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TOPOLOGICAL MODULI SPACE FOR GERMS OF HOLOMORPHIC FOLIATIONS III: COMPLETE FAMILIES

DAVID MARÍN, JEAN-FRANÇOIS MATTEI AND ELIANE SALEM

January 19, 2022

Abstract. In this work we use our previous results on the topological classification of generic singular foliation germs on $(\mathbb{C}^2, 0)$ to construct complete families: after fixing the semi-local topological invariants we prove the existence of a minimal family of foliation germs that contain all the topological classes and such that any equisingular global family with parameter space an arbitrary complex manifold factorizes through it.

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1. Introduction

This paper is the outcome of a series of three works on the topological classification of germs of singular foliations in the complex plane. In [8], after fixing the topological invariants already known [7], we have constructed a moduli space of topological classes. Then in [9], we have studied small perturbations of a generic

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foliation by proving the existence of a "topologically universal" deformation and by representing the "deformation functor". In the present paper we rely on these two results and in particular on the algebraic structure of the moduli space highlighted in [8] to obtain a "complete family". Here "complete" means a family that contains all the topological classes and such that any equisingular global family with parameter space an arbitrary complex manifold factorizes through it via a multivalued map.

We shall only consider germs of foliations on $(\mathbb{C}^2, 0)$ that are "**generalized curves**" [1], in the sense that on the **exceptional divisor** $\mathcal{E}_{\mathcal{F}} := E_{\mathcal{F}}^{-1}(0)$ of the reduction of singularities map $E_{\mathcal{F}} : M_{\mathcal{F}} \to \mathbb{C}^2$, there are no singularities of the **reduced foliation** $\mathcal{F}^\sharp := E_{\mathcal{F}}^{-1}(\mathcal{F})$ of **saddle-node** type[1], cf. [2, 5]. The foliation $\mathcal{F}^\sharp$ may have **dicritical** (i.e. non-invariant) irreducible components of $\mathcal{E}_{\mathcal{F}}$ and also **nodal**[2] singularities.

We will say that two germs of foliations $\mathcal{F}$ and $\mathcal{G}$ have **same SL-type** if there exists a homeomorphism $\varphi : \mathcal{E}_{\mathcal{F}} \xrightarrow{\sim} \mathcal{E}_{\mathcal{G}}$ satisfying:

(SL1) $\varphi(\mathrm{Sing}(\mathcal{F}^\sharp)) = \mathrm{Sing}(\mathcal{G}^\sharp)$ and $\varphi(D) \cdot \varphi(D') = D \cdot D'$,

(SL2) if $D \subset \mathcal{E}_{\mathcal{F}}$ is $\mathcal{F}^\sharp$-invariant and $p \in D$ is a singular point of $\mathcal{F}^\sharp$, then the Camacho-Sad indices of $\mathcal{F}^\sharp$ at p along D and of $\mathcal{G}^\sharp$ at $\varphi(p)$ along $\varphi(D)$, are equal,

(SL3) if $\mathcal{H}_D^{\mathcal{F}} : \pi_1(D \setminus \mathrm{Sing}(\mathcal{F}^\sharp), \cdot) \to \mathrm{Diff}(\mathbb{C}, 0)$ denotes the holonomy morphism of $\mathcal{F}^\sharp$ along an invariant component $D \subset \mathcal{E}_{\mathcal{F}}$ and φ_* denotes the morphism induced by φ at the fundamental groups level, then up to composition by an inner automorphism of $\mathrm{Diff}(\mathbb{C}, 0)$, $\mathcal{H}_D^{\mathcal{F}} \circ \varphi_*^{-1}$ is the holonomy morphism of $\mathcal{G}^\sharp$ along $\varphi(D)$.

When all foliations of a family have same SL-type, the family will be called **equisingular**. This notion, which presupposes local equireduction of the considered family, is specifically defined in Section 2.1. For an equisingular family $\mathcal{F}_P$ with parameter space a complex manifold P, we will denote by $\mathcal{F}_P(t_0) := \mathcal{F}_P|_{t=t_0}$ the **fiber** over a particular value $t_0 \in P$ and by $\mathcal{F}_{P,t_0}$ the germ of this family along the fiber $\mathcal{F}_P(t_0)$.

The notion of **tame foliation** will be defined in section 4.1. If we exclude some exceptional configurations on $\mathcal{E}_{\mathcal{F}}$, the set of differential forms defining tame foliations contains a Krull open dense set. Our main result asserts the existence in this context of an equisingular global family which is "topologically complete". We also describe the (minimal) redundancy of their topological classes.

Theorem A. *Let $\mathcal{F}$ be a tame foliation. Then there exist $\tau \in \mathbb{N}$, a quotient $\mathbf{D}$ of a finite product of totally disconnected subgroups of $\mathbb{U}(1)$ and an equisingular global family $\underline{\mathcal{F}}_{\mathbf{U}}$ over $\mathbf{U} = \mathbb{C}^\tau \times \mathbf{D}$ such that*

(1) for any foliation $\mathcal{G}$ with same SL-type as $\mathcal{F}$, there exists $u_0 \in \mathbf{U}$ such that $\mathcal{G}$ is $\mathcal{C}^0$-conjugated to the fiber $\mathcal{F}_{\mathbf{U}}(u_0)$,

(2) if P is a connected and simply connected manifold, $t_0 \in P$ and $\underline{\mathcal{G}}_P$ is an equisingular global family whose fiber $\mathcal{G}_P(t_0)$ is $\mathcal{C}^0$-conjugated to a fiber $\mathcal{F}_{\mathbf{U}}(u_0)$, then there exists a holomorphic map $\lambda : P \to \mathbf{U}$ such that $\lambda(t_0) = u_0$, and for any $t \in P$ the germs of families $\underline{\mathcal{G}}_{P,t}$ and $\lambda^* \underline{\mathcal{F}}_{\mathbf{U},\lambda(t)}$ over the germ of manifold (P, t), are $\mathcal{C}^0$-conjugated,

[1] i.e. locally defined by a vector field germ whose linear part has exactly one non-zero eigenvalue.

[2] i.e. locally defined by a vector field germ such that the ratio of the eigenvalues of its linear part is strictly positive.

(3) there exist $p \in \mathbb{N}$, a holomorphic action $*$ of $\mathbb{Z}^p$ on $\mathbf{U}$ and an action $\star$ of a finite product $\mathbf{I}$ of braid groups on the quotient $(\mathbf{U}/*\mathbb{Z}^p)$, such that $\mathcal{F}_{\mathbf{U}}(u_1)$ and $\mathcal{F}_{\mathbf{U}}(u_2)$ are $\mathcal{C}^0$-conjugated if and only if there exists $g \in \mathbf{I}$ such that $g \star (\mathbb{Z}^p * u_1) = \mathbb{Z}^p * u_2$.

Tame foliations have the remarkable property that any two $\mathcal{C}^0$-conjugated tame foliations are also conjugated by an **excellent homeomorphism**, i.e. one that lifts through the reduction of singularities which is holomorphic at the non-nodal singular points, cf. Theorem 4.1. This result extends to equisingular families of tame foliations:

Theorem B. *Let $\underline{\mathcal{F}}_Q$ and $\underline{\mathcal{G}}_Q$ be two equisingular global families of foliations over a complex manifold Q, whose fibers are tame. Then the following properties are equivalent:*

(1) for any $u \in Q$ the fibers $\mathcal{F}_Q(u)$ and $\mathcal{G}_Q(u)$ are $\mathcal{C}^0$-conjugated,
(2) for any $u \in Q$ the fibers $\mathcal{F}_Q(u)$ and $\mathcal{G}_Q(u)$ are $\mathcal{C}^{\mathrm{ex}}$-conjugated,
(3) the global families $\underline{\mathcal{F}}_Q$ and $\underline{\mathcal{G}}_Q$ are locally $\mathcal{C}^{\mathrm{ex}}$-conjugated.

A **$\mathcal{C}^{\mathrm{ex}}$-conjugacy** of families is a $\mathcal{C}^0$-conjugacy of families that lifts through the local equireduction maps as a homeomorphism which is holomorphic at the non-nodal singularities, cf. §2.1.

Theorems A and B will follow from analogous results in the context of marked foliations that allow to use the moduli space of $\mathcal{C}^{\mathrm{ex}}$-conjugacy classes of marked foliations constructed in [8]. A **marking** of $\mathcal{F}$ by a marked divisor $(\mathcal{E}, \Sigma, \cdot)$ is a homeomorphism $f : \mathcal{E} \to \mathcal{E}_{\mathcal{F}}$ such that $f(\Sigma) = \mathrm{Sing}(\mathcal{F}^\sharp)$ and $f(D) \cdot f(D') = D \cdot D'$, cf. §2.2. When $\mathcal{F}$ and $\mathcal{G}$ are endowed with markings $f : \mathcal{E} \to \mathcal{E}_{\mathcal{F}}$ and $g : \mathcal{E} \to \mathcal{E}_{\mathcal{G}}$ by a common marked divisor $(\mathcal{E}, \Sigma, \cdot)$, we will say that the marked foliations $(\mathcal{F}, f)$ and $(\mathcal{G}, g)$ have **same marked SL-type** if conditions (SL1)-(SL3) are fulfilled and moreover $g^{-1} \circ \varphi \circ f$ is isotopic to the identity of $\mathcal{E}$ relatively to Σ. The following analogue of Theorem A in the marked setting holds for the larger class of **finite type foliations** introduced in [8, §6] and specified in [9, §5]; in this context we have the uniqueness of the factorization $\lambda : P \to \mathbf{U}$ of a marked family under a weaker topological condition on its parameter space P.

Theorem C. *Let $\mathcal{F}^\diamond = (\mathcal{F}, f)$ be a marked finite type foliation which is a generalized curve. Then there exists a marked equisingular global family of foliations $\underline{\mathcal{F}}^\diamond_{\mathbf{U}}$ over $\mathbf{U} = \mathbb{C}^\tau \times \mathbf{D}$ such that*

(0) $\mathbf{D}$ is a quotient of a finite product of totally disconnected subgroups of $\mathbb{U}(1)$ and τ is the dimension of the cohomological space $H^1(\mathsf{A}_{\mathcal{F}}, \mathcal{T}_{\mathcal{F}})$ (that we recall in (4)) whose finiteness characterizes the finite type of $\mathcal{F}$, cf. [9, Theorem 5.15],
(1) if $\mathcal{G}^\diamond = (\mathcal{G}, g)$ is a marked foliation with same marked SL-type as $\mathcal{F}^\diamond$, there exists $u_0 \in \mathbf{U}$ such that $\mathcal{G}^\diamond$ is $\mathcal{C}^{\mathrm{ex}}$-conjugated to $\mathcal{F}^\diamond_{\mathbf{U}}(u_0)$,
(2) if P is a connected manifold satisfying $H_1(P, \mathbb{Z}) = 0$, $t_0 \in P$ and $\underline{\mathcal{G}}^\diamond_P$ is a marked equisingular global family whose fiber $\mathcal{G}^\diamond_P(t_0)$ is $\mathcal{C}^{\mathrm{ex}}$-conjugated to a fiber $\mathcal{F}^\diamond_{\mathbf{U}}(u_0)$ as marked foliations, then there exists a unique holomorphic map $\lambda : P \to \mathbf{U}$ such that $\lambda(t_0) = u_0$ and for any $t \in P$ the germs of marked families $\underline{\mathcal{G}}^\diamond_{P,t}$ and $\lambda^* \underline{\mathcal{F}}^\diamond_{\mathbf{U},\lambda(t)}$, over the germ of manifold (P, t), are $\mathcal{C}^{\mathrm{ex}}$-conjugated.

An analogue of Theorem B in the marked setting will be given in Theorem 4.4.

We will also compare the conjugation notions of local families and deformations. A **deformation of** $\mathcal{F}$ is the data of a family $\underline{\mathcal{F}}_{P,t_0}$ over a germ of holomorphic manifold P at a point t_0 and a biholomorphism that identifies $\mathcal{F}$ with the fiber $\mathcal{F}_P(t_0)$. A **conjugacy of deformations** of $\mathcal{F}$ is a conjugacy of the associated families compatible with the corresponding biholomorphisms, cf. §2.1. In Theorem 4.5 we show that this compatibility condition is automatically fulfilled in the context of marked germs of families.

The central point of the paper is Theorem 2.9. It states the $\mathcal{C}^{\text{ex}}$-universality of the germ at any point of the parameter space of the global family $\underline{\mathcal{F}}_{\mathbf{U}}$ constructed in [8] that contains all the topological types in a fixed SL-class. This property will be proven by explicitly computing the Kodaira-Spencer map of this family at each point that provides an infinitesimal characterization of $\mathcal{C}^{\text{ex}}$-universality.

