Statistics and optimization of random pan stacking

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Abstract. We consider a minimal model of random pan stacking. A single pan consists of a V-shaped object characterized by its internal angle α. The stack is constructed by piling up N pans with different angles in a given, random order. The set of pans is generated by sampling from various kinds of distributions of the pan angles: discrete or continuous, uniform or optimized. For large N the mean height depends principally on the average of the distance between the bases of two consecutive pans, and the effective compaction of the stack, compared with the unstacked pans, is $2\log 2/\pi$. We also obtain the discrete and continuous distributions that maximize the mean stack height. With only two types of pans, the maximum occurs for equal probabilities, while when many types of pans are available, the optimum distribution strongly favours those with the most acute and the most obtuse angles. With a continuous distribution of angles, while one never finds two identical pans, the behaviour is similar to a system with a large number of discrete angles.

Keywords: pan stacking, optimization, random processes

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1. Introduction

The stacking of objects is a familiar operation. For example, a common conundrum of daily life is how to arrange a collection of pots and pans so that the stack will fit in a kitchen cupboard. Because of the varying form and convex nature of the pans, the height will vary depending on the order in which they are stacked. Other examples in the macroscopic realm, which typically involve identical objects, include the stacking of bowls, cups and chairs. Despite its ubiquitousness, there are relatively few published studies of stacking. Among these, Guendelman et al. [1] developed a simulation methodology to model non-convex rigid body stacking and Li et al. [2] introduced the concept of stackabilization, or how to design 3D objects so that they are more amenable to stacking. Thanavelu et al. [3] proposed a method for building structures from stacked, irregularly shaped objects.

At the microscopic level, stacking is associated with self-assembly. Columnar formation has been observed in bowlics [4], a kind of liquid crystal, and colloidal nanodisks [5]. Simulations of bowl-like structures [6] and [7] showed that the phase behavior is determined by a competition between the isotropic-nematic transition and stacking. Under some conditions red blood cells may stack to form rouleaux [8].

Stacking also features in processes involving granular materials. Edwards famously introduced the concept of powder compactivity by analogy with macroscopic thermodynamics [9]. More recently, the focus has shifted to shape-anisotropic granular particles [**?**] and in particular Gravish et al. [10] studied geometrically induced cohesion in ensembles of granular "U" particles that mechanically entangle through particle interpenetration.

There are few simple mathematical models of stacking. One well-known exception is a model of overhang: how far off the edge of a table can one reach by stacking N homogeneous, frictionless blocks [11]? Rather surprisingly, if one imposes a one-on-one restriction the overhang is asymptotic to $\log N/2$, i.e., there is no limit. Various generalizations of this original model have also been considered [12, 13]. We also note studies of the height of rectangle packings [**?**] and their connection with random walks [**?**].

In the method proposed to build structures from randomly stacked irregular objects [3] it may be desirable to maximize the height. In several children's games (e.g. Wood Bottle Challenge, TOMY Pile Up Pirates Stacking, Cat Pile, etc.), sport exhibitions, art installations and environmental art, an objective is to build the highest possible stack.

A natural question is then: what is the mean height of a stack produced by randomly stacking similar objects which can be inserted into each other? Also, does there exist a particular distribution of the object properties which maximizes the mean height of the pile? It is rather intuitive that the best strategy cannot consist in choosing only the highest individual objects, since such similar objects fit perfectly when inserted into each other. It is, however, much more subtle to guess what kind of object diversity is

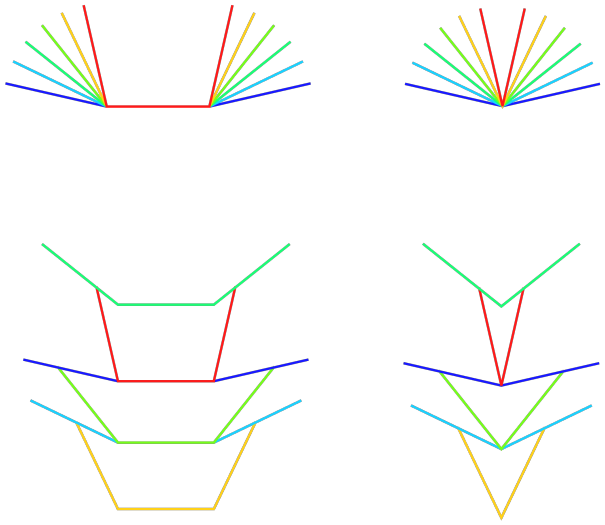


Figure 1. Pan stacking models. Stacks consisting of six pans formed from pans with uniformly spaced angles. The left figures show a three-line representation including a flat bottom, while the ones on the right show a simplified representation using two lines. For a given configuration, both representations have the same height. Each type of pan is represented using a single colour: red for the narrowest and blue for the broadest. The upper and lower figures show the shortest and tallest stacks respectively.

needed to counterbalance the decreasing nesting effect.

In this article, we propose a minimal model for the one-dimensional, random stacking of distinct objects with varying degrees of interpenetrability. We focus on the statistical properties of the stacks averaged over multiple realizations of the pan set when the pan angles are sampled from discrete and uniform distributions, and we seek the distributions that optimize the mean stack height.

2. Model

We can represent a pan as a two-dimensional object consisting of a flat bottom and two walls that rise at an angle α from the bottom. See figure 1. However, it is possible to use a simplified representation consisting of only two lines and this what we do in the remainder of the article. A pan consists of two fused line segments, each of length one, with an internal angle 2α. $\alpha = 0$ corresponds to a vertical line segment of length one and $\alpha = \pi/2$ corresponds to a horizontal line of length two (see figure 2). For simplicity we will assume that the pan thickness is negligible compared with its dimensions.

