HAL open science

Singular perturbation analysis for a coupled KdV-ODE system 1

Swann Marx, Eduardo Cerpa

▶ To cite this version:

Swann Marx, Eduardo Cerpa. Singular perturbation analysis for a coupled KdV-ODE system 1. 2022. hal-03842789v1

HAL Id: hal-03842789

https://hal.science/hal-03842789v1

Preprint submitted on 7 Nov 2022 (v1), last revised 18 Nov 2022 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Singular perturbation analysis for a coupled KdV-ODE system[1]

Swann Marx[2] and Eduardo Cerpa[3]

November 8, 2022

Abstract

Asymptotic stability is with no doubts an essential property to be studied for any system. This analysis often becomes very difficult for coupled systems and even harder when different time-scales appear. The singular perturbation method allows to decouple a full system into what are called the reduced order system and the boundary layer system, to get simpler stability conditions for the original system. In the infinite-dimensional setting, we do not have a general result making sure this strategy works. This papers is devoted to this analysis for some systems coupling the Korteweg-the Vries equation and an ordinary differential equation with different time-scales. More precisely, We obtain stability results and Tikhonov-type theorems.

Keywords: Dispersive systems, time scales, perturbation, stability

1 Introduction

This paper is devoted to the stability analysis of a system composed by a Korteweg-de Vries (for short KdV) equation coupled with a scalar ordinary differential equation (ODE) with different time scales. Such a situation may appear when the control (appearing in the ODE) can only be used through a dynamics (given by the ODE), and when one of the equations is faster than the other one. More precisely, we are interested in the system

[1]This work has been partially supported by ANID Millennium Science Initiative Program through Millennium Nucleus for Applied Control and Inverse Problems NCN19-161 and STIC-Amsud project C-CAIT.

[2]LS2N, École Centrale de Nantes & CNRS UMR 6004, F-44000 Nantes, France. E-mail: swann.marx@ls2n.fr

[3]Instituto de Ingeniería Matemática y Computacional, Facultad de Matemáticas, Pontificia Universidad Católica de Chile, Avda. Vicuña Mackenna 4860, Macul, Santiago, Chile. E-mail: eduardo.cerpa@uc.cl

$$\begin{cases} \varepsilon y_t + y_x + y_{xxx} = 0, \ (t,x) \in \mathbb{R}_+ \times [0,L], \\ y(t,0) = y(t,L) = 0, \ t \in \mathbb{R}_+, \\ y_x(t,L) = az(t), \ t \in \mathbb{R}_+, \\ y(0,x) = y_0(x), \ x \in [0,L], \\ \dot{z}(t) = bz(t) + cy_x(t,0), \ t \in \mathbb{R}_+, \\ z(0) = z_0, \end{cases} \tag{1}$$ {eq:fast_KdV}

and the system

$$\begin{cases} y_t + y_x + y_{xxx} = 0, \ (t,x) \in \mathbb{R}_+ \times [0,L], \\ y(t,0) = y(t,L) = 0, t \in \mathbb{R}_+, \\ y_x(t,L) = az(t), \ t \in \mathbb{R}_+, \\ y(0,x) = y_0(x), \ x \in [0,L] \\ \varepsilon \dot{z}(t) = bz(t) + cy_x(t,0), \ t \in \mathbb{R}_+, \\ z(0) = z_0, \end{cases} \tag{2}$$ {eq:fast_ODE}

where $a, b, c \in \mathbb{R}$ and $\varepsilon > 0$. The parameter ε is supposed to be small, meaning that in (1) the KdV equation is faster than the ODE, and in (2), the ODE is faster than the KdV equation. To analyze these systems from an asymptotic stability viewpoint, we will follow techniques borrowed from the singular perturbation literature (see e.g., [16,17] for the finite-dimensional case, [9,30,31] for the infinite-dimensional case). Roughly speaking, this technique proposes to decouple the full system into two approximated systems assuming that ε is sufficiently small. The approximated slow system is called the reduced order system while the approximated fast one is called the boundary layer system. It is known that, in the finite-dimensional case, if both systems are asymptotically stable, then the full-system is asymptotically stable as well for sufficiently small ε. In general, this is no longer the case in the infinite dimensional case, as illustrated in [8, 30] for some hyperbolic equations coupled with an ODE. Therefore, the singular perturbation techniques become very challenging for infinite-dimensional systems, even in the linear case.

Regarding the partial differential part of our systems, we note that even in the case where the KdV equation is not coupled with any ODE, the asymptotic stability analysis is not trivial at all. Indeed, if $L \in \mathcal{N}$, with

$$\mathcal{N} := \left\{ 2\pi\sqrt{\tfrac{k^2+kl+l^2}{3}} : k, l \in \mathbb{N} \right\}, \tag{3}$$

then the equilibrium point 0 of the KdV equation becomes stable, but not attractive, while, if $L \notin \mathcal{N}$, 0 is exponentially stable. In fact, this is linked to a lack of observability. With Neumann boundary control (i.e., a control that is acting on $y_x(t,L)$), the system is not controllable if $L \in \mathcal{N}$, as shown in [26]. However, when looking at the nonlinear version of the KdV equation, one has better controllability results for any $L \in \mathcal{N}$ [3, 6, 11, 12] and better stability results for some $L \in \mathcal{N}$ [10, 14, 23, 28]. In addition to these interesting results, let us mention [1, 5, 13, 19, 29] which propose to apply the backstepping method on the KdV

equation with various boundary control, [7] where a feedback-law is designed thanks to a Gramian methodology, [20] which deals with a saturated distributed control, [4, 27] which propose both a survey about the Kdv equation, or [2] where a PI controller is designed to achieve output regulation. This latter article is interesting because it is based on the forwarding method (see e.g., [21] for the finite-dimensional case, and [18, 32, 33] for some extensions to the infinite-dimensional case), which requires the existence of an ISS-Lyapunov functional (see e.g., [22] for an introduction on ISS). In [2], an ISS-Lyapunov functional is built thanks to some strictification technique borrowed from [25] at the price of assuming that $L \notin \mathcal{N}$. This Lyapunov functional, which was not available before [2] will be crucial to analyze (1) and (2) following the classical procedure of the singular perturbation analysis. Hence, all along the paper, we will assume that

$$L \notin \mathcal{N}. \tag{4}$$ {eq:assume}

In this article, we have several contributions. First, for each coupled system (1) and (2), we propose some conditions on the parameters a, b and c so that the exponential stability is ensured for any $\varepsilon > 0$. For each of the systems, different conditions will be given, because we are going to use different Lyapunov functionals for (1) and (2). Second, for each coupled system (1) and (2), we apply the singular perturbation analysis to find he boundary layer system and the reduced order system. The stability of these subsystems will imply the stability of the original system as soon as ε is small enough. Third, for each coupled system (1) and (2), we provide an analysis of the asymptotic behavior of the solutions with respect to ε by obtaining some Tikhonov theorems. To the best of our knowledge, this is the first time that a singular perturbation analysis is applied on a KdV equation, from a control viewpoint.

This article is divided into five sections. Section 2 is devoted to state and prove the well-posedness and stability results for (1) and (2) for any value of the parameter ε. In Section 3 and Section 4 we provide an asymptotic analysis of (1) and (2), respectively, by applying singular perturbation analysis for small values of the parameter ε. Section 5 collects some concluding remarks. Appendix A recalls a crucial result borrowed from [2] for the KdV equation subject to disturbances.

2 Analysis for any value of ε

sec:well-posed

2.1 Well-posedness

This short section deals with the well-posedness of (1) and (2) for any parameter a, b, c and ε. We state and prove that there exists a unique solution to both equations. Our proof relies on classical semigroup arguments. Without loss of generality, we assume that $\varepsilon = 1$, because, in the well-posedness proof, this parameter does not play any role. Thus, we can deal in a

unified way with both systems (1) and (2) studying

$$\begin{cases} y_t + y_x + y_{xxx} = 0, \ (t,x) \in \mathbb{R}_+ \times [0,L] \\ y(t,0) = y(t,L) = 0, \ t \in \mathbb{R}_+ \\ y_x(t,L) = az(t), \ t \in \mathbb{R}_+ \\ y(0,x) = y_0(x), \ x \in [0,L] \\ \dot{z}(t) = bz(t) + cy_x(t,0), \ t \in \mathbb{R}_+ \\ z(0) = z_0. \end{cases} \tag{5}$$ {eq:single}

Theorem 1. *Let $a, b, c \in \mathbb{R}$. For any initial condition $(y_0, z_0) \in H^3(0,L) \times \mathbb{R}$ satisfying the compatibility conditions $y_0(0) = y_0(L) = 0$ and $y_0'(L) = az_0$, there exists a unique strong solution $y \in C(\mathbb{R}_+; H^3(0,L)) \cap C^1(\mathbb{R}_+; L^2(0,L))$ of (5). Additionally, for any initial condition $(y_0, z_0) \in L^2(0,L) \times \mathbb{R}$ there exists a unique weak solution $y \in C(\mathbb{R}_+; L^2(0,L))$ to system (5).*

Proof. Applying [15, Corollary 2.2.3], we will prove the well-posedness of (5). To do so, we focus on the operator

$$\mathcal{A} : \ D(\mathcal{A}) \subset L^2(0,L) \times \mathbb{R} \to L^2(0,L) \times \mathbb{R},$$

where $D(\mathcal{A}) := \{(y,z) \in H^3(0,L) \times \mathbb{R} \mid y(0) = y(L), \ y'(L) = az\}$ and

$$\mathcal{A} \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} -y' - y''' \\ bz + cy'(0) \end{pmatrix}. \tag{6}$$

Our goal is to prove that there exists $\omega > 0$ such that $\mathcal{A} - \omega \mathrm{I}_{L^2(0,L)}$ and its adjoint operator generate a strongly continuous semigroup of contractions, where $\mathrm{I}_{L^2(0,L)}$ denotes the identity operator in $L^2(0,L)$. As explained in [15, Corollary 2.3.3], and noticing moreover that $\mathcal{A}$ is a closed operator, such a condition is sufficient to prove that $\mathcal{A}$ generates a strongly continuous semigroup. Consider in $L^2(0,L) \times \mathbb{R}$ the scalar product

$$\left\langle \begin{pmatrix} y_1 \\ z_1 \end{pmatrix}, \begin{pmatrix} y_2 \\ z_2 \end{pmatrix} \right\rangle = \int_0^L y_1 y_2 dx + z_1 z_2. \tag{7}$$