In Chapter 3 we look at the problem of existence of factorizations of global families through $\underline{\mathcal{F}}_{\mathbf{U}}$. Since Theorem 2.9 provide local factorizations, obtaining a global factorization is reduced to a gluing problem; the group structure of the moduli space obtained in [8] allows to translate it into a cohomological problem that can be solved under weak topological assumptions on the parameter space of the global family.

All the study in Chapters 2 and 3, leading to Theorem C, is made for marked families modulo $\mathcal{C}^{\text{ex}}$-conjugacy and only under the finite type assumption. But Theorems A and B concern non-marked global families and $\mathcal{C}^0$-conjugacies. To work with $\mathcal{C}^0$-conjugacies we require additional (Krull generic) hypothesis defining tame foliations in §4.1, which allow to prove Theorem A. The proof of Theorem B in Section 4.2 is based again in the group structure on the moduli space using the fact that the mapping class group of the exceptional divisor is countable.

Through all the paper a $\mathcal{C}^0$-conjugacy between two foliations is a homeomorphism sending leaves into leaves and preserving the orientation of the ambient spaces as well as the orientation of the leaves.

2. Locally universal family

2.1. **Equisingular global families and deformations.** We call **global family of (germs of) foliations over a complex manifold** Q (not necessarily connected), the data

$$\underline{\mathcal{F}}_Q := (M, \pi, \theta, \mathcal{F}_Q)$$

of a complex manifold M with $\dim(M) = \dim(Q) + 2$, a holomorphic surjective submersion $\pi : M \to Q$, a holomorphic section $\theta : Q \to M$ of π, and a germ along $\theta(Q)$ of a one dimensional holomorphic foliation $\mathcal{F}_Q$ on M whose leaves are contained in the fibers of π. We say that (M, π) is a **manifold over** Q. For each $u \in Q$ we consider in the fiber of π over u the germ of foliation at $\theta(u)$ obtained by restricting $\mathcal{F}_Q$:

$$M(u) := \pi^{-1}(u)\,, \quad \mathcal{F}_Q(u) := \mathcal{F}_Q|_{(M(u),\theta(u))}\,.$$

The family is **equireducible** if $\theta(Q)$ is the singular locus of $\mathcal{F}_Q$ and for any point $u_0 \in Q$ there is an open **trivializing neighborhood** $W \ni u_0$ and a map called

(minimal) equireduction map over W

$$E_{\mathcal{F}_W} : M_{\mathcal{F}_W} \to M_W := \pi^{-1}(W)$$

that is defined by a sequence of blow-ups with etale centers over W, and whose restriction to each fiber

$$M_{\mathcal{F}_W}(u) := \pi^{\sharp\,-1}(u)\,, \quad \pi^\sharp := \pi \circ E_{\mathcal{F}_W}\,, \quad u \in W\,,$$

is exactly the minimal reduction map of $\mathcal{F}_Q(u)$, and moreover the singular locus of the reduced foliation $\mathcal{F}_W^\sharp$ in $M_{\mathcal{F}_W}$ is also etale over W. A more detailed definition of this notion is given in [9, §2.2] or in [8, Chapter 10, step (vi)]. Up to shrinking the neighborhood W of u_0, the exceptional divisor $\mathcal{E}_{\mathcal{F}_W} = E_{\mathcal{F}_W}^{-1}(\theta(W))$ and the singular locus of the reduced foliation $\mathcal{F}_W^\sharp$ in $M_{\mathcal{F}_W}$ are topologically trivial: there exists a **trivializing homeomorphism over** W

$$\Psi_W : \mathcal{E}_{\mathcal{F}_W} \xrightarrow{\sim} \mathcal{E}_{\mathcal{F}_W}(u_0) \times W\,, \quad \mathrm{pr}_W \circ \Psi_W = \pi^\sharp_{|\mathcal{E}_{\mathcal{F}_W}}\,, \quad \mathcal{E}_{\mathcal{F}_W}(u) := \mathcal{E}_{\mathcal{F}_W} \cap \pi^{\sharp\,-1}(u)\,, \quad (1)$$

that sends the singular locus of $\mathcal{F}_W^\sharp$ on the product $\mathrm{Sing}(\mathcal{F}_W^\sharp(u_0)) \times W$, with

$$\mathcal{F}_W^\sharp(u) := \mathcal{F}_W^\sharp|_{\pi^{\sharp\,-1}(u)}\,.$$

Restricted to the fiber of $u \in W$, Ψ_W provides a homeomorphism that identifies the exceptional divisor of the reduction of $\mathcal{F}_W(u)$, with that of $\mathcal{F}_W(u_0)$,

$$\Psi_u : \mathcal{E}_{\mathcal{F}_W}(u) \xrightarrow{\sim} \mathcal{E}_{\mathcal{F}_W}(u_0)\,.$$

Thus the **holonomy** of the foliation $\mathcal{F}_W^\sharp(u)$ along an invariant component $D_u = \Psi_u^{-1}(D_{u_0})$ may be considered as a morphism $\mathcal{H}_{D_u}$ from the fundamental group of $D_{u_0} \setminus \mathrm{Sing}(\mathcal{F}_W(u_0))$ into the group $\mathrm{Diff}(\mathbb{C}, 0)$ of germs of biholomorphisms of $(\mathbb{C}, 0)$.

Definition 2.1. *We say that an equireducible family $\underline{\mathcal{F}}_Q$ is **equisingular at** $u_0 \in Q$ if there is a trivializing neighborhood W of u_0 such that for any invariant irreducible component $D_{u_0} \subset \mathcal{E}_{\mathcal{F}_W}(u_0)$ and for any point $m_0 \in \mathrm{Sing}(\mathcal{F}_W(u_0)) \cap D_{u_0}$, we have:*

(a) there exist biholomorphisms $g_u \in \mathrm{Diff}(\mathbb{C}, 0)$ depending holomorphically of $u \in W$ such that $g_u \circ \mathcal{H}_{D_u} \circ g_u^{-1} = \mathcal{H}_{D_{u_0}}$,

*(b) the **Camacho-Sad function** from W to $\mathbb{C}$:*

$$u \mapsto \mathrm{CS}(\mathcal{F}_W(u), D_u, m_u), \quad D_u := \Psi_W^{-1}(D_{u_0} \times \{u\})\,, \quad m_u := \Psi_W^{-1}(m_0, u)\,,$$

is constant.

A $\mathcal{C}^0$**-conjugacy** between two global equireducible families $\underline{\mathcal{F}}_Q = (M, \pi, \theta, \mathcal{F}_Q)$ and $\underline{\mathcal{F}}'_Q = (M', \pi', \theta', \mathcal{F}'_Q)$ over the same parameter space Q, is a germ of homeomorphism $\Phi : (M, \theta(Q)) \xrightarrow{\sim} (M', \theta'(Q))$ satisfying $\Phi(\mathcal{F}_Q) = \mathcal{F}'_Q$ and $\pi' \circ \Phi = \pi$. We also assume that Φ preserves the orientation of the ambient spaces and the orientation of the leaves. We will say that Φ is **excellent** or **of class** $\mathcal{C}^{\mathrm{ex}}$, if its lifting $\Phi^\sharp$ through any local equireduction maps $E_{\mathcal{F}_W}$ and $E_{\mathcal{F}'_W}$, $E_{\mathcal{F}'_W} \circ \Phi^\sharp = \phi \circ E_{\mathcal{F}_W}$, extends to the exceptional divisors, providing a homeomorphism

$$\Phi^\sharp : (M_{\mathcal{F}_W}, \mathcal{E}_{\mathcal{F}_W}) \xrightarrow{\sim} (M_{\mathcal{F}'_W}, \mathcal{E}_{\mathcal{F}'_W})$$

which is holomorphic at any singular point of the exceptional divisor that is a non nodal singular point of the foliation. We will also say that $\Phi^\sharp$ **is excellent**.

Let $\mu : P \to Q$ be a holomorphic map and let $\underline{\mathcal{F}}_Q = (M, \pi, \theta, \mathcal{F}_Q)$ be a global family of foliations over Q. We consider the fibered product $\mu^*M = M \times_Q P \subset M \times P$ with the projection $\mu^*\pi : M \times_Q P \to P$,

$$\begin{array}{ccc} \mu^*M & \xrightarrow{\rho_\mu} & M \\ \downarrow{\scriptstyle \mu^*\pi} & & \downarrow{\scriptstyle \pi} \\ P & \xrightarrow{\mu} & Q \end{array} \qquad \begin{array}{ccc} (\mu^*M)(t) & \xrightarrow[\rho_\mu]{\sim} & M(\mu(t)) \\ \downarrow & & \downarrow \\ t & \longmapsto & \mu(t) \end{array}$$

and the section $\mu^*\theta = (\theta \circ \mu) \times \mathrm{id}_P : P \to M \times_Q P$. Since the restrictions to each fiber of the canonical submersion ρ_μ are biholomorphisms, there is a unique one-dimensional foliation germ $\mu^*\mathcal{F}_Q$ on μ^*M along $(\mu^*\theta)(P)$, tangent to the fibers of $\mu^*\pi$, such that ρ_μ sends the leaves of $\mu^*\mathcal{F}_Q$ into the leaves of $\mathcal{F}_Q$. We will call $\mu^*\underline{\mathcal{F}}_Q = (\mu^*M, \mu^*\pi, \mu^*\theta, \mu^*\mathcal{F}_Q)$ the **pull-back** of the global family $\underline{\mathcal{F}}_Q$ by the map $\mu : P \to Q$. Since the equisingularity is a local property in the parameters, by [9, Proposition 3.7] it is preserved by pull-back. Moreover, if two global equisingular families $\underline{\mathcal{F}}_Q$ and $\underline{\mathcal{G}}_Q$ are $\mathcal{C}^0$ (resp. $\mathcal{C}^{\mathrm{ex}}$) conjugated by a homeomorphism Φ then so are $\mu^*\underline{\mathcal{F}}_Q$ and $\mu^*\underline{\mathcal{G}}_Q$ by $\mu^*\Phi = \Phi \times \mathrm{id}_P$.

Let u_0 be a point of Q and let $\mathcal{F}$ be a germ of foliation at a point m_0 of a two dimensional complex manifold M_0. An **equisingular deformation of $\mathcal{F}$ over the germ of manifold** (Q, u_0) is the data $(\underline{\mathcal{F}}_{Q,u_0}, \iota)$ of the germ at $\theta(u_0)$ of an equisingular family $\underline{\mathcal{F}}_Q = (M, \pi, \theta, \mathcal{F}_Q)$ together with the germ of an embedding $\iota : (M_0, m_0) \hookrightarrow (M, \theta(u_0))$ that sends $\mathcal{F}$ to the restricted foliation germ $\mathcal{F}_Q(u_0)$ on the **special fiber** $M(u_0)$. A **conjugacy between two equisingular deformations** $(\underline{\mathcal{F}}_{Q,u_0}, \iota)$ and $(\underline{\mathcal{F}}'_{Q,u_0}, \iota')$ is a conjugacy Φ between their associated families, $\Phi(\underline{\mathcal{F}}_{Q,u_0}) = \underline{\mathcal{F}}'_{Q,u_0}$, such that $\Phi \circ \iota = \iota'$.

If $\mu : (P, t_0) \to (Q, u_0)$ is a holomorphic map germ and $(\underline{\mathcal{F}}_{Q,u_0}, \iota)$ is an equisingular deformation of $\mathcal{F}$ over (Q, u_0), then $(\mu^*\underline{\mathcal{F}}_{Q,u_0}, \mu^*\iota)$ is an equisingular deformation of $\mathcal{F}$ over (P, t_0) where $\mu^*\iota$ is defined by $\rho_\mu \circ \mu^*\iota = \iota$ (recall that the restriction of ρ_μ to the fiber over t_0 is a biholomorphism onto the fiber over u_0).

