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Infinite horizon for symetric strategy population game

Meziane Privat

Abstract : To predict the behavior of a population game when time becomes very long, the process that characterizes the evolution of our game dynamics must be reversible. Known games satisfying this are 2 strategy games as well as potential games with an exponential protocol. We will try to extend the study of infinite horizons for what are called symetric strategy games.

Key words : Population Games, Stochastic Evolutionary Models, Infinite-Horizon.

AMS subject classification : 91A15, 91A22, 92D25

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1 The case one population

1.1 Informal presentation with an example

First, we will consider the case where there is only one population.

Imagine a population of N cats playing a three-state game: no eating, eating one mouse, or eating two. This is a game with one population and three strategies, but it can be transformed into a game with three populations and two strategies. Imagine three populations of N cats each (as if we were cloning the initial population), the first population would play a two-state game which would be: do not eat or eat, the second would play: eat a mouse or do not eat a mouse, and the third would play: eat two mice or not eat two mice. The possible states would only be those where N cats choose an "odd" strategy and $2N$ cats choose an "even" strategy. We have thus transformed the game into a two-strategy game that we know is reversible. To switch from the initial game to the new game, you have to make small changes.

Let A be an object concerning the initial game, its transformation for the new game will be A^*

In this case,

- $x = (x_1, x_2, x_3) \in X$ becomes $x^* = (x_1, 1-x_1, x_2, 1-x_2, x_3, 1-x_3) \in X^*$
- $S = \{a, b, c\}$ becomes $S^* = \{a, \bar{a}, b, \bar{b}, c, \bar{c}\}$
- $\mathfrak{p} = 1$ becomes $\mathfrak{p}^* = 3$
- $F(x) = (\alpha, \beta, \gamma)$ becomes $F^*(x^*) = (\alpha, \frac{x_2\beta+x_3\gamma}{x_2+x_3}, \beta, \frac{x_1\alpha+x_3\gamma}{x_1+x_3}, \gamma, \frac{x_1\alpha+x_2\beta}{x_1+x_2})$
or
 $F^*(x^*) = (\alpha, 0, \beta, 0, \gamma, 0)$
- R becomes $R^* = (R, R, R)$
-

$$\rho(\pi, x) = \begin{pmatrix} \rho_{a,a}(\pi, x) & \rho_{a,b}(\pi, x) & \rho_{a,c}(\pi, x) \\ \rho_{b,a}(\pi, x) & \rho_{b,b}(\pi, x) & \rho_{b,c}(\pi, x) \\ \rho_{c,a}(\pi, x) & \rho_{c,b}(\pi, x) & \rho_{c,c}(\pi, x) \end{pmatrix}$$

Becomes

$$\begin{aligned}
\rho^*(\pi^*, x^*) &= \begin{pmatrix} \rho_{a,a}^*(\pi^*, x^*) & \rho_{a,\bar{a}}^*(\pi^*, x^*) & 0 & 0 & 0 & 0 \\ \rho_{\bar{a},a}^*(\pi^*, x^*) & \rho_{\bar{a},\bar{a}}^*(\pi^*, x^*) & 0 & 0 & 0 & 0 \\ 0 & 0 & \rho_{b,b}^*(\pi^*, x^*) & \rho_{b,\bar{b}}^*(\pi^*, x^*) & 0 & 0 \\ 0 & 0 & \rho_{\bar{b},b}^*(\pi^*, x^*) & \rho_{\bar{b},\bar{b}}^*(\pi^*, x^*) & 0 & 0 \\ 0 & 0 & 0 & 0 & \rho_{c,c}^*(\pi^*, x^*) & \rho_{c,\bar{c}}^*(\pi^*, x^*) \\ 0 & 0 & 0 & 0 & \rho_{\bar{c},c}^*(\pi^*, x^*) & \rho_{\bar{c},\bar{c}}^*(\pi^*, x^*) \end{pmatrix} \\
&= \begin{pmatrix} \rho_{a,a}(\pi, x) & \rho_{a,b}(\pi, x) + \rho_{a,c}(\pi, x) \\ \frac{1}{2}(\rho_{b,a}(\pi, x) + \rho_{c,a}(\pi, x)) & \frac{1}{2}(\rho_{b,b}(\pi, x) + \rho_{b,c}(\pi, x) + \rho_{c,b}(\pi, x) + \rho_{c,c}(\pi, x)) \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \rho_{b,b}(\pi, x) & \rho_{b,a}(\pi, x) + \rho_{b,c}(\pi, x) \\ \frac{1}{2}(\rho_{a,b}(\pi, x) + \rho_{c,b}(\pi, x)) & \frac{1}{2}(\rho_{a,a}(\pi, x) + \rho_{a,c}(\pi, x) + \rho_{c,a}(\pi, x) + \rho_{c,c}(\pi, x)) \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \rho_{c,c}(\pi, x) & \rho_{c,a}(\pi, x) + \rho_{c,b}(\pi, x) \\ \frac{1}{2}(\rho_{a,c}(\pi, x) + \rho_{b,c}(\pi, x)) & \frac{1}{2}(\rho_{a,a}(\pi, x) + \rho_{a,b}(\pi, x) + \rho_{b,a}(\pi, x) + \rho_{b,b}(\pi, x)) \end{pmatrix}
\end{aligned}$$

