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# Infinite horizon for symmetric strategy population game

Meziane Privat

**Abstract :** To predict the behavior of a population game when time becomes very long, the process that characterizes the evolution of our game dynamics must be reversible. Known games satisfying this are 2 strategy games as well as potential games with an exponential protocol. We will try to extend the study of infinite horizons for what are called symmetric strategy games.

**Key words :** Population Games, Stochastic Evolutionary Models, Infinite-Horizon.

**AMS subject classification :** 91A15, 91A22, 92D25

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# 1 The case one population

## 1.1 Informal presentation with an example

First, we will consider the case where there is only one population.

Imagine a population of  $N$  cats playing a three-state game: no eating, eating one mouse, or eating two. This is a game with one population and three strategies, but it can be transformed into a game with three populations and two strategies. Imagine three populations of  $N$  cats each (as if we were cloning the initial population), the first population would play a two-state game which would be: do not eat or eat, the second would play: eat a mouse or do not eat a mouse, and the third would play: eat two mice or not eat two mice. The possible states would only be those where  $N$  cats choose an "odd" strategy and  $2N$  cats choose an "even" strategy. We have thus transformed the game into a two-strategy game that we know is reversible. To switch from the initial game to the new game, you have to make small changes.

Let  $A$  be an object concerning the initial game, its transformation for the new game will be  $A^*$

In this case,

- $x = (x_1, x_2, x_3) \in X$  becomes  $x^* = (x_1, 1-x_1, x_2, 1-x_2, x_3, 1-x_3) \in X^*$
- $S = \{a, b, c\}$  becomes  $S^* = \{a, \bar{a}, b, \bar{b}, c, \bar{c}\}$
- $\mathfrak{p} = 1$  becomes  $\mathfrak{p}^* = 3$
- $F(x) = (\alpha, \beta, \gamma)$  becomes  $F^*(x^*) = (\alpha, \frac{x_2\beta+x_3\gamma}{x_2+x_3}, \beta, \frac{x_1\alpha+x_3\gamma}{x_1+x_3}, \gamma, \frac{x_1\alpha+x_2\beta}{x_1+x_2})$   
or  
 $F^*(x^*) = (\alpha, 0, \beta, 0, \gamma, 0)$
- $R$  becomes  $R^* = (R, R, R)$
- 

$$\rho(\pi, x) = \begin{pmatrix} \rho_{a,a}(\pi, x) & \rho_{a,b}(\pi, x) & \rho_{a,c}(\pi, x) \\ \rho_{b,a}(\pi, x) & \rho_{b,b}(\pi, x) & \rho_{b,c}(\pi, x) \\ \rho_{c,a}(\pi, x) & \rho_{c,b}(\pi, x) & \rho_{c,c}(\pi, x) \end{pmatrix}$$

Becomes

$$\begin{aligned}
\rho^*(\pi^*, x^*) &= \begin{pmatrix} \rho_{a,a}^*(\pi^*, x^*) & \rho_{a,\bar{a}}^*(\pi^*, x^*) & 0 & 0 & 0 & 0 \\ \rho_{\bar{a},a}^*(\pi^*, x^*) & \rho_{\bar{a},\bar{a}}^*(\pi^*, x^*) & 0 & 0 & 0 & 0 \\ 0 & 0 & \rho_{b,b}^*(\pi^*, x^*) & \rho_{b,\bar{b}}^*(\pi^*, x^*) & 0 & 0 \\ 0 & 0 & \rho_{\bar{b},b}^*(\pi^*, x^*) & \rho_{\bar{b},\bar{b}}^*(\pi^*, x^*) & 0 & 0 \\ 0 & 0 & 0 & 0 & \rho_{c,c}^*(\pi^*, x^*) & \rho_{c,\bar{c}}^*(\pi^*, x^*) \\ 0 & 0 & 0 & 0 & \rho_{\bar{c},c}^*(\pi^*, x^*) & \rho_{\bar{c},\bar{c}}^*(\pi^*, x^*) \end{pmatrix} \\
&= \begin{pmatrix} \rho_{a,a}(\pi, x) & \rho_{a,b}(\pi, x) + \rho_{a,c}(\pi, x) \\ \frac{1}{2}(\rho_{b,a}(\pi, x) + \rho_{c,a}(\pi, x)) & \frac{1}{2}(\rho_{b,b}(\pi, x) + \rho_{b,c}(\pi, x) + \rho_{c,b}(\pi, x) + \rho_{c,c}(\pi, x)) \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \rho_{b,b}(\pi, x) & \rho_{b,a}(\pi, x) + \rho_{b,c}(\pi, x) \\ \frac{1}{2}(\rho_{a,b}(\pi, x) + \rho_{c,b}(\pi, x)) & \frac{1}{2}(\rho_{a,a}(\pi, x) + \rho_{a,c}(\pi, x) + \rho_{c,a}(\pi, x) + \rho_{c,c}(\pi, x)) \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \rho_{c,c}(\pi, x) & \rho_{c,a}(\pi, x) + \rho_{c,b}(\pi, x) \\ \frac{1}{2}(\rho_{a,c}(\pi, x) + \rho_{b,c}(\pi, x)) & \frac{1}{2}(\rho_{a,a}(\pi, x) + \rho_{a,b}(\pi, x) + \rho_{b,a}(\pi, x) + \rho_{b,b}(\pi, x)) \end{pmatrix}
\end{aligned}$$