Definition 2.2. *Let $(\underline{\mathcal{F}}_{Q^\cdot}, \iota)$ be an equisingular deformation over a germ of manifold $Q^\cdot := (Q, u_0)$, of a foliation $\mathcal{F}$. We say that $(\underline{\mathcal{F}}_{Q^\cdot}, \iota)$ is a* ***$\mathcal{C}^{\mathrm{ex}}$-universal deformation of*** *$\mathcal{F}$ if for any germ of manifold $P^\cdot = (P, t_0)$ and any equisingular deformation $(\underline{\mathcal{G}}_{P^\cdot}, \delta)$ of $\mathcal{F}$ over $P^\cdot$, there exists a unique germ of holomorphic map $\lambda : P^\cdot \to Q^\cdot$ such that the deformations $(\underline{\mathcal{G}}_{P^\cdot}, \delta)$ and $\lambda^*(\underline{\mathcal{F}}_{Q^\cdot}, \iota)$ of $\mathcal{F}$ are $\mathcal{C}^{\mathrm{ex}}$-conjugated.*

For each germ of manifold $Q^\cdot$ and each generalized curve foliation $\mathcal{F}$ let us denote by $\mathrm{Def}_{\mathcal{F}}^{Q^\cdot}$ set of $\mathcal{C}^{\mathrm{ex}}$-conjugacy classes of equisingular deformations of $\mathcal{F}$ over $Q^\cdot$.

Remark 2.3. Notice that if $\mu : Q'^\cdot \to Q^\cdot$ is a germ of biholomorphism, the $\mathcal{C}^{\mathrm{ex}}$-universality of $(\underline{\mathcal{F}}_{Q^\cdot}, \iota)$ and of $\mu^*(\underline{\mathcal{F}}_{Q^\cdot}, \iota)$ are clearly equivalent. On the other hand, it directly results from the definition that the $\mathcal{C}^{\mathrm{ex}}$-universality of $(\underline{\mathcal{F}}_{Q^\cdot}, \iota)$ only depends on its $\mathcal{C}^{\mathrm{ex}}$-class of conjugacy $[\underline{\mathcal{F}}_{Q^\cdot}, \iota] \in \mathrm{Def}_{\mathcal{F}}^{Q^\cdot}$. We will then say that $[\underline{\mathcal{F}}_{Q^\cdot}, \iota]$ is **$\mathcal{C}^{\mathrm{ex}}$-universal**. □

Theorem 2.4 ([9, Theorem 3.11 and Corollary 6.8])**.** *Let $\mathcal{F}$ and $\mathcal{G}$ be foliations of finite type which are generalized curves and let $\phi : \mathcal{G} \to \mathcal{F}$ be an excellent conjugacy of foliations. If $(\underline{\mathcal{F}}_{Q^\cdot}, \iota)$ is an equisingular deformation of $\mathcal{F}$ over $Q^\cdot$, there is an*

equisingular deformation $(\underline{\mathcal{G}}_{Q^\cdot}, \delta)$ *of* $\mathcal{G}$ *over* $Q^\cdot$ *and an excellent conjugacy of families* $\Phi : \underline{\mathcal{G}}_{Q^\cdot} \to \underline{\mathcal{F}}_{Q^\cdot}$ *such that* $\Phi \circ \delta = \iota \circ \phi$*. Moreover, the map*

$$\phi^* : \mathrm{Def}_{\mathcal{F}}^{Q^\cdot} \to \mathrm{Def}_{\mathcal{G}}^{Q^\cdot}\ , \quad [\underline{\mathcal{F}}_{Q^\cdot}, \iota] \mapsto [\underline{\mathcal{G}}_{Q^\cdot}, \delta]$$

is well defined, bijective and sends the class of a $\mathcal{C}^{\mathrm{ex}}$*-universal deformation of* $\mathcal{F}$ *to the class of a* $\mathcal{C}^{\mathrm{ex}}$*-universal deformation of* $\mathcal{G}$*.*

2.2. **Marked foliations and families.** Now, we fix for all the sequel a **marked divisor** $\mathcal{E}^\diamond = (\mathcal{E}, \Sigma, \cdot)$ in the sense of [8, §2.1], i.e. a connected compact complex curve with normal crossings $\mathcal{E}$, endowed with a finite subset Σ of $\mathcal{E}$ and a symmetric map $\mathrm{Comp}(\mathcal{E})^2 \to \mathbb{Z}$, $(D, D') \mapsto D \cdot D'$, where $\mathrm{Comp}(\mathcal{E})$ denotes the set of irreducible components of $\mathcal{E}$. The components of $\mathcal{E}$ without any point of Σ are called **dicritical components**, the others being called **invariant components**.

A **marked by** $\mathcal{E}^\diamond$ **foliation** is a pair $\mathcal{F}^\diamond = (\mathcal{F}, f)$ where

- $\mathcal{F}$ is a germ (at m_0) of a holomorphic foliation on a 2-dimensional manifold (M_0, m_0),
- f is a homeomorphism, called **marking of** $\mathcal{F}$, from $\mathcal{E}$ to the exceptional divisor $\mathcal{E}_{\mathcal{F}}$ of the reduction of $\mathcal{F}$ such that: $f(\Sigma)$ is the singular set $\mathrm{Sing}(\mathcal{F}^\sharp)$ of the reduced foliation $\mathcal{F}^\sharp$, and $D \cdot D'$ is equal to the intersection number of $f(D)$ with $f(D')$ in $M_{\mathcal{F}}$ for any components D, D' of $\mathcal{E}$. Moreover we will also suppose that f is holomorphic at each point of $\Sigma \cup \mathrm{Sing}(\mathcal{E})$.

We assume that there exists a foliation that can be marked by $\mathcal{E}^\diamond$, consequently the dual graph $\mathsf{A}_{\mathcal{E}}$ of $\mathcal{E}$ is necessarily a tree.

Two markings f and g of $\mathcal{F}$ by $\mathcal{E}^\diamond$ will be called **equivalent** if the homeomorphism $g^{-1} \circ f$ is isotopic to the identity map of $\mathcal{E}$ by a **good isotopy**, i.e. an isotopy leaving fixed Σ. A $\mathcal{C}^{\mathrm{ex}}$**-conjugacy between two marked by** $\mathcal{E}^\diamond$ **foliations** $\mathcal{F}^\diamond = (\mathcal{F}, f)$ and $\mathcal{G}^\diamond = (\mathcal{G}, g)$ is a germ φ of $\mathcal{C}^{\mathrm{ex}}$-conjugacy between these foliation germs, $\varphi(\mathcal{F}) = \mathcal{G}$, such that g and $\varphi^\sharp \circ f$ are equivalent markings by $\mathcal{E}^\diamond$ of $\mathcal{G}^\diamond$, $\varphi^\sharp$ being the lifting of φ through the reduction maps.

A **pre-marking by** $\mathcal{E}^\diamond$ **of an equireducible global family** $\underline{\mathcal{F}}_Q$ is a collection

$$(f_u)_{u \in Q}, \qquad f_u : \mathcal{E} \xrightarrow{\sim} \mathcal{E}_{\mathcal{F}_Q(u)}\ ,$$

of markings f_u for each foliation $\mathcal{F}_Q(u)$. Two pre-markings $(f_u)_{u \in Q}$ and $(g_u)_{u \in Q}$ of $\underline{\mathcal{F}}_Q$ of the same global family will be called **equivalent** if for each $u \in Q$ the markings f_u and g_u of $\mathcal{F}_Q(u)$ are equivalent. A **marking of an equireducible global family** $\underline{\mathcal{F}}_Q$ is a pre-marking that satisfies the following **local coherence property**: at any point $u_0 \in Q$ there is an equireduction neighborhood W of u_0 and a trivializing homeomorphism Ψ_W as in (1) such that the pre-marking $(f_u)_{u \in W}$ of $\underline{\mathcal{F}}_Q$ over W is equivalent to the pre-marking $(\Psi_u^{-1} \circ f_{u_0})_{u \in W}$, where $\Psi_u : \mathcal{E}_{\mathcal{F}_Q(u)} \xrightarrow{\sim} \mathcal{E}_{\mathcal{F}_Q(u_0)}$ is the restriction of Ψ_W to the fiber over u. A **marked by** $\mathcal{E}^\diamond$ **global family over a manifold** Q is the data

$$\underline{\mathcal{F}}_Q^\diamond = (\underline{\mathcal{F}}_Q, (f_u)_{u \in Q})$$

of an equireducible global family over Q and a marking by $\mathcal{E}^\diamond$ of this family. The **fiber at** $u \in Q$ of $\underline{\mathcal{F}}_Q^\diamond$ is the marked by $\mathcal{E}^\diamond$ foliation $\mathcal{F}_Q^\diamond(u) := (\mathcal{F}_Q(u), f_u)$.

Remark 2.5. One can check that the set over Q of the equivalence classes of markings by $\mathcal{E}^\diamond$ of the foliations $\mathcal{F}_Q(u)$, $u \in Q$, can be endowed with a topology such that

it becomes a covering over Q (the local coherence property being equivalent to the existence of continuous local sections) and the markings of $\underline{\mathcal{F}}_Q$ are continuous global sections. In particular:

(a) when Q is connected, two markings of $\underline{\mathcal{F}}_Q$ are equivalent as soon as, up to a good isotopy, they coincide at some point $u_0 \in Q$,
(b) when Q is connected and simply connected, any marking f_{u_0} of the foliation $\mathcal{F}_Q(u_0)$ for some $u_0 \in Q$, extends to a marking $(f_u)_{u\in Q}$ of $\underline{\mathcal{F}}_Q$, that is unique up to equivalence.

□

A $\mathcal{C}^{\mathrm{ex}}$-conjugacy between two marked by $\mathcal{E}^\diamond$ global families $(\underline{\mathcal{F}}_Q, (f_u)_{u\in Q})$ and $(\underline{\mathcal{G}}_Q, (g_u)_{u\in Q})$ is a $\mathcal{C}^{\mathrm{ex}}$-conjugacy of global families $\Phi(\underline{\mathcal{F}}_Q) = \underline{\mathcal{G}}_Q$ such that the restriction Φ_u of Φ to each fiber is a $\mathcal{C}^{\mathrm{ex}}$-conjugacy between the corresponding marked by $\mathcal{E}^\diamond$ foliations, i.e. $g_u^{-1} \circ \Phi_u^\sharp \circ f_u$ is isotopic to the identity map of $\mathcal{E}$ relatively to Σ.

For a marked by $\mathcal{E}^\diamond$ foliation $\mathcal{F}^\diamond = (\mathcal{F}, f)$ and D an invariant component of $\mathcal{E}$, we will denote by $[\mathcal{H}_D^{\mathcal{F}^\diamond}]$ the class, up to composition by inner automorphisms of $\mathrm{Diff}(\mathbb{C}, 0)$, of the group morphism

$$\mathcal{H}_D^{\mathcal{F}^\diamond} : \pi_1(D^*, o_D) \to \mathrm{Diff}(\mathbb{C}, 0)\,, \quad o_D \in D^* := D \setminus \mathrm{Sing}(\mathcal{F}^\sharp)\,,$$

where $\mathcal{H}_D^{\mathcal{F}^\diamond}(\dot\gamma)$ is the holonomy of the foliation $\mathcal{F}^\sharp$ along the loop $f \circ \gamma$ in $f(D)$. We also call **Camacho-Sad index of** $\mathcal{F}^\diamond$ at a point $m \in D$ and we write $\mathrm{CS}(\mathcal{F}^\diamond, D, m)$ the Camacho-Sad index of $\mathcal{F}^\sharp$ along $f(D)$ at the point $f(m)$.

