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# Directed Virtual Path Layouts in ATM Networks<sup>\*</sup>

## (Extended Abstract)

Jean-Claude Bermond, Nausica Marlin <sup>\*\*</sup>, David Peleg <sup>\*\*\*</sup>, and Stéphane Perennes<sup>\*\*</sup>

**Abstract.** This article investigates the problem of designing virtual di-paths (VPs) in a directed ATM model, in which the flow of information in the two directions of a link are not identical. On top of a given physical network we construct directed VPs. Routing in the physical network is done using these VPs. Given the capacity of each physical link (the maximum number of VPs that can pass through the link) the problem consists in defining a set of VPs to minimize the diameter of the virtual network formed by these VPs (the maximum number of VPs traversed by any single message). For the most popular types of simple networks, namely the path, the cycle, the grid, the tori, the complete k-ary tree, and the general tree, we present optimal or near optimal lower and upper bounds on the virtual diameter as a function of the capacity.

**Keywords :** ATM, Virtual path layout, diameter, Embedding,

## 1 Introduction

The advent of fiber optic media has changed the classical views on the role and structure of digital communication networks. Specifically, the sharp distinction between telephone networks, cable television networks, and computer networks has been replaced by a unified approach. The most prevalent solution for this new network challenge is *Asynchronous Transfer Mode* (ATM for short), which is thoroughly described in the literature [6, 15]. The transfer of data in ATM is based on packets of fixed length, termed *cells*. Each cell is routed independently, based on two routing fields at the cell header, called *virtual channel identifier* (VCI) and *virtual path identifier* (VPI). This method effectively creates two types of predetermined simple routes in the network, namely, routes which are based on VPIs (called *virtual paths* or VPs) and routes based on VCIs and VPIs (called *virtual channels* or VCs). VCs are used for connecting network users (e.g., a telephone call); VPs are used for simplifying network management - routing of VCs in particular. Thus the route of a VC may be viewed as a concatenation of complete VPs.

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A major problem in this framework is the one of defining the set of VPs in such a way that some good properties are achieved.

1. A capacity (or bandwidth) is assigned to each VP. The sum of the capacities of the VPs that share a physical link constitutes the *load* of this link. Naturally, this load must not exceed the link's capacity, namely, the amount of data it can carry. The sum of the capacities of all the physical links is a major component in the cost of the network, and should be kept as low as possible.
2. The maximum number of VPs in a virtual channel, termed *hop count* in the literature, should also be kept as low as possible so as to guarantee low set up times for the virtual channels and high data transfer rates.

In its most general formulation, the *Virtual Path Layout (VPL)* problem is an optimization problem in which, given a certain communication demand between pairs of nodes and constraints on the maximum load and hop count, it is first required to design a system of virtual paths satisfying the constraints and then minimizing some given function of the load and hop count.

We employ a restricted model similar to the one presented by Cidon, Gerstel and Zaks in [12]. In particular, we assume that all VPs have equal capacities, normalized to 1. Hence the load of a physical link is simply the number of VPs that share this link.

Although links based on optical fibers and cables are directed, traditional research uses an undirected model. Indeed, this model imposes the requirement that if there exists a VP from  $u$  to  $v$  then there exists also a VP from  $v$  to  $u$ . In fact, that is the way ATM networks are implemented at the present time. However, the two VPs (the one from  $u$  to  $v$  and the one in the other direction) do not need to have the same capacity. Indeed, in many applications the flows on the VPs are not equal. For example, in a video application where  $u$  is a server and  $v$  a consumer there is a VP from  $u$  to  $v$  using a large capacity (transmission of video data) and a VP from  $v$  to  $u$  used only for control or acknowledgments with a very small capacity which can be considered as negligible. Therefore, it seems more reasonable to use a directed model like the one introduced by Chanas and Goldschmidt in [5]. This would allow us to model the situation described above by a single VP of capacity 1 in the main direction.

We focus on the all-to-all problem (all pairs of nodes are equally likely to communicate). Thus, the resulting maximum hop count can be viewed as the *diameter* of the graph induced by the VPs.

More formally, given a communication network, the VPs form a virtual directed graph on the top of the physical one, with the same set of vertices but with a different set of arcs. (Specifically, a VP from  $u$  to  $v$  is represented by an arc from  $u$  to  $v$  in the virtual digraph.) This virtual digraph provides a *directed virtual path layout (DVPL)* for the physical graph. Each VC can be viewed as a simple dipath in the virtual digraph. Therefore, a central problem is to find a tradeoff between the maximum load and the virtual diameter. In this article, we consider the following problem:

*Given a capacity on each physical arc, minimize the diameter of an admissible virtual graph (a virtual digraph that doesn't load an arc more than its capacity)*

**Related Work** The problem has been considered in the undirected case, for example, in [12, 11, 20, 10, 16, 8]. The problem of minimizing the maximum load over all VPL with bounded hop-count is studied in [9, 1], and minimizing also the average load is considered in [11]. The one-to-many problem is handled in [9, 11], where the focus is on minimizing the eccentricity of the virtual graph from a special point called the root

(this problem is the *rooted virtual path layout problem*) rather than minimizing the diameter of the virtual graph. A duality in the chain network between the problem of minimizing the hop-count knowing the maximum load, and the one of minimizing the load, knowing the maximum hop-count, is established in [9]. The reader can find an excellent survey of the results in the undirected model in [21].

The techniques involved in our constructions bear a certain resemblance to various embedding techniques used previously in the context of parallel computing, in order to implement a useful virtual architecture on a given practical machine topology (cf. [18, 14]). The parameters of interest in such embeddings are the number of virtual processors mapped onto any physical processor, the load on the physical links, and the dilation of the embedding, namely, the maximum length of the physical path corresponding to a given virtual link. The relevant concerns in our context are somewhat different, as dilation is of no consequence, and on the other hand, we have the freedom of designing the virtual topology as desired, in order to optimize its diameter.