We then know by theorem 11.2.3 of [San10] what shape should have the stationary distribution of the game and therefore its infinite horizon.

1.2 Formal presentation

In this part we will lay the mathematical foundations to have access to the behavior when the time becomes very long for a game with 3 strategies.

Définition 1.1. *A is a symmetric game if for all $i, j \in S$ we have $\rho_{ij}(\pi, x) = \rho_{ji}(\pi, x)$ and $R_{ij} = R_{ji}$.*

Let A be a game of a population with 3 strategies that we will call the initial game, in connection with this game we then pose:

- X the state space of A
- $S = \{1, 2, 3\}$ the set of 3 strategies in A
- We have $\mathbf{p} = 1$
- Let F be its gain function
- We set ρ its full support revision protocol
- R its revision rate
- N the number of agents in the population

Proposition 1.1. *If for all $i, j \in S$ we have $\rho_{ij}(\pi, x) = \rho_{ji}(\pi, x)$ and $R_{ij} = R_{ji}$, then for any population game at 3 strategies there is a unique population game with two strategies and 3 population describing the same game.*

Proof. To demonstrate this we must first build this new game, ie show its existence, and then show its uniqueness.

Existence: Let A^* be a two-strategy game of n population defined as follows:

- $X^* = \{x^* = (x_1, 1-x_1, x_2, 1-x_2, x_3, 1-x_3) \text{ such that } \exists x = (x_1, x_2, x_3) \in X\}$ its state space
- $S^* = \{1, \bar{1}, 2, \bar{2}, 3, \bar{3}\}$ the set of $6(2 \times 3)$ strategies in A^*
- We have $\mathbf{p}^* = 3$ populations
- Let $x \in X$ such that $F(x) = y = (y_1, y_2, y_3)$ then $F^*(x^*) = y^* = (y_1, 0, y_2, 0, y_3, 0) \in \mathbb{R}^6$
- Let $N^* = (N, N, N) \in (\mathbb{N}^*)^3$ be the number of agents in each population (which is N for each)
-

$$\rho^*(\pi^*, x^*) = \begin{pmatrix} \rho_{1,1}^*(\pi^*, x^*) & \rho_{1,\bar{1}}^*(\pi^*, x^*) & 0 & 0 & 0 & 0 \\ \rho_{\bar{1},1}^*(\pi^*, x^*) & \rho_{\bar{1},\bar{1}}^*(\pi^*, x^*) & 0 & 0 & 0 & 0 \\ 0 & 0 & \rho_{2,2}^*(\pi^*, x^*) & \rho_{2,\bar{2}}^*(\pi^*, x^*) & 0 & 0 \\ 0 & 0 & \rho_{\bar{2},2}^*(\pi^*, x^*) & \rho_{\bar{2},\bar{2}}^*(\pi^*, x^*) & 0 & 0 \\ 0 & 0 & 0 & 0 & \rho_{3,3}^*(\pi^*, x^*) & \rho_{3,\bar{3}}^*(\pi^*, x^*) \\ 0 & 0 & 0 & 0 & \rho_{\bar{3},3}^*(\pi^*, x^*) & \rho_{\bar{3},\bar{3}}^*(\pi^*, x^*) \end{pmatrix}$$

