Design of fault tolerant on board networks with priorities

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We consider on-board networks in satellites interconnecting entering signals (inputs) to amplifiers (outputs). The connections are made via expensive switches with four links available. The paths connecting inputs to outputs should be link-disjoint. Among the input signals, some of them, called priorities, must be connected to the amplifiers which provide the best quality of service (that is to some specific outputs). In practice, amplifiers are subject to faults that cannot be repaired. Therefore we need to add extra outputs to ensure the existence of sufficient many valid ones. Given n inputs, p priorities and k faults, the problem consists in designing a low cost network (i. e. with the minimum number of switches) where it is possible to route the p priorities to the p best quality amplifiers and the other inputs to some valid amplifiers, for any sets of k faulty and p best quality amplifiers. Let $N(n,p,k)$ be the minimum number of switches of such a network. We prove here that $N(n,p,0) \leq n - p + \frac{n}{2}\lceil \log_2 p \rceil$ and give exact values of $N(n,p,k)$ for small p and k.

Keywords: switching networks, fault tolerance

Modern telecommunications satellites are very complex to design and an important industrial issue is to provide robustness at the lowest possible cost. A key component of telecommunication satellites is an interconnection network which allows to redirect signals received by the satellite to a set of amplifiers where the signals will be retransmitted. The network is made of expensive switches ; so we want to minimise their number subject to the following conditions : Each input and output is adjacent to exactly one link ; each switch is adjacent to exactly four links ; there are n inputs (signals) and $n+k$ outputs (amplifiers) ; among the $n+k$ outputs, k can fail permanently; and finally all the input signals should be sent to valid outputs via link-disjoint paths. Designing such networks is a complex problem that was proposed by Alcatel Space Industries and solutions are given in [3] and [4]. It was proposed also to consider the case where each signal needs a specific amplifier ; that leads to design a network realizing any permutation from the inputs to the outputs (see [1], [2]). Here we consider a new problem asked recently by Alcatel Space Industries. Among the n input signals, p of them called priorities must be connected to the amplifiers providing the best quality of service (that is to some specific outputs) and the others signals should be sent to other amplifiers. Note that the priority signals are given, but the amplifiers providing the quality of service change according the position of the satellite and so the networks should be able to route the signals for any set of k failed outputs and route the p priorities to any set of p valid outputs.

This problem can be restated more formally as follows:

Definition 1 A (n,p,k)-*network* G is a graph (V,E) where the vertex set V is partitioned into four subsets P, I, O and S called respectively the *priorities*, the *ordinary inputs*, the *outputs* and the *switches*, satisfying the following constraints:

- there are p priorities, $n-p$ ordinary inputs and $n+k$ outputs;
- each priority, each ordinary input and each output is connected to exactly one switch;
- switches have degree 4.

A (n,p,k)-network is *valid* if for any disjoint subsets F and B of O with $|F| = k$ and $|B| = p$, there exist in G, n edge-disjoint paths, p of them from P to B and the $n-p$ others joining I to $O \setminus (B \cup F)$. The set F corresponds to set of failures and B to the set of amplifiers providing the best quality of service. We

denote $N(n,p,k)$ the minimum number of switches (i.e. cardinality of S) of a valid (n,p,k)-network. A valid (n,p,k) – *network* with $N(n,p,k)$ switches will be called minimum.

Problem 1 Determine $N(n,p,k)$ and construct minimum (or almost minimum) valid networks.

As indicated above, the problem with no priority (that is $p = 0$) has been considered in [3] and almost solved in [4]. In these papers, the authors called $(n,0,k)$-networks (n,k) networks and denoted $N(n,0,k)$ by $N(n,k)$. In [3], it is shown that $N(n,0,2) = n$. In [4], it is proved that $\frac{3n}{2} - O(\frac{n}{k}) \leq N(n,0,k) \leq \frac{3n}{2} + O(k)$. The following values for small k are also given:

