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On the modeling of dependence between univariate Lévy wear processes and impact on the reliability function

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Abstract

Univariate Lévy processes have become quite common in the reliability literature for modeling accumulative deterioration. In case of correlated deterioration indicators, several possibilities have been suggested for modeling their dependence. The point of this paper is the study of three different dependence models: use of a regular copula, superposition of independent univariate Lévy processes and use of a Lévy copula. The three methods are first presented and analysed. In this way, it is shown that the multivariate process constructed through an ordinary copula cannot have independent increments in general, that is, it is not a Lévy process. The impact of a wrong choice for the model is next explored, based on data simulated from one model and next adjusted to all three models. It is shown that a wrong model can lead to either overestimate or underestimate the reliability function, which could be problematic in an application context.

Keywords: Reliability ; Multivariate degradation ; Lévy process ; Copula ; Lévy copula

1 Introduction

In reliability theory, a classical way for modeling the univariate deterioration accumulated by a system over time is to consider univariate Lévy processes, such as gamma processes [31], inverse gaussian processes [34] or Wiener processes [36]; please see [13] for more references.

Based on the development of online monitoring, several deterioration indicators are nowadays often monitored at the same time, leading to the need for multivariate deterioration models. This need has been paid much attention in the recent reliability literature, please see Section 1.2. in [33] for a very good and recent review on the subject. We here focus on three specific multivariate models from this literature.

A first model assume the multivariate process to be a Lévy process, with dependence modeled through a Lévy copula. Indeed, it is well-known that all

the dependence between the marginal processes of a multivariate Lévy process can be captured through a Lévy copula (see [14] for technical details about Lévy copulas). Among the reliability papers using this model, one can quote for instance [1], [16], and [17]. As will be seen later on, if this model allows to catch any range of dependence between the marginal processes of a multivariate Lévy process, its use entails a high technicality. In particular, the joint distribution of the process at time t is generally not available in full form and approximate Monte-Carlo simulations are hence needed for its numerical assessment [6]. This seems to have been an obstacle for a generalized use of bivariate Lévy processes in reliability, especially because these Monte-Carlo simulations are maybe not that intuitive at first sight. That is why we think that these simulation procedures deserve to be studied further. Note that some papers can be found in the reliability literature which provide other tools to facilitate the practical use of Lévy copulas. Among them, one can quote [11] for an example of estimation procedure in a model governed by a Clayton Lévy copula with gamma processes as margins, and [28] for the use of Fokker-Planck equations in order to obtain the Laplace transforms of the reliability function of a Lévy copula model.

Another way for modeling multivariate Lévy processes is to construct them through superposition of independent univariate Lévy processes. See, e.g., [18, 20] or [22, Section 4] for the bivariate case in a reliability context. See also [2] for the multivariate case in a general context (and also for other constructions of multivariate non negative Lévy processes). The construction by superposition is simple, which makes it easy to use. As will be seen later on, this model however suffers from limitations concerning the range of possible dependence between the components.

Finally, several papers from the recent literature suggest to consider time-independent regular copulas for modeling the dependence between univariate Lévy processes (details further). As a reminder, let us recall that regular copulas allows to capture all the dependence between random variables (not between stochastic processes). See, e.g., [23] for more details about regular copulas. See, e.g., [7, 8, 9, 19, 24, 25, 35, 37] for some papers considering regular copulas for modeling multivariate deterioration. See also [36, Section 7], which highlights the fact that this model has become more and more common in the reliability literature.

The first goal of the paper is to present and analyse in details the three previous models, that is the model based on superposition, and those two based on regular or Lévy copulas. On that aspect, we try to highlight the main advantage and drawback of each method and try to provide all required technical tools for their practical use. As for the model based on regular copulas, we show that this model is not coherent as, generally, such a construction leads to a process with dependent increments. The use of this model in many engineering papers seems therefore inappropriate. For the model based on Lévy copulas, we compare two different methods from the literature for their numerical assessment. Finally, we investigate the possible dependence range of each model and compare them.

The second goal of the paper is the study of the impact of a wrong choice for the model in a practical context. With that aim, data are generated from one

model, which are used to fit the three models (including the right one). Next, the reliability of a two-component parallel system is computed, considering the three adjusted models and the results are compared.

The paper is organized as follows: the three models are described in Section 2 and the impact of a wrong modeling on the reliability function is next explored in Section 3. Conclusive remarks and perspective end this paper in Section 4.

Please note that the present study is limited to the case of non negative bivariate processes for sake of simplicity, but the results and conclusion would remain valid in the multivariate case.

2 The three models

Let us first remind that a bivariate process $(\mathbf{Y}(t) = (Y_1(t), Y_2(t)))_{t\geq 0}$ is said to be a Lévy process, as soon as:

- $(\mathbf{Y}(t))_{t\geq 0}$ is stochastically continuous,
- $\mathbf{Y}(0) = (0,0)$ almost surely,
- $(\mathbf{Y}(t))_{t\geq 0}$ has independent increments: for all $m \geq 2$, for all $0 < t_1 < \cdots < t_m$, the random vectors $\mathbf{Y}(t_1), \mathbf{Y}(t_2) - \mathbf{Y}(t_1), \cdots, \mathbf{Y}(t_m) - \mathbf{Y}(t_{m-1})$ are (mutually) independent,
- $(\mathbf{Y}(t))_{t\geq 0}$ is a right-continuous with left-side limits (càdlàg) process.

Please note that the last assumption is not mandatory, but it is not a restriction as such a càdlàg version of the process always exists, please see [27] for more details. In all the paper, we consider non negative Lévy processes, which are often called subordinators in the literature, please see [2] for more details on such processes.

2.1 The model based on a regular copula

2.1.1 Description of the model

Let us first remind that, from Sklar's Theorem [29], if $\mathbf{F}$ is a 2-dimensional distribution function with continuous marginal cumulative distribution functions F_1 and F_2, then there exists a unique function $\mathbf{C} : [0,1]^2 \to [0,1]$ such that

$$\mathbf{F}(x_1, x_2) = \mathbf{C}(F_1(x_1), F_2(x_2)), \quad \forall x = (x_1, x_2) \in \mathbb{R}^2.$$

The function $\mathbf{C}$ is called a regular copula and it is a 2-dimensional cumulative distribution function with standard uniform margins. The function $\mathbf{C}$ captures all the dependence between the two components of the distribution F.

We now come to the model used for instance in [7, 8, 9, 19, 24, 25, 35, 37] (considering only the two-dimensional case as explained in the introduction), which is based on the use of such regular copulas to model the dependence between two univariate Lévy processes.

Starting from two univariate non negative Lévy processes $(Y_1(t))_{t\geq 0}$ and $(Y_2(t))_{t\geq 0}$, we set $\mathbf{Y}(t) = (Y_1(t), Y_2(t))_{t\geq 0}$ to be the joint bivariate process and for each $(t_1, \cdots, t_m)$ such that $0 = t_0 < t_1 < \cdots < t_m$, let

$$\begin{aligned}
\Delta Y_{ij} &= Y_i(t_j) - Y_i(t_{j-1}), \\
\mathbf{Y}_j &= \mathbf{Y}(t_j) = (Y_1(t_j), Y_2(t_j)), \\
\Delta \mathbf{Y}_j &= (\Delta Y_{1j}, \Delta Y_{2j}) = \mathbf{Y}(t_j) - \mathbf{Y}(t_{j-1})
\end{aligned}$$

and let $F_{\Delta Y_{ij}}$, $\mathbf{F}_{\mathbf{Y}_j}$ and $\mathbf{F}_{\Delta \mathbf{Y}_j}$ be the corresponding cumulative distribution functions (CDF), for all $j = 0, \cdots, m$ and $i = 1, 2$.

In order to model the dependence between the univariate processes $(Y_i(t))_{t\geq 0}$, $i = 1, 2$, the authors of the quoted papers suggest to consider a regular copula $\mathbf{C}$ and assume that $\mathbf{C}$ is such that

$$\mathbf{F}_{\Delta \mathbf{Y}_j}(y_1, y_2) = \mathbf{C}\left(F_{\Delta Y_{1j}}(y_1), F_{\Delta Y_{2j}}(y_2)\right) \tag{1}$$

for all $j = 0, \cdots, m$, all $0 = t_0 < t_1 < \cdots < t_m$ and all $y_1, \cdots, y_d \geq 0$. This means that the regular copula which models the dependence between the random increments ΔY_{1j} and ΔY_{2j} on $(t_{j-1}, t_j]$ is assumed to be *independent* on t_{j-1} and t_j.

In the same papers, assuming the two processes $(Y_i(t))_{t\geq 0}$, $i = 1, 2$, to be jointly observed at times $t_1 < \cdots < t_m$, the likelihood function is next written as the product of the probability density functions of the bivariate increments $\Delta \mathbf{Y}_1$, $\cdots$, $\Delta \mathbf{Y}_m$. This means that the likelihood function is written as if the bivariate increments $\Delta \mathbf{Y}_1$, $\cdots$, $\Delta \mathbf{Y}_m$ were independent. Please see [8, Equation (11)], to refer to one single paper.

However, in [14], the authors remark that for a bivariate (or multivariate) Lévy process, the regular copula governing the dependence between the random variables $Y_i(t)$, $i = 1, 2$, usually depends on t (say $\mathbf{C}_t$). The assumption of a time-independent copula as in (1) hence seems questionable for the process $(\mathbf{Y}(t))_{t\geq 0}$ to have independent increments (that is to be a Lévy process).

In the next section, we explore some properties that should fulfill the multivariate process constructed through assumption (1) if it had independent increments, and check whether it is true.

2.1.2 Coherence of the model

Let $(\mathbf{Y}(t))_{t\geq 0}$ be a bivariate (non negative) process constructed through a regular copula as in (1). Using the notations of the previous section and considering $m = 2$, we have

$$\begin{aligned}
\mathbf{Y}_1 &= \mathbf{Y}(t_1) = (Y_1(t_1), Y_2(t_1)), \\
\mathbf{Y}_2 &= \mathbf{Y}(t_2) = (Y_1(t_2), Y_2(t_2)), \\
\Delta \mathbf{Y}_2 &= \mathbf{Y}(t_2) - \mathbf{Y}(t_1) = (Y_1(t_2) - Y_1(t_1), Y_2(t_2) - Y_2(t_1))
\end{aligned}$$

with $0 < t_1 < t_2$.

Based on assumption (1), the corresponding CDF are

$$\mathbf{F}_{\mathbf{Y}_1}(y_1, y_2) = \mathbb{P}(Y_1(t_1) \leq y_1, Y_2(t_1) \leq y_2) = \mathbf{C}\left(F_{Y_1(t_1)}(y_1), F_{Y_2(t_1)}(y_2)\right), \tag{2}$$

$$\mathbf{F}_{\mathbf{Y}_2}(y_1, y_2) = \mathbb{P}(Y_1(t_2) \leq y_1, Y_2(t_2) \leq y_2) = \mathbf{C}\left(F_{Y_1(t_2)}(y_1), F_{Y_2(t_2)}(y_2)\right) \tag{3}$$

and

$$\begin{aligned}\mathbf{F}_{\Delta\mathbf{Y}_2}(y_1, y_2) &= \mathbb{P}(Y_1(t_2) - Y_1(t_1) \leq y_1, Y_2(t_2) - Y_2(t_1) \leq y_2) \\ &= \mathbf{C}\left(F_{Y_1(t_2)-Y_1(t_1)}(y_1), F_{Y_2(t_2)-Y_2(t_1)}(y_2)\right),\end{aligned} \tag{4}$$

respectively, for all $0 < t_1 < t_2$ and all $y_1, y_2 \in \mathbb{R}_+$.