Doing some integrations by parts, one obtains, for all $(y,z) \in D(\mathcal{A})$

$$\left\langle \mathcal{A} \begin{pmatrix} y \\ z \end{pmatrix}, \begin{pmatrix} y \\ z \end{pmatrix} \right\rangle = 2a^2 z^2 - 2y'(0)^2 + 2bz^2 + 2czy'(0). \tag{8}$$

Using Young's Lemma one obtains, for $(y,z) \in D(\mathcal{A})$

$$\left\langle \mathcal{A} \begin{pmatrix} y \\ z \end{pmatrix}, \begin{pmatrix} y \\ z \end{pmatrix} \right\rangle \leq 2a^2 z^2 - 2y'(0)^2 + 2bz^2 + 2\frac{1}{\alpha} c^2 z^2 + 2\alpha y'(0)^2. \tag{9}$$

If one takes $\alpha = \frac{1}{4}$, one can prove easily that there exists a positive constant C such that, for

all $(y,z) \in D(\mathcal{A})$

$$\left\langle \mathcal{A}\begin{pmatrix} y \\ z \end{pmatrix}, \begin{pmatrix} y \\ z \end{pmatrix} \right\rangle \leq C(\|y\|^2_{L^2(0,L)} + z^2). \tag{10}$$

Then, for any $\omega > C$, one has that $\mathcal{A} - \omega \mathrm{I}_{L^2(0,L)}$ is dissipative.

One can prove that the adjoint operator of $\mathcal{A}$, denoted by $\mathcal{A}^*$, is defined as

$$\mathcal{A}^* : \ D(\mathcal{A}^*) \subset L^2(0,L) \times \mathbb{R} \to L^2(0,L) \times \mathbb{R},$$

where $D(\mathcal{A}^*) := \{(y,z) \in H^3(0,L) \times \mathbb{R} \mid y(0) = y(L),\ y'(0) = cz\}$ and

$$\mathcal{A}^* \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} -y' - y''' \\ bz + ay'(L) \end{pmatrix}. \tag{11}$$

Using the same scalar product than before, and performing some integrations by parts, one has, for all $(y,z) \in D(\mathcal{A}^*)$ that

$$\left\langle \mathcal{A}^* \begin{pmatrix} y \\ z \end{pmatrix}, \begin{pmatrix} y \\ z \end{pmatrix} \right\rangle = 2c^2z^2 - 2y'(L)^2 + 2bz^2 + 2azy'(L). \tag{12}$$

Again, thanks to the Young's inequality, one can prove that

$$\left\langle \mathcal{A}^* \begin{pmatrix} y \\ z \end{pmatrix}, \begin{pmatrix} y \\ z \end{pmatrix} \right\rangle \leq 2c^2z^2 - 2y'(L)^2 + 2bz^2 + \frac{2}{\alpha}a^2z^2 + 2\alpha y'(L)^2. \tag{13}$$

Setting $\alpha = \frac{1}{4}$, one can prove that there exists a positive constant C such that, for all $(y,z) \in D(\mathcal{A}^*)$

$$\left\langle \mathcal{A}^* \begin{pmatrix} y \\ z \end{pmatrix}, \begin{pmatrix} y \\ z \end{pmatrix} \right\rangle \leq C(\|y\|^2_{L^2(0,L)} + z^2). \tag{14}$$

Then, for any $w > C$, one can prove that $\mathcal{A}^* - \omega \mathrm{I}_{L^2(0,L)}$ is dissipative. Then, applying the Lumer-Phillips Theorem [24, Corollary 4.4, Chapter 1], one can deduce the result. □

2.2 Stability conditions for system (1)

Here we fix any $\varepsilon > 0$ and give some conditions on a, b and c such that the origin is globally exponentially stable for system (1). To do so, we have to first introduce a Lyapunov functional, inspired by [2]. This Lyapunov functional has been built thanks to the forwarding method, first designed for finite-dimensional systems [21], and later extended to some infinite-dimensional systems [2, 18, 32, 33]. It is defined as

$$V_1(y,z) = \varepsilon W(y) + \frac{1}{2}\left(\varepsilon \int_0^L M(x)y(t,x)dx - z(t)\right)^2, \tag{15}$$ {eq:Lyapunov-coupled}

where W comes from [2, Theorem 2.3] and we recall in the Appendix A. The function M is the solution to the boundary value problem

$$\begin{cases} M'''(x) + M'(x) = 0, \quad x \in (0, L), \\ M(0) = M(L) = 0, \quad M'(0) = -c, \end{cases} \tag{16}$$ {eq:bdv-M}

for which we know an explicit solution

$$M(x) = 2c \frac{\sin\left(\frac{x}{2}\right) \sin\left(\frac{L-x}{2}\right)}{\sin\left(\frac{L}{2}\right)} \in C^\infty([0, L]). \tag{17}$$ {eq:explicit-M}

This function M is defined through a Sylvester equation, as explained in [2]. Roughly speaking, the idea of the Lyapunov functional defined in (15) is to use that the fast system (i.e. the KdV equation) is already exponentially stable without coupling terms and to add a term such that z converges to the L^2-norm of y (modulo the function M, suitably chosen). This corresponds exactly to the forwarding method.

As proved in [2], this Lyapunov functional is equivalent to the usual norm, i.e. one has the following lemma whose proof is given for the sake of completeness.

lem:V1-norm **Lemma 2.** *There exist $\overline{\nu}_1$, $\underline{\nu}_1 > 0$ such that*

$$\underline{\nu}_1(\|y\|^2_{L^2(0,L)} + |z|^2) \leq V_1(y, z) \leq \overline{\nu}_1(\|y\|^2_{L^2(0,L)} + |z|^2), \tag{18}$$

with $\overline{\nu}_1 = \max\left(\varepsilon\overline{c} + \varepsilon^2\|M\|^2_{L^2(0,L)}, 1\right)$ and $\underline{\nu}_1 = \min\left(\frac{\underline{c}\varepsilon}{2}, \frac{1}{2}\frac{\underline{c}\varepsilon}{\varepsilon^2\|M\|^2_{L^2(0,L)} + \underline{c}\varepsilon}\right)$. Moreover, for $\varepsilon \leq 1$, one has the existence of a constant $C > 0$ such that

$$\varepsilon(\|y\|^2_{L^2(0,L)} + |z|^2) \leq V_1(y, z) \leq C(\|y\|^2_{L^2(0,L)} + |z|^2). \tag{19}$$

Proof. First, using Proposition 10 in Appendix A and Young's Lemma we have

$$V_1(y, z) \leq (\varepsilon\overline{c} + \varepsilon^2\|M\|^2_{L^2(0,L)})\|y\|_{L^2(0,L)} + |z|^2. \tag{20}$$

Second, using again Proposition 10 and Young's Lemma we get

$$V_1(y, z) \geq \underline{c}\varepsilon\|y\|^2_{L^2(0,L)} + \frac{1}{2}\left(1 - \frac{1}{\alpha}\right)\varepsilon^2 \int_0^L M(x)^2 y(t, x)^2 + \frac{1}{2}(1 - \alpha)z(t)^2. \tag{21}$$

Choose $\alpha = \frac{\varepsilon^2\|M\|^2_{L^2(0,L)}}{\varepsilon^2\|M\|^2_{L^2(0,L)} + \underline{c}\varepsilon}$. Then, $1 - \frac{1}{\alpha} < 0$, and one has

$$\begin{aligned} &V_1(y, z) \geq \\ &\quad \underline{c}\varepsilon\|y\|^2_{L^2(0,L)} - \frac{1}{2}\left(\frac{\underline{c}\varepsilon}{\varepsilon^2\|M\|_{L^2(0,L)}}\right)\varepsilon^2\|M\|^2_{L^2(0,L)}\|y\|^2_{L^2(0,L)} + \frac{1}{2}\frac{\underline{c}\varepsilon}{\varepsilon^2\|M\|^2_{L^2(0,L)} + \underline{c}\varepsilon}z^2. \end{aligned} \tag{22}$$

This concludes the proof. □

We are now ready to state and prove our stability result.

prop:coupled **Proposition 3.** *For any $\varepsilon > 0$, there exist positive constants a_*, k_1, k_2 such that, if $a < a_*$ and b, c satisfy $0 < k_1 < -(b - ac) < k_2$, then the origin is globally exponentially stable for system* (1).

