Data Analysis - Lecture 9 :Time series analysis, Hidden Markov Models

Jérémie Sublime

Data Analysis - Lecture 9

Time series analysis: Hidden Markov Models

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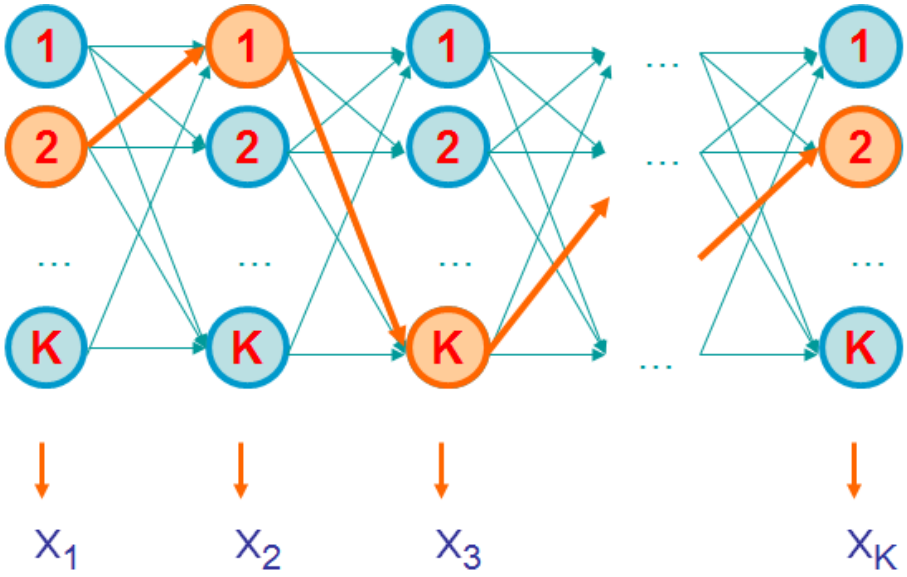
Plan

Outline

Introduction to HMM

- Hidden Markov models (HMMs) are a ubiquitous tool for modeling time series data.
- Far from being limited to time series, they are widely used with all sorts of sequential data:
  - Speech recognition systems
  - Genome sequence analysis
  - Pattern recognition
  - Object tracking and computer vision
  - Etc.

Introduction to HMM



Definition

A hidden Markov model is a tool for representing probability distributions over sequences of observations. It is assumed that each observation X_t was generated by some process whose state $S_t \in [1..K]$ is unknown (hence the name hidden).

Introduction to HMM

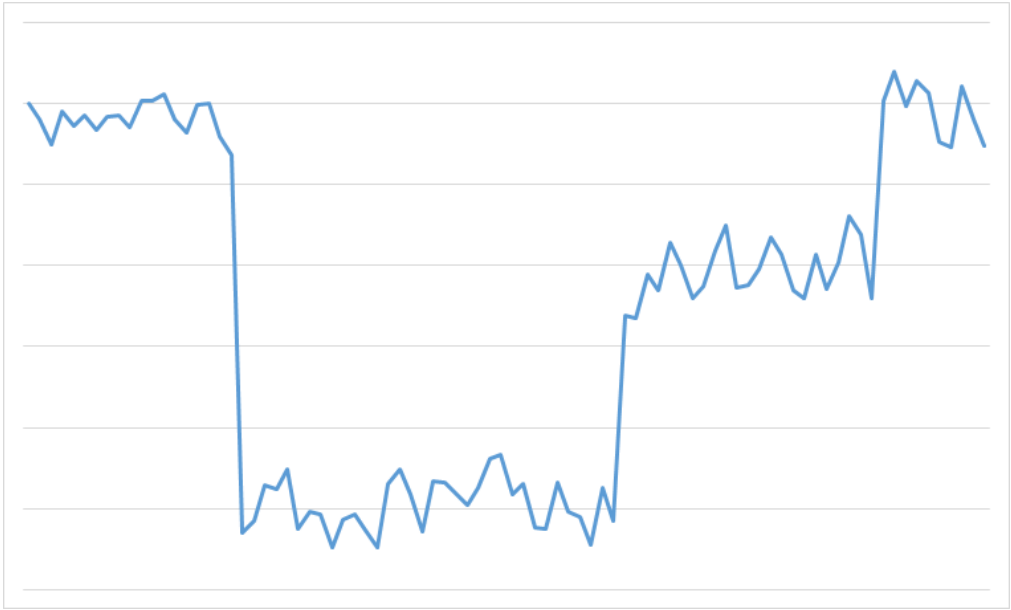


Figure: Example of a univariate sequence with 3 hidden states

HMM: Properties

- Each observation X_t depends only on the current state Q_t
- Markov property: Given the value of the state Q_{t-1}, the current state Q_t is independent from all states prior to $t-1$.

Definition of a HMM

A HMM is made of 5 key elements:

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- **An alphabet** $\Sigma = \{o_1, \cdots, o_M\}$ which defines the form that the observations can take:
  - $\Sigma = \mathbb{R}^d, d \in \mathbb{N}^*$ for multi-dimensional observations
  - $\Sigma = \{1, 2, 3, 4, 5, 6\}$ for dice rolls
  - etc.

