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Simple mechanical systems and the Galilean transformation

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Abstract

Simple mechanical systems such as idealised pendulums are quite easy to comprehend both qualitatively and quantitatively. The Galilean transformation does not pose any particular conceptual challenge either. Yet students often have problems with the application of such coordinate transformations to simple mechanical systems. With three examples I will highlight various pitfalls, thus confirming that it is very instructive to analyse a system from different frames of reference.

1. Simple mechanical systems

You will hardly find any physics classes at high schools or universities that do not deal with elementary idealised mechanical systems such as swinging pendulums or the planetary system. So, apparently these systems have already been discussed thoroughly and do not pose any didactic challenges.

2. The Galilean transformation

This is a supposedly simple chapter: at moderate velocities \vec{w} the laws of nature do not change under the Galilean coordinate transformation

$$\vec{r}' = \vec{r} - \vec{q} - \vec{w} t \quad \{1\}$$

from one inertial system to another. In equation {1} the constant vector \vec{q} induces a shift of the coordinate system. The term $\vec{w} t$ describes the velocity portion of the Galilean transformation (boost). We do not discuss a possible additional rotation here, and we leave out time translations. When examining one-dimensional movements we can simplify to

$$x' = x - q - w t.$$

Obviously, the above-mentioned simple mechanical systems can be analysed from different inertial systems without expecting any difficulties in the calculations. While this is true, at least surprises and pitfalls – for students as well as for teachers – can show up, as I will demonstrate in the following examples.

3. Example 1: A simple pendulum

We consider an idealised horizontally oscillating spring pendulum in the system of the resting Earth (see Figure 1).

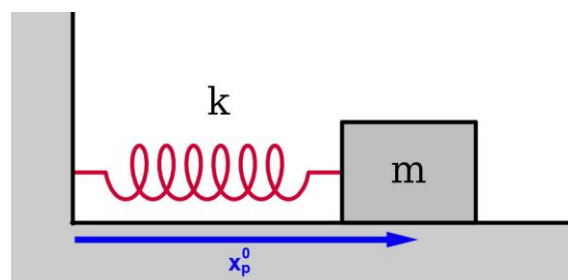


Fig. 1: Idealised spring pendulum in rest position

We idealise the Earth as an inertial system by ignoring its rotation and movement relative to the Sun. The pendulum suspension is firmly attached to our planet, friction does not occur, the coil spring is elastic and massless. The total energy E of the pendulum must be constant in time:

$$E = \frac{1}{2} k (x_p - x_p^0)^2 + \frac{1}{2} m v_p^2.$$

Here, m is the mass of the pendulum, x_p the time-dependent spatial coordinate, x_p^0 the rest position, $v_p = \dot{x}_p$ its velocity and k the spring constant.

Starting from the conservation of energy, we obtain from

$$\frac{dE}{dt} = 0$$

the oscillator equation

$$m \ddot{x}_p + k (x_p - x_p^0) = 0.$$

Did we get it right? Is it not also necessary to consider the Earth (mass M , without pendulum mass) and its oscillatory motion (with time-dependent velocity v_E)? The conservation of momentum requires

$$m v_p + M v_E = \text{const}$$

in every inertial system. We place ourselves here in the centre of mass system of Earth plus pendulum, so that

$$m v_p + M v_E = 0$$

applies.

The additional term in the energy formula describing the kinetic energy of the Earth thus becomes

$$\frac{1}{2} M v_E^2 = \frac{1}{2} \frac{m}{M} m v_p^2.$$

If we assume that the pendulum mass is of the order of 1 kg, the factor m/M is tiny because of $M = 2 \cdot 10^{30}$ kg. Therefore, the kinetic energy of the Earth's movement is completely negligible and does not need to be taken into account.

So far, so good. Now we consider the pendulum in a system moving with constant velocity w relative to the Earth. The direction of w should correspond to the line of the pendulum's motion, so that we can continue to calculate one-dimensionally:

$$x'_p = x_p - w t \quad v'_p = v_p - w.$$

Note that the new coordinate system is – like the Earth – at best „approximately“ an inertial system. Let the magnitude of w and of v'_p be 1 m/s, so the non-relativistic treatment is justified. Obviously, the total energy E' must also be constant in the new reference frame:

$$\begin{aligned} E' &= \frac{1}{2} k (x'_p - x'^0_p)^2 + \frac{1}{2} m v'^2_p \\ &= \frac{1}{2} k (x_p - x_p^0)^2 + \frac{1}{2} m v_p^2 + \frac{1}{2} m w^2 - m v_p w \\ E' &= E + \frac{1}{2} m w^2 - m v_p w. \end{aligned}$$

