The Exponential Function: A Mathemagical Hub

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Abstract

The exponential function, including its real- and complex-valued forms, constitutes one of the most important concepts in mathematics. In this work, the exponential function is approached in a progressive manner. After some brief historical remarks, we start our presentation of the exponential function with its conceptual understanding and definition, following with a succession of several related topics including its properties, related functions, statistical aspects, the complex exponential, geometrical structures, the Fourier series and transform, the concept of instantaneous frequency, the generalization of the Fourier transform into the Laplace transform, the matrix exponential, the logarithm of a matrix, linear dynamic systems, discrete-time linear dynamic systems, and Markov chains. A graph obtained by representing the main relationships addressed in the present work then suggests that the exponential function constitutes its respective hub.

"... the *exponential function* is to the *product* as the *identity function* is to the *sum*."

Excerpt from the present work.

1 Introduction

The field of mathematics, one of the earliest along the history of science, is characterized by the objective quantification and representation of diverse concepts, structures, and dynamics in terms mainly of values (numbers), relationships between these values (e.g. order or identity), transformations between these values as typically implemented by functions, as well as the organization of these values into structures such as constants, variables, vectors, matrices, graphs, etc.

Complemented by logical and computational resources, the above concepts and constructs have provided most of the basis for *scientific modeling*, which is primarily aimed at better understanding and predicting the dynamics of diverse real-world world phenomena. In a sense, mathematics can be understood as providing a framework that is employed to represent and simulate entities and dynamics from the real world, as well as a wide range of interesting theoretical abstractions.

As could be expected, the several mathematical concepts that have been developed along centuries can often be interrelated. For instance, the integer product can be understood in terms of repeated scalar products and sums, a polynomial on a variable x is composed of terms involving powers of x, and the sine and cosine can be derived from the complex exponential. The existence of these multiple relationships is related to possible *ontological frameworks* underlying mathematics. For instance, starting with some basic concepts and hypothesis, other mathematical concepts can then be successively derived through respective relationships, such as in the three above examples.

One particular interesting aspect motivated by the development of possible mathematical ontologies concerns the possible identification of more central, or fundamental core concepts from which several other concepts can be related and derived. Interestingly, the possible identification of these core concepts can not only help our understanding of mathematics (and therefore science), but also provide subsidies for respective effective implementations as well as for unifying seemingly diverse mathematical entities.

Needless to say, the identification of possible mathematical ontologies has constituted a substantial challenge when taken in a systematic manner, aimed at incorporating most mathematical results. One example is Hilbert's program, which was aimed at developing a fully compre-

hensive framework accounting for the whole of mathematics. Though interesting in itself, this type of approach has been constrained by the identification of Gödel's incompleteness theorems, which showed that there are limits to the self-consistency and formal verification of formal mathematical systems.

The present work develops along the above presented motivation of interrelating mathematical concepts, especially the exponential function in its real and complex forms. However, the approach henceforth described in this work is not aimed to be completely formal or totally comprehensive. Instead, it has a simpler and more pragmatic objective that consists in relating the *exponential function* to several other mathematical concepts and results. Though we shall present several such interrelationships, it is by no means implied that the exponential function accounts for most of mathematics.

At the same time, as already observed above, the identification of these interrelationships can contribute in several manners to better understanding and applying mathematics. These potential benefits include a more comprehensive appreciation of mathematical concepts, the development of systems (including hardware) founded on effective implementation and calculation of the exponential function, as well as possibly contributing as a didactic resource in which important mathematical results can be more effectively assimilated through the identified interrelationships.

As presented along the present work, the exponential function bears an almost unending number of relationships with other key mathematical concepts and structures. These interrelationships will be here presented in a progressive, successive manner, starting with the description of the exponential function in its real and complex form, followed by the presentation of its relationship with a number of other mathematical concepts including several basic concepts, series, other functions (including powers, logarithms, sine and cosine), difference and differential equations, analytical geometry, number theory, statistics and distributions, dynamic systems, Fourier series and transforms, the Laplace transform, as well as Markov chains.

2 Some Brief Historic Remarks

The development of the exponential and logarithm concepts extends along a considerable period of time and involved a substantial number of contributors (e.g. [1, 2, 3]). Here we can only provide an extremely brief summary of some of the respective developments that are somehow more directly related to the subjects to be described in the present work.

Possibly, the first approaches to the subjects of logarithms, also including trigonometric functions, was described by the Scottish mathematician John Napier (1550–1617) in his 1614 work *Mirifici logarithmorum canonis descriptio*. However, the value e, now known as the *Euler constant*, was a later contribution by Jacob Bernoulli (1655–1705). In a short period of time, these contributions started being widely adopted as means to facilitate numerical calculations, especially of products, with the help of logarithmic tables. Interestingly, the mechanical device developed by Napier, and known as *Napier's bones* constitutes a resource for calculation of products and quotients between values that is *not* based on the concept of logarithms.

After preliminary related developments by the English mathematician Roger Cotes (1682–1716), the important result now amply known as the *Euler's formula* was described by Leonard Euler (1707–1783) in 1748 in his two-volume work *Introduction in anaysin infinitorum*. The concepts of logarithm and exponential functions, in particular, contributed substantially for establishing relationships with the concept and calculation of the powers and roots, including concerning complex values, especially thanks to developments by Augustin-Louis Cauchy (1789–1857), in his *Cours d'analyze* (1821).

The Fourier series was developed mainly by Jean-Baptiste Joseph Fourier (1768–1830) as a means to solve the heat equation (diffusion) on a metal plate, which he described in his reference work *Mémoire sur la propagation de la chaleur dans les corps solides* (1807).

The development of matrix algebra was to a great extent pioneered by the British mathematician Arthur Cayley (1821-1895), who also employed matrices as resources for addressing linear systems of equations. Cayley focus on pure mathematics included also important contributions to analytic geometry, group theory, as well as in graph theory.

One of the first systematic approaches to the application of matrices to dynamics and differential equations has been developed in the book *Elementary matrices and some applications to dynamics and differential equations* [4], whose first 155 pages present a treatise on matrices, including infinite series of matrices and differential operators. The remainder of the book described the solution of differential equations by using matrices, as well as applications to dynamics of airplanes.

3 The Arithmetic and Geometric Progressions

A real-valued *progression* is a sequence of numbers x_k indexed by a non-negative value $k = 0, 1, 2, \ldots$. There are

two special types of progressions: *arithmetic* and *geometric*.

An *arithmetic progression* is such that:

$$x_k = k\Delta x + a = x_0, x_1, x_2, \ldots \tag{1}$$

where $\Delta x, a \in \mathbb{R}$. Observe that $x_0 = a$.

In other words, each successive element in the sequence can be obtained from the previous by adding a constant amount Δx.

For instance, in case $\Delta x = 2$ and $a = 3$, we have:

$$x_k = 2k + 3 = 3, 5, 7, 9, \ldots \tag{2}$$

Given N elements of an arithmetic progression, the respective *arithmetic series* can be defined as:

$$\sum_{k=0}^{N} x_k = x_0 + x_1 + x_2 + \ldots + x_N \tag{3}$$

The sum of the $(N+1)$ first elements of an arithmetic progression can be immediately calculated as:

$$S_N = \frac{(N+1)(x_N + x_0)}{2} \tag{4}$$

So, in the case of the above example, for $N = 3$, it follows that:

$$S_3 = 3 + 5 + 7 + 9 = \frac{4(9+3)}{2} = 24 \tag{5}$$

A *geometric progression* can be similarly defined respectively to the product as:

$$x_k = r^k \; a = x_0, x_1, x_2, \ldots \tag{6}$$

where $a, r, \in \mathbb{R}$ and $x_0 = a$.

Therefore, in a geometric progression each next element is obtained from the previous by multiplying the latter by a fixed ratio r.

As an example, for $r = 3$ and $a = 2$ we have:

$$x_k = r^k \; a = 3^k \; 2 = 2, 6, 18, \ldots \tag{7}$$

As with arithmetic progressions, *geometric progressions* can be associated respectively to *geometric series*. The sum of the $N+1$ first elements of a geometric progression can be calculated as:

$$S_N = \frac{a\left(1 - r^{N+1}\right)}{1 - r} = \frac{a\left(r^{N+1} - 1\right)}{r - 1} \tag{8}$$

for $r \neq 1$. In case $N \to \infty$, convergence requires that $|r| < 1$.

In the case of the example above, for $N = 2$:

$$S_3 = 2 + 6 + 18 = \frac{2\left(3^3 - 1\right)}{3 - 1} = 26 \tag{9}$$

4 The Exponential Function

The *exponential function* of a generic real-valued variable x is typically expressed as:

$$f(x) = \exp(x) = e^x > 0, \quad x \in \mathbb{R} \tag{10}$$

where $e = 2.71828\ldots$ is the *Euler number*.

Though it is sometimes conceptualized as a function on itself, as suggested by the form $exp(x)$, the exponential function can also be understood as a particular instance of its generalized form:

$$g(x) = c^x, \quad \forall c, x \in \mathbb{R} \tag{11}$$

also with $g(x) = e^x > 0$.

This function, which shall be henceforth referred to as the *general exponential function*, should not be confounded with the *power function* $p(x) = x^c$. Observe that the only difference between Equations 10 and 11 is the fact that the constant c is forced to be equal to e in the former case.

Figure 1 illustrates several examples of the general exponential functions obtained for equally spaced values of c centered at $c = e$. The exponential function for $c = e$ is highlighted in red.

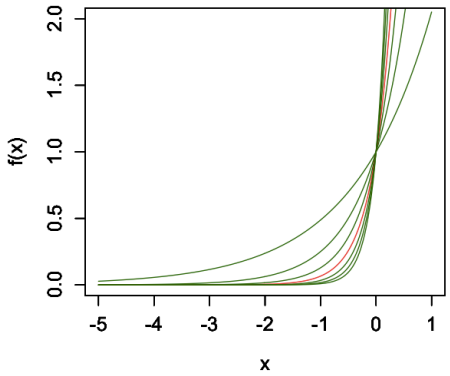

Figure 1: Several instances of the general exponential function obtained for 7 values of the constant c equally distributed (between -2 and 2) around the value $c = e$. The exponential function for $c = e$ is highlighted in red.

The exponential function should not be confounded with the *power function*, which is defined as:

$$h(x) = x^b \tag{12}$$

the main difference being that, in the power function the variable x is taken to the power of b, while in the exponential function the Euler constant is taken to the power of x.

Given a generic function, an immediately related aspect concerns its respective *inverse* $f^{-1}(x)$. The inverses of the

exponential functions correspond respectively to:

$$f^{-1}(x) = \ln x \tag{13}$$
$$g^{-1}(x) = \log_c x \tag{14}$$

which applies for $b \neq 1$, $b, x \in \mathbb{R}^+$, where R^+ is the set of real values x so that $x > 0$. The functionln() is called the *natural logarithm.*

Observe that it is the fact that $f(x) = e^x > 0$ for $x \in \mathbb{R}$ which restricts the domain of inverse of the exponential function to $\mathbb{R}^+$.