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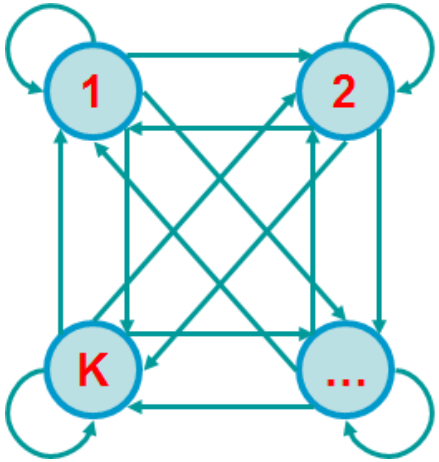

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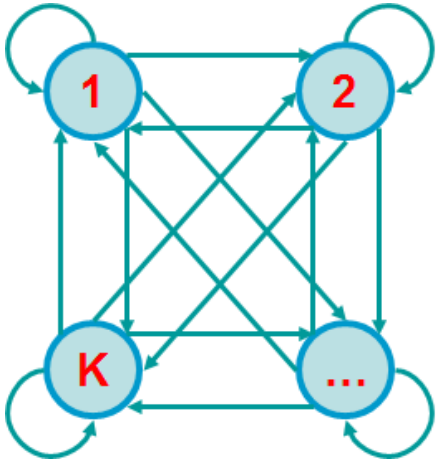

Figure: Example of a state transition diagram

- **A transition probability matrix** $A = (a_{ij})_{K \times K}, \quad \forall i \sum_j a_{ij} = 1$
  - a_{ij} is the probability of transition from state i to state j.

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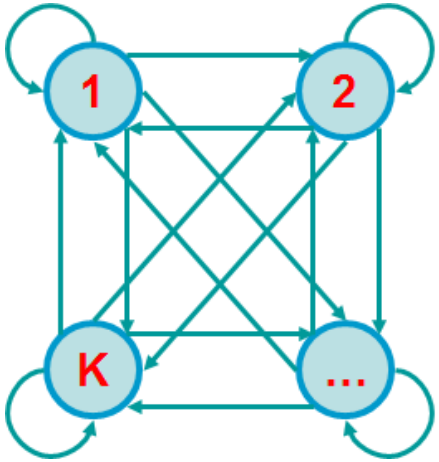

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- **Emission probabilities** within each state:

$$e_i(x) = P(x|Q = i), \quad \forall i \in [1..K] \sum_{x \in \Sigma} e_i(x) = 1$$

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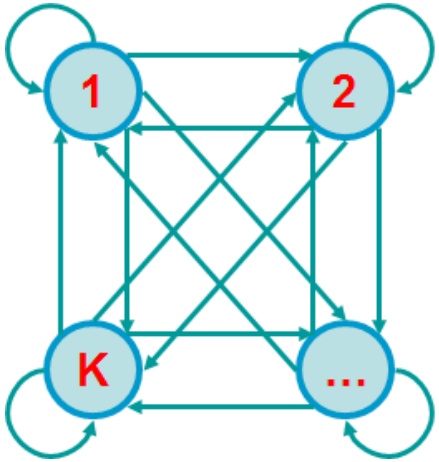

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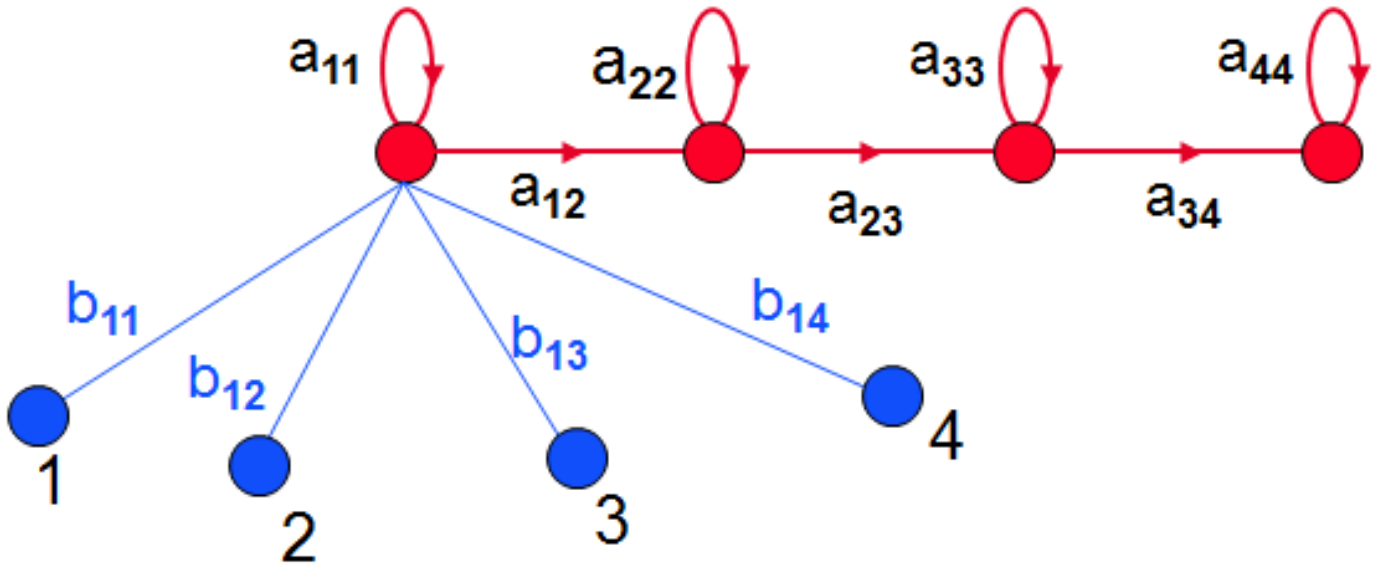
- **Starting probabilities**: $\pi_1, \cdots, \pi_K, \quad \sum_{i=1}^{K} \pi_i = 1$

