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Data Analysis - Lecture 8

Time series analysis: ARIMA models

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Plan

- 1 Fundamentals of time series analysis
- 2 The ARMA model
- 3 The ARIMA model
- 4 Useful statistical tools

Outline

- 1 Fundamentals of time series analysis
- 2 The ARMA model
- 3 The ARIMA model
- 4 Useful statistical tools

Time data & Time series

Time data contain one or several attributes that describe when the observations took place: year, month, day, hour, elapsed time since the beginning of an experiment, timestamp, etc.

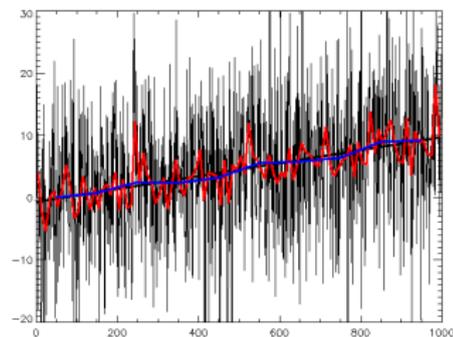
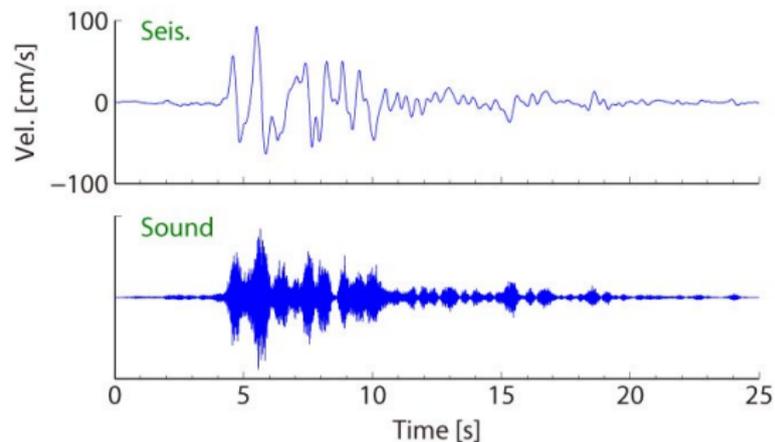
Times series: Definition

A time series is a series of data points indexed in time order.

- Most commonly, a time series is a sequence taken at successive equally spaced points in time.
- A time series usually describes the same observation evolving through time.

Time data & Time series

Time series can be used for a large number of applications: statistics, signal processing, weather forecast, earthquake prediction, finance analysis, budget predictions, tidal predictions, astrophysics, astronomy, tidal analysis, electroencephalography, control engineering, or any domain in science that involves time measurements.



Preliminary analysis of Time data

Processing time data first require to answer the following questions:

- Does my time series deals with one or several objects in time ?
- Are my observations equally spaced in time ? If not, can I fill in the gaps ?
 - If the observations are not equally spaced in time, most analysis are impossible to do.
- What am I looking for ?
 - Global trends ? Cyclic events ? Abnormalities ? Specific events ?
 - What is the time scale of interest for these data ?
- Should I differentiate some of the variables ?

Differentiating the variables ?

- When time is a parameter, the other variables can be differentiated with respect to time.

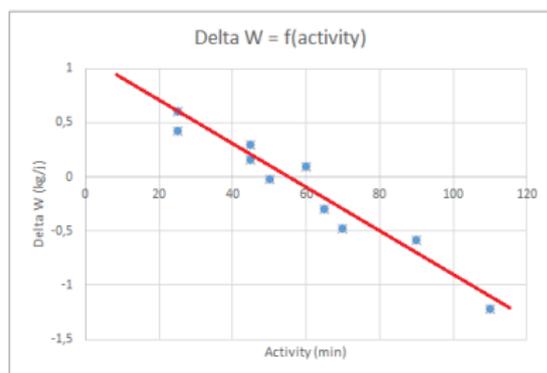
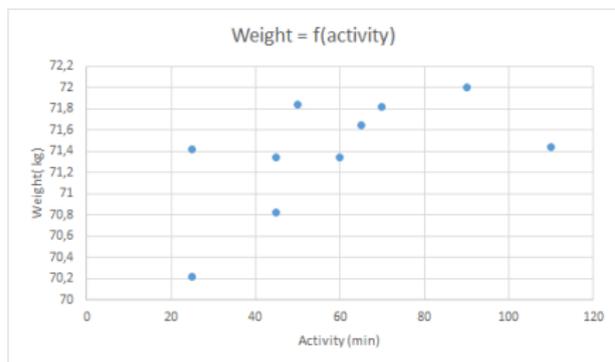
$$\Delta Y = Y_{t+1} - Y_t$$

Why differentiating ?

- Sometimes the new variables can contain information that are more useful than the old ones.
- One must consider which is more interesting between the raw value of a variable at a point in time and its evolution between two points in time.

Differentiating the variables ?

Day	Sport (min)	Weight (kg)	Δ Weight
14/05	90	72.0	-0.58
15/05	25	71.42	0.42
16/05	50	71.84	-0.02
17/05	70	71.82	-0.48
18/05	45	71.34	0.3
19/05	65	71.64	-0.3
20/05	60	71.34	0.1
21/05	110	71.44	-1.22
22/05	25	70.22	0.6
23/05	45	70.82	0.16



Times series analysis

While creating new attributes using the time variable may lead to finding new correlations and could help analyzing the data, it is not time series analysis.

Times series analysis: Definition

Times series analysis is the study of how one or several variables, or even objects behave with respect to time. Time series analysis can have several goals.

- Finding and describing trends in a time series.
- Building predictive models from a time series.

Time series vs regular data

Cross-Sectional Data

$X_1, X_2, \dots, X_N \sim iid \Rightarrow LLN: \frac{1}{N} \sum_i X_i = \mathbb{E}[X]$

- Draws from a fixed distribution
- No ordering

Time series Data

$x_1, x_2, \dots, x_t, \dots, x_T$ with t the time index, and T the size of the sample

- each realization x_t is a draw from a random variable X_t
- Natural ordering \Rightarrow Conditional models: $x_t | x_{t-1}, x_{t-2}$

Why do we need stationarity ?

Stochastic process	x_1	x_2	\dots	x_t	\dots	x_T
Realization 1	x_1^1	x_2^1	\dots	x_t^1	\dots	x_T^1
\vdots						
Realization m	x_1^m	x_2^m	\dots	x_t^m	\dots	x_T^m
\vdots						
Realization M	x_1^M	x_2^M	\dots	x_t^M	\dots	x_T^M

Without stationarity, these two measures are different:

- Ensemble Mean: $\widehat{\mathbb{E}(x_t)} = \frac{1}{M} \sum_{m=1}^M x_t^m$
- Time average of a realized sample path: $\bar{x}_T = \frac{1}{T} \sum_{i=1}^T x_i$

Stationarity: Definitions

Strict Stationarity

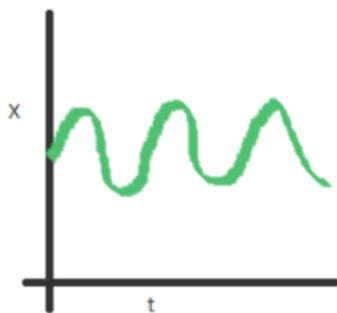
A time series is said to be strictly stationary if all its observations are drawn from the same distribution: the joint probability does not change in time.

Weak Stationarity

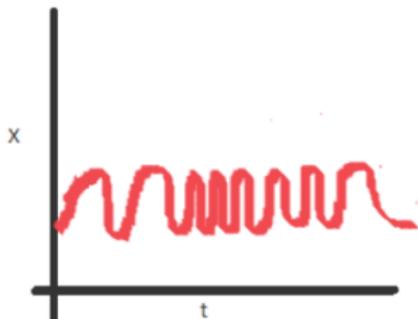
We do not require that each draw comes from the exact same distribution, only that the distributions have the same mean and variance (all of them not a function of time).