Proof. Using Proposition 10, setting $d_1 = 0$ and $d_2 = z$, the time derivative of V along the strong solutions to (1) yields

$$\frac{d}{dt}V(y,z) \leq -\lambda\|y\|^2_{L^2(0,L)} + \kappa_2 a^2 z(t)^2 + \left(\int_0^L M(y_x(t,x) + y_{xxx}(t,x)) - bz(t) - cy_x(t,0)\right)\left(\varepsilon\int_0^L My(t,x)dx - z(t)\right). \quad (23)$$

After some integration by parts, and using in particular that $M'(L) = c$ thanks to (17), one obtains that for all strong solutions to (1) we have

$$\begin{aligned}\frac{d}{dt}V_1(y,z) &\leq -\lambda\|y\|^2_{L^2(0,L)} + \kappa_2 a^2 z(t)^2 - (b-ac)z(t)\left(\varepsilon\int_0^L M(x)y(t,x)dx - z(t)\right)\\ &\leq -\lambda\|y\|^2_{L^2(0,L)} + \kappa_2 a^2 z(t)^2 + (b-ac)z(t)^2 - \varepsilon(b-ac)z(t)\int_0^L M(x)y(t,x)dx. \quad (24)\end{aligned}$$

Using Young's Lemma, one obtains that, for all strong solutions to (1)

$$\frac{d}{dt}V_1(y,z) \leq (-\lambda + \alpha\varepsilon^2\|M\|^2_{L^2(0,L)})\|y\|^2_{L^2(0,L)} + \left(\frac{(b-ac)^2}{\alpha} + (b-ac) + \kappa_2 a^2\right)z(t)^2. \quad (25)$$

Let us choose $\alpha = \frac{\lambda}{2\|M\|^2_{L^2(0,L)}\varepsilon^2}$. One has therefore

$$\frac{d}{dt}V_1(y,z) \leq -\frac{\lambda}{2}\|y\|^2_{L^2(0,L)} + \left(\frac{(b-ac)^2}{\alpha} + (b-ac) + \kappa_2 a^2\right)z(t)^2. \quad (26)$$

Let us consider the polynomial $\frac{X^2}{\alpha} - X + \kappa_2 a^2$. If $a^2 < \frac{\alpha}{4\kappa_2}$, this polynomial admits two square roots, defined by

$$X_1 = \frac{\alpha\left(1 - \sqrt{1 - \frac{4\kappa_2 a^2}{\alpha}}\right)}{2},\ X_2 = \frac{\alpha\left(1 + \sqrt{1 - \frac{4\kappa_2 a^2}{\alpha}}\right)}{2}.$$

Then, if $b - ac$ satisfies

$$X_1 < -(b-ac) < X_2,$$

then, there exists a positive constant μ such that, for all strong solutions to (1) we have

$$\frac{d}{dt}V_1(y,z) \leq -\mu V_1(y,z). \quad (27)$$

Using Lemma 2 we conclude the proof. □

One might see this result as an extension of the one provided in [2] where one has $b = 0$ and $c = \varepsilon = 1$, which corresponds to the case where an integrator is added. In [2], it is proved that, for a sufficiently small a, the origin of (1) (with $b = 0$, $c = \varepsilon = 1$) is exponentially stable. Therefore, Proposition 3 seems to follow the same line, since a has to be sufficiently small.

2.3 Stability conditions for system (2)

In this subsection, a sufficient conditions on a, b and c will be found to ensure the stability of (2) for any $\varepsilon > 0$. To do so, we use the Lyapunov functional

$$V_2(y, z) := -\frac{\varepsilon\kappa_2 a^2}{b} z^2 + W(y), \tag{28}$$ {eq:Lyapunov-fastODE}

where W is the ISS-Lyapunov functional given in Proposition 10. The following Lemma states that this Lyapunov functional is equivalent to the usual norm.

ov-fastODE-equi **Lemma 4.** *For any $b < 0$, defining $\overline{\nu}_2 := \max\left(\bar{c}, -\frac{\varepsilon\kappa_2 a^2}{b}\right)$ and $\underline{\nu}_2 := \min\left(\underline{c}, -\frac{\varepsilon\kappa_2 a^2}{b}\right)$, the Lyapunov functional defined in* (28) *satisfies*

$$\underline{\nu}_2(\|y\|^2_{L^2(0,L)} + |z|^2) \leq V_2(y, z) \leq \overline{\nu}_2(\|y\|^2_{L^2(0,L)} + |z|^2). \tag{29}$$

Proof. Using Proposition 10, one first has

$$V_2(y, z) \leq \overline{c}\|y\|^2_{L^2(0,L)} - \frac{\varepsilon\kappa_2 a^2}{b} \leq \overline{\nu}_2(\|y\|^2_{L^2(0,L)} + |z|^2), \tag{30}$$

where $\overline{\nu}_2 = \max\left(\bar{c}, -\frac{\varepsilon\kappa_2 a^2}{b}\right)$. Using again Proposition 10, one obtain

$$V_2(y, z) \geq \underline{c}\|y\|^2_{L^2(0,L)} - \frac{\varepsilon\kappa_2 a^2}{b} \geq \underline{\nu}_2(\|y\|^2_{L^2(0,L)} + |z|^2), \tag{31}$$

where $\underline{\nu}_2 = \min\left(\underline{c}, -\frac{\varepsilon\kappa_2 a^2}{b}\right)$. This concludes the proof. □

We have now the following result, which states that, for any $\varepsilon > 0$, and under suitable conditions on a, b, c, the origin is exponentially stable for system (2). As explained later on, these conditions differ from the ones collected in Proposition 3.

prop:coupled2 **Proposition 5.** *Let $\varepsilon > 0$. If $b < 0$ and $\frac{a^2c^2}{b^2} < \frac{\kappa_3}{4\kappa_2}$, then the origin is exponentially stable for system* (2)*.*

Proof. Note that, due to the condition $b < 0$, the Lyapunov functional defined in (28) is equivalent to the usual norm, invoking Lemma 4. Using Proposition 10 with $d_2 = az$, its derivative along (2) yields, for all strong solutions to (2)

$$\frac{d}{dt}V_2(y, z) = -\lambda\|y\|^2_{L^2(0,L)} + \kappa_2 a^2|z(t)|^2 - \kappa_3|y_x(t, 0)|^2 - 2\kappa_2 a^2 z^2 - 2\frac{a^2c}{b}\kappa_2 y_x(t, 0)z(t). \tag{32}$$

Using Young's Lemma, one obtains

$$\frac{d}{dt}V_2(y,z) \leq -\lambda\|y\|^2_{L^2(0,L)} - \kappa_2(1-2\alpha)a^2z^2 - \left(\kappa_3 - 2\frac{a^2c^2\kappa_2}{b^2\alpha}\right)|y_x(t,0)|^2 \tag{33}$$

Setting $\alpha = \frac{1}{4}$, one obtains:

$$\frac{d}{dt}V_2(y,z) \leq -\lambda\|y\|^2_{L^2(0,L)} - \frac{\kappa_2}{2}a^2z^2 - \left(\kappa_3 - \frac{8a^2c^2\kappa_2}{b^2}\right)|y_x(t,0)|^2 \tag{34}$$

Then, if $\frac{a^2c^2}{b^2} < \frac{\kappa_3}{4\kappa_2}$, and using Lemma 4, the desired result holds true, concluding therefore the proof. □

Note that the conditions given in Proposition 3 are quite different from the ones introduced in Proposition 5. Indeed, in contrast with Proposition 3, Proposition 5 assumes, with the hypothesis $b < 0$, that the ODE is already exponentially stable. As it will be illustrated later on, similar conditions will appear when looking at the reduced order system and the boundary layer system.

3 Fast KdV equation coupled with a slow ODE

sec:fast_KdV

3.1 Stability for small ε

The singular perturbation method proposes the decoupling of the different time-scales appearing in the system in order to get some subsystems that hopefully can be studied separately in order to conclude properties of the full system. Thus, we are going to compute the subsystems, namely the reduced order system and the boundary layer system, that are approximations of the KdV equation and the ODE when ε is closed to 0. We further prove that the stability conditions for those two systems apply for the full system (1) as soon as ε is small enough.

Reduced order system. Finding the reduced order system needs us to suppose that $\varepsilon = 0$. One has therefore to study this system

$$\begin{cases} h_x(t,x) + h_{xxx}(t,x) = 0, \quad t \in \mathbb{R}_+, x \in (0,L), \\ h(t,0) = h(t,L) = 0, \quad t \in \mathbb{R}_+, \\ h_x(t,L) = az(t), \quad t \in \mathbb{R}_+, \end{cases} \tag{35}$$ {eq:equi-point}

which corresponds to the KdV equation given in (1) when $\varepsilon = 0$. There exists an explicit solution to the latter equation given by

$$h(t,x) = -2az(t)\frac{1}{\sin\left(\frac{L}{2}\right)}\sin\left(\frac{x}{2}\right)\sin\left(\frac{L-x}{2}\right), \ \forall (t,x) \in \mathbb{R}_+ \times [0,L].$$

One can easily check that $h(t,0) = h(t,L) = 0$, for all $t \geq 0$. Moreover, one has, for all $(t,x) \in \mathbb{R}_+ \times [0,L]$

$$h_x(t,x) = -az(t)\frac{1}{\sin\left(\frac{L}{2}\right)}\cos\left(\frac{x}{2}\right)\sin\left(\frac{L-x}{2}\right) + az(t)\frac{1}{\sin\left(\frac{L}{2}\right)}\sin\left(\frac{x}{2}\right)\cos\left(\frac{L-x}{2}\right).$$

One has $h_x(t,0) = -az(t)$ and $h_x(t,L) = az(t)$, for all $t \geq 0$. Moreover, by definition of h, one has $h_x(t,x) + h_{xxx}(t,x) = 0$, for all $(t,x) \in \mathbb{R}_+ \times [0,L]$. In the following, we will use the following notation

$$h(t,x) := -f(x)z(t), \quad \text{where } f(x) := 2a\frac{1}{\sin\left(\frac{L}{2}\right)}\sin\left(\frac{x}{2}\right)\sin\left(\frac{L-x}{2}\right).$$

Therefore, since $h_x(t,0) = -az(t)$, the reduced order system is given by

$$\begin{cases} \dot{\bar{z}}(t) = (b-ac)\bar{z}(t), \quad t \in \mathbb{R}_+, \\ \bar{z}(0) = \bar{z}_0. \end{cases} \tag{36}$$

{eq:reduced-order}

In consequence, if $(b-ac) < 0$, then the origin of (36) is exponentially stable.

Boundary layer system. Consider $\tau = \frac{t}{\varepsilon}$ and $\bar{y}(\tau,x) := y(\tau,x) + z(\tau)f(x)$, for all $(\tau,x) \in \mathbb{R}_+ \times [0,L]$. One can check that

$$\bar{y}_\tau(\tau,x) = y_\tau(\tau,x) + \varepsilon\frac{d}{dt}f(x)z(t),$$

for all $(\tau,x) \in \mathbb{R}_+ \times [0,L]$. Setting $\varepsilon = 0$ yields $\bar{y}_\tau(\tau,x) = y_\tau(\tau,x)$. One can check also easily that $\bar{y}_x(\tau,x) + \bar{y}_{xxx}(\tau,x) = y_x(\tau,x) + y_{xxx}(\tau,x)$, for all $(\tau,x) \in \mathbb{R}_+ \times [0,L]$. One has also $\bar{y}(\tau,0) = \bar{y}(\tau,L) = 0$ and $\bar{y}_x(\tau,L) = 0$. Finally, the boundary layer is written as

$$\begin{cases} \bar{y}_\tau(\tau,x) + \bar{y}_x(\tau,x) + \bar{y}_{xxx}(\tau,x) = 0, \quad \tau \in \mathbb{R}_+, x \in (0,L), \\ \bar{y}(\tau,0) = \bar{y}(\tau,L) = 0, \quad \tau \in \mathbb{R}_+, \\ \bar{y}_x(\tau,L) = 0, \quad \tau \in \mathbb{R}_+, \\ \bar{y}(0,x) = \bar{y}_0(x) \quad x \in (0,L). \end{cases} \tag{37}$$

{eq:boundary-layer}

Since $L \notin \mathcal{N}$, the origin is always exponentially stable for system (37).

