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Imperfect condition-based maintenance for a gamma degradation process in presence of unknown parameters

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Abstract

A system subject to degradation is considered. The degradation is modelled by a gamma process. A condition-based maintenance policy with perfect corrective and an imperfect preventive actions is proposed. The maintenance cost is derived considering a Markov-renewal process. The statistical inference of the degradation and maintenance parameters by the maximum likelihood method is investigated. A sensibility analysis to different parameters is carried out and the perspectives are detailed.

Keywords

Degradation modelling, gamma process, condition-based maintenance, imperfect maintenance, statistical inference, Maximum likelihood Estimation

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Introduction

Since few decades, more sensor data are available and complex systems monitoring has received a lot of attention. Complex systems undergo degradation and their health condition evolves in time. Monitoring these latter permits to make maintenance decisions according to the usage, operation conditions and degradation level of the system. In order to make sensible maintenance decisions, it is important to be able to predict the future evolution of the system. The prediction allows to decide whether the system should be maintained correctively or preventively. For such predictions and therefore maintenance decisions, the degradation of the system should be modelled according to expert knowledge or historical data. To this aim the development of a mathematical model accurately describing the system's behaviour is essential. However, to be useful, the chosen model should be credible, incorporate the major features of collected data and evolve with time. It is important to remind that health or degradation indicators, are usually influenced by several factors, which are not necessarily known, and their impacts cannot be modelled through a deterministic expression. Since the degradation undergoes different unknown factors due to the environmental conditions, usage, etc. this phenomenon can be considered as random. Stochastic processes, such as Lévy processes, Markov processes, counting processes, birth processes are complex and exhaustive models to describe the degradation indicator evolution in time (1; 2). Frequently, gamma and Wiener processes are considered to model the evolution of gradual degradation for monotone and non-monotone phenomenon respectively (3; 4; 5). In the framework of this paper, a gamma process is adopted for monotonous degradation modelling. Once the behaviour of a deteriorating system is modelled, its future failure time can be estimated and it is possible to plan efficient maintenance rules.

In presence of degradation model and data, maintenance decision making is an important issue which can be addressed correctly by analysing maintenance historical data. Maintenance actions can be corrective or preventive. In this paper, a condition-based maintenance policy is considered where maintenance operation is triggered if the monitored health condition indicator exceeds a critical threshold, refer to (6; 7; 8; 9; 10; 11; 12). One considers both perfect ("as good as new") and imperfect maintenance which means after maintenance, the

system is not necessarily restored to as good as new. For instance, steel structures such as tanks or bridges, are covered by an organic coating in order to protect them from corrosion due to outdoor conditions. The coating layer deteriorates in time, maintenance actions are carried out to improve the condition of the coating, refer to (13). The main concern in imperfect maintenance policies is to model the impact of maintenance actions on the system. Most of the imperfect maintenance models lead to the reduction of age or impact the failure intensity, refer to (14; 15; 16). Imperfect maintenance models reducing the degradation level are still a broad field to explore. Kijima in (17) propose a cumulative damage shock model with imperfect periodic maintenance actions reducing the degradation level by a percentage of the total damage. An extension of this model is developed later in Kijima 1992 (18). Since then, there are maintenance models considering a deterministic maintenance efficiency for degradation reduction (19; 20). Authors in (21; 13; 22; 23; 24; 25; 26) consider imperfect maintenance impacting the degradation level with a random maintenance efficiency modelled by a random variable with a uniform, beta, exponential, gamma, gaussian, truncated gamma or gaussian distribution. Authors in Ponchet et al 2009 (27) consider degradation based imperfect maintenance where the efficiency is random and depends on the residual life time. Authors in (28; 29) give a large panel of important measures and other properties for an imperfectly repaired gamma deteriorating system.

Long run average maintenance cost calculation in the framework of imperfect maintenance is not an easy task. Since the preventive actions are not replacements, the renewal theory cannot be applied to preventive cycles, (30; 31). To bypass this problem dynamic programming (13) or Markov renewal theory (21) is used. These calculation methods are possible if the degradation parameters as well as maintenance efficiency are known. In presence of historical data, first the parameters should be estimated and then propose an optimal maintenance rule.

Statistical Inference, in presence of large set of monitoring data very often rely on Maximum Likelihood Method. Inference in presence of degradation data has already received a lot of attention (32; 5; 33; 34). However, inference in the framework of condition-based imperfect maintenance is not extensively studied. For instance, Authors in (25; 35; 26) consider statistical inference in the case of imperfect maintenance

where Wiener, diffusion or jump diffusion process are considered as degradation model. The case of gamma process and imperfect maintenance is scarcely studied, refer to (36; 37; 38). Authors in (36) focus on semi-parametric estimation of the maintenance efficiency. In (37) an exponential distributed degradation reduction is considered and the parameter of the exponential distribution is estimated via MLE method based on degradation data just after maintenance.

In this paper, a condition-based maintenance policy is considered. The preventive imperfect maintenance under consideration reduces the degradation level of the system. The maintenance efficiency is not random but is considered as unknown. That is why statistical inference is considered. The main contribution of the paper is as follows:

- Point out the robustness of maintenance decision rules in presence of unknown degradation and maintenance parameters.
- Parameter estimation in presence of degradation data with imperfect maintenance.
- Sensitivity analysis of the maintenance long run average cost to degradation and maintenance parameters.

The remainder of the paper is as follows. The first part of the paper is devoted to the model description and maintenance cost calculation. The next section discusses the statistical inference in presence degradation data with imperfect maintenance. Eventually based on numerical examples, the properties of a maintenance policy based on parameter estimates are discussed.

Degradation process and maintenance model

Degradation process

A system undergoing a monotonous degradation is considered. It is assumed that the degradation process is a gamma process, $\{X_t, t \geq 0\}$, with shape parameter $a(t) = \alpha t^{\beta}$, $\alpha > 0$, $\beta > 0$, and scale parameter $b > 0$. The gamma process $\{X_t, t \geq 0\}$ is a continuous-time stochastic process such that:

- $X(0) = 0$ with probability one,

- $X(t) - X(s) \sim \mathcal{G}(a(t) - a(s), b)$ for all $t > s \geq 0$, with density function $\forall x \in \mathbb{R}^+$:

$$f_{a(t)-a(s),b}(x) = \frac{b^{a(t)-a(s)}}{\Gamma(a(t) - a(s))} x^{a(t)-a(s)-1} e^{-bx}, \tag{1}$$

- $\{X_t, t \geq 0\}$ has independent non-overlapping increments.

Maintenance model

Let $\{Y_t, t \geq 0\}$ denotes the stochastic process of the maintained system. It is considered that the system is inspected at times $\tau, 2\tau, \ldots$. Let us denote by $\{t_j = j\tau, j \in \mathbb{N}\}$ the set of inspection times. It is referred to ${t_j}^-$ and ${t_j}^+$ as the time just before and just after the inspection time t_j respectively. Let be M the failure threshold beyond which the system does not fulfil correctly its missions. The failure is not self announced and can be only detected by inspection. Let L be the preventive threshold triggering the preventive action, $0 < L < M$. At each inspection t_j^-, a decision is taken respect to the observed degradation level:

- if $Y_{t_j^-} \geq M$, then a Corrective Maintenance (CM) is performed and the system is replaced by a new one (As Good As New): in such a case, both the degradation level and the time set to zero;
- if $L \leq Y_{t_j^-} < M$, then a Preventive Maintenance (PM) is performed. We consider here an imperfect PM, and more precisely, an ARD_∞ model, parameterized by ρ, i.e. the degradation level just after the inspection (at time t_j) is reduced by a proportional quantity of the degradation level just before the inspection: $Y_{t_j{}^+} \approx (1 - \rho) Y_{t_j{}^-}$;
- if $Y_{t_j^-} < L$, no action is performed $Y_{t_j{}^+} \approx Y_{t_j{}^-}$.

There is a difference between a CM and a perfect PM: in the second case, only the degradation level is set to zero.

Let us denote $U_i = \mathbb{1}_{L \leq Y_{(i\tau)^-} < M}$. Between the inspection at time $(i-1)\tau$ and the inspection at time $i\tau$, the system is assumed to evolve according to i.i.d. copies $(X_t^{(i)})_{t \geq 0}$ of the gamma process $(X_t)_{t \geq 0}$.

Thus, before the first inspection, we have $\forall t \in [0, \tau)$, $Y_t = X_t^{(1)}$. At the first inspection, we could have:

- if $Y_{\tau^-} < L$, no action is performed and then $Y_\tau = Y_{\tau^-} = X_{\tau^-}^{(1)}$.

- if $L \leq Y_{\tau^-} < M$, an imperfect PM (ARD_∞) is performed and then the degradation level is reduced of $\rho X_\tau^{(1)}$ and thus $Y_\tau = (1-\rho)X_{\tau^-}^{(1)}$.
- if $Y_\tau^- > M$ then the system is replaced by a new one.