Definition of a HMM

Notations

A HMM model is denoted $M = \{A, B, \pi\}$, where:

- A is the transition probability matrix
- B contains the emissions probability laws $e_i(x)$
- π the starting probabilities.



Question 1: Evaluating

The Evaluation Problem

- **Given**: a sequence of observations $X = \{x_1, \cdots, x_L\}$ and a model $M = \{A, B, \pi\}$
- **Question**: How do we efficiently compute $P(X|M)$, the probability of observing this sequence X knowing the model M.

- The probability $P(X|M)$ can be viewed as a measure of quality to evaluate the model M.
- It can be used to discriminate or select among alternative models $M_1, M_2, M_3, ...$

Question 2: Decoding

The Decoding Problem

- **Given**: a sequence of observations $X = \{x_1, \cdots, x_L\}$ and a model $M = \{A, B, \pi\}$
- **Question**: How do we compute the most probable sequence(s) of states $\boldsymbol{Q} = \{Q_1, \cdots, Q_L\}$?

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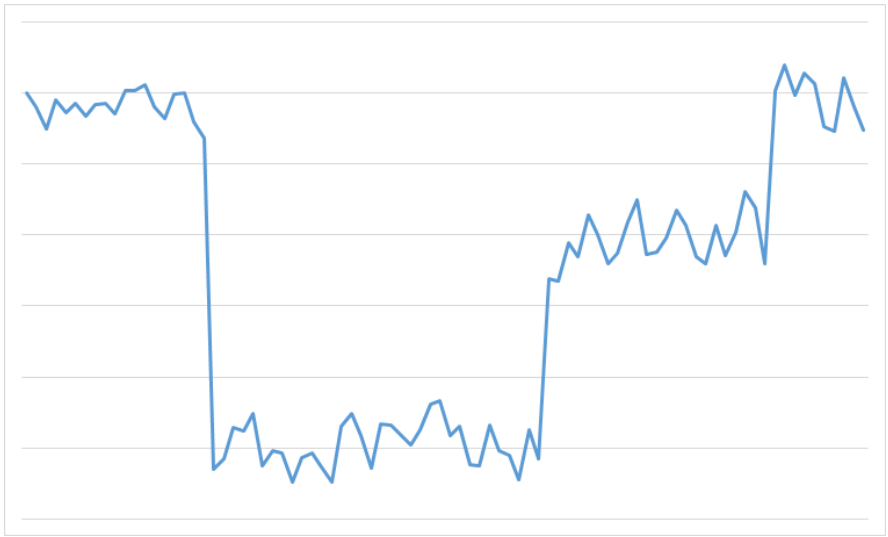