Definition 2.6. *We denote by*

- $\mathrm{SL}(\mathcal{F}^\diamond)$ *the collection of all marked by* $\mathcal{E}^\diamond$ *foliations* $\mathcal{G}^\diamond = (\mathcal{G}, g)$ *having the same marked SL-type as* $\mathcal{F}^\diamond$*, i.e. satisfying*
$$[\mathcal{H}_D^{\mathcal{F}^\diamond}] = [\mathcal{H}_D^{\mathcal{G}^\diamond}] \quad \textit{and} \quad \mathrm{CS}(\mathcal{F}^\diamond, D, m) = \mathrm{CS}(\mathcal{G}^\diamond, D, m)$$
for any invariant component D *of* $\mathcal{E}$ *and any point* $m \in D$*,*
- $\mathrm{SL}_Q(\mathcal{F}^\diamond)$ *the collection of all marked by* $\mathcal{E}^\diamond$ *equisingular global families* $\underline{\mathcal{F}}_Q^\diamond$ *over* Q *such that any fiber* $\underline{\mathcal{F}}_Q^\diamond(u)$*,* $u \in Q$*, belongs to* $\mathrm{SL}(\mathcal{F}^\diamond)$*.*

Remark 2.7. When Q is connected, a marked equisingular global family $\underline{\mathcal{F}}_Q^\diamond$ belongs to $\mathrm{SL_Q}(\mathcal{F}^\diamond)$ as soon as one of its fibers $\mathcal{F}_Q^\diamond(u_0)$ belongs to $\mathrm{SL}(\mathcal{F}^\diamond)$. Indeed the Camacho-Sad indices of $\mathcal{F}_Q^\diamond(u)$ depend continuously on u and they are determined up to $2\pi i\mathbb{Z}$ by the holonomy maps around the singular points. The constancy of $u \mapsto \left[\mathcal{H}_D^{\mathcal{F}_Q^\diamond(u)}\right]$ follows from the equisingularity of $\underline{\mathcal{F}}_Q$ and the coherence property of the marking. □

2.3. **Local universality.** Let us suppose now that $\mathcal{F}^\diamond = (\mathcal{F}, f)$ is a marked by $\mathcal{E}^\diamond$ foliation with $\mathcal{F}$ a finite type generalized curve, on an ambient space (M_0, m_0). Theorem D in [8, §2.6] gives a description of the collection

$$\mathrm{Mod}([\mathcal{F}^\diamond]) := \{[\mathcal{G}^\diamond]\ ;\ \mathcal{G}^\diamond \in \mathrm{SL}(\mathcal{F}^\diamond)\}$$

of all $\mathcal{C}^{\mathrm{ex}}$-conjugacy classes $[\mathcal{G}^\diamond]$ of marked by $\mathcal{E}^\diamond$ foliations $\mathcal{G}^\diamond \in \mathrm{SL}(\mathcal{F}^\diamond)$: it is a set endowed with an abelian group structure given by an exact sequence

$$0 \to \mathbb{Z}^p \xrightarrow{\alpha} \mathbb{C}^\tau \xrightarrow{\Lambda} \mathrm{Mod}([\mathcal{F}^\diamond]) \xrightarrow{\beta} \mathbf{D} \to 1\,, \tag{2}$$

where

(i) $\tau = \dim_{\mathbb{C}} H^1(\mathsf{A}_{\mathcal{F}}, \mathcal{T}_{\mathcal{F}})$ is the dimension of a cohomological space (that we recall in (4)) whose finiteness characterizes the finite type of $\mathcal{F}$, cf. [9, §5.5].

(ii) $\mathbf{D}$ is a quotient of a finite product of totally disconnected subgroups of $\mathbb{U}(1) := \{|z| = 1\} \subset \mathbb{C}$, that according to [12] can be uncountable when $\mathcal{F}^\sharp$ possesses a singularity which is non-linearizable and non-resonant, cf. [8, Example 4, §8].

Definition 2.8. *We call **moduli map** of a marked global family $\underline{\mathcal{F}}^\diamond_Q \in \mathrm{SL}_Q(\mathcal{F}^\diamond)$ the map*

$$\mathrm{mod}_{\underline{\mathcal{F}}^\diamond_Q} : Q \to \mathrm{Mod}([\mathcal{F}^\diamond]) \,, \quad u \mapsto [\mathcal{F}^\diamond_Q(u)] \,.$$

We proved in [8] that for any map $\zeta : \mathbf{D} \to \mathrm{Mod}([\mathcal{F}^\diamond])$ such that $\beta \circ \zeta = \mathrm{id}_{\mathbf{D}}$, there exists a marked equisingular global family

$$\underline{\mathcal{F}}^\diamond_{\mathbf{U}} = (\underline{\mathcal{F}}_{\mathbf{U}}, (f_{z,d})_{z,d}) \in \mathrm{SL}_{\mathbf{U}}(\mathcal{F}^\diamond) \,, \quad \underline{\mathcal{F}}_{\mathbf{U}} = (M_0 \times \mathbf{U}, \pi, \theta, \mathcal{F}_{\mathbf{U}}) \,,$$

$$\mathbf{U} := \mathbb{C}^\tau \times \mathbf{D} \,, \quad \pi : M_0 \times \mathbf{U} \to \mathbf{U} \,, \quad \pi(m, z, d) := (z, d) \,, \quad \theta(z, d) := (m_0, z, d) \,,$$

where $\mathbf{D}$ is endowed with the discrete topology, such that if we denote by the dot $\cdot$ the group operation in $\mathrm{Mod}([\mathcal{F}^\diamond])$, we have:

$$\mathrm{mod}_{\underline{\mathcal{F}}^\diamond_{\mathbf{U}}}(z, d) = \Lambda(z) \cdot \zeta(d) \,. \tag{3}$$

The goal of this section is to prove that this global family satisfies a local universal property:

Theorem 2.9. *Let $\mathcal{F}^\diamond = (\mathcal{F}, f)$ be a marked by $\mathcal{E}^\diamond$ foliation of finite type which is a generalized curve. Let ζ be a section of the map $\beta : \mathrm{Mod}([\mathcal{F}^\diamond]) \to \mathbf{D}$ in the exact sequence (2). Then there exists a marked global family $\underline{\mathcal{F}}^\diamond_{\mathbf{U}} = (\underline{\mathcal{F}}_{\mathbf{U}}, (f_{z,d})_{z,d}) \in \mathrm{SL}_{\mathbf{U}}(\mathcal{F}^\diamond)$, $\mathbf{U} := \mathbb{C}^\tau \times \mathbf{D}$, such that*

(1) the moduli map $\mathrm{mod}_{\underline{\mathcal{F}}^\diamond_{\mathbf{U}}}$ is surjective and relation (3) is satisfied,

(2) for any point $\tilde{u} \in \mathbf{U}$, the deformation $(\underline{\mathcal{F}}_{\mathbf{U},\tilde{u}}, \iota_{\tilde{u}})$ of the foliation $\underline{\mathcal{F}}_{\mathbf{U}}(\tilde{u})$ over the germ of manifold $(\mathbf{U}, \tilde{u})$, given by the germ of $\underline{\mathcal{F}}_{\mathbf{U}}$ at $\tilde{u}$ and the embedding $\iota_{\tilde{u}}(m) := (m, \tilde{u})$, is $\mathcal{C}^{\mathrm{ex}}$-universal.

To prove this result we will see that the marked global family $\underline{\mathcal{F}}^\diamond_{\mathbf{U}}$ constructed in [8, §10, Step (vii)], which fulfills property (1), also satisfies the assertion (2). For this we will use the following cohomological criterion of universality proven in [9].

Theorem 2.10 ([9, Theorem 6.7])**.** *Let $(\underline{\mathcal{F}}_{Q,u_0}, \iota)$ be an equisingular deformation of a finite type generalized curve $\mathcal{F}$. The following properties are equivalent:*

(1) $(\underline{\mathcal{F}}_{Q,u_0}, \iota)$ is $\mathcal{C}^{\mathrm{ex}}$-universal,

(2) the Kodaira-Spencer map $\frac{\partial \mathcal{F}_Q}{\partial u}\Big|_{u=u_0} : T_{u_0}Q \to H^1(\mathsf{A}_{\mathcal{F}}, \mathcal{T}_{\mathcal{F}})$ is an isomorphism.

Let us explain first what is the cohomological space $H^1(\mathsf{A}_{\mathcal{F}}, \mathcal{T}_{\mathcal{F}})$ and the Kodaira-Spencer map of an equisingular deformation.

The **dual graph** $\mathsf{A}_{\mathcal{E}}$ of the divisor $\mathcal{E}$ is the tree with vertex set $\mathsf{Ve}_{\mathsf{A}_{\mathcal{E}}}$ formed by the irreducible components of $\mathcal{E}$ and edge set $\mathsf{Ed}_{\mathsf{A}_{\mathcal{E}}}$ consisting in unordered pairs $\langle D, D' \rangle$ of distinct irreducible components of $\mathcal{E}$ with $D \cap D' \neq \emptyset$. We also consider the set of oriented edges $\mathcal{I}_{\mathsf{A}_{\mathcal{E}}}$ consisting in pairs $(D, \mathsf{e}) \in \mathsf{Ve}_{\mathsf{A}_{\mathcal{E}}} \times \mathsf{Ed}_{\mathsf{A}_{\mathcal{E}}}$ with $D \in \mathsf{e}$.

Let us denote by $\underline{\mathcal{X}}_{\mathcal{F}} \subset \underline{\mathcal{B}}_{\mathcal{F}}$ the sheaves of tangent and basic holomorphic vector fields of $\mathcal{F}^\sharp$ on $M_{\mathcal{F}}$. We will use the cohomology $H^1(\mathsf{A}_{\mathcal{F}}, \mathcal{T}_{\mathcal{F}})$ of the group-graph

(cf. [8, §3]) $\mathcal{T}_{\mathcal{F}}$ of infinitesimal transverse symmetries over the dual graph $\mathsf{A}_{\mathcal{F}}$ of the exceptional divisor $\mathcal{E}_{\mathcal{F}}$, introduced in [9, Definition 5.8]. The group-graph $\mathcal{T}_{\mathcal{F}}$ is given by

$$\mathcal{T}_{\mathcal{F}}(D) = \varprojlim_{D\subset U} H^0(U, \underline{\mathcal{B}}_{\mathcal{F}}/\underline{\mathcal{X}}_{\mathcal{F}})$$

if $D \subset \mathcal{E}_{\mathcal{F}}$ is not a dicritical component of $\mathcal{F}$ and zero otherwise,

$$\mathcal{T}_{\mathcal{F}}(\langle D, D'\rangle) = \varprojlim_{D\cap D'\subset U} H^0(U, \underline{\mathcal{B}}_{\mathcal{F}}/\underline{\mathcal{X}}_{\mathcal{F}})$$

if $D \cap D'$ is not reduced to a nodal singularity of $\mathcal{F}^\sharp$ and zero otherwise, where U runs over all the open sets of $M_{\mathcal{F}}$ containing D or $D\cap D'$. By definition, $H^1(\mathsf{A}_{\mathcal{F}}, \mathcal{T}_{\mathcal{F}})$ is the 1-cohomology vector space of the following complex

$$\bigoplus_{D\in \mathsf{Ve}_{\mathsf{A}_{\mathcal{F}}}} \mathcal{T}_{\mathcal{F}}(D) \xrightarrow{\partial^0} \bigoplus_{(D,\langle D,D'\rangle)\in\mathcal{I}_{\mathsf{A}_{\mathcal{F}}}} \mathcal{T}_{\mathcal{F}}(\langle D, D'\rangle) \xrightarrow{\partial^1} \bigoplus_{\langle D,D'\rangle\in \mathsf{Ed}_{\mathsf{A}_{\mathcal{F}}}} \mathcal{T}_{\mathcal{F}}(\langle D, D'\rangle)), \quad (4)$$

where $\partial^0((X_D)_D) = (X_{D'} - X_D)_{(D,\langle D,D'\rangle)}$,

$$\partial^1((X_{D,\langle D,D'\rangle})_{(D,\langle D,D'\rangle)}) = (X_{D,\langle D,D'\rangle} + X_{D',\langle D,D'\rangle})_{\langle D,D'\rangle}.$$

If $\mathcal{F}$ is of finite type then $H^1(\mathsf{A}_{\mathcal{F}}, \mathcal{T}_{\mathcal{F}})$ is of finite dimension by [9, Theorem 5.15] and we can give an explicit basis by applying Remark 5.10, Proposition 5.12 and Theorem 2.15 as in the proof of the last assertion of Theorem 6.4 of [9]. For this we fix a marking $f : \mathcal{E} \to \mathcal{E}_{\mathcal{F}}$ and we consider the set

$$S_{\mathcal{F}^\diamond} = \{\star \in \mathsf{Ve}_{\mathsf{A}_{\mathcal{E}}} \cup \mathsf{Ed}_{\mathsf{A}_{\mathcal{E}}} : \dim_{\mathbb{C}} \mathcal{T}_{\mathcal{F}}(f(\star)) = 1\}$$

and we choose a subset

$$\mathcal{A}'' \subset \mathcal{A}' = \{\langle D, D'\rangle \in S_{\mathcal{F}^\diamond} \,|\, D \text{ or } D' \notin S_{\mathcal{F}^\diamond}\}$$

obtained by removing from $\mathcal{A}'$ an element in each connected component of $S_{\mathcal{F}^\diamond}$ not reduced to a single edge, cf [9, §2.6]. Finally we choose for each $\mathsf{e} \in \mathcal{A}''$ one vertex $D \in \mathsf{e}$. This gives us an orientation for each edge $\mathsf{e} \in \mathcal{A}''$. Let us denote by

$$\mathcal{A} = \{(D, \mathsf{e}) : \mathsf{e} \in \mathcal{A}''\} \subset \mathcal{I}_{\mathsf{A}_{\mathcal{E}}}.$$

Such a set will be called **set of active oriented edges** for $\mathcal{F}^\diamond$.