Let $\alpha_1, \alpha_2, \cdots, \alpha_N$ denote the angles of the pans in a given configuration from bottom to top. The same mathematical model also describes a stack of cones with

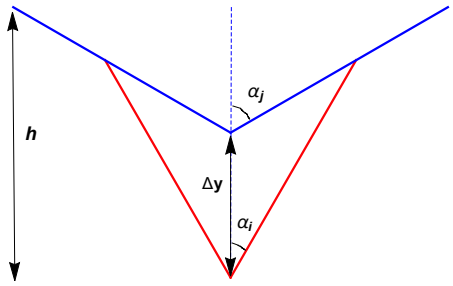


Figure 2. Pan stacking geometry. A pan of angle α_j is stacked on top of a narrower pan of angle $\alpha_i < \alpha_j$. h is the total height of the 2-pan stack and Δy is the inter-pan distance.

internal angles α_i in three dimensions. The height of the stack is given by

$$h_N(\alpha_1, \alpha_2, \cdots, \alpha_N) = \sum_{i=1}^{N-1} \Delta y(\alpha_i, \alpha_{i+1}) + \cos \alpha_N \tag{1}$$

where

$$\Delta y(\alpha_i, \alpha_j) = \begin{cases} \cos \alpha_i - \sin \alpha_i \cot \alpha_j, & \alpha_i \leq \alpha_j \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

is the height difference between the nadirs of pan j stacked on top of pan i. The minimum height increment is zero. This occurs with two consecutive, identical pans ($\alpha_j = \alpha_i$) or if the upper pan is more acute than the lower one ($\alpha_j < \alpha_i$).

We also define a compaction ratio ‡ as

$$\phi = \frac{h_N(\alpha_1, \alpha_2, \cdots, \alpha_N)}{\sum_i h_1(\alpha_i)} \tag{3}$$

where $h_1(\alpha_i) = \cos \alpha_i$ is the height of the individual pan i.

We consider stacks formed from pans whose angles are sampled from either a continuous or discrete distribution.

3. Stack statistics for uniformly distributed pan angles

Let us assume that a stack is constructed of pans with a statistical distribution of angles characterized by the normalised distribution $\psi(\alpha)$. The mean height, averaged over all realizations, is

$$\langle h_N \rangle = \int_0^{\pi/2} d\alpha_1 \cdots \int_0^{\pi/2} d\alpha_N \; \psi(\alpha_1) \cdots \psi(\alpha_N) h(\alpha_1, \cdots, \alpha_N) \tag{4}$$

Substituting (1) and introducing the symmetrized function $g(\alpha_1, \alpha_2) = (\Delta y(\alpha_1, \alpha_2) + \Delta y(\alpha_2, \alpha_1))/2$ gives

$$\langle h_N \rangle = (N-1)\langle g(\alpha_1, \alpha_2) \rangle + \langle \cos \alpha \rangle \tag{5}$$

‡ Not to be confused with the Edwards compactivity [9]

where $\langle g(\alpha_1,\alpha_2)\rangle = \int_0^{\pi/2} d\alpha_1 \int_0^{\pi/2} d\alpha_2 \quad \psi(\alpha_1)\psi(\alpha_2)g(\alpha_1,\alpha_2)$ and $\langle\cos\alpha\rangle = \int_0^{\pi/2} d\alpha\psi(\alpha)\cos\alpha$. The explicit form of $g(\alpha_1,\alpha_2)$ is

$$g(\alpha_1,\alpha_2) = \begin{cases} \dfrac{\sin(\alpha_2-\alpha_1)}{2\sin\alpha_2}, & \alpha_1 < \alpha_2 \\ \dfrac{\sin(\alpha_1-\alpha_2)}{2\sin\alpha_1}, & \alpha_1 > \alpha_2 \\ 0, & \alpha_1 = \alpha_2 \end{cases} \tag{6}$$

This function gives the mean distance between the bases of two pans stacked both ways.

For a uniform distribution of angles, $\psi(\alpha) = \frac{2}{\pi},\ 0 \le \alpha < \pi/2$, the mean height is

$$\langle h_N\rangle = (N-1)\langle g\rangle + \langle\cos\alpha\rangle = (N-1)\frac{4\log 2}{\pi^2} + \frac{2}{\pi} \tag{7}$$

and the effective compaction, or the ratio of the stack height to the sum of the height of the pans considered separately, in the large N limit is

$$\bar{\phi} = \frac{\langle h_N\rangle}{N\langle\cos\alpha\rangle} = \frac{2}{\pi}\log 2 = 0.4413 \tag{8}$$

That is, an average stacking of a large number of pans is 44% as high as the sum of the height of the individual pans.

With more effort one can calculate the second moment:

$$\begin{aligned} \langle h_1^2\rangle &= \frac{1}{2} \\ \langle h_2^2\rangle &= \frac{1}{2} + \frac{2}{\pi^2}(5 - 4\log 2) \\ \langle h_N^2\rangle &= \frac{1}{2} + \left(\frac{2}{\pi}\right)^2 (3 + N - 4\log 2) + \left(\frac{2}{\pi}\right)^3 (4G + \pi(1-3\log 2) + 2\log 2)(N-2) \\ &\quad + \left(\frac{2}{\pi}\right)^4 (\log 2)^2 (N-2)(N-3),\ N \ge 3 \end{aligned} \tag{9}$$

where $G = 0.915966$ is the Catalan constant. The variance behaves as

$$\langle h_N^2\rangle - \langle h_N\rangle^2 = c_1 N + c_0,\ N \ge 3 \tag{10}$$

with $c_1 = (32G\pi + \pi^2(10 - 24\log 2) - 48(\log 2)^2)/\pi^4 = 0.03625$ and $c_0 = (-64G\pi - 14\pi^2 - 16\pi\log 2 + 40\pi^2\log 2 + 80(\log 2)^2)/\pi^4 + 1/2 = 0.036985$. The fact that the variance is not directly proportional to the number of pans is due to the correlation between two adjacent pairs, e.g. $\langle g_{12}g_{23}\rangle \neq \langle g_{12}^2\rangle$. For large N, however, this correlation is insignificant.

We examined the height distribution of an ensemble of 20-pan stacks numerically. To do this a single stack of 20 pans with angles sampled from a uniform distribution was generated. The height was recorded and the procedure was repeated 10,000 times to obtain a sampling of the height distribution shown in the histogram of Figure 3. With this number of pans, the distribution is accurately described by a gaussian distribution with mean given by (7) and standard deviation $\sqrt{c_1 N}$.