After the first inspection without replacement, we have $\forall t \in [\tau, 2\tau)$,

$$Y_t = Y_\tau + X_t^{(1)} - X_\tau^{(1)} = (1-\rho)^{U_1} X_\tau^{(1)} + X_t^{(2)} - X_\tau^{(2)}$$

At the second inspection at time 2τ, we have

- if $Y_{2\tau^-} < L$, no action is performed, and

$$Y_{2\tau} = Y_{2\tau^-} = (1-\rho)^{U_1} X_\tau^{(1)} + X_{2\tau}^{(2)} - X_\tau^{(2)}.$$

- if $L \leq Y_{2\tau^-} < M$, an imperfect PM (ARD_∞) is performed and then the degradation level is reduced of $\rho(X_{2\tau}^{(1)} - X_\tau^{(1)})$ and thus

$$\begin{aligned} Y_{2\tau} &= Y_{2\tau^-} - \rho(X_{2\tau^-}^{(2)} - X_\tau^{(2)}) \\ &= (1-\rho)^{U_1} X_\tau^{(1)} + (1-\rho)(X_{2\tau^-}^{(2)} - X_\tau^{(2)}) \end{aligned}$$

- if $Y_{2\tau^-} > M$ then the system is replaced.

By induction, we can conclude that after i inspections without replacement, we have $\forall t \in [i\tau, (i+1)\tau)$

$$Y_t = Y_{i\tau} + (X_t^{(i+1)} - X_{i\tau}^{(i+1)})$$

and if we denote $i = \lfloor t/\tau \rfloor$, then we get

$$Y_t = \sum_{j=1}^{i} (1-\rho)^{U_j} (X_{j\tau}^{(j)} - X_{(j-1)\tau}^{(j)}) + X_t^{(i+1)} - X_{i\tau}^{(i+1)}$$

Finally, we obtain

$$\begin{aligned} Y_{(i+1)\tau} &= Y_{i\tau} + (1-\rho)^{U_{i+1}} (X_{(i+1)\tau}^{(i+1)} - X_{i\tau}^{(i+1)}) \\ &= \sum_{j=1}^{i+1} (1-\rho)^{U_j} (X_{j\tau}^{(j)} - X_{(j-1)\tau}^{(j)}) \end{aligned} \tag{2}$$

Maintenance cost evaluation

Let C_I, C_P and C_C denote respectively the cost of an inspection, a preventive maintenance and a corrective maintenance. The maintenance cost at time t is given as follows:

$$C(t) = C_I[\frac{t}{\tau}] + C_P \sum_{i=1}^{[t/\tau]} \mathbb{1}_{L \leq Y_{i\tau^-} < M} + C_C \sum_{i=1}^{[t/\tau]} \mathbb{1}_{Y_{i\tau^-} \geq M}$$

As the corrective maintenance actions are perfect (replacement), by the renewal theory we can compute the long run average maintenance cost as follows:

$$\lim_{t\to\infty} \frac{C(t)}{t} = \frac{\mathbb{E}(C(T))}{\mathbb{E}(T)} = \frac{\mathbb{E}(C_C + [T/\tau]C_I + \sum_{i=1}^{T/\tau} \mathbb{1}_{L \leq Y_{i\tau^-} < M} C_p)}{\mathbb{E}(T)}$$
$$\frac{C_c + \mathbb{E}([T/\tau])C_i + \sum_{i=1}^{T/\tau} \mathbb{P}(L \leq Y_{i\tau} < M)C_P}{\mathbb{E}(T)} \tag{3}$$

where T is the length of a renewal cycle which means the period between two corrective maintenance actions:

$$T = \min\{t_i > 0,\, Y_{t_i} \geq M\}$$

and the expectation of the length of a renewal cycle is:

$$\mathbb{E}(T) = \sum_{k=1}^{\infty} k\tau \mathbb{P}(Y_{k\tau} > M) \tag{4}$$

However, the due to presence of imperfect maintenance actions, deriving the probabilities in equations (3) and (4) is not an easy task. To bypass this problem, the markovian renewal property of the maintained degradation process between two inspections is used to calculate the long run average maintenance cost. Let's consider the Markov renewal cycle $[t_{i-1}^-, t_i^-]$, x and y respectively the degradation level of maintained system at the beginning and the end of the cycle. Let's consider the Markov process $(Z_i = (Y_{t_i}, t_i^-))_{i\in\mathbb{N}}$ the process describing the system state just before each maintenance action. If the Markov process is stationary, let π be its stationary distribution. The long run average maintenance cost can be derived as follows:

$$\lim_{t\to\infty}\frac{C(t)}{t}=\frac{\mathbb{E}_\pi(C(\tau))}{\mathbb{E}_\pi(\tau)}=\frac{C_I+C_p\mathbb{P}_\pi(L\leq Y_{\tau^-}<M)+C_C\mathbb{P}_\pi(Y_{\tau^-}\geq M)}{\mathbb{E}_\pi(\tau)}$$
$$=\frac{C_I+C_p\mathbb{P}_\pi(L\leq Y_{\tau^-}<M)+C_C\mathbb{P}_\pi(Y_{\tau^-}\geq M)}{\tau}. \quad (5)$$

where $\mathbb{E}_\pi$ and $\mathbb{P}_\pi$ are the expectation and the probability under the steady-state distribution π respectively. Under π the probability of corrective and preventive maintenance during a Markov renewal cycle are as follows:

$$\mathbb{P}_\pi(L\leq Y_{\tau^-}<M)=\int_L^M\int_{[0,M]\times\mathbb{R}^+}f_{a(\tau+t)-a(t),b}(dy-x)\pi(dx,dt) \quad (6)$$

$$\mathbb{P}_\pi(Y_{\tau^-}\geq M)=\int_0^L\int_{[M,+\infty[\times\mathbb{R}^+}f_{a(\tau+t)-a(t),b}(dy-x)\pi(dx,dt). \quad (7)$$

The stationary distribution is obtained by numerical iteration and the integrals are calculated by Monte Carlo simulations.

Stationary distribution of the Maintained System

For the Markov process $(Z_i=(Y_{t_i},t_i))_{i\in\mathbb{N}}$ the underlying stationary distribution, π is the solution of :

$$\pi(\cdot)=\int_{[0,L]\times\mathbb{R}^+}p(\cdot,x,s)\pi(dx,ds) \quad (8)$$

where p is the transition density of the Markov process. Let us denote by $p(dz|u)$ the transition kernel for state $u=(x,s)$ to $dz=(dy,dt)$ which can be written by :

$$p(B,t,x,s)=\mathbb{P}(Y_t\in B|Y_s=x)=\int_B f_{a(t)-a(s),b}(y-x)\mathbb{1}_{x<L}dy$$
$$+\int_B f_{a(t)-a(s),b}(y-(1-\rho)x)\mathbb{1}_{L\leq x<M}dy+\int_B f_{a(\tau),b}(y)\mathbb{1}_{x>M}dy, \quad (9)$$

where $f_{a,b}$ is the density function of gamma distribution with shape parameter a and scale parameter b defined in equation (1).

Let us consider the homogeneous case, $\beta = 1$, let us denote by $\pi(dx, \tau) = \pi_\tau(dx)$. Thus, we have

$$\begin{aligned}\pi_\tau(dy) = &\int_0^{\min(L,y)} \pi_\tau(dx) f_{\alpha\tau,b}(dy - x))dx \\ &+ \int_{\min(\frac{y}{1-\rho},L)}^{\min(\frac{y}{1-\rho},M)} \pi_\tau(dx) f_{\alpha\tau,b}(dy - (1-\rho)x)dx \\ &+ f_{\alpha\tau,b}(y) \int_M^{+\infty} \pi_\tau(dx) \end{aligned} \tag{10}$$

We can solve this last equation by fixed-point iteration algorithm by considering that $\pi(x)$ is the solution of $\pi_\tau(y) = g(\pi_\tau(y))$, where g is a continuous function. Thus, for all y, we approximate eq. (10) at iteration k by

$$\begin{aligned}w_k(y) = &\int_0^{\min(L,y)} w_{k-1}(x) f_{\alpha\tau,b}(y - x)\, dx + f_{\alpha\tau,b}(y) \int_M^{+\infty} w_{k-1}(x)\, dx \\ &+ \int_{\min(\frac{y}{1-\rho},L)}^{\min(\frac{y}{1-\rho},M)} w_{k-1}(x) f_{\alpha\tau,b}(y - (1-\rho)x)\, dx. \end{aligned} \tag{11}$$

We set the initial value of w_k as $w_1(x) = x\exp(-x)$.