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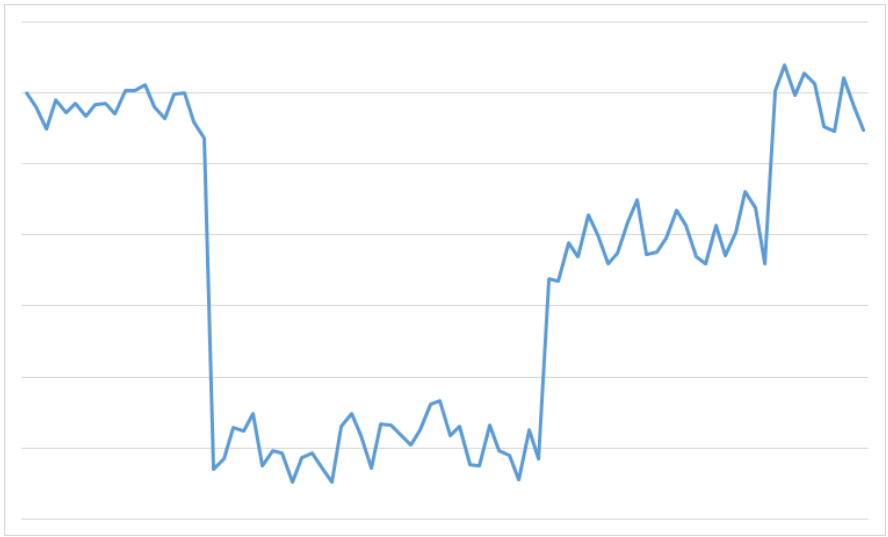
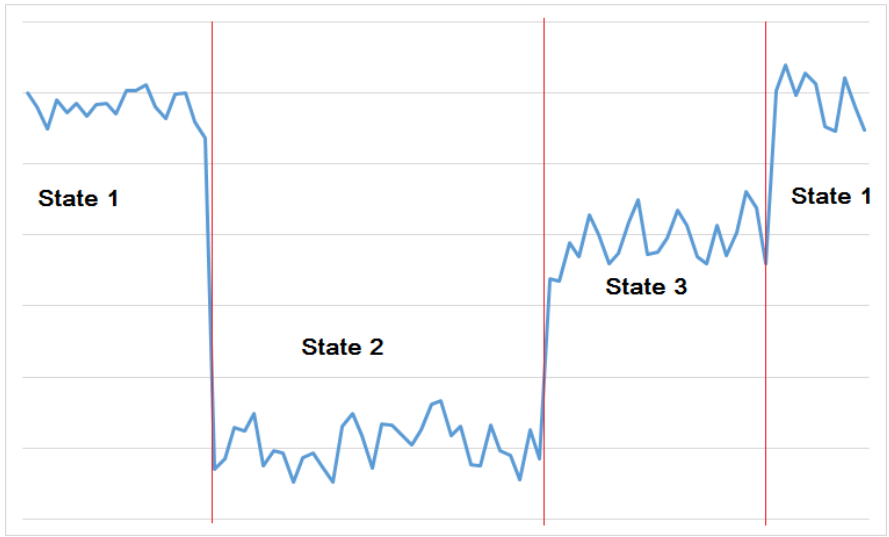
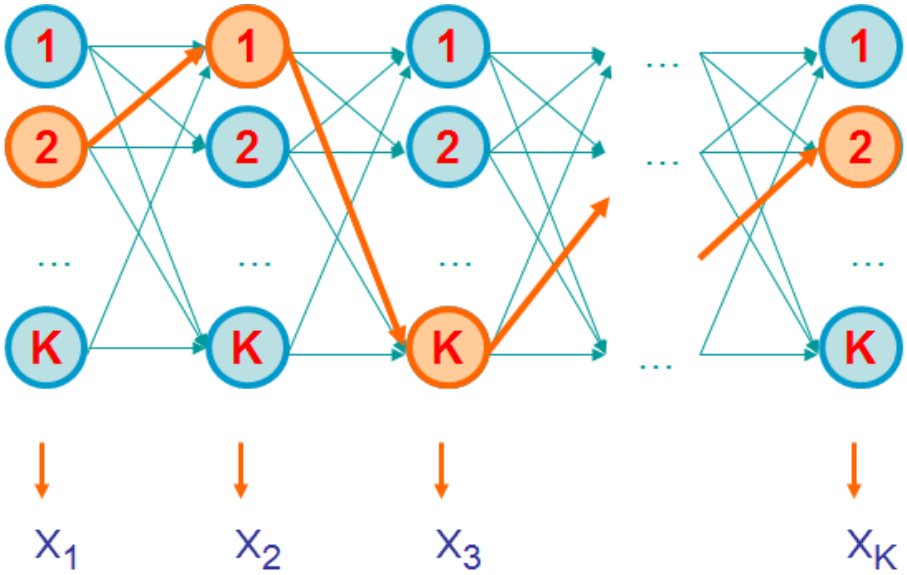




Question 3: Learning

The Learning Problem

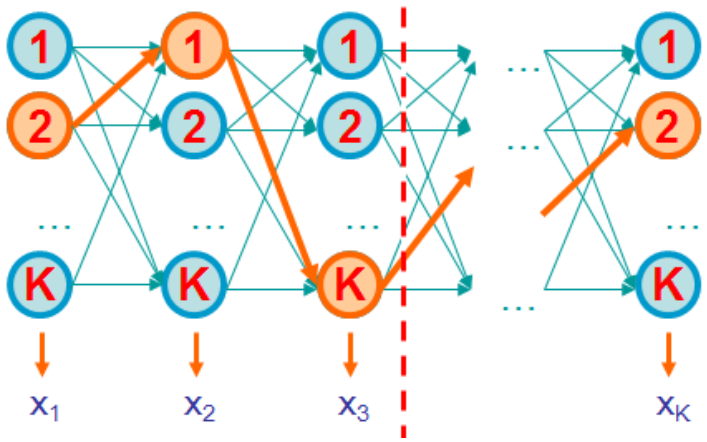
- **Given**: Just the sequence of observations $X = \{x_1, \cdots, x_L\}$
- **Question**: How do we learn $M = \{A, B, \pi\}$ and then figure out Q ?



Outline

The decoding problem

- Finding the sequence $\boldsymbol{Q} = \{Q_1, \cdots, Q_N\}$ of hidden states is the most common task with HMM.
- Given the observations $X = \{x_1, \cdots, x_N\}$, we want to find $\boldsymbol{Q}$ that maximizes $P(X, \boldsymbol{Q})$



Let us denote $V_k(t)$ the optimal sequence of state knowing that at time t we are in the state k:

$$V_k(t) = max_{Q_1 \cdots Q_{t-1}} P(x_1 \cdots x_t, Q_1 \cdots Q_{t-1}, Q_t = k)$$

Decoding: Main Idea

Viterbi algorithm: Main idea

The idea of the Viterbi algorithm is to recursively build the $V_k(i)$

- Knowing that $V_k(t) = max_{Q_1 \cdots Q_{t-1}} P(x_1 \cdots x_t, Q_1 \cdots Q_{t-1}, Q_t = k)$, how do we define $V_l(t+1)$?