Full system. Next result will say that the conditions for the reduced order system and the boundary layer system to be exponentially stable are sufficient for the full-system as soon as ε is sufficiently small. It is useful to introduce the variable

$$\tilde{y}(t,x) = y(t,x) + f(x)z(t), \quad \forall (t,x) \in \mathbb{R}_+ \times [0,L].$$

Noticing that $\tilde{y}_x(t,0) = y_x(t,0) + az(t)$, its dynamics together with the one of z is given by

$$\begin{cases} \varepsilon\tilde{y}_t + \tilde{y}_x + \tilde{y}_{xxx} = -\varepsilon((b-ac)z(t) + c\tilde{y}_x(t,0))f(x) \\ \tilde{y}(t,0) = \tilde{y}(t,L) = 0 \\ \tilde{y}_x(t,L) = 0 \\ \tilde{y}(0,x) = \tilde{y}_0(x) \\ \dot{z} = (b-ac)z(t) + c\tilde{y}_x(t,0) \\ z(0) = z_0. \end{cases} \tag{38}$$

{eq:sys-fastKdVthm1}

We can now state and prove the next result.

thm:fast-KdV1

Theorem 6. *For any $a, b, c \in \mathbb{R}$ such that $(b-ac) < 0$, there exists $\varepsilon^* > 0$ such that, for every $\varepsilon \in (0, \varepsilon^*)$ the origin is exponentially stable for system* (1).

Proof. We consider the Lyapunov functional (15). Applying Proposition 10 with $d_1(t,x) = -\varepsilon((b-ac)z(t) + c\tilde{y}_x(t,0))f(x)$ and $d_2(t) = 0$, one obtains that all strong solutions to (38) satisfy

$$\begin{aligned} \frac{d}{dt}V_1(\tilde{y},z) \leq& -\lambda\|\tilde{y}\|^2_{L^2(0,L)} + \varepsilon^2\kappa_1\|f\|^2_{L^2(0,L)}((b-ac)z(t) + c\tilde{y}_x(t,0))^2 - \kappa_3\tilde{y}_x(t,0)^2 \\ &+ (-(b-ac)z(t) + K\varepsilon((b-ac)z(t) + c\tilde{y}_x(t,0))) \cdot \left(\varepsilon\int_0^L M(x)\tilde{y}(t,x)dx - z(t)\right), \end{aligned} \tag{39}$$

where $K := \int_0^L M(x)f(x)dx$. Using Young's Lemma several times one obtains, that for all strong solutions to (38)

$$\begin{aligned} \frac{d}{dt}V_1(\tilde{y},z) \leq& \left(-\lambda + \alpha_1\|M\|^2_{L^2(0,L)} + \alpha_2\varepsilon^2\|M\|^2_{L^2(0,L)}\right)\|\tilde{y}\|^2_{L^2(0,L)} \\ &+ \left((b-ac) + (b-ac)^2\left(2\varepsilon^2\kappa_1\|f\|^2_{L^2(0,L)} + \frac{\varepsilon^2}{\alpha_1} + 2K^2\varepsilon^2\left(\frac{1}{\alpha_2} + \frac{1}{\alpha_3}\right)\right) + \alpha_3\right)z(t)^2 \\ &+ \left(2\varepsilon^2\kappa_1\|f\|^2_{L^2(0,L)} + K^2c^2\varepsilon^2\left(\frac{2}{\alpha_2} + \frac{2}{\alpha_3}\right) - \kappa_3\right)\tilde{y}_x(t,0)^2. \end{aligned} \tag{40}$$

Selecting $\alpha_3 = -\frac{-(b-ac)}{2}$, setting $\alpha_4 = \frac{1}{\alpha_2} + \frac{1}{\alpha_3}$, and choosing ε and α_1 and α_2 satisfying

$$\varepsilon^2 < \min\left(\frac{\kappa_3}{2\kappa_1\|f\|^2_{L^2(0,L)} + 2K^2c^2\alpha_4}, \frac{1}{2(ac-b)\left(2\kappa_1\|f\|^2_{L^2(0,L)} + \frac{1}{\alpha_1} + 2K^2\varepsilon^2\alpha_4\right)}\right) \tag{41}$$

and

$$-\lambda + \alpha_1\|M\|^2_{L^2(0,L)} + \alpha_2\varepsilon^2\|M\|^2_{L^2(0,L)} < 0, \tag{42}$$

one obtains that there exists $\mu > 0$ such that

$$V_1(\tilde{y},z) \leq e^{-\mu t}V(\tilde{y}_0, z_0), \quad \forall t \geq 0. \tag{43}$$

Using Lemma 2, one deduces the desired result. □

3.2 Tikhonov theorem

The most relevant part of the singular perturbation method is to use the obtained subsystems in order to approximate the dynamics of the full system. This section is devoted to this more precise analysis of the asymptotic behavior of the solutions with respect to the variable ε. To do so, we will follow the Tikhonov strategy that has been used for instance in [9, 31] for partial differential equations. We introduce the error solutions

$$\hat{z}(t) = z(t) - \bar{z}(t) \tag{44}$$

and

$$\hat{y}(t,x) = y(t,x) + f(x)\bar{z}(t) - \bar{y}\left(\frac{t}{\varepsilon}, x\right). \tag{45}$$

Using the solutions of (1), (36), (35) and (37). One can verify that

$$\dot{\hat{z}}(t) = bz(t) + cy_x(t,0) - (b-ac)\bar{z}(t).$$

Noticing that $\hat{y}_x(t,x) = y_x(t,x) + f'(x)\bar{z}(t) - \bar{y}_x\left(\frac{t}{\varepsilon}, x\right)$, one has

$$\hat{y}_x(t,0) = y_x(t,0) - a\bar{z}(t) - \bar{y}_x\left(\frac{t}{\varepsilon}, 0\right)$$

and because $\dot{\hat{z}}(t) = b(\bar{z}(t) - z(t)) + c\bar{y}_x\left(\frac{t}{\varepsilon}, 0\right)$, we get

$$\dot{\hat{z}}(t) = b\hat{z}(t) + c\hat{y}_x(t,0) + c\bar{y}_x\left(\frac{t}{\varepsilon}, 0\right). \tag{46}$$ {eq:EDO-Tikhonov}

Moreover,

$$\varepsilon\hat{y}_t(t,x) = \varepsilon y_t(t,x) + \varepsilon f(x)(b-ac)\bar{z}(t) - \bar{y}_t\left(\frac{t}{\varepsilon}, x\right). \tag{47}$$

Using the dynamics of y (given in (1)) and the one of $\bar{y}$ (given in (37)), one obtains

$$\varepsilon\hat{y}_t(t,x) = -y_x - y_{xxx} + \bar{y}_x\left(\frac{t}{\varepsilon}, x\right) + \bar{y}_{xxx}\left(\frac{t}{\varepsilon}, x\right) + \varepsilon f(x)(b-ac)z(t). \tag{48}$$

Note that

$$\hat{y}_x + \hat{y}_{xxx} = y_x + y_{xxx} - \bar{y}_x\left(\frac{t}{\varepsilon}, x\right) - \bar{y}_{xxx}\left(\frac{t}{\varepsilon}, x\right) + \bar{z}(t)(f'(x) + f'''(x)).$$

Recall that $h(t,x) = -\bar{z}(t)f(x)$ and that h solves (35), i.e. $\bar{z}(t)(f'(x) + f'''(x)) = 0$. Hence, one has

$$\varepsilon\hat{y}_t(t,x) = -\hat{y}_x - \hat{y}_{xxx} + \varepsilon f(x)(b-ac)\bar{z}(t). \tag{49}$$

Using the boundary conditions given in (1), (37) and (35), one has

$$\hat{y}(t,0) = \hat{y}(t,L) = 0, \ \forall t \geq 0.$$

Having in mind that $\hat{y}_x(t,L) = y_x(t,L) + f'(L)\bar{z}(t) - \bar{y}_x\left(\frac{t}{\varepsilon}, L\right) = az(t) - a\bar{z}(t) = a\hat{z}(t)$, one can write the system

$$\begin{cases} \varepsilon\hat{y}_t + \hat{y}_x + \hat{y}_{xxx} = \varepsilon f(x)(b-ac)\bar{z}(t), \quad t \in \mathbb{R}_+, x \in (0,L), \\ \hat{y}(t,0) = \hat{y}(t,L) = 0, \quad t \in \mathbb{R}_+, \\ \hat{y}_x(t,L) = a\hat{z}(t), \quad t \in \mathbb{R}_+, \\ \hat{y}(0,x) = y_0(x) - \bar{y}_0(x) + f(x)\bar{z}(0), \quad x \in (0,L), \\ \dot{\hat{z}} = b\hat{z}(t) + c\hat{y}_x(t,0) + c\bar{y}_x(t/\varepsilon,0), \quad t \in \mathbb{R}_+, \\ \hat{z}(0) = z_0 - \bar{z}_0. \end{cases} \tag{50}$$

{eq:Tikhonov-KdV}

We are now in position to state our first Tikhonov theorem.