For the non-homogeneous case, as explained in (39; 40), under some specific hypotheses the process will be K-ergodic or ergodic and therefore an asymptotic regularity behavior can be considered and by equations (8) and (9) a distribution can be derived. Since, for some cases, these hypotheses are valid for gamma kernel distribution the proposed cost calculation methodology can be applied for maintenance policy optimisation in specific cases proposed in this paper.

Statistical inference

It is supposed that maintenance policy proposed in the previous section is applied on a system for a given operational period. Consequently some data and information are available. According to the information level in disposal, statistical inference methods can be applied on available data. In this paper, it is considered that for statistical inference, the

type of maintenance action and the inspection times are known but the maintenance efficiency and the degradation dynamic are unknown. This section is devoted to the estimation of the different parameters involved in this model: on one side, the parameters α, β and b which correspond to the degradation part and, on the other side, the parameter ρ which corresponds to the maintenance part. Here it is assumed that the delay τ between two successive inspections is known. In addition, in our setting, it is assumed that only one system is observed and its degradation is measured n times (n inspections), just before performing, eventually, a maintenance action. It means that the degradation level after an imperfect maintenance is not observed, and thus the efficiency of this action is not measured. In other words, the observations are $Y_{t_0^-} = 0, Y_{t_1^-}, Y_{t_2^-}, Y_{t_3^-}, \ldots, Y_{t_n^-}$ with $t_j = j\tau$ for any $j \in \{0, \ldots, n\}$. There exists other possible sample schemes, for instance, unknown maintenance, only observation after maintenance actions are available, etc.

In order to estimate the four parameters, we will write the likelihood function associated to the observations from sampling scheme considered here. Since there is no explicit solution for the maximum likelihood estimator, it has to be computed numerically. Since most of numerical optimization procedures are iterative, some initial values for the parameters are provided. Numerical illustrations are presented in the next Section.

Maximum likelihood function

We observe a system subject to the maintenance policies as described above. We assume that the system is observed at calendar times $t_j = j\tau$ for $j \in \{0, \ldots, n\}$. We will denote by $y_0 = 0$ and $y_1, \ldots, y_n$ the observed degradation levels corresponding to $Y_{t_1^-}, \ldots, Y_{t_n^-}$. Let us recall that, when a corrective maintenance (CM) is performed, the system is replaced by a new one (both degradation level and age reset to zero). We will denote by N the number of CM performed between 0 and $n\tau$. In addition, let us denote $t_{j_1}, \ldots, t_{j_N}$ the instants when a CM is applied. We set $j_0 = 0$ and (except in the special case where j_N is already equal to n, meaning that the last observed degradation level is above the threshold M). The fact to replace the system by a fully new one after a CM leads to consider a modified time scale. Here now the observed age is $(s_1, \ldots, s_n)$ defined as follows. If $N = 0$ (no CM is performed on the observed time span),

$s_j = t_j$ for all $j \in \{0, \ldots, n\}$ and, if $N \geqslant 1$,

$$\forall k \in \{0, \ldots, N\}, \forall j \in \{j_k, \ldots, j_{k+1}\}, \qquad s_j = t_j - t_{j_k}.$$

It follows that, from any $j \in \{1, \ldots, n\}$, the random variable $\Delta Y_j = Y_{t_j^-} - Y_{t_{j-1}^+}$ is gamma distributed with shape parameter with shape parameter

$$\begin{aligned} \Delta a_j &= a(s_j) - a(s_{j-1}) \\ &= \alpha \left(s_j^\beta - s_{j-1}^\beta \right) \\ &= \alpha \sum_{k=0}^{N} \left\{ (t_j - t_{j_k})^\beta - (t_{j-1} - t_{j_k})^\beta \right\} \mathbb{I}_{j_k \leqslant j \leqslant j_{k+1}} \end{aligned}$$

and scale parameter b. The probability density function of ΔY_j is denoted by $f_{\Delta a_j, b}$. In addition, these increments are independent. While $Y_{t_j^-}$ is observed (and y_j corresponds to its realization), it is not the case for $Y_{t_{j-1}^+}$. However, we have enough information of $Y_{t_{j-1}^-}$ that can be used to overcome it. If $Y_{t_{j-1}^-} \leqslant L$, no maintenance action is performed and thus, thanks to the stochastic continuity of the gamma process, we have $\mathbb{P}[Y_{t_{j-1}^+} = Y_{t_{j-1}^-}] = 1$ and hence $Y_{t_{j-1}^+} \approx Y_{t_{j-1}^-}$. If $L \leqslant Y_{t_{j-1}^-} \leqslant M$, an imperfect maintenance action is performed. Similarly as above, we can write that $Y_{t_{j-1}^+} \approx (1-\rho) Y_{t_{j-1}^-}$. If $Y_{t_{j-1}^-} \geqslant M$, a perfect maintenance action is performed and thus the system restart as a new one, so that $Y_{t_{j-1}^+} = 0$. From these observations and using the independence of the increments, the likelihood function L can be written as follows:

$$L(\alpha, \beta, b, \rho; s_0, y_0, s_1, y_1, \ldots, s_n, y_n) = \prod_{j=1}^{n} f_{\Delta a_j, b}(y_j - (1-\rho_{j-1}) y_{j-1})$$

where

$$\rho_{j-1} = \begin{cases} 0 & \text{if } y_{j-1} \leqslant L \\ \rho & \text{if } L \leqslant y_{j-1} \leqslant M \\ 1 & \text{if } y_{j-1} \geqslant M. \end{cases}$$

We can next derive the log-likelihood function $\ell = \log L$ and expand the expression using the expression of the probability density function of the

gamma distribution :

$$\ell(\alpha,\beta,b,\rho;s_0,y_0,\ldots,s_n,y_n) = \alpha\log b\sum_{j=1}^{n}\left(s_j^\beta - s_{j-1}^\beta\right) - \sum_{j=1}^{n}\log\Gamma\left(\alpha\left(s_j^\beta - s_{j-1}^\beta\right)\right) - b\sum_{j=1}^{n}\left(y_j-(1-\rho_{j-1})y_{j-1}\right) + \sum_{j=1}^{n}\left(\alpha\left(s_j^\beta - s_{j-1}^\beta\right)-1\right)\log\left(y_j-(1-\rho_{j-1})y_{j-1}\right).$$

No closed-form expression for the maximum likelihood estimator can be expected, but it could be computed numerically through a well-suited numerical procedure. Notice that these above computations can be easily extended to the case where the degradation of several systems is observed. It can also be adapted for other kind parametric shape function or for the framework of a semi-parametric approach.

Initial guess

We propose here an initial guess for the estimation parameters when using an iterative numerical method to compute the maximum of the likelihood function, if an initial point is required. For the degradation part, we propose to start with an homogeneous gamma process. Hence, we propose $\widehat{\beta}_{init} = 1$. The two other parameters α and b can be estimated by applying a kind of moments method (see (5), for instance) to all inspection times t_j for which the degradation level is lower than L (nothing is done) or larger than M (replacement by a new system). In this way, these observations do not depend on the parameter ρ. Upon the assumption of a homogeneous gamma process, for any $j \in \{1,\ldots,n\}$, we have :

$$\mu = \mathbb{E}[\Delta Y_j] = \frac{\alpha\tau}{b} \qquad \text{and} \qquad \sigma^2 = \text{var}[\Delta Y_j] = \frac{\alpha\tau}{b^2}.$$

We can then express both α and b in function of μ and σ^2 as follows :

$$\alpha = \mu/\sigma^2 \qquad \text{and} \qquad b = \mu^2/\sigma^2.$$

We can then express both α and b in function of μ and σ^2 as follows :

$$\alpha = \mu/\sigma^2 \qquad \text{and} \qquad b = \mu^2/\sigma^2.$$

Thus we can get a simple estimator for a and b as follows:

$$\widehat{\alpha}_{init} = \widehat{\mu}/\widehat{\sigma}^2 \qquad \text{and} \qquad \widehat{b}_{init} = \widehat{\mu}^2/\widehat{\sigma}^2$$

where

$$\widehat{\mu} = \frac{1}{\tilde{n}} \sum_{j=1}^{n} \omega_j \Delta Y_j \qquad \text{and} \qquad \widehat{\sigma}^2 = \frac{1}{\tilde{n}} \sum_{j=1}^{n} \omega_j \left(\Delta Y_j - \widehat{\mu}\right)^2$$

where $\omega_j = \mathbb{I}_{Y_{t_{j-1}^-} \notin [L,M]}$ and where $\tilde{n}$ is the number of inspection times during which no action or a perfect maintenance is done :

$$\tilde{n} = \#\left\{j \in \{1,\ldots,n\} \text{ such that } Y_{t_{j-1}^-} \notin [L,M]\right\} = \sum_{j=1}^{n} \omega_j.$$

We now consider the parameter ρ (maintenance part of the model). For any inspection time t_j such that $Y_{t_{j-1}^-} \in [L,M]$ (imperfect maintenance), we have considered the increment $Y_{t_j^-} - (1-\rho)Y_{t_{j-1}^-}$.