However, because of the last term E' is time-dependent! The error in our calculation lies in the fact that here we must not ignore the Earth's movement! The correct calculation in the centre of mass system of Earth plus pendulum, which is a true inertial system, is, using $v'_E = v_E - w$:

$$\begin{aligned} E' &= E + \frac{1}{2} m w^2 - m v_p w + \frac{1}{2} M v'^2_E \\ &= E + \frac{1}{2} m w^2 - m v_p w + \frac{1}{2} M v_E^2 - M v_E w + \frac{1}{2} M w^2 \\ &= E + \frac{1}{2} m w^2 + \frac{1}{2} M w^2 = \text{const}. \end{aligned}$$

In the last step we have again ignored the numerically very small term $\frac{1}{2} M v_E^2$. Now the equation is correct: the total energy of a closed system is constant in time. What we have learned is that the basic calculations need to be exact. Afterwards we may make suitable approximations. The reverse order, however, usually does not work.

Incidentally, the by far largest term in the above equation is the last one, $\frac{1}{2} M w^2$. It has, however, no effect on the movement, as it is constant in time.

Frequently, in the context of pendulum analyses, the Earth's mass is assumed explicitly or implicitly infinite from the outset. In that case we can no longer accommodate the mass M in the equations at all, and may not find an explanation for the apparently time-dependent energy E' .

4. Example 2: Car trip

4.1 Braking

It is common knowledge that we should not drive too fast, since the braking distance s of a vehicle increases with the square of the velocity v_0 , because

$$\frac{1}{2} m v_0^2 = -F s = |F| s . \quad \{2\}$$

This is the conservation of energy, considered in the rest system of Earth, in which we are now going to analyse this process, which brings the car to a standstill. v_0 denotes the initial speed relative to the ground, m the vehicle mass, and F is the negative frictional force between the sliding car and the road, assumed to be constant during the full braking process. F disappears before and after braking. The braking distance is

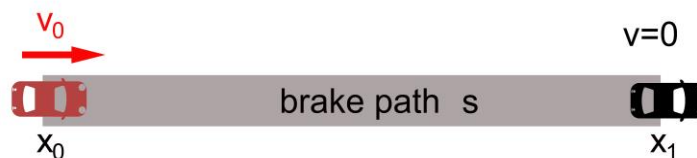


Fig. 2: Car immediately before (left) and after complete braking (right)

We ignore air friction and other complications. The braking distance s is the difference of the car's coordinates at the end (x_1) and at the beginning (x_0) of the braking process, see Fig. 2:

$$s = x_1 - x_0 .$$

Let the corresponding times be t_1 and t_0 . The kinetic energy of the car after the complete deceleration from v_0 to $v_1=0$ is zero, therefore there is no corresponding term in the above equation {2}. If $0 \leq v_1 < v_0$, then we must generalise to

$$E_0 = \frac{1}{2} m v_0^2 + 0 = \frac{1}{2} m v_1^2 - F s = E_1 .$$

On the left side is the sum of kinetic energy and "friction work" (here =0) before the braking process, E_0 . On the right side there are the energies at the end of the braking process, adding up to a total of E_1 . Here, $F < 0$ still applies.

Now we change the reference frame and look at the full braking process from there. The obvious choice is the viewpoint of the driver of a second car, initially going next to the first one at the same speed v_0 but not braking. This represents an inertial system, too. At the start of the braking process let $t'_0 = t_0 = 0$ and $x'_0 = x_0$. Now $v'_0 = 0$ and $v'_1 = -v_0$ apply to the braking car in the new reference frame. There are no fundamental changes in the calculation:

$$0 = E_1 - E_0 = \frac{1}{2} m (-v_0)^2 - F x'_1 - 0 + F x'_0$$

or

$$\frac{1}{2} m (-v_0)^2 = F (x'_1 - x'_0) .$$

Note that the force, which according to Newton is proportional to the acceleration, does not change in Galilean transformations, i.e. $F' = F < 0$. Therefore, we set $s' = x'_1 - x'_0$ in the new system, which results in

