  $\mathcal{C}$  are then all the paths connecting the top to the bottom of the tree, the energy  $E(\mathcal{C})$  being the sum of the  $N$  edges  $b$  visited by the path. One main difference between the REM and the directed polymer on a tree is that the average free energy in (1.10) becomes

$$\langle \log Z(\beta) \rangle = N\beta\beta_c J^2 - \frac{3\beta}{2\beta_c} \log N + R_{\text{tree}}(\beta) \quad (1.33)$$

so that the  $1/2$  in (1.10) is replaced by a  $3/2$  in the  $\log N$  correction. This  $3/2$  factor has the same origin as the well known Bramson  $3/2$  logarithmic correction in the position of a travelling wave in the Fisher-KPP equation [20, 19] and is also present in other random energy models with logarithmic correlations [25]. On the other hand, in contrast to (1.11), the explicit expression of  $R_{\text{tree}}(\beta)$  is not known.

The other difference [39] is that the first large  $N$  correction to the 1 step RSB (1.17,1.18) is for any fixed  $0 < q < 1$

$$P(q) \simeq \frac{1}{\sqrt{N}} \frac{\beta_c}{\beta} \sqrt{\frac{1}{4\pi \log 2}} (q(1-q))^{-\frac{3}{2}} \quad \text{for } \beta > \beta_c \quad (1.34)$$

where the overlap  $q(\mathcal{C}, \mathcal{C}')$  is just the fraction of their length that two paths  $\mathcal{C}, \mathcal{C}'$  have in common. This shows that the first finite size correction transforms a 1 step RSB into a full RSB.

Also like for the extremal points of the branching Brownian motion and for the GREM, the lowest energy levels can be represented by a decorated Poisson process with an exponential density [21, 22, 1, 2, 3, 83].

**1.8. Conclusion.** The random energy models (REM and GREM) are simple spin glass models which exhibit a number of features common to many other spin glass models (cusp in the magnetic susceptibility, de Almeida Thouless line, replica symmetry breaking and non-selfaveraging effects, slow dynamics and aging at low temperature). Compared to other spin glass models they have the advantage of being exactly solvable by elementary mathematical methods.

They can also be solved using the Parisi replica approach. Doing so, one recovers the correct phase diagrams and the extensive part of the free energy. It remains however an open question on how to recover this way more subtle properties such as finite size effects like (1.34) or expressions of the overlaps between two temperatures or for discrete energies. This seems to require to allow the blocks sizes in the replica approach to fluctuate.

The random energy models have also been equipped by dynamical rules and their dynamics have motivated a large number of studies (see [29, 65, 84, 82, 12, 70, 66, 58, 68, 67] and references therein).

Beyond their interest as spin glass models, they have been useful in a number of other contexts ranging from problems of protein folding [23, 63] or evolution [52] in biology to error-correcting codes [86, 46, 47].

## 2. THE ARCSINE LAW AS A UNIVERSAL AGING SCHEME FOR ACTIVATED DYNAMICS OF RANDOM ENERGY MODELS

**2.1. Introduction.** In glassy dynamics, the concept of *aging* refers to out-of-equilibrium relaxation behaviours that are dominated by ever slower transients and, as such, are history dependent. *Mean-field spin glasses* such as the REM [32] and the  $p$ -spin SK models ( $p \geq 2$ ) [85, 30] endowed with a suitable dynamics have been advocated since the early theoretical work on aging as fruitful testing grounds for the understanding of this phenomenon, giving rise to a host of often drastically simplified theoretical approaches, among which *trap models* have, and continue, to play a key role (see [11, 10, 13]).

In the past two decades, the questions of understanding aging in activated dynamics attracted great interest in the mathematics community. From the bulk of works carried out on mean-field and other models, a *universal aging mechanism* could be identified that links aging of the classical two-time correlation functions to the arcsine law for  $\alpha$ -stable subordinators in the theory of Lévy processes. The purpose of this chapter is to spell out this aging scheme while providing key tools and ideas that enable its effective implementation in the context of mean-field spin glass dynamics.