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$$V_l(t+1) = max_{Q_1 \cdots Q_t} P(x_1 \cdots x_{t+1}, Q_1 \cdots Q_t, Q_{t+1} = l)$$

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$$\begin{aligned} V_l(t+1) &= max_{Q_1 \cdots Q_t} P(x_1 \cdots x_{t+1}, Q_1 \cdots Q_t, Q_{t+1} = l) \\ &= max_{Q_1 \cdots Q_t} P(x_{t+1}, Q_{t+1} = l | x_1 \cdots x_t, Q_1 \cdots Q_t) P(x_1 \cdots x_t, Q_1 \cdots Q_t) \end{aligned}$$

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&= max_k \left[P(x_{t+1}, Q_{t+1} = l | Q_t = k) max_{Q_1 \cdots Q_{t-1}} P(x_1 \cdots x_t, Q_1 \cdots Q_{t-1}, Q_t = k) \right]
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&= max_k \left[P(x_{t+1} | Q_{t+1} = l) P(Q_{t+1} = l | Q_t = k) V_k(t) \right]
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&= max_k \left[P(x_{t+1} | Q_{t+1} = l) P(Q_{t+1} = l | Q_t = k) V_k(t) \right] \\
&= e_l(x_{t+1}) max_k \left[a_{kl} V_k(t) \right]
\end{aligned}
$$

The Viterbi Algorithm

Initialization:

- $V_0(0) = 1$
- $\forall k > 0 \quad V_k(0) = 0$

Iteration:

- $V_k(t) = e_j(x_t) max_q \left[a_{qk} V_q(t-1)\right]$
- $Ptr_k(t) = argmax_q \left[a_{qk} V_q(t-1)\right]$

Termination:

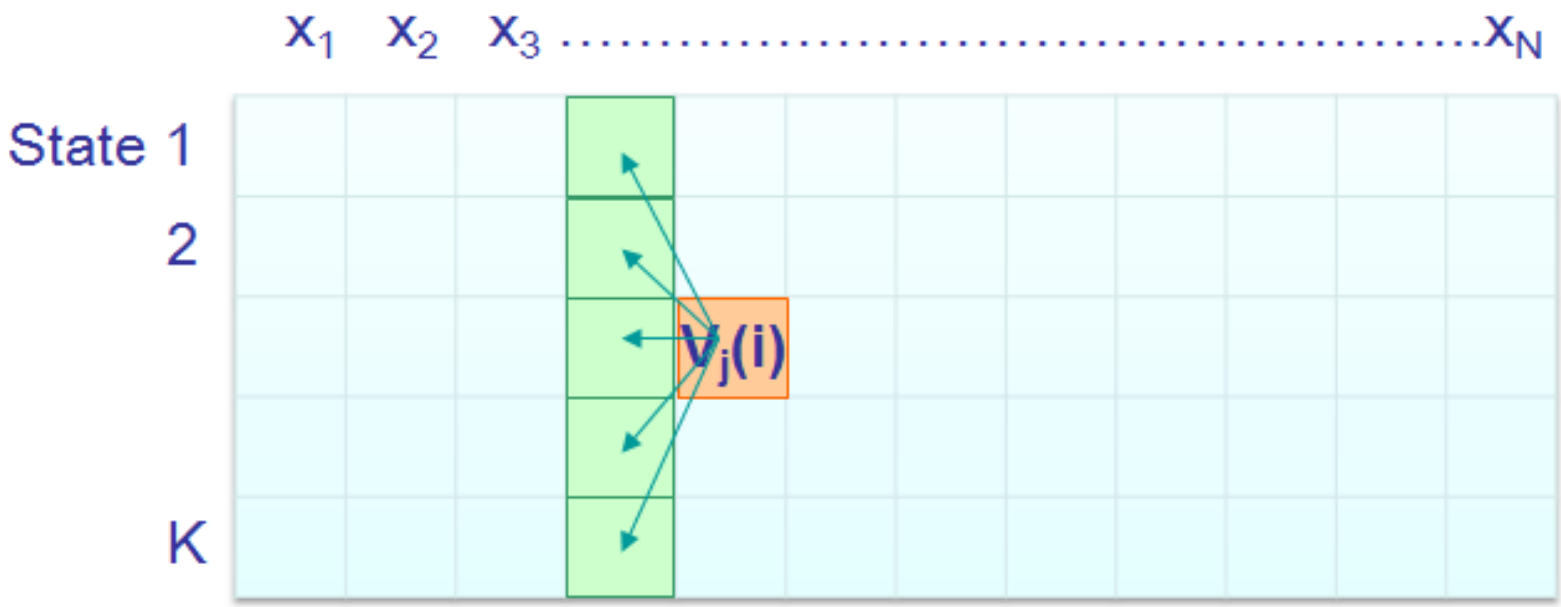
$$P(X, \boldsymbol{Q}^*) = max_k V_k(N)$$

Traceback:

$Q_N^* = argmax_k V_k(N)$

$Q_{t-1}^* = Ptr_{Q_t}(t)$

The Viterbi Algorithm



complexity of the Viterbi algorithm

- Time complexity $O(K^2N)$
- Space complexity $O(KN)$

The Viterbi Algorithm

An important remark on a practical detail

Underflows are a major problem with the Viterbi algorithm.

$$P(x_1, \cdots, x_t, Q_1, \cdots, Q_t) = \pi_{Q_1} \cdot a_{Q_1,Q_2} \cdot a_{Q_2,Q_3} \cdots a_{Q_{t-1},Q_t} \cdot e_{Q_1}(x_1) \cdots e_{Q_t}(x_t)$$

These numbes are extremely small, thus leading to a quick underflow.