Theorem 2.11 ([9]). *If $\mathcal{F}^\diamond = (\mathcal{F}, f)$ is a marked by $\mathcal{E}^\diamond$ foliation of finite type which is a generalized curve and $\mathcal{A} \subset \mathcal{I}_{\mathsf{A}_{\mathcal{E}}}$ is a set of active oriented edges for $\mathcal{F}^\diamond$ then the map*

$$\delta_{\mathcal{A}}^{\mathcal{F}^\diamond} : \bigoplus_{(D,\langle D,D'\rangle)\in\mathcal{A}} \mathcal{T}_{\mathcal{F}}(\langle f(D), f(D')\rangle) \to H^1(\mathsf{A}_{\mathcal{F}}, \mathcal{T}_{\mathcal{F}})$$

defined by $\delta_{\mathcal{A}}^{\mathcal{F}^\diamond}((X_{D,\langle D,D'\rangle})_{\mathcal{A}}) = [(Y_{D,\langle D,D'\rangle})_{\mathcal{I}_{\mathsf{A}_{\mathcal{F}}}}]$, where

$$Y_{D,\langle D,D'\rangle} = \begin{cases} X_{f^{-1}(D,\langle D,D'\rangle)} & \text{if } f^{-1}(D, \langle D, D'\rangle) \in \mathcal{A}, \\ -X_{f^{-1}(D',\langle D,D'\rangle)} & \text{if } f^{-1}(D', \langle D, D'\rangle) \in \mathcal{A}, \\ 0 & \text{otherwise,} \end{cases}$$

is an isomorphism, $f^{-1}((D, \langle D, D'\rangle))$ denoting $(f^{-1}(D), \langle f^{-1}(D), f^{-1}(D')\rangle)$.

Remark 2.12. Let $\underline{\mathcal{F}}_Q^\diamond = (M, \pi, \theta, \mathcal{F}_Q, (f_u)_{u\in Q})$ be a global equisingular family of foliations over a connected manifold Q endowed with a marking $(f_u : \mathcal{E} \to \mathcal{E}_{\mathcal{F}_Q(u)})_{u\in Q}$ by $\mathcal{E}^\diamond$. Assume that some fiber $\mathcal{F}_Q(u_0)$ of $\mathcal{F}_Q$ is of finite type then all the fibers $\mathcal{F}_Q(u)$ are of finite type because the definition (cf. [8, Definition 6.7]) only depends

on their holonomies. For the same reason the set $S_{\mathcal{F}_Q^\diamond(u)}$ does not depend on $u \in Q$. Consequently the map $\delta_{\mathcal{A}}^{\mathcal{F}_Q^\diamond(u)}$ is an isomorphism for any $u \in Q$. □

The simplest example of equisingular family over Q is the **constant family** $\mathcal{F}_Q^{\mathrm{ct}}$ that is obtained in the ambient space $M_0 \times Q$ by lifting a given foliation $\mathcal{F}$ on M_0 to each submanifold $M_0 \times \{u\}$ through the projection map $M_0 \times Q \to M_0$. We now make more precise the fact that every equisingular family after reduction is equivalent to a constant family along each irreducible component of the exceptional divisor.

Theorem 2.13 ([9, Theorem 3.8])**.** *Let $\underline{\mathcal{F}}_Q$ be an equisingular family of foliations over Q and $u_0 \in Q$. If the fiber $\mathcal{F}_{u_0} = \mathcal{F}_Q(u_0)$ is a generalized curve, then we can associate to each $D \in \mathsf{Ve}_{\mathsf{A}_{\mathcal{F}_{u_0}}}$, a homeomorphism germ*

$$\Psi_D : (M_{\underline{\mathcal{F}}_{Q,u_0}}, D) \xrightarrow{\sim} (M_{\mathcal{F}_{u_0}} \times Q, D \times \{u_0\}),$$

so that:

(i) Ψ_D is a map over Q, i.e. $\mathrm{pr}_Q^\sharp \circ \Psi_D = \pi^\sharp$, the identity map over u_0 and it is excellent: Ψ_D is holomorphic at each point of $\mathrm{Sing}(\mathcal{E}_{\mathcal{F}_{u_0}}) \cup \mathrm{Sing}(\mathcal{F}_{u_0}^\sharp)$ except perhaps at the singular points of $\mathcal{E}_{\mathcal{F}_{u_0}}$ that are nodal singularities of $\mathcal{F}_{u_0}^\sharp$,

(ii) Ψ_D conjugates the foliation $\mathcal{F}_{Q,u_0}^\sharp$ to the foliation $\mathcal{F}_{Q,u_0}^{\mathrm{ct}\,\sharp}$ obtained after equireduction of the constant deformation $\mathcal{F}_{Q,u_0}^{\mathrm{ct}}$,

(iii) the germ of $\Psi_D \circ \Psi_{D'}^{-1}$ at the intersection point $\{s_{DD'}\} = (D \cap D') \times \{u_0\}$ of two irreducible components D and D', is the identity when either $s_{DD'}$ is a nodal singular point of $\mathcal{F}_{u_0}^\sharp$ or $s_{DD'}$ is a regular point of $\mathcal{F}_{u_0}^\sharp$.

The collection $(\Psi_D)_D$ of these homeomorphisms indexed by the components of $\mathcal{E}$ is called **good trivializing system** for $\underline{\mathcal{F}}_{Q,u_0}$.

For each $(D, \langle D, D'\rangle) \in \mathcal{I}_{\mathsf{A}_\mathcal{E}}$ the composition $\phi_{D,\langle D,D'\rangle} := \Psi_{D'} \circ \Psi_D^{-1}$ is a local biholomorphism at $(D \cap D') \times \{u_0\}$ preserving the constant family $(\mathcal{F}_{u_0}^\sharp)_{Q,u_0}^{\mathrm{ct}}$. For every $v \in T_{u_0}Q$ the Lie derivative $L_v(\mathrm{pr}_{M_{\mathcal{F}_{u_0}}} \circ \phi_{D,\langle D,D'\rangle})$ defines a germ of basic vector field of $\mathcal{F}_{u_0}^\sharp$ at $D \cap D'$. It is shown in [9, §6.2] that the **Kodaira-Spencer map**

$$\frac{\partial \mathcal{F}_Q}{\partial u}\Big|_{u=u_0} : T_{u_0}Q \to H^1(\mathsf{A}_{\mathcal{F}_{u_0}}, \mathcal{T}_{\mathcal{F}_{u_0}})$$

given by $v \mapsto [(L_v(\mathrm{pr}_{M_{\mathcal{F}_{u_0}}} \circ \phi_{D,\langle D,D'\rangle}))_{D,\langle D,D'\rangle}]$ does not depend on the choice we have made of a good trivializing system for $\mathcal{F}_{Q,u_0}$.

Proof of Theorem 2.9. In a first step we recall the construction of $\underline{\mathcal{F}}_{\mathbf{U}}^\diamond$. In the second step we fix $\tilde{u} \in \mathbf{U}$ and we will determine a good trivializing system $(\Upsilon_D)_D$ for $\underline{\mathcal{F}}_{\mathbf{U},\tilde{u}}$. Finally, in the last step, we compute the Kodaira-Spencer map of $\underline{\mathcal{F}}_{\mathbf{U},\tilde{u}}$ and we apply Theorem 2.10 to end the proof.

-Step 1. Let us fix $d \in \mathbf{D}$ and a marked foliation $\mathcal{G}^\diamond = (\mathcal{G}, g)$ belonging to $\zeta(d)$. First, let us recall that there are germs of $\mathcal{C}^{\mathrm{ex}}$-homeomorphisms compatible with the markings

$$\psi_D : (M_\mathcal{G}, g(D)) \xrightarrow{\sim} (M_\mathcal{F}, f(D)), \quad D \in \mathsf{Ed}_{\mathsf{A}_\mathcal{E}}, \tag{5}$$

that conjugate $\mathcal{G}^\sharp$ to $\mathcal{F}^\sharp$. The biholomorphism germs

$$\varphi_{D,\langle D,D'\rangle} := \psi_{D'} \circ \psi_D^{-1} : (M_{\mathcal{F}}, s_{\langle D,D'\rangle}) \xrightarrow{\sim} (M_{\mathcal{F}}, s_{\langle D,D'\rangle}) \,, \tag{6}$$

with

$$\{s_{\langle D,D'\rangle}\} := f(D \cap D') \,, \quad \langle D, D'\rangle \in \mathsf{Ed}_{\mathsf{A}_{\mathcal{E}}} \,,$$

leave $\mathcal{F}^\sharp$ invariant. Thanks to [8, §11.5] we may require that $\varphi_{D,\langle D,D'\rangle}$ is the identity map when $s_{\langle D,D'\rangle}$ is a nodal singular point of $\mathcal{F}^\sharp$ or a singular point of the divisor belonging to a dicritical component.

We fix a set of active oriented edges $\mathcal{A} \subset \mathcal{I}_{\mathsf{A}_{\mathcal{E}}}$ for $\mathcal{F}^\diamond$, we choose for each $(D, \mathsf{e}) \in \mathcal{A}$ a germ at $f(s_{\mathsf{e}})$ of basic and not tangent holomorphic vector field X_{e} to $\mathcal{F}^\sharp$ on $M_{\mathcal{F}}$ and we fix a bijection $\kappa : \mathcal{A} \to \{1, \dots, \tau\}$. Now, $d \in \mathbf{D}$ being fixed, in all the sequel we will simply denote by $\underline{\mathcal{F}}^\diamond_{\mathbb{C}^\tau}$ the marked global family $\underline{\mathcal{F}}^\diamond_{\mathbf{U}}$ restricted to $\mathbb{C}^\tau \times \{d\}$.

The ambient space of $\underline{\mathcal{F}}_{\mathbb{C}^\tau}$ is constructed by gluing neighborhoods $U_D \times \mathbb{C}^\tau$ of $f(D) \times \mathbb{C}^\tau$ in $M_{\mathcal{F}} \times \mathbb{C}^\tau$, using an appropriate family of biholomorphisms

$$\mathfrak{u} := (\Phi_{D,\mathsf{e}})_{D \in \mathsf{e} \in \mathsf{Ed}_{\mathsf{A}}} \,,$$

$$\Phi_{D,\mathsf{e}} : (U_D \times \mathbb{C}^\tau, \{s_{\mathsf{e}}\} \times \mathbb{C}^\tau) \xrightarrow{\sim} (U_{D'} \times \mathbb{C}^\tau, \{s_{\mathsf{e}}\} \times \mathbb{C}^\tau) \,, \quad \text{with } \mathsf{e} = \langle D, D'\rangle \,.$$

Writing $(m, t) \mapsto \exp(Z)[t](m)$ the flow at the time t of a vector field Z, we set:

- $\Phi_{D,\mathsf{e}}(m, z) = (\varphi_{D,\mathsf{e}}(m), z)$, $z = (z_1, \dots, z_\tau)$, if $(D, \mathsf{e}), (D', \mathsf{e}) \notin \mathcal{A}$,
- $\Phi_{D,\mathsf{e}}(m, z) = (\varphi_{D,\mathsf{e}} \circ \exp(X_{\mathsf{e}})[z_{\kappa(\mathsf{e})}](m), z)$, if $(D, \mathsf{e}) \in \mathcal{A}$,
- $\Phi_{D,\mathsf{e}} = \Phi^{-1}_{D',\mathsf{e}}$ if $(D', \mathsf{e}) \in \mathcal{A}$,

where $\varphi_{D,\mathsf{e}}$ are the biholomorphism germs in (6). We consider the following germ of manifold

$$(M_{\mathfrak{u}}, \mathcal{E}_{\mathfrak{u}}) := \Big(\bigcup_{D \in \mathsf{Ve}_{\mathsf{A}_{\mathcal{E}}}} U_D \times \mathbb{C}^\tau \times \{D\} \Big/ \sim_{\mathfrak{u}} \,, \bigcup_{D \in \mathsf{Ve}_{\mathsf{A}_{\mathcal{E}}}} f(D) \times \mathbb{C}^\tau \times \{D\} \Big/ \sim_{\mathfrak{u}} \Big) \,,$$

the equivalence relation $\sim_{\mathfrak{u}}$ being defined by:

$$U_D \times \mathbb{C}^\tau \times \{D\} \ni (m, z, D) \sim_{\mathfrak{u}} (\Phi_{D,\langle D,D'\rangle}(m, z), D') \in U_{D'} \times \mathbb{C}^\tau \times \{D'\}$$

when (m, z) belongs to the domain of $\Phi_{D,\langle D,D'\rangle}$ and with the single relation $(m, z, D) \sim_{\mathfrak{u}} (m, z, D)$ otherwise. As the biholomorphisms $\Phi_{D,\mathsf{e}}$ leave invariant the projections $U_D \times \mathbb{C}^\tau \to \mathbb{C}^\tau$ and the constant family $\mathcal{F}^{\sharp\,\mathrm{ct}}_{\mathbb{C}^\tau}$, the gluing process provides a holomorphic submersion $\pi_{\mathfrak{u}} : M_{\mathfrak{u}} \to \mathbb{C}^\tau$ and a foliation tangent to the fibers of $\pi_{\mathfrak{u}}$, which we denote by $\mathcal{F}_{\mathfrak{u}}$.