One interesting manner to look at the exponential function and its logarithm inverse is as providing a bridge to numeric values that are *extremely large or small*, constituting a kind of telescope/microscope to these realms. For instance we have that:

$$10^{15} = 1.000.000.000.000.000$$
$$log_{10}(1.000.000.000.000.000) = 15$$
$$10^{-21} = 0.000000000000000000001$$
$$log_{10}(0.000000000000000000001) = -21$$

Given a generic logarithmic function $\log_c(x)$, the constant c is said to constitutes its respective *basis*. Therefore, the Napierian logarithm has basis e, with respective logarithm being expressed as:

$$\ln x = \log_e x \tag{15}$$

Recall that logarithms can have their base changed according to the following rule:

$$\log_c x = \frac{\log_b x}{\log_b c} = \frac{1}{\log_b c}\ \log_b x = (constant)\ \log_b x \tag{16}$$

The above results allow us to relate the exponential function e^x to its generic counterpart c^x. More specifically, for $c \in \mathbb{R}^+$:

$$g(x) = c^x > 1, \quad \forall x \in \mathbb{R} \Longrightarrow$$
$$\Longrightarrow \log_c g(x) = x = \frac{\log_e g(x)}{\log_e c} = \frac{1}{\ln c}\ \ln g(x) \Longrightarrow$$
$$\Longrightarrow \ln g(x) = x\ \ln c \Longrightarrow g(x) = e^{x\ \ln c} \Longrightarrow$$

which leads to:

$$\boxed{c^x = e^{x\ \ln c}} \tag{17}$$

This is a particularly important relationship that should be kept in mind. In addition, it can be shown that the exponential function satisfies the following relationship:

$$(e^x)^y = (e^y)^x = e^{x\ y} \tag{18}$$

where $x, y \in \mathbb{R}$. An analogue property holds for the general exponential function, i.e.:

$$(b^x)^y = (b^y)^x = b^{x\ y} \tag{19}$$

It is important to take into account that, in general:

$$(b^x)^y \neq b^{(x^y)} \tag{20}$$

which means that the expression b^{x^y} is intrinsically ambiguous unless some preliminary convention is assumed regarding the order of the powers.

The identity in Equation 17 also allows us, assuming $x \in \mathbb{R}^+$, to relate the exponential and power functions as:

$$x^b = e^{b \ln x} \tag{21}$$

Another particularly relevant property, more formally known as *group homomorphism* (e.g. [5, 6, 7, 8]) established between the additive and multiplicative binary operations, which is satisfied by both the exponential functions, is as follows:

$$\boxed{f(a+b) = f(a)\ f(b) \iff e^{a+b} = e^a\ e^b} \tag{22}$$
$$\boxed{g(a+b) = g(a)\ g(b) \iff c^{a+b} = c^a\ c^b} \tag{23}$$

where $a, b, c \in \mathbb{R}$.

The group property also holds for the respective inverse (logarithmic) functions:

$$\ln(a+b) = (\ln a)(\ln b) \tag{24}$$
$$\log_c(a+b) = (\log_c a)(\log_c b) \tag{25}$$

where $a, b, c \in \mathbb{R}^+$.

We can employ the group property, combined with the property in Equation 18, to better understand the operation $(e^a\ e^b)^c$:

$$(e^a\ e^b)^c = \left(e^{a+b}\right)^c = e^{c(a+b)} = e^{ca}\ e^{cb} \tag{26}$$

where $a, b, c \in \mathbb{R}$.

This means that the power by c is distributive respectively to the product of exponentials. Actually, we also have the more general property that:

$$(a\ b)^c = a^c\ b^c \tag{27}$$

with $a, b, c \in \mathbb{R}$.

Before the advent of electronic computers, this property that was extensively used for calculating products, which was performed by consulting respective tables of logarithms and calculating the simpler sum instead of the potentially much more demanding original product.

The exponential function also has important features which distinguish it from its more general counterpart. One of them concerns its derivative:

$$\frac{d\ e^x}{dx} = e^x \tag{28}$$

$$\frac{d\ c^x}{dx} = \frac{d\ e^{x\ \ln c}}{dx} = (\ln c)\ c^x \tag{29}$$

for $c \in \mathbb{R}^+$.

Therefore, the exponential function has an intrinsically simpler form than its more general counterpart, the latter requiring a non-unit constant.

The intrinsic properties of the exponential function can be better appreciated by considering graphical constructions such as that shown in Figure 2. Here, we have the exponential function e^x in both its continuous form $f(x)$ (blue) as well as being discretized as $F_k = f(x_k)$ (red) at equally spaced points $x_k = x_0 + k\Delta x$, $\forall \Delta x > 0$ and $k = 0, 1, 2, \ldots$, along its domain after having started at an arbitrary position x_0.

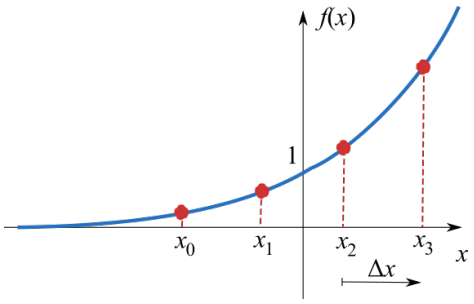

Figure 2: The continuous exponential function $f(x) = e^x$ and a possible respective discretization F_k (red points) defined by abscissae values $x_k = x_0 + k\Delta x$, $\Delta x > 0$ and $k = 0, 1, 2, \ldots$

We have from the property in Equation 22 that:

$$F_k = e^{x_0 + k\Delta x} = e^{x_0}\ e^{k\Delta x} \tag{30}$$

which leads to the following interactions:

$$\begin{aligned}
&F_0 = e^{x_0} \\
&F_1 = e^{x_0+\Delta x} = e^{x_0}\ e^{\Delta x} = F_0\ e^{\Delta x} \\
&F_2 = e^{x_0+2\Delta x} = e^{x_0}\ e^{\Delta x+\Delta x} = F_1\ e^{\Delta x} \\
&F_3 = e^{x_0+3\Delta x} = e^{x_0}\ e^{2\Delta x+\Delta x} = F_2\ e^{\Delta x} \\
&\ldots \\
&F_k = F_{k-1}\ e^{\Delta x} = r\ F_{k-1}
\end{aligned} \tag{31}$$

with $k > 0$ and $r = e^{\Delta x}$.

Equation 31 can be understood as a *geometrical progression* with ratio $r = e^{\Delta x}$, meaning that each successive element along the discretized exponential can be obtained interactively from the previous instance through a simple real multiplication by r.

The sum of the values of the above progression can be readily calculated by the following expression, which involves the own exponential function:

$$S = \sum_{k=0}^{n} F_k = F_0 \frac{r^{n+1} - 1}{r - 1} = e^{x_0} \frac{e^{\Delta X(n+1)} - 1}{e^{\Delta X} - 1} \tag{32}$$

By taking logarithms at both sides of Equation 31, we derive the following *interactive equation*, which holds irrespectively of the choice of $x_0 \in \mathbb{R}$:

$$\ln(F_k) - \ln(F_{k-1}) = \Delta x \tag{33}$$

of which the time-discretized exponential function, by the above developed construction, is intrinsically a solution.

It can be verified that all the above results also hold for $\Delta x < 0$ or $r < 0$, which can also be understood as mirroring of the exponential function with respect to the $y-$ axis.

We have therefore seen in this section that the exponential function, through its basic property in Equation 22, has the intrinsic characteristic, for equally spaced domain points, of defining a respective geometric progression. This also means that each new value of the discretized exponential can be conveniently obtained through interactive real products with the respective ratio $r = e^{\Delta x}$.

This result means, among other things, that the exponential has a relatively simple construction rule that can be understood in direct analogy with another fundamental mapping, namely the *identity function* $h(x) = x$, i.e.:

$$H_k = k\Delta x + H_0 = H_{k-1} + \Delta x \tag{34}$$

with $H_0 = x_0$ and $k = 1, 2, \ldots$.

In this sense, it could be informally said that the *exponential function* is to *geometric series* as the *identity function* is to *arithmetic series*. This would also mean that the *exponential function* is to the *product* as the *identity function* is to the *sum*.

5 Inferring the Exponential Function

In the developments described in the previous section, we started with the exponential function being domain-discretized and then derived the absolute and interactive rules in Equation 22 and 31, respectively. In this section, we will proceed the other way round. More specifically, we will seek for a generic, as yet unknown, function $g(x)$ from which a domain-discrete function $G_k = g(x_k)$ can be obtained, with

$$x_k = k\Delta x + x_0, \quad x_0 \in \mathbb{R}, k = 0, 1, \ldots \tag{35}$$

so that G_k satisfies the following interactive rule for any x_0 and Δx:

$$G_k = r\ G_{k-1}, \quad \forall x \in \mathbb{R}, k = 1, 2, \ldots \tag{36}$$

for $\forall r \in \mathbb{R}^+$ and $G_0 = g(x_0)$.

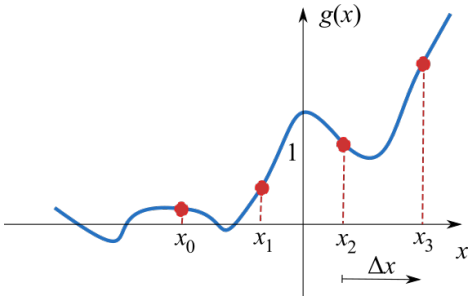

Figure 3: A generic function $g(x)$ (in blue) so that its values G_k (in red) follow a geometric progression, i.e. $G_k = r\ G_{k-1}, \quad \forall x \in \mathbb{R}, k = 1, 2, \ldots$, with $G_0 = g(x_0)$.

It follows immediately from Equation 36 that:

$$G_k = rG_{k-1} = r\,(rG_{k-2}) = \ldots = r^k\ G_0 \tag{37}$$

which holds for $\forall x \in \mathbb{R}, k = 1, 2, \ldots$.
We have from Equation 18 that:

$$G_k = r^k\ G_0 = e^{k \ln r}\ G_0 \tag{38}$$

From Equation 35, it follows that:

$$k = \frac{x_k - x_0}{\Delta x} \tag{39}$$

By substituting this identity into Equation 38, we have:

$$\boxed{G_k = g(x_k) = e^{(\frac{x_k - x_0}{\Delta x}) \ln r}\ G_0} \tag{40}$$

which constitutes the solution, valid for $x_k = k\Delta x + x_0$, of the interactive Equation 36 specifically to x_0 and Δx.