$$F s' = + \frac{1}{2} m v_0^2 . \quad \{3\}$$

This result equals that of {2} – however, $s' < 0$ because of $F < 0$. We cannot just remove the wrong sign of s' – but instead an alarm bell should be set ringing. We have to be aware that the braking distance is only defined in the rest system of planet Earth. In other reference systems, the difference $s' = x'_1 - x'_0$ can take on any value and, in particular, can become negative – which makes no sense for a braking distance. The braking distance s therefore only has a well-defined meaning in the reference system Earth. So, how can we now calculate the required braking distance s from {3}? Obviously, we will have to transform the coordinates accordingly back to the Earth system:

$$x' = x - w t \quad x = x' + w t ,$$

with $w = v_0$. Therefore:

$$s = x_1 - x_0 = (x'_1 + w t_B) - (x'_0 + w \cdot 0) .$$

If we calculate the Galilean invariant braking duration

$$t_B = t'_1 - t'_0 = t_1 - t_0 = -v_0 \frac{m}{F} = v_0 \frac{m}{|F|}$$

and insert it, we arrive at

$$s = x_1 - x_0 = s' - v_0^2 \frac{m}{F} = -\frac{1}{2} v_0^2 \frac{m}{F}$$

This is the expected result - with the correct sign!

4.2 Accelerating

Now the car is to be accelerated from $\mathbf{v} = 0$ to $\mathbf{v} = \mathbf{v}_0$ – initially again in the Earth's system of rest. We assume that the engine power $\mathbf{P} = \mathbf{F} \cdot \mathbf{v}$ is constant, with F being the time-dependent accelerating force and v the current velocity relative to the Earth. Apparently, the force F decreases as the car speeds up – even without air friction, which we again neglect. This is why you can accelerate strongly when starting off, but not at high speed on the motorway, regardless of air resistance. With our assumptions this means that the acceleration process is by no means a braking process running backwards in time, because for the latter we had assumed a constant force, according to the known laws for sliding friction.

If we change to another reference frame moving with fixed velocity w relative to the Earth, we get different values for the velocities, $v' = v - w$. We can calculate the change in kinetic energy caused by an acceleration in both reference systems - and find a similar paradox as with the spring pendulum: the increase in kinetic energy during the acceleration process depends on w ! The solution to this puzzle is similar to that of the pendulum: here, too, the movement of the Earth must be taken into account for the result to be correct.

5. Example 3: Angular momentum of point masses

With respect to the origin of the coordinate system, the orbital angular momentum for a single point mass is defined by

$$\vec{L} = \vec{r} \times \vec{p} ,$$

with \vec{r} denoting the position vector pointing from the zero point of the coordinate system to the point mass and $\vec{p} = m \vec{v}$ the particle momentum.

5.1 Linear movement

The simplest case is a force-free point mass that consequently moves uniformly in a straight line in an inertial system, at constant velocity \vec{v} and fixed momentum $\vec{p} = m \vec{v}$:

$$\vec{r} = \vec{r}_0 + \vec{v} t .$$

At this point already, students often have problems understanding, because although "nothing is rotating here", the angular momentum is generally not zero. The second difficulty is that, although \vec{r} is time-dependent, the cross product $\vec{r} \times \dot{\vec{r}}$ and thus the angular momentum are constant in time – as is to be expected for a closed system. The value of the angular momentum depends on the reference point of the position vector \vec{r} , which can be identified with the origin of the coordinate system:

$$\vec{L} = m \vec{r}_0 \times \vec{v} .$$

Note that the term $\vec{v} t \times \vec{p}$ vanishes.

Shifting the coordinates by the fixed vector \vec{q} results in a new value for the angular momentum:

$$\vec{L}' = (\vec{r} - \vec{q}) \times \vec{p} = \vec{L} - \vec{q} \times \vec{p} .$$

Due to the conservation of (linear) momentum, this angular momentum \vec{L}' is also constant in time.

With a „Galilei boost“ of velocity \vec{w} (and $\vec{q} = \vec{0}$) the result in the new reference system is

$$\begin{aligned} \vec{L}' &= (\vec{r} - \vec{w} t) \times m (\vec{v} - \vec{w}) \\ &= \vec{L} + m t \vec{v} \times \vec{w} - m \vec{r} \times \vec{w} \\ &= \vec{L} - m \vec{r}_0 \times \vec{w} = m \vec{r}_0 \times (\vec{v} - \vec{w}) . \end{aligned}$$

The angular momenta \vec{L}' and \vec{L} in the two inertial systems therefore differ by an additive constant vector – which, of course, is allowed. So with \vec{L} also \vec{L}' is time-independent.