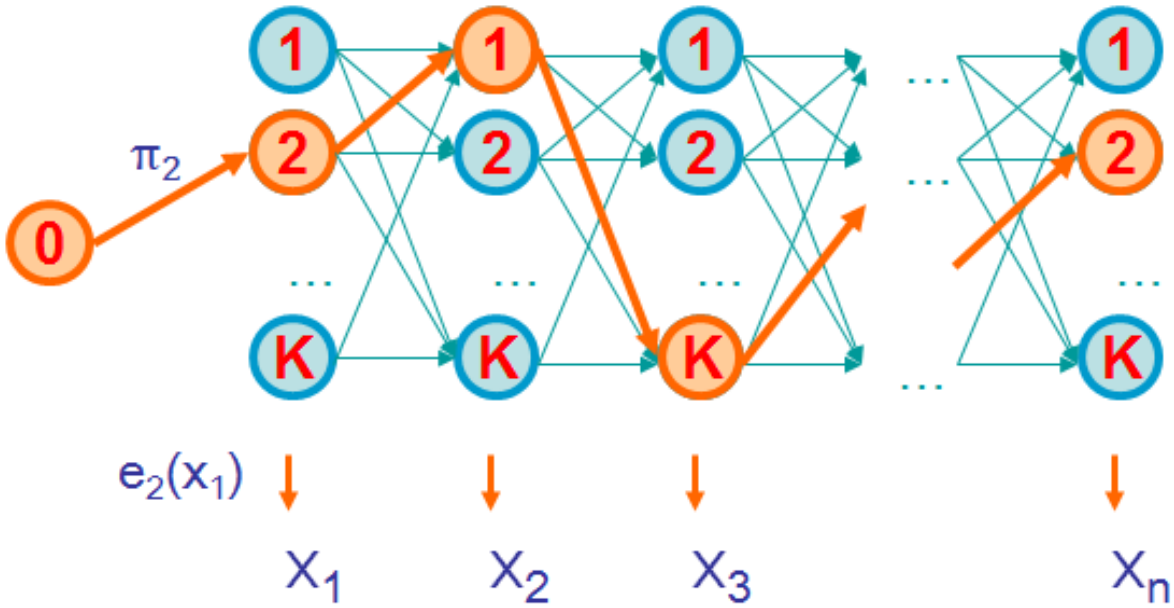
A good solution is to use the logarithm of all values, e.g:

$$\log V_k(t) = \log e_j(x_t) + \log \left(max_q \left[a_{qk} \times V_q(t-1) \right] \right)$$

Generating a sequence by the model

Given a HMM, we can generate a sequence of length n as follows:

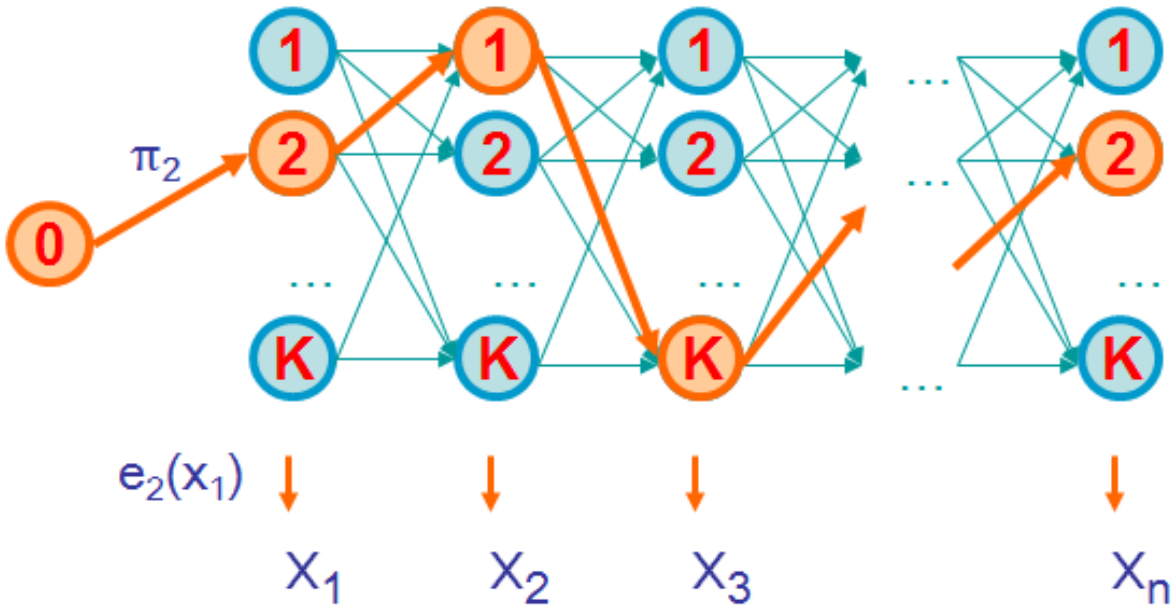
1. Start at state Q_1 according to π_k
2. Emit observation x_1 according to $e_{Q_1}(x_1)$
3. Go to state Q_2 according to a_{Q_1,Q_2}
4. ... until emitting x_n