with $\forall i \in S :$

$$\begin{aligned}
\text{(i)} \quad & \rho_{i,i}^*(\pi^*, x^*) = \rho_{i,i}(\pi, x) \\
\text{(ii)} \quad & \rho_{i,\bar{i}}^*(\pi^*, x^*) = \sum_{\substack{j \in S \\ j \neq i}} \rho_{i,j}(\pi, x) \\
\text{(iii)} \quad & \rho_{\bar{i},i}^*(\pi^*, x^*) = \frac{1}{2} \sum_{\substack{j \in S \\ j \neq i}} \rho_{j,i}(\pi, x) \\
\text{(iv)} \quad & \rho_{i,\bar{i}}^*(\pi^*, x^*) = \frac{1}{2} \left(\sum_{\substack{j \in S \\ j \neq i}} \sum_{\substack{k \in S \\ k \neq i \\ k \neq j}} \rho_{j,k}(\pi, x) + \sum_{\substack{j \in S \\ j \neq i}} \rho_{j,j}(\pi, x) \right)
\end{aligned}$$

- Let R^* the new revision rate the same as ρ^*

Unicity: We now need to show that from each element previously constructed for A^* we can find the element of A used, in other words, that there is a bijection between each element of A^* and the element of A which allowed its construction.

- To go from N^* to N , just take the first coordinate of N^*
- To go from \mathbf{p}^* to \mathbf{p} just set $\mathbf{p} = \frac{\mathbf{p}^*}{3}$
- Let $f : \mathbb{R}^6 \rightarrow \mathbb{R}^3$ such that $f(x_1, \dots, x_6) = (x_1, x_3, x_5)$, then $f(X^*) = X$ and $f(F(X^*)) = F(X)$
- By writing $S^* = \{1, \dots, 6\}$ we can find S by setting $S = \{1, 3, 5\}$
- Recall that $\forall i, j \in S, \rho_{ij}(\pi, x) = \rho_{ji}(\pi, x)$
 - $\forall i \in S, \rho_{ii}(\pi, x) = \rho_{ii}^*(\pi^*, x^*)$
 - $\forall i, j, k \in S$ with $i \neq j \neq k$, we have:

$$\begin{aligned}
\rho_{ij}(\pi, x) + \rho_{ji}(\pi, x) &= (\rho_{ij}(\pi, x) + \rho_{ik}(\pi, x)) + (\rho_{ji}(\pi, x) + \rho_{jk}(\pi, x)) \\
&\quad - (\rho_{ik}(\pi, x) + \rho_{jk}(\pi, x)) \\
&= \rho_{i,\bar{i}}^*(\pi^*, x^*) + \rho_{j,\bar{j}}^*(\pi^*, x^*) - 2\rho_{\bar{k},k}^*(\pi^*, x^*)
\end{aligned}$$

we then have $\forall i, j, k \in S$ with $i \neq j \neq k$:

$$\rho_{ij}(\pi, x) = \frac{\rho_{i,\bar{i}}^*(\pi^*, x^*) + \rho_{j,\bar{j}}^*(\pi^*, x^*) - 2\rho_{\bar{k},k}^*(\pi^*, x^*)}{2}$$

- For R^* we prove uniqueness as for the previous point.

□

We will call A^* the transformation of A .

Proposition 1.2. *If for all $i, j \in S$ we have $\rho_{ij}(\pi, x) = \rho_{ji}(\pi, x)$, and $R_{ij} = R_{ji}$ then for any population game from N agents to 3 strategies with a full support revision protocol one can define a stationary distribution for the evolutionary process (X_t^N) on \mathcal{X}^N .*

Proof. Let A be a population game of N agents with 3 strategies with a satisfactory full-support revision protocol for all $i, j \in S$ that $\rho_{ij}(\pi, x) = \rho_{ji}(\pi, x)$. Then by applying Proposition 3.1, we can define the game A^* which is the transformation of A . We can then apply theorem 11.2.3 of [San10] which gives us the stationary distribution of the set A^* which is for its population $i \in \llbracket 1, 3 \rrbracket$:

$$\frac{\mu_{\mathcal{X}_i}^{N_i}}{\mu_0^{N_i}} = \prod_{j=1}^{N_{\mathcal{X}_i}} \frac{N-j-1}{j} \cdot \frac{\rho_{i,i}^*(F^{*i}(\frac{j-1}{N}), \frac{j-1}{N})}{\rho_{i,i}^*(F^{*i}(\frac{j}{N}), \frac{j}{N})} \text{ with } \mathcal{X}_i \in \{0, \frac{1}{N}, \frac{2}{N}, \dots, 1\} \text{ and } \sum_{i=1}^3 \mathcal{X}_i = 1$$

with $\mu_0^{N_i}$ determined by the requirement that $\sum_{\mathcal{X}_i \in \mathcal{X}^{N_i}} \mu_{\mathcal{X}_i}^{N_i} = 1$.