We then know by theorem 11.2.3 of [San10] what shape should have the stationary distribution of the game and therefore its infinite horizon.

## 1.2 Formal presentation

In this part we will lay the mathematical foundations to have access to the behavior when the time becomes very long for a game with 3 strategies.

**Définition 1.1.** *A is a symmetric game if for all  $i, j \in S$  we have  $\rho_{ij}(\pi, x) = \rho_{ji}(\pi, x)$  and  $R_{ij} = R_{ji}$ .*

Let  $A$  be a game of a population with 3 strategies that we will call the initial game, in connection with this game we then pose:

- $X$  the state space of  $A$
- $S = \{1, 2, 3\}$  the set of 3 strategies in  $A$
- We have  $\mathfrak{p} = 1$
- Let  $F$  be its gain function
- We set  $\rho$  its full support revision protocol
- $R$  its revision rate
- $N$  the number of agents in the population

**Proposition 1.1.** *If for all  $i, j \in S$  we have  $\rho_{ij}(\pi, x) = \rho_{ji}(\pi, x)$  and  $R_{ij} = R_{ji}$ , then for any population game at 3 strategies there is a unique population game with two strategies and 3 population describing the same game.*

*Proof.* To demonstrate this we must first build this new game, ie show its existence, and then show its uniqueness.

**Existence:** Let  $A^*$  be a two-strategy game of  $n$  population defined as follows:

- $X^* = \{x^* = (x_1, 1-x_1, x_2, 1-x_2, x_3, 1-x_3) \text{ such that } \exists x = (x_1, x_2, x_3) \in X\}$  its state space
- $S^* = \{1, \bar{1}, 2, \bar{2}, 3, \bar{3}\}$  the set of  $6(2 \times 3)$  strategies in  $A^*$
- We have  $\mathfrak{p}^* = 3$  populations
- Let  $x \in X$  such that  $F(x) = y = (y_1, y_2, y_3)$  then  $F^*(x^*) = y^* = (y_1, 0, y_2, 0, y_3, 0) \in \mathbb{R}^6$
- Let  $N^* = (N, N, N) \in (\mathbb{N}^*)^3$  be the number of agents in each population (which is  $N$  for each)

$$\rho^*(\pi^*, x^*) = \begin{pmatrix} \rho_{1,1}^*(\pi^*, x^*) & \rho_{1,\bar{1}}^*(\pi^*, x^*) & 0 & 0 & 0 & 0 \\ \rho_{\bar{1},1}^*(\pi^*, x^*) & \rho_{\bar{1},\bar{1}}^*(\pi^*, x^*) & 0 & 0 & 0 & 0 \\ 0 & 0 & \rho_{2,2}^*(\pi^*, x^*) & \rho_{2,\bar{2}}^*(\pi^*, x^*) & 0 & 0 \\ 0 & 0 & \rho_{\bar{2},2}^*(\pi^*, x^*) & \rho_{\bar{2},\bar{2}}^*(\pi^*, x^*) & 0 & 0 \\ 0 & 0 & 0 & 0 & \rho_{3,3}^*(\pi^*, x^*) & \rho_{3,\bar{3}}^*(\pi^*, x^*) \\ 0 & 0 & 0 & 0 & \rho_{\bar{3},3}^*(\pi^*, x^*) & \rho_{\bar{3},\bar{3}}^*(\pi^*, x^*) \end{pmatrix}$$

with  $\forall i \in S :$

$$\begin{aligned}
\text{(i)} \quad & \rho_{i,i}^*(\pi^*, x^*) = \rho_{i,i}(\pi, x) \\
\text{(ii)} \quad & \rho_{i,\bar{i}}^*(\pi^*, x^*) = \sum_{\substack{j \in S \\ j \neq i}} \rho_{i,j}(\pi, x) \\
\text{(iii)} \quad & \rho_{\bar{i},i}^*(\pi^*, x^*) = \frac{1}{2} \sum_{\substack{j \in S \\ j \neq i}} \rho_{j,i}(\pi, x) \\
\text{(iv)} \quad & \rho_{\bar{i},\bar{i}}^*(\pi^*, x^*) = \frac{1}{2} \left( \sum_{\substack{j \in S \\ j \neq i}} \sum_{\substack{k \in S \\ k \neq i \\ k \neq j}} \rho_{j,k}(\pi, x) + \sum_{\substack{j \in S \\ j \neq i}} \rho_{j,j}(\pi, x) \right)
\end{aligned}$$