Now, note that $\mathbf{Y}(t_2) = \mathbf{Y}(t_1) + (\mathbf{Y}(t_2) - \mathbf{Y}(t_1))$, or equivalently $\mathbf{Y}_2 = \mathbf{Y}_1 + \Delta\mathbf{Y}_2$.

Let us assume that the process $(\mathbf{Y}_t)_{t\geq 0}$ has independent increments.

In that case, $\mathbf{Y}_1$ and $\Delta\mathbf{Y}_2$ should be independent, and the distribution of $\mathbf{Y}_2$ should be the convolution of the distributions of $\mathbf{Y}_1$ and $\Delta\mathbf{Y}_2$.

Assuming that $\mathbf{Y}_1$ admits a probability density function (PDF) with respect to Lebesgue measure (for simplicity but the results would be similar otherwise), one should have

$$\begin{aligned}\mathbf{F}_{\mathbf{Y}_2}(y_1, y_2) &= \left(\mathbf{F}_{\Delta\mathbf{Y}_2} * f_{\mathbf{Y}_1}\right)(y_1, y_2) \\ &= \int_{\mathbb{R}_+^2} \mathbf{F}_{\Delta\mathbf{Y}_2}(x_1, x_2) f_{\mathbf{Y}_1}(y_1 - x_1, y_2 - x_2)\, dx_1\ dx_2\end{aligned}$$

for all $(y_1, y_2) \in \mathbb{R}_+^2$.

Integrating over y_1 and y_2, one should then have

$$\int_{[0,z_1]\times[0,z_2]} \mathbf{F}_{\mathbf{Y}_2}(y_1, y_2)\, dy_1\ dy_2 \tag{5}$$

$$= \int_{[0,z_1]\times[0,z_2]} \left(\int_{\mathbb{R}_+^2} \mathbf{F}_{\Delta\mathbf{Y}_2}(x_1, x_2) f_{\mathbf{Y}_1}(y_1 - x_1, y_2 - x_2)\, dx_1\ dx_2\right) dy_1\ dy_2$$

$$= \int_{\mathbb{R}_+^2} \mathbf{F}_{\Delta\mathbf{Y}_2}(x_1, x_2) \left(\int_{[0,z_1]\times[0,z_2]} f_{\mathbf{Y}_1}(y_1 - x_1, y_2 - x_2)\, dy_1\ dy_2\right) dx_1\ dx_2$$

$$= \int_{[0,z_1]\times[0,z_2]} \mathbf{F}_{\Delta\mathbf{Y}_2}(x_1, x_2)\, \mathbf{F}_{\mathbf{Y}_1}(z_1 - x_1, z_2 - x_2)\, dx_1\ dx_2 \tag{6}$$

for all $(z_1, z_2) \in \mathbb{R}_+^2$.

Now let $\mathbf{G}_{t_1,t_2}(z_1, z_2)$ stand for the difference between (5) and (6). Substituting $\mathbf{F}_{\mathbf{Y}_1}$, $\mathbf{F}_{\mathbf{Y}_2}$ and $\mathbf{F}_{\Delta\mathbf{Y}_2}$ by their expressions with respect to $\mathbf{C}$ (see $(2-4)$),

we get

$$\begin{aligned}\mathbf{G}_{t_1,t_2}(z_1,z_2) = &\int_{[0,z_1]\times[0,z_2]} \mathbf{C}\left(F_{Y_1(t_2)}(y_1), F_{Y_2(t_2)}(y_2)\right) dy_1\ dy_2 \\ &- \int_{[0,z_1]\times[0,z_2]} \mathbf{C}\left(F_{Y_1(t_2)-Y_1(t_1)}(x_1), F_{Y_2(t_2)-Y_2(t_1)}(x_2)\right) \\ &\times \mathbf{C}\left(F_{Y_1(t_1)}(z_1-x_1), F_{Y_2(t_1)}(z_2-x_2)\right) dx_1\ dx_2\end{aligned}$$

for all $(z_1, z_2) \in \mathbb{R}_+^2$.

Conclusion: Based on $(5-6)$, *if the process* $\mathbf{Y}$ *had independent increments,* one should have

$$\mathbf{G}_{t_1,t_2}(z_1,z_2) = 0 \tag{7}$$

for all $(z_1, z_2) \in \mathbb{R}_+^2$.

We next check whether Equation (7) is true on two different univariate non negative Lévy processes. For both cases, a Clayton copula is considered, with

$$\mathbf{C}(u_1,u_2) = \left[\max\left(u_1^{-\theta} + u_2^{-\theta} - 1, 0\right)\right]^{-\frac{1}{\theta}}$$

where $\theta \in [-1,\infty)\setminus\{0\}$, and where $\theta \to 0$ corresponds to independence, $\theta \to +\infty$ to the upper Fréchet-Hoeffding bound (complete dependence) and $\theta = -1$ to the lower Fréchet-Hoeffding bound, please see [23] for more details. The function $\mathbf{G}_{t_1,t_2}(z_1,z_2)$ depends on θ through the copula $\mathbf{C}$ and it is denoted by $\mathbf{G}_{t_1,t_2,\theta}(z_1,z_2)$.

Example 1 *As a first example, gamma processes are considered, with* $Y_i(t) \sim \Gamma(t,1)$ *for* $i = 1,2$. *The function* $\mathbf{G}_{t_1,t_2,\theta}(z_1,z_2)$ *is plotted with respect to* (z_1,z_2) *in Figure 1 for* $t_1 = 1$, $t_2 = 2$ *and* $\theta = 1$ *(left plot),* $\theta = -0.5$ *(right plot). As can be seen, even in the case of homogeneous univariate processes and with* $t_2 - t_1 = t_1 = 1$, $\mathbf{G}_{t_1,t_2,\theta}(z_1,z_2)$ *can be positive or negative, but it is not a null function (that is we do not have* $\mathbf{G}_{t_1,t_2,\theta}(z_1,z_2) = 0$ *for all* $z_1, z_2 \in \mathbb{R}_+$*), which means that the bivariate process does not have independent increments.*

The function $\mathbf{G}_{t_1,t_2,\theta}(z_1,z_2)$ *is next plotted with respect to* θ *for* $t_1 = 1$, $t_2 = 2$ *and* $(z_1,z_2) = (2,2)$ *in Figure 2, where we can see that it is positive or negative, except from the two cases* $\theta \to 0$ *and* $\theta \to +\infty$, *when it tends towards 0. These two cases correspond to the cases of independence and complete dependence, respectively, and it can be easily checked that indeed, in these two specific cases,* $(\mathbf{Y}(t))_{t\geq 0}$ *has independent increments.*

Example 2 *As a second example, inverse gaussian processes are considered, with* $Y(t) \sim IG\left(t, bt^2\right)$ *and*

$$f_t(y) = t\sqrt{\frac{b}{2\pi y^3}}\ \exp\left[-\frac{b(y-t)^2}{2y}\right],$$

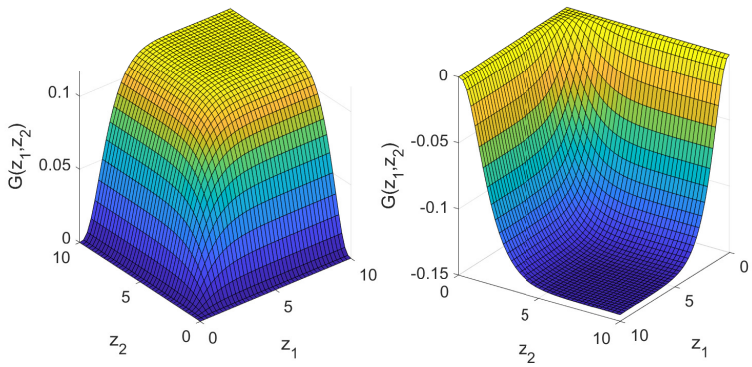

Figure 1: The function $\mathbf{G}_{t_1,t_2,\theta}(z_1,z_2)$ with respect to (z_1,z_2) for $t_1=1$, $t_2=2$ with $\theta=1$ (left plot) and $\theta=-0.5$ (right plot), Gamma case (Example 1).

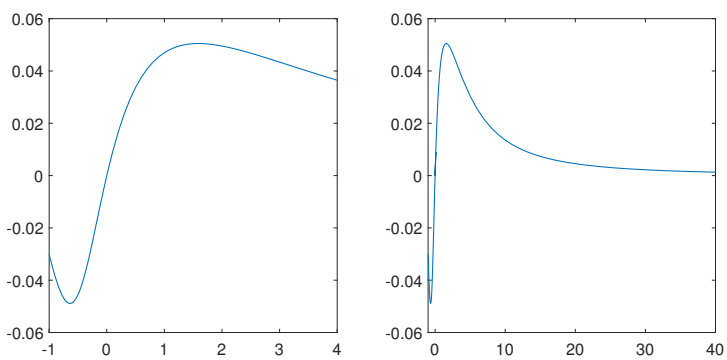

Figure 2: The function $G_{t_1,t_2,\theta}(2,2)$ with respect to θ for $t_1=1$, $t_2=2$ and $(z_1,z_2)=(2,2)$. The left plot is a zoomed version of the right plot.

following the parameterization of [34]. Here again, the function $\mathbf{G}_{t_1,t_2,\theta}(z_1,z_2)$ *is plotted with respect to* (z_1,z_2) *in Figure 3 for* $t_1=1$, $t_2=2$ *and* $\theta=1$ *(left plot),* $\theta=-0.5$ *(right plot), where we can see that* $\mathbf{G}_{t_1,t_2,\theta}(z_1,z_2)$ *is not the null function either.*

Even if this study is restricted to non negative Lévy processes, note that we have also tested the case of Gaussian processes for which the results are similar, that is the construction of a bivariate process through a regular copula and univariate Gaussian processes does not lead to a bivariate Lévy process.