Now, in order the above equation does not depend on Δx, and considering that we are free to chose any value for $r \in \mathbb{R}+$, without loss of generality we can make:

$$r = e^{\Delta x} \Longrightarrow \ln r = \Delta x \tag{41}$$

which leads to the following possible solution of the original problem:

$$G_k = g(x_k) = e^{(\frac{x_k - x_0}{\Delta x})\Delta x} G_0 = e^{x_k - x_0}\ G_0 \Longrightarrow$$
$$\Longrightarrow G_k = g(x_k) = e^{x_k} \frac{G_0}{e^{x_0}} \tag{42}$$

with $k = 1, 2, \ldots$.

Though we know from the start that $g(x_0) = G_0$, we still do not have an explicit form for $g(x_0)$ because the above equation does not cover the case $k > 0$. As we are looking for a generic solution $g(x)$ that does not depend on the choice of x_0, we can take the *ansatz* $G_0 = e^{x_0}$, allowing Equation 42 to be rewritten as:

$$G_k = g(x_k) = e^{x_k} \tag{43}$$

As this result does not depend on x_0 or Δ, we obtain the following generic solution to the initially sought function $g(x)$:

$$g(x) = e^x \tag{44}$$

from which any respectively obtained discrete function G_k will satisfy both Equations 36 and 37 which, as developed in Section 4, turns indeed to be the case.

Though in this section we developed the concept of the exponential function while starting from the geometric progression, there are other pathways leading to that same function. For instance, we could start with the question of which is the function that leads to itself under the important operations of differentiation and integration. The answer would be the exponential function. In addition, since:

$$\frac{de^x}{dx} = e^x \tag{45}$$

it also follows that the exponential function is infinitely differentiable, i.e.:

$$\frac{d^n e^x}{dx^n} = e^x \tag{46}$$

with $n = 1, 2, \ldots$.

6 Some Basic Properties

Given a complex (therefore, including real) function $f(x)$ that has infinite derivatives at a point x_0, its respective *Taylor series* expansion around x_0 is as follows:

$$f(x) = f(x_0) + \frac{\dot{f}(x_0)}{1!}(x - x_0) +$$
$$+ \frac{\ddot{f}(x_0)}{2!}(x - x_0) + \frac{\dddot{f}(x_0)}{3!}(x - x_0) + \ldots \tag{47}$$

where $\dot{f}$ is the first derivative, and so on. When $x_0 = 0$, the Taylor series is often called *McLaurin series*.

The exponential function $f(x) = e^x$ can be expressed in terms of its McLaurin series:

$$f(x) = \exp(x) = e^x =$$
$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \ldots = \sum_{k=0}^{\infty} \frac{x^k}{k!} \tag{48}$$

with infinite radius of convergence, i.e. $\forall x \in \mathbb{R}$, meaning that the exponential function is *entire* (i.e. *holomorphic* in the complex plane).

The series above provides a means to numerically calculate the exponential function in terms of the simpler sum and multiplication operations.

The power series for the *logarithm function* $g(x) \ln(1+x)$ is as follows:

$$g(x) = \ln(x) = \\ = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \ldots = \\ = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{(x-1)^k}{k} \tag{49}$$

which holds for $x \in (0, 2)$, therefore not being *entire* as the exponential function is.

The exponential function has several useful properties. Some of them are presented as follows including, for reference purposes, some already seen in this work¿

$$e^0 = 1 \tag{50}$$
$$e^x > 0, \quad \forall x \tag{51}$$
$$\lim_{x \to -\infty} e^x = 0 \tag{52}$$
$$\lim_{x \to \infty} e^x = \infty \tag{53}$$
$$e^{\ln x} = \ln\left(e^x\right) = x, \quad \forall x \tag{54}$$
$$e^{\alpha(a+b)} = e^{\alpha a} e^{\alpha b}, \quad \forall \alpha, a, b \in \mathbb{R} \tag{55}$$
$$\ln(ab) = (\ln a) + (\ln b), \quad \forall a, b, \in \mathbb{R}^+ \tag{56}$$
$$e^{x \ln b} = b^x, \quad \forall x \in \mathbb{R},\ b \in \mathbb{R}^+ \tag{57}$$
$$e^{ab} = \left(e^b\right)^a = \left(e^a\right)^b, \quad \forall a, b \in \mathbb{R} \tag{58}$$
$$\frac{de^x}{dx} = e^x, \quad \forall x \tag{59}$$
$$\int_{-\infty}^{x} e^x dx = e^x, \quad \forall x \tag{60}$$

An interesting graphical property of the exponential function can is as follows. Let us start with the generic exponential function:

$$g(x) = a\ e^{bx} \tag{61}$$

with $a, b \in \mathbb{R}$.

If we take the natural logarithm at both sides, we obtain:

$$\ln(g(x)) = h(x) = \ln\left(a\ e^{bx}\right) = \ln(a) + bx \tag{62}$$

which corresponds to a *linear function* $h(x)$ on x having *intercept* equal to $\ln(a)$ and *slope* b. That is why the *semi-log* plot of the generic exponential function in Equation 61 will necessarily yield a *straight line*.

Now, let us consider the following generic power function of x, also known as a *power law*:

$$p(x) = a\ x^b \tag{63}$$

In case $x \in \mathbb{R}^+$ (i.e. a positive real value), the previous equation can be rewritten as:

$$p(x) = a\ e^{\ln(x)b} \tag{64}$$

Now, if we take the logarithm at both sides:

$$\ln(p(x)) = \ln(a) + \ln(x)b = c + b \ln(x) \tag{65}$$

That is why the power law in Equation 63 yields a straight line when represented as a *log-log* plog.

7 Some Related Functions

In addition to its several roles in the form $g(x) = be^{ax}$, the exponential function also integrates several other important functions including, but by no means limited to: sigmoid, gaussian, and gamma. Some of these functions are briefly presented in this section. Given that the gamma function involves the complex form of the exponential function, it will be presented in Section 9.

We start with the following function:

$$f(x) = \begin{cases} 1 - e^{-ax}, & x \geq 0 \\ 0, & x < 0 \end{cases} \tag{66}$$

Given that the exponential function $g(x) = e^{ax}$ takes values in the interval $(0, 1]$ for $x \geq 0$, the function $f(x)$ above, henceforth called *complemented exponential*, will also take values in the same interval $(0, 1]$. Figure 4 illustrates this function for several values of a.

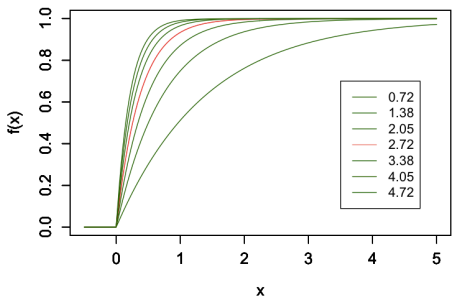

Figure 4: The *complemented exponential function* for seven values of the parameter a, centered at $a = e$. The higher the value of a, the most abrupt the transition from 0 to 1.

Another particularly important function that involves the exponential is the *gaussian*, which can be written as:

$$g_\sigma(x) = a\ e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \tag{67}$$

The parameters μ and σ controls the position of the function along the $x-$axis and its width, respectively.

Observe that the gaussian can be understood as an exponential whose argument has been taken to the second power, therefore involving the *composition* of these two functions. Figure 5 depicts a gaussian function being obtained as the composition of a negative parabola into an exponential function.

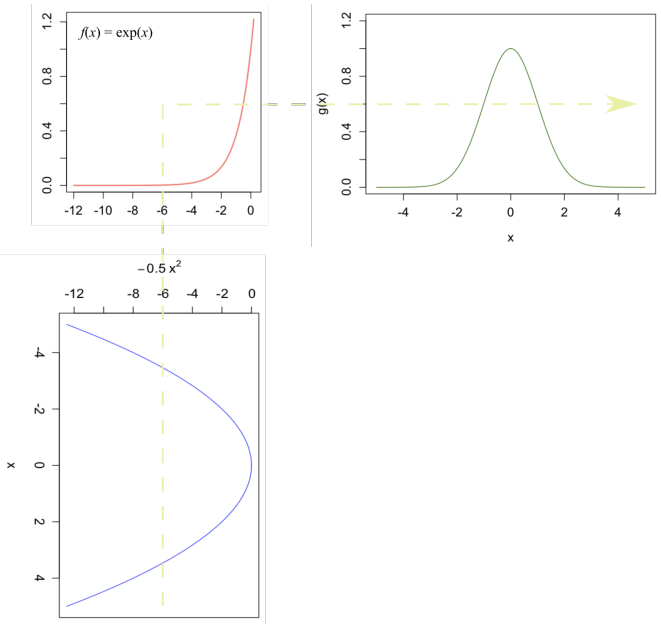

Figure 5: Graphical representation of obtaining a Gaussian function $g(x)$ as the *composition* of the negative parabola $-0.5x^2$ into the exponential function $f(x) = \exp(x)$.

Figure 6 illustrates the gaussian function for $a = 1$, $\mu = 1$, and several values of σ. It should be kept in mind that the area of the gaussian function is not necessarily equal to 1, as is the case with the closely related *normal distribution* to be addressed in Section 8.

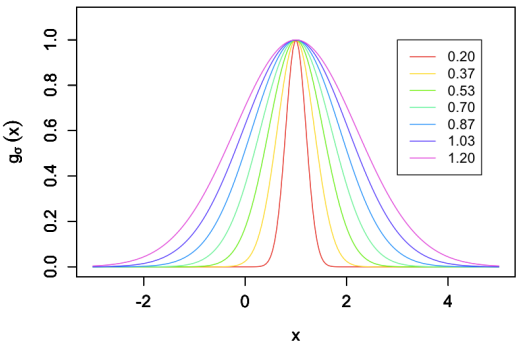

Figure 6: The *gaussian function* for $a = 1$, $\mu = 1$, and several values of σ.

Yet another important function obtained from the exponential function is one of the types of the the *sigmoid function*, which can be expressed as:

$$s(x) = \frac{1}{1 + e^{-ax}} \tag{68}$$

where the parameter a controls how abrupt the transition from 0 to 1 underwent by this function is. Figure 7

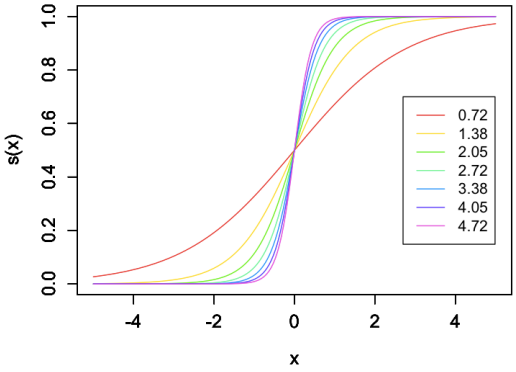

Figure 7: One of the types of sigmoid function for several values of its parameter a, which controls how abrupt the transition from 0 to 1 is.