Theorem 7. *There exist positive constants a_*, k_1, k_2 and ε^* such that if $a < a_*$, b, c satisfy $0 < k_1 < -(b-ac) < k_2$ and $\varepsilon < \varepsilon^*$, then for any initial conditions $(y_0, z_0), (\bar{y}_0, \bar{z}_0) \in L^2(0,L) \times \mathbb{R}$ such that*

$$\|y_0 - \bar{y}_0 + fz_0\|_{L^2(0,L)} + |z_0 - \bar{z}_0| = O(\varepsilon^{\frac{3}{2}}), \qquad |\bar{z}_0| = O(\varepsilon^{\frac{1}{2}}), \quad \|\bar{y}_0\|_{L^2(0,L)} = O(\varepsilon^{\frac{3}{2}}),$$

we have that the solutions of (1) *satisfy for some $\mu > 0$ that*

$$\|y(t,\cdot) - \bar{y}(t/\varepsilon,\cdot) + f(\cdot)z(t)\|_{L^2(0,L)} + |z(t) - \bar{z}(t)| = O(\varepsilon)e^{-\mu t}. \tag{51}$$

Proof. Let us consider the Lyapunov functional (15). Its derivative along strong solutions to (50) yields

$$\begin{aligned} \frac{d}{dt}V_1(\hat{y},\hat{z}) \leq & -\lambda\|\hat{y}\|^2_{L^2(0,L)} + \kappa_1\varepsilon^2\|f\|^2_{L^2(0,L)}(b-ac)^2\bar{z}(t)^2 + \kappa_2 a^2\hat{z}(t)^2 \\ & + (K\varepsilon(b-ac)\bar{z}(t) - (b-ac)\hat{z}(t) + c\bar{y}_x(t,0))\left(\varepsilon\int_0^L M(x)\hat{y}(t,x)dx - \hat{z}(t)\right), \end{aligned}$$

where $K := \int_0^L f(x)M(x)dx$. Using Young's Lemma several times, one obtains for all strong solutions of (50) that

$$\begin{aligned} \frac{d}{dt}V_1(\hat{y},\hat{z}) \leq & (-\lambda + (\alpha_1\varepsilon^2 + \alpha_3 + \alpha_4)\|M\|^2_{L^2(0,L)})\|\hat{y}\|^2_{L^2(0,L)} \\ & + \left(\kappa_2 a^2 + (b-ac) + \alpha_2 + \alpha_5 + (b-ac)^2\frac{\varepsilon^2}{\alpha_3}\right)\hat{z}(t)^2 \\ & + K^2\varepsilon^2(b-ac)^2\left(\frac{1}{\alpha_1} + \frac{1}{\alpha_2}\right)\bar{z}(t)^2 + c^2\left(\frac{\varepsilon^2}{\alpha_4} + \frac{1}{\alpha_5}\right)\bar{y}_x(t,0)^2. \end{aligned}$$

One selects α_1, α_3 and α_4 such that

$$-\lambda + (\alpha_1\varepsilon^2 + \alpha_3 + \alpha_4)\|M\|^2_{L^2(0,L)} < 0$$

and α_2 and α_5 such that

$$\alpha_2 + \alpha_5 = -\frac{b-ac}{2}.$$

Then, setting $\mu_1 := \lambda - (\alpha_1\varepsilon^2 + \alpha_3 + \alpha_4)\|M\|^2_{L^2(0,L)} < \lambda$, one has

$$\frac{d}{dt}V(\hat{y},\hat{z}) \leq -\mu_1\|\hat{y}\|^2_{L^2(0,L)} + \left(\kappa_2 a^2 + \frac{(b-ac)}{2} + (b-ac)^2\frac{\varepsilon^2}{\alpha_3}\right)\hat{z}(t)^2 \\ + K^2\varepsilon^2(b-ac)^2\left(\frac{1}{\alpha_1}+\frac{1}{\alpha_2}\right)\bar{z}(t)^2 + c^2\left(\frac{\varepsilon^2}{\alpha_4}+\frac{1}{\alpha_5}\right)\bar{y}_x(t,0)^2.$$

Let us now consider the polynomial $P(X) = \kappa_2 a^2 - \frac{1}{2}X + \frac{\varepsilon^2}{\alpha_3}X^2$. If a is such that $a^2 < \frac{\alpha_3}{16\varepsilon^2\kappa_2}$, $P(X)$ admits two square roots

$$X_1 = \alpha_3\frac{1-\sqrt{1-\frac{16\varepsilon^2\kappa_2 a^2}{\alpha_3}}}{4\varepsilon^2},\ X_2 = \alpha_3\frac{1+\sqrt{1-\frac{16\varepsilon^2\kappa_2 a^2}{\alpha_3}}}{4\varepsilon^2}. \tag{52}$$

Replacing X by $ac-b$, one has $P(ac-b) < 0$ if

$$X_1 < ac - b < X_2.$$

Then, there exists $\mu_2 > 0$ such that, for all strong solutions to (50)

$$\frac{d}{dt}V(\hat{y},\hat{z}) \leq -\mu_1\|\hat{y}\|^2_{L^2(0,L)} - \mu_2\hat{z}(t)^2 + K^2\varepsilon^2(b-ac)^2\left(\frac{1}{\alpha_1}+\frac{1}{\alpha_2}\right)\bar{z}(t)^2 \\ + c^2\left(\frac{\varepsilon^2}{\alpha_4}+\frac{1}{\alpha_5}\right)\bar{y}_x\left(\frac{t}{\varepsilon},0\right)^2. \tag{53}$$ {theo:Tikhonov-before-in

Using Proposition 10, the Grönwall's Lemma and setting $\mu_3 := \min\left(\frac{\mu_1}{\overline{\nu}_1}, \frac{\mu_2}{\overline{\nu}_1}\right)$, one obtains for all $t \geq 0$ that

$$V_1(\hat{y},\hat{z}) \leq e^{-\mu_3 t}V(\hat{y}_0,\hat{z}_0) + O(\varepsilon^2)\int_0^t e^{-\mu_3(t-s)}|\bar{z}(s)|^2 ds + O(1)\int_0^t e^{-\mu_3(t-s)}\bar{y}_x^2\left(\frac{t}{\varepsilon},0\right) \tag{54}$$ {eq:inequalityV_1-Tikhon

Let us estimate the integrals appearing in (54). Using (36), one has

$$\bar{z}(t)^2 \leq e^{-(b-ac)t}\bar{z}_0^2. \tag{55}$$ {eq:inequality-barz}

Moreover, using the Lyapunov functional given in Proposition 10, one obtains for all strong solutions to (37) that

$$\frac{d}{d\tau}W(\bar{y}) \leq -\frac{\lambda}{\overline{c}}W(\bar{y}) - \kappa_3\bar{y}_x(\tau,0),$$

with $\tau = \frac{t}{\varepsilon}$. Notice that

$$\mu_3 \leq \frac{\mu_1}{\overline{\nu}_1} \leq \frac{\mu_1}{\varepsilon\overline{c} + \varepsilon^2\|M\|^2_{L^2(0,L)}} \leq \frac{\mu_1}{\varepsilon\overline{c}} \leq \frac{\lambda}{\varepsilon\overline{c}} \tag{56}$$

where we have used the definition of ν_1 given in Lemma 2 and the definition of μ_1 and μ_3. Then, one has for all strong solutions to (37) that

$$\frac{d}{d\tau}W(\bar{y}) \leq -\varepsilon\mu_3 W(\bar{y}) - \kappa_3 \bar{y}_x(\tau, 0).$$

Hence, using the Grönwall's Lemma again one obtains for all $\tau \geq 0$ that

$$W(\bar{y}) \leq e^{-\varepsilon\mu_3\tau} W(\bar{y}_0) - \int_0^\tau e^{-\varepsilon\mu_3(\tau-s)} \bar{y}_x(\tau, 0)^2 d\tau.$$

One can conclude that

$$\int_0^\tau e^{-\varepsilon\mu_3(\tau-s)} y_x(s, 0)^2 ds \leq \bar{c} e^{-\varepsilon\mu_3\tau} \|\bar{y}_0\|^2_{L^2(0,L)}. \tag{57}$$ {eq:hidden-regularity1}

Setting $\bar{s} = \varepsilon s$ and using the definition of τ, one obtains

$$\int_0^{\frac{t}{\varepsilon}} e^{-\mu_3(t-\bar{s})} \bar{y}_x\left(\frac{\bar{s}}{\varepsilon}, 0\right)^2 ds \leq \bar{c} e^{-\mu_3 t} \|\bar{y}_0\|^2_{L^2(0,L)}. \tag{58}$$ {eq:hidden-regularity2}

Then, using (55) and (58), and noticing that, since $\varepsilon \leq 1$, one has $\frac{t}{\varepsilon} \geq t$, one obtains finally that for all $t \geq 0$

$$V_1(\hat{y}, \hat{z}) \leq O(1) e^{-\mu_3 t} V_1(\hat{y}_0, \hat{z}_0) + O(\varepsilon^2) e^{-(b-ac)t} |\bar{z}_0|^2 + O(1) e^{-\mu_3 t} \|\bar{y}_0\|^2_{L^2(0,L)}. \tag{59}$$

Taking $\mu_4 = \min((b - ac), \mu_3)$, and using the smallness condition on the initial conditions, one obtains that

$$V_1(\hat{y}, \hat{z}) \leq e^{-\mu_4 t} O(\varepsilon^3). \tag{60}$$

Using Lemma 2, one has $V_1(\hat{y}, \hat{z}) > O(\varepsilon)(\|\hat{y}\|_{L^2(0,L)} + |\hat{z}|)^2$, concluding thus the proof. □

4 Fast ODE coupled with a slow KdV equation

sec:fast_ODE

4.1 Stability for small ε

Following the steps in the singular perturbation method, we are going to compute the reduced order system and the boundary layer system for (2). The exponential stability conditions will be drastically different, which explains why we used a different Lyapunov functional. In addition to this different Lyapunov functional, these conditions will hold at the price of considering strong solutions to (2).