- Let  $R^*$  the new revision rate the same as  $\rho^*$

**Unicity:** We now need to show that from each element previously constructed for  $A^*$  we can find the element of  $A$  used, in other words, that there is a bijection between each element of  $A^*$  and the element of  $A$  which allowed its construction.

- To go from  $N^*$  to  $N$ , just take the first coordinate of  $N^*$
- To go from  $\mathbf{p}^*$  to  $\mathbf{p}$  just set  $\mathbf{p} = \frac{\mathbf{p}^*}{3}$
- Let  $f : \mathbb{R}^6 \rightarrow \mathbb{R}^3$  such that  $f(x_1, \dots, x_6) = (x_1, x_3, x_5)$ , then  $f(X^*) = X$  and  $f(F(X^*)) = F(X)$
- By writing  $S^* = \{1, \dots, 6\}$  we can find  $S$  by setting  $S = \{1, 3, 5\}$
- Recall that  $\forall i, j \in S, \rho_{ij}(\pi, x) = \rho_{ji}(\pi, x)$ 
  - $\forall i \in S, \rho_{ii}(\pi, x) = \rho_{ii}^*(\pi^*, x^*)$
  - $\forall i, j, k \in S$  with  $i \neq j \neq k$ , we have:

$$\begin{aligned}
\rho_{ij}(\pi, x) + \rho_{ji}(\pi, x) &= (\rho_{ij}(\pi, x) + \rho_{ik}(\pi, x)) + (\rho_{ji}(\pi, x) + \rho_{jk}(\pi, x)) \\
&\quad - (\rho_{ik}(\pi, x) + \rho_{jk}(\pi, x)) \\
&= \rho_{i,\bar{i}}^*(\pi^*, x^*) + \rho_{j,\bar{j}}^*(\pi^*, x^*) - 2\rho_{\bar{k},k}^*(\pi^*, x^*)
\end{aligned}$$

we then have  $\forall i, j, k \in S$  with  $i \neq j \neq k$  :

$$\rho_{ij}(\pi, x) = \frac{\rho_{i,\bar{i}}^*(\pi^*, x^*) + \rho_{j,\bar{j}}^*(\pi^*, x^*) - 2\rho_{\bar{k},k}^*(\pi^*, x^*)}{2}$$

- For  $R^*$  we prove uniqueness as for the previous point.

□

We will call  $A^*$  the transformation of  $A$ .

**Proposition 1.2.** *If for all  $i, j \in S$  we have  $\rho_{ij}(\pi, x) = \rho_{ji}(\pi, x)$ , and  $R_{ij} = R_{ji}$  then for any population game from  $N$  agents to 3 strategies with a full support revision protocol one can define a stationary distribution for the evolutionary process  $(X_t^N)$  on  $\mathcal{X}^N$ .*

*Proof.* Let  $A$  be a population game of  $N$  agents with 3 strategies with a satisfactory full-support revision protocol for all  $i, j \in S$  that  $\rho_{ij}(\pi, x) = \rho_{ji}(\pi, x)$ . Then by applying Proposition 3.1, we can define the game  $A^*$  which is the transformation of  $A$ . We can then apply theorem 11.2.3 of [San10] which gives us the stationary distribution of the set  $A^*$  which is for its population  $i \in \llbracket 1, 3 \rrbracket$ :

$$\frac{\mu_{\mathcal{X}_i}^{N_i}}{\mu_0^{N_i}} = \prod_{j=1}^{N \setminus \mathcal{X}_i} \frac{N-j-1}{j} \cdot \frac{\rho_{i,\bar{i}}^*(F^{*i}(\frac{j-1}{N}), \frac{j-1}{N})}{\rho_{i,i}^*(F^{*i}(\frac{j}{N}), \frac{j}{N})} \text{ with } \mathcal{X}_i \in \{0, \frac{1}{N}, \frac{2}{N}, \dots, 1\} \text{ and } \sum_{i=1}^3 \mathcal{X}_i = 1$$

with  $\mu_0^{N_i}$  determined by the requirement that  $\sum_{\mathcal{X}_i \in \mathcal{X}^{N_i}} \mu_{\mathcal{X}_i}^{N_i} = 1$ .