- Constant mean: $\mathbb{E}(x_t) = \mu$
- Constant variance: $\text{Var}(x_t) = \gamma_0$
- Constant co-variance: $\text{Cov}(x_t, x_{t-h}) = \gamma_h \quad \forall h \in [1..T]$

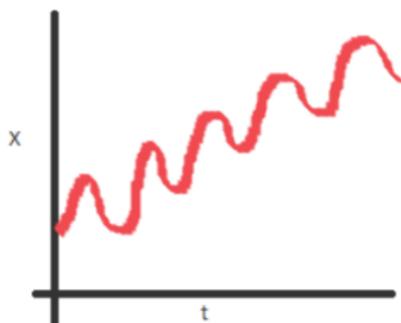
Stationarity: Examples



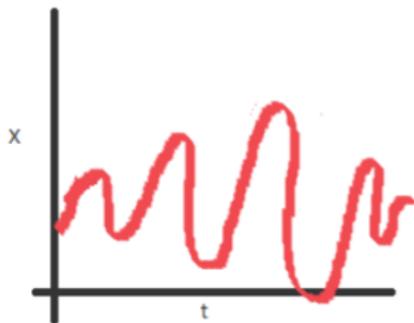
Stationary series



Non-Stationary series



Non-Stationary series



Non-Stationary series

Stationarity and weak dependence

Weak dependence hypothesis

x_t and x_{t+h} are approximately independent from each other when $h \rightarrow \infty$.

- Each observation contain new information about the distribution.

Under the weak dependence hypothesis and the weak stationarity hypothesis, we have: $\bar{x}_T \rightarrow \mathbb{E}(x_t)$

And with this, we can get information from the sample about the underlying distribution.

Remark

The assumptions of weak stationarity and weak dependence replaces the i.i.d hypothesis from cross-sectional data.

- Weak dependence \Rightarrow independence
- Weak stationarity \Rightarrow identically distributed

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From regression to auto-regressive processes

While regression analysis can be used for time series analysis, it can only catch global trends and has the following weaknesses:

- It cannot be applied to several variables.
- It cannot detect cyclic or seasonal events.
- It cannot be used to predict "aftershock effects" after random shocks in a time line.
- It is ill adapted to segment a time series into several events.

From regression to auto-regressive processes

- In linear regressions, we try to find regressions coefficients to solve an equation of form $y = a \cdot x + b + \epsilon$.
- For time series predictions, the model is more like $x_t = \rho x_{t-1} + \epsilon_t$
 - ρ is a coefficient to be found
 - ϵ_t is an error term between the prediction and the real value.

This is what we call an **auto-regressive process** of order one: **AR(1)**

Auto-regressive processes

Auto-regressive process of order p

An auto-regressive process of order p is defined as follows:

$$AR(p) : \quad X_t = \sum_{i=1}^p \rho_i X_{t-i} + \epsilon_t$$

- the ρ_i are parameters
- ϵ_t is a white noise term (random noise i.i.d($0, \sigma^2$))

Auto-regressive process of order 1

$$AR(1) : \quad X_t = c + \rho X_{t-1} + \epsilon_t$$

Auto-regressive processes of order 1: Example (1/2)

Let us consider an oil price problem. The oil price change at a moment t ΔO_t is linked to the oil price change at the instant $t - 1$:

$$\Delta O_t = 0.7 \times \Delta O_{t-1} + \epsilon_t$$

Auto-regressive processes of order 1: Example (1/2)

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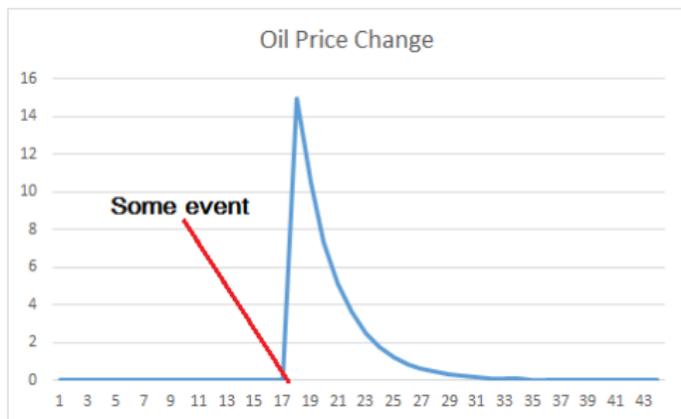
- If some event occurs, the oil price will suddenly change, resulting in a spike in the oil price change (ϵ spikes).

Auto-regressive processes of order 1: Example (1/2)

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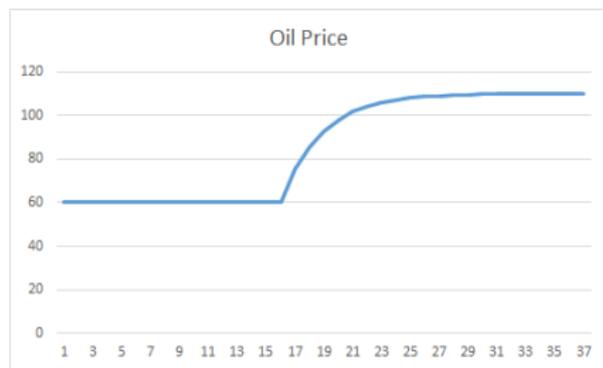
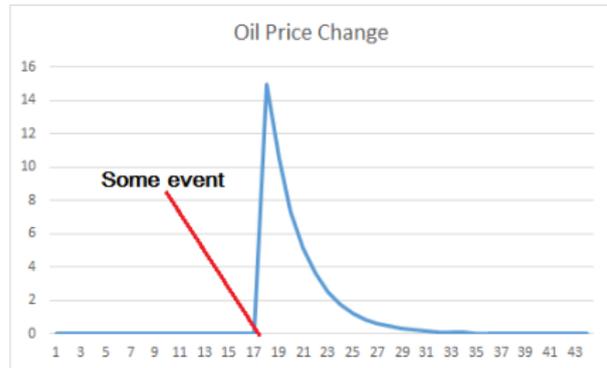
- If some event occurs, the oil price will suddenly change, resulting in a spike in the oil price change (ϵ spikes).
- Then following this model, the event has some persistent effect until the market stabilizes and the price stops changing.



Auto-regressive processes of order 1: Example (2/2)

The oil price change at a moment t ΔO_t is linked to the oil price change at the instant $t - 1$:

$$\Delta O_t = 0.7 \times \Delta O_{t-1} + \epsilon_t$$



Remark

This is an AR(1) model for the Oil Price Change, but since we use the first derivation of the Oil Price, we will see later that this model is an ARIMA(1,1,0) model of the Oil Price.

Stationarity conditions of an AR(1) Process 1/2

$$X_t = \rho X_{t-1} + \epsilon_t \text{ with } \epsilon_t \sim iid(0, \sigma^2)$$

Stationarity conditions of an AR(1) Process 1/2

$$X_t = \rho X_{t-1} + \epsilon_t \text{ with } \epsilon_t \sim iid(0, \sigma^2)$$

$$X_t = \rho(\rho X_{t-2} + \epsilon_{t-1}) + \epsilon_t$$

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$$\Rightarrow X_t = \rho^t X_0 + \sum_{i=0}^{t-1} \rho^i \epsilon_{t-i}$$

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$$\mathbb{E}[X_t] = \rho^t \mathbb{E}[X_0] + \sum_{i=0}^{t-1} \rho^i \mathbb{E}[\epsilon_{t-i}] = \rho^t \mathbb{E}[X_0]$$

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Condition on the mean

For this term to be constant, we need $\mathbb{E}[X_0] = 0$ and therefore $\mathbb{E}[X_t] = 0$.

Stationarity conditions of an AR(1) Process 2/2

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- We want the variance to be constant: $Var(X_t) = Var(X_{t-1})$

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- We want the variance to be constant: $Var(X_t) = Var(X_{t-1})$
- We substitute in the previous expression, and we get:

$$Var(X_t) = \rho^2 Var(X_t) + \sigma^2$$

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$$Var(X_t) = \rho^2 Var(X_t) + \sigma^2 \quad \Rightarrow \quad (1 - \rho^2)Var(X_t) = \sigma^2$$

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Conditions on the variance and on ρ

$$Var(X_t) = \frac{\sigma^2}{1 - \rho^2}$$

- Since we don't want a negative variance or a null denominator, we deduce that $|\rho| < 1$

Stationary covariance and weak dependence of an AR(1) process

From our previous calculi, we know that: $X_{t+h} = \rho^h X_t + \sum_{i=0}^{h-1} \rho^i \epsilon_{t+h-i}$

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$$\text{Cov}(X_t, X_{t+h}) = \text{Cov}\left(X_t, \rho^h X_t + \sum_{i=0}^{h-1} \rho^i \epsilon_{t+h-i}\right)$$

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$$\text{Cor}(X_t, X_{t+h}) = \frac{\text{Cov}(X_t, X_{t+h})}{\text{Var}(X_t)} = \rho^h$$

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Weak dependence condition

We want $\lim_{h \rightarrow \infty} \text{Cor}(X_t, X_{t+h}) = 0$, so once we find once again that we need $|\rho| < 1$.