The ambient space of $\underline{\mathcal{F}}_{\mathbb{C}^\tau}$ is the manifold over $\mathbb{C}^\tau$, obtained by contracting $\mathcal{E}_{\mathfrak{u}}$ to a τ-dimensional manifold $S_{\mathfrak{u}}$:

$$E_{\mathfrak{u}} : (M_{\mathfrak{u}}, \mathcal{E}_{\mathfrak{u}}) \longrightarrow (M^\flat_{\mathfrak{u}}, S_{\mathfrak{u}}) \,, \quad \pi^\flat_{\mathfrak{u}} : (M^\flat_{\mathfrak{u}}, S_{\mathfrak{u}}) \longrightarrow \mathbb{C}^\tau \,, \quad \pi^\flat_{\mathfrak{u}} \circ E_{\mathfrak{u}} := \pi_{\mathfrak{u}} \,.$$

Restricted to $S_{\mathfrak{u}}$ the submersion $\pi^\flat_{\mathfrak{u}}$ is a biholomorphism, we will denote by $\theta_{\mathfrak{u}}$ its inverse. Finally the foliation $\mathcal{F}_{\mathbb{C}^\tau}$ constructed in [8, §10, Step (vii)] is the direct image $E_{\mathfrak{u}}(\mathcal{F}_{\mathfrak{u}})$ and we have an equisingular global family over $\mathbb{C}^\tau$

$$\underline{\mathcal{F}}_{\mathbb{C}^\tau} := \Big(M^\flat_{\mathfrak{u}}, \pi^\flat_{\mathfrak{u}}, \theta_{\mathfrak{u}}, \mathcal{F}_{\mathbb{C}^\tau} \Big) \,.$$

By a classical property of blow-ups, there is a germ F of biholomorphism that conjugates the reduced foliation $\mathcal{F}^\sharp_{\mathbb{C}^\tau}$ to $\mathcal{F}_{\mathfrak{u}}$:

$$F : (M_{\underline{\mathcal{F}}_{\mathbb{C}^\tau}}, \mathcal{E}_{\mathcal{F}_{\mathbb{C}^\tau}}) \xrightarrow{\sim} (M_{\mathfrak{u}}, \mathcal{E}_{\mathfrak{u}}) \,, \quad F(\mathcal{F}^\sharp_{\mathbb{C}^\tau}) = \mathcal{F}_{\mathfrak{u}} \,, \tag{7}$$

In order to endow this family with a marking by $\mathcal{E}^\diamond$ we highlight that we have:

$$\Phi_{D,\mathsf{e}}(m, 0) = (\varphi_{D,\mathsf{e}}(m), 0) \,, \quad D \in \mathsf{e} \in \mathsf{Ed}_{\mathsf{A}_{\mathcal{E}}} \,.$$

Therefore, according to relations (6) the maps ψ_D introduced in (5) glue as a germ of $\mathcal{C}^{\mathrm{ex}}$-homeomorphism $\Psi_{\mathcal{G}}$ which conjugates $\mathcal{G}^\sharp$ to the foliation $\mathcal{F}^\sharp_{\mathbb{C}^\tau}(0)$ obtained by restricting $\mathcal{F}^\sharp_{\mathbb{C}^\tau}$ to the fiber $M_{\mathfrak{u}}(0) := \pi_{\mathfrak{u}}^{-1}(0)$:

$$\Psi_{\mathcal{G}} : (M_{\mathcal{G}}, \mathcal{E}_{\mathcal{G}}) \xrightarrow{\sim} (M_{\mathfrak{u}}(0), \mathcal{E}_{\mathfrak{u}}(0)) , \quad \Psi_{\mathcal{G}}(\mathcal{G}^\sharp) = \mathcal{F}^\sharp_{\mathbb{C}^\tau}(0) ,$$

with $\mathcal{E}_{\mathfrak{u}}(0) := \mathcal{E}_{\mathfrak{u}} \cap \pi_{\mathfrak{u}}^{-1}(0)$. The homeomorphism $\Psi_{\mathcal{G}} \circ g : \mathcal{E} \to \mathcal{E}_{\mathfrak{u}}(0)$ defines a marking of $\mathcal{F}_{\mathbb{C}^\tau}(0)$, that extends to a marking $(f_z)_z$ of the global family $\underline{\mathcal{F}}_{\mathbb{C}^\tau}$ thanks to property (b) of Remark 2.5. Since $S_{\mathfrak{u}}$ is Stein, up to a biholomorphism over $\mathbb{C}^\tau$, we can assume by classical arguments that $M^\flat_{\mathfrak{u}}$ is a neighborhood of $\{m_0\} \times \mathbb{C}^\tau \subset M_0 \times \mathbb{C}^\tau$ and $\pi^\flat_{\mathfrak{u}}$ is the projection map onto $\mathbb{C}^\tau$.

-Step 2. Now we also fix $\tilde{z} \in \mathbb{C}^\tau$ and we identify $\underline{\mathcal{F}}_{\mathbf{U},\tilde{u}}$, $\tilde{u} := (\tilde{z}, d)$, to the deformation $(\underline{\mathcal{F}}_{\mathbb{C}^\tau,\tilde{z}}, \iota_{\tilde{z}})$ over the manifold germ $(\mathbb{C}^\tau, \tilde{z})$ of the fiber $\mathcal{F}_{\mathbb{C}^\tau}(\tilde{z}) = \mathcal{F}_{\mathbf{U}}(\tilde{u})$, defined by the germ $\underline{\mathcal{F}}_{\mathbb{C}^\tau,\tilde{z}}$ at $\tilde{z}$ of the family $\underline{\mathcal{F}}_{\mathbb{C}^\tau}$ and the inclusion map $\iota_{\tilde{z}} : \pi_{\mathfrak{u}}^{\flat\,-1}(\tilde{z}) \hookrightarrow M^\flat_{\mathfrak{u}}$.

If we restrict the map germ (7) to the fiber over $\tilde{z}$, we obtain a biholomorphism germ $F_{\tilde{z}}$ between the ambient space $M_{\mathcal{F}_{\mathbb{C}^\tau}(\tilde{z})}$ of $\mathcal{F}^\sharp_{\mathbb{C}^\tau}(\tilde{z})$ and the manifold germ $(\pi_{\mathfrak{u}}^{-1}(\tilde{z}), \mathcal{E}_{\mathfrak{u}} \cap \pi_{\mathfrak{u}}^{-1}(\tilde{z}))$. This manifold is also the manifold germ

$$\big(M_{\mathfrak{u}(\tilde{z})}, \mathcal{E}_{\mathfrak{u}(\tilde{z})} \big) := \Big(\bigcup_{D \in \mathsf{Ve}_{\mathsf{A}_{\mathcal{E}}}} U_D \times \{D\} / \sim_{\mathfrak{u}(\tilde{z})}, \bigcup_{D \in \mathsf{Ve}_{\mathsf{A}_{\mathcal{E}}}} f(D) \times \{D\} / \sim_{\mathfrak{u}(\tilde{z})} \Big) ,$$

defined by the gluing process given by the equivalence relation $\sim_{\mathfrak{u}(\tilde{z})}$ defined by the family:

$$\mathfrak{u}(\tilde{z}) := (\Phi^{\tilde{z}}_{D,\mathsf{e}})_{D \in \mathsf{e} \in \mathsf{Ed}_{\mathsf{A}_{\mathcal{E}}}} ,$$

$$\Phi^{\tilde{z}}_{D,\mathsf{e}} : (U_D, s_{\mathsf{e}}) \xrightarrow{\sim} (U_{D'}, s_{\mathsf{e}}) , \quad m \mapsto \Phi_{D,\mathsf{e}}(m, \tilde{z}) , \quad \mathsf{e} = \langle D, D' \rangle .$$

Clearly $F_{\tilde{z}}$ conjugates $\mathcal{F}^\sharp_{\mathbb{C}^\tau}(\tilde{z})$ to the foliation $\mathcal{F}_{\mathfrak{u}(\tilde{z})}$ obtained by gluing $\mathcal{F}^\sharp$ restricted to each U_D, i.e.

$$F_{\tilde{z}} : \big(M_{\mathcal{F}_{\mathbb{C}^\tau}(\tilde{z})}, \mathcal{E}_{\mathcal{F}_{\mathbb{C}^\tau}(\tilde{z})}\big) \xrightarrow{\sim} \big(M_{\mathfrak{u}(\tilde{z})}, \mathcal{E}_{\mathfrak{u}(\tilde{z})} \big) , \quad F_{\tilde{z}}(\mathcal{F}^\sharp_{\mathbb{C}^\tau}(\tilde{z})) = \mathcal{F}_{\mathfrak{u}(\tilde{z})} .$$

Let us denote by $\{U_D \times \mathbb{C}^\tau\} \subset M_{\mathfrak{u}}$ the image of the canonical embedding $U_D \times \mathbb{C}^\tau \hookrightarrow M_{\mathfrak{u}}$, and by

$$g_D : \{U_D \times \mathbb{C}^\tau\} \subset M_{\mathfrak{u}} \longrightarrow U_D \times \mathbb{C}^\tau$$

the inverse of this embedding. We have the following relations of "change of charts"

$$g_{D'} = \Phi_{D,\langle D,D' \rangle} \circ g_D . \tag{8}$$

Similarly, $\{U_D\}$ denoting the image of the canonical embedding $U_D \hookrightarrow M_{\mathfrak{u}(\tilde{z})}$ and

$$g^{\tilde{z}}_D : \{U_D\} \subset M_{\mathfrak{u}(\tilde{z})} \longrightarrow U_D$$

denoting its inverse, we also have:

$$g^{\tilde{z}}_{D'} = \Phi^{\tilde{z}}_{D,\langle D,D' \rangle} \circ g^{\tilde{z}}_D . \tag{9}$$

Notice that g_D conjugates the foliation $\mathcal{F}_{\mathfrak{u}}$ restricted to $\{U_D \times \mathbb{C}^\tau\}$ to the constant deformation $(\mathcal{F}^\sharp)^{\mathrm{ct}}_{\mathbb{C}^\tau}$ of $\mathcal{F}^\sharp$, restricted to $U_D \times \mathbb{C}^\tau$. Similary $g^{\tilde{z}}_D$ conjugates $\mathcal{F}_{\mathfrak{u}(\tilde{z})}$ restricted to $\{U_D\}$ to $\mathcal{F}^\sharp$ restricted to U_D. Hence $g^{\tilde{z}}_D \times \mathrm{id}_{\mathbb{C}^\tau}$ conjugates the constant deformation of $\mathcal{F}_{\mathfrak{u}(\tilde{z})}$ on $\{U_D\} \times \mathbb{C}^\tau$ over $\mathbb{C}^\tau$, denoted by $(\mathcal{F}_{\mathfrak{u}(\tilde{z})})^{\mathrm{ct}}_{\mathbb{C}^\tau}$, to the constant deformation $(\mathcal{F}^\sharp)^{\mathrm{ct}}_{\mathbb{C}^\tau}$ of $\mathcal{F}^\sharp$ restricted to $U_D \times \mathbb{C}^\tau$. If we write $[U_D \times \mathbb{C}^\tau] := F^{-1}(\{U_D \times$