Evaluation

We want an algorithm that can compute:

- $P(X)$ the probability of X given the model
- $P(x_i \cdots x_j)$ the probability of a substring of X given the model
- $P(Q_i = k|X)$ the posterior probability that the i^{th} state is k, given X



The Forward Algorithm

We want to calculate $P(X)$, the probability of the whole sequence given the HMM.
We can sum all possible ways of generating X:

$$P(X) = \sum_{\boldsymbol{Q}} P(X, \boldsymbol{Q}) = \sum_{\boldsymbol{Q}} P(X|\boldsymbol{Q})P(\boldsymbol{Q})$$

To avoid summing over an exponential number of paths $\boldsymbol{Q}$, we define the **forward probability**:

$$f_k(t) = P(x_1 \cdots x_t, Q_t = k)$$

It is the probability of observing the sequence $x_1 \cdots x_t$ and having the t^{th} state being k.

The Forward Algorithm: Derivation

$$f_k(t) = P(x_1 \cdots x_t, Q_t = k)$$

The Forward Algorithm: Derivation

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f_k(t) &= P(x_1 \cdots x_t, Q_t = k) \\
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&= \sum_{l} P(x_1 \cdots x_{t-1}, Q_{t-1} = l) a_{lk} e_k(x_t) \\
&= e_k(x_t) \sum_{l} f_l(t-1) a_{lk}
\end{aligned}
$$

The Forward Algorithm

We can compute all the $f_k(t)$ using dynamic programming:

Initialization:

- $f_0(0) = 1$
- $\forall k > 0 \quad f_k(0) = 0$

Iteration:

$$f_k(i) = e_k(x_t) \sum_l f_l(t-1) a_{lk}$$

Termination:

$$P(X) = \sum_k f_k(N)$$

The Backward Algorithm

Motivation

To unlock the sequence of hidden states, want to compute $P(Q_t = k|X)$ the probability distribution on the t^{th} position given X.

Since we have $P(Q_t = k|X) = \frac{P(Q_t=k,X)}{P(X)}$, we start by computing $P(Q_t = k, X)$:

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Let us note $b_k(t)$ the backward probability.

The Backward Algorithm: Derivation

$$b_k(t) = P(x_{t+1} \cdots x_N | Q_t = k)$$

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\begin{aligned}
b_k(t) &= P(x_{t+1} \cdots x_N | Q_t = k) \\
&= \sum_{Q_{t+1} \cdots Q_N} P(x_{t+1} \cdots x_N, Q_{t+1} \cdots Q_N | Q_t = k)
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&= \sum_{l} P(x_{t+1} | Q_{t+1} = l) a_{kl} \sum_{Q_{t+2} \cdots Q_N} P(x_{t+2} \cdots x_N, Q_{t+2} \cdots Q_N | Q_{t+1} = l)
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&= \sum_{l} e_l(x_{t+1}) a_{kl} b_l(t+1)
\end{aligned}
$$

The Backward Algorithm

We can compute all the $b_k(t)$ using dynamic programming:

Initialization:

- $\forall k \quad b_k(N) = 0$

Iteration:

$$b_k(i) = \sum_l e_l(x_{t+1}) a_{kl} b_l(t+1)$$

Termination:

$$P(X) = \sum_l \pi_l e_l(x_1) b_l(1)$$

Computational complexity

Complexity analysis for the Forward and Backward algorithms

- Time complexity $O(K^2N)$
- Space complexity $O(KN)$

Like for the Viterbi algorithm, underflows can be a problem:

- Method 1: Use sums of log
- Method 2: Rescale every few positions by multiplying by a constant

Posterior decoding

Using both Forward and backward algorithms, we can now calculate:

$$P(Q_t = k|X) = \frac{f_k(t)b_k(t)}{P(X)}$$

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$$\widehat{Q_t} = argmax_k P(Q_t = k|X)$$