Stationary VS non Stationary AR(1)

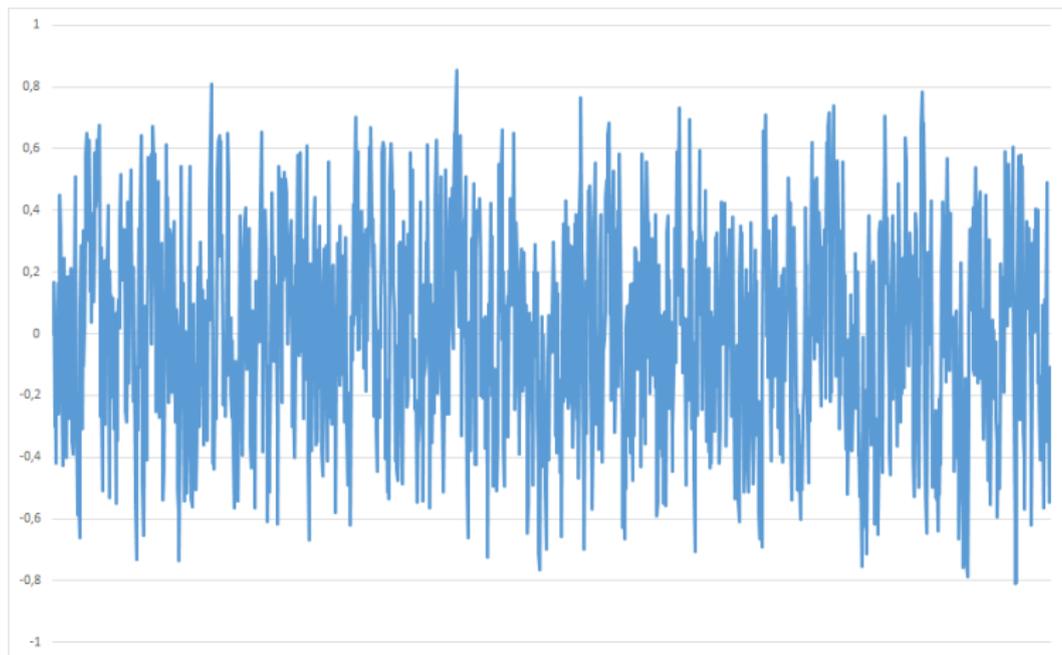


Figure: $\rho = 0.5$

Stationary VS non Stationary AR(1)

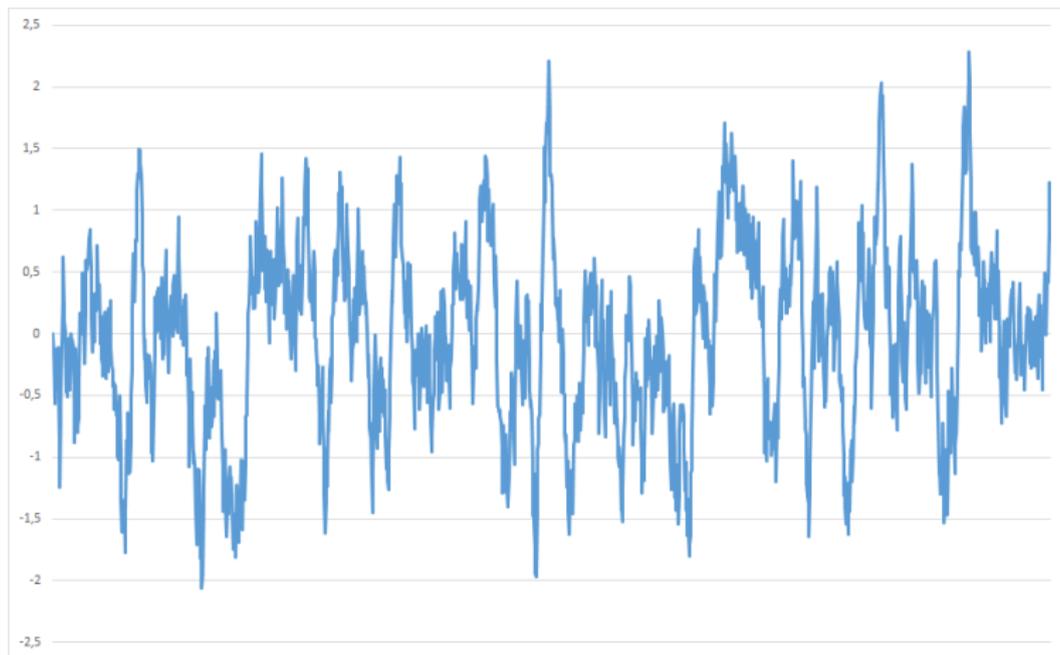


Figure: $\rho = 0.95$

Stationary VS non Stationary AR(1)

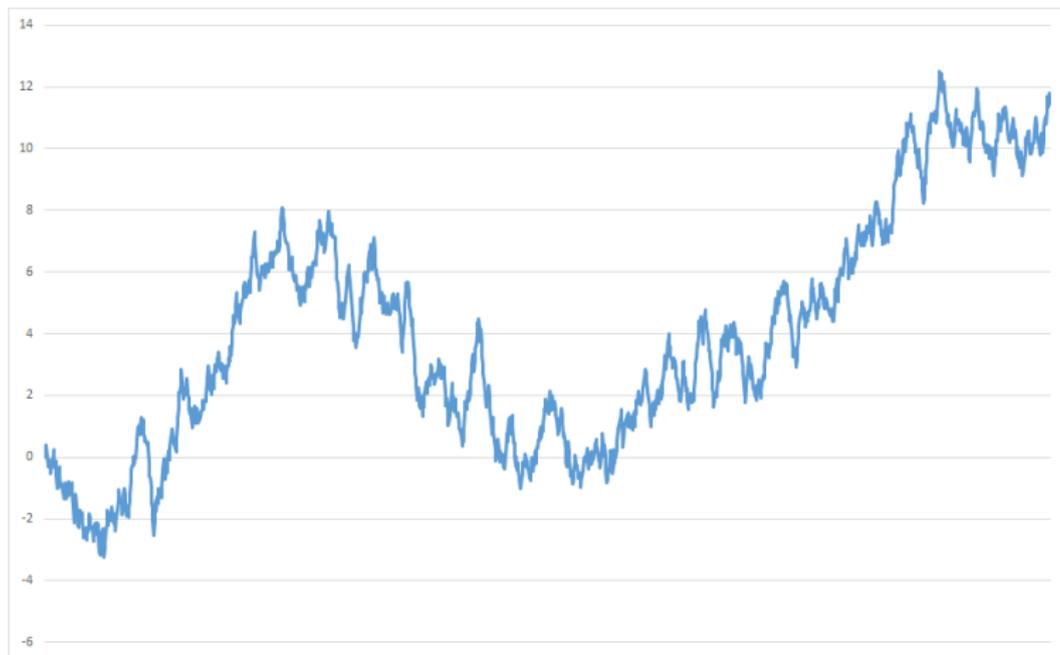


Figure: $\rho = 1$

Stationary VS non Stationary AR(1)