$$N(n,0,4) = n + \left\lceil \frac{n}{4} \right\rceil$$

$$N(n,0,6) = n + \frac{n}{4} + \sqrt{\frac{n}{8}} + O(1)$$

$$N(n,0,8) = n + \frac{n}{3} + \frac{2}{3}\sqrt{\frac{n}{3}} + O(\sqrt[4]{n})$$

$$N(n,0,12) = n + \frac{3n}{7} + O(\sqrt{n})$$

Theorem 1 $N(n,1,0) = n - 1$

Proof. In a minimum $(n,1,0)$-network, there is no switch connected to 2 ordinary inputs or more. Indeed, either there is an output connected to such a switch s and putting this output in B gives a contradiction; otherwise removing such a switch s and connecting its two adjacent ordinary inputs to its two others neighbours we obtain a smaller valid network. Thus $N(n,1,0) \geq n - 1$. Let G be the graph with one priority p, ordinary input set $\{i_1,i_2,\ldots,i_{n-1}\}$, output set $\{o_1,o_2,\ldots,o_n\}$ and switch set $\{s_1,s_2,\ldots,s_{n-1}\}$ such that $(p,s_1,s_2,\ldots,s_{n-1},o_n)$ is a path and for $1 \leq l \leq n-1$, i_l and o_l are adjacent to s_l. (See Figure 1.) Clearly, G

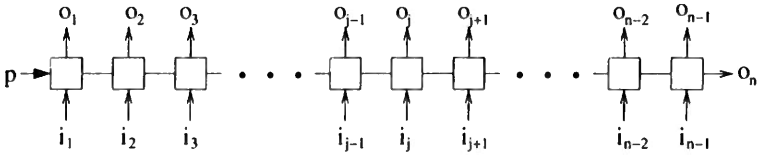

FIG. 1 – *Minimum* (8,1,0)*-network.*

is valid. Indeed, suppose that o_j is the priority output. Since o_{n-1} and o_n are equivalent, we may suppose that $j \neq n$. Then $P_0 = (p,s_1,s_2,\ldots,s_j,o_j)$, $P_j = (i_j,s_j,s_{j+1},\ldots,s_{n-1},o_n)$ and for $l \neq j$, and $P_l = (i_l,s_l,o_l)$ are the desired paths. □

Theorem 2 *For* $n \geq p \geq 1$,

$$\begin{aligned}
&(i) && N(2n,2p,0) && \leq && 2N(n,p,0) + n - 1\\
&(ii) && N(2n+1,2p,0) && \leq && N(n+1,p,0) + N(n,p,0) + n\\
&(iii) && N(2n,2p+1,0) && \leq && N(n,p+1,0) + N(n,p,0) + n\\
&(iv) && N(2n+1,2p,0) && \leq && N(n+1,p+1,0) + N(n,p,0) + n
\end{aligned}$$

Proof. (i) Let G^1 and G^2 be two valid $(n,p,0)$-networks. For $i = 1,2$, let I^i, P^i, $O^i = \{o_1^i,o_2^i,\ldots,o_n^i\}$ and S^i be the ordinary input set, priority set, output set and switch set of G^i. Let H be the graph defined as follows: Its ordinary input set is $I = I^1 \cup I^2$, its priority set is $P = P^1 \cup P^2$, its switch set is $S^1 \cup S^2 \cup S$ with $S = \{s_1,s_2,\ldots,s_{n-1}\}$ and its output set is $O = \{o_1,o_2,\ldots,o_{n-1}\} \cup \{o_1',o_2',\ldots,o_{n-1}'\} \cup \{o_n^1,o_n^2\}$. And for every $1 \leq j \leq n-1$, s_j is linked to o_j, o_j', the neighbour of o_j^1 in G_1 and the neighbour of o_j^2 in G_2. (See Figure 2)

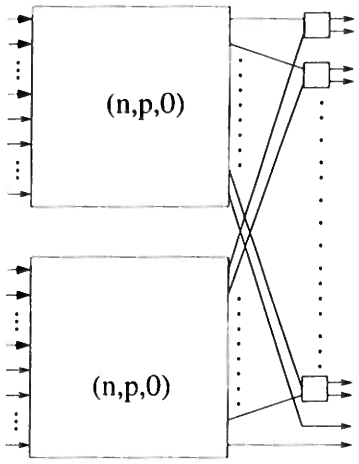

FIG. 2 – *Construction of networks from smaller ones.*

We claim that H is a valid $(2n,2p,0)$-network. Indeed, let B be a $2p$-subset of O. Clearly, one can partition O in two n-sets O_1 and O_2 such that for $i = 1,2$, $o_n^i \in O_i$, $|B \cap O_i| = p$ and for all $1 \leq j \leq n-1$, $|O_i \cap \{o_j, o'_j\}| = 1$. For $i = 1,2$, let $B_i = B \cap O_i$. Since G^1 and G^2 are valid $(n,p,0)$-networks, there are edge-disjoint paths joining P^i to B_i and I^i to $O_i \setminus B_i$. Hence, there are edge-disjoint paths joining P to B and I to $O \setminus B$.