It is also interesting to examine the height distribution of the extreme stacks. We did this as follows. First, a set of N pans was generated by sampling the angles from a

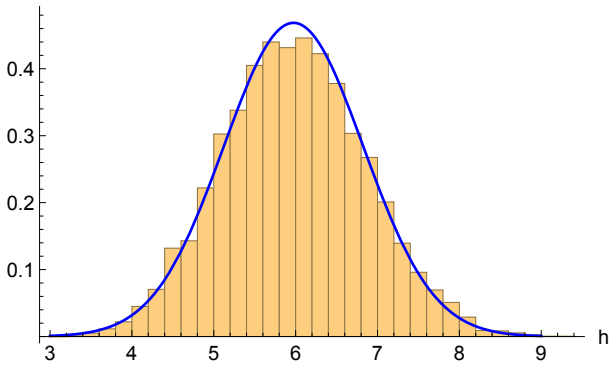


Figure 3. Height distribution in 10000 realizations of a 20 pan stack. The solid curve shows a gaussian distribution with the mean given by (7) and standard deviation $\sqrt{c_1 N}$.

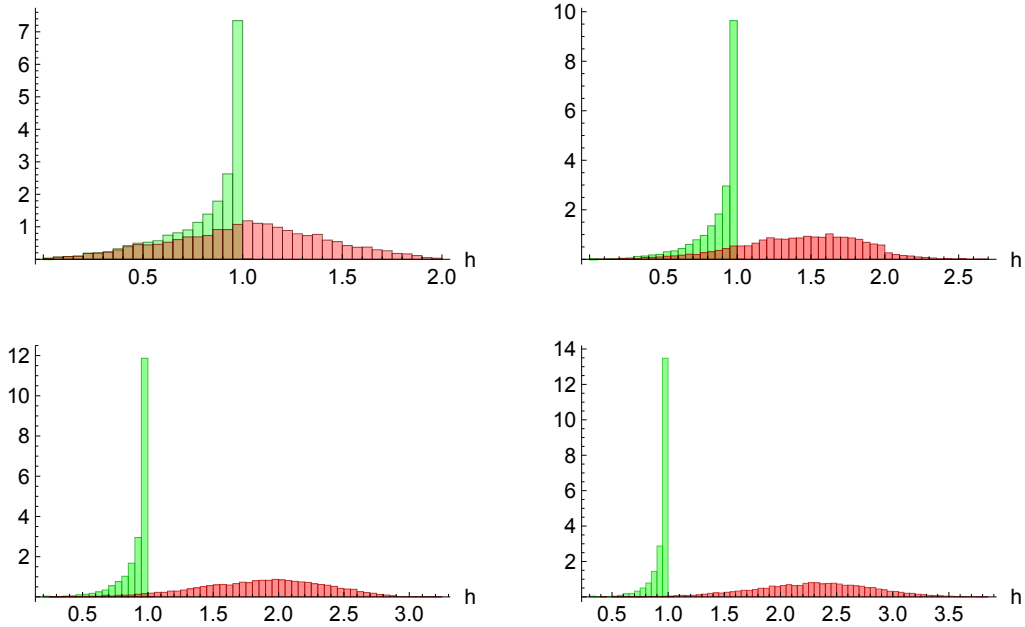


Figure 4. Height distribution of the lowest (green) and highest (red) stacks in 10000 realizations of an N-pan stack. The pan angles are sampled from a uniform distribution in the range $0 < \alpha < \pi/2$ and $N = 2, 3, 4, 5$, left to right, top to bottom.

uniform distribution. For this set all N! stacks corresponding to all possible permutations of the pan order were computed. The height of the lowest and highest stacks were then recorded. This procedure was repeated 10,000 times to generate histograms of the distribution of the lowest and highest stacks. Results for $N = 2, 3, 4, 5$ are shown in Fig. 4.

The stack with the lowest height always corresponds to the ordering $\alpha_1 > \alpha_2 > \cdots \alpha_{N-1} > \alpha_N$, i.e., the pans are arranged from the bottom in order of decreasing angle.

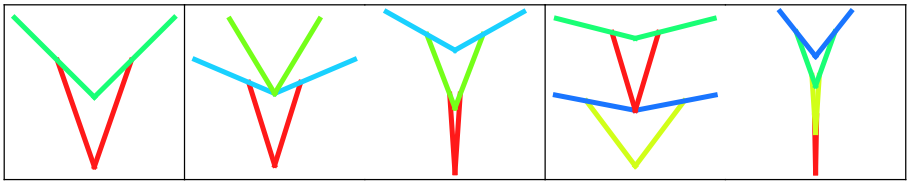


Figure 5. Examples of the highest stack. For $N = 2$ the highest stack is always ordered 1-2; For $N = 3$ the highest stack may be ordered 1-2-3 or 1-3-2, while for $N = 4$ the highest stack is most often 2-4-1-3 but may be 1-2-3-4. The pan angles are selected from a uniform, continuous distribution in the range $0 < \alpha < \pi/2$.

The stack with the greatest height *may* have the configuration $\alpha_1 < \alpha_2 < \cdots \alpha_{N-1} < \alpha_N$, but other configurations are possible and may occur with much higher frequency. Also the maximum height may correspond to degenerate configurations.

For $N = 2$ the highest stack is always ordered 12. Even in this case, however, it appears difficult to obtain an analytical expression for the height distribution. One observes from the numerical simulations that it is broader than the shortest stack. It is almost triangular with a maximum at one. The distributions are more rounded for $N > 2$. For $N = 3$, one notes the presence of a discontinuity at $h = 2$. For $N = 2$ there is considerable overlap between the two distributions, while for $N = 4$ the overlap is considerably reduced. As N increases further the distributions become increasingly separated.