Let us now introduce the main objects that come into play.

The models. The Hamiltonians we consider are centered Gaussian processes  $(H_N(\sigma), \sigma \in \mathcal{V}_N)$  indexed by the vertices  $\sigma$  (or configurations) of the discrete hypercube  $\mathcal{V}_N = \{-1, 1\}^N$ . The choice of the covariance completely determines the model. Given an integer  $p \geq 2$ , the  $p$ -spin SK models have covariance

$$\mathbb{E}H_N(\sigma)H_N(\eta) = N [R_N(\sigma, \eta)]^p, \quad (2.1)$$

where  $R_N(\sigma, \eta) \equiv \frac{1}{N} \sum_{i=1}^N \sigma_i \eta_i$  denotes the usual normalised overlap. We can see from (2.1) that as  $p$  increases the correlations weaken. Formally taking the limit  $p \rightarrow \infty$  yields a collection of independent Gaussians that defines the Random Energy Model (REM),

$$\mathbb{E}H_N(\sigma)H_N(\eta) = N\delta(\sigma, \eta). \quad (2.2)$$

These sequences of random Hamiltonians are defined on some abstract probability space,  $(\Omega, \mathcal{F}, \mathbb{P})$ .  $\mathbb{E}$  denotes the expectation with respect to  $\mathbb{P}$ . We refer to this space as the *disorder* (also called the *random environment*).

What dynamics to choose? Relevant models of spin glass dynamics are Glauber dynamics on state space  $\mathcal{V}_N$ , reversible with respect to the Gibbs measure at inverse temperature  $\beta > 0$  associated to the random Hamiltonian of the model considered. Typical examples of such dynamics are single spin-flip continuous-time Markov jump processes  $(X_N(t), t > 0)$  on  $\mathcal{V}_N$  whose jump rates  $\lambda_N(\eta, \sigma)$  obey the detailed balance equation

$$e^{-\beta H_N(\sigma)} \lambda_N(\sigma, \eta) = e^{-\beta H_N(\eta)} \lambda_N(\eta, \sigma) \quad \text{if } \sigma \sim \eta \quad (2.3)$$

and  $\lambda_N(\sigma, \eta) = 0$  else, where  $\sigma \sim \eta$  if the configurations  $\sigma$  and  $\eta$  differ in exactly one coordinate. This still leaves considerable freedom of choice. The one that has drawn the most attention and publications to date is the *random hopping dynamics* whose jump rates,

$$\lambda_N(\sigma, \eta) = \frac{1}{N} e^{+\beta H_N(\sigma)} \quad \text{if } \sigma \sim \eta \quad (2.4)$$

only depend on the initial configuration,  $\sigma$ . Although physically unrealistic (the trajectories of this process are independent of the random Hamiltonian), this choice has played an

important role in understanding activated aging. A choice that is advocated in physics as being more satisfactory is that of *Metropolis dynamics* whose rates are given by

$$\lambda_N(\sigma, \eta) = \frac{1}{N} e^{-\beta[H_N(\eta) - H_N(\sigma)]^+} \quad \text{if } \sigma \sim \eta \quad (2.5)$$

and  $\lambda_N(\sigma, \eta) = 0$  else, where  $a+ = \max\{a, 0\}$ . The trajectories of the process now do depend on the random Hamiltonian and are biased against increasing the energy.

Note that the law of the dynamics depends on the disorder. In the sequel we call  $\mathcal{P}_{\pi_N}$  the law of  $X_N$  conditional on the  $\sigma$ -algebra  $\mathcal{F}$ , i.e., for fixed realisation of the disorder, started in the initial condition  $\pi_N$ .