Maybe have to make a mixture model to return to dimension 1, to see

□

2 The case \mathfrak{p} populations

In this part we will show that proposition 1.2 is true for several populations, for this we will show a more general result by showing that we can find an infinite horizon that the number of strategy of the game of each population is 2 or 3 .

Proposition 2.1. *Let A be a game with \mathfrak{p} populations such that the population p plays a game with n^p strategies ($n^p = 2$ or 3), that its revision protocol is with full support and that for all $i, j \in S$ we have $\rho_{ij}^p(\pi, x) = \rho_{ji}^p(\pi, x)$ and $R_{ij} = R_{ji}$ then we can define a stationary distribution for the evolutionary process (X_t^N) on \mathcal{X}^N .*

Proof. Let $p \in \llbracket 1, \mathfrak{p} \rrbracket$ A^p be the game played by the population p . Applying Proposition 1.2, let $p \in \llbracket 1, \mathfrak{p} \rrbracket$ A^{*p} be the transformation into a 2-strategy game of the game played by the population p ($A^{*p} = A$ if $n^p = 2$). By setting A^* the set of these games, we then end up with a game A^* which is

a 2-strategy game with $\sum_{k=1}^p n^k = n$ populations, and A^* represents the same game as A . We can then apply Theorem 11.2.3 of [San10] which gives us the stationary distribution of the set A^* which is for its population $i \in \llbracket 1, n \rrbracket$ such that $\exists p \in \llbracket 1, \mathfrak{p} \rrbracket$ such that $i \in \llbracket \sum_{k=0}^{p-1} n^k + 1, \sum_{k=0}^p n^k \rrbracket$ (with $n^0 = 0$):

$$\frac{\mu_{\mathcal{X}_i}^{N_i}}{\mu_0^{N_i}} = \prod_{j=1}^{N^p \mathcal{X}_i} \frac{N^{p-j-1}}{j} \cdot \frac{\rho_{i,i}^*(F^{*i}(\frac{j-1}{N^p}), \frac{j-1}{N^p})}{\rho_{i,i}^*(F^{*i}(\frac{j}{N^p}), \frac{j}{N^p})} \text{ with } \mathcal{X}_i \in \{0, \frac{1}{N^p}, \frac{2}{N^p}, \dots, 1\} \text{ and}$$

$$\sum_{i=n^{p-1}+1}^{n^p} \mathcal{X}_i = 1 \text{ and } \sum_{j=1}^n \mathcal{X}_j = \mathfrak{p}$$

with $\mu_0^{N_i}$ determined by the requirement that $\sum_{\mathcal{X}_i \in \mathcal{X}^{N_i}} \mu_{\mathcal{X}_i}^{N_i} = 1$ □

3 Transformation of an n-strategy symmetric game into a 2-strategy game

To go from a game with n strategies to a game with 2 strategies, we will show that for a game with n strategies there exists a unique game with $n-1$ strategies which represents the same game, then by repeating this we come across a game with 2 strategies.

Définition 3.1. *A is a symmetric game if for all $i, j \in S$ we have $\rho_{ij}(\pi, x) = \rho_{ji}(\pi, x)$ and $R_{ij} = R_{ji}$.*

Let A be a game of a population with n strategies that we will call the initial game, in connection with this game we then pose:

- X the state space of A
- $S = \{1, 2, \dots, n\}$ the set of n strategies in A
- We have $\mathfrak{p} = 1$
- Let F be its gain function
- We set ρ its full support revision protocol
- R its revision rate
- N the number of agents in the population

Proposition 3.1. *For any population game with n strategies ($n > 3$) there exists a unique population game with n population each playing a game with $n - 1$ strategies describing the same game .*

Proof. To demonstrate this we must first build this new game, ie show its existence, and then show its uniqueness.