The height distribution of the shortest and tallest stacks are shown in figure 4. For all N the height distribution of the lowest stack is restricted to the interval $(0, 1)$. It rises rapidly from zero to a maximum at $h = 1$ where it falls discontinuously to zero. Using the property that in the lowest stack the pans must be strictly ordered in decreasing angle, i.e., $\alpha_1 > \alpha_2 > \cdots > \alpha_N$ one can show (see Appendix) that the distribution is given exactly by

$$f_0(x) = \frac{2N(1 - \frac{2}{\pi}\arccos x)^{N-1}}{\pi\sqrt{1 - x^2}} \tag{11}$$

As N increases it approaches a dirac delta function, $\lim_{N\to\infty} f_0(x) = \delta(x - 1)$.

4. Optimized pan angle distributions

Domestic engineers and warehouse managers are routinely confronted with the problem of how to order a collection of objects so that the height of the stack is minimized. Here we consider a related, but different problem where the stack is formed by sequentially and randomly selecting objects according to a given probability distribution. The minimum mean height is obtained when all the objects are the same, i.e. the distribution is a delta function. In contrast, the distribution that *maximizes* the mean height is more subtle. This may be useful in a number of applications. For example, it has been proposed to generate structures by randomly stacking objects [3]. In the remainder of this section we consider both discrete and continuous distributions.

4.1. Discrete distribution of angles

Here we consider a pool of n different types of pans with angles $\theta_1 < \theta_2 \cdots \theta_{n-1} < \theta_n$. A stack is generated by selecting N pans, with probability p_i of selecting a pan with angle θ_i. We will focus on the large N limit so we can neglect the contribution of the $\cos\alpha_N$ term in (7). Then the mean height of stacks per pan is given by:

$$\langle g \rangle = \sum_{i=1}^{n} \sum_{j=i+1}^{n} 2 p_i p_j g(\theta_i, \theta_j) \tag{12}$$

For example, for $n = 2$, $\theta_1 = \pi/6, \theta_2 = \pi/3$ with probabilities $p_1, p_2 = 1 - p_1$ we find

$$\langle g \rangle = \frac{p_1 p_2}{\sqrt{3}} \tag{13}$$

If only one component is present, the height of the stack is zero (neglecting the $\cos\alpha_N$ term). A maximum mean height of $1/(4\sqrt{3}) = 0.144338$ is obtained when each pan is selected with equal probability $p_1 = p_2 = 1/2$.

Now let us consider a three-component system with angles $\theta_1 = \pi/6, \theta_2 = \pi/4, \theta_3 = \pi/3$ and probabilities p_1, p_2, p_3. The mean height is

$$\langle g \rangle = \frac{\sqrt{3}-1}{2} p_1 p_2 + \frac{1}{\sqrt{3}} p_1 p_3 + \frac{1}{\sqrt{2}} \left(1 - \frac{1}{\sqrt{3}}\right) p_2 p_3 \tag{14}$$

A contour plot of this function is shown in figure 6. A unique maximum mean height of 0.1469 occurs for a composition $p_1 = 0.4481, p_2 = 0.1176, p_3 = 0.4344$.

For $n \in \mathbb{N}$ the problem of finding the optimum of the p_i quadratic polynomials of (12) with the constraint $\sum_i p_i = 1$ is mathematically well defined. The optimum compositions and corresponding mean heights are shown in figures 7 and 8, respectively. We note that the extreme pans with the smallest and largest angles are increasingly favoured as the number of components increases. There is a rapid drop in the value of p_i after the component with the smallest angle. This is followed by a nearly linear region with a negative slope followed by a rapid rise from p_{n-1} to p_n. The mean optimum height increases monotonically with the number of components. There is a rather large increase between $n = 2$ to $n = 3$, while for $n > 8$ the height increases very slowly.

As n increases further, the pans with the most acute and least acute angles become increasingly present in the optimum stack. Figure 8 shows that the coefficients vary as $1/n$ for large n and approach a finite value as $n \to \infty$. The optimum mean height increases as the number of components increases and approaches an asymptotic value $g_\infty - \langle g \rangle \sim 1/n$. To understand this behaviour, let us consider a simplified stacking model for which the spacing between two objects of types i and j is given by $g_{ij} = 1 - \delta_{ij}$. The mean value is then $\langle g \rangle = 2 \sum_{i=1}^{n} \sum_{j=i+1}^{n} p_i p_j g_{ij}$. If all objects occur with equal probability $p_i = 1/n$ so that $\langle g \rangle = (1 - 1/n)$. This shows that the asymptotic increase of the mean inter-pan spacing with increasing number of components is due to reduction of the probability of finding two adjacent, identical pans with a zero height increment.

The optimum composition results from a competition between two opposing effects. First, since the function $g(\theta_i, \theta_j)$ increases monotonically with θ_j, it reaches a maximum

for $i = 1$ and $j = n$. This implies that the term $2g(\theta_1, \theta_n)p_1p_n$ in the expression for $\langle g \rangle$, (12), is dominant, and this in turn implies that both p_1 and p_n are necessarily the dominant probabilities in the optimum distribution. This pair of pans has the largest inter-pan distance. But, as discussed above, since the pans are randomly selected in stacking process, it is possible that two consecutive pans are identical, leading to a zero inter-pan distance and a waste of pans for increasing the stack height. To counterbalance this effect, the stack must contain some intermediate pans that reduce the probability of consecutive, identical pans.

To gain insight into the shape of the downward sloping distribution between the two extreme values p_1 and p_n, one can devise two oversimplified models for which the spacing between two objects of type i and j is given either by $j - i$ or by $(j - i)/j$. The first case corresponds to extremely flat pans such that $\alpha_i \sim \alpha_j \sim \pi/2$, which leads to the *linear* height variation $\Delta y(\alpha_i, \alpha_j) \sim \alpha_j - \alpha_i$. The second case corresponds to extremely acute pans such that $\alpha_i \sim \alpha_j \sim 0$, which leads to a *non-linear* height variation $\Delta y(\alpha_i, \alpha_j) \sim 1 - \alpha_i/\alpha_j$. For each of these cases, the optimization of the mean height of the stack provides a different regime, similar to those of figures 7 and 12, but with a more regular behaviour. For the flat case (SEE FIGURE AND APPENDIX), one gets a pure binary distribution with $p_1 \simeq p_n \simeq 0.5$ and a quasi-zero distribution in the middle. On the contrary, for the acute (non-linear) case, one obtains a regularly decreasing distribution between p_1 and p_{n-1} before a sharp increase for the last value p_n. This extreme, oversimplified model of acute pans captures the main points of the downward sloping distribution of the article's model. We clearly see that it is due to the non-linear behaviour of $\Delta y(\alpha_i, \alpha_j)$ near $\alpha_i \sim \alpha_j \sim 0$. We can go further and note that the global behaviour of figures 7 and 12 can be approximately reconstituted by summing up the two extreme previous cases: the acute case giving the $\theta \simeq 0$ behaviour and the flat case giving the $\theta \simeq \pi/2$ behaviour.