Two-time correlation functions. A natural observable with which to quantify aging is the overlap between the states of the process at two far distant points in time,  $c_N t$  and  $c_N(t+s)$ , where  $c_N$  is the time-scale on which we observe the process. Such an overlap is a random variable that depends both on the disorder and on the process  $X_N$ . Instead of averaging, we fix a realisation of the disorder and choose for two-time correlation function the probability

$$\mathcal{C}_{N,\varepsilon}(t, s) \equiv \mathcal{P}_{\pi_N} \left( R_N(X_N(c_N t), X_N(c_N(t+s))) \geq 1 - \varepsilon \right) \quad (2.6)$$

where  $0 \leq \varepsilon < 1$ . Our aim is then to study the asymptotic behaviour of  $\mathcal{C}_N(t, s)$  when  $N \rightarrow \infty$ , trying to obtain results that are valid for typical realisations of the disorder, namely,  $\mathbb{P}$ -almost surely (i.e., *quenched results* in the terminology of physics). Depending on the model, substitutes for (2.6) can be used, such as the *no-jump* correlation function from trap models

$$\Pi_N(t, s) = \mathcal{P}_{\pi_N} \left( X_N(c_N t) = X_N(c_N(t+u)) \quad \forall t \leq u < t+s \right). \quad (2.7)$$

We will now endow our models with different dynamics and study their correlation functions when the initial distribution  $\pi_N$  is the uniform measure on  $\mathcal{V}_N$  (to mimic a deep quench) and for time-scales  $c_N$  that diverge exponentially fast with  $N$  (this puts us in the setting of *activated dynamics*).

**2.2. The random hopping dynamics of the REM.** Consider the Hamiltonian (2.2) equipped with the dynamics (2.4) and define the one-parameter sequence

$$c_N(\gamma) = \exp \left\{ \beta \gamma N - \frac{\beta}{2\gamma} (\ln(\gamma^2 N/2) + \ln 4\pi) \right\}, \quad \gamma > 0. \quad (2.8)$$

The dynamical phase diagram of the REM on time-scale  $c_N = c_N(\gamma)$  is fully understood for all  $\gamma, \beta > 0$ . Denote by  $\text{Asl}_\alpha$  the probability distribution function of the generalised arcsine law of parameter  $\alpha$ ,  $0 < \alpha < 1$ ,

$$\text{Asl}_\alpha(u) \equiv \frac{\sin \alpha \pi}{\pi} \int_0^u (1-x)^{-\alpha} x^{\alpha-1} dx, \quad 0 \leq u \leq 1. \quad (2.9)$$

**Theorem 2.1.** *Given  $\gamma > 0$  let  $c_N = c_N(\gamma)$ . For all  $\beta$  and  $\gamma$  satisfying*

$$0 < \gamma < \min \left( \beta, \sqrt{2 \ln 2} \right) \quad (2.10)$$

*we have that for all  $\varepsilon \in [0, 1)$ ,  $t > 0$  and  $s > 0$ ,  $\mathbb{P}$ -almost surely*

$$\lim_{N \rightarrow \infty} \mathcal{C}_{N,\varepsilon}(t, s) = \text{Asl}_{\gamma/\beta}(t/(t+s)). \quad (2.11)$$

*The same result holds true for the function  $\Pi_N(t, s)$  defined in (2.7).*

Theorem 2.1 was proved in [60]. It improves on previous results of [8] obtained in a sub-region of (2.10). Furthermore, it was proved in the seminal papers [6, 7] that in the case  $\gamma = \sqrt{2 \ln 2}$  and  $\beta > \sqrt{2 \ln 2}$ , (2.11) remains true albeit in  $\mathbb{P}$ -probability only, on time-scales of the form  $c_N = \rho c_N(\sqrt{2 \ln 2})$ , and taking the double limit  $N \uparrow \infty$  first and  $\rho \downarrow 0$  next. These are the longest time-scales before stationarity. When  $\gamma > \min(\beta, \sqrt{2 \ln 2})$  the process  $X_N$  is stationary. We refer the reader to [60] for exhaustive references and a complete description of the dynamical phase diagram, including its transition lines.