8 The Exponential Function in Statistics

Jointly with *random variables*, the concept of *distribution functions* plays a key role in probability and statistics (e.g. [9, 10, 11, 12]). More specifically, a function $p(x)$ that satisfies the following requisites can be called a distribution function, or *distribution* for short:

$$p(x) \geq 0, \quad \forall x \in \mathbb{R} \tag{69}$$

$$\int_{-\infty}^{\infty} p(x)dx = 1 \tag{70}$$

where X is the name of the associated random variable, whose values are represented as x.

It follows that any function $g(x) > 0$ with finite area can be, in principle, normalized as a respective distribution by making:

$$p(x) = \frac{g(x)}{\int_{-\infty}^{\infty} g(x)dx} \tag{71}$$

Given a distribution $p(x)$, its *expectancy* can be calculated as:

$$E[X] = \int_{-\infty}^{\infty} x\ p(x)\ dx \approx \mu \tag{72}$$

where μ is the *average* of x as estimated from respective sample. For simplicity's sake, we shall henceforth assume that the average equals the expectancy, i.e. $\mu = E[X]$.

The second central moment of $p(x)$, also called *variance*, can be written as:

$$Var\{x\} = E\left[(X-\mu)^2\right] = E\left[X^2\right] - E\left[X\right]^2 \approx \sigma^2 \quad (73)$$

For simplicity's sake, it is henceforth understood that the estimated variance σ^2 is equal to the second central moment, i.e. $\sigma^2 = Var\{x\}$.

The quantity $\sigma = +\sqrt{\sigma^2}$ corresponds to the *standard deviation* of X.

As could be expected, the exponential function restricted to $x \in R^+$ can be found in several important statistical distributions, some of which are briefly described as follows.

We start by the *exponential distribution*, which can be defined as:

$$p_\alpha(x) = \begin{cases} \alpha\ e^{-\alpha x}, & x \geq 0 \\ 0, & x < 0 \end{cases} \quad (74)$$

$$\mu = \frac{1}{\alpha} \quad (75)$$

$$\sigma^2 = \frac{1}{\alpha^2} \quad (76)$$

Figure reffig:expons illustrates several instances of the exponential distribution respectively to varying values of its parameter α.

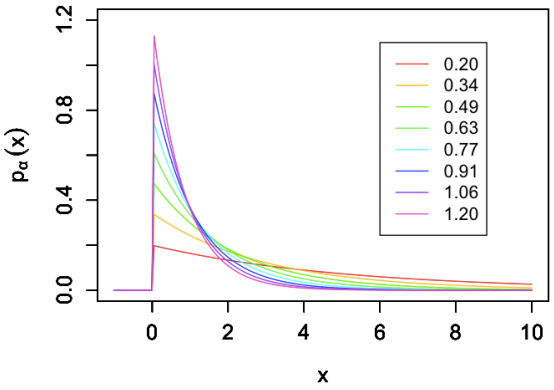

Figure 8: Examples of several exponential distributions for varying values of its parameter *alpha*.

One of the most important statistical distributions, if not the most important, constitutes the *normal distribution*, which can be expressed as:

$$g_\sigma(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad (77)$$

$$(78)$$

The parameters μ and σ are related to the *mean* and *standard deviation* of the respectively associated random variable X.

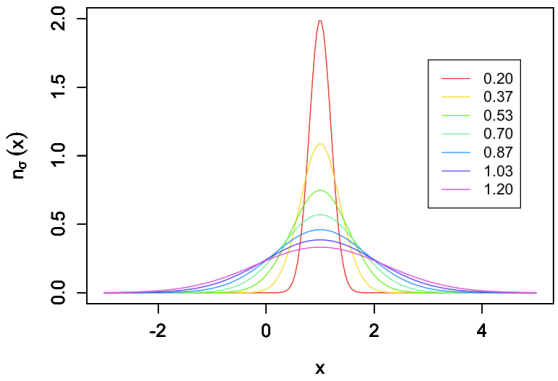

Figure 9: Examples of several normal distributions assuming $\mu = 1$ and varying the parameter σ. All these distributions have area equal to 1. It is interesting to compare these distributions with the respective unnormalized versions in Fig. 6.

Figure reffig:normals shows several instances of the normal distribution respectively to $\mu = 1$ and varying values of its parameter σ.

Interestingly, the *Dirac delta* can be expressed as:

$$\delta(x) = \lim_{\sigma \to 0} n_\sigma(x) \quad (79)$$

this illustrates why the Dirac delta can be understood as having unit area, as this property is preserved as $\sigma \to 0$.

9 The Complex Exponential

The incorporation of complex numbers (e.g. [13, 14, 10, 15]) into mathematics constitutes one of the most important related developments to this area.

It can be shown that the complex exponential adheres to the following power series, which can be understood as its definition:

$$g(z) = g(x,y) = \exp(z) = e^z = \\ = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24} + \ldots = \sum_{k=0}^{\infty} \frac{z^k}{k!} \quad (80)$$

where and $i = \sqrt{-1}$ and x and y are called the real and imaginary parts of the complex number z.

As with the real-valued exponential function, the group property also holds:

$$g(z) = \exp(z) = e^z = e^{x+iy} = e^x\ e^{iy} \quad (81)$$

The *complex power function* is defined in terms of the complex exponential, respectively to a complex value z and $b \in \mathbb{R}$, as:

$$z^b = \exp(b\ \ln(z)),\ \text{for}\ z \neq 0 \quad (82)$$

It should be borne in mind that both the complex logarithm and complex power functions are multiple valued (e.g. [13, 14]).

The *Euler's formula* establishes an important relationship between the complex exponential function and the real-valued *sine* and *cosine* functions:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta) \tag{83}$$

This relation can be readily used to express a generic complex number $z = x + iy$, in terms of its polar representation:

$$z = x + iy = \rho e^{i\theta} = \rho\left[\cos(\theta) + i\sin(\theta)\right] \tag{84}$$

with $\rho \in \mathbb{R}^+$ and $\theta \in \mathbb{R}$ corresponding to:

$$\rho = \sqrt{x^2 + y^2} \tag{85}$$

$$\theta = \arctan\left(\frac{y}{x}\right) \tag{86}$$

where θ is sometimes understood as the *phase* of z. In case we have:

$$u = \rho e^{i(\theta+\phi)} \tag{87}$$

the term ϕ can be understood as a *phase lag*, *phase delay* or *relative phase* (e.g. [16]).

It is interesting to observe that the quantity $\rho = \sqrt{x^2 + y^2}$ is intrinsically related to the concept of *Euclidian distance*, which plays a special role in geometric spaces.

The sine and cosine functions can themselves be derived from the complex exponential function as follows:

$$\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2} \tag{88}$$

$$\sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i} \tag{89}$$

It is also possible to consider the complex exponential defined respectively to a parametrized complex argument $z(t) = x(t) + iy(t)$, with $t = [t_{min}, t_{max}]$ corresponding to the respective *parameter*:

$$\exp(z(t)) = e^{x(t)}\, e^{iy(t)} = e^{x(t)}\left[\cos(y(t)) + i\sin(y(t))\right] \tag{90}$$

In particular, we can start with:

$$\exp(z(t)) = \rho(t)\, e^{i\omega(t)} \tag{91}$$

and make $\rho(t) = 1$ and $\omega(t) = 2\pi f_o t$, which leads to the following complex parametric function:

$$h(t) = e^{i\omega(t)} \tag{92}$$

Interestingly, the Euler relationships indicates that this parametric function $w(t)$ corresponds to an helix with amplitude 1 and frequency f_0 along the $t-$axis. By projecting this complex helix onto the Argand (or complex) plane — with the $x-$ and $y-$ axes corresponding to the real and imaginary portions of the complex values — we get the complex circle with radius 1, which is illustrated in Figure 15.

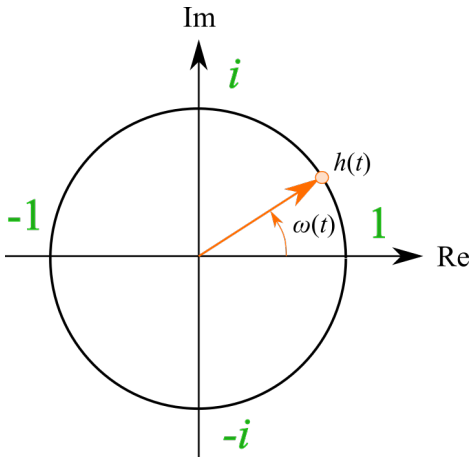

Figure 10: The *complex circle* can be understood as the projection onto the Argand plane of the complex parametric function $h(t) = e^{i\omega(t)}$, which is periodic with period $T_0 = 1/f_0$.

It is interesting to consider the group property of the complex exponential, which accounts for much of its especially important characteristics. Let us start with the following complex values:

$$u = e^{\sigma_u + i\theta_u} = e^{\sigma_u}\, e^{i\theta_u} = x_u + iy_u$$
$$v = e^{\sigma_v + i\theta_v} = e^{\sigma_v}\, e^{i\theta_v} = x_v + iy_v$$

with $\sigma_u, \theta_u, \sigma_v, \theta_v \in \mathbb{R}$. It follows that:

$$u\ v = e^{\sigma_u + i\theta_u + \sigma_v + i\theta_v} = e^{(\sigma_u+\sigma_v)+i(\theta_u+\theta_v)}$$

The resulting complex value is also a complex exponential whose exponent argument has real part corresponding to the sum of the original correlates of u and v, and imaginary part given as the sum of the respective imaginary correlates.

Let us know separate the complex value v into two respective magnitude and phase parts, i.e.:

$$v_M = e^{\sigma_v} = \rho_v \in \mathbb{R}$$
$$v_P = e^{i\theta_v} \in \mathbb{C}$$

with $\sigma_v, \theta_v \in \mathbb{R}$. Now we have from the group property that:

$$u\ v_M = e^{(\sigma_u+\sigma_v)+i\theta_u} = e^{(\sigma_u+\sigma_v)} e^{i\theta_u} \tag{93}$$

Thus, the product of a complex exponential u by a magnitude component v_M has as effect the modification only of the real part of the the former, in a manner that is directly analogous to the real exponential. More specifically, this product will change only the *magnitude* of u.

Similarly, the product of u by the phase component v_P yields:

$$u\ v_P = e^{\sigma_u + i(\theta_u + \theta_v)} = e^{\sigma_u} e^{i(\theta_u + \theta_v)} \tag{94}$$

implying that only the imaginary part of u is changed, without any alteration of its magnitude $\rho_u = e^{\sigma_u}$. More specifically, the phase of u corresponds to the sum of the original phases.

We can therefore conclude that the group property respective to the complex exponential can be understood as two complementary effects corresponding to two complementary effects regarding change of magnitude (analogous to the real exponential) and phase (exclusive to the complex exponential).

Now, let us consider the solution of the following equation, implying the complex circle to cross the $x-$axis at value 1:

$$\begin{aligned} &e^{i\omega(t)} = 1 \Longleftrightarrow \omega(t) = 2k\pi = 2\pi f_0 t, \\ &k = \ldots, -1, 0, 1, \ldots \end{aligned} \tag{95}$$

which is satisfied for $t = k/f_0$, yielding the *period* T_0 of the complex parametric function $h(t)$. This equation highlights the intrinsic relationship between the complex exponential and the fundamental constant π.