Existence: Let A_{n-1}^* be a set of n populations each having access to $n - 1$ strategies defined as follows:

- $X_{n-1}^* = \{x_{n-1}^* = (x_1, x_2, \dots, x_{n-2}, 1 - x_{n-1} - x_n, x_2, x_3, \dots, x_{n-1}, 1 - x_1 - x_n, \dots, x_n, x_1, x_2, \dots, x_{n-3}, 1 - x_{n-1} - x_{n-2}) \text{ such that } \exists x = (x_1, \dots, x_n) \in X\}$ its state space
- $S_{n-1}^* = \{1, 2, \dots, n - 2, a_1, 2, 3, \dots, n - 1, a_2, \dots, n, 1, 2, \dots, n - 3, a_n\}$ the set of $n - 1 \times n$ strategies in A_{n-1}^*
- We have $\mathfrak{p}_{n-1}^* = n$ populations
- Let $x \in X$ such that $F(x) = y = (y_1, \dots, y_n)$ then $F_{n-1}^*(x_{n-1}^*) = y_{n-1}^* = (y_1, y_2, \dots, y_{n-2}, 0, y_2, y_3, \dots, y_{n-1}, 0, \dots, y_n, y_1, y_2, \dots, y_{n-3}, 0)$
- Let $N_{n-1}^* = (N, \dots, N) \in (\mathbb{N}^*)^n$ be the number of agents in each population (which is N for each)
-

$$\rho_{n-1}^*(\pi_{n-1}^*, x_{n-1}^*) = \begin{pmatrix} A_1^{n-1} & 0_{\mathcal{M}_{n-1}} & \cdots & 0_{\mathcal{M}_{n-1}} \\ 0_{\mathcal{M}_{n-1}} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0_{\mathcal{M}_{n-1}} \\ 0_{\mathcal{M}_{n-1}} & \cdots & 0_{\mathcal{M}_{n-1}} & A_n^{n-1} \end{pmatrix} \in \mathcal{M}_{n(n-1)}$$

with $\forall i \in S$:

$$A_i^{n-1} = \begin{pmatrix} \rho_{(i+0) \equiv n, (i+0) \equiv n} & \rho_{(i+0) \equiv n, (i+1) \equiv n} & \cdots & \rho_{(i+0) \equiv n, (i+n-3) \equiv n} & \rho_{n-1}^*_{(i+0) \equiv n, a_i} \\ \rho_{(i+1) \equiv n, (i+0) \equiv n} & \rho_{(i+1) \equiv n, (i+1) \equiv n} & \cdots & \rho_{(i+1) \equiv n, (i+n-3) \equiv n} & \rho_{n-1}^*_{(i+1) \equiv n, a_i} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \rho_{(i+n-3) \equiv n, (i+0) \equiv n} & \rho_{(i+n-3) \equiv n, (i+1) \equiv n} & \cdots & \rho_{(i+n-3) \equiv n, (i+n-3) \equiv n} & \rho_{n-1}^*_{(i+n-3) \equiv n, a_i} \\ \rho_{n-1}^*_{a_i, (i+0) \equiv n} & \rho_{n-1}^*_{a_i, (i+1) \equiv n} & \cdots & \rho_{n-1}^*_{a_i, (i+n-3) \equiv n} & \rho_{n-1}^*_{a_i, a_i} \end{pmatrix} \in \mathcal{M}_{n-1}$$

with $\forall j \in S$:

- $\rho_{n-1,j,a_i}^* = \rho_{j,(i+n-2)\equiv n} + \rho_{j,(i+n-1)\equiv n}$
- $\rho_{n-1,a_i,j}^* = \rho_{(i+n-2)\equiv n,j} + \rho_{(i+n-1)\equiv n,j}$
- $\rho_{n-1,a_i,a_i}^* = \rho_{(i+n-2)\equiv n,(i+n-2)\equiv n} + \rho_{(i+n-2)\equiv n,(i+n-1)\equiv n} \rho_{(i+n-1)\equiv n,(i+n-2)\equiv n} + \rho_{(i+n-1)\equiv n,(i+n-1)\equiv n}$

where $\forall a \in \mathbb{N}$ $a_{\equiv n}$ is a modulo n .

- We set R_{n-1}^* the new revision rate in the same way as ρ_{n-1}^*

Unicity: We now need to show that from each element previously constructed for A^* we can find the element of A used, in other words, that there is a bijection between each element of A^* and the element of A which allowed its construction.