Reduced order system. Setting $\varepsilon = 0$, one obtains that $z(t) = -\frac{c}{b} y_x(t, 0)$. The reduced order system, whose state is denoted by $\bar{y}$, satisfies

$$\begin{cases} \bar{y}_t + \bar{y}_x + \bar{y}_{xxx} = 0, \ (t,x) \in \mathbb{R}_+ \times [0,L], \\ \bar{y}(t,0) = \bar{y}(t,L) = 0, \ t \in \mathbb{R}_+ \\ \bar{y}_x(t,L) = -\frac{ac}{b}\bar{y}_x(t,0), \ t \in \mathbb{R}_+ \\ \bar{y}(0,x) = \bar{y}_0(x), \ x \in [0,L]. \end{cases} \tag{61}$$

{eq:reduced_order1}

Using the Lyapunov functional given in Proposition 10 with $d_2(t) = -\frac{ac}{b}y_x(t,0)$, one has

$$\frac{d}{dt}W(\bar{y}) \leq -\lambda W + \left(\kappa_2 \frac{a^2c^2}{b^2} - \kappa_3\right) y_x(t,0)^2. \tag{62}$$

Hence, if $\frac{a^2c^2}{b^2} < \frac{\kappa_3}{\kappa_2}$, then the origin of (64) is ensured to be exponentially stable. Note that this conditions looks like the one given in [34]. However, in this latter paper, one requires that $\left|-\frac{ac}{b}\right| < 1$. This is surely associated to the fact that the Lyapunov approach is more conservative than the one followed in [34].

Boundary layer system. Consider $\tau = \frac{t}{\varepsilon}$ and $\bar{z}(\tau) = z(\tau) + \frac{c}{b}y_x(t,0)$. One has $\dot{\bar{z}}(\tau) = \dot{z}(\tau) + \varepsilon\frac{d}{dt}\frac{c}{b}y_{xt}(t,0)$. With $\varepsilon = 0$, one obtains that

$$\dot{\bar{z}}(\tau) = \dot{z}(\tau) = bz(\tau) + cy_x(\tau,0) = b(z(\tau) + \frac{c}{b}y_x(\tau,0)) = b\bar{z}(\tau).$$

Then, the boundary layer system is defined by

$$\dot{\bar{z}}(\tau) = b\bar{z}(\tau), \tag{63}$$

{eq:boundary-layer1}

which means that its origin is exponentially stable if $b < 0$.

Full system. Consider $\tilde{z}(t) = z(t) + \frac{c}{b}y_x(t,0)$. Then, y solves the following equation

$$\begin{cases} y_t + y_x + y_{xxx} = 0, \ (t,x) \in \mathbb{R}_+ \times [0,L], \\ y(t,0) = y(t,L) = 0, \ t \in \mathbb{R}_+, \\ y_x(t,L) = a\tilde{z}(t) - \frac{ac}{b}y_x(t,0), \ t \in \mathbb{R}_+ \\ y(0,x) = y_0(x), \ x \in [0,L]. \end{cases} \tag{64}$$

{eq:reduced_order1}

One has $\varepsilon\dot{\tilde{z}} = \varepsilon\dot{z} + \varepsilon\frac{c}{b}y_{tx}(t,0) = bz(t) + \frac{c}{b}y_x(t,0) + \varepsilon\frac{c}{b}y_{tx}(t,0)$, i.e.,

$$\varepsilon\dot{\tilde{z}}(t) = b\tilde{z}(t) + \varepsilon\frac{c}{b}y_{tx}(t,0). \tag{65}$$

{eq:fast_ODE-difference}

Since the dynamics of $\tilde{z}$ introduces the time-derivative of $y_x(t,0)$, one needs more regularity on y. We consider therefore $v = \varepsilon y_t$. The use of the parameter ε in the variable v is due to

the fact that, in (64) it appears the state $\tilde{z}$. The dynamics of v is given by

$$\begin{cases} v_t + v_x + v_{xxx} = 0, \ (t,x) \in \mathbb{R}_+ \times [0,L], \\ v(t,0) = v(t,L) = 0, \ t \in \mathbb{R}_+, \\ v_x(t,L) = ab\tilde{z}(t), \ t \in \mathbb{R}_+ \\ v(0,x) = -\varepsilon y_0' - \varepsilon y_0''', \ x \in [0,L], \\ \varepsilon \dot{\tilde{z}}(t) = b\tilde{z}(t) + \varepsilon \frac{c}{b} v_x(t,0), \ t \in \mathbb{R}_+ \\ \tilde{z}(0) = z_0 + \frac{c}{b} y_0'. \end{cases} \tag{66}$$

{eq:reduced_order_regula

Well-posedness of (66) is given by our results in Section 2. Because we work in L^2-regularity for v and H^3-regularity for y, some compatibility conditions appear on the intitial data. We are in position to state the following result.

th:singular2

Theorem 8. *For any $a, b, c \in \mathbb{R}$ such that $\frac{a^2c^2}{b^2} < \frac{\kappa_3}{\kappa_2}$, where κ_2 and κ_3 are defined in Proposition 10, there exists $\varepsilon^* > 0$ such that for any $\varepsilon \in (0, \varepsilon^*)$, the origin of (2) is exponentially stable for any initial conditions $(y_0, z_0) \in H^3(0,L) \times \mathbb{R}$ such that*

$$y_0(0) = y_0(L) = 0, \quad y_0'(L) = ab(z_0 + \frac{c}{b} y_0'(0)).$$

Proof. To prove this result, we consider the following Lyapunov functional

$$V_3(v, \tilde{z}) = W(v) - \varepsilon \kappa_2 a^2 b \tilde{z}^2, \tag{67}$$

{eq:V_3}

where W is the Lyapunov functional given in Proposition 10. Using the same proof as in Lemma 4, one has that for any $b < 0$, we can define $\overline{\nu}_3 = \max\left(\overline{c}, -\varepsilon\kappa_2^2 a^2 b\right)$ and $\underline{\nu}_3 = \min\left(\underline{c}, -\varepsilon\kappa_2 a^2 b\right)$, where $\kappa_2 > 0$ comes from Proposition 10 such that the Lyapunov functional (67) satisfies

$$\underline{\nu}_3(\|v\|_{L^2(0,L)} + |\tilde{z}|^2) \leq V_3(v, \tilde{z}) \leq \overline{\nu}_3(\|v\|_{L^2(0,L)} + |\tilde{z}|^2). \tag{68}$$

{ex-lema}

Time derivative of (67) along the strong solutions to (66) yields

$$\frac{d}{dt} V_3(v,z) \leq -\lambda \|v\|^2_{L^2(0,L)} + \kappa_2 a^2 b^2 \tilde{z}^2 - \kappa_3 v_x(0)^2 - 2\kappa_2 a^2 b^2 \tilde{z}^2 + 2\kappa_2 a^2 \left(b\tilde{z}\varepsilon \frac{c}{b} v_x(t,0) \right). \tag{69}$$

Using Young's Lemma, one gets

$$\frac{d}{dt} V_3(v,z) \leq -\lambda \|v\|^2_{L^2(0,L)} - \kappa_2 a^2 b^2 \tilde{z}^2 - \kappa_3 v_x(0)^2 + 2\alpha\kappa_2 a^2 b^2 \tilde{z}^2 + \frac{2}{\varepsilon\alpha} \frac{a^2c^2\kappa_2}{b^2} v_x(t,0)^2 \tag{70}$$

and setting $\alpha = \frac{1}{4}$ one obtains

$$\frac{d}{dt} V_3(v,z) \leq -\lambda \|v\|^2_{L^2(0,L)} - \kappa_2 \frac{a^2b^2}{2} \tilde{z}^2 + \left(8\varepsilon^2 \kappa_2 \frac{a^2c^2}{b^2} - \kappa_3 \right) v_x(0)^2. \tag{71}$$

If one takes $\varepsilon < \frac{1}{8}$, $b < 0$ and $\frac{a^2c^2}{b^2} < \frac{\kappa_3}{\kappa_2}$, then one obtains the desired result using (68). $\square$

4.2 Tikhonov theorem

This subsection is devoted to the asymptotic analysis of (2) with respect to ε. As before, such an analysis requires to consider strong solutions to (2). Let us introduce the two variables

$$\hat{z}(t) = z(t) + \frac{c}{b} y_x(t,0) - \bar{z}\left(\frac{t}{\varepsilon}\right), \ \hat{y}(t) = y(t) - \bar{y}(t). \tag{72}$$

One can check that

$$\varepsilon \dot{\hat{z}}(t) = b\hat{z}(t) + \frac{\varepsilon c}{b}\left(\hat{y}_{tx}(t,0) + \bar{y}_{tx}(t,0)\right) \tag{73}$$ {eq:EDO-Tikhonov2}

and

$$\begin{cases} \hat{y}_t + \hat{y}_x + \hat{y}_{xxx} = 0, \ (t,x) \in \mathbb{R}_+ \times [0,L], \\ \hat{y}(t,0) = \hat{y}(t,L) = 0, \ t \in \mathbb{R}_+, \\ \hat{y}_x(t,L) = a\left(\hat{z}(t) + \bar{z}\left(\frac{t}{\varepsilon}\right)\right), \ t \in \mathbb{R}_+, \\ \hat{y}(0,x) = \hat{y}_0(x), \ x \in [0,L]. \end{cases} \tag{74}$$

Since (73) introduces the time-derivative of $y_x(t,0)$, let us consider $\hat{v} = \varepsilon \hat{y}_t$, where ε is introduced because the boundary condition makes appear z and $\bar{z}$. Its dynamics satisfies the following system

$$\begin{cases} \hat{v}_t + \hat{v}_x + \hat{v}_{xxx} = 0, \ (t,x) \in \mathbb{R}_+ \times [0,L], \\ \hat{v}(t,0) = \hat{v}(t,L) = 0, \ t \in \mathbb{R}_+, \\ \hat{v}_x(t,L) = ab\hat{z}(t) + \varepsilon \frac{ac}{b}\left(v_x(t,0) + \bar{y}_{tx}(t,0)\right)) + ab\bar{z}\left(\frac{t}{\varepsilon}\right), \ t \in \mathbb{R}_+, \\ \hat{v}(0,x) = \hat{v}_0(x), \ x \in [0,L]. \end{cases} \tag{75}$$ {eq:EDP-Tikhonov2}

Let us consider the Lyapunov functional

$$V_4(\hat{v},\hat{z}) = W(\hat{v}) - 3\varepsilon\kappa_2 a^2 b|\hat{z}|^2. \tag{76}$$ {eq:V_4}

We can prove, as before, the following. For any $b < 0$, defining $\overline{\nu}_4 := \max(\overline{c}, -3\varepsilon\kappa_2 a^2 b)$ and $\underline{\nu}_4 := \min(\underline{c}, -3\varepsilon\kappa_2 a^2 b)$, where κ_2 comes from Proposition 10, the Lyapunov functional (76) satisfies

$$\underline{\nu}_4(\|\hat{v}\|^2_{L^2(0,L)} + |\hat{z}\|^2) \leq V_4(\hat{v},\hat{z}) \leq \overline{\nu}_4(\|\hat{v}\|^2_{L^2(0,L)} + |\hat{z}|^2). \tag{77}$$ {lem:V_4}

Moreover, if $\varepsilon \leq 1$, then there existe $C > 0$ such that

$$\varepsilon(\|\hat{v}\|^2_{L^2(0,L)} + |\hat{z}|^2) \leq V_4(\hat{v},\hat{z}) \leq C(\|\hat{v}\|^2_{L^2(0,L)} + |\hat{z}|^2). \tag{78}$$

We are now in position to state and prove our Tikhonov theorem for (2).