5.2 Planetary system

Frequently, there are discussions of a point mass in a central field, for example in connection with our planetary system and the gravitational field of the sun. We consider the motion of a planet in an inertial system in which the centre of gravity of the sun and planet is at rest and defines the origin of the coordinate system. If we assume that the mass of the sun generating the gravitational field is infinitely large, the sun does not move and the angular momentum is constant in time, because

$$\dot{\vec{L}} = m \dot{\vec{r}} \times \vec{v} + m \vec{r} \times \dot{\vec{v}} = \vec{0} + \vec{0} .$$

This last step takes into account that we are dealing with a „central“ force $m\vec{r}$ proportional to \vec{r} , so that the torque $\vec{r} \times \vec{F}$ and the cross product $\vec{r} \times \ddot{\vec{r}} = \vec{r} \times \dot{\vec{v}}$ disappear.

So that's it!? No, because just as one may analyse the *angular momentum* in *rectilinear* motion, one may also consider the *linear momentum* \vec{p} in the *orbital motion* in question, which, however, is obviously not constant in time for a planet! Referring to the acting forces does not provide an explanation, because we are dealing with a closed system in which both momentum and angular momentum are conserved! This is not the only problem: If the reference point = origin of the coordinate system is shifted, one obtains

$$\vec{L}' = (\vec{r} - \vec{q}) \times \vec{p} = \vec{L} - \vec{q} \times \vec{p} .$$

But contrary to \vec{L} , \vec{L}' is not constant in time! Is it possible that the question of conservation of angular momentum depends on the reference point?

The solution is similar to that in Chapter 3: The solar mass M must be explicitly taken into account in the calculations, even if the planet with mass m is much "lighter", i.e. $m/M \ll 1$. Now the total angular momentum in the centre of mass system of the two celestial bodies (which represents an "exact" inertial system) is given by

$$\vec{L} = M \vec{R} \times \vec{V} + m \vec{r} \times \vec{v} . \quad \{4\}$$

with the capitalized quantities denoting the mass, position vector and velocity of the sun. The centre of mass system is defined by

$$M \vec{R} + m \vec{r} = \vec{0} .$$

Consequently, we obtain conservation of momentum:

$$M \vec{V} + m \vec{v} = \vec{0} .$$

The total angular momentum {4} is constant in time, as can easily be calculated by forming the time derivative. Note that the planetary motion (and not the solar one) makes the dominant contribution to \vec{L} , because

$$M \vec{R} \times \vec{V} = -m \vec{r} \times \vec{V} = \frac{m}{M} m \vec{r} \times \vec{v} .$$

The next step consists in checking what happens when the coordinate system is shifted by \vec{q} (without boost):

$$\vec{L}' = M (\vec{R} - \vec{q}) \times \vec{V} + m (\vec{r} - \vec{q}) \times \vec{v}$$

Using the conservation of momentum, we obtain

$$\vec{L}' = \vec{L} + [M \vec{V} + m \vec{v}] \times \vec{q} = \vec{L}$$

The equations are consistent now.

5.3 Planetary system and Galilean boost

Now that the basic questions have been answered, we finally examine how Galilean transformations with $\vec{w} \neq \vec{0}$ change the angular momentum. For the sake of simplicity we put $\vec{q} = \vec{0}$. In the boosted reference frame we now have

$$\vec{L}' = M(\vec{R} - \vec{w}t) \times (\vec{V} - \vec{w}) + m(\vec{r} - \vec{w}t) \times (\vec{v} - \vec{w}).$$

Combining similar terms leads to

$$\vec{L}' = \vec{L} - [M\vec{R} + m\vec{r}] \times \vec{w} + [M\vec{V} + m\vec{v}] \times \vec{w}t = \vec{L}.$$

Here again we have taken advantage of the fact that the quantities \vec{R}, \vec{r} and \vec{V}, \vec{v} are defined in the centre of mass system.

In particular, \vec{L}' as well as \vec{L} are constant in time. This only “works” because we have not omitted the terms proportional to M !

Finally, one may wonder why, so far, we have always found $\vec{L}' = \vec{L}$ for the planetary system. Can that hold for any Galilean transformation? The answer is no – if one applies {1}, i.e. a coordinate shift and a boost at the same time, there will be additional terms proportional to $\vec{q} \times \vec{w}$, which are constant in time.

6. Conclusion

By comparing the calculations carried out in different reference systems, it becomes clear that their complexities vary: A wise choice of reference system makes life much easier! On the other hand, by comparing the analyses in different reference systems, one often achieves a deeper understanding.

7. Acknowledgement

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