*Our Results* The following table summarizes the results, giving lower and upper bounds on the virtual diameter (the minimum diameter of an admissible virtual digraph) as a function of the number of vertices  $n$  in the physical graph, its diameter  $D_G$ , its maximum in- and out-degree  $d$ , and the capacity  $c$  considered as a constant. Some of the proofs are omitted and will appear in the full version, and other proofs are included in the appendix. The results mentioned in the table for the path in the special case of  $c = 1$  are due to [4, 3].

Graph $G$	Capacity	Lower Bound	Upper Bound
General Graph	$c = \mathcal{O}(1)$	$\frac{\log n}{\log(cd)} - 1$	$D_G$
Path $P_n$	$c = 1$	$\frac{n}{2} + \log n$	$\frac{n}{2} + \log n + \mathcal{O}(1)$
	$c = \mathcal{O}(1)$	$\Omega(n^{\frac{1}{2c-1}})$	$\mathcal{O}(c \cdot n^{\frac{1}{2c-1}})$
Cycle $C_n$	$c = 1$	$2\sqrt{2n} + \mathcal{O}(1)$	$2\sqrt{2n} + \mathcal{O}(1)$
	$c = \mathcal{O}(1)$	$\Omega(n^{\frac{1}{2c}})$	$\mathcal{O}(c \cdot n^{\frac{1}{2c}})$
Torus $TM(a, b), a \leq b$	$c = \mathcal{O}(1)$	$\Omega((a \cdot b)^{1/2ac})$	$\mathcal{O}(a \cdot b^{1/2ac})$
Mesh $M(a, b), \log b \leq a \leq b$	$c = \mathcal{O}(1)$	$\Omega(\log n)$	$\mathcal{O}(\log n)$
Arbitrary Tree $T$	$c = 2$	$1/2 \cdot n^{1/3}$	$\tilde{D}(P_{D_T}, c)$ $32 \cdot n^{1/3}$ $D_G^{1/3} \cdot \log n$
	$c = \mathcal{O}(1)$	$\Omega(D_G^{1/(2c-1)})$	$8c \cdot n^{1/(2c-1)}$ $D_G^{1/(2c-1)} \cdot \log n$
Complete $k$ -ary Tree $T$	$c = 2, k = 2$	$D_G/2 - 1$	$D_G/2 + 2$
	$c = \mathcal{O}(1)$	$2 \left\lfloor \frac{h-1}{\lceil \log_k c \rceil + 1} \right\rfloor + 1$	

## 2 The model

The physical network is represented by a strongly connected weighted digraph  $G = (V, E, c)$ . The number of vertices is denoted by  $n = |V|$ . The vertex set  $V$  represents the network switches and end-users, and the arc set  $E$  represents the set of physical

directed links. The parameter  $c$  is the capacity (or weight) function, assigning to each arc  $e$  its capacity  $c(e)$ . For simplicity, given  $c_0 \in \mathbb{N}^+$ , we denote by  $c_0$  the constant capacity function in which  $c(e) = c_0$  for all  $e$ .

The network formed by the VPs is represented by a strongly connected digraph  $H = (V, E')$  and a function  $P$  assigning to each arc  $e' = (x, y) \in E'$  a simple dipath  $P(e')$  connecting  $x$  to  $y$  in  $G$ . In our terminology, the pair  $(H, P)$  is a *virtual digraph on  $G$* , an arc of  $H$  is a *virtual arc*, and the dipath  $P(e')$  in  $G$  associated with a virtual arc  $e'$  is a *virtual dipath (VP)*. To a dipath  $Q = (e'_1, \dots, e'_l)$  in  $H$  is associated a *route* in  $G$  consisting of the concatenation of  $P(e'_1), \dots, P(e'_l)$ .

The *load* of an arc  $e$  of  $G$  is the number of dipaths  $P(e')$  of  $G$  associated to an arc  $e'$  of  $H$  (or virtual dipaths) that contain the arc  $e$ , that is,  $l(e) = |\{e' \in E' \text{ s.t. } e \in P(e')\}|$ . A virtual digraph  $(H, P)$  satisfying the requirement  $\forall e \in E, l(e) \leq c(e)$  is referred as a  *$c$ -admissible directed virtual paths layout of  $G$* , shortly denoted  *$c$ -DVPL of  $G$* . The aim is to design  $c$ -DVPL of  $G$  with minimum hop-count. This corresponds to minimizing the diameter of the virtual digraph. For any digraph  $F$ ,  $d_F(x, y)$  denotes the distance from  $x$  to  $y$  in  $F$ , and  $D_F$  denotes  $F$ 's diameter. The *virtual diameter*,  $\tilde{D}(G, c)$ , of the digraph  $G$  with respect to the capacity  $c$ , is the minimum of  $D_H$  over all the  $c$ -DVPL  $H$  of  $G$ .

Note that both problems are *NP-hard*. (Indeed, deciding the question of whether  $\tilde{D}(G, c_0) = 1$  is equivalent to determining the arc-forwarding index of  $G$ , which is known to be *NP-hard*, see also [4].)

### 3 General Bounds

For most digraphs it turns out that  $\tilde{D}(G, c_0)$  is logarithmic even for  $c_0 = 1$ . Hence the ratio  $\frac{\tilde{D}(G, c_0)}{\log n}$  is of importance. For  $d$ -bounded degree digraphs<sup>1</sup>, a classical result states that  $\frac{\log n}{\log(c_0 d)} - 1 \leq \tilde{D}(G, c_0)$ . It is obtained by applying the Moore bound to the virtual digraph with  $n$  nodes, degree at most  $c_0 d$ , and diameter  $\tilde{D}(G, c_0)$  (see [16, 20]). Note also that  $\tilde{D}(G, 1) \leq D_G$ .