The proofs of (ii), (iii) and (iv) are similar. □

Corollary 1 $N(n,p,0) \leq n - p + \frac{n}{2}\lceil \log_2 p \rceil$

Theorem 3 $N(n,2,0) = \lceil \frac{3n}{2} - 3 \rceil$

Proof. Theorem 2 gives $N(n,2,0) \leq \lceil \frac{3n}{2} - 3 \rceil$.

Let G be a minimum $(n,2,0)$-network G. An *ordinary switch* is a switch that is adjacent to one ordinary input. A switch adjacent to no ordinary input is a *usual switch*. Let s be the number of usual switches. A *cluster* of G is a maximal connected induced subgraph containing no usual switches. Let C be a cluster. Its degree $deg(C)$ is the number of edges joining a vertex of C to a vertex of $V(G) \setminus C$. Because C is connected one can check that : $deg(C) \leq in(C) + 2 - out(C) - pr(C)$ $(\star_1)$. Moreover $deg(C) \geq in(C) + pr(C) - out(C)$ $(\star_2)$ because $in(C) + pr(C) - out(C)$ paths must leave C. It follows that $pr(C) \leq 2$.

Let us now prove that C has at most one output. Indeed, suppose that $out(C) \geq 2$. Put two outputs of C in B, then $in(C) - out(C) + 2$ ordinary paths must leave C and $2 - pr(C) \geq 1$ priority paths must enter C. Thus $deg(C) \geq in(C) + 2 - out(C) + 1$. This contradicts $(\star_1)$.

Suppose now that $out(C) = 1$. If the output of C is in B then $in(C)$ ordinary paths must leave C and $1 - pr(C)$ priority paths must enter C. Thus $deg(C) \geq in(C) + 1 - pr(C)$. Thus, by $(\star_1)$, we obtain $deg(C) = in(C) + 1 - pr(C)$.

Hence, G has two kind of clusters, 1-clusters containing one output with degree $in(C) + 1 - pr(C)$ and 0-clusters with no output and degree at least $in(C)$ by $(\star_2)$. Since every input and output is contained in a cluster, there are $\sum_{C \text{ cluster}} deg(C) \geq 2n - 4$ edges joining clusters to usual switches. Thus, $4s \geq 2n - 4$, so $s \geq (n-2)/2$. Since there are $n - 2$ ordinary switches, we obtain $N(n,2,0) \geq \frac{3}{2}(n-2)$ □

Strengthening the proof of Theorem 3, we obtain the following lower bounds and some constructions give us the following upper bounds which improve the one of Corollary 1.

Theorem 4

$$N(n,1,1) = \left\lceil \frac{3n}{2} - 1 \right\rceil$$

$$\left\lceil \frac{3n}{2} - 1 \right\rceil \leq N(n,1,2) \leq \left\lceil \frac{3n}{2} + \frac{7}{2} \right\rceil$$

$$\frac{3n}{2} + \frac{k}{4} - \frac{3}{2} \leq N(n,1,k) \quad \text{for } k \geq 3$$

$$\frac{3n}{2} + \frac{\sqrt{n}}{2} - \frac{15}{4} \leq N(n,2,1) \leq \frac{3n}{2} + \sqrt{n} + O(1)$$

$$\frac{3n}{2} + \sqrt{\left\lfloor \frac{k+5}{2} \right\rfloor} \frac{\sqrt{n}}{\sqrt{2}} + \Theta(1) \leq N(n,2,k) \quad \text{for } k \geq 3$$

$$\frac{3n}{2} + \frac{\sqrt{n}}{2} - 6 \leq N(n,3,0) \leq \frac{3n}{2} + \sqrt{n} + O(1)$$

$$\frac{7n}{4} - 2p + \frac{5k}{8} \leq N(n,p,k) \quad \text{if } (p,k) \geq (3,1)$$

$$\frac{15n}{8} - \frac{11p}{8} + \frac{13k}{16} \leq N(n,p,k) \quad \text{if } (p,k) \geq (5,1)$$

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