As a conclusion, the construction of the bivariate process $(\mathbf{Y}(t))_{t\geq 0}$ *based on a regular copula as in* (1) *generally does not lead to a process with independent increments. Then, providing the distributions of the bivariate increments* $\Delta\mathbf{Y}_1,\cdots,\Delta\mathbf{Y}_m$ *through* (1) *does not allow to recover the joint distribution of*

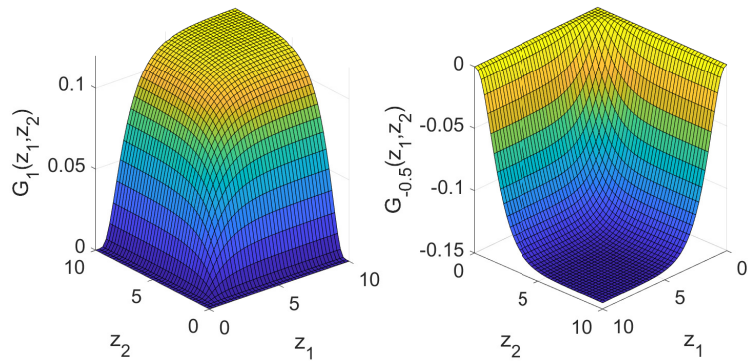

Figure 3: The function $\mathbf{G}_{t_1,t_2,\theta}(z_1,z_2)$ with respect to (z_1,z_2) for $t_1=1$, $t_2=2$ with $\theta=1$ (left plot) and $\theta=-0.5$ (right plot), Inverse Gaussian case (Example 2).

successive increments $(\Delta\mathbf{Y}_1,\cdots,\Delta\mathbf{Y}_m)$*, nor the joint finite-dimensional distribution of* $(\mathbf{Y}(t_1),\cdots,\mathbf{Y}(t_m))$ *with* $0<t_1<\cdots<t_m$*. Remembering that the family of such finite-dimensional distributions is required to fully characterize the distribution of the bivariate process* $(\mathbf{Y}(t))_{t\geq 0}$ *(see e.g. [4, Chapter 7]), it means that the distribution of the bivariate process is not fully characterized by* (1) *and should be further specified.*

2.2 Bivariate Lévy processes constructed through superposition

2.2.1 The model

We here consider a (non negative) bivariate process constructed through superposition as in [18, 20, 21], with

$$\begin{cases} Y_1(t)=X_1(t)+X_3(t), \\ Y_2(t)=X_2(t)+X_3(t) \end{cases}$$

for all $t\geq 0$, where $(X_i(t))_{t\geq 0}$, $i=1,2,3$ are independent univariate non negative Lévy processes (please see [2, §2.1] for the extension to the multivariate case). It is well known that the joint process $(\mathbf{Y}(t)=(Y_1(t),Y_2(t)))_{t\geq 0}$ is a Lévy process. Also, the joint CDF of $\mathbf{Y}(t)$ is available in full form, with

$$\begin{aligned}\mathbf{F}_{\mathbf{Y}(t)}(y_1,y_2)&=\mathbb{P}(X_1(t)+X_3(t)\leq y_1, X_2(t)+X_3(t)\leq y_2)\\ &=\int_{\mathbb{R}_+}F_{X_1(t)}(y_1-v)\,F_{X_2(t)}(y_2-v)\,f_{X_3(t)}(v)\,dv\end{aligned}$$

for all $y_1,y_2\in\mathbb{R}_+$ (see, e.g., [20]), where $F_{X_i(t)}$ and $f_{X_i(t)}$ are the CDF and PDF of $X_i(t)$, $i=1,2,3$, respectively.

For each $t > 0$, the regular copula modeling the dependence between the random variables $Y_1(t)$ and $Y_2(t)$ is denoted by $\mathbf{C}_t$.

Based on [23, Corollary 2.3.7.], we know that

$$\begin{aligned}\mathbf{C}_t(z_1, z_2) &= \mathbf{F}_{\mathbf{Y}(t)}\left(F^{-1}_{Y_1(t)}(z_1), F^{-1}_{Y_2(t)}(z_2)\right) \\ &= \int_{\mathbb{R}_+} F_{X_1(t)}\left(F^{-1}_{Y_1(t)}(z_1) - v\right) F_{X_2(t)}\left(F^{-1}_{Y_2(t)}(z_2) - v\right) f_{X_3(t)}(v)\, dv\end{aligned}$$

for all $z_1, z_2 \in [0, 1]$.

We next look at the dependence of $\mathbf{C}_t$ with respect to t.

2.2.2 On the time dependence of $\mathbf{C}_t$ with respect to t

To analyse the dependence of $\mathbf{C}_t$ with respect to t, we consider gamma processes with $X_i(t) \sim \Gamma(t, 1)$ for $i = 1, 2, 3$. In that case, we have $Y_i(t) \sim \Gamma(2t, 1)$ for $i = 1, 2$, and $(\mathbf{Y}_t)_{t \geq 0}$ is a bivariate Lévy process, with gamma processes as margins.

The copula function $\mathbf{C}_t$ is plotted in Figure 4 for $t = 1$ (that is $\mathbf{C}_1$) and for $t = 2$ (that is $\mathbf{C}_2$), as well as the difference $\mathbf{C}_1 - \mathbf{C}_2$. The plots of $\mathbf{C}_1$ and $\mathbf{C}_2$ look very similar (left and middle plots) but when looking at the difference $\mathbf{C}_1 - \mathbf{C}_2$ (right plot), it can be seen that it can be both positive and negative, and $\mathbf{C}_1 - \mathbf{C}_2$ is not the null function.

Hence, even for such a simple bivariate Lévy process, the regular copula function modeling the dependence between $Y_1(t)$ and $Y_2(t)$ depends on t.

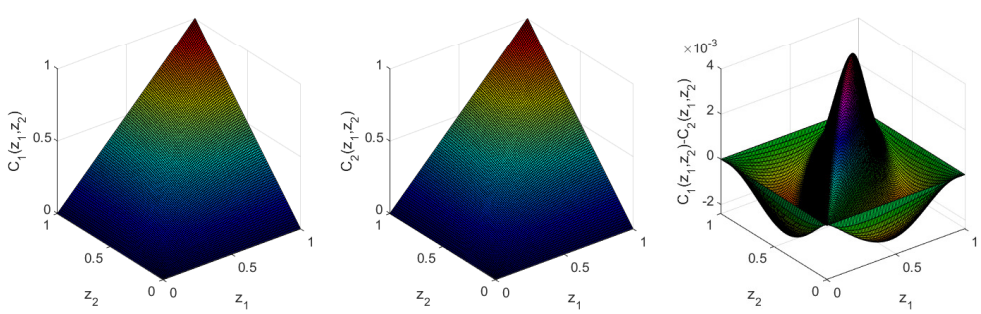

Figure 4: The functions $C_1(z_1, z_2)$ (left), $C_2(z_1, z_2)$ (middle) and $C_1(z_1, z_2) - C_2(z_1, z_2)$ (right) with respect to (z_1, z_2).

Note however that if we extend the study to Lévy processes on the whole $\mathbb{R}$ line, this result would not remain valid as it is easy to check that considering a bivariate Gaussian process constructed though superposition of independent univariate Gaussian processes leads to a time-independent regular copula for $(Y_1(t), Y_2(t))$.

2.3 Model based on a Lévy copula

2.3.1 The model

We finally consider a bivariate non-negative Lévy processes $(\mathbf{Y}(t) = (Y_1(t), Y_2(t)))$ (without drift). The process $(\mathbf{Y}(t))_{t\geq 0}$ is non-decreasing (component-wise) and hence has bounded variation. It is a pure jump process and the jumps arrival times and sizes are governed by its Lévy measure $\mu(d\mathbf{y})$ on $\mathbb{R}_+^2 \setminus \{\mathbf{0}\}$, which is such that

$$\iint_{\mathbb{R}_+^2} (\min(|\mathbf{y}|, 1))\, \mu(d\mathbf{y}) < \infty,$$

(see [3, 27] for details).

The corresponding bivariate tail integral function is defined by

$$\mathbf{U}(\mathbf{y}) = \mathbf{U}(y_1, y_2) = \iint_{[y_1,+\infty)\times[y_2,+\infty)} \mu(d\mathbf{y}) \tag{8}$$

for all $\mathbf{y} = (y_1, y_2) \in \mathbb{R}_+^2 \setminus \{\mathbf{0}\}$, with $\mathbf{U}(y_1, \infty) = \mathbf{U}(\infty, y_2) = 0$ for $y_1, y_2 > 0$ and $\mathbf{U}(\mathbf{0}) = +\infty$, see [6, 14].

For $i = 1, 2$, the process $(Y_i(t))_{t\geq 0}$ is a univariate Lévy process with Lévy measure μ_i where $\mu_1(dy_1) = \mu(dy_1 \times \mathbb{R}_+)$ and $\mu_2(dy_2) = \mu(\mathbb{R}_+ \times dy_2)$. The corresponding tail integral function is U_i, with

$$U_i(x_i) = \int_{[x_i,+\infty)} \mu_i(dy)$$

for $x_i > 0$, $U_i(0) = +\infty$ and $U_i(+\infty) = 0$.

Based on Sklar's theorem for Lévy processes, it is known that all the dependence between the univariate processes $(Y_1(t))_{t\geq 0}$ and $(Y_2(t))_{t\geq 0}$ is governed by a Lévy copula $\mathbf{L}$, which links $\mathbf{U}$ and U_i, $i = 1, 2$ through

$$\mathbf{U}(\mathbf{y}) = \mathbf{L}(U_1(y_1), U_2(y_2))$$

for all $\mathbf{y} = (y_1, y_2) \in \mathbb{R}_+^2 \setminus \{\mathbf{0}\}$ (see [6, 14] for details).

Lévy copulas allow to model any kind of dependence between the marginal processes of a bivariate Lévy process. As explained in the introduction, the joint distribution of $\mathbf{Y}_t$ is generally not available in full form. Approximate Monte-Carlo simulations are hence used for their numerical assessment, which are presented in the next section. (Note that numerical schemes could also be used, see e.g. [6, Chapter 12]).

In the following, we will make use of Pearson correlation coefficient ρ, which is known to be independent on t and is given by

$$\rho = \rho_{\mathbf{Y}(t)} = \frac{\iint_{\mathbb{R}_+^2} \mathbf{U}(u_1, u_2)\, du_1\, du_2}{\sigma_{Y_1(1)}\sigma_{Y_2(1)}} = \frac{\iint_{\mathbb{R}_+^2} \mathbf{L}(U_1(u_1), U_2(u_2))\, du_1\, du_2}{\sigma_{Y_1(1)}\sigma_{Y_2(1)}} \tag{9}$$

for all $t > 0$ and also of the Laplace transform of $\mathbf{Y}(t)$, with

$$\begin{aligned}\mathcal{L}_{\mathbf{Y}(t)}(\mathbf{s}) &= \mathbb{E}\left(e^{-s_1 Y_1(t) - s_2 Y_2(t)}\right) \\ &= \mathcal{L}_{Y_1(t)}(s_1)\,\mathcal{L}_{Y_2(t)}(s_2)\,e^{t s_1 s_2 \mathcal{L}_{\mathbf{U}}(s_1,s_2)}\end{aligned} \tag{10}$$

and

$$\begin{aligned}\mathcal{L}_{\mathbf{U}}(s_1,s_2) &= \iint\limits_{\mathbb{R}_+^2} e^{-s_1 y_1 - s_2 y_2}\mathbf{U}(y_1,y_2)\,dy_1\;dy_2 \\ &= \iint\limits_{\mathbb{R}_+^2} e^{-s_1 y_1 - s_2 y_2}\mathbf{L}(U_1(y_1),U_2(y_2))\,dy_1\;dy_2\end{aligned}$$

for all $t > 0$ and all $\mathbf{s} = (s_1, s_2) \in \mathbb{R}_+^2$. As we could not find any proof in the literature, some details are given in the Appendix for formulas (9) and (10).

Standard Lévy copulas are positive Archimedean Lévy copulas which are of the shape

$$L(\mathbf{u}) = \Phi\left(\Phi^{-1}(u_1) + \Phi^{-1}(u_2)\right) \tag{11}$$

for all $\mathbf{u} = (u_1, u_2) \in \mathbb{R}_+^2 \setminus \{\mathbf{0}\}$, where Φ is the generator function.