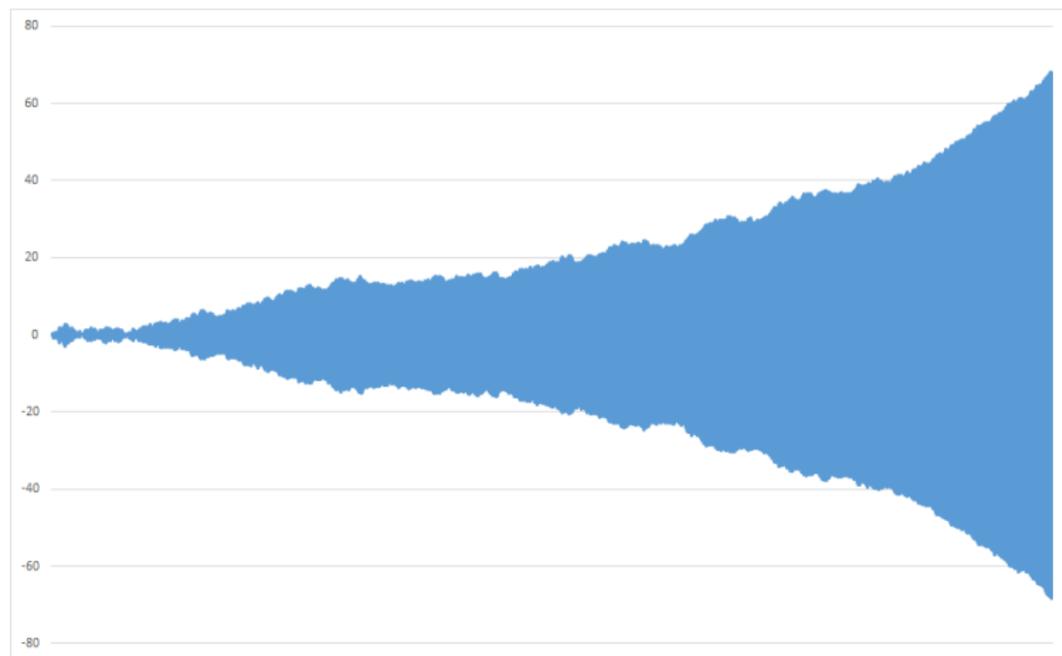


Figure: $\rho = -1.002$

Moving average processes

A moving average model is a common approach for modeling univariate time series. In this model, the output variable depends linearly on the current and various past values of a stochastic (imperfectly predictable term).

Moving average model of order q : $MA(q)$

$$MA(q) : \quad X_t = \mu + \epsilon_t + \sum_{i=1}^q \theta_i \cdot \epsilon_{t-i}$$

- μ is the mean of the series (often assumed to be 0)
- The θ_i are the parameters
- $\epsilon_t, \dots, \epsilon_{t-q}$ are white noise error terms.

Moving average model of order 1: $MA(1)$

$$MA(1) : \quad X_t = \mu + \epsilon_t + \theta \cdot \epsilon_{t-1}$$

Example of MA processes 1/2

Let us consider an example in which we want a model for the daily demand for soda bottles in a grocery store:

$$\text{Demand} = 25 + \epsilon_t - 0.5 \cdot \epsilon_{t-1}$$

- 25 is the average number of bottles usually sold in a day.
- In this example, ϵ_t could be the change in temperature: $\epsilon_t = \Delta_{temp}(t)$

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- The explanation for the “ $-0.5 \cdot \epsilon_{t-1}$ ” term is that, if the temperature already increased yesterday, the customers already bought soda and don't need to buy more today.

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- The explanation for the “ $-0.5 \cdot \epsilon_{t-1}$ ” term is that, if the temperature already increased yesterday, the customers already bought soda and don't need to buy more today.
- Further terms could be added:

$$\text{Demand} = 25 + \epsilon_t - 0.5 \cdot \epsilon_{t-1} - 0.25\epsilon_{t-2}$$

Example of MA processes 2/2

We go back to an oil price example:

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Remark

Unlike in the previous oil price example, we study the oil price, and not the oil price change ! It is not the same model.

MA(1) process: stationarity and weak dependency 1/2

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Stationarity: conditions on the mean and variance

The condition for a weak stationarity are always respected since we have both the mean and the variance that are constant.

MA(1) process: stationarity and weak dependency 2/2

For the third stationarity hypothesis, we want $\text{Cov}(X_t, X_{t+h})$ to be a function of h independent from t .

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From there we can infer that:

$$\forall \tau > 1 \quad \text{Cov}(X_t, X_{t-\tau}) = \text{Cov}(\epsilon_t + \theta \cdot \epsilon_{t-1}, \epsilon_{t-\tau} + \theta \cdot \epsilon_{t-1-\tau}) = 0$$

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Stationarity and dependency

With the 3 conditions being respected, we can conclude that an MA(1) process is always stationary. Furthermore, with the third condition, the weak dependence hypothesis is easily verified.

MA(1) process: Example

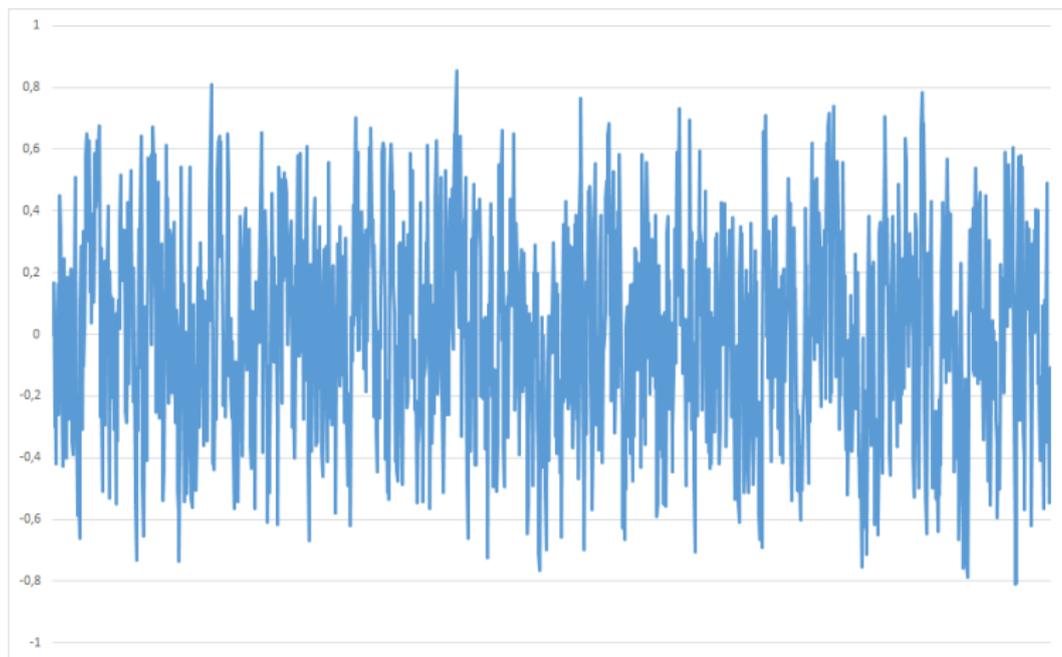


Figure: Example of a MA(1) process ($\theta = -0.3$)

Telling MA(1) from AR(1)

- Sometimes, just looking at the time series curb is not enough to tell whether it follows an AR(1) model, a MA(1) model, or something else.
- The best solution to guess MA(1) from AR(1) is to look at the correlation $Corr(X_t, X_{t+h})$

$$MA(1) : \quad Corr(X_t, X_{t+h}) = \begin{cases} \frac{\theta}{1-\theta^2} & \text{if } h = 1 \\ 0 & \text{if } h > 1 \end{cases}$$

$$AR(1) : \quad Corr(X_t, X_{t+h}) = \rho^h$$

ARMA model

ARMA(p,q)

An ARMA(p,q) process, is a model that contain an AR(p) process and and MA(q) process:

$$X_t = \mu + \epsilon_t + \sum_{i=1}^p \rho_i X_{t-i} + \sum_{i=1}^q \theta_i \cdot \epsilon_{t-i}$$

- The constant μ is the expectation of X_t (often assumed to be 0 and not taken into consideration in the regressive term).
- The ρ_i and θ_i are the parameters from the auto-regressive and moving average process respectively.
- The ϵ_i are white error terms.