Let us now outline the strategy of the proof of Theorem 2.1.

*Clock process and jump chain.* All Markov jump processes  $X_N$  can be constructed as the time change of a *jump chain* by the inverse of a *clock process*: the former describes the trajectories of  $X_N$  and the latter the time elapsed along them. In the random hopping dynamics, the jump chain  $(J_N(k), k \in \mathbb{N})$  is the simple random walk (SRW) on  $\mathcal{V}_N$  and, setting

$$\tau_N(\sigma) = \exp(-\beta H_N(\sigma)), \quad (2.12)$$

the *clock process* is the partial sum process

$$\tilde{S}_N(k) = \sum_{i=1}^k \tau_N(J_N(i)) e_{N,i}, \quad k \in \mathbb{N}, \quad (2.13)$$

where  $(e_{N,i}, n \in \mathbb{N}, i \in \mathbb{N})$  is a family of independent mean one exponential random variables, independent of  $J_N$ . Then

$$X_N(t) = J_N(\tilde{S}_N^{-1}(t)), \quad t > 0. \quad (2.14)$$

*The arcsine law as a universal aging mechanism.* One readily sees that the two-time correlation function (2.7) can be rewritten as

$$\Pi_N(t, s) = \mathcal{P}_{\pi_N}(\mathcal{R}_N \cap (t, (t+s)) = \emptyset), \quad t, s > 0, \quad (2.15)$$

where  $\mathcal{R}_N \equiv \{c_N^{-1} \tilde{S}_N(k), k \in \mathbb{N}\}$  is the range of the time re-scaled clock process (2.13). This observation places the clock process in the limelight as it is clear that the predicted slowdown of the time de-correlations can only be attributed to some asymptotic ( $N \rightarrow \infty$ ) properties of the clock.

Among the known possible limits, stable subordinators of exponent  $0 < \alpha < 1$  play a special role. These are non-negative processes  $S$  with independent and stationary increments that increase only by jumps, the pairs of their *jump times* and *jump sizes* being given by the points  $\{(t_k, \xi_k)\}$  of a Poisson point process of intensity measure  $dt \times d\nu$ ,

$$S(t) = \sum_{t_k \leq t} \xi_k, \quad t > 0, \quad (2.16)$$

and  $\nu$ , the so-called Lévy measure, takes the specific form

$$\nu(u, \infty) = cu^{-\alpha}, \quad u > 0, \quad (2.17)$$

where  $c > 0$  is a constant and  $0 < \alpha < 1$ . A simple aging mechanism is then provided by the arcsine law, which states that if  $S$  is a stable subordinator of exponent  $\alpha \in (0, 1)$  then, denoting by  $\mathcal{R}$  its range and recalling (2.9)

$$\mathcal{P}(\mathcal{R} \cap (t, (t+s)) = \emptyset) = \text{Asl}_\alpha \left( \frac{t}{t+s} \right), \quad t, s > 0. \quad (2.18)$$

Convergence of clock processes. Our immediate goal is thus to find criteria for the convergence of the clock process (2.13) under proper scaling, i.e., we ask whether there are sequences,  $a_N, c_N$ , such that the process

$$S_N(t) \equiv c_N^{-1} \tilde{S}_N(\lfloor a_N t \rfloor) = c_N^{-1} \sum_{i=1}^{\lfloor a_N t \rfloor} \tau_N(J_N(i)) e_{N,i}, \quad t > 0, \quad (2.19)$$

converges to a subordinator. Note that  $c_N$  is the time-scale on which we observe the process  $X_N$  while  $a_N$  is an auxiliary time-scale associated to the number of steps the jump chain,  $J_N$ , takes. This links aging to a classical and well studied problem of probability theory: given an arbitrary array of positive random variables  $\{Z_{N,i}, i \geq 1, N \geq 1\}$ , find sequences,  $a_N, c_N$ , such that the sequence of partial sum processes