The *gamma function* constitutes another example of an important complex function that involves the complex exponential function, through its related complex power form u^{z-1}. More specifically, we have:

$$\Gamma(z) = \int_0^\infty u^{z-1} e^{-u} du \tag{96}$$

where z is a complex number so that $Re(z) > 0$, and u is a real-valued integration variable.

In particular, for a non-negative integer value $n \in \mathbb{R}^+$, we have the following important relationship being established between the gamma function and the *factorial function*:

$$\Gamma(n) = (n-1)! \tag{97}$$

This relationship illustrates the fact that the gamma function can be understood as a generalization of the factorial function to complex values $z | Re(z) > 0$.

Figure 11 presents the factorial of non-negative integer values shown as points overlain onto the continuous gamma function $\Gamma(x)$ restricted to real values $x \in (0, 5.5]$. Observe that, though we have by definition that $0! = 1$, this particular case is not covered by the gamma extension of the factorial concept.

Together with the complex exponential function, the gamma function allows us to approach an expression for

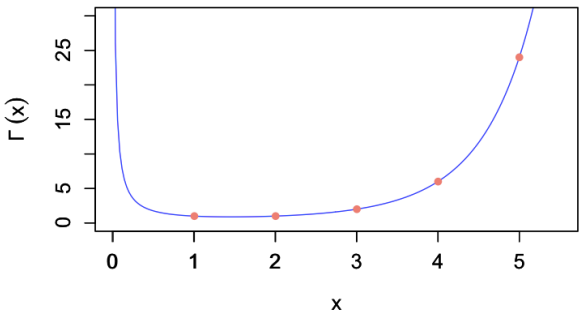

Figure 11: The gamma function $\Gamma(x)$ restricted to real values x along the interval $(0, 5.5]$ and respectively overlain integer-valued factorials (points) of non-negative integers $n = 1, 2, \ldots$ considering $x \in (0, 5.5]$. This construction emphasizes the fact that the gamma function can be used to generalized the factorial function to continuous values (and also to complex numbers).

the following infinite complex series, known as the *Riemann zeta function*:

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \ldots \tag{98}$$

where $s = \sigma + it$.

More specifically, in case $\sigma > 1$, we have that the Riemann zeta function can be calculated as:

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{u^{s-1}}{e^u - 1} du \tag{99}$$

As figured out by Leonard Euler, the Riemann zeta function is related to the concept of *prime number* (e.g. [17]), which plays a central role in *Number Theory* (e.g. [18, 19]), through the following relationship:

$$\zeta(s) = \prod_{p \text{ is prime}} \frac{1}{1 - p^s} \tag{100}$$

Therefore, we have that the complex exponential function is also closely related to fundamental mathematical concepts such as those studied in Number Theory.

10 Exponential Geometry

The exponential function in its complex form can be associated to *parametric curves* (e.g. [20, 21]) in respective geometrical spaces.

A parametric curve $\vec{\gamma}(t)$ can be understood as the mapping of a continuous parameter $t \in [a, b]$ into a continuous succession of points (x, y) in the $\mathbb{R}^2$ space. More specifically, we can write:

$$\vec{\gamma}(t) : t \in [a, b] \longrightarrow \vec{\gamma}(t) = (x(t), y(t)) \tag{101}$$

Given a complex function $g(z(t)) = x(t) + iy(t)$, we can always make the following bijective association:

$$g(z(t)) \longleftrightarrow \vec{\gamma}(t)$$
$$x(t) + iy(t) \longleftrightarrow (x(t), y(t)) \tag{102}$$

For instance, in the case of the helix $h(z(t)) = h(t) = e^{i2\pi f_0 t}$, we would have:

$$h(z(t)) \longleftrightarrow \vec{\gamma}(t)$$
$$\cos(2\pi f_0 t) + i\sin(2\pi f_0 t) \longleftrightarrow (\cos(2\pi f_0 t), \sin(2\pi f_0 t)) \tag{103}$$

Figure 12 illustrates this function for $t \in [0, 4]$ and $f_0 = 2$.

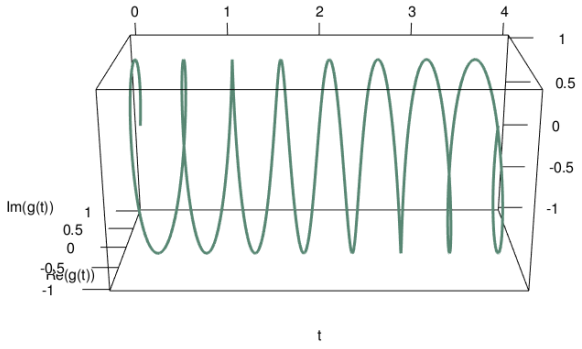

Figure 12: The complex helix $h(t) = e^{i2\pi f_0 t}$ represented as a parametric curve in the $\mathcal{R}^2 \times t$ space for $t \in [0, 4]$ and $f_0 = 2$.

Now, consider the complex parametric function $g(z) = \rho(t)e^{\omega(t)}$, with:

$$\rho(t) = e^{-at} \tag{104}$$
$$\omega(t) = \omega(t) = 2\pi f_0 t \tag{105}$$

This yields a parametric curve in $\mathcal{R}^2 \times t$ which is *modulated* by the varying amplitude corresponding to a decaying exponential $\rho(t) = e^{-1t}$, as shown in Figure 13 respectively to $a = 1$ and $f_0 = 2$.

Figure 14 depicts the same function as in the previous example, but now projected into the $\mathcal{R}^2$ plane.

Given a parametric curve, it is possible to define its respective derivatives for $k = 1, 2, \ldots$ as:

$$\frac{d^k\vec{\gamma}(t)}{dt^k} = \left(\frac{d^k x(t)}{dt^k}, \frac{d^k y(t)}{dt^k}\right) \tag{106}$$

As an example, in the case of the helix in Equation 103, we would have:

$$\frac{d\vec{\gamma}(t)}{dt} = 2\pi f_0 \left(-\sin(2\pi f_0 t), \cos(2\pi f_0 t)\right) \tag{107}$$

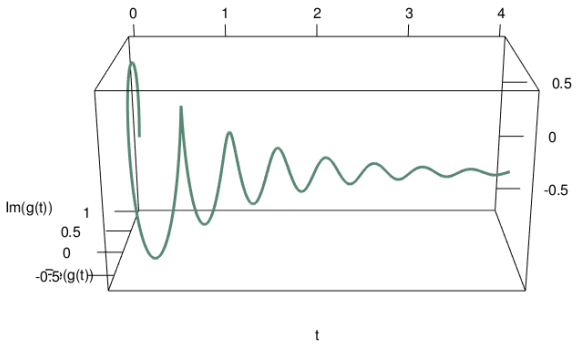

Figure 13: The parametric function defined by the complex parametric function $g(z) = \rho(t)e^{\omega(t)}$ for $a = 1$ and $f_0 = 2$.

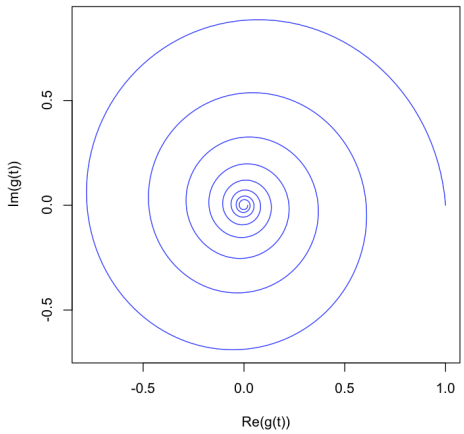

Figure 14: A spiral is obtained by projecting the parametric curve of Fig. 13 into the Argand plane (associated to the $\mathcal{R}^2$ plane).

11 The Fourier Series and Transform

Let $g(t)$ be a real (or complex) function with period T. Then, in case its *Fourier Series* (e.g. [22, 23, 24]) exist, it can be expressed as:

$$c_k = \frac{1}{T}\int_T g(t)\exp\left(-i2\pi kt\right)dt \tag{108}$$
$$g(t) \approx \sum_{k=-N}^{N} c_k \exp\left(i\frac{2\pi kt}{T}\right) \tag{109}$$

where c_k are often called the *Fourier coefficients* respectively to the represented function $g(t)$. The complex Fourier series in Expression 109 provides a means to approximate (and sometimes fully recover) the original function $g(t)$ in terms of a linear combination of its respective coefficients.

The Fourier transform (e.g. [22, 24]) of a non-necessarily periodic function $g(t)$, as well as its inverse,

in case they exist, can be calculated as:

$$G(f) = \mathcal{F}\{g(t)\} = \int_{-\infty}^{\infty} g(t)\exp(-i2\pi ft)dt \tag{110}$$

$$g(t) = \mathcal{F}^{-1}\{G(f)\} = \int_{\infty}^{\infty} G(f)\exp(i2\pi ft)df \tag{111}$$

Given two complex functions $g(t)$ and $h(t)$, their *Hermitian product* can be expressed as:

$$\langle g(t), h(t)\rangle = \int_{-\infty}^{\infty} g(t)\ h^*(t)dt \tag{112}$$

It is interesting to observe an intrinsic analogy between the coefficients c_k in the Fourier series and the Fourier transform $G(f)$. In fact, both these expressions can be rewritten in terms of the Hermitian inner product as:

$$c_k = \frac{1}{T}\langle g(t), \exp(i2\pi kt)\rangle \tag{113}$$

$$G(f) = \langle g(t), \exp(i2\pi ft)\rangle \tag{114}$$

As the real-valued inner product, its Hermitian counterpart can be understood as a measurement of the *similarity* between its two arguments, however influenced by the magnitudes of the respectively compared functions (e.g. [25]). In this sense, the Fourier series and transform can be understood as approximating the original function by linearly combining complex exponentials with diverse frequencies with weights corresponding to the similarity between each of those components and the original function $g(t)$.

Of particular interest is the fact that the basic components used for the linear combinations involved in both the Fourier series and transform correspond to *complex exponential*. The effectiveness of this basis for representation of functions stems from the fact that these complex exponential components are *orthogonal* one another.

As a Fourier transform calculation example, given the truncated (or causal) exponential function:

$$g(t) = \begin{cases} e^{-\sigma t}, & t \geq 0 \\ 0, & t < 0 \end{cases} \tag{115}$$

with $\sigma \geq 0$:

Let us calculate its respective Fourier transform:

$$\begin{aligned} G(f) &= \int_0^{\infty} e^{-\sigma t}e^{-i2\pi ft}dt = \\ &= \int_0^{\infty} e^{-(\sigma+i2\pi f)t}dt = \\ &= \frac{-1}{\sigma+i2\pi f}\ e^{-(\sigma+i2\pi f)t}\Big|_0^{\infty} = \frac{1}{\sigma+i2\pi f} = \frac{1}{s} \end{aligned} \tag{116}$$

with $s = \sigma + i2\pi f = \sigma + i\omega$.