In the following, we consider Clayton-Lévy (CL) copulas with

$$\begin{aligned}\Phi_{\theta,CL}(t) &= t^{-\frac{1}{\theta}}, \\ \mathbf{L}_{\theta,CL}(u_1,u_2) &= \left(u_1^{-\theta} + u_2^{-\theta}\right)^{-\frac{1}{\theta}},\end{aligned}$$

and Gumbel-Lévy (GL) copulas with

$$\begin{aligned}\Phi_{\theta,GL}(t) &= \exp\left(t^{-\frac{1}{\theta}}\right) - 1, \\ \mathbf{L}_{\theta,GL}(u_1,u_2) &= \exp\left(\left[(\ln(1+u_1))^{-\theta} + (\ln(1+u_2))^{-\theta}\right]^{-\frac{1}{\theta}}\right) - 1\end{aligned}$$

for all $t > 0$ and all $\mathbf{u} = (u_1, u_2) \in \mathbb{R}_+^2 \setminus \{\mathbf{0}\}$, where $\theta > 0$. Note that the case $\theta \to 0^+$ corresponds to independence and $\theta \to \infty$ to complete dependence for both CL and GL copulas.

2.3.2 Approximate Monte-Carlo simulation

Non-negative bivariate Lévy processes (without drift) are pure jump processes that can be represented through series of the shape

$$\mathbf{Y}(t) = \sum_{n=1}^{\infty} \mathbf{Z}_n\;\mathbf{1}_{\{V_n \le t\}},$$

where V_n, $n = 1, 2, \cdots$ are the jumps times and $\mathbf{Z}_n$, $n = 1, 2, \cdots$ the bivariate jump sizes (see [26]). Approximate Monte-Carlo (MC) simulation methods of

such processes are based on the truncation of these series, removing the (infinitely many) smallest jumps and keeping only the (finitely many) largest ones.

A first simulation method based on series representation can be find in [6]. This method relies on the simulation of the jumps times and sizes of one component (say the first component), and next on the simulation of the jumps sizes of the other component, conditionally on the jump size on the first component. This method is called Conditional Method in [10]. As was already observed in several papers, this method induces some dissymmetry between the two components and "suffers from convergence problems when the components are weakly dependent" [30]. We hence prefer consider another simulation method, suggested in [30]. Setting T to be the horizon time, it writes down the following way:

Let

$$\begin{aligned}
F_{\theta,CL}\left(u_2|u_1\right) &= \frac{\partial \mathbf{L}_{\theta,CL}\left(u_1,u_2\right)}{\partial u_1} = \left(1+\left(\frac{u_2}{u_1}\right)^{-\theta}\right)^{-\frac{1+\theta}{\theta}}, \\
F_{\theta,GL}(u_2|u_1) &= \frac{\partial \mathbf{L}_{\theta,GL}\left(u_1,u_2\right)}{\partial u_1} \\
&= \frac{1}{1+u_1}\left(1+\left(\frac{\ln\left(1+u_2\right)}{\ln\left(1+u_1\right)}\right)^{-\theta}\right)^{-\frac{1+\theta}{\theta}} \\
&\times \exp\left\{\left[\left(\ln\left(1+u_1\right)\right)^{-\theta}+\left(\ln\left(1+u_2\right)\right)^{-\theta}\right]^{-\frac{1}{\theta}}\right\}
\end{aligned}$$

and let $F^{-1}_{\theta,\cdot}(u_2|u_1)$ be the inverse function with respect to u_2 of $F_{\theta,\cdot}\left(u_2|u_1\right)$, where the point in the notation $F_{\theta,\cdot}$ refers to CL or GL. Then

$$F^{-1}_{\theta,CL}\left(u_2|u_1\right) = u_1\left(u_2^{-\frac{\theta}{1+\theta}}-1\right)^{-1/\theta}$$

for the Clayton-Lévy case. As for the Gumbel-Lévy copula, there is no closed-form expression for the inverse function and a numerical inversion procedure is used. More precisely, for fixed u_1, u_2 and θ, $F^{-1}_{\theta,GL}\left(u_2|u_1\right)$ is obtained as the root of the function $x \mapsto F_{\theta,GL}\left(x|u_1\right)-u_2$, using the R function *uniroot*.

Let $(\Gamma_i^{11})_{i\in\mathbb{N}^*}$ and $(\Gamma_i^{22})_{i\in\mathbb{N}^*}$ be two independent sequences of jump times of independent standard Poisson processes with rate T. Let also $(W_i^j)_{i\in\mathbb{N}^*}$, $j=1,2$, $(Q_i^j)_{i\in\mathbb{N}^*}$, $j=1,2$ be four independent sequences of i.i.d. uniform random variables on $[0,1]$ and let $(V_i)_{i\in\mathbb{N}^*}$ be an independent sequence of i.i.d. uniform random variables on $[0,T]$. Define:

$$\Gamma_i^{12} = F^{-1}_{\theta,\cdot}(Q_i^1|\Gamma_i^{11}) \text{ and } \Gamma_i^{21} = F^{-1}_{\theta,\cdot}(Q_i^2|\Gamma_i^{22})$$

for all $i \geq 1$. Then, an approximation of $Y_1(t)$ and $Y_2(t)$ is obtained through

the following formulas:

$$Y_1^k(t) = \sum_{i=1}^{k} U_1^{-1}(\Gamma_i^{11})\mathbf{1}_{\{n_i^{1k}W_i^1 \le 1\}}\mathbf{1}_{[0,t]}(V_i) + \sum_{i=1}^{k} U_1^{-1}(\Gamma_i^{21})\mathbf{1}_{\{n_i^{2k}W_i^2 \le 1\}}\mathbf{1}_{[0,t]}(V_i) \tag{12}$$

$$Y_2^k(t) = \sum_{i=1}^{k} U_2^{-1}(\Gamma_i^{22})\mathbf{1}_{\{n_i^{2k}W_i^2 \le 1\}}\mathbf{1}_{[0,t]}(V_i) + \sum_{i=1}^{k} U_2^{-1}(\Gamma_i^{12})\mathbf{1}_{\{n_i^{1k}W_i^1 \le 1\}}\mathbf{1}_{[0,t]}(V_i) \tag{13}$$

for all $t \in [0,T]$, with

$$n_i^{1k} = \#\left\{j = 1,2 : \Gamma_i^{1j} \le \frac{k}{T}\right\}, \quad n_i^{2k} = \#\left\{j = 1,2 : \Gamma_i^{2j} \le \frac{k}{T}\right\},$$

where k is an integer (to be chosen large enough).

Remark 3 *In such approximations, the k first points are considered for both Poisson processes $(\Gamma_i^{11})_{i\in\mathbb{N}^*}$ and $(\Gamma_i^{22})_{i\in\mathbb{N}^*}$, leading to mostly $2k$ jumps for $Y_j^k(t)$, $j = 1,2$ (in practice, a little less as some of them disappear due to the indicators on $\left\{n_i^{jk}W_i^j \le 1\right\}$, $j = 1,2$ in (12) and (13)). An alternative would be to consider a threshold level, say τ, as proposed by [30], and consider all points of the two Poisson processes in $[0,\tau]$. The choice of the k first points for both Poisson processes as in (12) and (13) allows an easier comparison with another simulation method which is considered just bellow.*

Formulas (12) and (13) lead to the following algorithm.

In the specific case of an Archimedean copula as in (11), the authors in [10] suggest another approximation:

$$Y_1^K(t) = \sum_{n=1}^{K} U_1^{-1}\left(\Phi\left(\frac{E_{1n}}{G^{-1}(\Gamma_n)}\right)\right)\mathbf{1}_{\{V_n \le t\}}, \tag{14}$$

$$Y_2^K(t) = \sum_{n=1}^{K} U_2^{-1}\left(\Phi\left(\frac{E_{2n}}{G^{-1}(\Gamma_n)}\right)\right)\mathbf{1}_{\{V_n \le t\}} \tag{15}$$

for all $t \ge 0$, where:

- K is an integer (to be chosen large enough),
- $(\Gamma_n)_{n\ge1}$ are the points of a Poisson process with rate T,
- E_{in}, $i = 1,2$, $n \ge 1$ are independent standard exponential distributions (with mean 1),
- $(V_n)_{n\ge1}$ are i.i.d. random variables uniformly distributed on $[0,T]$,

Algorithm 1 (Tankov)

- Chose an integer k;
- Generate Γ_k^{11} and Γ_k^{22} independently, according to the gamma distribution with shape parameter k and rate T;
- For $j = 1, 2$, generate i.i.d. $\Gamma_1^{jj}, \cdots, \Gamma_{k-1}^{jj}$ independently according to the uniform distribution on $\left[0, \Gamma_k^{jj}\right]$ (no need for sorting the Γ_i^{jj}, $i = 1, \cdots, k-1$);
- Generate i.i.d. V_i, $i = 1, \cdots, k$ independently according to the uniform distribution on $[0, T]$;
- Generate i.i.d. W_i^1, W_i^2, Q_i^1, Q_i^2, $i = 1, \cdots, k$ independently according to the uniform distribution on $[0, 1]$;
- Compute $\Gamma_i^{12} = F_{\theta,\cdot}^{-1}(Q_i^1|\Gamma_i^{11})$ and $\Gamma_i^{21} = F_{\theta,\cdot}^{-1}(Q_i^2|\Gamma_i^{22})$ for $i = 1, \cdots, k$;
- Compute n_i^{1k} and n_i^{2k} for $i = 1, \cdots, k$;
- Compute $Y_1^k(t)$ and $Y_2^k(t)$ according to (12) and (13).

with Γ_n's, V_n's and E_{ij}'s independent, and where $G = \mathcal{LS}^{-1}(\Phi)$ is the inverse Laplace-Stieltjes transform of Φ.

In the specific case of a Clayton-Lévy copula, we have

$$\Phi_{\theta,CL}(s) = s^{-\frac{1}{\theta}} = \int_{\mathbb{R}_+} e^{-su} \frac{u^{\frac{1}{\theta}-1}}{\Gamma\left(\frac{1}{\theta}\right)} du = \int_{\mathbb{R}_+} e^{-su} dG(u) = \mathcal{LS}_G(s)$$

with

$$G(u) = \frac{\theta\ u^{\frac{1}{\theta}}}{\Gamma\left(\frac{1}{\theta}\right)} = \frac{u^{\frac{1}{\theta}}}{\Gamma\left(1 + \frac{1}{\theta}\right)}$$

and

$$G^{-1}(t) = \left(\frac{t}{\theta}\Gamma(1/\theta)\right)^{\theta} = \left(t\ \Gamma\left(1 + \frac{1}{\theta}\right)\right)^{\theta}.$$

In the case of a Gumbel-Lévy copula, the computation of G^{-1} would require two successive numerical procedures, first for the computation of the inverse Laplace-Stieltjes transform G of Φ, and next for its inversion. Based on this numerical complexity, this simulation procedure is not considered further for the Gumbel-Lévy copula.

Formulas (14) and (15) lead to the following algorithm (which is used for the Clayton-Lévy copula).

Remark 4 *For comparison purpose between Algorithms 1 and 2, we take $K = 2k$ in order to get (mostly) the same number of jumps in the two simulation methods, which leads to very similar computation times.*

Algorithm 2 (Grothe-Hofert)

- Chose an integer K;
- Generate Γ_K according to the gamma distribution with shape parameter K and rate T;
- Generate i.i.d. $\Gamma_1, \cdots, \Gamma_{K-1}$ according to the uniform distribution on $[0, \Gamma_K]$ (no need for sorting the Γ_i, $i = 1, \cdots, K-1$);
- Generate i.i.d. E_{in} according to the standard exponential distribution for $1 \leq n \leq K$ and $i = 1, 2$;
- Generate i.i.d. $V_1, \cdots, V_K$ according to the uniform distribution on $[0, T]$;
- Compute $Y_1^K(t)$ and $Y_2^K(t)$ according to (14) and (15).