$$S_N(t) = c_N^{-1} \sum_{i=1}^{\lfloor a_N t \rfloor} Z_{N,i}, \quad t > 0 \quad (2.20)$$

converges to a Lévy processes. Among all known convergence criteria, those obtained in [49] are specially well suited to situations where there is a temporal structure as is the case in clock processes. There are three conditions<sup>1</sup>. The first two of them are sufficient conditions for the point process  $\{(i/a_N, Z_{N,i}/c_N)\}$  to converge in a suitable sense to a Poisson point process  $\{(t_k, \xi_k)\}$  with intensity measure  $dt \times d\nu$ . The third one is needed to guarantee that the sum (2.20) converges to the corresponding sum of these Poisson points, and so, to a process of the form (2.16) if the limiting measure  $\nu$  in (2.21) is of the form (2.17).

To illustrate our strategy, let us consider the first of these conditions which contains the most interesting information from the point of view of theoretical physics (see, e.g., [15] for a complete statement or the three conditions). It asks that for all  $t > 0$  and all  $u > 0$ , as  $N \rightarrow \infty$ , in  $\mathcal{P}_{\pi_N}$ -probability,

$$\lim_{N \uparrow \infty} \sum_{i=1}^{\lfloor a_N t \rfloor} \mathcal{P}_{\pi_N} (c_N^{-1} Z_{N,i} > u | \mathcal{F}_{N,i-1}) = t\nu(u, \infty), \quad (2.21)$$

where  $\mathcal{F}_{N,i}$  denotes the  $\sigma$ -algebra generated by the  $Z_{N,j}$ 's with  $j \leq i$ . Going back to the REM, here is how we can implement it. In view of (2.19) we naturally take

$$Z_{N,i} = \tau_N(J_N(i)) e_{N,i}. \quad (2.22)$$

These  $Z_{N,i}$ 's are random variables that depend both on the simple random walk  $J_N$  and on the disorder. We deal with these two randomness as follows. We observe first that  $J_N$  is “fast-mixing”. Indeed, setting  $\ell_N = 2N^2$  and denoting by  $P_{\pi_N}$  the law of  $J_N$  started in the uniform distribution, we have that for any  $\sigma, \eta \in \mathcal{V}_N$  and any  $i \geq 0$  (note that  $J_N$  is 2-periodic),

$$\left| \sum_{q=0}^1 P_{\pi_N} (J_N(\ell_N + i + q) = \eta, J_N(0) = \sigma) - 2\pi_N(\sigma)\pi_N(\eta) \right| \leq 2^{-3n+1}.$$

Assuming that  $a_N \gg \ell_N$ , this enables us to replace our chain-dependent condition (2.21), through an ergodic theorem, by a condition that now only depends on the disorder. Doing

<sup>1</sup>The results of [49] are more general than those we present, which we limit to the convergence to stable subordinators that interest us most.

this, (2.21) becomes

$$[a_N t] \sum_{\sigma \in \mathcal{V}_N} \pi_N(\sigma) Q_N^u(\sigma) \rightarrow t\nu(u, \infty), \quad u > 0, \quad (2.23)$$

where  $Q_N^u(\sigma) \equiv \mathcal{P}_\sigma(Z_{N,i} > c_N u)$  is the tail distribution of  $Z_{N,i}$  when the process  $X_N$  starts in  $\sigma$ . The highlight of this approach is that since  $Q_N^u(\sigma)$  now only depends on the disorder, (2.23) takes the form of a law of large numbers under the law  $\mathbb{P}$ , and this makes it possible to try to obtain almost sure (quenched) results. This in turn will condition the choice of the  $a_N$  and  $c_N$ . Observing that for  $a_N = e^{N\gamma^2/2}$  and  $c_N = c_N(\gamma)$  the re-scaled increments (2.22) are heavy tailed,<sup>2</sup>

$$a_N \mathbb{P}(\tau_N(\sigma) e_{N,i} > c_N u) \sim u^{\gamma/\beta}, \quad (2.24)$$

we readily obtain that, under the assumptions of Theorem 2.1, convergence in (2.23) holds  $\mathbb{P}$ -almost surely to the limiting measure

$$\nu(u, \infty) = u^{\gamma/\beta} (\gamma/\beta) \Gamma(\gamma/\beta), \quad u > 0. \quad (2.25)$$

The remaining two convergence conditions are dealt with in a similar way.