This must be positive and thus we have the following lower bound for ρ :

$$\rho \geqslant \rho^* = \max\left\{0; \max\left\{\frac{Y_{t_{j-1}^-} - Y_{t_j^-}}{Y_{t_{j-1}^-}} \ ; \ j \text{ such that } Y_{t_{j-1}^-} \in [L,M]\right\}\right\}.$$

Since 1 is an obvious upper bound, a good initial guess for ρ can be:

$$\widehat{\rho}_{init} = \frac{\rho^\star + 1}{2}.$$

Numerical implementation

In this Section, we illustrate the different developments achieved above throughout numerical computations and simulations.

Optimal maintenance policy

In this first subsection, we have computed the optimal threshold L for several cases reflecting different situations. Before reporting these results, we will illustrate the stationary distribution as described above, since the computations of the cost of a policy depend on this latter.

Stationary distribution For a given set of parameters $M = 50$, $\rho = 0.8$, $\alpha = 1$, $\beta = 1$, $b = 1$, in Figure 1, the stationary distribution of the underlying Markov renewal process is depicted. In left hand side of Figure 1, for a given preventive threshold $L = 40$, the stationary distributions associated to three different values of inspection interval $\tau \in \{1, 5, 10\}$ are illustrated. The right hand side figure shows different stationary distributions associated to $\tau = 1$ and six different values of preventive threshold $L \in \{20, 25, 30, 35, 40, 45\}$.

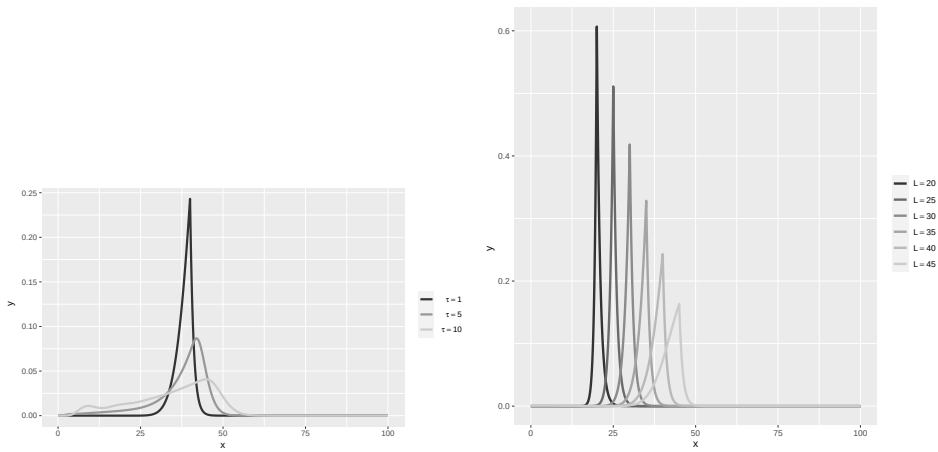

Figure 1. The stationary distribution in function of the inspection interval τ (left) and the preventive threshold L (right) for $\alpha = 1$, $\beta = 1$, $b = 1$, $\rho = 0.5$

Computations of the optimal preventive threshold For different scenarios of degradation, maintenance and cost parameters, according to the method described in previous sections the long run average maintenance cost is optimised and the optimal values of the preventive thresholds are obtained. In Figure 2, for two different parameters setting the long run average costs are depicted and the optimal preventive threshold can be deduced. Since in comparison to the corrective cost, the

preventive and inspection costs are low, therefore the preventive threshold is high and very close to the corrective threshold. Indeed, in this case, with frequent inspections, the risk of failure between two inspections is very low and a high preventive threshold is far enough to avoid failure.

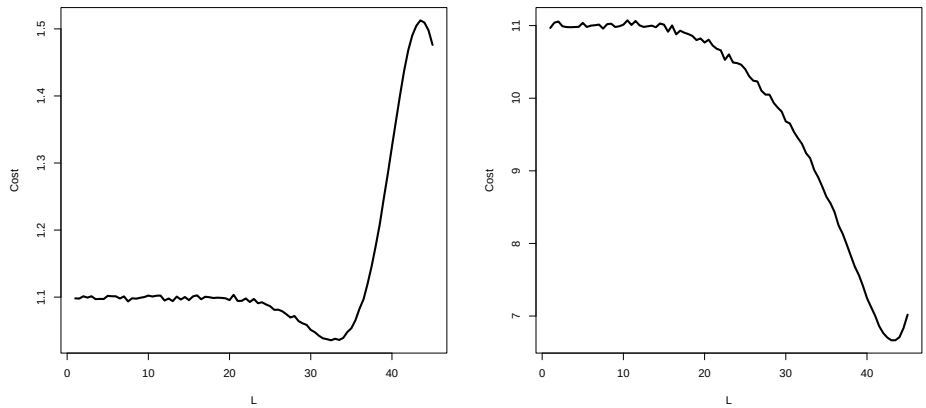

Figure 2. The optimal long run average maintenance cost in function of the preventive threshold, $\alpha = 0.5$, $\beta = 1$, $b = 1$, $\tau = 10$,$\rho = 0.5$ (Left), $\alpha = 2$, $\beta = 1$, $b = 1$, $\tau = 1$, $\rho = 0.8$ (Right) for $C_i = 1$, $C_p = 10$ and $C_c = 50$.

In the following paragraphs, for different parameters a sensitivity analysis is carried out.

Sensitivity analysis to the inspection interval In Figure 3, for two cases of slow and fast degradation, the optimal long run average maintenance cost is depicted in function of the inspection interval. The figure in the right hand side corresponds to $\alpha = 1$, $\beta = 1$, $b = 1$, whereas the figure in the left hand side corresponds to $\alpha = 2$, $\beta = 1$, $b = 1$. When the degradation is slow the maintenance actions rare and the total cost is mostly impacted by the inspection cost. That is why the optimal cost reduces with less frequent inspections (larger inspection interval). It can be noticed that when the degradation rate is high, it is possible to find a trade-off between inspections and maintenance operations. For a large ratio of corrective and preventive cost the maintenance policy can be optimised according to the inspection interval.

Sensitivity analysis to degradation parameters In Table 1 the variations of the optimal long run average maintenance cost and the

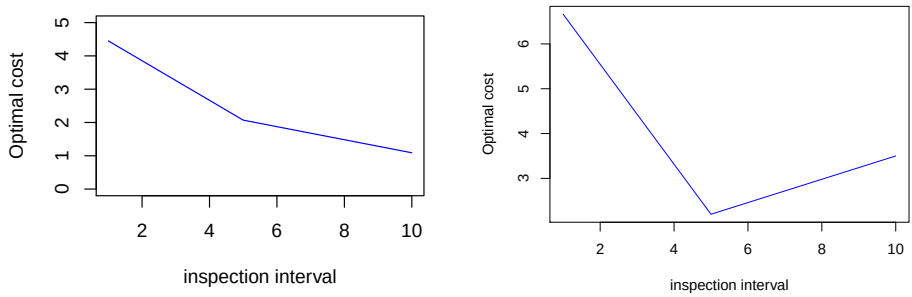

Figure 3. The optimal long run average maintenance cost versus the inspection interval, $\alpha = 1$, $\beta = 1$, $b = 1$ (Left), $\alpha = 2$, $\beta = 1$, $b = 1$ (Right) for $C_i = 1$, $C_p = 10$ and $C_c = 50$.

optimal preventive threshold for different values of α are presented. It can be noticed that the optimal long run average maintenance cost is an increasing function of α and therefore an increasing function of the degradation rate. For small values of α, where the degradation is slow and the risk of failure is low, especially with frequent inspections the optimal preventive threshold is very close to the corrective threshold. In these cases, the maintenance cost units could be key determinant. This latter will be studied in the next paragraphs.

α \ τ	1	5	10
0.5	0.46(43)	0.57(45)	1.04(32)
1	4.45(50)	2.07(32.5)	1.09(7)
2	6.66(43)	2.19(7)	3.5(7.5)

Table 1. The optimal maintenance cost (the optimal preventive threshold) for $\beta = 1$, $b = 1$, $\rho = 0.8$, $C_i = 1$, $C_p = 10$ and $C_c = 50$.

In Figure 4 the variations of the optimal long run average maintenance cost in function of the degradation parameter β are depicted. It can be noticed that the optimal long run average maintenance cost is a non-decreasing function of β. Indeed as β increases the degradation rate increases which leads to higher probability of maintenance actions and induces more costs.