$\mathbb{C}^\tau\}$) and $[U_D] := F_{\tilde z}^{-1}(\{U_D\})$, then $F_{\tilde z} \times \mathrm{id}_{\mathbb{C}^\tau}$ conjugates the constant deformation of $\mathcal{F}^\sharp_{\mathbb{C}^\tau}(\tilde z)$ over $\mathbb{C}^\tau$, denoted by $(\mathcal{F}^\sharp_{\mathbb{C}^\tau}(\tilde z))^{\mathrm{ct}}_{\mathbb{C}^\tau}$, to $(\mathcal{F}_{\mathsf{u}(\tilde z)})^{\mathrm{ct}}_{\mathbb{C}^\tau}$:

$$\begin{array}{ccccccccc} [U_D\times\mathbb{C}^\tau] & \xrightarrow[\sim]{F} & \{U_D\times\mathbb{C}^\tau\} & \xrightarrow[\sim]{g_D} & U_D\times\mathbb{C}^\tau & \xrightarrow[\sim]{(g_D^{\tilde z}\times\mathrm{id}_{\mathbb{C}^\tau})^{-1}} & \{U_D\}\times\mathbb{C}^\tau & \xrightarrow[\sim]{(F_{\tilde z}\times\mathrm{id}_{\mathbb{C}^\tau})^{-1}} & [U_D]\times\mathbb{C}^\tau\,. \\ \mathcal{F}^\sharp_{\mathbb{C}^\tau} & \mapsto & \mathcal{F}_{\mathsf{u}} & \mapsto & (\mathcal{F}^\sharp)^{\mathrm{ct}}_{\mathbb{C}^\tau} & \mapsto & (\mathcal{F}_{\mathsf{u}(\tilde z)})^{\mathrm{ct}}_{\mathbb{C}^\tau} & \mapsto & (\mathcal{F}^\sharp_{\mathbb{C}^\tau}(\tilde z))^{\mathrm{ct}}_{\mathbb{C}^\tau} \end{array}$$

The homeomorphism

$$\Upsilon_D := (F_{\tilde z}\times\mathrm{id}_{\mathbb{C}^\tau})^{-1}\circ(g_D^{\tilde z}\times\mathrm{id}_{\mathbb{C}^\tau})^{-1}\circ g_D\circ F : [U_D\times\mathbb{C}^\tau]\xrightarrow{\sim}[U_D]\times\mathbb{C}^\tau\,,$$

is a $\mathcal{C}^{\mathrm{ex}}$-trivialization of $\mathcal{F}^\sharp_{\mathbb{C}^\tau}$ as deformation of $\mathcal{F}^\sharp_{\mathbb{C}^\tau}(\tilde z)$, i.e. a $\mathcal{C}^{\mathrm{ex}}$-conjugacy from $\mathcal{F}^\sharp_{\mathbb{C}^\tau}$ to the constant deformation of $\mathcal{F}^\sharp_{\mathbb{C}^\tau}(\tilde z)$. The collection $(\Upsilon_D)_{D\in\mathsf{Ve}_{\mathsf{A}_{\mathcal{E}}}}$ is a good trivializing system for $\underline{\mathcal{F}}_{\mathbb{C}^\tau,\tilde z}$. Consider the germ of biholomorphism

$$\Upsilon_{D,\mathsf{e}} : \big(([U_D]\cap[U_{D'}])\times\mathbb{C}^\tau, s_{\mathsf{e}}\big)\xrightarrow{\sim}\big(([U_D]\cap[U_{D'}])\times\mathbb{C}^\tau, s_{\mathsf{e}}\big)\,,$$

$$\Upsilon_{D,\mathsf{e}} := \Upsilon_{D'}\circ\Upsilon_D^{-1}\,,\quad \mathsf{e}=\langle D,D'\rangle\,.$$

If we compute the expression of $\Upsilon_{D,\mathsf{e}}$ in the following "chart"

$$\chi := (g_D^{\tilde z}\times\mathrm{id}_{\mathbb{C}^\tau})\circ(F_{\tilde z}\times\mathrm{id}_{\mathbb{C}^\tau}) : [U_D]\times\mathbb{C}^\tau\xrightarrow{\sim}U_D\times\mathbb{C}^\tau$$

we get:

$$\widetilde{\Upsilon}_{D,\mathsf{e}} := \chi\circ\Upsilon_{D,\mathsf{e}}\circ\chi^{-1} = (g_D^{\tilde z}\times\mathrm{id}_{\mathbb{C}^\tau})\circ((g_{D'}^{\tilde z})^{-1}\times\mathrm{id}_{\mathbb{C}^\tau})\circ g_{D'}\circ g_D^{-1}$$

and, thanks to (8) and (9) we obtain:

$$\widetilde{\Upsilon}_{D,\mathsf{e}} = (\Phi^{\tilde z}_{D,\mathsf{e}}\times\mathrm{id}_{\mathbb{C}^\tau})^{-1}\circ\Phi_{D,\mathsf{e}}\,.$$

Using the explicit expressions of $\Phi_{D,\mathsf{e}}$ and $\Phi^{\tilde z}_{D,\mathsf{e}}$, we finally have, writing $z=(z_1,\dots,z_\tau)\in\mathbb{C}^\tau$ and $\tilde z=(\tilde z_1,\dots,\tilde z_\tau)$,

(i) $\widetilde{\Upsilon}_{D,\mathsf{e}}(m,z) = (\exp(X_{\mathsf{e}})[z_{\kappa(\mathsf{e})}-\tilde z_{\kappa(\mathsf{e})}](m), z)$, if $(D,\mathsf{e})\in\mathcal{A}$,
(ii) $\widetilde{\Upsilon}_{D,\mathsf{e}} = \widetilde{\Upsilon}^{-1}_{D',\mathsf{e}}$, if $(D',\mathsf{e})\in\mathcal{A}$,
(iii) $\widetilde{\Upsilon}_{D,\mathsf{e}}(m,z) = (m,z)$, if $(D,\mathsf{e}),(D',\mathsf{e})\notin\mathcal{A}$.

-*Step 3.* From the previous expression of $\widetilde{\Upsilon}_{D,\mathsf{e}}$ we deduce the following partial derivatives:

$$\left.\frac{\partial\mathrm{pr}_{U_D}\circ\widetilde{\Upsilon}_{D,\mathsf{e}}}{\partial z_k}\right|_{z=\tilde z} = \begin{cases} X_{\mathsf{e}} & \text{if } (D,\mathsf{e})\in\mathcal{A} \text{ and } k=\kappa((D,\mathsf{e})),\\ -X_{\mathsf{e}} & \text{if } (D',\mathsf{e})\in\mathcal{A} \text{ and } k=\kappa((D',\mathsf{e})),\\ 0 & \text{otherwise.}\end{cases}$$

Since $\Upsilon_{D,\mathsf{e}} = [(g_D^{\tilde z}\circ F_{\tilde z})\times\mathrm{id}_{\mathbb{C}^\tau}]^{-1}\circ\widetilde{\Upsilon}_{D,\mathsf{e}}\circ[(g_D^{\tilde z}\circ F_{\tilde z})\times\mathrm{id}_{\mathbb{C}^\tau}]$, we deduce that

$$\left.\frac{\partial\mathrm{pr}_{M_{\mathcal{F}_{\tilde z}}}\circ\Upsilon_{D,\mathsf{e}}}{\partial z_k}\right|_{z=\tilde z} = \begin{cases} X^{\tilde z}_{\mathsf{e}} & \text{if } (D,\mathsf{e})\in\mathcal{A} \text{ and } k=\kappa((D,\mathsf{e})),\\ -X^{\tilde z}_{\mathsf{e}} & \text{if } (D',\mathsf{e})\in\mathcal{A} \text{ and } k=\kappa((D',\mathsf{e})),\\ 0 & \text{otherwise,}\end{cases}$$

where $X^{\tilde z}_{\mathsf{e}} = (g_D^{\tilde z}\circ F_{\tilde z})^* X_{\mathsf{e}}$ is a basis of the one-dimensional vector space $\mathcal{T}_{\mathcal{F}_{\tilde z}}(f_{\tilde z}(s_{\mathsf{e}}))$, $\mathcal{F}^\diamond_{\tilde z}$ denoting the marked foliation $(\mathcal{F}_{\mathbb{C}^\tau}(\tilde z), f_{\tilde z})$, because $g_D^{\tilde z}\circ F_{\tilde z} : [U_D]\to U_D$ is a biholomorphism conjugating $\mathcal{F}^\sharp_{\tilde z}$ and $\mathcal{F}^\sharp$. Hence the linear map $\gamma_{\tilde z} : T_{\tilde z}\mathbb{C}^\tau \to \bigoplus_{(D,\mathsf{e})\in\mathcal{A}}\mathcal{T}_{\mathcal{F}_{\tilde z}}(f_{\tilde z}(s_{\mathsf{e}}))$ defined by $\gamma_{\tilde z}\left(\frac{\partial}{\partial z_{\kappa(D,\mathsf{e})}}\right) = X^{\tilde z}_{\mathsf{e}}$ is an isomorphism. By construction, and according to Theorem 2.11 and Remark 2.12, the Kodaira-Spencer map $\left.\frac{\partial\mathcal{F}_{\mathbb{C}^\tau}}{\partial z}\right|_{z=\tilde z} = \delta^{\mathcal{F}^\diamond_{\tilde z}}_{\mathcal{A}}\circ\gamma_{\tilde z}$ is also an isomorphism. By Theorem 2.10 we conclude

that the deformation $(\underline{\mathcal{F}}_{\mathbf{U},\tilde{u}}, \iota_{\tilde{u}})$ of $\mathcal{F}_{\tilde{u}}$ is $\mathcal{C}^{\mathrm{ex}}$-universal. This achieves the proof of Theorem 2.9. $\square$

3. Factorization properties of the locally universal family

3.1. **Local factorization property.** We will now prove that the global family $\underline{\mathcal{F}}^{\diamond}_{\mathbf{U}}$ of Theorem 2.9 is complete in a similar meaning to that given by Kodaira-Spencer in [4] in the context of complex manifolds:

Theorem 3.1. *Let $\mathcal{F}^{\diamond} = (\mathcal{F}, f)$ be a marked by $\mathcal{E}^{\diamond}$ foliation of finite type which is a generalized curve and let $\underline{\mathcal{G}}^{\diamond}_{P}$ be a marked global family in $\mathrm{SL}_P(\mathcal{F}^{\diamond})$. Let us consider $t_0 \in P$ and $\tilde{u} \in \mathbf{U}$ such that the marked foliation $\mathcal{G}^{\diamond}_{P}(t_0)$ is $\mathcal{C}^{\mathrm{ex}}$-conjugated to the fiber $\mathcal{F}^{\diamond}_{\mathbf{U}}(\tilde{u})$ of the marked global family $\underline{\mathcal{F}}^{\diamond}_{\mathbf{U}}$ given by Theorem 2.9. Then there exists a unique germ of holomorphic map $\lambda : (P, t_0) \to (\mathbf{U}, \tilde{u})$ such that the germ of $\underline{\mathcal{G}}^{\diamond}_{P}$ at t_0 is $\mathcal{C}^{\mathrm{ex}}$-conjugated, as marked family, to the germ at t_0 of $\lambda^* \underline{\mathcal{F}}^{\diamond}_{\mathbf{U}}$.*

Proof. Let ϕ be an $\mathcal{C}^{\mathrm{ex}}$-homeomorphism such that

$$\phi(\mathcal{G}^{\diamond}_{P}(t_0)) = \mathcal{F}^{\diamond}_{\mathbf{U}}(\tilde{u})\,.$$

We will denote by $(\underline{\mathcal{G}}_{P,t_0}, \delta)$ the deformation of $\mathcal{G}_P(t_0)$ over the germ of manifold (P, t_0) defined by the germ of $\underline{\mathcal{G}}_P$ at t_0 and by the embedding δ given by the inclusion map of the ambient space of $\mathcal{G}_P(t_0)$ in that of $\mathcal{G}_P$. According to Theorem 2.9 and Theorem 2.4 the deformation $(\underline{\mathcal{F}}_{\mathbf{U},\tilde{u}}, \iota_{\tilde{u}})$ is $\mathcal{C}^{\mathrm{ex}}$-universal and any deformation

$$(\underline{\mathcal{G}}_{\mathbf{U},\tilde{u}}, \kappa) \in \phi^*([\underline{\mathcal{F}}_{\mathbf{U},\tilde{u}}, \iota_{\tilde{u}}]) \;\in\; \mathrm{Def}^{(\mathbf{U},\tilde{u})}_{\mathcal{G}_P(t_0)}$$

is a $\mathcal{C}^{\mathrm{ex}}$-universal deformation of $\mathcal{G}_P(t_0)$. There exist a holomorphic map germ $\lambda : (P, t_0) \to (\mathbf{U}, \tilde{u})$ and a $\mathcal{C}^{\mathrm{ex}}$-conjugacy Φ_{P,t_0} from the deformation $(\underline{\mathcal{G}}_{P,t_0}, \delta)$ to $\lambda^*(\underline{\mathcal{G}}_{\mathbf{U},\tilde{u}}, \kappa)$. By definition of ϕ^*, the associated families $\underline{\mathcal{G}}_{\mathbf{U},\tilde{u}}$ and $\underline{\mathcal{F}}_{\mathbf{U},\tilde{u}}$ are $\mathcal{C}^{\mathrm{ex}}$-conjugated as germs of families over $(\mathbf{U}, \tilde{u})$, by a $\mathcal{C}^{\mathrm{ex}}$-homeomorphism $\Phi_{\mathbf{U},\tilde{u}}$ which is equal to ϕ over $\tilde{u}$. Since ϕ is compatible with the markings of $\mathcal{G}^{\diamond}_{P}(t_0)$ and $\mathcal{F}^{\diamond}_{\mathbf{U}}(\tilde{u})$, it follows from Remark 2.5 that the homeomorphism $\lambda^*\Phi_{\mathbf{U},\tilde{u}} \circ \Phi_{P,t_0}$ that conjugates the associated family $\underline{\mathcal{G}}_{P,t_0}$ to $\lambda^*\underline{\mathcal{F}}_{\mathbf{U},\tilde{u}}$, is compatible with the markings of the families $\underline{\mathcal{G}}^{\diamond}_{P,t_0}$ and $\lambda^*\underline{\mathcal{F}}^{\diamond}_{\mathbf{U},\tilde{u}}$. The uniqueness of λ results from the following lemma. $\square$