Back to two-time correlation functions. The above strategy enables us to prove that under the assumptions of Theorem 2.1,  $\mathbb{P}$ -almost surely

$$S_N \Rightarrow_{J_1} V_{\gamma/\beta} \quad (2.26)$$

where  $S_N$  is defined in (2.19) and  $V_{\gamma/\beta}$  is the stable subordinator of parameter  $\gamma/\beta$ . The symbols  $\Rightarrow_{J_1}$  denote weak convergence in Skorokhod space equipped with the  $J_1$ -topology. This space is designed to describe the trajectories of stochastic processes admitting jumps, such as those of Poisson processes and Lévy processes. Among the topologies on this space,  $J_1$  is the finest. It guarantees that each jump in the limiting process correspond to exactly one jump before the limit. No other topology fulfils this condition, which is essential if one wants to be able to relate the convergence (2.25) to that of the two-time correlation function (2.15) and, a fortiori, to (2.6).

In the REM, convergence in the  $J_1$ -topology easily follows from the facts that (i) the leading contributions to the clock (2.19) come from the visits of  $J_N$  to configurations such that  $\tau_N(\sigma) \sim c_N$ , which can be seen as *deep traps*, (ii) these deep traps are typically far apart from each other and (iii) after visiting one of them, the simple random walk typically does not revisit it: each jump of the limiting subordinator then corresponds to exactly one such visit.

These and other fine properties of the simple random walk will finally imply that (2.6) and (2.7) are equal in the limit  $N \rightarrow \infty$ .

**2.3. The random hopping dynamics of the  $p$ -spin SK model.** We will now see that the same aging behaviour as that found in the REM is also present in the  $p$ -spin SK model (2.1) equipped with the dynamics (2.4) when  $p \geq 3$ . Let  $c_N(\gamma)$  be as in (2.8).

**Theorem 2.2.** *Given  $\gamma > 0$  let  $c_N = c_N(\gamma)$ . For any  $p \geq 3$ , there exists a function  $\zeta(p)$  such that for all  $\beta$  and  $\gamma$  satisfying*

$$0 < \gamma < \min(\beta, \zeta(p)) \quad (2.27)$$

<sup>2</sup>Note that for this choice,  $c_N$  is the size of the maximum of  $\tau_N$  along trajectories of  $J_N$  of length  $a_N$ . This explains the specific form of (2.8).



we have that for all  $\varepsilon \in (0, 1)$ ,  $t > 0$  and  $s > 0$ ,

$$\lim_{N \rightarrow \infty} \mathcal{C}_{N,\varepsilon}(t, s) = \text{Asl}_{\gamma/\beta}(t/(t+s)). \quad (2.28)$$

Convergence holds  $\mathbb{P}$ -almost surely if  $p > 4$ , and in  $\mathbb{P}$ -probability if  $p = 3, 4$ .

The function  $\zeta(p)$  is increasing and it satisfies

$$\zeta(3) \simeq 1.0291 \quad \text{and} \quad \lim_{p \rightarrow \infty} \zeta(p) = \sqrt{2 \log 2}. \quad (2.29)$$

This quenched result was proved in [15]. In [5] an analogous result was obtained, with the same constants  $\zeta(p)$  and  $K_p$ , but convergence there is in law with respect to the disorder. These results do not cover the case of the SK model ( $p = 2$ ) which seems to belong to a different universality class.