Here we derive a tighter bound related to the expansion-congestion parameters of  $G$ . First we recall three standard definitions: A *routing* for  $G$  is a mapping associating to each pair of vertices  $(x, y)$  a *route* (i.e. a dipath in  $G$ ) from  $x$  to  $y$ ; the *congestion* of a routing is the maximal *load* of an arc of  $G$  (i.e., the maximum number of routes going through an arc); the *arc-forwarding index* of  $G$ , denoted  $\pi(G)$ , is the minimum congestion of a routing.

The parameter  $\pi(G)$  has been extensively studied and many relations exist between  $\pi$  and other parameters like bisection or expansion, see [17, 19, 13]. There are strong relationships between  $\pi(G)$  and the DVPL issue. A routing for  $G$  is a DVPL of  $G$  where  $H$  is the complete digraph, and so  $\pi(G)$  is the smallest integer  $c_0$  such that  $\tilde{D}(G, c_0) = 1$ .

**Proposition 1.** *Let  $G$  be a  $d$ -bounded digraph, then*

$$\frac{\log \pi(G)}{\log(c_0 d)} - 1 + O(\log \tilde{D}) \leq \tilde{D}(G, c_0).$$

<sup>1</sup> where both the in- and out- degrees are upper-bounded by  $d$

*Proof.* With every  $c_0$ -DVPL  $H$  of  $G$  one can associate a routing for  $G$  as follows. Note that for any ordered pair of vertices  $(x, y)$  there exists at least one dipath in  $H$  from  $x$  to  $y$  with length smaller than  $D_H$ . We select one such dipath and choose the associated real dipath as the route from  $x$  to  $y$ . Due to the capacity constraint, at most  $c_0 d$  virtual dipaths enter (resp., leave) any given vertex of  $G$ ; one can easily check that the number of dipaths in  $H$  of length  $k$  that use an arc is at most  $k c_0 (c_0 d)^{k-1}$ . Hence the congestion of our routing is upper-bounded by  $M = c_0 + 2c_0(c_0 d) + 3c_0(c_0 d)^2 + \dots + D_H c_0 (c_0 d)^{D_H-1}$ .

By definition,  $\pi \leq M$ ; as  $M \leq c_0 \frac{D_H (c_0 d)^{D_H}}{c_0 d - 1}$ , taking the logarithm we obtain the result.

*Remark 1.* The lower bound of proposition 1 is quite similar to the one derived on the gossip time of a network under WDM or wormhole models [7, 2]. In both cases one must construct a route between any pair of vertices: for gossip problems the route is built along  $T$  time steps, whereas in the context of VPL design it is constructed by using  $\tilde{D}(G, c)$  jumps.

The following proposition indicates that for bounded  $c_0$ , one can expect  $\tilde{D}(G, c_0)$  to be logarithmic only if  $D_G$  is not too large. The result is valid for (distance-) symmetric digraphs (namely, such that  $d(x, y) = d(y, x)$ ).

**Proposition 2.** *Let  $G$  be a symmetric bounded degree digraph with  $\log D_G = \Omega(\log n)$ .*

$$\text{If } \tilde{D}(G, c_0) = \Theta(\log n), \text{ then } c_0 = \Omega\left(\frac{D_G \log n}{n}\right).$$

*In particular, if  $c_0$  is constant and  $\tilde{D}(G, c_0) = \Theta(\log n)$  then  $D_G = O\left(\frac{n}{\log n}\right)$ .*

We will see in proposition 8 that the above lower bound is tight in some sense.

*Proof.* The idea is that the design of an efficient DVPL is prevented by the existence of a long geodesic dipath contained in  $G$ . Let us first formalize the notion that a digraph "contains" some bad sub-structure.

Define a *retraction* of a digraph  $G$  as a digraph  $G'$  such that there exist a mapping  $f$  from  $V(G)$  onto  $V(G')$  satisfying a *contraction* condition:  $d_G(x, y) \geq d_{G'}(f(x), f(y))$ . Define the *total load* of  $G$  for virtual diameter  $D_0$  as  $\mathcal{L}(G, D_0) = \min(\sum_{e \in E} l(e))$  where the minimum is taken on all DVPL such that  $D_H \leq D_0$ .

Due to the contraction condition, for any retraction  $G'$  of  $G$  we have  $\mathcal{L}(G, D_0) \geq \mathcal{L}(G', D_0)$ . Moreover, denoting the number of arcs of  $G$  by  $|E|$ , the maximum load is greater than or equal to the average load. Hence we have proven the following.

**Lemma 1.** *If  $G'$  is a retraction of  $G$  then  $\pi(G, D_0) \geq \frac{\mathcal{L}(G, D_0)}{|E|} \geq \frac{\mathcal{L}(G', D_0)}{|E|}$ .*

First, we claim that the path  $P_{D_G}$  of length  $D_G$  is a retraction of  $G$ . To prove this, consider the following mapping. Label the vertices of  $P_{D_G}$  by  $0, 1, \dots, D_G$ , and choose a pair of vertices  $(x, y)$  of  $G$  such that  $d(x, y) = d(y, x) = D$ ; then map any vertex at distance  $i$  from  $x$  onto vertex  $i$  of the path. Due to the triangle inequality, and to symmetry, the mapping is contracting.

Now, suppose that we are given a bounded degree digraph  $G$  with  $\log D_G = \Theta(\log n)$ , and the capacity function  $c_0$ . Consider any DVPL with diameter  $D_H = \Theta(\log n)$ . By lemma 1 we have  $c_0 \geq \frac{\mathcal{L}(L_{D_G}, D_H)}{|E|}$ . we also know that if  $D_0 \sim \log D_G$  then  $\mathcal{P}(L_{D_G}, D_0) \sim D_G \log D_G$  [21]; it follows that  $c_0 \geq \frac{D_G \log D_G}{|E|}$ . As  $|E| \leq nd$ , we obtain  $c_0 \geq \frac{D_G \log n}{dn}$ .