Maybe have to make a mixture model to return to dimension 1, to see

□

## 2 The case $\mathfrak{p}$ populations

In this part we will show that proposition 1.2 is true for several populations, for this we will show a more general result by showing that we can find an infinite horizon that the number of strategy of the game of each population is 2 or 3 .

**Proposition 2.1.** *Let  $A$  be a game with  $\mathfrak{p}$  populations such that the population  $p$  plays a game with  $n^p$  strategies ( $n^p = 2$  or  $3$ ), that its revision protocol is with full support and that for all  $i, j \in S$  we have  $\rho_{ij}^p(\pi, x) = \rho_{ji}^p(\pi, x)$  and  $R_{ij} = R_{ji}$  then we can define a stationary distribution for the evolutionary process  $(X_t^N)$  on  $\mathcal{X}^N$ .*

*Proof.* Let  $p \in \llbracket 1, \mathfrak{p} \rrbracket$   $A^p$  be the game played by the population  $p$ . Applying Proposition 1.2, let  $p \in \llbracket 1, \mathfrak{p} \rrbracket$   $A^{*p}$  be the transformation into a 2-strategy game of the game played by the population  $p$  ( $A^{*p} = A$  if  $n^p = 2$ ). By setting  $A^*$  the set of these games, we then end up with a game  $A^*$  which is

a 2-strategy game with  $\sum_{k=1}^p n^k = n$  populations, and  $A^*$  represents the same game as  $A$ . We can then apply Theorem 11.2.3 of [San10] which gives us the stationary distribution of the set  $A^*$  which is for its population  $i \in \llbracket 1, n \rrbracket$  such that  $\exists p \in \llbracket 1, \mathfrak{p} \rrbracket$  such that  $i \in \llbracket \sum_{k=0}^{p-1} n^k + 1, \sum_{k=0}^p n^k \rrbracket$  (with  $n^0 = 0$ ):

$$\frac{\mu_{\mathcal{X}_i}^{N_i}}{\mu_0^{N_i}} = \prod_{j=1}^{N^p \mathcal{X}_i} \frac{N^{p-j-1}}{j} \cdot \frac{\rho_{i,i}^*(F^{*i}(\frac{j-1}{N^p}), \frac{j-1}{N^p})}{\rho_{i,i}^*(F^{*i}(\frac{j}{N^p}), \frac{j}{N^p})}$$

with  $\mathcal{X}_i \in \{0, \frac{1}{N^p}, \frac{2}{N^p}, \dots, 1\}$  and

$$\sum_{i=n^{p-1}+1}^{n^p} \mathcal{X}_i = 1 \text{ and } \sum_{j=1}^n \mathcal{X}_j = \mathfrak{p}$$

with  $\mu_0^{N_i}$  determined by the requirement that  $\sum_{\mathcal{X}_i \in \mathcal{X}^{N_i}} \mu_{\mathcal{X}_i}^{N_i} = 1$  □

### 3 Transformation of an n-strategy symmetric game into a 2-strategy game

To go from a game with  $n$  strategies to a game with 2 strategies, we will show that for a game with  $n$  strategies there exists a unique game with  $n - 1$  strategies which represents the same game, then by repeating this we come across a game with 2 strategies.

**Définition 3.1.** *A is a symmetric game if for all  $i, j \in S$  we have  $\rho_{ij}(\pi, x) = \rho_{ji}(\pi, x)$  and  $R_{ij} = R_{ji}$ .*

Let  $A$  be a game of a population with  $n$  strategies that we will call the initial game, in connection with this game we then pose:

- $X$  the state space of  $A$
- $S = \{1, 2, \dots, n\}$  the set of  $n$  strategies in  $A$
- We have  $\mathfrak{p} = 1$
- Let  $F$  be its gain function
- We set  $\rho$  its full support revision protocol
- $R$  its revision rate
- $N$  the number of agents in the population



**Proposition 3.1.** *For any population game with  $n$  strategies ( $n > 3$ ) there exists a unique population game with  $n$  population each playing a game with  $n - 1$  strategies describing the same game .*

*Proof.* To demonstrate this we must first build this new game, ie show its existence, and then show its uniqueness.