The method of proof is that of Section 2.2. The jump chain and clock process representation (2.13)-(2.14) of  $X_N$  is unchanged. However, because  $H_N$  in (2.12) now is defined through (2.1), the  $\tau_N(\sigma)$ 's are correlated random variables with respect to the disorder, and we no longer can expect the leading contributions to the re-scaled clock process (2.19) to come from widely separated single configurations (deep traps) as in the REM. Instead, such contributing terms form clusters (deep valleys). The idea to deal with this situation is to coarse-grain  $S_N$  over suitable blocks. Namely, we introduce a new scale,  $\theta_N \ll a_N$ , and instead of (2.19) consider the process

$$S_N^b(t) = c_N^{-1} \sum_{i=1}^{\lfloor [a_N t]/\theta_N \rfloor} Z_{N,i}, \quad t > 0, \quad (2.30)$$

where the increments are the block variables

$$Z_{N,i} \equiv \sum_{j=\theta_N(i-1)+1}^{\theta_N i} \tau_N(J_N(i)) e_{N,i}, \quad i \geq 1. \quad (2.31)$$

The point of this procedure is that, using the fast-mixing property of  $J_N$  and choosing  $2N^2 \ll \theta_N \ll a_N$ , the random variables  $Z_{N,i}$  can be made close to independent and identically distributed. The strategy presented in Section 2.2 for proving convergence of clock processes then carries forward unchanged, the main difference being that the distribution  $Q_N^u(\sigma)$  of the increments (2.31) that enters condition (2.23) is now much more complex, and is a random variable depending on the disorder. Nevertheless, along the trajectories of  $J_N$  the shape and width of the deep valleys that give the leading contributions to  $S_N^b$  can be described quite precisely, and the distribution of the  $\tau_N(\sigma)$ 's across these valleys can be well approximated using Gaussian interpolation techniques. This enables us to prove that under the assumptions of Theorem 2.2, choosing  $a_N$  and  $c_N$  as in (2.24) the increments (2.31) are heavy tailed, and  $S_N^b$  converges to a stable subordinator of exponent  $\gamma/\beta$  in Skorokhod space equipped with the  $J_1$ -topology,  $\mathbb{P}$ -almost surely if  $p > 4$ , and in  $\mathbb{P}$ -probability if  $p = 3, 4$ .

Finally, we must prove that the convergence of the coarse-grained process  $S_N^b$  allows to control the two-time correlation function (2.6). This will follow from the fact that in a block of length  $\theta_N$ , essentially all the time is spent in a single visit to quite small ‘‘valley’’, within which the process does not make more than  $o(N)$  steps.

**2.4. Metropolis dynamics of the REM.** The following result proved in [58] establishes that despite their widely different microscopic structure, Metropolis dynamics of the REM with jump rates (2.5) and the random hopping dynamics exhibit the same aging behaviour in the same domain of temperature and time-scale.

**Theorem 2.3.** *Let  $c_N = c_N(\gamma)$ . For all  $\beta$  and  $\gamma$  satisfying*

$$0 < \gamma < \min\left(\beta, \sqrt{2 \ln 2}\right) \quad (2.32)$$

*we have that for  $\varepsilon = 0$  and all  $t > 0$  and  $s > 0$ ,  $\mathbb{P}$ -almost surely*

$$\lim_{N \rightarrow \infty} \mathcal{C}_{N,\varepsilon}(t, s) = \text{Asl}_{\gamma/\beta}(t/(t+s)). \quad (2.33)$$

Although still valid, the jump chain and clock process representation (2.13)-(2.14) of  $X_N$  is no longer efficient in the framework of Metropolis dynamics. Indeed, it was shown in [57] for a truncated REM that due to metastable trapping in local valleys of the energy landscape, the jump chain of Metropolis dynamics is itself an aging process that presents the same complexity as  $X_N$ . In particular, its mixing time is exponential in  $N$ . This calls for a complete rethink of the process representation scheme.