ARMA(1,1)

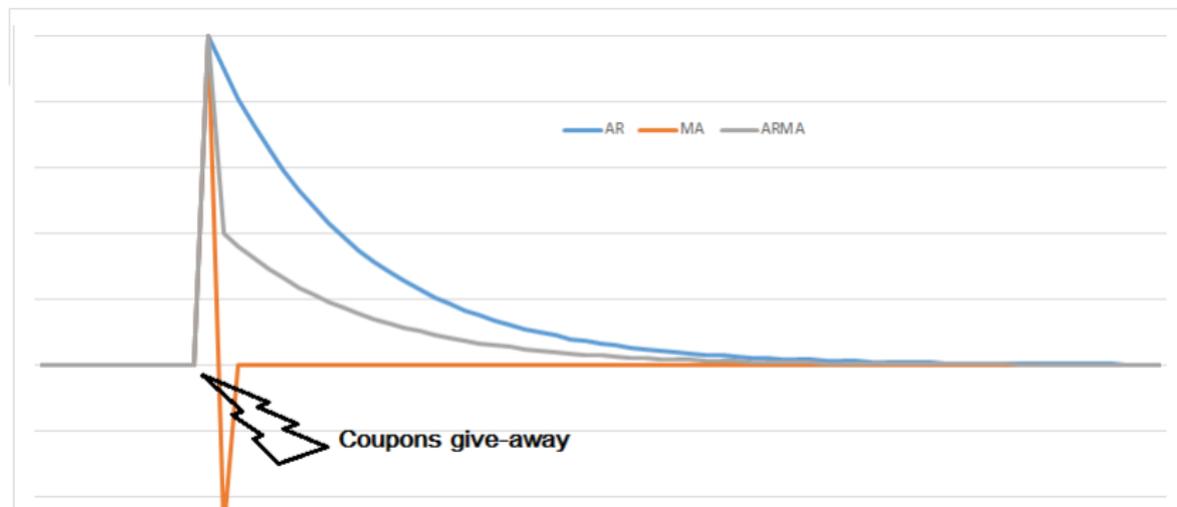
$$ARMA(1,1) : \quad X_t = \mu + \epsilon_t + \rho X_{t-1} + \theta \cdot \epsilon_{t-1}$$

ARMA(1,1), Example 1

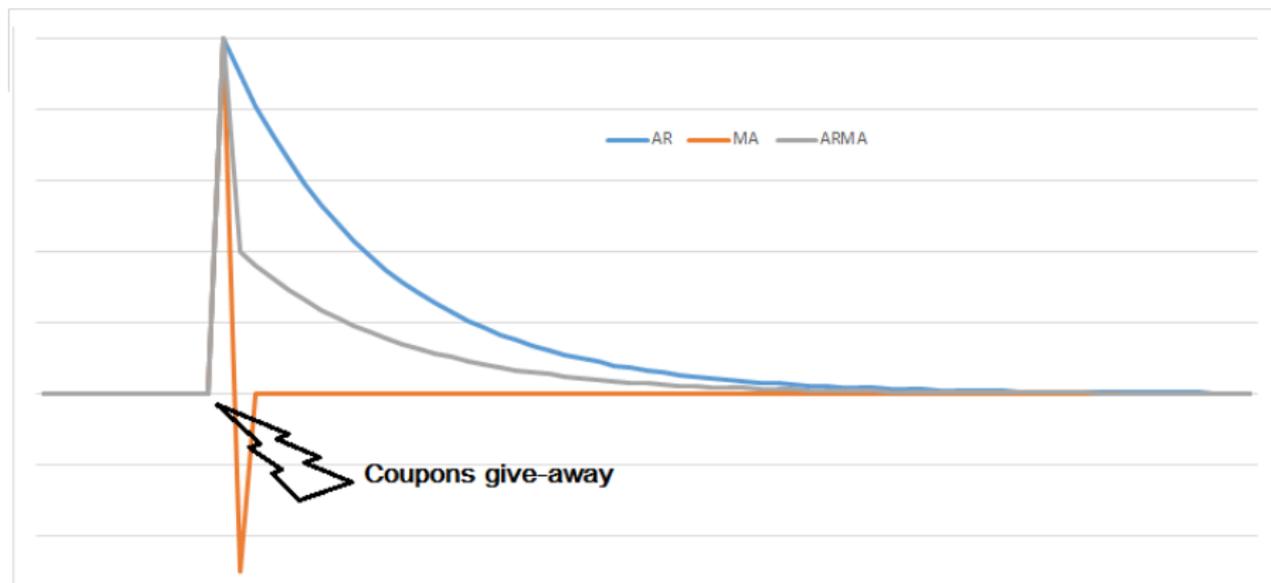
Let us consider a good sales problem:

$$Sales_t = \epsilon_t + \rho Sales_{t-1} + \theta \cdot \epsilon_{t-1}$$

- ρ is a loyalty effect from the customer to the good. Let's take $\rho = 0.9$
- ϵ_t represents free coupons given at a time t .
- θ is a negative effect due to the fact that people already bought the product at time $t - 1$. Let's take $\theta = -0.3$



ARMA(1,1), Example 1



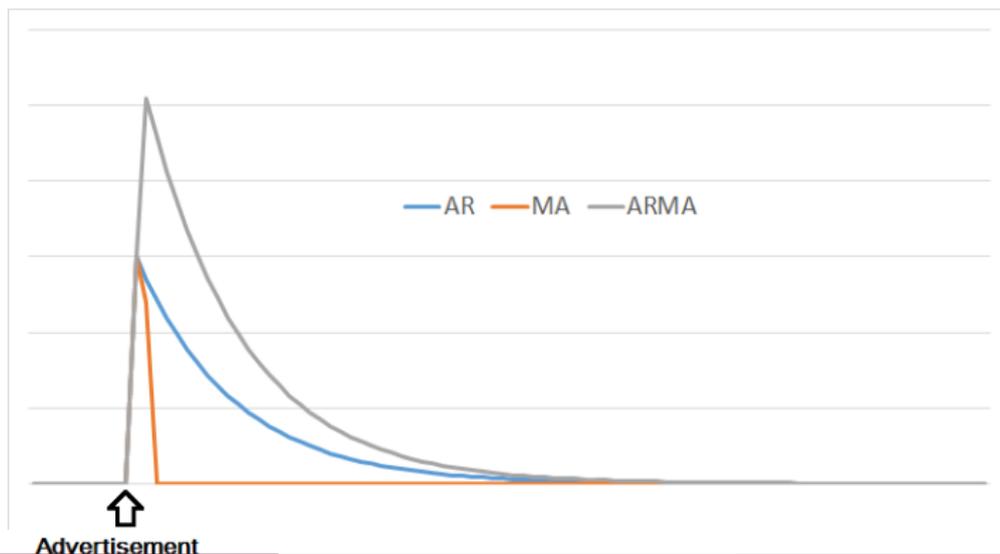
- We see that at the beginning the MA process causes a rapide decline in sales.
- Then, once the MA effect is over, the sales decrease following an AR process.

ARMA(1,1), Example 2

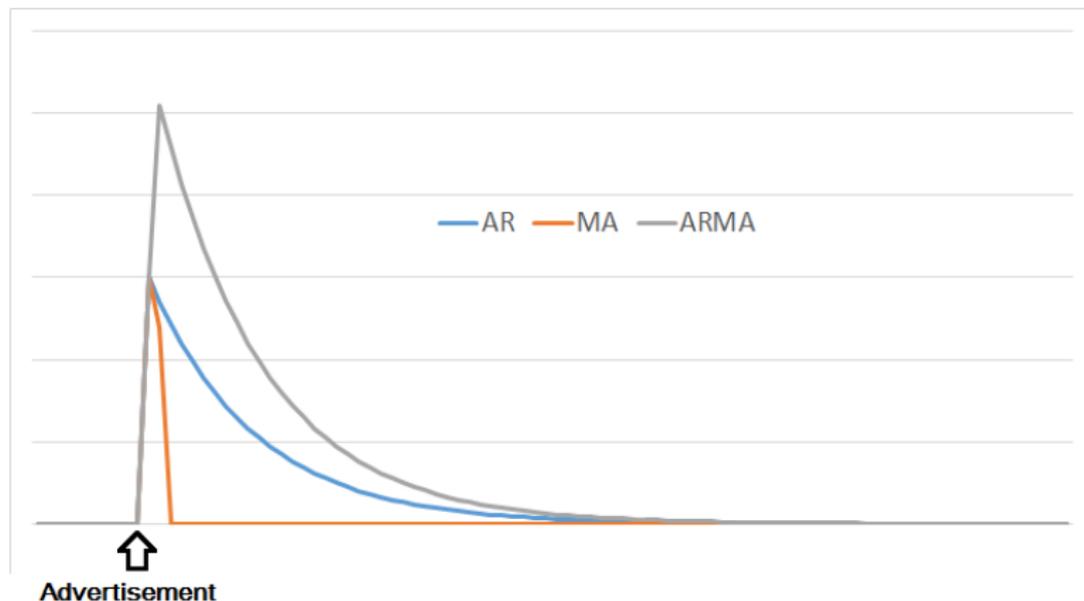
Let us consider the same good sales problem:

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- ρ is a loyalty effect from the customer to the good. Let's take $\rho = 0.9$
- ϵ_t is this time advertisement for the product at a time t .
- θ becomes therefore a positive effect. Let's take $\theta = 0.8$.