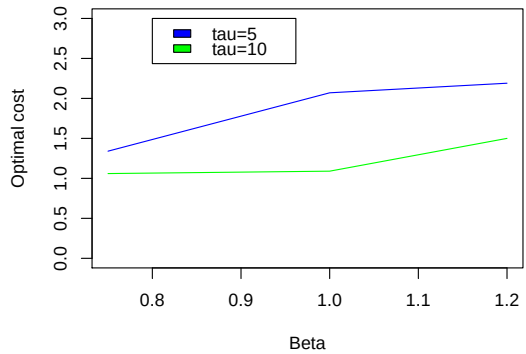

Figure 4. The optimal long run average maintenance cost versus the parameter β, $\alpha = 1$, $b = 1$, $C_i = 1$, $C_p = 10$ and $C_c = 50$.

Sensitivity analysis to the maintenance efficiency For different cases of inspection interval and $\alpha \leq 1$, the variations of the optimal long run average maintenance cost in function of the maintenance efficiency is depicted in Figure 5. It can be noticed that in these cases, the variations of the maintenance efficiency don't impact the optimal maintenance cost and preventive maintenance threshold. Indeed in this case, the system is inspected very often and the degradation level increase between two inspections is not very significant, therefore the risk of corrective action is not meaningful. The optimal maintenance threshold is very close to the failure threshold and even with high corrective cost the optimal maintenance cost remains insensible to maintenance efficiency. If the preventive cost was dependent to the maintenance efficiency, more variations could have been noticed. For $\alpha > 1$ and large inspection intervals the variations of the optimal long run average maintenance cost in function of the maintenance efficiency are given in Table 2. It can be noticed that in these cases, the variations of the maintenance efficiency impact the optimal maintenance cost and preventive maintenance threshold. For a small maintenance efficiency, the optimal policy does not recommend the preventive maintenance. Indeed, when $\alpha > 1$ the degradation is fast and the risk of exceeding the failure threshold between two inspections is very high. Therefore, if the preventive maintenance does not decrease the degradation level

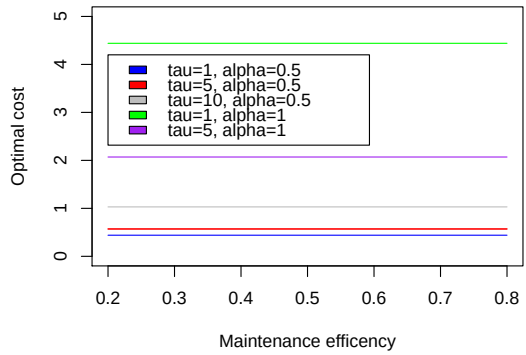

Figure 5. The optimal long run average maintenance cost versus the maintenance efficiency, $\beta = 1$, $b = 1$, $C_i = 1$, $C_p = 10$ and $C_c = 50$.

significantly, the maintenance operations will be mainly corrective actions.

(τ, α) \ ρ	0.2	0.5	0.8
$(10, 1)$	$2.1(50)$	$1.13(3.5)$	$1.09(7)$
$(5, 2)$	$4.2(50)$	$1.26(3.5)$	$2.19(7)$

Table 2. The optimal long run average maintenance cost (the optimal preventive threshold) versus the inspection interval and α for $\beta = 1$, $b = 1$, $C_i = 1$, $C_p = 10$ and $C_c = 50$.

Sensitivity analysis to cost units In Table 3, The optimal long run average maintenance costs and the optimal preventive thresholds for different cost ratio variations are given. As expected by increasing the corrective cost the optimal long run average maintenance cost increase.

Since the probability of corrective or preventive maintenance don't depend on the inspection cost, the corrective to inspection cost ratio does not impact the optimal preventive threshold. However, the preventive to corrective cost ratio impact significantly the costs and the optimal preventive threshold. In cases depicted in Table 3 the probability of corrective maintenance is maximum 0.1 during an inspection cycle and the preventive maintenance does not reduce significantly the deterioration.

	$C_i = \frac{C_c}{20}$	$C_i = \frac{C_c}{50}$	$C_i = \frac{C_c}{100}$	$C_i = \frac{C_c}{500}$
$C_c = 5C_p$	$11.82(45.5)$	$4.4(45)$	$7.82(45.5)$	$35.11(45.5)$
$C_c = 10C_p$	$8.6(44)$	$28.27(44)$	$4.65(44)$	$19, 24(44)$
$C_c = 50C_p$	$28.79(42)$	$13.79(42)$	$8.79(42)$	$8.8(42)$

Table 3. the optimal long run average maintenance costs (the optimal preventive thresholds) for different cost ratio when $\alpha = 1$, $\beta = 1$, $b = 1$,$\rho = 0.2$.

Parameter Estimation

First, we will introduce the different cases we have studied. These nine cases aim at recovering various situations with respect to the gamma process and with respect to preventive maintenance actions. Next we present the results we have obtained.

Benchmark cases In order to study the quality of the maximum likelihood estimator, we have defined a set of nine cases. More precisely, we have set $\alpha = 1$ and $b = 1$ for all cases, but $\beta \in \{0.75, 1, 1.2\}$ in order to recover various cases for the shape function (concave, linear and convex) and $\rho \in \{0.2, 0.5, 0.8\}$ in order to consider different degrees of the efficiency for the maintenance action. The different cases are summarized in the table 4. The level for a perfect maintenance has been set to $M = 10$. The characteristic of the maintenance action scheme was : $L = 5$ (level for performing an imperfect maintenance) and $\tau = 1$ (the delay between two successive inspection times).

Case	1	2	3	4	5	6	7	8	9
α	1.0	1.0	1.0	1.00	1.0	1.0	1.0	1.0	1.0
β	1.0	1.0	1.0	0.75	0.75	0.75	1.2	1.2	1.2
b	1.0	1.0	1.0	1.00	1.0	1.0	1.0	1.0	1.0
ρ	0.2	0.5	0.8	0.20	0.5	0.8	0.2	0.5	0.8

Table 4. Benchmark for simulation

Applications For each cases, we have simulated $N = 1,000$ sample paths of the degradation process subject to the maintenance policy as defined in Section 2. For each of simulation, we have estimated the four parameters. Hence, for each cases, we get $1,000$ estimates of the parameters. We have plotted the empirical distributions on Figures 6 to 14.

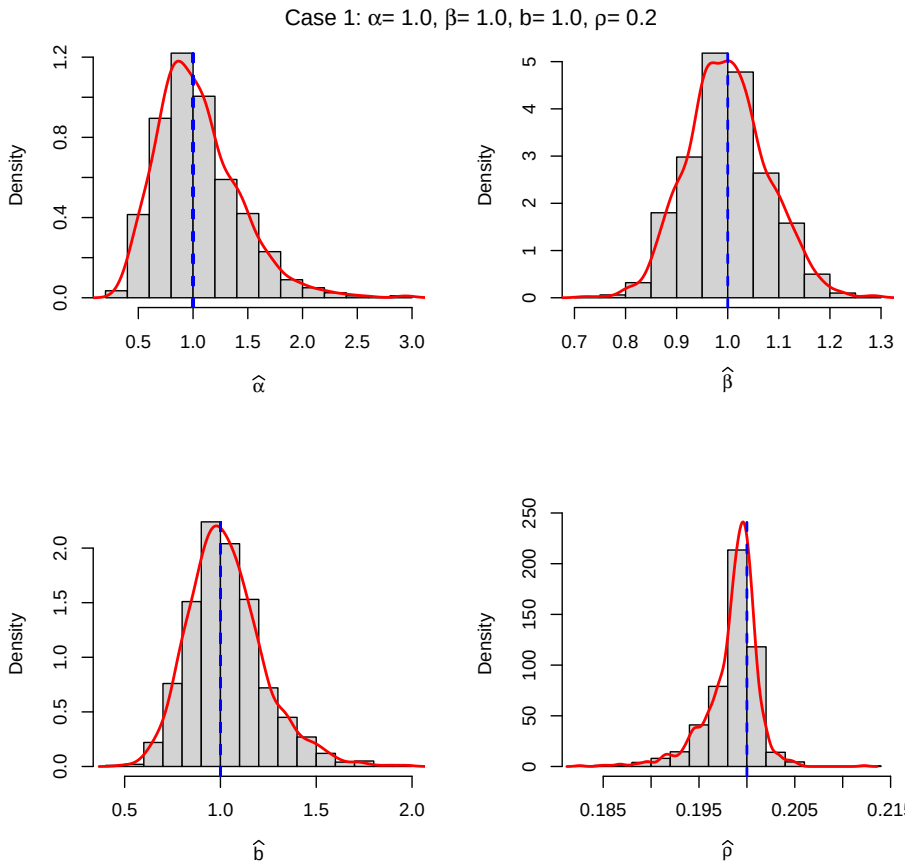

Figure 6. Estimation results for case 1: histogram of the estimator and the true value in blue