## 4 The Cycle $C_n$

In this section the physical digraph  $G$  is  $C_n$ , the symmetric directed cycle of length  $n$ . We choose arbitrarily a sense of direction on  $C_n$ . For concreteness, consider as positive, or forward (resp., negative or backward) the clockwise (resp., counterclockwise) direction. We assume that  $\forall e \in E$ ,  $c(e) = c^+$  if  $e$  is a forward arc and  $c(e) = c^-$  if  $e$  is a backward arc.

It turns out that our bounds can be expressed as functions of  $\sigma = c^+ + c^-$ . It is then convenient to define  $ub_C(n, \sigma)$  (resp.,  $lb_C(n, \sigma)$ ) as an upper bound (resp., lower bound) on  $\tilde{D}(C_n, c)$  valid if  $c$  satisfies  $c^+ + c^- = \sigma$ . By the definition,  $lb_C(n, \sigma) \leq \tilde{D}(C_n, c) \leq ub_C(n, \sigma)$ .

**Proposition 3.**

$$\frac{n^{\frac{1}{\sigma}}}{2} \leq \tilde{D}(C_n, c) < 2\sigma \left(\frac{n}{2}\right)^{\frac{1}{\sigma}} + 1.$$

In particular, if  $c = c_0$  then

$$\frac{n^{\frac{1}{2c_0}}}{2} \leq \tilde{D}(C_n, c_0) < 4c_0 \left(\frac{n}{2}\right)^{\frac{1}{2c_0}} + 1.$$

Upper and lower bounds are both proved by induction from the next two lemmas (to be proved later):

**Lemma 2.**  $lb_C(n, \sigma) \geq \min_{p \in N^+} \{\max(\frac{n}{2p}, lb_C(p, \sigma - 1))\}$ .

**Lemma 3.**  $ub_C(n, \sigma) \leq \min_{p \in N^+} \{2(p - 1) + ub_C(\lceil \frac{n}{p} \rceil, \sigma - 1)\}$

*Proof (proposition 3).* First we consider the lower bound. We prove by induction on  $\sigma$  that  $lb_C(n, \sigma) \geq \frac{1}{2}n^{\frac{1}{\sigma}}$ . For the initial case we have  $lb_C(n, 1) = n - 1 \geq \frac{n}{2}$ . Now to go from  $\sigma - 1$  to  $\sigma$  we use lemma 2 which states that  $lb_C(n, \sigma) \geq \min_{p \in N^+} \max(\frac{n}{2p}, \frac{1}{2}p^{\frac{1}{\sigma-1}})$ . An elementary analysis shows that  $\max(\frac{n}{2p}, \frac{1}{2}p^{\frac{1}{\sigma-1}}) \geq \frac{1}{2}n^{\frac{1}{\sigma}}$ . Hence  $lb_C(n, \sigma) \geq \frac{1}{2}n^{\frac{1}{\sigma}}$  and the proof is completed.

Now, we prove the upper bound. First we show by induction on  $\sigma$  that for  $n = 2a^\sigma$ ,  $a \in N$ ,  $ub_C(n, \sigma) \leq 2\sigma \left(\frac{n}{2}\right)^{\frac{1}{\sigma}} - 2\sigma + 1 = 2\sigma a - 2\sigma + 1$ . For  $\sigma = 1$ ,  $ub_C(n, 1) \leq n - 1$ . For the inductive step from  $\sigma - 1$  to  $\sigma$ , we apply lemma 3 with  $p = a$ , getting  $ub_C(n, \sigma) \leq 2(a - 1) + ub_C(2a^{\sigma-1}, \sigma - 1)$ . By induction,  $ub_C(2a^{\sigma-1}, \sigma - 1) = 2(\sigma - 1)a - 2(\sigma - 1) + 1$ ; so we get the expected result.

For other values of  $n$ , the claim is proved as follows. Let  $a$  be such that  $2(a - 1)^\sigma < n \leq 2a^\sigma$ , that is  $a - 1 < \left(\frac{n}{2}\right)^{\frac{1}{\sigma}} \leq a$ . By applying lemma 3 with  $x = a$  for  $\sigma$  times, we obtain  $ub_C(n, \sigma) \leq 2\sigma a - 2\sigma + 1$ . As  $a \leq \left(\frac{n}{2}\right)^{\frac{1}{\sigma}} + 1$ , this implies  $ub_C(n, \sigma) < 2\sigma \left(\frac{n}{2}\right)^{\frac{1}{\sigma}} + 1$ .

*Proof (lemma 2).* Let  $H$  be an optimal  $c$ -DVPL of  $C_n$  and let  $[x_1, y_1]$  be the dipath consisting of all the vertices of  $C_n$  between  $x_1$  and  $y_1$  in the positive direction. Let  $d(x_1, y_1)$  denote the number of arcs in  $[x_1, y_1]$ . We say that  $[x_1, y_1]$  is covered by  $H$  if (the VP corresponding to) some virtual arc contains  $[x_1, y_1]$ .

First we prove that if  $[x_1, y_1]$  is covered then  $D_H \geq lb_C(d(x_1, y_1), \sigma - 1)$ . For this, we shorten the cycle by identifying all the nodes in  $[y_1, x_1]$  with  $x_1$ , obtaining a cycle  $C'$

of length  $d(x_1, y_1)$ . Virtual arcs are just transformed according to this graph quotient. As example a virtual arc from  $x \in [x_1, y_1]$  to  $y \in [x_1, y_1]$  is left unchanged; and a virtual arc from  $x \in [x_1, y_1]$  to  $y \in [y_1, x_1]$  is transformed into the arc  $(x, x_1)$ . Note that the virtual arc containing the positive arcs of  $[x_1, y_1]$  (see figure 1) is transformed into a loop. We also remove loops or multiple virtual dipaths in order to get a simple DVPL on  $C'$ .