- To go from N_{n-1}^* to N , just take the first coordinate of N_{n-1}^*
- To go from \mathfrak{p}_{n-1}^* to \mathfrak{p} just set $\mathfrak{p} = \frac{\mathfrak{p}_{n-1}^*}{n}$
- Let's set $f : \mathbb{R}^{n^2-n} \rightarrow \mathbb{R}^n$ such that $f(x_1, x_2, \dots, x_{n^2-n}) = (x_1, \dots, x_{pn-p-1}, \dots, x_{n^2-2n-1})$ with $p \in \llbracket 1, n-1 \rrbracket$, then $f(X^*) = X$ and $f(F(X^*)) = F(X)$
- By writing $S_{n-1}^* = \{1, \dots, n^2-n\}$ we can find S by setting $S = \{1, \dots, pn-p-1, \dots, n^2-2n-1\}$ with $p \in \llbracket 1, n-1 \rrbracket$
- (i) $\forall i \in S, \rho_{ii}(\pi, x) = \rho_{n-1(i-1)n-(i-1)+1, (i-1)n-(i-1)+1}^*(\pi_{n-1}^*, x_{n-1}^*)$ (this is the first coordinate of $A_i^{n-1}(\pi_{n-1}^*, x_{n-1}^*)$)
- (ii) $\forall i \in S$ and $j \in \llbracket (i+1)_{\equiv n}, (i+n-3)_{\equiv n} \rrbracket$ such that $(i+a)_{\equiv n} = j$,
 $\rho_{ij}(\pi, x) = \rho_{n-1(i-1)n-(i-1)+1, (i-1)n-(i-1)+1+a}^*(\pi_{n-1}^*, x_{n-1}^*)$
- (iii) $\forall i \in S$ and $j = (i+n-1)_{\equiv n}$, $\rho_{ij}(\pi, x) = \rho_{n-1(j-1)n-(j-1)+2, (j-1)n-(j-1)+1}^*(\pi_{n-1}^*, x_{n-1}^*)$
- (iv) $\forall i \in S$ and $j = (i+n-2)_{\equiv n}$,

$$\rho_{ij}(\pi, x) = \rho_{n-1(i-1)n-(i-1)+1, (i-1)n-(i-1)+n-1}^*(\pi_{n-1}^*, x_{n-1}^*) - \rho_{n-1(j-1)n-(j-1)+2, (j-1)n-(j-1)+1}^*(\pi_{n-1}^*, x_{n-1}^*)$$
- For R_{n-1}^* we prove uniqueness as for the previous point.

□

Now that we know how to go from a game with n strategies to a game with $n - 1$ strategies, we can explain how to go from a game with n strategy to a game with m strategy with $m \leq n$.

Proposition 3.2. *For any population game with n strategies ($n > 3$) there exists a population game with n population each playing a game with m strategies ($m \leq n$) describing the same game.*

Proof. Let A be the initial game with n strategies, let A_{n-1}^* be its transformation into a game with $n - 1$ strategies given by Proposition 1.1. Applying proposition 1.1 again to the game A_{n-1}^* we find a game B_{n-1}^* defined as follows:

- $X_{B_{n-1}^*}^* = \{x_{B_{n-1}^*}^* = (x_1, x_2, \dots, x_{n-3}, 1-x_{n-2}-x_{n-1}-x_n, \dots, x_n, x_1, x_2, \dots, x_{n-4}, 1-x_{n-1}-x_{n-2}-x_{n-3}) \in \mathbb{R}^{n \times (n-1) \times (n-2)} \text{ such that } \exists x = (x_1, \dots, x_n) \in X\}$ its state space
- $S_{B_{n-1}^*}^* = \{1, 2, \dots, n-3, a_1, 2, 3, \dots, n-2, a_2, \dots, n, 1, 2, \dots, n-4, a_n\}$ the set of $(n-2) \times (n-1) \times n$ strategies in A_{n-1}^*
- We have $\mathfrak{p}_{B_{n-1}^*}^* = n \times (n-1)$ populations
- Let $x \in X$ such that $F(x) = y = (y_1, \dots, y_n)$ then $F_{B_{n-1}^*}^*(x_{B_{n-1}^*}^*) = y_{B_{n-1}^*}^* = (y_1, y_2, \dots, y_{n-3}, 0, y_2, y_3, \dots, y_{n-2}, 0, \dots, y_n, y_1, y_2, \dots, y_{n-4}, 0)$
- We set $N_{B_{n-1}^*}^* = (N, \dots, N) \in (\mathbb{N}^*)^{n \times n-1}$ the number of agents in each population (which is N for each)
-