We can observe that the empirical distribution for $\widehat{\alpha}$ is quite asymmetric and right-tailed, whatever the case. For $\widehat{\beta}$, its empirical distribution is always relatively symmetric. For the two other parameters, it depends on the case. The empirical distribution for $\widehat{b}$ is in general symmetric, except when the shape function of the gamma process is concave (cases 4 to 6), while the empirical distribution for $\widehat{\rho}$ is mostly asymmetric, except when the shape function of the gamma process is convex (cases 7 to 9). In order to get more symmetric distributions with a reduced variance, one can apply

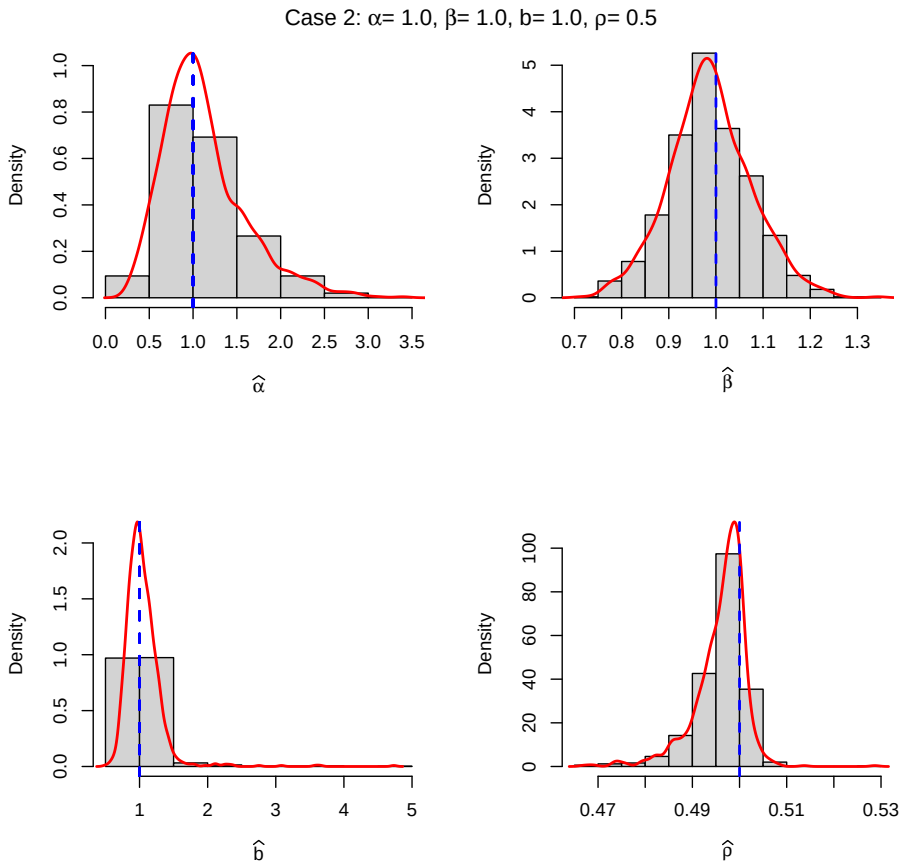

Figure 7. Estimation results for case 2: histogram of the estimator and the true value in blue

a suitable transformation to the estimator. For instance, for $\widehat{\alpha}$, $\widehat{\beta}$ and $\widehat{b}$, one can use a logarithmic transformation while for $\widehat{\rho}$, one can apply a logit or a probit transformation. Notice that, for some cases and for some parameters, there are some atypical estimations which affect naturally the empirical distributions.

To analyze the performance and the quality of the estimators, we have computed the empirical mean square errors (MSE) of the estimators, see Table 5. The results point out the very good performance of estimators specially in the case of $\widehat{\beta}$ and $\widehat{\rho}$. The efficiency of estimates permits to rely

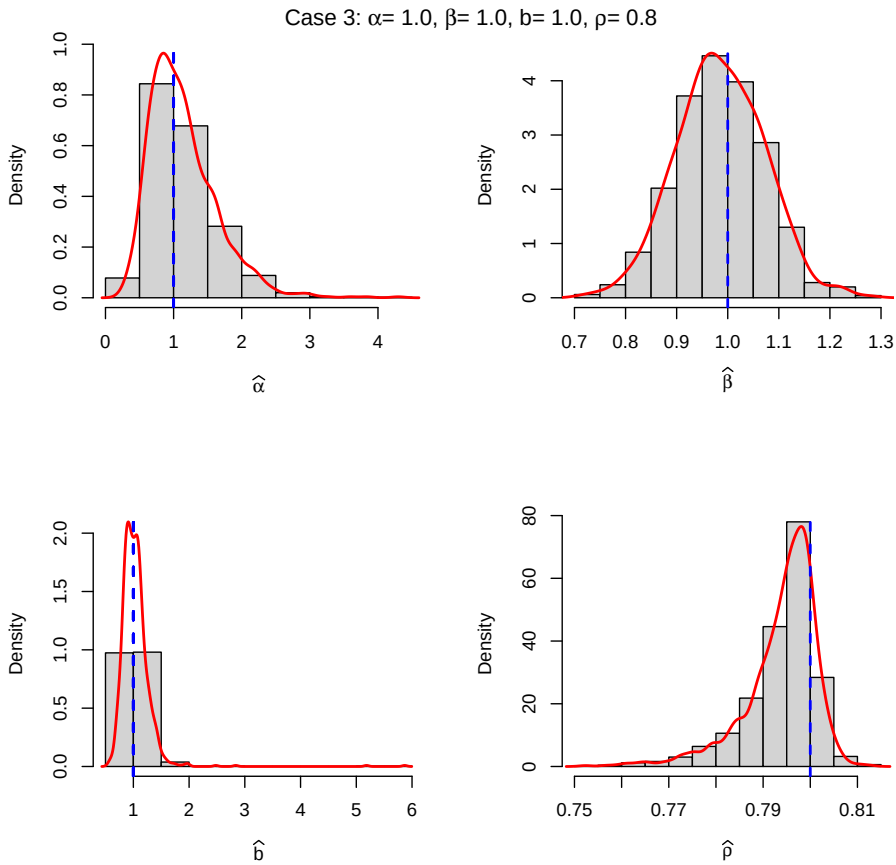

Figure 8. Estimation results for case 3: histogram of the estimator and the true value in blue

on the parameters estimates in presence of maintenance data in order to propose an optimal preventive threshold.

Estimation-based maintenance optimisation

This section gives a short study of a maintenance policy based on parameter estimators.

Optimal preventive threshold evaluation based on ρ and α estimator

With the large variability of $\widehat{\alpha}$ noticed in the previous examples, even for $\alpha = 1$ there is a high probability that $\widehat{\alpha}$ takes values lower than one.

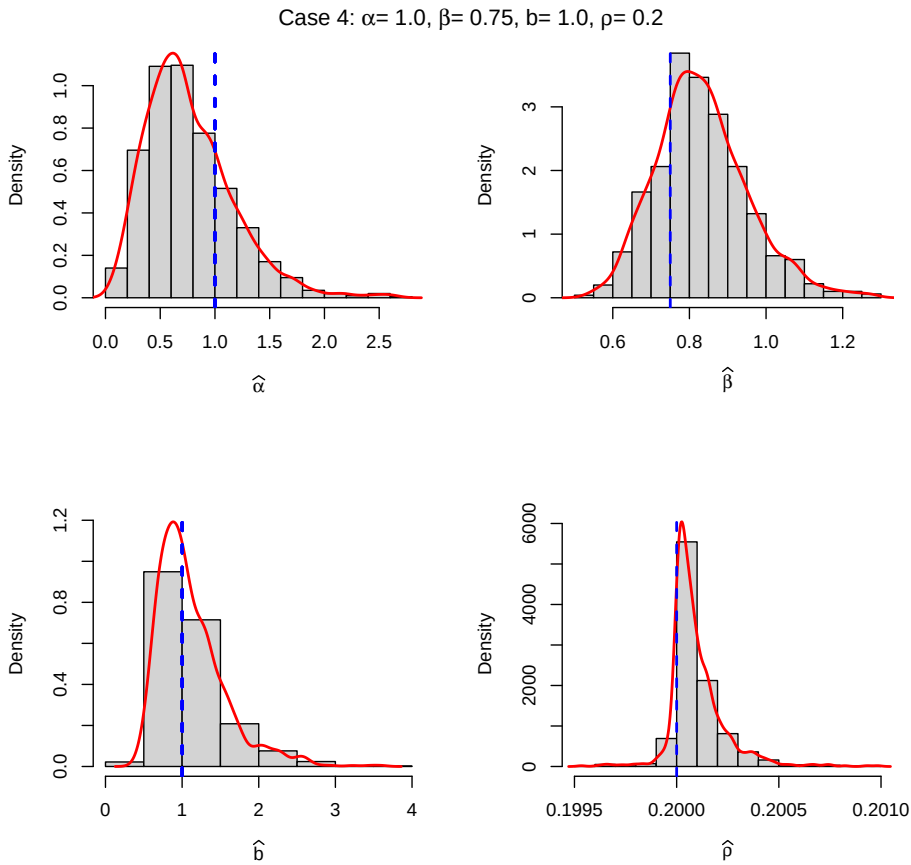

Figure 9. Estimation results for case 4: histogram of the estimator and the true value in blue

In these cases as noticed in our sensitivity analysis the optimal costs and thresholds are insensitive to the maintenance efficiency. Whereas the optimal results can vary for different maintenance efficiency when $\widehat{\alpha}$ takes values higher than 1. Therefore according to estimation error of α, the impact of the estimation error of the maintenance efficiency can be very different from negligible to very significant.