ARMA(1,1), Example 1



- We see that with $\theta > 0$, there is a huge initial boost on the sale.
- It is again followed by an AR type decay process which takes more time due to the initial higher boost effect.

ARMA(1,1) behavior examples

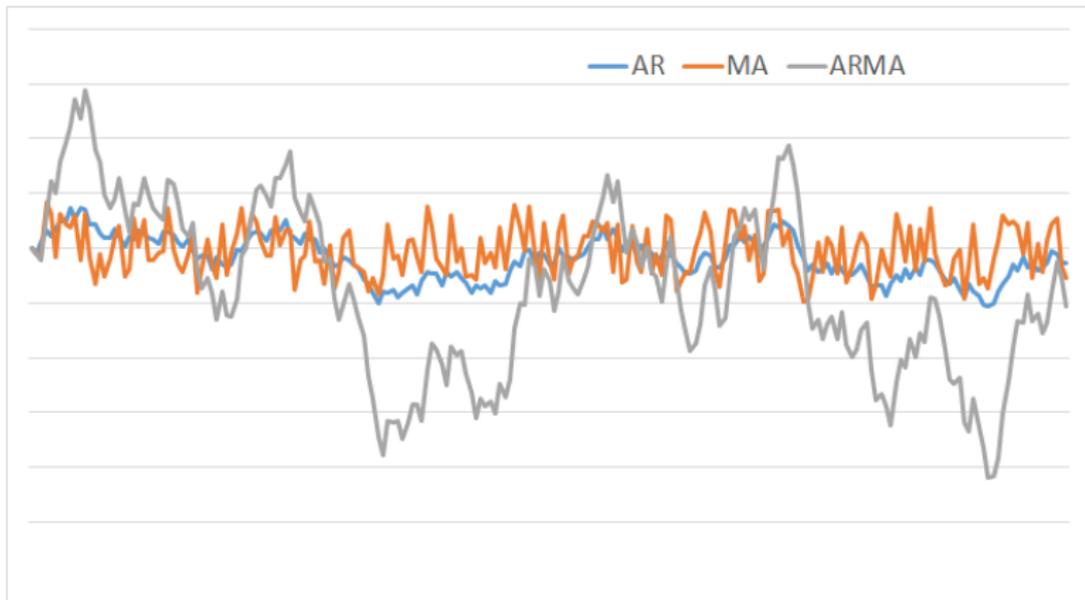


Figure: $\rho = 0.95$ and $\theta = 3$

ARMA(1,1) behavior examples

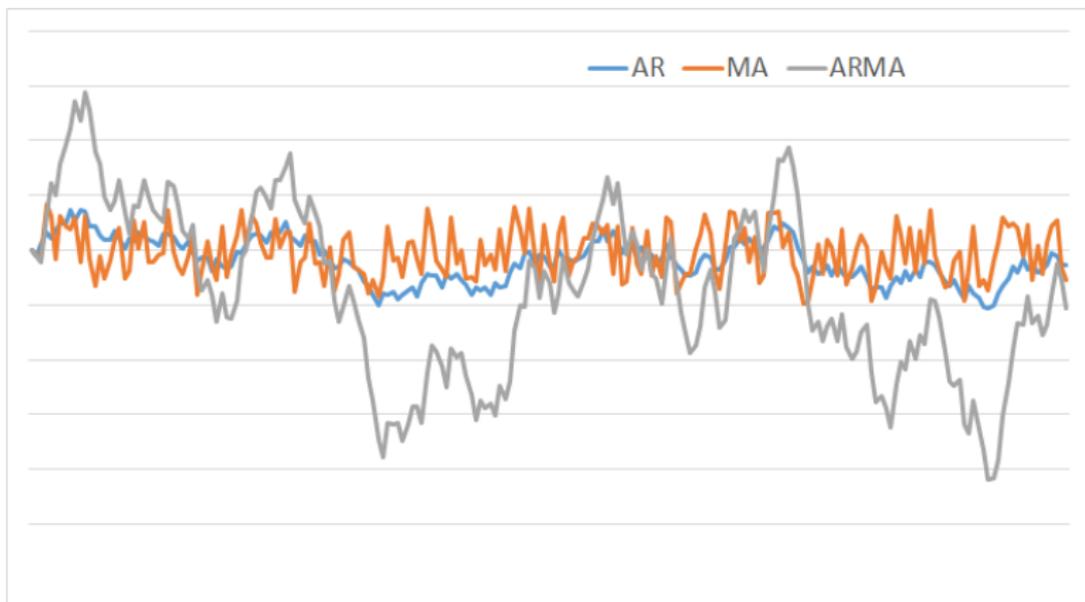


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ARMA(1,1) behavior examples

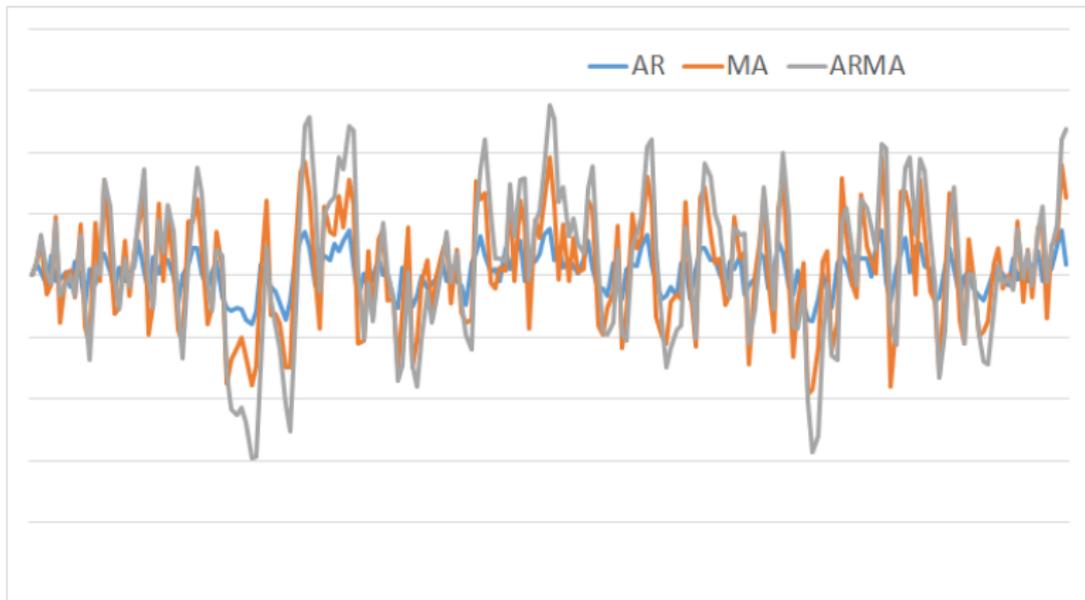


Figure: $\rho = 0.5$ and $\theta = 3$

ARMA(1,1) behavior examples

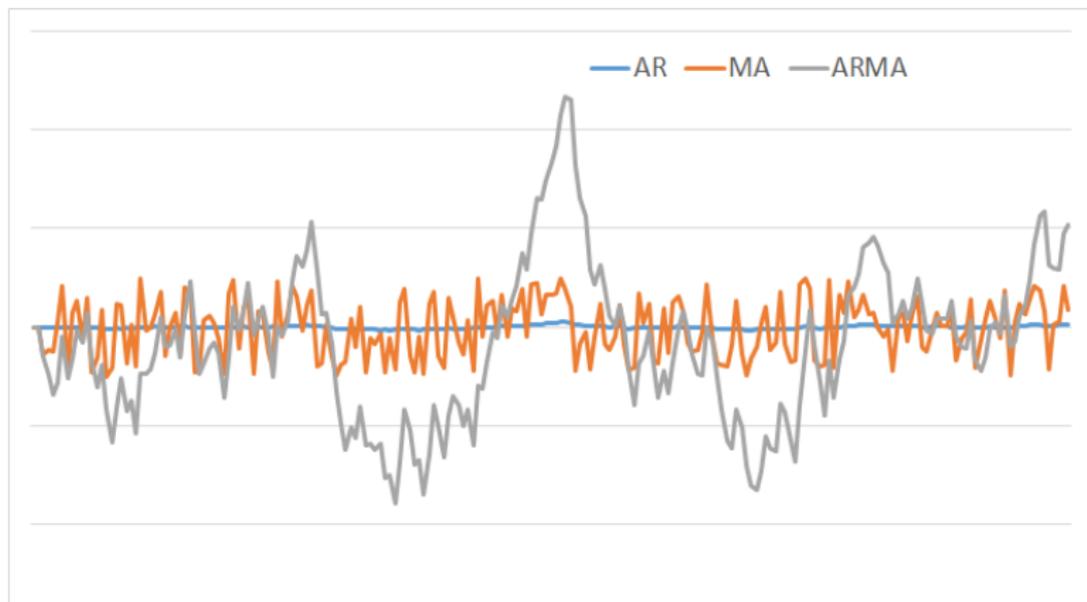


Figure: $\rho = 0.90$ and $\theta = 50$

ARMA(1,1) properties

ARMA processes have the same properties than AR processes when it comes to stationarity and the weak dependence hypothesis:

- $|\rho| < 1$ is required
- The mean value of the series should be 0.

Guessing the parameters of any ARMA(p,q) model (1/5)

Lagged values

- The lagged values of a time series are its values of $t - x$ considered at time t .
- Lagged values are often used for correlation analysis between a series at time t and $t - x$.

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PACF: Partial autocorrelation functions

- The partial autocorrelation is like the autocorrelation, but controlled for smaller lags. For example: the PACF at $t - 2$ discards any correlation that might have shown at $t - 1$.
- ACF and PACF are usually similar for the first lag, but diverge afterwards.

Guessing the parameters of any ARMA(p,q) model (2/5)

The values p and q of any ARMA model can be guessed using two tools:

- The autocorrelogram for the q of the MA model.
- The partial autocorrelogram for p of the AR model.

Partial autocorrelation

In time series analysis, the **partial autocorrelation function (PACF)** gives the partial correlation of a time series with its own lagged values, controlling for the values of the time series at all shorter lags. It contrasts with the **autocorrelation function**, which does not control for other lags.