By separating the real and imaginary part, we obtain:

$$\frac{1}{\sigma+i\omega} = \frac{\sigma}{\sigma^2+\omega^2} - i\frac{\omega}{\sigma^2+\omega^2} \tag{117}$$

When the direct and inverse Fourier transforms of a function $g(t)$ exist, we can write the respectively defined *Fourier transform pair* as:

$$g(t) \longleftrightarrow G(g) \tag{118}$$

Some examples of these pairs include:

$$e^{i2\pi f_0 t} \longleftrightarrow \delta(f_0) \tag{119}$$

$$\cos(2\pi f_0 t) \longleftrightarrow 0.5\,[\delta(-f_0) + \delta(f_0)] \tag{120}$$

$$\sin(2\pi f_0 t) \longleftrightarrow 0.5i\,[\delta(-f_0) - \delta(f_0)] \tag{121}$$

where $\delta(f_0) = \delta(f - f_0)$ is the Dirac delta function shifted to the position f_0.

Of particular interest is the Fourier transform property:

$$\frac{d^\beta g(t)}{dt^\beta} \longleftrightarrow (i2\pi f)^\beta\, G(f) \tag{122}$$

which not only allows the generalization of the derivative to complex orders $\beta \in \mathbb{C}$, but also establishes an intrinsic relation between the complex exponential, through its power form, and the derivative of complex functions.

12 The Discrete Fourier Transform – DFT

Let us start with the complex exponential component with period $T_0 = 1/f_0$:

$$h(t) = e^{-i2\pi f_0 t} \tag{123}$$

By quantizing the time parameter t as $t = k\Delta t$, $k = 0, 1, \ldots, N-1$, with $\Delta t = \frac{T_0}{N} = \frac{1}{f_0 N}$:

$$h_{1,k} = h(t = k\Delta t) = e^{-i\frac{2\pi k}{N}} \tag{124}$$

Now, we can define additional complex exponential components h_j with periods $T_j = T_0/j$ that are integer fractions of T_0, which leads to $f_j = jf_0$ and $f_j = j/T_0$. We can now write:

$$h_{j,k} = e^{-i2\pi f_j t} = e^{-i2\pi j f_0 t} = e^{-i\frac{2\pi jk}{N}} \tag{125}$$

for $j = 2, \ldots, N-1$.

Jointly with the constant component $h_{0,k} = 1$, the above components $h_{j,k}$ constitutes the *basis* of the discrete Fourier transform, which can be combined into an $N \times N$ matrix W_N.

Figure 15 illustrates the DFT basis element $h_{1,k}$ for $N = 4$.

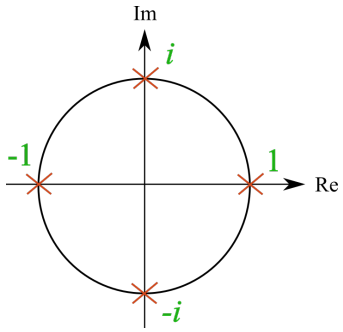

Figure 15: The basis elements $h_{1,k}$, with $k = 0, 1, 2, \ldots N-1$ for $N = 4$.

The DFT matrix for $N = 4$ corresponds to:

$$W_4 = \begin{bmatrix} \leftarrow & h_{0,k} & \rightarrow \\ \leftarrow & h_{1,k} & \rightarrow \\ \leftarrow & h_{2,k} & \rightarrow \\ \leftarrow & h_{3,k} & \rightarrow \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & i \end{bmatrix} \tag{126}$$

Observe that:

$$W_4 = W_4^T \tag{127}$$

which is verified for any N.

The DFT $\vec{G}$ of an N-elements time-discrete function $g(t)$, represented in terms of a respective vector $\vec{g}$, as well as its respective inverse (IDFT), can now be respectively expressed in terms of the following matrix equations (e.g. [22, 23, 26, 27]):

$$\vec{G} = W_N \ \vec{g} \tag{128}$$

$$\vec{g} = W_N^{-1} \ \vec{G} \tag{129}$$

These two expressions, which stem directly from the respective Equations 108 and 110, can again be understood in terms of Hermitian inner products between the components of the Fourier basis (i.e. the complex exponential $e^{i2\pi ft}$) and the time-discrete signal $\vec{g}$ to be analysed/transformed. More specifically, each element j of $\vec{G}$ is obtained by multiplying the respective row j of W_N (which is the conjugate complex of the Fourier components, as required by the Hermitian inner product) and the discrete-time signal $\vec{g} = [g_k] = g(k\Delta t)$.

Given a complex matrix A, it is said to be *unitary* if and only if:

$$A^{-1} = (A^T)^* = (A^*)^T \tag{130}$$

In the case of the DFT matrix W_4, it follows that:

$$W_4 \left[W_4^T\right]^* = W_4 \left[W_4\right]^* = \tag{131}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & i \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & -i \end{bmatrix} = \tag{132}$$

$$= \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} = 4 \ \mathrm{I} \tag{133}$$

from which it is said that W_4 is a *quasi-unitary* matrix. It can be verified that this property applies to any non-negative integer value N, i.e.:

$$W_N \left[W_N^T\right]^* = W_N \left[W_N^*\right]^T = W_N W_N^* = N \ \mathrm{I} \tag{134}$$

The fact of W_N being a quasi-unitary and symmetric matrix allows the effective determination of its inverse as:

$$W_N \left[W_N\right]^* = N \ \mathrm{I} \Longrightarrow \tag{135}$$

$$\Longrightarrow \left[W_N\right]^* = N \ W_N^{-1} \ \mathrm{I} = N \ W_N^{-1} \Longrightarrow \tag{136}$$

$$\Longrightarrow W_N^{-1} = \frac{1}{N} \left[W_N\right]^* \tag{137}$$

Therefore, we can calculate the DFT and IDFT as:

$$\vec{G} = W_N \ \vec{g} \tag{138}$$

$$\vec{g} = \frac{1}{N} \left[W_N\right]^* \ \vec{G} \tag{139}$$

13 Instantaneous Frequency

The concept of *instantaneous frequency* (e.g. [28, 29, 30]) provides and interesting and useful extension of the more traditional concept of frequency of a signal. The definition of the instantaneous frequency of a signal $s(t)$, a purely real function of an independent variable f having $S(f)$ as its respective Fourier transform, relies on the respective *analytical representation* of $s(t)$ as:

$$s_a(t) = \mathcal{F}^{-1} \left\{2 \ h(f) S(f)\right\} \tag{140}$$

where $h(f)$ is the *heaviside function*:

$$h(f) = \begin{cases} 0, & f < 0 \\ 1/2, & f = 0 \\ 1, & f > 0 \end{cases} \tag{141}$$

The *instantaneous amplitude* and *instantaneous phase* of $s(t)$ can now be defined as:

$$\begin{cases} A_i(t) = |s_a(t)| \\ \phi_i(t) = \arctan \left\{ \frac{Im(s_a(t))}{Re(s_a(t))} \right\} \end{cases} \tag{142}$$

from which the *instantaneous frequency* of $s(t)$ can be expressed as:

$$f_i(t) = \frac{1}{2\pi}\frac{d}{dt}\ \phi_i(s(t)) \tag{143}$$

As an example of calculation of the instantaneous frequency of a function, consider $g(t) = \cos(2\pi f_0 t)$. First, we obtain the respective analytical representation as:

$$g_a(t) = \mathcal{F}^{-1}\left\{2\ h(f)G(f)\right\} = \mathcal{F}^{-1} 0.5\delta(f_0) = 0.5e^{i2\pi f_0 t} \tag{144}$$

from which the respective *instantaneous phase* can be obtained as:

$$\phi_i(t) = \arctan\left\{\frac{Im(s_a(t))}{Re(s_a(t))}\right\} = \frac{\sin(2\pi f_o t)}{\cos(2\pi f_0 t)} = 2\pi f_0 t \tag{145}$$

leading to the following instantaneous frequency:

$$f_i(t) = \frac{1}{2\pi}\frac{d}{dt}\ \phi_i(s(t)) = \frac{1}{2\pi}\frac{d}{dt}\ 2\pi f_0 t = f_0 \tag{146}$$

Therefore, as could be expected, the cosine function $g(t) = \cos(2\pi f_0 t)$ has instantaneous frequency f_0 at any value of $t \in \mathbb{R}$. However, the instantaneous frequency of more general signals will often yield varying values (e.g. [30]), being closely related to the concept of *amplitude modulation* – AM, as used in telecommunications.

14 The Laplace Transform

Given a complex function $g(s)$ of a complex variable s, its respectively *Laplace transform* (e.g. [31]) can be defined as follows:

$$\mathcal{L}\left\{f(t)\right\}(s) = F(s) = \int_0^\infty g(s)\ e^{-st} dt \tag{147}$$

where $s(t) = \sigma + i\omega$, with $\sigma, \omega \in \mathbb{R}$.

Observe that the Laplace transform basis components can be rewritten as:

$$e^{-st} = e^{-\sigma t}\ e^{-i\omega t} \tag{148}$$

thus indicating that the Laplace transform constitutes a generalization of the Fourier transform incorporating components with exponential decay $e^{-\sigma t}$, which contributes to the effective representation of a wider range of possible signals.

In case $g(s) = g(t)$ is a function of a a non-negative real argument t so that:

$$g(t) = \begin{cases} g(t), & t \geq 0 \\ 0, & t < 0 \end{cases} \tag{149}$$

we can write:

$$\begin{aligned}\mathcal{L}\left\{g(t)\right\}(s) &= \int_0^\infty g(t)\ e^{-\sigma t}\ e^{-i\omega t} dt = \\ &= \int_0^\infty \left[g(t)\ e^{-\sigma t}\right]\ e^{-i\omega t} dt = \\ &= \int_{-\infty}^\infty \left[g(t)\ e^{-\sigma t}\right]\ e^{-i\omega t} dt = \\ &= \mathcal{F}\left\{g(t)\ e^{-\sigma t}\right\}(f)\end{aligned} \tag{150}$$

As an example, let us calculate the Laplace transform of the truncated exponential function:

$$g(t) = \begin{cases} e^{-at}, & t \geq 0 \\ 0, & t < 0 \end{cases} \tag{151}$$

By applying the definition:

$$\begin{aligned}\mathcal{L}\left\{f(t)\right\}(s) &= \int_0^\infty e^{-at}\ e^{-st} dt = \\ &= \int_0^\infty e^{-(a+s)t} dt = \\ &= \frac{-1}{a+s}\ e^{-(a+s)t}\Big|_0^\infty = \frac{1}{s+a}\end{aligned} \tag{152}$$

for $Re(s) < -a$.