2.3.3 Comparison of the two simulation approaches

In this section, a bivariate Lévy process $(\mathbf{Y}_t)_{t\geq 0}$ with Clayton-Lévy copula L_θ is considered, with gamma processes as margins and marginal parameters $(A_i, 1)$, $i = 1, 2$. It is called a bivariate Gamma-Clayton-Lévy (GCL) process with parameters $(A_1, 1, A_2, 1, \theta)$. The objective is to compare the two simulation methods, as described by Algorithms 1 and 2. Remembering that the joint PDF of $\mathbf{Y}_t$ is not available in full form but fully characterized by its joint Laplace transform (available in full form), the accuracy of the simulation methods is evaluated by comparing the empirical Laplace transform of $\mathbf{Y}(t)$ with its theoretical counterpart, which is given by (10) (in a general setting). For a better account, the marginal empirical Laplace transforms and Pearson correlation coefficient are also compared with their theoretical counterparts, which are given by (24) and (9), respectively.

In the specific case of a process with gamma processes as margins, we have

$$U_i(u_i) = A_i \int_{u_i}^{\infty} \frac{e^{-y_i}}{y_i} dy_i = A_i \operatorname{Ei}(u_i) \tag{16}$$

for $i = 1, 2$ and $u_i > 0$, where Ei is the exponential integral function.

Formula (9) may then be written in the following way:

$$\begin{aligned}\rho_{\theta,CL} &= \frac{\iint\limits_{\mathbb{R}_+^2} \left((A_1 \operatorname{Ei}(u_1))^{-\theta} + (A_2 \operatorname{Ei}(u_2))^{-\theta} \right)^{-\frac{1}{\theta}} \, du_1 \, du_2}{\sqrt{A_1 A_2}} \\ &= \iint\limits_{\mathbb{R}_+^2} \left(\alpha^{-\frac{\theta}{2}} \operatorname{Ei}(u_1)^{-\theta} + \alpha^{\frac{\theta}{2}} \operatorname{Ei}(u_2)^{-\theta} \right)^{-\frac{1}{\theta}} \, du_1 \, du_2 \end{aligned} \tag{17}$$

with

$$\alpha = \frac{A_1}{A_2}.$$

Formulas (24) and (10) write down:

$$\mathcal{L}_{Y_i(t)}(s_i) = \exp\left(t\int_{\mathbb{R}_+}\left(e^{-s_i y_i}-1\right)A_i\frac{e^{-y_i}}{y_i}dy_i\right) = \left(\frac{1}{1+s_i}\right)^{A_i t} \quad \text{for } i=1,2,$$

$$\mathcal{L}_{\mathbf{Y}(t)}(\mathbf{s}) = \left(\frac{1}{1+s_1}\right)^{A_1 t}\left(\frac{1}{1+s_2}\right)^{A_2 t} e^{t s_1 s_2 \mathcal{L}_{\mathbf{U}}(s_1,s_2)}$$

with

$$\mathcal{L}_{\mathbf{U}}(\mathbf{s}) = \iint\limits_{\mathbb{R}_+^2} e^{-s_1 y_1 - s_2 y_2}\left((A_1\,\mathrm{Ei}\,(u_1))^{-\theta} + (A_2\,\mathrm{Ei}\,(u_2))^{-\theta}\right)^{-\frac{1}{\theta}} dy_1\, dy_2.$$

Then, for each set of parameters (A_1, A_2, θ, K, t), the following procedure is used. Firstly, $N = 5000$ trajectories $(\mathbf{Y}^{(n)}(u))_{u\in[0,t]}$, $n = 1,\ldots,N$ of the process $\mathbf{Y}$ are simulated. Next, we have to chose the points where to compare the empirical and theoretical Laplace transforms of $Y_i(t)$, $i = 1,2$ and we take $s_{i,j} = \mathcal{L}^{-1}_{Y_i(t)}(0.01 * j)$ for $i = 1,2$ and $j = 1,\ldots,100$, which allows to mostly cover the range of $\mathcal{L}_{Y_i(t)}$ for $i = 1,2$. For each $i = 1,2$, the relative error on the marginal Laplace transform is then computed by:

$$E_L^{(i)}(t) = \frac{1}{100}\sum_{j=1}^{100}\frac{\left|\mathcal{L}_{Y_i(t)}(s_{i,j}) - \bar{\mathcal{L}}_{Y_i(t)}(s_{i,j})\right|}{\mathcal{L}_{Y_i(t)}(s_{i,j})}, \tag{18}$$

where $\bar{\mathcal{L}}_{Y_i(t)}(.)$ is the empirical Laplace transform for the i-th margin, with

$$\bar{\mathcal{L}}_{Y_i(t)}(s_{i,j}) = \frac{1}{N}\sum_{n=1}^{N} e^{-s_{i,j}Y_i^{(n)}(t)}.$$

For the bivariate process, the theoretical and empirical Laplace transforms ($\mathcal{L}_{\mathbf{Y}(t)}(\boldsymbol{s})$ and $\bar{\mathcal{L}}_{\mathbf{Y}(t)}(\boldsymbol{s})$, respectively) are evaluated on a regular grid of 100 points $\boldsymbol{s} = (s_1, s_2)$, which mostly allows to cover the range of $\mathcal{L}_{\mathbf{Y}(t)}$, as previously. The relative error $\mathbf{E}_L(t)$ is next computed in a similar manner as in Equation (18).

Finally, an empirical linear correlation coefficient $\hat{\rho}$ is computed between the series $Y_1^{(n)}(t)$ and $Y_2^{(n)}(t)$, $n = 1,\ldots,N$ and compared to the theoretical value ρ.

This experiment, which provides three relative errors (for margins 1 and 2, and for the bivariate distribution) and the empirical correlation coefficient, is repeated $B = 100$ times. The means and standard deviations of the three relative errors together with the mean and standard deviation of the estimates of the linear correlation coefficient are then computed.

Setting $A_1 = 0.5$, $A_2 = 1$, $t = 1$, several values are considered for K $(50, 100, 500$ and $1000)$ and for θ $(0.2, 0.5, 1$ and $10)$. The means and standard

Table 1: Mean (and standard deviation between brackets) of the relative errors $E_L^{(1)}(1)$ for the first margin

K	θ	Grothe-Hofert	Tankov
50	0.2	0.812 (0.025)	0.012 (0.006)
100	0.2	0.502 (0.019)	0.014 (0.007)
500	0.2	0.126 (0.016)	0.013 (0.007)
1000	0.2	0.057 (0.014)	0.011 (0.006)
50	0.5	0.017 (0.008)	0.012 (0.006)
100	0.5	0.012 (0.006)	0.012 (0.007)
500	0.5	0.013 (0.006)	0.013 (0.007)
1000	0.5	0.012 (0.006)	0.011 (0.006)
50	1.0	0.012 (0.007)	0.012 (0.006)
100	1.0	0.012 (0.006)	0.012 (0.007)
500	1.0	0.013 (0.007)	0.013 (0.007)
1000	1.0	0.012 (0.007)	0.012 (0.006)
50	10.0	0.012 (0.007)	0.012 (0.006)
100	10.0	0.013 (0.006)	0.013 (0.007)
500	10.0	0.012 (0.007)	0.012 (0.007)
1000	10.0	0.012 (0.007)	0.012 (0.006)

deviations of the relative errors for the Laplace transforms of the first margin and bivariate distribution are provided in Tables 1 and 2, respectively. The results for the second margin are very close to those for the first margin, and they are hence omitted. The mean and standard deviation of the correlation estimator $\hat{\rho}$ are provided in Table 3, as well as the true correlation value ρ.

From Tables 1 to 3, it can be noted that the two generation approaches perform similarly for large ($\theta = 1$) and very large ($\theta = 10$) dependencies between the two components of the process $(\mathbf{Y}_t)_{t\geq 0}$ and that the convergence with respect to K is very fast. However, for a moderate correlation $\theta = 0.5$ (which leads to $\rho \simeq 0.410$), the generation method of Grothe and Hofert seems to need a larger value for K than Tankov's one, as for the relative error on the Laplace transforms. For very small correlation ($\theta = 0.2$, leading to a quasi-independence situation), we can observe a clear advantage of Tankov's method. This phenomenon is not observed for the correlation coefficient, where both methods lead to mostly similar results (but the correlation coefficient is not sufficient to fully characterize the distribution, contrary to the bivariate Laplace transform).

Note that other experiments have been conducted, varying the value of A_1 (with A_1 equal to A_2 and A_1 much larger than A_2). The results were very similar to those of Tables 1 to 3 and they are hence omitted. They however confirm that the approach proposed by Tankov is very efficient, whatever the level of dependency, whereas the method proposed by Grothe and Hofert needs larger values for K as θ is decreasing.

2.4 Comparison of the dependence range

We here look at the possible dependence range implied by the three models, by comparing the maximal values of the corresponding Pearson correlation

Table 2: Mean (and standard deviation in brackets) of the relative errors $\mathbf{E}_L(1)$ for the bivariate distribution

K	θ	Grothe-Hofert	Tankov
50	0.2	2.426 (0.075)	0.026 (0.015)
100	0.2	1.457 (0.058)	0.028 (0.018)
500	0.2	0.358 (0.040)	0.029 (0.016)
1000	0.2	0.174 (0.030)	0.029 (0.018)
50	0.5	0.079 (0.044)	0.045 (0.029)
100	0.5	0.039 (0.026)	0.040 (0.029)
500	0.5	0.035 (0.023)	0.040 (0.027)
1000	0.5	0.041 (0.029)	0.036 (0.022)
50	1.0	0.034 (0.026)	0.034 (0.022)
100	1.0	0.033 (0.020)	0.033 (0.020)
500	1.0	0.033 (0.019)	0.031 (0.022)
1000	1.0	0.034 (0.021)	0.031 (0.020)
50	10.0	0.028 (0.019)	0.028 (0.018)
100	10.0	0.030 (0.017)	0.033 (0.021)
500	10.0	0.027 (0.018)	0.028 (0.021)
1000	10.0	0.028 (0.021)	0.028 (0.021)

coefficients (which depend on the models).

As for the model based on a regular copula, the maximal dependence between $Y_1(t)$ and $Y_2(t)$ is obtained for the upper Fréchet bound, that is when

$$\mathbf{C}(u_1, u_2) = \min(u_1, u_2)$$

for all $u_1, u_2 > 0$, or equivalently, when the random variables $Y_1(t)$ and $Y_2(t)$ are comonotonic (see for example Section 2.9 in [12]).