Theorem 9. *There exist ε^* and $\mu > 0$ such that for any $\varepsilon \in (0,\varepsilon^*)$, for any $b < 0$, for any $a, c \in \mathbb{R}$ such that $\frac{a^2c^2}{b^2} < \frac{\kappa_3}{44\kappa_2\varepsilon^2}$, where κ_2 and κ_3 come from Proposition 10, and for any initial conditions $(y_0, z_0) \in H^3(0,L) \times \mathbb{R}$ satisfying the compatibility conditions*

$$y_0(0) = y_0(L) = 0, \quad y'(L) = az_0$$

and the smallness conditions

$$\begin{aligned}&\|y_0-\bar{y}_0\|_{H^3(0,L)}+\left|z_0+\frac{c}{b}y_0'(0)-\bar{z}_0\right|=O(\varepsilon^{5/2})\\&|\bar{z}_0|=O(\varepsilon^{5/2}),\quad \|\bar{y}_0\|_{H^3(0,L)}=O(\varepsilon^{3/2}),\end{aligned}\tag{79}$$

then for all $t\geq 0$

$$\|y(t,\cdot)-\bar{y}(t,\cdot)\|_{H^3(0,L)}+\left|z(t)+\frac{c}{b}y_x(t,0)-\bar{z}(t/\varepsilon)\right|=O(\varepsilon)e^{-\mu t}.\tag{80}$$

Proof. Using Proposition 10 with $d_1=0$, $d_2=ab\hat{z}(t)+\varepsilon\frac{ac}{b}\left(v_x(t,0)+\bar{y}_{tx}(t,0)\right))+ab\bar{z}\left(\frac{t}{\varepsilon}\right)$, the time derivative of V_4 along strong solutions to (73)-(75) yields

$$\begin{aligned}\frac{d}{dt}V_4(\hat{v},\hat{z})\leq&-\lambda\|\hat{v}\|^2_{L^2(0,L)}+\kappa_2\left(ab\hat{z}(t)+\varepsilon\frac{ac}{b}\left(v_x(t,0)+\bar{y}_{tx}(t,0)\right)+ab\bar{z}\left(\frac{t}{\varepsilon}\right)\right)^2\\&-3\kappa_2a^2b^2\hat{z}^2-\frac{3\varepsilon\kappa_2a^2}{c}\frac{1}{b}(\hat{v}_x(t,0)+\bar{y}_{tx}(t,0))b\hat{z}(t).\end{aligned}\tag{81}$$

Using Young's Lemma we get

$$\begin{aligned}&\kappa_2\left(ab\hat{z}(t)+\frac{ac}{b}\left(v_x(t,0)+\bar{y}_{tx}(t,0)\right)+ab\bar{z}\left(\frac{t}{\varepsilon}\right)\right)^2\\&\qquad\leq 2\kappa_2a^2b^2\hat{z}(t)^2+4\kappa_2\varepsilon^2\frac{a^2c^2}{b^2}\left(v_x(t,0)+\bar{y}_{tx}(t,0)\right)^2+4\kappa_2a^2b^2\bar{z}\left(\frac{t}{\varepsilon}\right)^2\\&\qquad\leq 2\kappa_2a^2b^2\hat{z}(t)^2+8\kappa_2\varepsilon^2\frac{a^2c^2}{b^2}\left(\hat{v}_x(t,0)^2+\bar{y}_{tx}^2(t,0)\right)+4\kappa_2a^2b^2\bar{z}\left(\frac{t}{\varepsilon}\right)^2\end{aligned}\tag{82}$$

and

$$\frac{3\kappa_2a^2}{c}\frac{1}{b}(\hat{v}_x(t,0)+\bar{y}_{tx}(t,0))b\hat{z}(t)\leq\frac{6\varepsilon^2\kappa_2a^2c^2}{\alpha b^2}(\hat{v}_x(t,0)^2+\bar{y}_{tx}^2(t,0))+3a^2b^2\alpha\kappa_2\hat{z}(t)^2.\tag{83}$$

Let us take $\alpha=\frac{1}{6}$ and gather all the above inequalities to get

$$\begin{aligned}\frac{d}{dt}V_4(\hat{v},\hat{z})\leq&-\lambda\|\hat{v}\|^2_{L^2(0,L)}-\frac{1}{2}\kappa_2a^2b^2\hat{z}^2+\left(\frac{44\varepsilon^2\kappa_2a^2c^2}{b^2}-\kappa_3\right)v_x(t,0)^2\\&+4\kappa_2a^2b^2\bar{z}\left(\frac{t}{\varepsilon}\right)^2+\frac{44\varepsilon^2\kappa_2a^2c^2}{b^2}\bar{y}_{tx}(t,0)^2.\end{aligned}\tag{84}$$

Choosing $\frac{a^2c^2}{b^2}\leq\frac{\kappa_3}{44\kappa_2\varepsilon^2}$ as in the statement of the theorem, one obtains for all solutions to (75)-(73)

$$\frac{d}{dt}V_4(\hat{v},\hat{z})\leq-\lambda\|\hat{v}\|^2_{L^2(0,L)}-\frac{1}{2}\kappa_2a^2b^2\hat{z}^2+4\kappa_2a^2b^2\bar{z}\left(\frac{t}{\varepsilon}\right)^2+\frac{44\varepsilon^2\kappa_2a^2c^2}{b^2}\bar{y}_{tx}^2(t,0).\tag{85}$$

Denoting by $\mu:=\min\left(\frac{\lambda}{\overline{c}},-\frac{1}{6\varepsilon b}\right)$, where $\overline{c}$ comes from Proposition 10, and using the Grönwall's

Lemma, one obtains, for all $t \geq 0$

$$V_4(\hat{v}, \hat{z}) \leq e^{-\mu t} V_4(\hat{v}_0, \hat{z}_0) + \int_0^t e^{-\mu(t-s)} \left(O(1)\bar{z}\left(\frac{s}{\varepsilon}\right)^2 + O(\varepsilon^2)\bar{y}_{tx}^2(s,0) \right) ds. \tag{86}$$

On one hand, one can prove that, for all $t \geq 0$

$$\int_0^t e^{-\mu(t-s)} \left| \bar{z}\left(\frac{s}{\varepsilon}\right) \right|^2 ds \leq O(1) e^{b\frac{t}{\varepsilon}} |\bar{z}_0|^2.$$

On the other hand, consider the variable $\bar{v}\,(t,x) = \varepsilon \bar{y}_t(t,x)$. It satisfies the following KdV equation

$$\begin{cases} \bar{v}_t + \bar{v}_x + \bar{v}_{xxx} = 0, \ (t,x) \in \mathbb{R}_+ \times [0,L], \\ \bar{v}(t,0) = \bar{v}(t,L) = 0, \ t \in \mathbb{R}_+, \\ \bar{v}_x(t,L) = -\frac{ac}{b}\bar{v}_x(t,0), \ t \in \mathbb{R}_+, \\ \bar{v}_x(0,x) = \bar{v}_0(x), \ x \in [0,L]. \end{cases} \tag{87}$$ {eq:reduced_order_regula

Using the ISS-Lyapunov functional given in Proposition 10 with $d_2(t) = -\frac{ac}{b}\bar{v}_x(t,0)$ along the solutions to (87), one obtains

$$\frac{d}{dt}W(\bar{v}) \leq -\lambda \|\bar{v}\|_{L^2(0,L)}^2 + \left(\frac{a^2c^2}{b^2}\kappa_2 - \kappa_3 \right) |\bar{v}_x(t,0)|^2. \tag{88}$$

Under the condition on a, c and b, there exists $\kappa_4 > 0$ such that

$$\frac{d}{dt}W(\bar{v}) \leq -\lambda \|\bar{v}\|_{L^2(0,L)}^2 - \kappa_4 \bar{|} v_x(t,0)|^2. \tag{89}$$

Using Proposition 10 and since $\mu \leq \frac{\lambda}{\bar{c}}$, one has, for all strong solution to (87)

$$\frac{d}{dt}W(\bar{v}) \leq -\mu W(\bar{v}) - \kappa_4 v_x(t,0)^2. \tag{90}$$

Thanks to Grönwall's Lemma, one gets

$$\int_0^t e^{-\mu(t-s)} |v_x(s,0)|^2 ds \leq \frac{1}{\kappa_4} e^{-\mu t} W(\bar{v}_0). \tag{91}$$

Therefore, using the smallness conditions given in the statement of the theorem, one has for all $t \geq 0$

$$\begin{aligned} V_4(\hat{v}, \hat{z}) \leq & e^{-\mu t} V_4(\hat{v}_0, \hat{z}_0) + O(1) e^{bt} |\bar{z}_0|^2 + O(\varepsilon^2) e^{-\mu t} \|\bar{y}_0\|_{H^3}^2 \\ \leq & e^{-\mu_1 t} O(\varepsilon^5), \end{aligned} \tag{92}$$

where $\mu_1 = \min(\mu, -b)$.