**Existence:** Let  $A_{n-1}^*$  be a set of  $n$  populations each having access to  $n - 1$  strategies defined as follows:

- $X_{n-1}^* = \{x_{n-1}^* = (x_1, x_2, \dots, x_{n-2}, 1 - x_{n-1} - x_n, x_2, x_3, \dots, x_{n-1}, 1 - x_1 - x_n, \dots, x_n, x_1, x_2, \dots, x_{n-3}, 1 - x_{n-1} - x_{n-2}) \text{ such that } \exists x = (x_1, \dots, x_n) \in X\}$  its state space
- $S_{n-1}^* = \{1, 2, \dots, n - 2, a_1, 2, 3, \dots, n - 1, a_2, \dots, n, 1, 2, \dots, n - 3, a_n\}$  the set of  $n - 1 \times n$  strategies in  $A_{n-1}^*$
- We have  $\mathfrak{p}_{n-1}^* = n$  populations
- Let  $x \in X$  such that  $F(x) = y = (y_1, \dots, y_n)$  then  $F_{n-1}^*(x_{n-1}^*) = y_{n-1}^* = (y_1, y_2, \dots, y_{n-2}, 0, y_2, y_3, \dots, y_{n-1}, 0, \dots, y_n, y_1, y_2, \dots, y_{n-3}, 0)$
- Let  $N_{n-1}^* = (N, \dots, N) \in (\mathbb{N}^*)^n$  be the number of agents in each population ( which is  $N$  for each)
- 

$$\rho_{n-1}^*(\pi_{n-1}^*, x_{n-1}^*) = \begin{pmatrix} A_1^{n-1} & 0_{\mathcal{M}_{n-1}} & \cdots & 0_{\mathcal{M}_{n-1}} \\ 0_{\mathcal{M}_{n-1}} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0_{\mathcal{M}_{n-1}} \\ 0_{\mathcal{M}_{n-1}} & \cdots & 0_{\mathcal{M}_{n-1}} & A_n^{n-1} \end{pmatrix} \in \mathcal{M}_{n(n-1)}$$

with  $\forall i \in S$  :

$$A_i^{n-1} = \begin{pmatrix} \rho^{(i+0) \equiv n, (i+0) \equiv n} & \rho^{(i+0) \equiv n, (i+1) \equiv n} & \cdots & \rho^{(i+0) \equiv n, (i+n-3) \equiv n} & \rho_{n-1}^*_{(i+0) \equiv n, a_i} \\ \rho^{(i+1) \equiv n, (i+0) \equiv n} & \rho^{(i+1) \equiv n, (i+1) \equiv n} & \cdots & \rho^{(i+1) \equiv n, (i+n-3) \equiv n} & \rho_{n-1}^*_{(i+1) \equiv n, a_i} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \rho^{(i+n-3) \equiv n, (i+0) \equiv n} & \rho^{(i+n-3) \equiv n, (i+1) \equiv n} & \cdots & \rho^{(i+n-3) \equiv n, (i+n-3) \equiv n} & \rho_{n-1}^*_{(i+n-3) \equiv n, a_i} \\ \rho_{n-1}^*_{a_i, (i+0) \equiv n} & \rho_{n-1}^*_{a_i, (i+1) \equiv n} & \cdots & \rho_{n-1}^*_{a_i, (i+n-3) \equiv n} & \rho_{n-1}^*_{a_i, a_i} \end{pmatrix} \in \mathcal{M}_{n-1}$$

with  $\forall j \in S$ :

- $\rho_{n-1, j, a_i}^* = \rho_{j, (i+n-2) \equiv n} + \rho_{j, (i+n-1) \equiv n}$
- $\rho_{n-1, a_i, j}^* = \rho_{(i+n-2) \equiv n, j} + \rho_{(i+n-1) \equiv n, j}$
- $\rho_{n-1, a_i, a_i}^* = \rho_{(i+n-2) \equiv n, (i+n-2) \equiv n} + \rho_{(i+n-2) \equiv n, (i+n-1) \equiv n} \rho_{(i+n-1) \equiv n, (i+n-2) \equiv n} + \rho_{(i+n-1) \equiv n, (i+n-1) \equiv n}$

where  $\forall a \in \mathbb{N}$   $a \equiv n$  is  $a$  modulo  $n$ .

- We set  $R_{n-1}^*$  the new revision rate in the same way as  $\rho_{n-1}^*$

**Unicity:** We now need to show that from each element previously constructed for  $A^*$  we can find the element of  $A$  used, in other words, that there is a bijection between each element of  $A^*$  and the element of  $A$  which allowed its construction.