Note that our transformation does not increase the load of any arc; furthermore the virtual arc from  $x_1$  to  $y_1$  disappears, so the congestion of any positive arc decreases. Moreover, our transformation does not increase the virtual diameter.

Consequently, we obtain a  $c'$ -DVPL of  $C'$  (a cycle of length  $d(x_1, y_1)$ ) with  $c'^+ + c'^- = \sigma - 1$ , and diameter at most  $D_H$ . It follows that

$$D_H \geq lb_C(d(x_1, y_1), \sigma - 1) \quad (1)$$

Now we argue that there exist vertices  $u$  and  $v$  with large  $d(u, v)$  such that  $[u, v]$  is covered. Let  $\mathcal{P}$  be the shortest dipath in  $H$  from 0 to  $n/2$ , and assume w.l.o.g. that  $\mathcal{P}$  contains the arcs of  $[0, n/2]$ . Let  $\mathcal{S}$  denote the set of vertices of  $\mathcal{P}$  between  $x$  and  $y$  in the positive direction. Then  $|\mathcal{S}| \leq D_H + 1$ , and therefore there exist vertices  $u$  and  $v$  such that  $[u, v]$  is covered and with

$$d(u, v) \geq \frac{n}{2D_H}. \quad (2)$$

Let  $p = \max\{d(u, v) \mid [u, v] \text{ is covered}\}$ . From (2) we have  $D_H \geq \frac{n}{2p}$ , and from (1) it follows that  $D_H \geq lb_C(p, \sigma - 1)$ .



Fig. 1. Collapsing a cycle

*Proof (lemma 9).* Let us construct a  $c$ -DVPL on  $C_n$ . Without loss of generality suppose that  $c^+ \geq c^-$ , so  $c^+ \neq 0$ . Let  $p \in N^+$ , we proceed as follows.

- Use  $n$  virtual arcs  $(i, i + 1)_{i \in [0, n-1]}$  of dilation 1 in the positive direction.
- Let  $S$  be the set of vertices  $\{0, p, 2p, \dots, (\lceil \frac{n}{p} \rceil - 1)p\}$ , and note that vertices of  $S$  form a cycle  $C_{\lceil \frac{n}{p} \rceil}$ .
- Use an optimal  $c'$ -DVPL for  $C_{\lceil \frac{n}{p} \rceil}$  with  $c'^+ = c^+ - 1$ , and  $c'^- = c^-$ , that is  $c'^+ + c'^- = \sigma - 1$ .

By construction, the diameter  $\Delta(S)$  of the set  $S$  (i.e., the maximal distance of two vertices in  $S$ ) is at most  $ub_C(\lceil \frac{n}{p} \rceil, \sigma - 1)$ ; moreover, for any vertex  $x$ , we have  $d(S, x) \leq p - 1$  and  $d(x, S) \leq p - 1$ . Hence  $d(x, y) \leq d(S, x) + d(y, S) + \Delta(S) \leq 2(p - 1) + ub_C(\lceil \frac{n}{p} \rceil, \sigma - 1)$ .

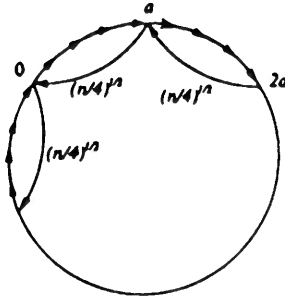


Fig. 2.  $C_n$  for  $c = 1$  ( $\sigma = 2$ )

In the case of capacity  $c_0 = 1$  we have been able to determine  $\tilde{D}(C_n, c)$  quite exactly.

**Proposition 4.**  $2\sqrt{2n} - \Theta(1) \leq \tilde{D}(C_n, 1) \leq 2\sqrt{2n} + 1$ .

The upper bound is the one given for the general case, We conjecture that  $\tilde{D}(C_n, 1) = 2\sqrt{2n} - 3$ , and wish for simpler argument that would extend to greater capacities. Note also that using lemma 2 from the starting condition  $lb_C(n, 2) \geq 2\sqrt{n} + \Theta(1)$  would slightly improve the lower bound on  $lb_C(n, \sigma)$ . the lower bound proof requires some care so the next section is devoted to its exposure.

#### 4.1 Proof of the lower bound for $c = 1$ in the cycle

Let  $H$  be an optimal virtual digraph on  $G$  with respect to the capacity 1.

**Definition 1.** – The forward successor of a vertex  $x$  is denoted  $x^+$ ,  
 –  $P_{C_n}^+(x, y)$  denotes the dipath from  $x$  to  $y$  in  $C_n$  in the positive direction,  
 – a path  $Q = (e'_1, \dots, e'_q)$  from  $x$  to  $y$  in  $H$  is said to be of type + if  $P_{C_n}^+(x, y) \subset W(Q)$ .

Definitions are given for the positive direction, but similar notions apply for the negative direction as well.

**Definition 2.** A circuit-bracelet of size  $n$  is a digraph  $A$  constructed as follows (see figure 3):

- The digraph is made of a set of cycles  $C_i, i \in I$  directed in a clockwise manner.
- For any  $i, C_i$  and  $C_{i+1 \bmod I}$  share a unique vertex  $v_{i+1 \bmod I}$ .
- The length of the dipath in  $C_i$  from  $v_{i-1}$  to  $v_i$  is denoted  $p_i$  and is called the positive length of  $C_i$ ; similarly, the length of the dipath in  $C_i$  from  $v_{i-1}$  to  $v_i$  is denoted  $n_i$  and is called the negative length of  $C_i$ .