Some realizations of stacks generated from pans selected from uniform and optimized distributions are shown in figure 9. A short movie showing the real time stacking of pans taken randomly from these two distributions, as well as one with only the two extreme pans, is also available in Supplementary Material.

4.2. Continuous distribution of angles

We have already considered the statistics of stacks generated by randomly selecting pans from a continuous, uniform distribution in section 3. We now seek a continuous distribution that maximizes the mean height of an N pan stack, again in the limit of large N. In contrast to the discrete distributions discussed above, all N pans are distinct and the probability of finding two, identical pans is zero.

More specifically, the problem is to find the distribution $\psi(\alpha)$, with $\alpha_{\min} \leq \alpha \leq \alpha_{\min}$, that maximizes

$$\langle g(\alpha_1, \alpha_2) \rangle = \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha_1 \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha_2 \; \psi(\alpha_1)\psi(\alpha_2)g(\alpha_1, \alpha_2) \tag{15}$$

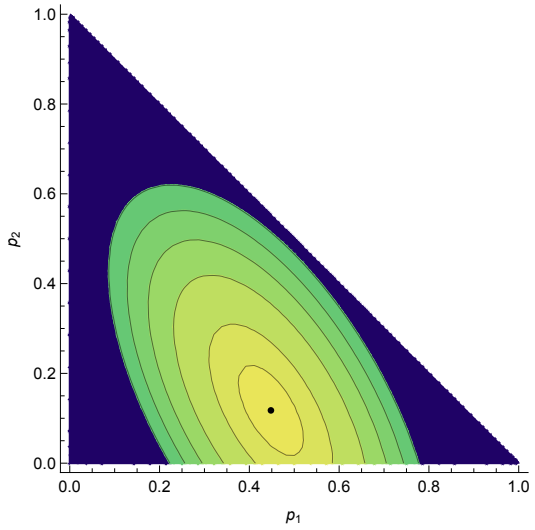


Figure 6. Contour plot of the stack height per pan as a function of the composition of a 3 component system with angles $\pi/6, \pi/4, \pi/3$. The point shows the composition that yields the greatest height. The contours show lines of constant height at intervals of 0.01.

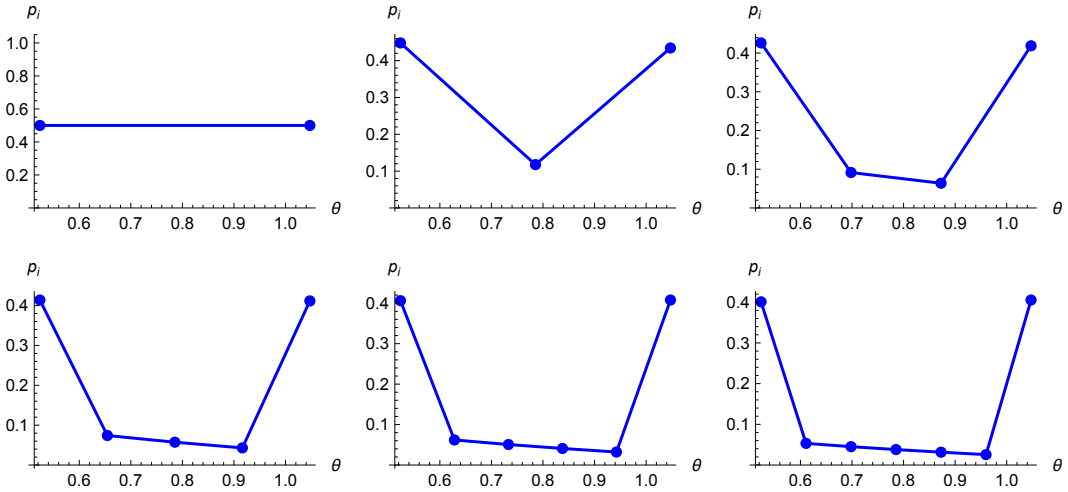


Figure 7. Composition that maximizes the mean height of a stack (in the large N limit) constructed from n types of pans with angles uniformly spaced between $\pi/6$ and $\pi/3$. In constructing the stack, a pan of type i is selected with probability p_i. $n = 2, 3, 4, 5, 6, 7$ left-to-right, top-to-bottom.

subject to constraint that $\int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha\ \psi(\alpha) = 1$. Applying standard calculus of variations methods gives the following Fredholm integral equation of the first kind

$$\int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha'\ \psi(\alpha') g(\alpha, \alpha') = \lambda, \tag{16}$$

where $\lambda > 0$ is a Lagrange multiplier. Multiplying this equation by $\psi(\alpha)$ and integrating over α shows that $\langle g(\alpha_1, \alpha_2) \rangle = \lambda$, i.e., the Lagrange multiplier is equal to the optimized

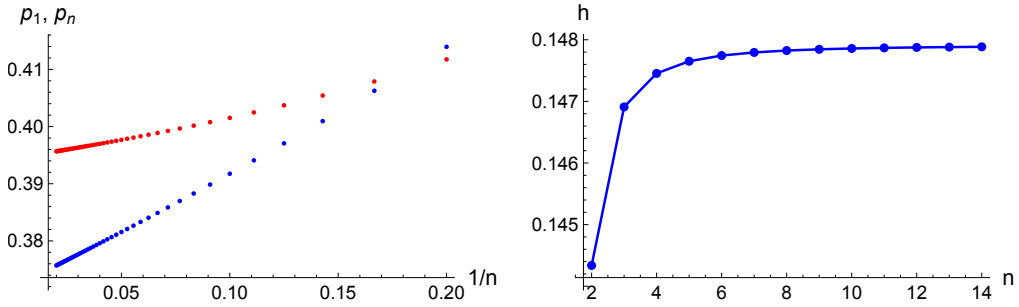


Figure 8. Left: Probabilities of the narrowest p_1 (top, in red) and broadest p_n (bottom, in blue) pans in the optimum composition as a function of the inverse of the number of components, $1/n$: Right: Mean height of the optimum stack formed from n components.