$$\rho_{B_{n-1}^*}^*(\pi_{n-1}^*, x_{n-1}^*) = \begin{pmatrix} B_1^{n-1} & 0_{\mathcal{M}_{n-2}} & \cdots & 0_{\mathcal{M}_{n-2}} \\ 0_{\mathcal{M}_{n-2}} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0_{\mathcal{M}_{n-2}} \\ 0_{\mathcal{M}_{n-2}} & \cdots & 0_{\mathcal{M}_{n-2}} & B_{n \times (n-1)}^{n-1} \end{pmatrix} \in \mathcal{M}_{n(n-1)(n-2)}$$

with $\forall i \in S$:

$$B_i^{n-1} = \begin{pmatrix} \rho(i+0)_{\equiv n, (i+0)_{\equiv n}} & \rho(i+0)_{\equiv n, (i+1)_{\equiv n}} & \cdots & \rho(i+0)_{\equiv n, (i+n-4)_{\equiv n}} & \rho_{B_{n-1}^*}^*(i+0)_{\equiv n, a_i} \\ \rho(i+1)_{\equiv n, (i+0)_{\equiv n}} & \rho(i+1)_{\equiv n, (i+1)_{\equiv n}} & \cdots & \rho(i+1)_{\equiv n, (i+n-4)_{\equiv n}} & \rho_{B_{n-1}^*}^*(i+1)_{\equiv n, a_i} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \rho(i+n-4)_{\equiv n, (i+0)_{\equiv n}} & \rho(i+n-4)_{\equiv n, (i+1)_{\equiv n}} & \cdots & \rho(i+n-4)_{\equiv n, (i+n-4)_{\equiv n}} & \rho_{B_{n-1}^*}^*(i+n-4)_{\equiv n, a_i} \\ \rho_{B_{n-1}^*}^*_{a_i, (i+0)_{\equiv n}} & \rho_{B_{n-1}^*}^*_{a_i, (i+1)_{\equiv n}} & \cdots & \rho_{B_{n-1}^*}^*_{a_i, (i+n-4)_{\equiv n}} & \rho_{B_{n-1}^*}^*_{a_i, a_i} \end{pmatrix} \in \mathcal{M}_{n-1}$$

- Let $R_{B_{n-1}^*}^* = (R, \dots, R)$ be the revision rate in each population (which is R for each)

Let us then set A_{n-2}^* a restriction of B_{n-1}^* such that its revision protocol is:

$$\begin{pmatrix} B_1^{n-1} & 0_{\mathcal{M}_{n-2}} & \cdots & 0_{\mathcal{M}_{n-2}} & \cdots & \cdots & 0_{\mathcal{M}_{n-2}} \\ 0_{\mathcal{M}_{n-2}} & \ddots & \ddots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0_{\mathcal{M}_{n-2}} & \cdots & \cdots & 0_{\mathcal{M}_{n-2}} \\ 0_{\mathcal{M}_{n-2}} & \cdots & 0_{\mathcal{M}_{n-2}} & B_{in-i-1}^{n-1} & 0_{\mathcal{M}_{n-2}} & \cdots & 0_{\mathcal{M}_{n-2}} \\ 0_{\mathcal{M}_{n-2}} & \cdots & \cdots & 0_{\mathcal{M}_{n-2}} & \cdots & \cdots & 0_{\mathcal{M}_{n-2}} \\ 0_{\mathcal{M}_{n-2}} & \ddots & \ddots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \cdots & \cdots & 0_{\mathcal{M}_{n-2}} \\ 0_{\mathcal{M}_{n-2}} & \cdots & 0_{\mathcal{M}_{n-2}} & 0_{\mathcal{M}_{n-2}} & 0_{\mathcal{M}_{n-2}} & \cdots & B_{n^2-n-1}^{n-1} \end{pmatrix} \in \mathcal{M}_{n(n-2)}$$

We then say that A_{n-2}^* is the transformation of A into a game with n populations playing a game with $n-2$ strategies, similarly, by iterating we can find A_m^* . \square

Proposition 3.3. *If for all $i, j \in S$ we have $\rho_{ij}(\pi, x) = \rho_{ji}(\pi, x)$ and $R_{ij} = R_{ji}$, then for any n strategy population game with a fully supported revision protocol one can define a stationary distribution for the evolutionary process (X_t^N) on \mathcal{X}^N .*