Denote by  $f(n)$  the minimal value of  $D_A$ , where  $A$  is any circuit-bracelet of size  $n$ . In the remaining of the section indices are taken modulo  $I$ .

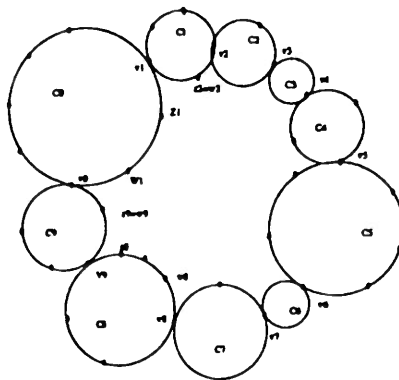
**Lemma 4.**  $f(n) = \tilde{D}(C_n, 1)$

*Proof.* Notice that if an arc  $e$  of  $G$  is not used by a virtual dipath  $P(e')$  with  $e' \in E'$ , we add a virtual arc  $e'$  such that  $P(e') = (e)$ . This transformation can only decrease the diameter of  $H$ , which is of no consequence since we only look for a lower bound on the virtual diameter. Using this manipulation, we know that  $\forall e \in E, \exists e' \in E'$  s.t.  $e \in P(e')$ . This implies

$$\sum_{e' \text{ arc of type } -} w(e') = \sum_{e' \text{ arc of type } +} w(e') = n. \quad (3)$$

Now, we show that : If  $e' = (x, y) \in E'$  is an arc of type  $+$  of weight  $w(e') > 3$  then all the arcs of type  $-$  between  $y^-$  and  $x^+$  are of weight 1.

Since the capacity of any arc of  $G$  is 1, and there is already a virtual arc of type  $+$  between  $x$  and  $y$ , there is no virtual arc of type  $+$  ending at any vertex between  $x^+$  and  $y^-$ . Since  $H = (V, E')$  is strongly connected, there is at least one arc ending at each one of these vertices. These arcs are of type  $-$ . For the same reasons of capacity and connectivity, these virtual arcs are of weight 1. Due to this property it is easy to see that there exists a digraph isomorphism from  $H$  to a circuit-bracelet of size  $n$  (see figure 3).



**Fig. 3.** A circuit-bracelet

We first prove the result for the special case of a *regular circuit-bracelet*, namely, a circuit-bracelet satisfying  $p_i = 1$  for every  $i$ . Let  $g(n)$  denote the minimal value of  $D_A$  where  $A$  is any regular circuit-bracelet of size  $n$ .

**Lemma 5.**  $g(n) = 2\sqrt{2n} + \Theta(1)$ .

*Proof.* We know from the construction of lemma 3 that there exists a regular circuit-bracelet with diameter  $D \leq \sqrt{2n} + O(1) = \Theta(\sqrt{n})$ , and prove that  $D \geq 2\sqrt{2n} + \Theta(1)$ . We assume that  $n$  is sufficiently large, and fix a positive integer  $p$ . Note that the circuits of a regular circuit-bracelet all consist of a single arc in the positive direction. Call a circuit *large* if its size is greater than  $\frac{D}{p}$ . Note that the size of any circuit is at most  $D + 1 = O(\sqrt{n})$ , hence there is at least  $\Omega(\sqrt{n})$  circuits. Moreover the total number

of circuits is  $O(\sqrt{n})$ , otherwise the distance between two antipodal vertices would be greater than  $D$ . As  $n \leq s \frac{D}{p} + bD$  with  $s + b = \Theta(\sqrt{n})$  it follows that :

$$\text{For sufficiently large } p : b = \Theta(\sqrt{n}) \quad (4)$$

Suppose that large circuits are ordered cyclically according to the circuit-bracelet structure:  $C_{i_0}, C_{i_1}, \dots, C_{i_{b-1}}$ , and that there are  $s$  small circuits. Let  $\delta$  denote the number of vertices in the small circuits. We denote the ancestor of  $v_i$  in  $C_i$  by  $w_i$ , and the successor of  $v_{i+1}$  in  $C_i$  by  $z_i$ . Let  $k \in \{0, 1, \dots, b-1\}$  and consider dipaths from  $z_{i_k}$  to  $w_{i_{k-p}}$ ; in the negative direction the cost is exactly  $d_1 = \sum_{j \in [k-p, k]} n_{i_j} - 2$ ; as these circuits are large,  $n_{i_j} \geq \frac{D}{p}$  and hence  $d_1 \geq \frac{p+1}{p} D - 2$ . So we must use the positive direction. The length is  $d_k = n_{i_k} + n_{i_{k-p}} + b + s - \# \text{ circuits in } [i_{k-p}, i_k]$ . Summing on all the  $k$ 's, each circuit is counted  $b-p$  times; moreover, all the vertices in large circuits are counted twice. Hence we have  $\sum_{k=0}^{b-1} d_k = 2(n - \delta) + b(b + s) - p(b + s) \leq bD$ . So  $\frac{2(n-\delta)}{b} + b + s - p - p \frac{s}{b} \leq D$ . Note now that  $\delta \leq s \cdot \frac{D}{p}$ , so :

$$\begin{aligned} \frac{2n}{b} + b - \frac{2sD}{pb} + s - p - p \frac{s}{b} &\leq D \\ \frac{2n}{b} + b + s \left(1 - \frac{2D}{bp} - \frac{p}{b}\right) - p &\leq D \quad (5) \end{aligned}$$

In (5) the left member is greater than  $\frac{2n}{b} + b - p$  when the factor with  $s$  is positive, that is :  $b \geq \frac{D}{p} + p$ . Due to claim (4) it is verified if  $p$  has been chosen large enough. The minimum of the left member in (5) is then  $2\sqrt{2n} + \Theta(1)$  and is achieved for  $s = 0, b = \sqrt{2n}$ . So we have  $g(n) = 2\sqrt{2n} + \Theta(1)$ .