$$\alpha(h) = \text{Cor}(x_{t+h} - P_{t,h}(x_{t+h}), x_t - P_{t,h}(x_t))$$

with $P_{t,h}(x)$ the projection of x onto the space spanned by $x_{t+1}, \dots, x_{t+h-1}$.

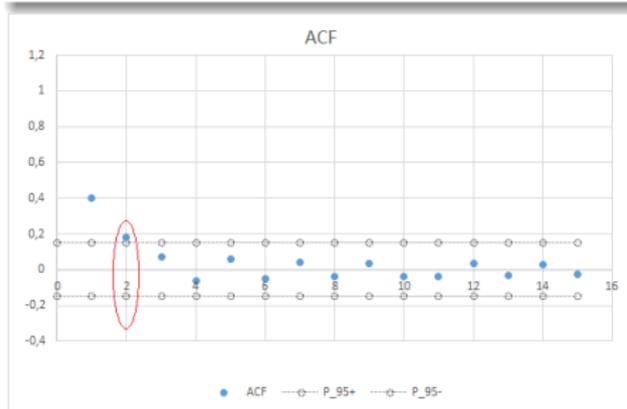
Guessing the parameters of any ARMA(p,q) model (3/5)

Autocorrelation and partial autocorrelation confidence intervals.

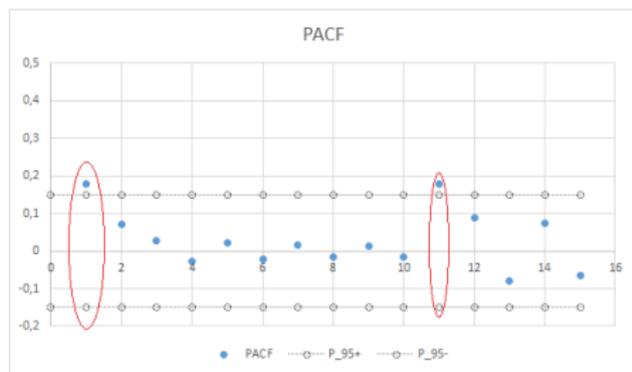
Under the null hypothesis that we expect 0 autocorrelation, the confidence interval for 95% is: $P_{0.95} = 0 \pm \frac{2}{\sqrt{T}}$

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Anything outside of these intervals denotes an autocorrelation and can be used to identify the type of ARMA(p,q) model.



(a) ACF showing MA(2)



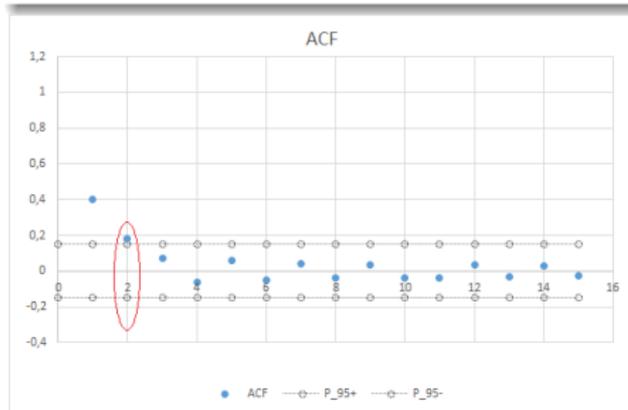
(b) PACF showing AR(1) or AR(11)

Figure: A possible ARMA(1,2) or ARMA(11,2)

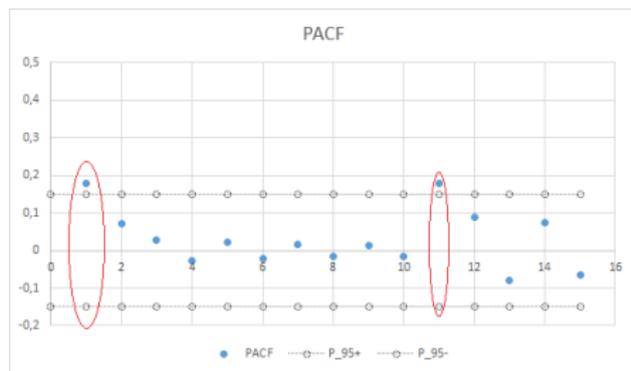
Guessing the parameters of any ARMA(p,q) model (4/5)

ACF and PACF: how to read ?

- The ACF of a stationary AR(p) model goes to zero at an exponential rate, while the PACF becomes zero after lag p.
- For an MA(q) model, it is the opposite: the ACF cuts off brutally after lag q and the PACF goes to zero relatively quickly.



(a) ACF showing MA(2)



(b) PACF showing AR(1) or AR(11)

Figure: A possible ARMA(1,2) or ARMA(11,2)

Guessing the parameters of any ARMA(p,q) model (5/5)

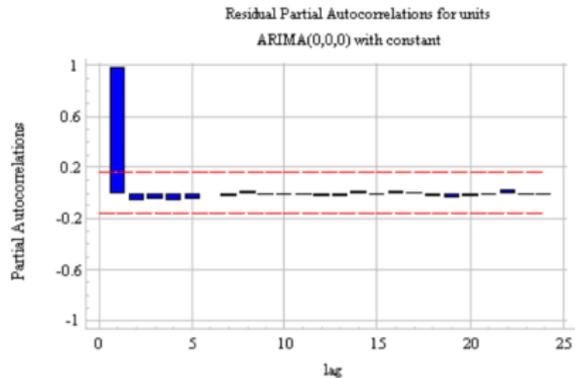
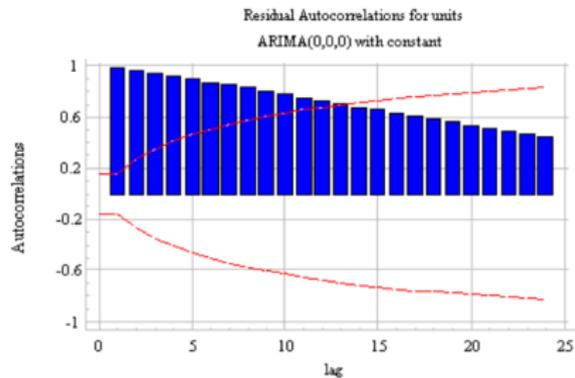
Tips to interpret PACF and ACF

- High values for p and q are highly suspicious. Check if your result makes sense depending on the application field.
- AR model may have *echo phenomena*: An AR(2) models may show echoes on lags 4 and 6. Once again, high values are suspicious.
- An isolated spike on a high value lag out of the confidence interval is probably just noise.