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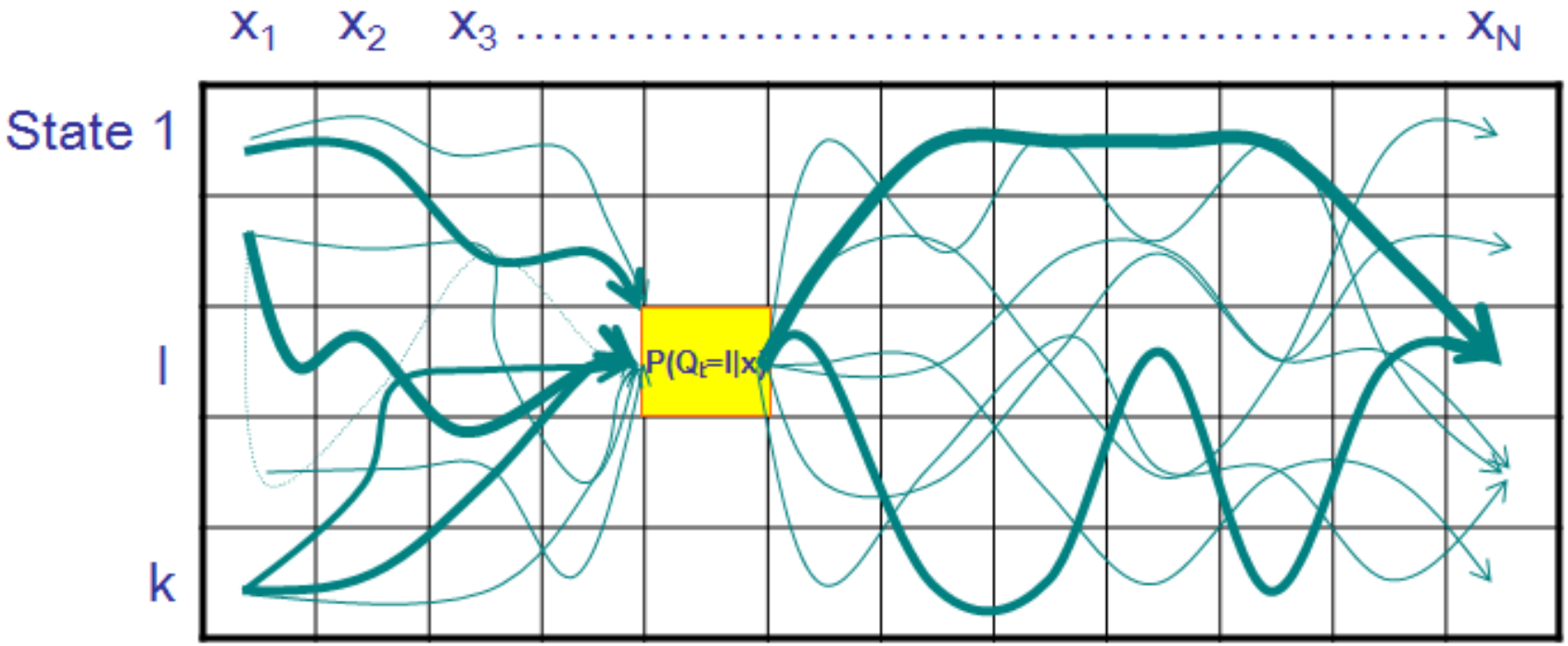
- From there, the posterior decoding gives us a likelihood curve of states for each position.
- This is sometimes more informative than Viterbi path $\boldsymbol{Q}^*$

Posterior decoding



Outline

The learning problem

- If the hidden states are known, it is easy to guess the parameters:
  - Count the transitions to guess A
  - Count or measure the distribution of the observations in each states to guess the emission parameters$B$.
- If the parameters A, B and π are known, the sequence of hidden states can be guessed using Viterbi or the Forward-Backward algorithm.

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- If the parameters A, B and π are known, the sequence of hidden states can be guessed using Viterbi or the Forward-Backward algorithm.
- We have a "Chicken and egg" problem.

The learning problem: Use the EM algorithm

This kind of problem can be solved using the EM algorithm (Cf. Lecture 6) to alternatively optimize both the model and the most likely path.

- **Initialization**: Guess the initial HMM parameters $M = \{A, B, \pi\}$
- Repeat until convergence:
  - **E-Step**: Compute the distribution overs paths.
  - **M-Step**: Compute the maximum likelihood parameters.

The Baum-Welch algorithm

The Forward-Backward algorithm can be adapted into an EM like procedure to learn the model. This is known as the Baum-Welch algorithm.
Repeat until convergence:

- Compute the probability of each state at each position using forward and backward probabilities
  - This gives the expected distribution of the observations for each state using Bayes Theorem.
- Compute the probability of each pair of states at each pair of consecutive positions t and $t+1$ using forward(t) and backward(t+1)
  - This gives the expected transition counts.

$$Count(k \rightarrow l) = \frac{\sum_i f_k(t) a_{kl} b_l(t+1)}{P(X)}$$

Outline

ARMA or HMM, when to use which ?

Type of data

- If your observations/data seem (or are known) to depend upon previous observations, use ARMA.
- If your observations/data are independent in time, use HMM.
- For HMM series, each state is assumed to have stationary properties. With ARMA series, it depends on the parameters.
- With cyclic data, ARMA models (and their evolved ARIMA version) are usually better.
- For stationary data with a single observable event, HMM are usualy a bit too much and ARMA models are enough.
- For multi-dimensional data, HMM should be preferred.

ARMA or HMM, when to use which ?

Type of tasks

- ARMA models are dedicated to modeling time dependent events and to predict the evolution of a phenomenon in time.
- HMM usually perform poorly in long term predictions.
- For time data segmentation and modeling, HMM should be used.
- Both ARMA and HMM can be used to fill in missing data.
- For data sequences analysis other than time data, HMM are usually preferred.