This provides:

$$\bar{F}_{\mathbf{Y}(t)}(u_1, u_2) \leq \min\left(\bar{F}_{Y_1(t)}(u_1), \bar{F}_{Y_2(t)}(u_2)\right)$$

and Pearson correlation coefficient of $\mathbf{Y}_t$ fulfills:

$$\rho_{\mathbf{Y}_t} \leq \rho_{\max}^{Fr}(t) = \frac{\iint\limits_{\mathbb{R}_+^2} \min\left(\bar{F}_{Y_1(t)}(u_1), \bar{F}_{Y_2(t)}(u_2)\right) \, du_1 \, du_2 - \mathbb{E}(Y_1(t))\,\mathbb{E}(Y_2(t))}{\sigma_{Y_1(t)}\sigma_{Y_2(t)}}. \tag{19}$$

As for the superposition model, it is easy to check that Pearson correlation coefficient of $\mathbf{Y}(t)$ is independent on t (as for any Lévy process) and it is given by

$$\begin{aligned}
\rho^S &= \rho_{\mathbf{Y}(1)} \\
&= \frac{var(X_3(1))}{\sigma(Y_1(1))\,\sigma(Y_2(1))} \\
&\leq \frac{\min(var(Y_1(1)), var(Y_2(1)))}{\sigma(Y_1(1))\,\sigma(Y_2(1))} \\
&= \rho_{\max}^S
\end{aligned}$$

Table 3: Mean (and standard deviation in brackets) of the correlation coefficient estimators $\hat{\rho}$

K	θ	ρ	Grothe-Hofert	Tankov
50	0.2	0.060	0.067 (0.021)	0.060 (0.017)
100	0.2	0.060	0.062 (0.016)	0.059 (0.016)
500	0.2	0.060	0.062 (0.018)	0.058 (0.017)
1000	0.2	0.060	0.059 (0.017)	0.054 (0.018)
50	0.5	0.410	0.410 (0.022)	0.405 (0.019)
100	0.5	0.410	0.409 (0.017)	0.409 (0.021)
500	0.5	0.410	0.410 (0.020)	0.410 (0.020)
1000	0.5	0.410	0.407 (0.019)	0.403 (0.021)
50	1.0	0.711	0.709 (0.012)	0.709 (0.012)
100	1.0	0.711	0.711 (0.010)	0.710 (0.014)
500	1.0	0.711	0.711 (0.012)	0.712 (0.012)
1000	1.0	0.711	0.710 (0.013)	0.708 (0.012)
50	10.0	0.970	0.970 (0.001)	0.970 (0.001)
100	10.0	0.970	0.970 (0.001)	0.970 (0.001)
500	10.0	0.970	0.970 (0.001)	0.970 (0.001)
1000	10.0	0.970	0.970 (0.001)	0.970 (0.001)

with

$$\rho_{\max}^{S} = \min\left(\frac{\sigma\left(Y_1\left(1\right)\right)}{\sigma\left(Y_2\left(1\right)\right)}, \left(\frac{\sigma\left(Y_1\left(1\right)\right)}{\sigma\left(Y_2\left(1\right)\right)}\right)^{-1}\right) \tag{20}$$

(see for example [12]). Note that the upper bound is reachable, taking $X_1(1) = 0$ or $X_2(1) = 0$ according to whether $\sigma\left(Y_1\left(1\right)\right) \leq \sigma\left(Y_2\left(1\right)\right)$ or the opposite, which leads to $\sigma\left(Y_1\left(1\right)\right) = \sigma\left(X_3\left(1\right)\right)$ or $\sigma\left(Y_2\left(1\right)\right) = \sigma\left(X_3\left(1\right)\right)$, respectively.

Finally, for the model based on a Lévy copula, the maximal dependence is obtained for the Lévy copula of complete dependence $\mathbf{L}^{\|}$, with

$$\mathbf{L}^{\|}\left(\mathbf{u}\right) = \min\left(u_1, u_2\right)$$

for $\mathbf{u} = (u_1, u_2) \in \mathbb{R}_+^2 \setminus \{\mathbf{0}\}$ (see [6]). Hence, the upper bound for ρ is given by

$$\rho \leq \rho_{\max}^{L} = \frac{\iint\limits_{\mathbb{R}_+^2} \min\left(U_1\left(u_1\right), U_2\left(u_2\right)\right)\ du_1\ du_2}{\sigma_{Y_1(1)}\sigma_{Y_2(1)}}. \tag{21}$$

For illustration purpose, we now assume that the marginal processes are gamma processes with $Y_i\left(t\right) \sim \Gamma\left(A_i t, 1\right)$, $i = 1, 2$. In that case, we have $\mathbb{E}\left(Y_i\left(t\right)\right) = var\left(Y_i\left(t\right)\right) = A_i t$ and $U_i\left(u_i\right) = A_i \operatorname{Ei}\left(u_i\right)$ for $u_i > 0$ and $i = 1, 2$ (see (16)). We set

$$\alpha = var\left(Y_1\left(t\right)\right)/var\left(Y_2\left(t\right)\right) = A_1/A_2.$$

For the superposition model, the maximum coefficient $\rho_{\max}^{S}$ is computed through (20) with

$$\rho_{\max}^{S} = \rho_{\max}^{S}\left(\alpha\right) = \min\left(\alpha^{\frac{1}{2}}, \alpha^{-\frac{1}{2}}\right).$$

For the Lévy copula model, the maximum coefficient given in (21) depends on the same α with

$$\rho_{\max}^{L}(\alpha) = \iint_{\mathbb{R}_+^2} \min\left(\alpha^{\frac{1}{2}} \operatorname{Ei}(u_1), \alpha^{-\frac{1}{2}} \operatorname{Ei}(u_2)\right) \, du_1 \, du_2.$$

For the regular copula model, the maximum coefficient $\rho_{\max}^{Fr}(t)$ is computed through (19), which depends on t, A_1 and A_2. For comparison purpose, we take $A_1 = \alpha$ and $A_2 = 1$ so that $\rho_{\max}^{Fr}(t)$ now only depends on t and α (which is still equal to A_1/A_2). It is denoted by $\rho_{\max}^{Fr}(t, \alpha)$.
The functions $\rho_{\max}^{L}(\alpha)$ and $\rho_{\max}^{S}(\alpha)$ are plotted in Figure 5, together with $\rho_{\max}^{Fr}(t, \alpha)$ for $t = 0.1$, $t = 1$, $t = 10$. We can notice on this figure that $\rho_{\max}^{L}(\alpha) \geq \rho_{\max}^{S}(\alpha)$. This is coherent with the fact that the superposition model is a specific Lévy process (that could be modeled through a Lévy copula) and hence exhibits some restrictive dependence range when compared to a general Lévy process modeled through a Lévy copula. Also, we have $\rho_{\max}^{Fr}(t, \alpha) \geq \rho_{\max}^{L}(\alpha)$ for $t = 0.1$, $t = 1$ and $t = 10$. This means that assuming the process $\mathbf{Y}$ to be a Lévy process induces some restriction on the possible dependence between $Y_1(t)$ and $Y_2(t)$, when compared to a general random vector $(Y_1(t), Y_2(t))$ without any specification for the structure of the process $\mathbf{Y}$. Here again, this was expected. Finally, it seems that $\rho_{\max}^{Fr}(t, \alpha)$ decreases with t (this has been checked for various values of t beyond $t = 0.1$, $t = 1$ and $t = 10$). We do not have clear explanations for that fact.

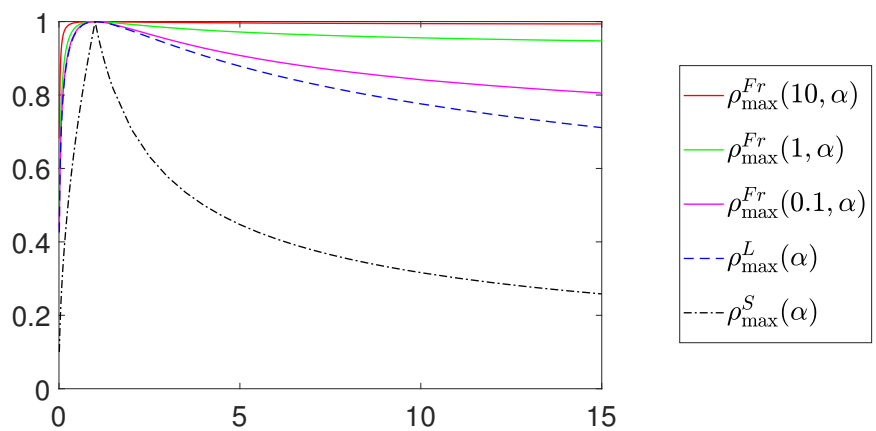

Figure 5: $\rho_{\max}^{L}(\alpha)$, $\rho_{\max}^{S}(\alpha)$ and $\rho_{\max}^{Fr}(t, \alpha)$ for $t = 0.1$, $t = 1$ and $t = 10$.

3 On the impact of a wrong modeling on the reliability function

3.1 Introduction

In this section, a two-component series system is considered, with joint deterioration level modeled by one of the three models from Section 2, where component j is considered as failed as soon as its deterioration level is beyond a given failure threshold L_j, $j = 1, 2$. The point here is to investigate the impact of a wrong modeling on the reliability function of the two-component series system, with

$$R(t) = \mathbb{P}(Y_1(t) < L_1, Y_2(t) < L_2) \tag{22}$$

for all $t > 0$.

With that aim, data are generated from one of the two coherent models (superposition or Lévy copula), which is considered to be the true model. Next, the parameters of the three models (including the true one) are estimated from a series of observations of the increments $\Delta Y_{ij} = Y_i(t_j) - Y_i(t_{j-1})$, $i = 1, 2$, $j = 1, ..., d$. To be more specific, based on the fact that our purpose is to investigate the impact of a wrong modeling of the dependence between the gamma marginal processes, the marginal parameters are assumed to be known and only the dependence parameters are estimated. Note that the point of the paper is not the comparison of estimation methods (as in [11] for instance), and large size samples are considered, in order the estimation results to be reliable. Once the dependence parameters have been estimated, the corresponding reliability functions are computed with the estimated parameters.

3.2 Estimation methods

3.2.1 Model based on regular copulas

For the model based on regular copulas described in Subsection 2.1, three copula families are considered (Clayton, Frank and Gümbel), with one single parameter $\theta > 0$ to be estimated for each of them. With that aim, we use the R package "copula": firstly the function *pobs* transforms the increments into pseudo-observations in the set $[0, 1] \times [0, 1]$, and then the function *fitCopula* gives the maximum pseudo-likelihood estimate.

The reliability function is next estimated by:

$$\hat{R}(t) = C_{\hat{\theta}}(F_{Y_1(t)}(L_1), F_{Y_2(t)}(L_2)).$$

3.2.2 Model based on superposition

Let $X_i(t) \sim \Gamma(a_i t, b)$ for $i = 1, 2, 3$ so that $Y_i(t) \sim \Gamma(A_i t, b)$ for $i = 1, 2$, with $A_1 = a_1 + a_3$, $A_2 = a_2 + a_3$. In this model, all the dependence is measured by parameter a_3, with

$$cov(Y_1(t), Y_2(t)) = var(X_3(t)) = \frac{a_3}{b^2} t.$$

An estimator based on the empirical covariance is used for a_3, with

$$\hat{a}_3 = \frac{b^2}{t_d} \sum_{j=1}^{d} \left(\Delta Y_{1,j} - \frac{A_1}{b} \Delta t_j \right) \left(\Delta Y_{2,j} - \frac{A_2}{b} \Delta t_j \right).$$

Note that this estimator is unbiased as

$$\begin{aligned} \mathbb{E}\left(\hat{a}_3\right) &= \frac{b^2}{t_d} \sum_{j=1}^{d} cov\left(\Delta Y_{1,j}, \Delta Y_{2,j}\right) \\ &= \frac{b^2}{t_d} \sum_{j=1}^{d} var\left(\Delta X_{3,j}\right) \\ &= \frac{b^2}{t_d} \sum_{j=1}^{d} \frac{a_3}{b^2} \Delta t_j \\ &= a_3. \end{aligned}$$

Then we set

$$\begin{aligned} \hat{a}_1 &= A_1 - \hat{a}_3, \\ \hat{a}_2 &= A_2 - \hat{a}_3, \end{aligned}$$

and the reliability function is estimated by

$$\hat{R}(t) = \int_{\mathbb{R}} \hat{F}_{X_1(t)}(L_1 - v) \hat{F}_{X_2(t)}(L_2 - v) \hat{f}_{X_3(t)}(v)\, dv,$$

where $\hat{f}_{X_3(t)}$ and $\hat{F}_{X_i(t)}, i = 1, 2$ stand for the PDF and CDF of the gamma distribution with parameters $(\hat{a}_3, b)$ and $(\hat{a}_i, b), i = 1, 2$ respectively.