- To go from  $N_{n-1}^*$  to  $N$ , just take the first coordinate of  $N_{n-1}^*$
- To go from  $\mathfrak{p}_{n-1}^*$  to  $\mathfrak{p}$  just set  $\mathfrak{p} = \frac{\mathfrak{p}_{n-1}^*}{n}$
- Let's set  $f : \mathbb{R}^{n^2-n} \rightarrow \mathbb{R}^n$  such that  $f(x_1, x_2, \dots, x_{n^2-n}) = (x_1, \dots, x_{pn-p-1}, \dots, x_{n^2-2n-1})$  with  $p \in \llbracket 1, n-1 \rrbracket$ , then  $f(X^*) = X$  and  $f(F(X^*)) = F(X)$
- By writing  $S_{n-1}^* = \{1, \dots, n^2-n\}$  we can find  $S$  by setting  $S = \{1, \dots, pn-p-1, \dots, n^2-2n-1\}$  with  $p \in \llbracket 1, n-1 \rrbracket$
- (i)  $\forall i \in S, \rho_{ii}(\pi, x) = \rho_{n-1, (i-1)n-(i-1)+1, (i-1)n-(i-1)+1}^*(\pi_{n-1}^*, x_{n-1}^*)$  (this is the first coordinate of  $A_i^{n-1}(\pi_{n-1}^*, x_{n-1}^*)$ )
- (ii)  $\forall i \in S$  and  $j \in \llbracket (i+1) \equiv n, (i+n-3) \equiv n \rrbracket$  such that  $(i+a) \equiv n = j$ ,  $\rho_{ij}(\pi, x) = \rho_{n-1, (i-1)n-(i-1)+1, (i-1)n-(i-1)+1+a}^*(\pi_{n-1}^*, x_{n-1}^*)$
- (iii)  $\forall i \in S$  and  $j = (i+n-1) \equiv n$ ,  $\rho_{ij}(\pi, x) = \rho_{n-1, (j-1)n-(j-1)+2, (j-1)n-(j-1)+1}^*(\pi_{n-1}^*, x_{n-1}^*)$
- (iv)  $\forall i \in S$  and  $j = (i+n-2) \equiv n$ , 
$$\rho_{ij}(\pi, x) = \rho_{n-1, (i-1)n-(i-1)+1, (i-1)n-(i-1)+n-1}^*(\pi_{n-1}^*, x_{n-1}^*) - \rho_{n-1, (j-1)n-(j-1)+2, (j-1)n-(j-1)+1}^*(\pi_{n-1}^*, x_{n-1}^*)$$
- For  $R_{n-1}^*$  we prove uniqueness as for the previous point.

□

Now that we know how to go from a game with  $n$  strategies to a game with  $n - 1$  strategies, we can explain how to go from a game with  $n$  strategy to a game with  $m$  strategy with  $m \leq n$ .

**Proposition 3.2.** *For any population game with  $n$  strategies ( $n > 3$ ) there exists a population game with  $n$  population each playing a game with  $m$  strategies ( $m \leq n$ ) describing the same game.*

*Proof.* Let  $A$  be the initial game with  $n$  strategies, let  $A_{n-1}^*$  be its transformation into a game with  $n - 1$  strategies given by Proposition 1.1. Applying proposition 1.1 again to the game  $A_{n-1}^*$  we find a game  $B_{n-1}^*$  defined as follows:

- $X_{B_{n-1}^*}^* = \{x_{B_{n-1}^*}^* = (x_1, x_2, \dots, x_{n-3}, 1-x_{n-2}-x_{n-1}-x_n, \dots, x_n, x_1, x_2, \dots, x_{n-4}, 1-x_{n-1}-x_{n-2}-x_{n-3}) \in \mathbb{R}^{n \times (n-1) \times (n-2)} \text{ such that } \exists x = (x_1, \dots, x_n) \in X\}$  its state space
- $S_{B_{n-1}^*}^* = \{1, 2, \dots, n-3, a_1, 2, 3, \dots, n-2, a_2, \dots, n, 1, 2, \dots, n-4, a_n\}$  the set of  $(n-2) \times (n-1) \times n$  strategies in  $A_{n-1}^*$
- We have  $\mathfrak{p}_{B_{n-1}^*}^* = n \times (n-1)$  populations
- Let  $x \in X$  such that  $F(x) = y = (y_1, \dots, y_n)$  then  $F_{B_{n-1}^*}^*(x_{B_{n-1}^*}^*) = y_{B_{n-1}^*}^* = (y_1, y_2, \dots, y_{n-3}, 0, y_2, y_3, \dots, y_{n-2}, 0, \dots, y_n, y_1, y_2, \dots, y_{n-4}, 0)$
- We set  $N_{B_{n-1}^*}^* = (N, \dots, N) \in (\mathbb{N}^*)^{n \times n-1}$  the number of agents in each population (which is  $N$  for each)
- 