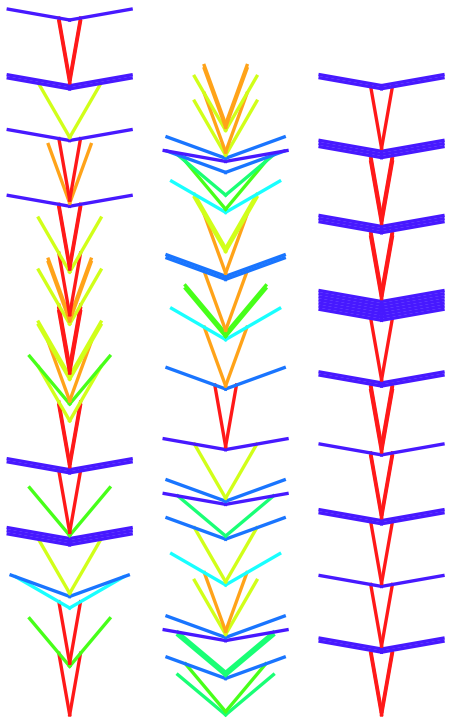


Figure 9. Three, $N = 40$ pan stacks constructed using $n = 8$ different types of pans with equally spaced angles in the range $\pi/6 < \alpha < \pi/3$. Left: the pans were selected from the optimised distribution; Middle: the pans were selected with equal probabilities, Right: the two extreme pans were selected with equal probabilities. For clarity, each pan is offset vertically by a small amount.

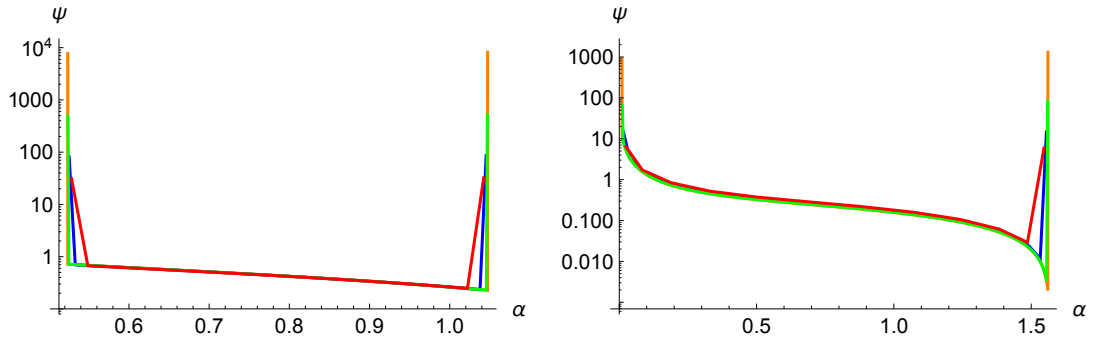


Figure 10. Distributions that maximize the height of a stack. The angle of the pans is restricted to the range $\pi/6 \le \alpha \le \pi/3$ (left) and $0.01 \le \alpha \le \pi/2 - 0.01$ (right), for 10 (red), 20 (blue), 50 (green) and 100 (orange) gaussian integration points. The optimum heights for 100 points are 0.1479 and 0.3716, respectively.

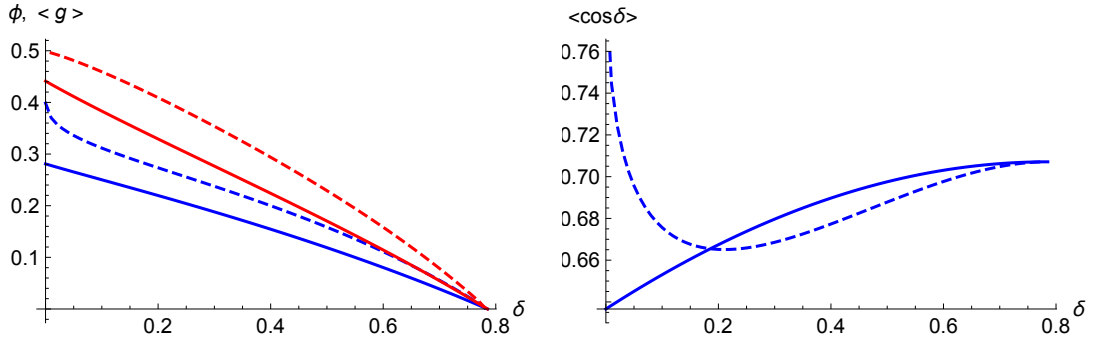


Figure 11. Left: Mean inter-pan distance $\langle g \rangle$ (blue) and compaction ratio ϕ (red) for a uniform distribution of pan angle in the range $\delta \le \alpha \le \pi/2 - \delta$ as a function of δ; Right: $\langle \cos\alpha \rangle$ for pan angles distributed in the range $\delta \le \alpha \le \pi/2 - \delta$ as a function of δ. In both cases, the solid and dashed lines show the results using the uniform and optimized distributions, respectively.

height.

We have been unable to find an analytic solution of this equation, but we solved it numerically using Gauss-Legendre quadrature. With this method the double integral in (15) is approximated by

$$\langle g \rangle = 2 \sum_{i=1}^{n} \sum_{j=i+1}^{n} w_i w_j \psi_i \psi_j g(x_i, x_j) \tag{17}$$

where $\psi_i = \psi(x_i)$ and w_i and x_i are the weights and abscissas of the gaussian quadrature, respectively. The resulting expression is maximized by finding the optimum values of $\psi_i, i = 1, n$. We verified the solution by confirming that it satisfies (16).