Similarly to the Fourier transform, the Laplace transform is also directly related to the operations of differentiation and integration:

$$\int_0^t f(t)dt \longleftrightarrow \frac{F(s)}{s} \tag{153}$$

$$\frac{df(t)}{dt} \longleftrightarrow sF(s) - f(0) \tag{154}$$

15 Ordinary Differential equations

Differential equations are at the core of physical modeling, as they provide an effective representation of several phenomena in terms of instantaneous properties such as position and velocity (e.g. [32]). Among the several types of differential equations, those called *ordinary differential equations* – ODEs – are frequently adopted in modeling approaches. Basically, this type of differential equations is characterized by the involved unknown elements being a function of a single variable, such as time or *position*.

In the present work, we shall focus on *linear ordinary differential equations*, which are characterized by satisfying the *superimposition principle* (e.g. [33]). In particular, consider the following linear ODE:

$$\dot{g}(t) = a\ g(t) \tag{155}$$

$$g(t = t_0) = g_0 \tag{156}$$

which means that the tendency (or velocity) of the increase of the variable $g(t)$ is proportional, through t the constant $a \in \mathbb{R}$, to its current value. In the case of $a > 0$, we would have that the quantity $g(t)$ tends to increase at a rate $dg(t)/dt$ proportional to its current value. The initial condition, $g(t = t_0) = g_0$, informs the value of the variable $g(t)$ at an initial time instant t_0.

Let us solve this equation as follows:

$$\begin{aligned}
&\frac{dg(t)}{dt} = a\ g(t) \implies \\
&\implies \frac{1}{g(t)}\ dg = a\ dt \implies \\
&\implies \int \frac{1}{g(t)}\ dg = \int a\ dt \implies \\
&\implies \ln(g(t)) = a\ t + c \implies \\
&\implies e^{\ln(g(t))} = e^{a\ t+c} \implies \\
&\implies g(t) = e^c\ e^{at} \qquad (157)
\end{aligned}$$

Observe that this *general* solution depends on the constant $c \in \mathbb{R}$, which is not yet determined. By taking into account the initial condition, we can write:

$$\begin{aligned}
&\implies g(t_0) = g_0 = e^c\ e^{at_0} \implies \\
&\implies e^c = g_0\ e^{-at_0} \implies \\
&\implies c = \ln\left(g_0\ e^{-at_0}\right) \implies \\
&\implies c = \ln\left(g_0\right) - at_0 \qquad (158)
\end{aligned}$$

Thus, the solution of the original differential equation can be expressed as:

$$\begin{aligned}
&g(t) = e^{\ln(g_0)-at_0}\ e^{at} \implies \\
&\implies g(t) = g_0\ e^{-at_0}\ e^{at} \implies \\
&\implies g(t) = g_0\ e^{a(t-t_0)} \qquad (159)
\end{aligned}$$

Therefore, we have that the solution of the possibly most basic linear ordinary differential equation involves the exponential function. In other words, the exponential function is such that its derivative is proportional to itself, therefore constituting a kind of *eigenfunction* associated to the original differential equation 155.

Interestingly enough, as we will see in Section 19, an analogue solution can be found for a system of linear ordinary differential equations. However, it is necessary to get acquainted with the useful concept of matrix exponential as a preparation for that, which provides the subject for the following section.

16 Exponential Directional Field

Given an ordinary differential equation on a single variable x, the concept of *directional field* provides a particularly interesting resource for visualizing, and better understanding, the characteristics of the given differential equation as well as its possible solutions.

Let us consider the following linear ODE:

$$\frac{dx(t)}{dt} = \dot{x}(t) = x(t) \qquad (160)$$

Its respective directional field can be obtained by plotting a short line segment having inclination corresponding to the derivative $\dot{x}(t, x)$ at each of a set of respective discrete lattice points (t, x) as determined by the original differential equation, therefore defining a respective tangent field. Figure 16 illustrates a directional field (set of green line segments) obtained for the above equation considering $t \in [-1, 1]$ and $x \in [-1, 1]$.

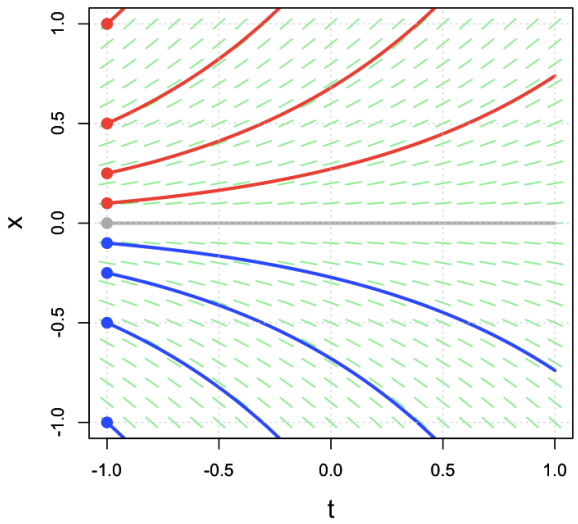

Figure 16: A directional field, corresponding to the set of green line segments, obtained for Eq. 160. The slope (inclination) of each of these segments corresponds to derivative $\dot{x}(t, x)$ of $x(t)$ at each of the considered discrete points (t, x). Given an initial condition (starting points at the left-hand side of the plot), the respective solution can be obtained by following the tangent field.

Observe that, in the case of this particular ODE, the derivatives $\dot{x}(t, x)$ at the points (t, x) do not depend of t, but only on x, in which case the differential equation is said to be *autonomous*.

Also shown in Figure 16 are several possible solutions of the considered differential equation, obtained respectively to each initial conditions specified by the left-hand points. These solutions are obtained by following the tangent field as one moves along the horizontal axis.

The solution shown in gray, respective to the initial condition $(x(t = -1) = 0)$, corresponds to the only equilibrium solution of the considered ODE. More specifically an *equilibrium solution* is such that $\dot{x} = 0$ for any value of t. In the case of this particular ODE, the only equilibrium solution can be said to be *unstable*, since any small perturbation along its trajectory will unavoidably cause it to

either increase or decrease, which it will do in exponential manner.

As illustrated by the above example, the directional field of a given ODE provides a direct visualization of all possible solutions in the considered window. Additional examples of directional fields, also including systems of two differential equations, can be found in [34].

17 The Matrix Exponential

In a direct analogy to Equation 48, it is possible to express the exponential of a *complex matrix* A, with dimension $N \times N$ in terms of the following power series:

$$e^A = 1 + x + \frac{A^2}{2} + \frac{A^3}{6} + \ldots = \sum_{k=0}^{\infty} \frac{A^k}{k!} \tag{161}$$

Observe that the exponential of the matrix A is itself a matrix with the same dimension as the matrix A from which it derives.

As an example, let:

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 2 & 2 & 2 & -1 \\ 0 & 0 & 0 & 3 \end{bmatrix} \tag{162}$$

From Equation 161, we have that:

$$e^A \approx \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 2 & 2 & 2 & -1 \\ 0 & 0 & 0 & 3 \end{bmatrix} + \ldots =$$

$$= \begin{bmatrix} 2.43 & 2.06 & 1.43 & 6.36 \\ 4.93 & 5.30 & 4.93 & 0 \\ 12.7 & 12.7 & 13.7 & -6.36 \\ 0 & 0 & 0 & 20.1 \end{bmatrix}$$

where $\approx$ stands for the fact of presenting truncated values in the result, and/or taking a finite number of power series terms.

As could be expected, the matrix exponential and the logarithm of a matrix (to be briefly addressed in Section 18) are related as:

$$b^A = e^{A \ln b} \tag{163}$$

The following property is observed by the matrix exponential:

$$e^{(A^T)} = \left[e^A\right]^T \tag{164}$$

In the case of two complex square matrices A and B with the same dimension, we also have that:

$$e^{A+B} = e^A \; e^B \tag{165}$$

which leads to:

$$e^A \; e^{-A} = e^{A-A} = I \tag{166}$$

from which we have that the inverse of matrix e^A is:

$$\left[e^A\right]^{-1} = e^{-A} \tag{167}$$

Interestingly, the matrix corresponding to the matrix exponentiation e^A is always invertible.

18 The Logarithm of a Matrix

Let A be a complex matrix. In case it is invertible, it is possible to obtain its respective *logarithm* in terms of the following power series:

$$g(A) = \ln(A) = \\ = (A - I) - \frac{(A-I)^2}{2} + \frac{(A-I)^3}{3} - \ldots = \\ = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{(A-I)^k}{k} \tag{168}$$

We have that:

$$e^{\ln(A)} = A \tag{169}$$

The properties of the logarithm of matrix are relatively more intricate than those of the logarithm function, often involving additional requirements as matrices being positive definite, etc.

19 Linear Dynamic Systems

Linear dynamic systems are characterized by obeying the superimposition principle. More specifically, if $\vec{x}$ and $\vec{y}$ are solutions of a given such type of system, $\vec{x} + \vec{y}$ will also be a solution.

Linear dynamic systems are often represented in terms of respective systems of (in our case ordinary) differential equations, such as the following system of homogeneous differential equations:

$$\begin{cases} \dot{x}_1 = a_{1,1}x_1 + a_{1,2}x_1 + \ldots + a_{1,N}x_N \\ \dot{x}_2 = a_{2,1}x_1 + a_{2,2}x_1 + \ldots + a_{2,N}x_N \\ \ldots \\ \dot{x}_N = a_{N,1}x_1 + a_{N,2}x_1 + \ldots + a_{N,N}x_N \end{cases} \tag{170}$$

which can be directly translated to the following matrix equation:

$$\dot{X}(t) = AX(t) \tag{171}$$

$$X(t_0) = X_0 \tag{172}$$

with:

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,N} \\ a_{2,1} & a_{2,2} & \dots & a_{2,N} \\ \dots & & & \\ a_{N,1} & a_{N,2} & \dots & a_{N,N} \end{bmatrix} \tag{173}$$

As with the linear ODE $\dot{g}(t) = ag(t)$ seen in Section 15, the solution of the above system can be obtained as:

$$\vec{x}(t) = e^{A(t-t_0)}\ \vec{x}_0 \tag{174}$$

Figure 17 presents the solution of the system of linear, homogenous differential equations specified by the following matrix and considering $\vec{x}(0) = [-1\ 1\ 0\ 2]^T$ for $t \in [0, 5]$:

$$\begin{bmatrix} -0.2 & -0.4 & -0.1 & 0.2 \\ -0.4 & -0.6 & -0.3 & -0.1 \\ -0.1 & -0.3 & 0.4 & -0.1 \\ 0.2 & -0.1 & -0.1 & -0.6 \end{bmatrix} \tag{175}$$

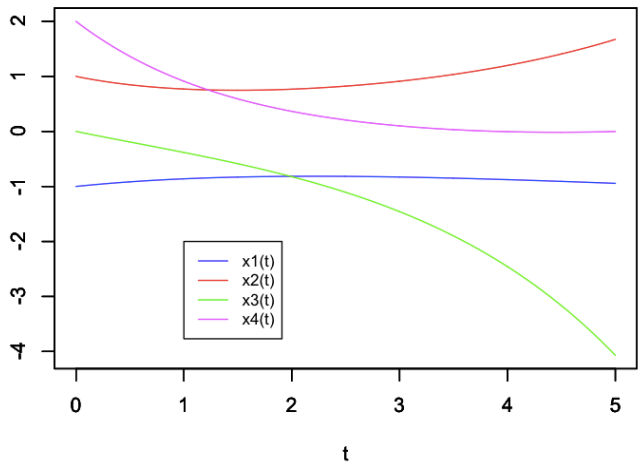

Figure 17: Solution of the system of linear, homogenous differential equations specified by the matrix in Eq. 175, obtained by using Eq. 174, considering $\vec{x}(0) = [-1\ 1\ 0\ 2]^T$ adn $t \in [0, 5]$.