$$\rho_{B_{n-1}^*}^*(\pi_{n-1}^*, x_{n-1}^*) = \begin{pmatrix} B_1^{n-1} & 0_{\mathcal{M}_{n-2}} & \cdots & 0_{\mathcal{M}_{n-2}} \\ 0_{\mathcal{M}_{n-2}} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0_{\mathcal{M}_{n-2}} \\ 0_{\mathcal{M}_{n-2}} & \cdots & 0_{\mathcal{M}_{n-2}} & B_{n \times (n-1)}^{n-1} \end{pmatrix} \in \mathcal{M}_{n(n-1)(n-2)}$$

with  $\forall i \in S$  :

$$B_i^{n-1} = \begin{pmatrix} \rho_{(i+0)\equiv n, (i+0)\equiv n} & \rho_{(i+0)\equiv n, (i+1)\equiv n} & \cdots & \rho_{(i+0)\equiv n, (i+n-4)\equiv n} & \rho_{B_{n-1}^*}^{*(i+0)\equiv n, a_i} \\ \rho_{(i+1)\equiv n, (i+0)\equiv n} & \rho_{(i+1)\equiv n, (i+1)\equiv n} & \cdots & \rho_{(i+1)\equiv n, (i+n-4)\equiv n} & \rho_{B_{n-1}^*}^{*(i+1)\equiv n, a_i} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \rho_{(i+n-4)\equiv n, (i+0)\equiv n} & \rho_{(i+n-4)\equiv n, (i+1)\equiv n} & \cdots & \rho_{(i+n-4)\equiv n, (i+n-4)\equiv n} & \rho_{B_{n-1}^*}^{*(i+n-4)\equiv n, a_i} \\ \rho_{B_{n-1}^*}^{*a_i, (i+0)\equiv n} & \rho_{B_{n-1}^*}^{*a_i, (i+1)\equiv n} & \cdots & \rho_{B_{n-1}^*}^{*a_i, (i+n-4)\equiv n} & \rho_{B_{n-1}^*}^{*a_i, a_i} \end{pmatrix} \in \mathcal{M}_{n-1}$$

- Let  $R_{B_{n-1}^*}^* = (R, \dots, R)$  be the revision rate in each population (which is  $R$  for each)

Let us then set  $A_{n-2}^*$  a restriction of  $B_{n-1}^*$  such that its revision protocol is:

$$\begin{pmatrix} B_1^{n-1} & 0_{\mathcal{M}_{n-2}} & \cdots & 0_{\mathcal{M}_{n-2}} & \cdots & \cdots & 0_{\mathcal{M}_{n-2}} \\ 0_{\mathcal{M}_{n-2}} & \ddots & \ddots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0_{\mathcal{M}_{n-2}} & \cdots & \cdots & 0_{\mathcal{M}_{n-2}} \\ 0_{\mathcal{M}_{n-2}} & \cdots & 0_{\mathcal{M}_{n-2}} & B_{in-i-1}^{n-1} & 0_{\mathcal{M}_{n-2}} & \cdots & 0_{\mathcal{M}_{n-2}} \\ 0_{\mathcal{M}_{n-2}} & \cdots & \cdots & 0_{\mathcal{M}_{n-2}} & \cdots & \cdots & 0_{\mathcal{M}_{n-2}} \\ 0_{\mathcal{M}_{n-2}} & \ddots & \ddots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \cdots & \cdots & 0_{\mathcal{M}_{n-2}} \\ 0_{\mathcal{M}_{n-2}} & \cdots & 0_{\mathcal{M}_{n-2}} & 0_{\mathcal{M}_{n-2}} & 0_{\mathcal{M}_{n-2}} & \cdots & B_{n^2-n-1}^{n-1} \end{pmatrix} \in \mathcal{M}_{n(n-2)}$$

We then say that  $A_{n-2}^*$  is the transformation of  $A$  into a game with  $n$  populations playing a game with  $n-2$  strategies, similarly, by iterating we can find  $A_m^*$ .  $\square$

**Proposition 3.3.** *If for all  $i, j \in S$  we have  $\rho_{ij}(\pi, x) = \rho_{ji}(\pi, x)$  and  $R_{ij} = R_{ji}$ , then for any  $n$  strategy population game with a fully supported revision protocol one can define a stationary distribution for the evolutionary process  $(X_t^N)$  on  $\mathcal{X}^N$ .*