Proof. Let A be a symmetric population game of N agents with n strategies with a fully supported revision protocol satisfying for all $i, j \in S$ that $\rho_{ij}(\pi, x) = \rho_{ji}(\pi, x)$. Then by applying propositions 4.1 and 4.2, we can define the game A_3^* which is the transformation of A into a game with n populations playing a game with 3 strategies. Then by applying Proposition

3.1, we can define the game A^* which is the transformation of A . We can then apply theorem 11.2.3 of [San] which gives us the stationary distribution of the set A^* which is for its population $i \in \llbracket 1, n \rrbracket$:

$$\frac{\mu_{\mathcal{X}_i}^{N_i}}{\mu_0^{N_i}} = \prod_{j=1}^{N\mathcal{X}_i} \frac{N-j-1}{j} \cdot \frac{\rho_{i,i}^*(F^{*i}(\frac{j-1}{N}), \frac{j-1}{N})}{\rho_{i,i}^*(F^{*i}(\frac{j}{N}), \frac{j}{N})} \text{ with } \mathcal{X}_i \in \{0, \frac{1}{N}, \frac{2}{N}, \dots, 1\} \text{ and } \sum_{i=1}^n \mathcal{X}_i = 1$$

with $\mu_0^{N_i}$ determined by the requirement that $\sum_{\mathcal{X}_i \in \mathcal{X}^{N_i}} \mu_{\mathcal{X}_i}^{N_i} = 1$ \square

Remark: The chosen restriction is natural because it mainly consists of removing coordinates that appear several times.

4 Symetric game with \mathfrak{p} populations

In this part we will show that proposition 3.3 is true for several populations, for this we will show a more general result by showing that we can find an infinite horizon whatever the number of strategies in the game of each population.

Proposition 4.1. *Let A be a symetric game with \mathfrak{p} populations such that the population p plays a game with n^p strategies, its revision protocol has complete support and for all $i, j \in S$ we have $\rho_{ij}^p(\pi, x) = \rho_{ji}^p(\pi, x)$ and $R_{ij} = R_{ji}$ then we can define a stationary distribution for the evolutionary process (X_t^N) on \mathcal{X}^N .*

Proof. Let $p \in \llbracket 1, \mathfrak{p} \rrbracket$ A^p be the game played by the population p . Applying Proposition 3.3, let $p \in \llbracket 1, \mathfrak{p} \rrbracket$ A^{*p} be the transformation into a 2-strategy game of the game played by the population p . By setting A^* the set of these games, we then end up with a game A^* which is a 2-strategy game with $\sum_{k=1}^{\mathfrak{p}} n^k = n$ populations, and A^* represents the same game as A . We can then apply Theorem 11.2.3 of [San10] which gives us the stationary distribution of the set A^* which is for its population $i \in \llbracket 1, n \rrbracket$ such that $\exists p \in \llbracket 1, \mathfrak{p} \rrbracket$ such that $i \in \llbracket \sum_{k=0}^{p-1} n^k + 1, \sum_{k=0}^p n^k \rrbracket$ (with $n^0 = 0$):

$$\frac{\mu_{\mathcal{X}_i}^{N_i}}{\mu_0^{N_i}} = \prod_{j=1}^{N^p \mathcal{X}_i} \frac{N^p - j - 1}{j} \cdot \frac{\rho_{i,i}^*(F^{*i}(\frac{j-1}{N^p}), \frac{j-1}{N^p})}{\rho_{i,i}^*(F^{*i}(\frac{j}{N^p}), \frac{j}{N^p})} \text{ with } \mathcal{X}_i \in \{0, \frac{1}{N^p}, \frac{2}{N^p}, \dots, 1\} \text{ and}$$

$$\sum_{i=n^{p-1}+1}^{n^p} \mathcal{X}_i = 1 \text{ and } \sum_{j=1}^n \mathcal{X}_j = \mathfrak{p}$$

with $\mu_0^{N_i}$ determined by the requirement that $\sum_{\mathcal{X}_i \in \mathcal{X}^{N_i}} \mu_{\mathcal{X}_i}^{N_i} = 1$ \square

Remark: Not all the properties of this article are subject to the symmetry condition of the protocol and the revision rate, so if the infinite horizon of 3-strategy games is found by applying the same properties, we will have the infinite horizons of all games on the condition of assuming the inertia and myopia of the players.

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