Moreover, as it has been pointed out in the previous paragraphs the optimal threshold is very sensitive to the value of α. As it can be noticed in Tables 6 and 7, the estimation errors α can induce a large difference in the

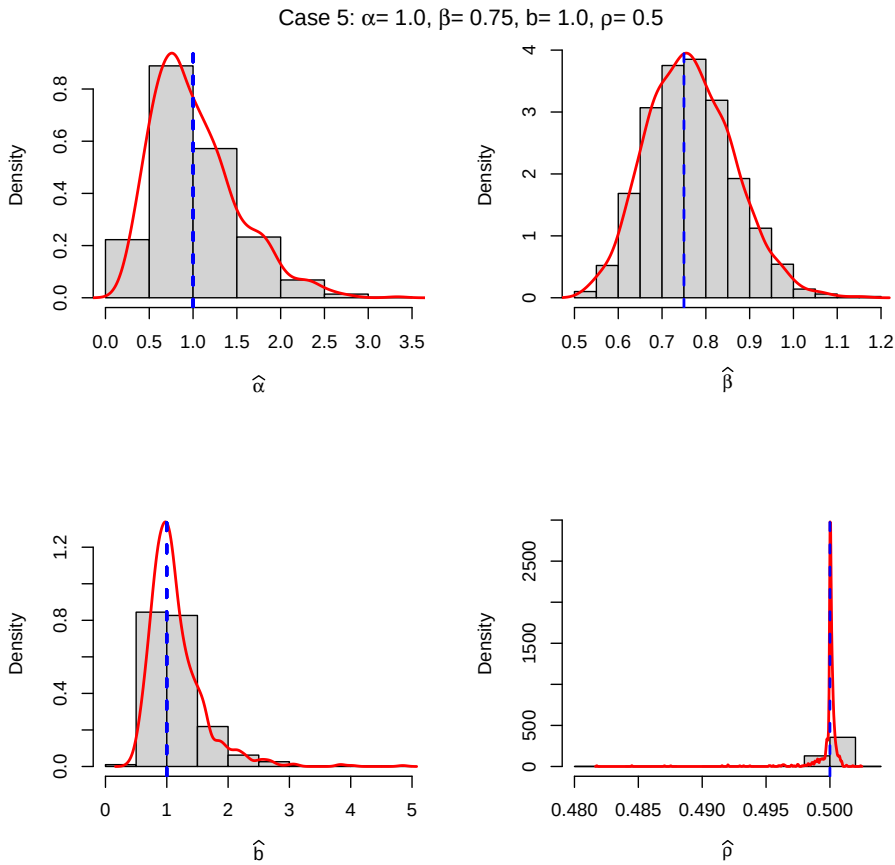

Figure 10. Estimation results for case 5: histogram of the estimator and the true value in blue

optimal preventive threshold but do not change significantly the optimal cost.

Since the $\widehat{\rho}$ has a relatively small variability and its influence on the long run average maintenance cost depends mainly on the values of α, one can consider that the estimations errors of ρ has not a substantial impact on the optimal cost.

Optimal preventive threshold evaluation based on β estimator The previous observations seem not always valid for the estimation error of β when $\beta > 1$. As it can be noticed in Tables 8 and 9, the variations of β can

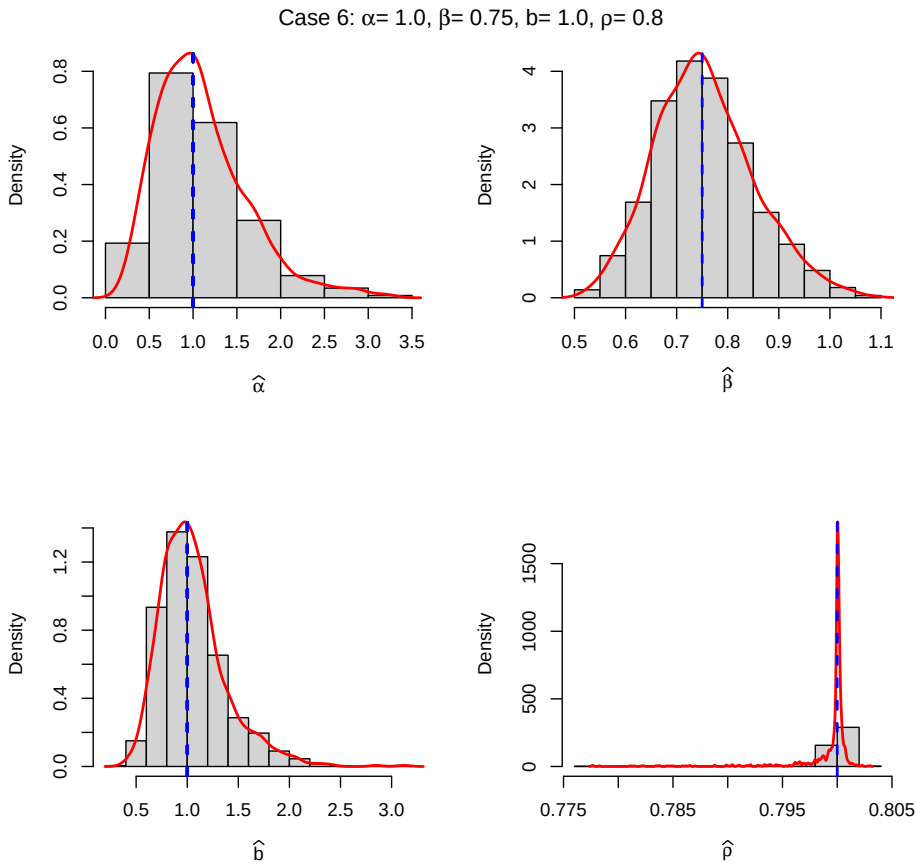

Figure 11. Estimation results for case 6: histogram of the estimator and the true value in blue

induce a large difference in the optimal optimal cost. Hence the impact of the estimation error is non negligible. For instance for $\beta = 1.2$ if $\widehat{\beta} = 1$, using the optimal threshold for $\beta = 1$ the optimal maintenance cost will double. Since the variability of $\widehat{\beta}$ seems relatively small, the maintenance policy based on the $\widehat{\beta}$ could still be reliable.

Conclusions

In this paper, a system deteriorating according to a gamma process is considered. In the framework of a condition-based maintenance policy,

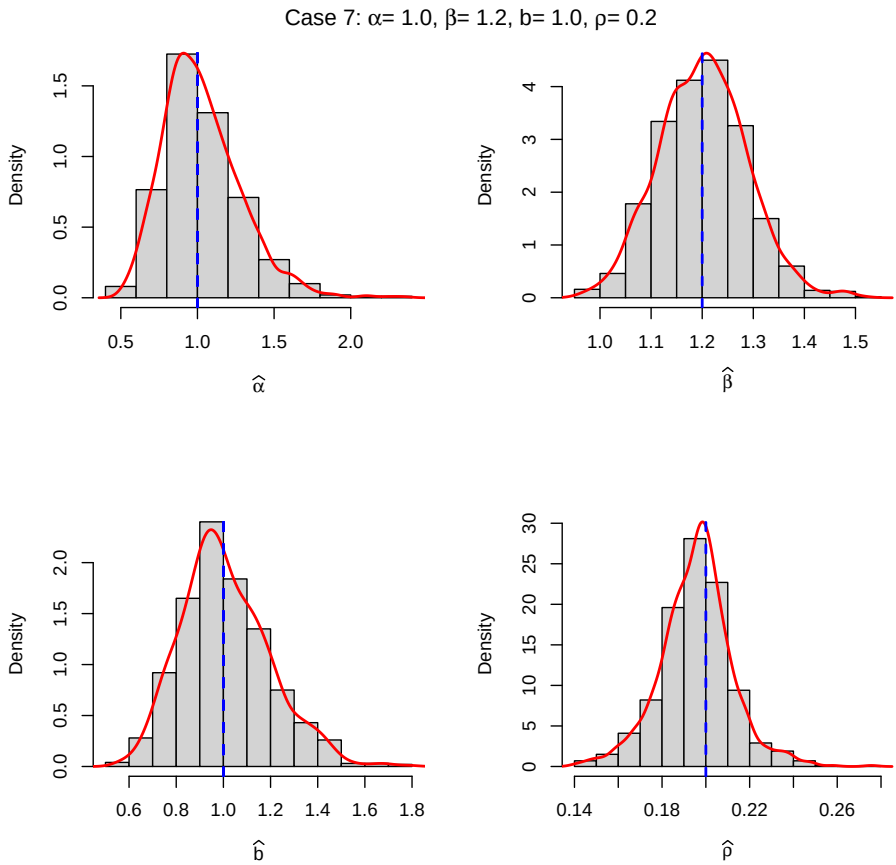

Figure 12. Estimation results for case 7: histogram of the estimator and the true value in blue

imperfect preventive maintenance actions and perfect corrective actions are studied. A maintenance optimisation procedure based on the cost is proposed and a sensitivity analysis is carried out. Parameter estimation in presence of maintenance data is considered and their efficiency is discussed through numerical implementations.