*Proof.* Let  $A$  be a symmetric population game of  $N$  agents with  $n$  strategies with a fully supported revision protocol satisfying for all  $i, j \in S$  that  $\rho_{ij}(\pi, x) = \rho_{ji}(\pi, x)$ . Then by applying propositions 4.1 and 4.2, we can define the game  $A_3^*$  which is the transformation of  $A$  into a game with  $n$  populations playing a game with 3 strategies. Then by applying Proposition

3.1, we can define the game  $A^*$  which is the transformation of  $A$ . We can then apply theorem 11.2.3 of [San] which gives us the stationary distribution of the set  $A^*$  which is for its population  $i \in \llbracket 1, n \rrbracket$ :

$$\frac{\mu_{\mathcal{X}_i}^{N_i}}{\mu_0^{N_i}} = \prod_{j=1}^{N_i} \frac{N-j-1}{j} \cdot \frac{\rho_{i,i}^*(F^{*i}(\frac{j-1}{N}), \frac{j-1}{N})}{\rho_{i,i}^*(F^{*i}(\frac{j}{N}), \frac{j}{N})} \text{ with } \mathcal{X}_i \in \{0, \frac{1}{N}, \frac{2}{N}, \dots, 1\} \text{ and } \sum_{i=1}^n \mathcal{X}_i = 1$$

with  $\mu_0^{N_i}$  determined by the requirement that  $\sum_{\mathcal{X}_i \in \mathcal{X}^{N_i}} \mu_{\mathcal{X}_i}^{N_i} = 1$   $\square$

**Remark:** The chosen restriction is natural because it mainly consists of removing coordinates that appear several times.

## 4 Symetric game with $\mathfrak{p}$ populations

In this part we will show that proposition 3.3 is true for several populations, for this we will show a more general result by showing that we can find an infinite horizon whatever the number of strategies in the game of each population.

**Proposition 4.1.** *Let  $A$  be a symmetric game with  $\mathfrak{p}$  populations such that the population  $p$  plays a game with  $n^p$  strategies, its revision protocol has complete support and for all  $i, j \in S$  we have  $\rho_{ij}^p(\pi, x) = \rho_{ji}^p(\pi, x)$  and  $R_{ij} = R_{ji}$  then we can define a stationary distribution for the evolutionary process  $(X_t^N)$  on  $\mathcal{X}^N$ .*

*Proof.* Let  $p \in \llbracket 1, \mathfrak{p} \rrbracket$   $A^p$  be the game played by the population  $p$ . Applying Proposition 3.3, let  $p \in \llbracket 1, \mathfrak{p} \rrbracket$   $A^{*p}$  be the transformation into a 2-strategy game of the game played by the population  $p$ . By setting  $A^*$  the set of these games, we then end up with a game  $A^*$  which is a 2-strategy game with  $\sum_{k=1}^{\mathfrak{p}} n^k = n$  populations, and  $A^*$  represents the same game as  $A$ . We can then apply Theorem 11.2.3 of [San10] which gives us the stationary distribution of the set  $A^*$  which is for its population  $i \in \llbracket 1, n \rrbracket$  such that  $\exists p \in \llbracket 1, \mathfrak{p} \rrbracket$  such that  $i \in \llbracket \sum_{k=0}^{p-1} n^k + 1, \sum_{k=0}^p n^k \rrbracket$  (with  $n^0 = 0$ ):

$$\frac{\mu_{\mathcal{X}_i}^{N_i}}{\mu_0^{N_i}} = \prod_{j=1}^{N^p \mathcal{X}_i} \frac{N^p - j - 1}{j} \cdot \frac{\rho_{i,i}^*(F^{*i}(\frac{j-1}{N^p}), \frac{j-1}{N^p})}{\rho_{i,i}^*(F^{*i}(\frac{j}{N^p}), \frac{j}{N^p})} \text{ with } \mathcal{X}_i \in \{0, \frac{1}{N^p}, \frac{2}{N^p}, \dots, 1\} \text{ and}$$

$$\sum_{i=n^{p-1}+1}^{n^p} \mathcal{X}_i = 1 \text{ and } \sum_{j=1}^n \mathcal{X}_j = \mathfrak{p}$$

with  $\mu_0^{N_i}$  determined by the requirement that  $\sum_{\mathcal{X}_i \in \mathcal{X}^{N_i}} \mu_{\mathcal{X}_i}^{N_i} = 1$   $\square$

**Remark:** Not all the properties of this article are subject to the symmetry condition of the protocol and the revision rate, so if the infinite horizon of 3-strategy games is found by applying the same properties, we will have the infinite horizons of all games on the condition of assuming the inertia and myopia of the players.

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## References

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