**Proposition 5.**  $\tilde{D}(C_n, 1) = f(n) = 2\sqrt{2n} + \Theta(1)$ .

*Proof.* Recall that  $D = \Theta(\sqrt{n})$ . Consider a circuit-bracelet, and note that  $n_i + p_i \leq D + 2$ , so that we can find an integer  $k$  such that  $C_1, C_2, \dots, C_k$  contains  $x \geq 2D + 2, x = \Theta(\sqrt{n})$  vertices. Consider the shortest dipath from  $v_1$  to  $v_{k+1}$  and suppose that it uses the positive direction, so that  $\sum_{i \in [1, k]} p_i \leq D$ . It follows that  $\sum_{i \in [1, k]} n_i > D$ . So, the dipath from  $v_k$  to  $v_1$  cannot use the negative direction, and must use the positive one. It follows that  $\sum_{i \in [1, k]} p_i \leq D$ . Globally,  $\sum p_i \leq 2D = \Theta(\sqrt{n})$ . If we remove this  $\Theta(\sqrt{n})$  vertices we obtain a regular circuit-bracelet with lesser diameter. It follows that  $f(n) \geq g(n - \Theta(\sqrt{n})) = 2\sqrt{2n} + \Theta(1)$ .

## 5 The Path $P_n$

In this section the physical digraph  $G$  is the  $n$ -vertex path  $P_n$ . Our bounds are valid for any capacity function  $c$  such that positive (resp., negative) arcs have capacity  $c^+$  (resp.,  $c^-$ ), with  $c^+ + c^- = \sigma \geq 2$ , and the additional requirement  $c^+ \geq 1, c^- \geq 1$ .

**Proposition 6.**

$$\frac{n^{\frac{1}{\sigma-1}}}{2} \leq \tilde{D}(P_n, c) \leq (2(\sigma-1)) \left(\frac{n-1}{2}\right)^{\frac{1}{\sigma-1}} + 2.$$

*Proof.* Will appear in the full version

## 6 Grids and Toroidal Meshes

First the physical digraph  $G$  is the toroidal mesh of dimensions  $a \times b$ ,  $TM(a, b)$ . Recall that  $TM(a, b) = C_a \square C_b$ , the Cartesian sum (also called product) of two cycles.

**Proposition 7.**

$$\tilde{D}(TM(a, b), c_0) \leq 2(4c_0) \cdot \left(\frac{a}{2}\right)^{1/2c_0-1} + 2a + 1 + (4ac_0 - 1) \left(\frac{b}{2}\right)^{1/2ac_0}.$$

Note that in order to get a graph  $G$  such that  $\tilde{D}(G, c_0) \sim \log n$  with  $c_0$  bounded we can use a toroidal mesh  $T(\log n \frac{n}{\log n})$ . Hence we have the following.

**Proposition 8.** *There exists an infinite family of digraphs with  $n$  vertices and diameter  $n/\log n$ , such that with bounded capacity  $c_0$ ,  $\tilde{D}(G, c_0) = \Theta(\log n)$ .*

Note that this is the counterpart of proposition 2 .

For  $\lceil \log_2 b \rceil \leq a \leq b$  we have :

**Proposition 9.**  $\tilde{D}(T(a, b), 1) = \Omega(\log n)$ .

Results for the grid  $M(a, b) = P_a \square P_b$  are similar and will be detailed in the full version.

## 7 The Complete Symmetric $k$ -ary Tree $T(k, h)$

Let us recall that in a complete  $k$ -ary tree, each vertex that is not a leaf has exactly  $k$  children. The *depth* of a vertex is its distance from the root. The depth of the tree is the maximum depth of its vertices. In this section the physical digraph  $G$  is  $T(k, h)$ , the directed symmetric complete  $k$ -ary tree of depth  $h$  rooted in  $r_0$ , the only vertex of degree  $k$ .  $T(k, h)$  has  $\frac{k^{h+1}-1}{k-1}$  vertices and diameter  $2h$ . For a vertex  $x$  of the graph  $T(k, h)$ , let  $f(x)$  denote its parent; vertex  $y$  is said to be *below*  $x$  if  $\exists i \geq 0$  s.t.  $f^i(y) = x$ . Note that  $x$  is below itself.

### 7.1 Lower Bound

**Proposition 10.**  $\tilde{D}(T(k, h), c) \geq 2 \left\lfloor \frac{h-1}{\lceil \log_k c \rceil + 1} \right\rfloor + 1$ .

*Proof.* Let  $H$  be an  $c$ -DVPL of  $T(k, h)$ . Let  $\gamma = \lceil \log_k c \rceil + 1$ . Let  $r$  be a vertex of depth  $d$ ,  $1 \leq d \leq h - \gamma$ ; Let  $B(\gamma, r)$  denote the complete  $k$ -ary subtree of  $T(k, h)$  of depth  $\gamma$  rooted in  $r$ . A leaf  $x$  of  $B(\gamma, r)$  is said to be *upward-bad* for  $r$  if there doesn't exist any virtual arc  $e'$  that starts below  $x$  and ends not below  $r$ . If there doesn't exist any virtual arc  $e'$  that starts not below  $r$  and ends below  $x$  then  $x$  is said to be *downward-bad* for  $r$ . We claim the following: *For any vertex  $r$  of depth  $d$ ,  $1 \leq d \leq h - \gamma$  there exist an upward-bad vertex and a downward-bad vertex for  $r$ .*

Indeed, suppose that all the  $k^\gamma$  leaves of  $B(\gamma, r)$  are not upward-bad. There exists a virtual arc that starts below each leaf and ends not below  $r$ . Then the load of the arc  $(r, f(r))$  is at least  $k^\gamma$ . Since the capacity of this arc is  $c < k^{\lceil \log_k c \rceil + 1}$ , there exists at least one leaf that is upward-bad for  $r$ . The same argument considering the load of arc  $(f(r), r)$  completes the proof of the claim.