This work can be extended with the following perspectives: statistical inference in the case of a random maintenance efficiency, maintenance optimisation considering the availability cost, maintenance optimisation with respect to three parameters, the maintenance efficiency, the

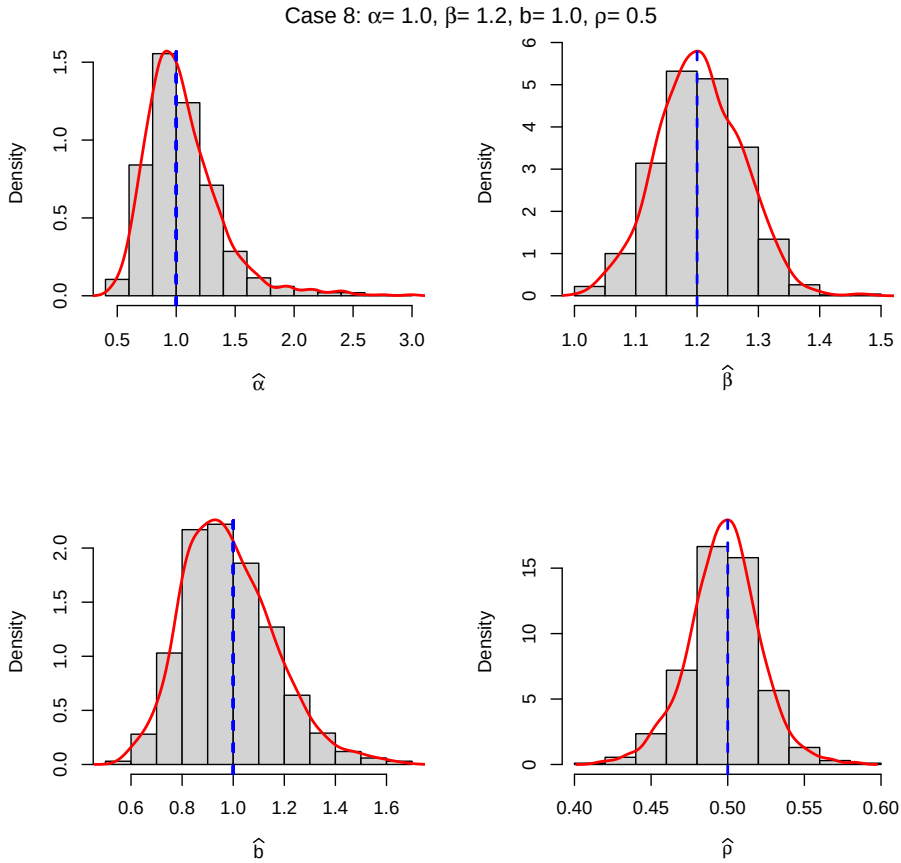

Figure 13. Estimation results for case 8: histogram of the estimator and the true value in blue

inspection interval and the preventive threshold, considering semi-parametric estimation results in maintenance optimisation, estimation uncertainty propagation on the optimal maintenance parameters.

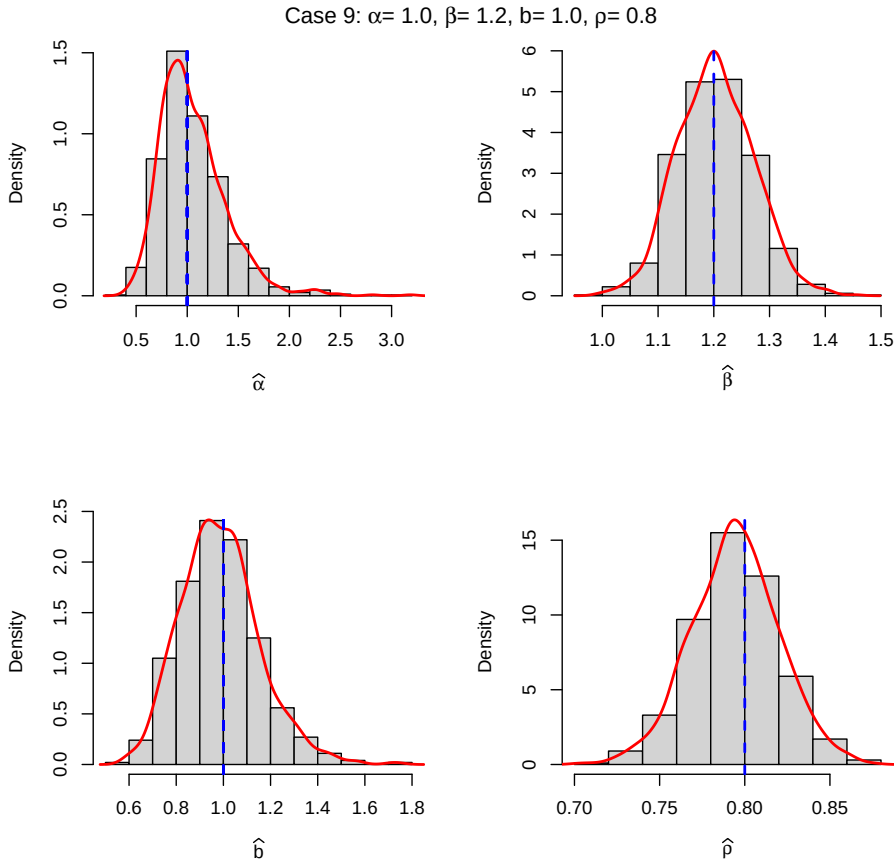

Figure 14. Estimation results for case 9: histogram of the estimator and the true value in blue

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Case	$\widehat{\alpha}$	$\widehat{\beta}$	$\widehat{b}$	$\widehat{\rho}$
1	0.14501960	0.00623174	0.04063781	0.00000940
2	0.23953934	0.00777192	0.08207026	0.00004514
3	0.26441177	0.00773402	0.08595129	0.00009757
4	0.21386479	0.02123256	0.20623311	0.00000003
5	0.23802037	0.00985090	0.22662213	0.00000162
6	0.28267709	0.00936353	0.11017564	0.00000811
7	0.06464378	0.00736223	0.03520747	0.00029121
8	0.10561423	0.00471150	0.03194594	0.00055110
9	0.10965295	0.00461064	0.02813155	0.00069049

Table 5. Empirical MSE

$\widehat{\alpha}$	C_{opt}	L_{opt}
0.5	1.10	32.5
1	1.06	45
2	1.10	7.5

Table 6. The optimal long run average maintenance cost (the optimal preventive threshold) versus estimated values of α for $\alpha = 1$, $\beta = 1$, $b = 1$, $\tau = 0$, $\rho = 0.8$ $C_i = 1$, $C_p = 10$ and $C_c = 50$.

$\widehat{\alpha}$	C_{opt}	L_{opt}
0.5	3.9	32.5
1	3.65	45
2	3.5	7.5

Table 7. The optimal long run average maintenance cost (the optimal preventive threshold) versus estimated values of α for $\alpha = 2$, $\beta = 1$, $b = 1$, $\tau = 0$, $\rho = 0.8$ $C_i = 1$, $C_p = 10$ and $C_c = 50$.

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$\widehat{\beta}$	C_{opt}	L_{opt}
0.75	1.06	45
1	1.06	45
1.2	1.10	7.5

Table 8. The optimal long run average maintenance cost (the optimal preventive threshold) versus estimated values of β for $\alpha = 1$, $\beta = 1$, $b = 1$, $\tau = 10$, $\rho = 0.8$ $C_i = 1$, $C_p = 10$ and $C_c = 50$.

$\widehat{\beta}$	C_{opt}	L_{opt}
0.75	2.9	45
1	2.9	45
1.2	1.51	7.5

Table 9. The optimal long run average maintenance cost (the optimal preventive threshold) versus estimated values of β for $\alpha = 1$, $\beta = 1.2$, $b = 1$, $\tau = 10$, $\rho = 0.8$ $C_i = 1$, $C_p = 10$ and $C_c = 50$.

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