A continuous-time representation scheme. We call *exploration process* any continuous-time Markov jump process  $Y_N$  on  $\mathcal{V}_N$  that has the same trajectories as  $X_N$ . Denoting by  $\tilde{\lambda}_N(\sigma, \eta)$  and  $\lambda_N(\sigma, \eta)$  the jump rates of  $Y_N$  and  $X_N$  respectively, and by  $\tilde{\lambda}_N(\sigma) \equiv \sum_{\eta \sim \sigma} \tilde{\lambda}_N(\sigma, \eta)$  and  $\lambda_N(\sigma) \equiv \sum_{\eta \sim \sigma} \lambda_N(\sigma, \eta)$  their mean holding times at  $\sigma$ , we have

$$X_N(t) = Y_N(\tilde{S}_N^{-1}(t)), \quad t > 0, \quad (2.34)$$

where the now *continuous-time clock process*  $\tilde{S}_N$  is given by

$$\tilde{S}_N(t) = \int_0^t \lambda_N^{-1}(Y_N(s)) \tilde{\lambda}_N(Y_N(s)) ds, \quad t > 0. \quad (2.35)$$

To bring us back to the setting of partial sum processes for which the convergence conditions of [49] apply, we proceed as in the  $p$ -spin SK model, i.e., introduce a new scale,  $\theta_N$ , and coarse-grain the time re-scaled clock process  $S_N(t) \equiv c_N^{-1} \tilde{S}_N(\lfloor a_N t \rfloor)$  over blocks of length  $\theta_N$ . The resulting clock,  $S_N^b(t)$ , takes the form (2.30) with increments

$$Z_{N,i} \equiv \int_{\theta_N(i-1)}^{\theta_N i} \lambda_N^{-1}(Y_N(s)) \tilde{\lambda}_N(Y_N(s)) ds. \quad (2.36)$$

The gist of this construction is that we are now free to choose the exploration process  $Y_N$ , the idea being to choose it in such a way that it mimics the jump chain  $J_N$  of the random hopping dynamics, i.e., we ask that (i) it be fast-mixing and that (ii) its invariant measure  $\mu_N$  resembles the uniform distribution  $\pi_N$  on  $\mathcal{V}_N$ . As before, the mixing condition serves to reduce the clock process convergence conditions to laws of large numbers under the law  $\mathbb{P}$  on the disorder. In particular, (2.21) with increments (2.36) takes the form (2.23) with  $\pi_N$  replaced by  $\mu_N$ . From there, the strategy is unchanged: prove the convergence of the clock  $S_N^b(t)$  in the fine  $J_1$ -topology and then prove that this enables us to control some useful two-time correlation function.

The proof of Theorem 2.3 then relies on detailed information on the properties of the exploration process combined with the knowledge of the valley structure of the energy landscape. This is in sharp contrast with the random hopping dynamics where it suffices

to describe the valley structure along the trajectories of the simple random walk  $J_N$ . Nevertheless, in the REM, questions about the valley structure of the energy landscape can be rephrased in terms of the well-studied graph structure of the percolation cloud in site percolation on  $\mathcal{V}_N$ , and thus, they can be answered fully.

**2.5. Conclusions and perspectives.** We have seen that subordinators arise as universal aging mechanisms within the emblematic class of the  $p$ -spin SK models, including the REM. Beyond models on the hypercube  $\mathcal{V}_N$ , it has also explained aging in a wide class of models and graphs that cover trap models on the complete graph, on hierarchical graphs and on  $\mathbb{Z}^d$  (see [56, 59, 61] and the references therein). Understanding the activated aging behavior of the  $p$ -spin SK model endowed with Metropolis dynamics remains an open problem. Here we are up against the difficulty that we do not as yet have a good enough understanding of the properties of the energy landscape, either rigorously or theoretically, although several recent papers tackle this problem through, in particular, numerical simulations [87, 88, 4, 64]. The subordinator based mechanism is also present in another important family of spin glasses, the GREM, whose analysis has been initiated in [51], paving the way for further progress in the area of models with hierarchically structured energy landscape.

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