Due to inequality (77), one has $V_4(\hat{v}, \hat{z}) \geq O(\varepsilon)(\|\hat{v}\|_{L^2(0,L)}^2 + |\hat{z}|^2)$. Moreover, recall that $\hat{v} = \varepsilon y_t = -\varepsilon(y_x + y_{xxx})$. Hence, $V_4(\hat{v}, \hat{z}) \geq O(\varepsilon^3)(\|\hat{y}\|_{H^3(0,L)}^2 + |\hat{z}|^2)$. Then, one can deduce the desired result concluding the proof. □

5 Conclusion

sec_conclusion

In this paper, we have provided a singular perturbation analysis for two coupled systems composed by a KdV equation and an ODE. In particular, we have proved that, the conditions for the reduced order system and the boundary layer system to be exponentially system also work for the full-system for ε small enough. Different Lyapunov functionals have been introduced for the cases where the KdV equation is faster or the ODE is faster. For both cases, the ISS Lyapunov functional built in [2] has been instrumental. It is also worth mentioning that, when the ODE is faster than the KdV equation, the perturbation analysis can be performed only for sufficiently smooth solutions.

A ISS-Lyapunov functional

sec_ISS

This appendix recalls a crucial result provided in [2], which proposes the construction of a ISS-Lyapunov functional for the KdV equation. This result will be instrumental all along this paper. To introduce it, let us focus on the following disturbed KdV equation

$$\begin{cases} y_t + y_x + y_{xxx} = d_1(t,x), \quad (t,x) \in \mathbb{R}_+ \times (0,L), \\ y(t,0) = y(t,L) = 0, \quad t \in \mathbb{R}_+, \\ y_x(t,L) = d_2(t), \quad t \in \mathbb{R}_+, \\ y(0,x) = y_0(x), \quad x \in [0,L], \end{cases} \tag{93}$$

{eq:KdV_disturbed}

where $d_1 \in L^2(0,T;L^2(0,L))$ and $d_2 \in L^2(0,T)$, for any $T \geq 0$. The well-posedness of (93) can be obtained using semigroup theory in standard way for strong or mild solutions, depending on the regularity of the data. According to [2, Theorem 2.3], one has the following result.

prop:ISS-Lyap

Proposition 10. *There exists an ISS-Lyapunov functional for* (93)*, i.e. there exists a function* $W : L^2(0,L) \to \mathbb{R}$ *and positive constants* $\lambda, \kappa_1, \kappa_2, \kappa_3, \overline{c}, \underline{c}$ *such that*

$$\underline{c}\|y\|^2_{L^2(0,L)} \leq W(y) \leq \overline{c}\|y\|^2_{L^2(0,L)} \tag{94}$$

and the derivative of W *along the solutions to* (93) *satisfies*

$$\frac{d}{dt}W(y) \leq -\lambda\|y\|^2_{L^2(0,L)} + \kappa_1\|d_1(t,\cdot)\|^2_{L^2(0,L)} + \kappa_2|d_2(t)|^2 - \kappa_3|y_x(t,0)|^2. \tag{95}$$

Note that the term $-\kappa_3|y_x(t,0)|^2$ does not appear in [2, Theorem 2.3], but following the proof in that paper, one can prove that such a term exists. It will be useful in our context.

References

2018exponential [1] H. Ayadi. Exponential stabilization of cascade ODE-linearized KdV system by boundary Dirichlet actuation. *Eur. J. Control*, 43:33–38, 2018.

balogoun2021iss [2] I. Balogoun, S. Marx, and D. Astolfi. ISS Lyapunov strictification via observer design and integral action control for a Korteweg-de-Vries equation. *arXiv preprint arXiv:2107.09541*, 2021.

cerpa2007 [3] E. Cerpa. Exact controllability of a nonlinear Korteweg-de Vries equation on a critical spatial domain. *SIAM J. Control Optim.*, 46(3):877–899, 2007.

cerpa2014 [4] E. Cerpa. Control of a Korteweg-de Vries equation: a tutorial. *Math. Control Relat. Fields*, 4(1):45–99, 2014.

cerpacoron [5] E. Cerpa and J.M. Coron. Rapid stabilization for a Korteweg-de Vries equation from the left Dirichlet boundary condition. *IEEE Trans. Automat. Control*, 58(7):1688–1695, 2013.

rpa2009boundary [6] E. Cerpa and E. Crépeau. Boundary controllability for the nonlinear Korteweg-de Vries equation on any critical domain. *Ann. Inst. H. Poincaré C Anal. Non Linéaire*, 26(2):457–475, 2009.

cerpaem [7] E. Cerpa and E. Crépeau. Rapid exponential stabilization for a linear Korteweg-de Vries equation. *Discrete Contin. Dyn. Syst. Ser. B*, 11(3):655–668, 2009.

erpa-prieur2017 [8] E. Cerpa and C. Prieur. Effect of time scales on stability of coupled systems involving the wave equation. In *2017 IEEE 56th Annual Conference on Decision and Control (CDC)*, pages 1236–1241, 2017.

rpa2019singular [9] E. Cerpa and C. Prieur. Singular perturbation analysis of a coupled system involving the wave equation. *IEEE Trans. Automat. Control*, 65(11):4846–4853, 2020.

u2015asymptotic [10] J. Chu, J-M. Coron, and P. Shang. Asymptotic stability of a nonlinear Korteweg–de Vries equation with critical lengths. *J. Differential Equations*, 259(8):4045–4085, 2015.

corcre [11] J.-M. Coron and E. Crépeau. Exact boundary controllability of a nonlinear KdV equation with critical lengths. *J. Eur. Math. Soc. (JEMS)*, 6(3):367–398, 2004.

coron2020small [12] J.-M. Coron, A. Koenig, and H.-M. Nguyen. On the small-time local controllability of a kdv system for critical lengths. *arXiv preprint arXiv:2010.04478*, 2020.

lucoron [13] J-M. Coron and Q. Lü. Local rapid stabilization for a Korteweg-de Vries equation with a Neumann boundary control on the right. *J. Math. Pures Appl. (9)*, 102(6):1080–1120, 2014.

coron2017local [14] J.-M. Coron, I. Rivas, and S. Xiang. Local exponential stabilization for a class of Korteweg–de Vries equations by means of time-varying feedback laws. *Anal. PDE*, 10(5):1089–1122, 2017.

020introduction [15] R. Curtain and H Zwart. *Introduction to infinite-dimensional systems theory. A state-space approach.* Texts in Applied Mathematics. Springer, New York, 2020.

il2002nonlinear [16] Hassan K. Khalil. *Nonlinear systems*. Macmillan Publishing Company, New York, 1992.

vic1999singular [17] P. Kokotović, H. K Khalil, and J. O'reilly. *Singular perturbation methods in control*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1999.

x2021forwarding [18] S. Marx, L. Brivadis, and D. Astolfi. Forwarding techniques for the global stabilization of dissipative infinite-dimensional systems coupled with an ODE. *Math. Control Signals Systems*, 33(4):755–774, 2021.

marxcerpa [19] S. Marx and E. Cerpa. Output feedback stabilization of the Korteweg–de Vries equation. *Automatica*, 87:210–217, 2018.

marx2017global [20] S. Marx, E. Cerpa, C. Prieur, and V. Andrieu. Global stabilization of a Korteweg–de Vries equation with saturating distributed control. *SIAM J. Control Optim.*, 55(3):1452–1480, 2017.

azenc1996adding [21] F. Mazenc and L. Praly. Adding integrations, saturated controls, and stabilization for feedforward systems. *IEEE Trans. Automat. Control*, 41(11):1559–1578, 1996.

chenko2020input [22] A. Mironchenko and C. Prieur. Input-to-state stability of infinite-dimensional systems: recent results and open questions. *SIAM Review*, 62(3):529–614, 2020.

nguyen2021 [23] H.-M. Nguyen. Decay for the nonlinear KdV equations at critical lengths. *J. Differential Equations*, 295:249–291, 2021.

pazy [24] A. Pazy. *Semigroups of Linear Operators and Applications to Partial Differential Equations*. Applied Mathematical Sciences, 1983.

praly2019 [25] L. Praly. Observers to the aid of "strictification" of Lyapunov functions. *Systems and Control Letters*, 134:104510, 2019.

rosier [26] L. Rosier. Exact boundary controllability for the Korteweg-de Vries equation on a bounded domain. *ESAIM Control Optim. Calc. Var.*, 2:33–55, 1997.

rosierzhang2009 [27] L. Rosier and B-Y. Zhang. Control and stabilization of the Korteweg-de Vries equation: recent progresses. *J. Syst. Sci. Complex.*, 22(4):647–682, 2009.

g2018asymptotic [28] S. Tang, J. Chu, P. Shang, and J-M. Coron. Asymptotic stability of a Korteweg–de Vries equation with a two-dimensional center manifold. *Advances in Nonlinear Analysis*, 7(4):497–515, 2018.

15stabilization [29] S. Tang and M. Krstic. Stabilization of linearized Korteweg-de Vries systems with anti-diffusion by boundary feedback with non-collocated observation. In *2015 American Control Conference (ACC)*, pages 1959–1964. IEEE, 2015.

ng2017stability [30] Y. Tang and G. Mazanti. Stability analysis of coupled linear ODE-hyperbolic PDE systems with two time scales. *Automatica*, 85:386–396, 2017.

ang2015tikhonov [31] Y. Tang, C. Prieur, and A. Girard. Tikhonov theorem for linear hyperbolic systems. *Automatica*, 57:1–10, 2015.

rrand2019adding [32] A. Terrand-Jeanne, V. Andrieu, V. Dos Santos Martins, and C.-Z. Xu. Adding integral action for open-loop exponentially stable semigroups and application to boundary control of PDE systems. *IEEE Trans. Automat. Control*, 65(11):4481–4492, 2019.

anghe2022output [33] N. Vanspranghe and L.. Brivadis. Output regulation of infinite-dimensional nonlinear systems: a forwarding approach for contraction semigroups. *arXiv preprint arXiv:2201.10146*, 2022.

ang1994boundary [34] B.-Y. Zhang. Boundary stabilization of the Korteweg-de Vries equation. In *Control and estimation of distributed parameter systems: nonlinear phenomena (Vorau, 1993)*, volume 118 of *Internat. Ser. Numer. Math.*, pages 371–389. Birkhäuser, Basel, 1994.