In case an $N \times N$ matrix A is symmetric and has N distinct real-valued eigenvalues λ_i associated to respective eigenvectors $\vec{v}_i$ (e.g. [35, 36]), the solution (e.g. [33]) of the respectively defined homogeneous system of linear differential equations can be written as:

$$\vec{x}(t) = \\ = c_1 e^{\lambda_1 (t-t_0)} \vec{v}_1 + c_2 e^{\lambda_2 (t-t_0)} \vec{v}_2 + \ldots + c_N e^{\lambda_N (t-t_0)} \vec{v}_N \tag{176}$$

where:

$$\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_N \end{bmatrix} = \begin{bmatrix} \uparrow & \uparrow & \dots & \uparrow \\ \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_N \\ \downarrow & \downarrow & \dots & \downarrow \end{bmatrix}^{-1} \begin{bmatrix} x_{0,1} \\ x_{0,2} \\ \vdots \\ x_{0,N} \end{bmatrix} \tag{177}$$

20 Discrete-Time Linear Dynamic Systems

Given a homogeneous linear dynamic system as in the previous section (Eq. 170), it is possible to write its (possibly approximate) interactive finite difference (order 1) equation as follows:

$$\begin{cases} x_1^{[t+\Delta t]} = \left(a_{1,1} x_1^{[t]} + \ldots + a_{1,N} x_N^{[t]}\right) \Delta t + x_1^{[t]} \\ x_2^{[t+\Delta t]} = \left(a_{2,1} x_1^{[t]} + \ldots + a_{2,N} x_N^{[t]}\right) \Delta t + x_2^{[t]} \\ \dots \\ x_N^{[t+\Delta t]} = \left(a_{N,1} x_1^{[t]} + \ldots + a_{N,N} x_N^{[t]}\right) \Delta t + x_N^{[t]} \end{cases}$$

assuming that the time values t are organized as $t^{[k+1]} = t^{[k\Delta t + t_0]}$. In addition, the initial state is represented as $X_0 = X^{[t=t_0]}$.

The above finite difference approximation can be obtained simply by considering that:

$$\frac{dx(t)}{dt} = \lim_{\Delta x \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t} \approx \frac{x(t+\Delta t) - x(t)}{\Delta t} \tag{178}$$

where tha approximation assumes small values of Δt.

The system of interactive equations in Equation 178, also called a *map*, can be compactly expressed in terms of the following matrix equation:

$$\vec{x}^{[t+\Delta t]} = A\vec{x}^{[t]}\Delta t + \vec{x}^{[t]} = (A\Delta t + I)\ \vec{x}^{[t]} \tag{179}$$

$$\vec{x}^{[t_0]} = \vec{x}_0 \tag{180}$$

By making $C = (A\Delta t + I)$:

$$\vec{x}^{[t+\Delta t]} = C\vec{x}^{[t]} \tag{181}$$

$$\vec{x}^{[t_0]} = \vec{x}_0 \tag{182}$$

Which has the following solution:

$$\vec{x}^{[t]} = e^{\ln C \frac{(t-t_0)}{\Delta t}}\ \vec{x}_0 = C^{\frac{t-t_0}{\Delta t}}\ \vec{x}_0 \tag{183}$$

21 Markov Chains

In case the homogeneous system of linear ODEs in the previous section has $\Delta t = 1$ and t_0 is an integer value, we can write:

$$\vec{x}^{[t+1]} = A\vec{x}^{[t]} + \vec{x}^{[t]} = (A + I)\vec{x}^{[t]} \tag{184}$$

$$\vec{x}^{[t_0]} = \vec{x}_0 \tag{185}$$

By making $B = A + I$, we get the mapping:

$$\vec{x}^{[t+1]} = B\vec{x}^{[t]} \tag{186}$$

$$\vec{x}^{[t_0]} = \vec{x}_0 \tag{187}$$

In case the square matrix B is a *stochastic matrix*, having non-negative entries so that the sum of respective columns are all equal to 1, the above mapping can be understood as a *Markovian chain* (e.g. [37, 38]), with the following solution:

$$\vec{x}^{[t]} = e^{\ln B(t-t_0)}\ \vec{x}_0 = B^{t-t_0}\ \vec{x}_0 \tag{188}$$

A stochastic matrix can be shown always to have an eigenvalue equal to 1. A Markov chain associated to an *irreducible* matrix B will converge to an equilibrium (or stationary) state given by the respective eigenvector associated to the eigenvalue 1. The matrix B will be said to be irreducible when the respectively derived digraph is not strongly connected, i.e. it is possible to reach any of its nodes through paths starting at any other possible node.

In case matrix P can be diagonalized, or eigendecomposed, we can write:

$$VB = V\Lambda \implies \tag{189}$$

$$\implies B = V\Lambda V^{-1} \tag{190}$$

where Λ is a diagonal matrices containing the eigenvalues λ_i of B in descending order and V is the matrix composed having the respective eigenvectors $\vec{v}_i$ as columns.

We have the following interesting property:

$$B^n = V\Lambda\ (V^{-1}V)\ \Lambda\ (V^{-1}V)\ \Lambda \ldots \Lambda\ (V^{-1}V)\ \Lambda V^{-1} = \\ = V\left[\Lambda\right]^n V^{-1} \tag{191}$$

with:

$$\Lambda^n = \begin{bmatrix} \lambda_1^n & 0 & \ldots & 0 \\ 0 & \lambda_2^n & \ldots & 0 \\ \ldots & & \ddots & \ldots \\ 0 & 0 & \ldots & \lambda_N^n \end{bmatrix} \tag{192}$$

This yields the following discrete-time solution of the Markov chain associated to B:

$$\vec{x}^{[t]} = \left[a_1\lambda_1^t\vec{v}_1 + a_2\lambda_2^t\vec{v}_2 + \ldots + a_N\lambda_N^t\vec{v}_N\right] \tag{193}$$

where the constants $a_i \in \mathbb{R}$ are determined from the initial state $x^{[0]}$. This interesting result confirms that the equilibrium dynamics is defined by the eigenvector $\vec{v}_1$ associated to the eigenvalue with maximum value ($\lambda_1 = 1$). It is also of interest to compare the equation above with that in Equation 176.

22 Concluding Remarks

In addition to its own theoretical importance, mathematics has provided the main foundation for modeling in the physical sciences. To a great extent, this key role has been allowed by the intrinsic potential of mathematical concepts and structures to represent entities and phenomenal from the physical world.

In this work, after developing the concept of the exponential function, we successively presented its direct implications and relationships with several important, or even central, mathematical concepts ranging from fundamental constants to the operations of complex differentiation and integration, while being related even to number theoretical concepts as prime numbers. Figure 18 presents a graph, or network, obtained by representing several of the concepts addressed in the present work as nodes, while many of their interrelationships have been indicated by respective links.

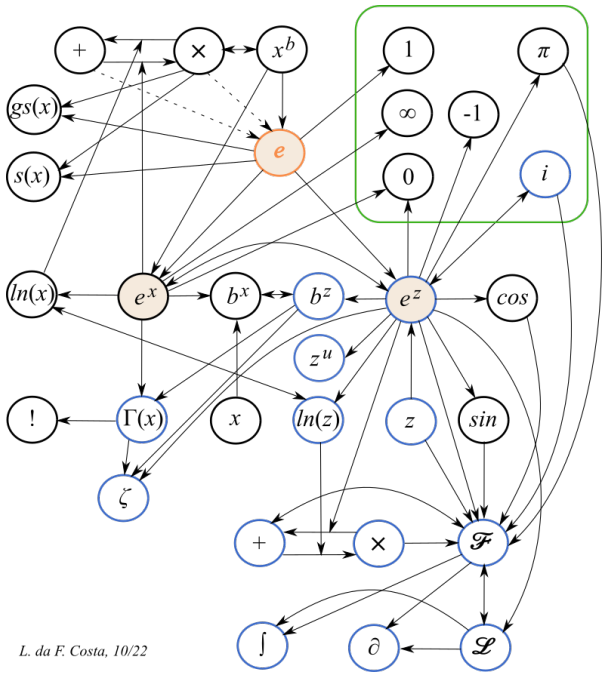

Figure 18: A graph illustrated some of the arithmetic, algebraic and functional relationships between the exponential function and several other mathematic concepts. Five fundamental constants, as well as the infinity are shown within the green box The Gaussian function is shown as $gs()$, the sigmoid function as $s()$, and the Fourier series (or transform) as $\mathcal{F}$. The zeta function is shown as ζ, the Laplace transform as $\mathcal{L}$, differentiation as ∂ and integration as $\int$. Complex values and functions are shown as blue nodes, and z and u are complex variables, while b and x are a real variables. The node associated to the complex exponential function e^z constitutes the main hub in the presented graph. Arguably, the exponential function directly relates to a considerable breadth of mathematical concepts.

Despite its relatively recent history, the exponential and its logarithm inverse are related to an impressive range of key mathematical concepts. For instance, several of the fundamental mathematical functions, shown within the green box in Figure 18, are directly related to the complex exponential function. In addition, several important

functions can be derived from the exponential function, as well as the Fourier and Laplace transforms, which allow important connections with the operations of complex differentiation and integration.

Thus, in a sense, a wide range of key mathematical concepts can be directly associated to the exponential function, which is reflected in the large number of connections attached to the three respective nodes and, in particular, to the complex exponential function, which can be identified as the main hub (i.e. node with the highest degree, or number of connections, e.g. [39, 40, 41]) of the graph in Figure 18.

Though the present work focused attention on mathematical and statistical concepts related to the exponential function, the latter has also plays a central role in respective *applications* to a wide range of areas, ranging from electrical engineering to data science, which is planned to be covered in subsequent works.

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Observations

As all other preprints by the author, the present work contains preliminary work subject to further revision and validation. Respective modification, commercial use, or distribution of any of its parts are not possible. Many of the preprints by the author are also available in Hal and/or arXiv. This work can also be cited by using the DOI number or article identification link. Thanks for reading.

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