Now we prove that  $D_H \geq 2(\lfloor \frac{h-1}{\gamma} \rfloor) + 1$ . Let  $i_0 = \lfloor (h-1)/\gamma \rfloor + 1$ . Define two sequences of vertices  $(l_i)_{i=1..i_0}$  and  $(r_i)_{i=1..i_0}$  as follows. Let  $l_1$  and  $r_1$  be the leftmost and the rightmost neighbors of the root of  $T(k, h)$ , respectively. If  $i \leq i_0 - 1$ , choose for  $l_{i+1}$  an upward-bad vertex for  $l_i$ . By induction, the depth of  $l_i$  is  $1 + \gamma(i-1)$  and if  $i \leq i_0 - 1$  it is less than  $h - \gamma$  so, from the claim,  $l_{i+1}$  exists. Symmetrically, we define the sequence  $(r_i)_{i=1..i_0}$  by choosing  $r_{i+1}$  as one of the downward-bad vertices for  $r_i$ .

Let us now consider the shortest path  $\mathcal{P}$  in  $H$  from  $l_{i_0}$  to  $r_{i_0}$ . Let  $y$  be the first vertex of  $\mathcal{P}$  not below  $l_1$ . By construction,  $\mathcal{P}$  uses at least  $i_0$  virtual arcs from  $l_{i_0}$  to  $y$ . Also  $x$ , the predecessor of  $y$  in  $\mathcal{P}$ , is below  $l_1$  and thus not below  $r_1$ . Hence,  $\mathcal{P}$  uses at least  $i_0$  virtual arcs from  $x$  to  $r_{i_0}$ . In summary,  $\mathcal{P}$  uses at least  $2i_0 - 1$  virtual arcs. So  $D_G \geq 2i_0 - 1$  that is  $2(\lfloor (h-1)/\gamma \rfloor) + 1$ .

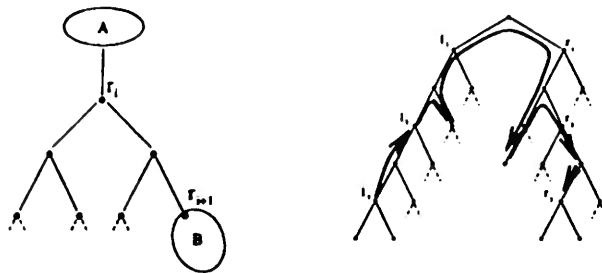


Fig. 4.  $k = 2, c = 2$  or  $3, \gamma = 2$ , there exist no arcs from A to B; (on the right)  $k = 2, \gamma = 2, h = 6$ , one cannot do better than 5 from  $l_3$  to  $r_3$

## 7.2 Upper Bound for $k = 2, c = 2$

Proposition 11.  $\tilde{D}(T(2, h), 2) \leq D/2 + 1$ .

*Proof.* The construction of an admissible virtual graph  $H$  on  $D(T(2, h))$  is undirected. In other words, for each VP from  $x$  to  $y$ , there is a VP from  $y$  to  $x$  using the opposite path. Place a VP of dilation 2 between each vertex of even depth and its grandparent and a VP of dilation 1 between each vertex of even depth and its parent. If  $h$  is odd then add a VP of dilation 1 between each leaf and its parent. This construction gives the upper bound.

## 8 Arbitrary Trees

In this section, the graph at hand is an arbitrary tree  $T$ . The lower bound follows from our bound for the  $n$ -vertex path  $P_n$ , upon noting the following.

Proposition 12.  $\tilde{D}(T, c) \geq \tilde{D}(P_{D_T}, c)$ .

One natural upper bound follows by embedding a cycle around the tree. Consider a cycle  $C_{2n-2}$  embedded around the tree  $T$  in a depth-first fashion. Let  $c^+ = \lceil c/2 \rceil$  and  $c^- = \lfloor c/2 \rfloor$ . An admissible graph  $H$  on  $C_{2n-2}$  with respect to  $c^+$  on positive arcs and  $c^-$  on negative ones gives us an admissible virtual graph on  $T$ . Thus,  $\tilde{D}(T, c) \leq \tilde{D}(C_{2n-2}, c/2) \leq 2c(n-1)^{1/c} + 1$ .

Another upper bound is derived by a recursive construction.

**Proposition 13.** *There exists a constant such that  $\dot{D}(T, c) \leq 8c \cdot n^{\frac{1}{2c-1}}$ .*

**Proposition 14.**  $\dot{D}(T, c) \leq \dot{D}(P_{D_T}, c) \cdot \log n$ .

## 9 Open problems and directions

Some of our bounds are not tight, and the remaining gaps may be narrowed. Establishing upper and lower bounds on  $\dot{D}$  for other families of graphs may also be interesting and useful.

Looking for the minimum diameter is reasonable when all the connections may be requested with roughly the same probability, which is also not always realistic. In case of non-uniform traffic, instead of studying  $\dot{D}$ , one may try to optimize its weighted counterpart,  $\sum r(i, j) \cdot d_H(i, j)$ , where  $r(i, j)$  denotes the traffic requirements between  $i$  and  $j$ ; such a target function may make it desirable to place the VPs between the node pairs which communicate the most.

Finally, there may be other parameters of the directed ATM model worth studying. One may also consider variations on the model with variable capacity of the VPs.

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