3.2.3 Model based on a Lévy copula

Let $(\mathbf{Y}_t)_{t \geq 0}$ be a bivariate Lévy process with gamma processes as margins, $Y_i(t) \sim \Gamma(A_i t, b_i)$, $i = 1, 2$, and dependence modeled by a Clayton-Lévy or Gumbel-Lévy copula with parameter θ. To determine the parameter θ of the Lévy copula, we use the following estimator for Pearson correlation coefficient ρ:

$$\hat{\rho} = \frac{b_1 b_2}{t_d \sqrt{A_1 A_2}} \sum_{j=1}^{d} \left(\Delta Y_{1j} - \frac{A_1}{b_1} \Delta t_j \right) \left(\Delta Y_{2j} - \frac{A_2}{b_2} \Delta t_j \right),$$

which can be seen to be an unbiased estimator as previously for a_3, in the specific setting of this paper where the marginal parameters are assumed to be known. Whenever the marginal parameters are unknown and have to be estimated, please see [20] for an unbiased estimator.

Then $\hat{\theta}$ is obtained by searching θ such that $\hat{\rho} = \rho_{\theta,CL}$ or $\hat{\rho} = \rho_{\theta,GL}$, where $\rho_{\theta,CL}$ is given in (17) and where

$$\rho_{\theta,GL} = \frac{b_1 b_2}{\sqrt{A_1 A_2}} \iint_{\mathbb{R}_+^2} \left(\exp\left\{ \left[\left(\ln\left(1+U_1\left(u_1\right)\right)\right)^{-\theta} + \left(\ln\left(1+U_2\left(u_2\right)\right)\right)^{-\theta} \right]^{-\frac{1}{\theta}} \right\} - 1 \right) du_1\, du_2$$

(see (9)).

Finally, the reliability is estimated through (22), considering $\hat{\theta}$ as parameter for the underlying Lévy copula and using Monte Carlo simulations with $J = 200\,000$ trajectories for the computation.

3.3 True model: Superposition

In this section, we suppose that the true model is the superposition model with $a_1 = a_2 = a_3 = b = 1$, which leads to $Y_i(t) \sim \Gamma(2t, 1)$ for $i = 1, 2$.

The parameters a_3 (for the superposition) and θ (for classic and Lévy copulas) are estimated from 250 repetitions of $d = 5000$ observations. We consider two sets of observation times. The first one with periodic observation times with period 1: $\Delta t_j = 1$, $j = 1, \cdots, d$. The second one with aperiodic observation times: $\{\Delta t_j, j = 1, \cdots, d\}$ are i.i.d. observations of the uniform distribution on $[1, 20]$.

The results for the estimations of the dependence parameters are provided for the six models and the two sets of observation times in Table 4, where the mean and standard deviation over the 250 repetitions are given, together with two empirical quantiles. Note that for the three regular copula models, the estimates highly depend on the type of observation times (periodic or not), which is not the case for the superposition model or models constructed through Lévy copulas. Also, considering periodic observations, we have tested different periodicity for the regular copula models and observed that the results also depend on the periodicity. (Example: for the Clayton copula, the estimates for the parameter θ are equal to 0.553 for $\Delta t_j = 0.1$, and 0.641 for $\Delta t_j = 10$, $j = 1, \cdots, d$).

We next consider the previously obtained mean estimates as dependence parameters, and compute the associated reliability function for failure thresholds $L_1 = L_2 = 10$. The reliability functions for the superposition model, for ordinary Clayton, Frank and Gumbel copula models, and for Clayton-Lévy and Gumbel-Lévy copula models are respectively denoted by R_S, R_C, R_F, R_G, R_{CL} and R_{GL}. The true superposition model (with the exact value for the dependence parameter) is also computed and denoted R_{True}. All these functions are plotted in Figures 6 and 7, for the two different sets of observation times. For a better understanding of the possible impact of the dependence model on the reliability function, we have also looked at the overall range of the reliability function considering the two extreme cases (independence and total dependence). Remark that, as the two components share the same distribution, the total dependence just means that $Y_1(t) = Y_2(t)$. The two extreme case have

Model	periodic observation times				aperiodic observation times			
	mean	std	$q_{0.25}$	$q_{0.975}$	mean	std	$q_{0.25}$	$q_{0.975}$
Superposition (a_3)	1.000	0.047	0.917	1.100	0.999	0.039	0.919	1.075
Clayton (θ)	0.607	0.028	0.558	0.661	4.144	0.082	4.001	4.326
Frank (θ)	3.375	0.113	3.143	3.608	15.679	0.224	15.289	16.126
Gümbel (θ)	1.469	0.019	1.438	1.510	3.350	0.050	3.252	3.453
Clayton-Lévy (θ)	0.595	0.029	0.546	0.659	0.593	0.024	0.546	0.642
Gumbel-Lévy (θ)	0.727	0.041	0.656	0.808	0.727	0.032	0.664	0.797

Table 4: Estimates of the dependence parameters for the six considered models when the true model is the superposition model

been added to the figures, where R_I (resp. R_{TD}) corresponds to independence (resp. total dependence).

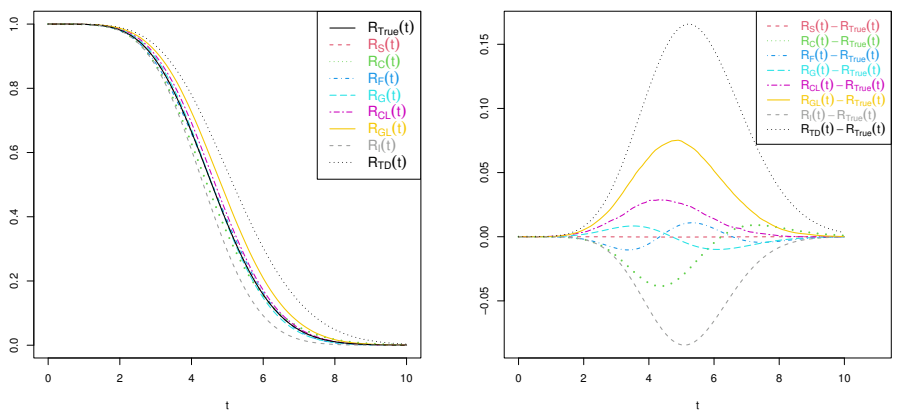

Figure 6: The reliability functions R_{True}, R_S, R_C, R_F, R_G, R_{CL}, R_{GL} (left) and the bias $R_S - R_{True}$, $R_C - R_{True}$, $R_F - R_{True}$, $R_G - R_{True}$, $R_{CL} - R_{True}$, $R_{GL} - R_{True}$ for periodic observation times when the true model is the superposition model

As expected and based on the fact that the parameter a_3 is very well estimated (see Table 4), the difference between the true reliability R_{True} and its estimate using the true model R_S is very close to 0 in the two figures. Now, if we focus on the two Lévy copula models, it seems that, for this example, the Clayton-Lévy family is a better (or less bad) choice than the Gumbel-Lévy family to fit the data. More important, we can see that these two models are not impacted at all by the choice of observation times (please look at the scale

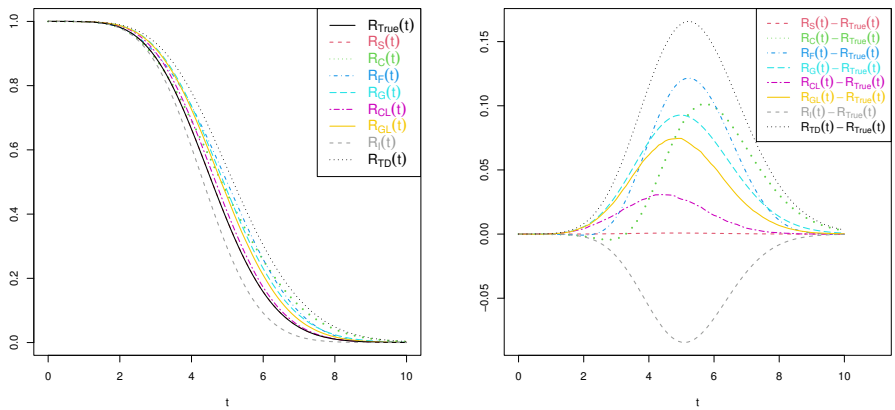

Figure 7: The reliability functions R_{True}, R_S, R_C, R_F, R_G, R_{CL}, R_{GL} (left) and the bias $R_S - R_{True}$, $R_C - R_{True}$, $R_F - R_{True}$, $R_G - R_{True}$, $R_{CL} - R_{True}$, $R_{GL} - R_{True}$ for aperiodic observation times when the true model is the superposition model

of the vertical axis on the two figures, which shows that the results are mostly similar for the two sets of observation times). This is not the case for ordinary copula models. Indeed, if the reliability functions R_S, R_C, R_F are quite close to R_{True} for periodic observation times (Figure 6), probably due to the fact that, in that case, the increments of the process are *iid* random variables, it is no longer the case with aperiodic observation times (Figure 7). It can be observed that the models with ordinary copulas are only slightly better than the total dependence extreme model, which confirms that they are far from the true model. This is mainly due to the incoherence of this type of model, as explained in the beginning of the paper. Finally, note also that the models based on Lévy copula does not catch that well the dependence induced by a superposition model. This must be due to the fact that the two Lévy copulas that have been considered here are of the Archimedean type, which implies that the corresponding bivariate Lévy measure is absolutely continuous with respect to Lebesgue measure (assuming the marginal Lévy measures to be absolutely continuous) whereas the bivariate Lévy measure for the superposition model is of the shape

$$\mu(d\mathbf{y}) = \mu_{X_1}(y_1)\, dy_1 \times \delta_0(dy_2) + \delta_0(dy_1) \times \mu_{X_2}(y_2)\, dy_2 + \mu_{X_3}(y_1)\, dy_1 \times \delta_{y_1}(dy_2)$$

and is hence not absolutely continuous with respect to Lebesgue measure. (The bivariate Lévy measure is easily obtained, noticing that the processes X_1 and X_2 never jump together, whereas the jumps of X_3 have just the same contribution in both Y_1 and Y_2). The superposition model which implies either

Model	mean	std	$q_{0.25}$	$q_{0.975}$
Superposition (a_3)	0.442	0.020	0.408	0.482
Clayton (θ)	2.479	0.051	2.385	2.579
Frank (θ)	10.103	0.160	9.794	10.417
Gümbel (θ)	2.484	0.034	2.422	2.559
Clayton-Lévy (θ)	0.528	0.021	0.492	0.572
Gumbel-Lévy (θ)	0.593	0.025	0.549	0.646

Table 5: Estimates of the dependence parameters for the six considered models when the true model is the Clayton-Lévy copula model

independent jumps (both in size and time location) for the two marginal processes, or simultaneous jumps with completely dependent jump sizes is hence not well represented by a Lévy copula model with a smooth Lévy copula such as an Archimedean one.