Some results are shown in figure 10 for pan angles restricted to the ranges $\pi/6 \le \alpha \le \pi/3$ and $0 \le \alpha \le \pi/2$. The behaviour is consistent with the results

presented earlier in figure 7 for a small, discrete set of pan angles and as the number of components increases (see figure 12) the resemblance becomes more striking. The optimized distribution strongly favours pans at the extremes of the allowed range. Just beyond $\alpha = \alpha_{\min}$, the probability drops precipitously and then continues to decrease more slowly in the intermediate range of α before rapidly rising just before $\alpha = \alpha_{\max}$. The difference between the highest and lowest values of $\psi(\alpha)$ increases with increasing range of allowed angles. Figure 11 compares the mean inter-pan spacing, $\langle\phi\rangle$, and the compaction ratio, ϕ, calculated using the optimized and uniform pan angle distributions. For the full range of angles ($0 \leq \alpha \leq \pi/2$), the inter-pan spacing is about 50% larger than with the uniform distribution, while the compaction ratio is about 25% larger.

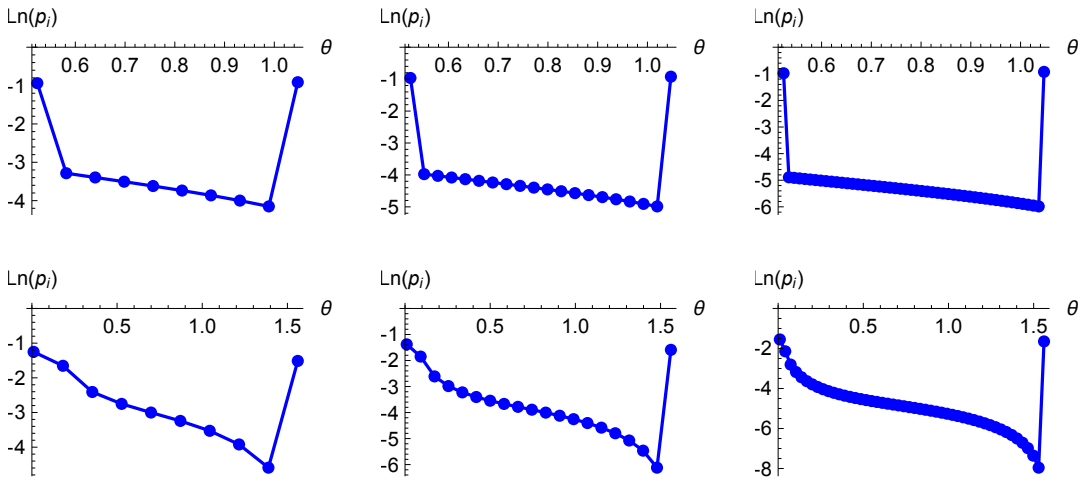
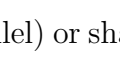
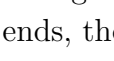


Figure 12. Discrete distributions that maximize the height of a stack. The angle of the pans is restricted to the range $\pi/6 \leq \alpha \leq \pi/3$ (top) and $0.01 \leq \alpha \leq \pi/2 - 0.01$ (bottom). For $n = 10$, 20 and 50 types of pans (left to right).

5. Discussion

Our model of random pan stacking is obviously highly idealized. It neglects at least three effects that may have important consequences.

The first arises from the thickness of the pans. If we want to go beyond the infinitely thin assumption and introduce a finite, homogeneous thickness w, identical for all pans (in the two-line model), we need to add a global factor $\sum_{i=1}^{N} w \csc \alpha_i$ to (1). This has the consequence of imposing a minimum allowed pan angle of $\alpha = \arctan w$. Moreover, two identical pans can no longer be completely inserted into each other so that they merge and become one. When $w << 1$, i.e., the thickness is small compared to the pan side, which is the usual situation, we expect the results to be qualitatively similar to the infinitely thin case. If, however, w is sufficiently large the strategy for maximizing the stack height may be reduced to piling up only the highest individual pans (those with

the smallest internal angles). The shape of the pans ends also has some effect. Three natural choices can be made: flat ends  (i.e., ends are orthogonal to the pan walls, which means that the top sides of the pan are shorter than the bottom sides), beveled ends  (i.e., all the pan sides are parallel) or sharp ends  (but in this case, each pan is characterized by two different internal angles and the analysis is slightly modified).

For example, with flat or beveled ends, the height of the stack becomes

$$h_N(\alpha_1, \alpha_2, \cdots, \alpha_N) = \sum_{i=1}^{N-1} \Delta y(\alpha_i, \alpha_{i+1}) + \cos\alpha_N + \delta w(\alpha_N) \tag{18}$$

with

$$\Delta y(\alpha_i, \alpha_j) = \begin{cases} w\csc\alpha_i + \cos\alpha_i - \sin\alpha_i \cot\alpha_j, & \alpha_i \leq \alpha_j \\ w\csc\alpha_i, & \text{otherwise} \end{cases} \tag{19}$$

where w is the pan thickness and $\delta w(\alpha_N)$ is the additional height increase due to the pan placed on top of the stack. This last term is model-dependent but is negligible in the large N limit.

The second effect is closely related to the first one and concerns the shape of the pans. In reality, the pans may have various shapes that can be quite different from the line segments considered here for simplicity. If the pan walls are concave, for example, the internal angle ceases to be the dominant parameter for optimizing the total stack height. If, on the contrary, the pan walls flare out, two different pans can perfectly insert into each other and the small angle pans may become ultra dominant. Finally, object specific features, e.g. handles, are expected to greatly modify the analysis.

The third effect occurs for objects that can move or be deformed during the stacking process. A well known effect is the rotation and sliding of cups when we attempt to pile them up in a kitchen cupboard. This rearrangement is not static and changes for each new cup in the pile. It also depends on the degrees of freedom of the cups, i.e. if the cups can turn and move along one or two directions. In the former case, an alternation of the cup orientations is quite intuitive, whereas one may reasonably expect a helicoidal structure in the latter case. Another effect arises for objects which can be deformed under the weight of the pans above. In this case, the number of pans becomes one of the dominant parameters in the problem of height optimization.