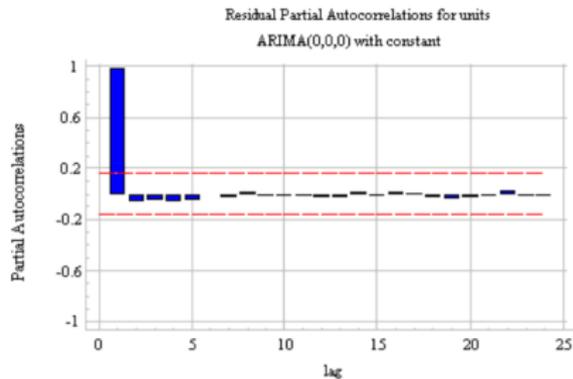
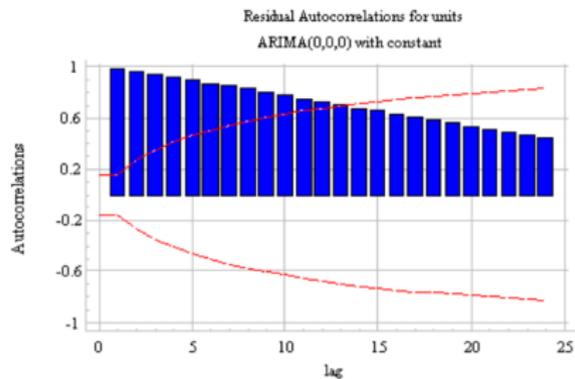
Important remarks

- Depending on your software, your ACF and PACF graphs may look very different and it can be confusing :
 - Lag 0 should have an auto-correlation of 1.
 - In R, the PACF starts at lag = the number of time the series was differentiated.
- ACF and PACF readings are not enough, and several models still need to be tested based on their likelihood and performances.

Guessing the parameters of any ARMA(p,q) model: Example

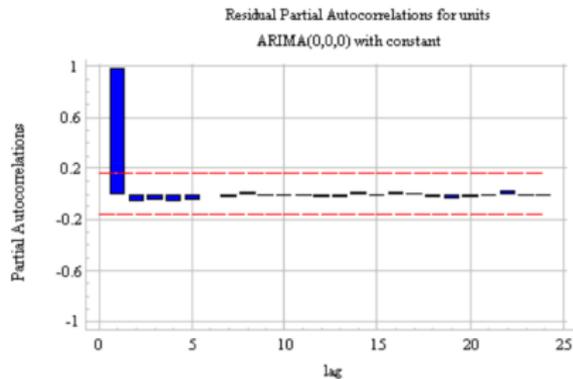
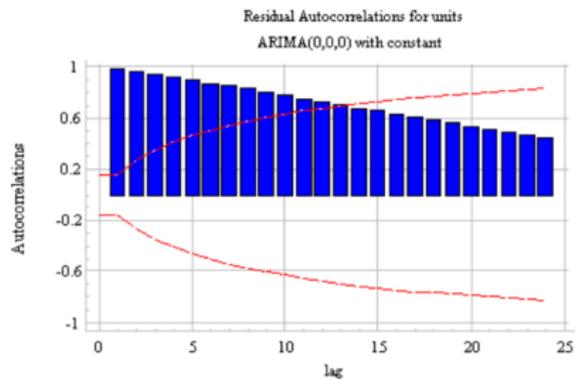


Guessing the parameters of any ARMA(p,q) model: Example



- The autocorrelations are significant for a large number of lags. But perhaps the autocorrelations beyond lag 1 are merely due to the propagation of autocorrelation at lag 1.
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- This is confirmed by the PACF: This could be an ARMA(1,0)
- Remark: The ACF does not go down exponentially which means that the series may need to be differenced (see ARIMA)

Outline

- 1 Fundamentals of time series analysis
- 2 The ARMA model
- 3 The ARIMA model**
- 4 Useful statistical tools

From ARMA to ARIMA

We have seen that with time series we are not always working directly with the original series, but quite often with its differentiated version.

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The $ARIMA(p,d,q)$ model is an $ARMA(p,q)$ model applied to the d -th derivation of a time series.

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- The parameter d is usually found by trying several values incrementally until fitting model is found.
- Parameters p and q are found using the same methods (acf and pacf) than for classical ARMA models on a given d -th derivation.

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Remark: Unless you are dealing with a very complicated phenomenon, d rarely goes beyond 2.

Rules to Guess the number d parameter in ARIMA (1/2)

- Rule 1 : If a series has a positive autocorrelation out to a high number of lags, then it probably needs a higher order of differencing.
- Rule 2 : If the lag-1 autocorrelation is 0 or negative, or the autocorrelations are all small and patternless, then the series does not need a higher order of differencing. If the lag-1 autocorrelation is -0.5 or more negative, the series may be overdifferenced. **BEWARE OF OVERDIFFERENCING !**
- Rule 3: The optimal order of differencing is often the order of differencing at which the standard deviation is lowest.

Rules to Guess the number d parameter in ARIMA (2/2)

- Rule 4:
 - A model without differencing assumes that the original series is stationary (mean-reverting).
 - A model with one order of differencing assumes that the original series has a constant average trend (e.g. a random walk or SES-type model, with or without growth).
 - A model with two orders of total differencing assumes that the original series has a time-varying trend (e.g. a random trend).
- Rule 5:
 - A model without differencing normally includes a constant term (which allows for a non-zero mean value). A model with two orders of total differencing normally does not include a constant term.
 - In a model with one order of total differencing, a constant term should be included if the series has a non-zero average trend.

Limits of the ARIMA model

ARIMA models have several limits:

- They can only handle univariate data
- The parameters are quite complex to guess and the properties difficult to assess when the parameters p and q grow beyond 2.
- These models do not allow for an in depth analysis of a time series (series segmentation event detection, state detection, etc.)

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Augmented Dickey Fuller Test

- Short name : ADF test
- Null Hypothesis H_0 ($p > 0.05$) : Presence of a root unit, meaning the time series is not stationary
- Alternate Hypothesis : The time series could be stationary

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What to do with the result ?

- If $p > 0.05$: we can't reject the null hypothesis and the time series is not stationary
- In the presence of a root unit (or the absence of stationnarity), differentiating the time series ($d = d + 1$ in the ARIMA model) should be considered before looking for the ARMA parameters using ACF and PACF
- **Remark:** The absence of a root unit (alternate hypothesis) implies that the time series could be stationary, but it does not imply that it definitely is.

Kwiatkowski-Phillips-Schmidt-Shin Test

- Short name : KPSS test
- Null Hypothesis H_0 ($p > 0.05$) : The time series is stationnary or trend-stationnary
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What to do with the result ?

- If $p > 0.05$: we can't reject the null hypothesis and the time series is stationnary
- If the series is stationnary, it is possible to run an ARMA analysis by looking at the ACF and PACF

Box-Pierce Test

- Short name : BP test
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When to use ?

- Can be used on a time series to confirm that searching for AR parameters is relevant (but it says nothing about MA parameters)
- Can be used on the residuals of a model: Good residuals should not have any auto-correlation.

LJung-Box Test

- Other name : LJung-Box Q Test
- Null Hypothesis H_0 ($p > 0.05$) : The data are independently distributed
- Alternate Hypothesis : The data exhibit serial correlation and are not independently distributed

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When to use ?

- Is roughly equivalent to the Box-Pierce test, but considered more reliable
- It can be used on a time series or on its residuals

Shapiro-Wilk Test

- Short name : SW test
- Null Hypothesis H_0 ($p > 0.05$) : The time series or the sample follows a gaussian distribution
- Alternate Hypothesis : It doesn't not follow a gaussian distribution

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When to use ?

- Can be used on a time series to confirm a Gaussian distribution
- Can be used on the residuals of a model to confirm that they follow a Gaussian distribution which makes them more likely to be white noise (desirable property).