3.4 True model: Clayton-Lévy copula

We now consider a Clayton-Lévy copula model as the true model and study again the effect of a wrong modeling. More precisely, $d = 5000$ observations of a Clayton-Lévy copula model with parameter $\theta = 0.5$ and margins $Y_i(t) \sim \Gamma(t, 1)$ for i=1,2 are considered. The parameters are estimated from 250 repetitions of the process considering aperiodic observation times $\{\Delta t_j, j = 1, \cdots, n\}$, simulated as i.i.d. observations of the uniform distribution on $[1, 20]$. The results of the estimation procedure are provided in Table 5 and the resulting reliability functions are represented in Figure 8 , again with the two extreme models, independence and total dependence. (Note that we have also considered the case of periodic observation times but the results were very similar, so that they are omitted).

Here again, we can see that the estimate of the reliability function considering the true model with the estimated parameter (R_{CL}) fits very well with the true reliability function (R_{True}). Also, the best results are obtained for the superposition model (R_S), closely followed by the model based on the Gumbel-Lévy copula (R_{GL}). Finally, note that the models based on the regular copulas (R_C, R_L, R_G) lead to a higher distortion with respect to the true reliability.

4 Conclusive remarks

As a conclusion, we have here considered three models for modeling the dependence between univariate wear Lévy processes. The model based on regular copulas has been shown to lead to a bivariate process that do not have independent increments, that is, it is not a Lévy process. The construction hence is questionable, as the distribution of successive increments (among other things) is not known and should be further specified to get a coherent model. The model

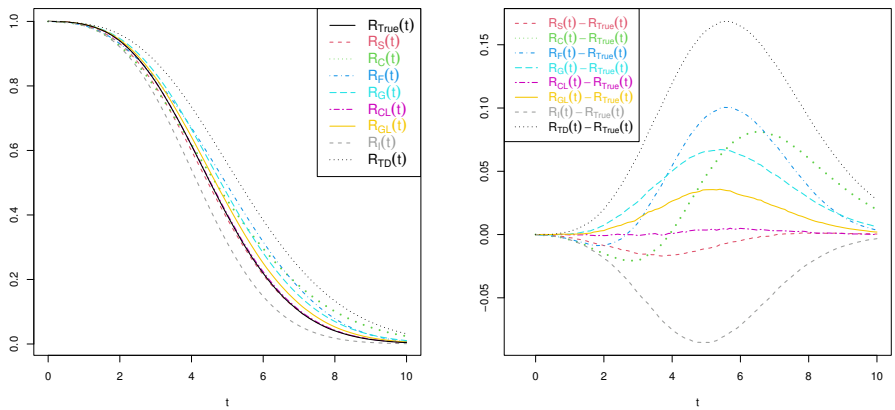

Figure 8: The reliability functions R_{True}, R_S, R_C, R_F, R_G, R_{CL}, R_{GL} (left) and the bias $R_S - R_{True}$, $R_C - R_{True}$, $R_F - R_{True}$, $R_G - R_{True}$, $R_{CL} - R_{True}$ $R_{GL} - R_{True}$ for aperiodic observation times when the true model is a Clayton-Lévy copula model with $\theta = 0.5$

constructed by superposition is a specific bivariate Lévy process that allows easy computations. It however suffers from a restrictive range of dependence when compared to a general Lévy process constructed through a Lévy copula. The model based on Lévy copulas allows any kind of dependence between the margins of a bivariate Lévy process. However, this model implies a higher technicality and requires approximate simulation methods for its numerical assessment (or the use of numerical schemes).

As for the impact on the reliability function of a wrong choice of model, we have seen that the model based on a regular copula, even though its incoherence, provides not that bad results when the underlying (true) process is a Lévy process (constructed either by superposition of through a Lévy copula) in case of periodic observation times. The results are less convincing when the observation times are not periodic. We also saw that, considering a Lévy copula as the true model, a wrong choice for the family of Lévy copulas has an impact on the reliability function and the superposition model can even provide better results than a model with a wrongly chosen Lévy copula (see Figure 8). Finally, the models based on a smooth Lévy copula do not catch that well the dependence implied by the superposition model.

Then, in practice, what could be suggested? At first, as there is no theoretical basis for the model based on regular copulas, we would suggest not to use this model (especially when the observation times are not periodic). Next, to chose between a Lévy process constructed by superposition or a Lévy copula, one can first estimate Pearson correlation coefficient. If it is beyond the upper

bound for the superposition model, then of course, this model cannot be used. If it is below, one can maybe refer to technical experts to better understand the interaction between the two marginal deterioration indicators. Indeed, in a superposition model, the dependence is modeled through simultaneous jumps with completely dependent jump sizes in the two marginal processes. This can be meaningful in an application context. Think for instance of the wings of a wind turbine where all wings are subject to the same wind that can have a similar impact on each wing and at the same time. Another point for reflection could be the development of goodness-of-fit tests, but this clearly is a challenging issue, as, to the best of our knowledge, we are not aware of any existing reliable goodness-of-fit test even for well-known univariate Lévy processes such as gamma processes.

Note also that the method for choosing the model between superposition and a parametric family of Lévy copulas highly depends on the data. In this article, we have considered so-called low-frequency observations, which means that the levels of deterioration are measured during inspections at isolated times. In case of continuous monitoring, it may be possible to have high-frequency observations which may allow to consider non parametric estimation procedures for the underlying Lévy copula, see for instance [5]. See also Section 4 in [30] for more references. This would allow not to make an a priori choice between the superposition and Lévy copula models (as the superposition model corresponds to a specific Lévy copula). Furthermore, even if we have focused, in this paper, on bivariate processes, it is important to note that in the case of a large number of components, due to the high technicality of Lévy copula models, it would probably be more reasonable to use simpler models, such as superposition models.

Finally, even if it does not provide a multivariate Lévy process, another promising option is to model the dependence between univariate Lévy processes through the use of a random effect, as it allows tractable computations, based on the conditional independent-increment property given the random effect. See for instance [15, 32]. We are not aware of any study of the possible range of dependence obtained through this method, when compared to that obtained through a Lévy copula or by superposition, but the question seems to be of interest due to the model's potential.

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5 Appendix

Proof of Formulas (10) and (9). Let $\mathbf{s} = (s_1, s_2) \in \mathbb{R}_+^2$. Based on Lévy-Khinchin formula [6, Corollary 3.1], it is well-known that

$$\mathcal{L}_{\mathbf{Y}(t)}(\mathbf{s}) = \mathbb{E}\left(e^{-s_1 Y_1(t) - s_2 Y_2(t)}\right) = \exp\left(t \iint_{\mathbb{R}_+^2} \left(e^{-s_1 y_1 - s_2 y_2} - 1\right) \mu(d\mathbf{y})\right) \quad (23)$$

Writing

$$e^{-s_1 y_1 - s_2 y_2} - 1 = \left(e^{-s_1 y_1} - 1\right)\left(e^{-s_2 y_2} - 1\right) + \left(e^{-s_1 y_1} - 1\right) + \left(e^{-s_2 y_2} - 1\right)$$

and noting that

$$\exp\left(t \iint_{\mathbb{R}_+^2} \left(e^{-s_1 y_1} - 1\right) \mu_X(d\mathbf{y})\right) = \exp\left(t \int_{\mathbb{R}_+} \left(e^{-s_1 y_1} - 1\right) \mu_1(dy_1)\right) = \mathcal{L}_{Y_1(t)}(s_1) \quad (24)$$

(with a similar expression for the second component), we get

$$\mathcal{L}_{\mathbf{Y}(t)}(\mathbf{s}) = \exp\left(t \iint_{\mathbb{R}_+^2} \left(e^{-s_1 y_1} - 1\right)\left(e^{-s_2 y_2} - 1\right) \mu(d\mathbf{y})\right) \mathcal{L}_{Y_1(t)}(s_1)\, \mathcal{L}_{Y_2(t)}(s_2).$$

Now let us write the double integral as

$$\begin{aligned}
&\iint_{\mathbb{R}_+^2} \left(e^{-s_1 y_1} - 1\right)\left(e^{-s_2 y_2} - 1\right) \mu(d\mathbf{y}) \\
&= s_1 s_2 \iint_{\mathbb{R}_+^2} \left(\int_0^{y_1} e^{-s_1 z_1} dz_1\right)\left(\int_0^{y_2} e^{-s_2 z_2} dz_2\right) \mu(d\mathbf{y}).
\end{aligned}$$

Using Fubini's theorem, we get

$$\begin{aligned}
&\iint_{\mathbb{R}_+^2} \left(e^{-s_1 y_1} - 1\right)\left(e^{-s_2 y_2} - 1\right) \mu(d\mathbf{y}) \\
&= s_1 s_2 \iint_{\mathbb{R}_+^2} e^{-s_1 z_1 - s_2 z_2} \left(\iint_{[y_1,+\infty)\times[y_2,+\infty)} \mu(d\mathbf{y})\right) dz_1\ dz_2 \\
&= s_1 s_2 \iint_{\mathbb{R}_+^2} e^{-s_1 z_1 - s_2 z_2}\ \mathbf{U}(z_1, z_2)\ dz_1\ dz_2 \\
&= s_1 s_2 \mathcal{L}_{\mathbf{U}}(s_1, s_2),
\end{aligned}$$

which provides the result for Formula (10).

As for Formula (9), it is well-known that $\rho_{\mathbf{Y}(t)}$ is independent of t for a Lévy process so that it is enough to take $t = 1$. Hence, the point is to compute the covariance between $Y_1(1)$ and $Y_2(1)$. Now, starting again from (23), it is easy to check that

$$\mathbb{E}(Y_1(1)) = -\frac{\partial \mathcal{L}_{\mathbf{Y}(1)}}{\partial s_1}\left(0^+, 0^+\right) = \iint_{\mathbb{R}_+^2} y_1 \mu(d\mathbf{y}) = \int_{\mathbb{R}^+} y_1 \mu_1(dy_1)$$

(the same for $\mathbb{E}(Y_2(1))$) and

$$\begin{aligned}
\frac{\partial^2 \mathcal{L}_{\mathbf{Y}(1)}}{\partial s_1 \partial s_2}\left(0^+, 0^+\right) &= \mathbb{E}(Y_1(1)\, Y_2(1)) \\
&= \iint_{\mathbb{R}_+^2} y_1 y_2\ \mu(d\mathbf{y}) + \iint_{\mathbb{R}_+^2} y_1 \mu(d\mathbf{y}) \iint_{\mathbb{R}_+^2} y_2 \mu(d\mathbf{y}) \\
&= \iint_{\mathbb{R}_+^2} y_1 y_2\ \mu(d\mathbf{y}) + \mathbb{E}(Y_1(1))\, \mathbb{E}(Y_2(1)).
\end{aligned}$$

Hence

$$
\begin{aligned}
cov\left(Y_1\left(1\right), Y_2\left(1\right)\right) &= \iint\limits_{\mathbb{R}_+^2} y_1 y_2 \ \mu(d\mathbf{y}) \\
&= \iint\limits_{\mathbb{R}_+^2} \left(\int_0^{y_1} dz_1\right)\left(\int_0^{y_2} dz_2\right) \ \mu(d\mathbf{y}) \\
&= \iint\limits_{\mathbb{R}_+^2} \left(\iint\limits_{[y_1,+\infty)\times[y_2,+\infty)} \mu\left(d\mathbf{y}\right)\right) dz_1 \ dz_2 \\
&= \iint\limits_{\mathbb{R}_+^2} \mathbf{U}\left(z_1, z_2\right) dz_1 \ dz_2
\end{aligned}
$$

using Fubini's theorem again for the third line, which achieves this proof.