Finally, let us stress the fact that the randomness occurs at two different levels in our analysis. First, the pans are sampled from a continuous or discrete distribution. Then they are piled up in a random order to form the stack whose properties, e.g. the height, are determined. A different procedure is to systematically change the order of a given set of pans in the stack. The statistics problem thus becomes a combinatorial one, and we expect to see the emergence of particular, regular structures that maximize the stack height. This question will be addressed in a future publication.

6. Acknowledgments

We thank Luc Tartar for helpful discussions concerning the integral equation.

Appendix

Matrix elements

For a uniform distribution of pan angle in the range $0 \leq \alpha \leq \pi/2$ the necessary matrix elements are:

$$\langle g(\alpha_1,\alpha_2)\rangle = \frac{4\log 2}{\pi^2} = 0.2809$$
$$\langle g(\alpha_1,\alpha_2)^2\rangle = \frac{1}{\pi^2} = 0.1013$$
$$\langle g(\alpha_1,\alpha_2)g(\alpha_2,\alpha_3)\rangle = \left[\frac{1}{2} + G + \frac{1}{96}\left[\pi^3 + \pi(12 - 84\log 2 + 6(\log 2)^2)\right]\right]\left(\frac{2}{\pi}\right)^3 = 0.08272$$
$$\langle g(\alpha_1,\alpha_2)\cos\alpha_1\rangle = \frac{1}{8} + \frac{2(1-\log 2)}{\pi^2} = 0.1872$$

We consider a more general expression for the mean inter-pan distance $\langle g\rangle$ for a uniform distribution of angles in the range $\delta \leq \alpha \leq \pi/2 - \delta$:

$$\langle g\rangle = \int_\delta^{\pi/2-\delta} d\alpha_1 \int_\delta^{\pi/2-\delta} d\alpha_2\ \psi(\alpha_1)\psi(\alpha_2)g(\alpha_1,\alpha_2)$$
$$= \frac{\log[\cot(\delta/2)\cot(\pi/4+\delta/2)] - \cos\delta\log(\cot\delta) - (\pi/2-2\delta)\sin\delta}{(\pi/2-2\delta)^2}$$

with the properties $\lim_{\delta\to 0}\langle g\rangle = 4\log 2/\pi^2$ and $\lim_{\delta\to\pi/4}\langle g\rangle = 0$. Figure 11 shows how $\langle g\rangle$ and $\bar{\phi}$ depend on δ.

Derivation of the height distribution of the shortest stack

The angle, α, of a single pan is uniformly distributed. To find distribution of the height of the pan we first calculate the cumulative distribution function $P_Z(z) = \text{Prob}(0 \leq Z \leq z)$

$$P(z) = \int_0^z \int_0^{\pi/2} \psi(\alpha)\delta(z' - \cos\alpha)d\alpha\ dz' \tag{20}$$
$$= \frac{2}{\pi}\int_0^{\pi/2} d\alpha\Theta(z - \cos\alpha) \tag{21}$$
$$= \frac{2}{\pi}\int_0^z \frac{du}{\sqrt{1-u^2}} \tag{22}$$

The probability density function of the height is

$$p(z) = P'(z) = \frac{2}{\pi\sqrt{1-z^2}} \tag{23}$$

Now for a collection of N pans, in the shortest stack the pans are arranged in the order $\alpha_1 < \alpha_2 < \cdots < \alpha_N$. We therefore want the distribution of $z_{\max} = \max\{z_1, z_2, \cdots, z_n\}$. Another way of thinking about this is that only the tallest pan contributes to the stack

height. The cumulative distribution of the maximum is given by

$$Q_N(z) = \text{Prob}[z_{\max} \leq z] = \text{Prob}[z_1 \leq z, z_2 \leq z, \cdots, z_N \leq z] \tag{24}$$

$$= \left[\int_0^z p(z')dz'\right]^N = \left[1 - \frac{2}{\pi}\arccos z\right]^N \tag{25}$$

Finally, the height distribution of the shortest stack is obtained by differentiating with cumulative distribution function:

$$f_0(z) = Q'_N(z) = \frac{2N(1 - \frac{2}{\pi}\arccos x)^{N-1}}{\pi\sqrt{1-x^2}} \tag{26}$$

A simplified model

Let us suppose that there are n types of pans with "heights" $h_i = i/n$. The interpan spacing between two pans i and j is given by

$$g_{ij} = (h_i - h_j)\Theta(i-j) \tag{27}$$

Note this is essentially the result obtained from (19) in the limit of nearly flat pans, i.e., by setting $\alpha_i = \pi/2 - \delta$. If all pans are present with equal probability then the mean spacing in a random stack is

$$\langle g \rangle = \frac{1}{n^3}\sum_{i=1}^{n-1}\sum_{j=0}^{i-1}(i-j) \tag{28}$$

and the probability of height g is

$$p(g) = \sum_{i=1}^{n-1}\sum_{j=0}^{i-1}(i-j)\delta_{i-j,g} \tag{29}$$

From this we can calculate the moment generating function

$$M(n,\lambda) = \sum_{g=0}^{n-1} e^{\lambda g}p(g) \tag{30}$$

$$M(n,\lambda) = \frac{2e^{(n+1)\lambda} + e^{2\lambda}n(n-1) + n(n-1) - 2e^{\lambda}(n^2+1)}{2(e^{\lambda}-1)^2 n^2} \tag{31}$$

With this function, the cumulants can be obtained from the cumulant generating function

$$K(n,\lambda) = \sum \kappa_i \frac{\lambda^i}{i!} = \log M(n,\lambda) \tag{32}$$

$$\kappa_1 = K''(t)|_{t\to 0} = \langle i \rangle = \frac{n^2-1}{6n} \tag{33}$$

$$\kappa_2 = K'(t)|_{t\to 0} = \langle i^2 \rangle - \langle i \rangle^2 = \frac{2n^4 - n^2 - 1}{36n^2} \tag{34}$$

etc.

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