Lemma 3.2. *Let $\underline{\mathcal{G}}^{\diamond}_{P}$ be a marked global family in $\mathrm{SL}_P(\mathcal{F}^{\diamond})$ over a connected complex manifold P. Two holomorphic liftings $\lambda, \mu : P \to \mathbf{U}$ of the moduli map of $\underline{\mathcal{G}}^{\diamond}_{P}$ through the moduli map of $\underline{\mathcal{F}}^{\diamond}_{\mathbf{U}}$,*

$$\mathrm{mod}_{\underline{\mathcal{F}}^{\diamond}_{\mathbf{U}}} \circ \lambda = \mathrm{mod}_{\underline{\mathcal{F}}^{\diamond}_{\mathbf{U}}} \circ \mu = \mathrm{mod}_{\underline{\mathcal{G}}^{\diamond}_{P}}\,, \tag{10}$$

$$\begin{tikzcd}
\mathbf{U} \arrow[rr, "\mathrm{mod}_{\underline{\mathcal{F}}^{\diamond}_{\mathbf{U}}}"] & & \mathrm{Mod}([\mathcal{F}^{\diamond}]) \\
P \arrow[u, bend left, "\lambda"] \arrow[u, bend right, "\mu"'] \arrow[urr, "\mathrm{mod}_{\underline{\mathcal{G}}^{\diamond}_{P}}"'] & &
\end{tikzcd}$$

coincide as soon as they take the same value at some point $t_0 \in P$.

Proof. Let $d \in \mathbf{D}$ be the image of $\lambda(t_0) = \mu(t_0)$ by the morphism β in (2). It follows from relation (3) and (10) that for any $t \in P$ we have: $\Lambda(\lambda(t)) \cdot \zeta(d) = \Lambda(\mu(t)) \cdot \zeta(d)$.

According to the exact sequence (2), there is $N_t \in \mathbb{Z}^p$ such that $\lambda(t) - \mu(t) = \alpha(N_t)$. The following sets

$$K_N := \{t \in P \; ; \; \alpha(N_t) = N\} \, , \quad N \in \mathbb{Z}^p \, ,$$

are closed analytic subsets of P given by the global equations $\lambda(t) - \mu(t) = N$. They cannot all be proper subsets of P, because $P = \cup_{N \in \mathbb{Z}^p} K_N$. Therefore there exists $N_0 \in \mathbb{Z}^p$ such that $\lambda(t) - \mu(t) = \alpha(N_0)$ for any $t \in P$. As $\lambda(t_0) = \mu(t_0)$, we have: $N_0 = 0$, which ends the proof. □

Corollary 3.3. *Let $\mathcal{F}^\diamond$ be a marked by $\mathcal{E}^\diamond$ foliation of finite type which is a generalized curve. For any marked global family $\underline{\mathcal{G}}_P^\diamond \in \mathrm{SL}_P(\mathcal{F}^\diamond)$ over a connected manifold P the map $\beta \circ \mathrm{mod}_{\underline{\mathcal{F}}_P^\diamond} : P \to \mathbf{D}$ is constant, where $\beta : \mathrm{Mod}([\mathcal{F}^\diamond]) \to \mathbf{D}$ still denotes the last group morphism in the exact sequence (2).*

Proof. It suffices to prove that the map $\beta \circ \mathrm{mod}_{\underline{\mathcal{F}}_{\mathbf{U}}^\diamond}$ is locally constant. Let t_0 be a point in P. There is $\tilde{u} \in \mathbf{U}$ such that $[\mathcal{F}_{\mathbf{U}}^\diamond(\tilde{u})] = [\mathcal{G}_P^\diamond(t_0)] \in \mathrm{Mod}([\mathcal{F}^\diamond])$. Theorem 3.1 provides a holomorphic map germ $\lambda : (P, t_0) \to (\mathbf{U}, \tilde{u})$ such that, according to (3), for $t \in P$ close to t_0, we have:

$$\mathrm{mod}_{\underline{\mathcal{G}}_P^\diamond}(t) = \mathrm{mod}_{\underline{\mathcal{F}}_{\mathbf{U}}^\diamond}(\lambda(t), d) = \Lambda(\lambda(t)) \cdot \zeta(d) \; \in \; \Lambda(\mathbb{C}^\tau) \cdot \zeta(d) = \beta^{-1}(d) \, .$$

□

3.2. **Global factorization property.** We are interested now in factorizing up to $\mathcal{C}^{\mathrm{ex}}$-conjugacy marked global families through the marked global family $\underline{\mathcal{F}}_{\mathbf{U}}^\diamond$ provided by Theorem 2.9. However there could already exist obstructions to the factorization of the ambient space of the family in the ambient space of $\underline{\mathcal{F}}_{\mathbf{U}}^\diamond$, which is trivial over $\mathbf{U}$. To obtain such a factorization we need to weaken the relation of conjugacy.

Definition 3.4. *Two equisingular (resp. marked equisingular) global families over a manifold Q are* ***locally $\mathcal{C}^{\mathrm{ex}}$-conjugated*** *if their germs at any point of Q are $\mathcal{C}^{\mathrm{ex}}$-conjugated as families (resp. as marked families, see Section 2.1).*

The object of this section is to prove the following theorem of factorization up to local $\mathcal{C}^{\mathrm{ex}}$-conjugacy.

Theorem 3.5. *Let $\mathcal{F}^\diamond$ be a marked by $\mathcal{E}^\diamond$ foliation of finite type which is a generalized curve and let $\underline{\mathcal{F}}_{\mathbf{U}}^\diamond$ be the marked equisingular global family given by Theorem 2.9. Let P be a connected manifold satisfying $H_1(P, \mathbb{Z}) = 0$ and let $\underline{\mathcal{G}}_P^\diamond$ be a global family in $\mathrm{SL}_P(\mathcal{F}^\diamond)$. Then for any $t_0 \in P$ and $(\tilde{z}, d) \in \mathbf{U}$ such that the marked foliations $\mathcal{G}_P^\diamond(t_0)$ and $\mathcal{F}_{\mathbf{U}}^\diamond(\tilde{z}, d)$ are $\mathcal{C}^{\mathrm{ex}}$-conjugated, there exists a unique holomorphic map $\lambda : P \to \mathbf{U}$ satisfying $\lambda(t_0) = (\tilde{z}, d)$, such that the marked global families $\underline{\mathcal{G}}_P^\diamond$ and $\lambda^* \underline{\mathcal{F}}_{\mathbf{U}}^\diamond$ are locally $\mathcal{C}^{\mathrm{ex}}$-conjugated.*

Proof. - Step 1: Construction of λ. According to Corollary 3.3, $\mathrm{mod}_{\underline{\mathcal{G}}_P^\diamond}$ takes values in $\beta^{-1}(d)$. Thus for any $t \in P$, there exist $z_t \in \mathbb{C}^\tau$ and a $\mathcal{C}^{\mathrm{ex}}$-homeomorphism ϕ_t such that $\phi_t(\mathcal{G}_P(t)) = \mathcal{F}_{\mathbf{U}}^\diamond(z_t, d)$. Let us denote by $\underline{\mathcal{G}}_{P,t}^\diamond$ the germ of $\underline{\mathcal{G}}_P^\diamond$ at t. According to Theorem 3.1 there exists a holomorphic map germ $\lambda_t : (P, t) \to (\mathbf{U}, (z_t, d))$ such that $\underline{\mathcal{G}}_{P,t}^\diamond$ and $\lambda_t^* \underline{\mathcal{F}}_{\mathbf{U}}^\diamond$ are $\mathcal{C}^{\mathrm{ex}}$-conjugated, as germs of families. Therefore there exist an open covering $(V_i)_{i \in I}$, $I \subset \mathbb{N}$, of P and holomorphic maps $\lambda_i : V_i \to \mathbb{C}^\tau$ such that the restriction of $\underline{\mathcal{G}}_P^\diamond$ to V_i are $\mathcal{C}^{\mathrm{ex}}$-conjugated to $(\lambda_i, d)^* \underline{\mathcal{F}}_{\mathbf{U}}^\diamond$. Thus we have:

$$\mathrm{mod}_{\underline{\mathcal{F}}_{\mathbf{U}}^\diamond} \circ (\lambda_i, d) = \mathrm{mod}_{\underline{\mathcal{G}}_P^\diamond}|_{V_i} \, .$$

We can also require that this covering is locally finite and that the open sets V_i and $V_i \cap V_j$, $i, j \in I$, are connected. When $V_i \cap V_j$ is non empty, the restrictions of (λ_i, d)

and (λ_j, d) to this open set are two factorizations of the moduli map of $\underline{\mathcal{G}}_P^\diamond$ through $\mathrm{mod}_{\underline{\mathcal{F}}_{\mathbf{U}}^\diamond}$. Fixing a point t_{ij} in $V_i \cap V_j$, we have:

$$[\mathcal{F}_{\mathbf{U}}^\diamond(\lambda_i(t_{ij}), d)] = [\mathcal{F}_{\mathbf{U}}^\diamond(\lambda_j(t_{ij}), d)] = [\mathcal{G}_P^\diamond(t_{ij})]\,.$$

The relation (3) gives $\Lambda(\lambda_i(t_{ij}))\cdot\zeta(d) = \Lambda(\lambda_j(t_{ij}))\cdot\zeta(d)$; thus $(\lambda_j(t_{ij})-\lambda_i(t_{ij}))$ belongs to the kernel of Λ and there exist $N_{ij} \in \mathbb{Z}^p$ such that $\lambda_j(t_{ij})-\lambda_i(t_{ij}) = \alpha(N_{ij})$. As by assumption we have: $H_1(P,\mathbb{Z}) = 0$, the Čech cohomology group $H^1(P,\mathbb{Z}^p)$ is trivial and there exist $N_i \in \mathbb{Z}^p$, $i \in I$, such that $N_j - N_i = N_{ij}$ as soon as $V_i \cap V_j \neq \emptyset$. Notice that the maps $(\alpha(N_i)+\lambda_i, d) : V_i \to \mathbf{U}$ are again liftings of $\mathrm{mod}_{\underline{\mathcal{G}}_P^\diamond}|_{V_i}$ through $\mathrm{mod}_{\underline{\mathcal{F}}_{\mathbf{U}}^\diamond}$. Indeed we have:

$$\begin{aligned}\mathrm{mod}_{\underline{\mathcal{F}}_{\mathbf{U}}^\diamond} \circ (\alpha(N_i) + \lambda_i, d) =& \Lambda(\alpha(N_i) + \lambda_i) \cdot \zeta(d)\\ =& \Lambda(\lambda_i) \cdot \zeta(d) = \mathrm{mod}_{\underline{\mathcal{F}}_{\mathbf{U}}^\diamond} \circ (\lambda_i, d) = \mathrm{mod}_{\underline{\mathcal{G}}_P^\diamond}|_{V_i}\,.\end{aligned}$$

Since $\alpha(N_i)+\lambda_i(t_{ij}) = \alpha(N_j)+\lambda_j(t_{ij})$, thanks to Lemma 3.2 and the connectedness of $V_i \cap V_j$, the maps $\alpha(N_i) + \lambda_i$, $i \in I$, glue to a global holomorphic map

$$\lambda : P \longrightarrow \mathbb{C}^\tau \times \{d\} \subset \mathbf{U}\,, \quad \mathrm{mod}_{\underline{\mathcal{F}}_{\mathbf{U}}^\diamond} \circ (\lambda, d) = \mathrm{mod}_{\underline{\mathcal{G}}_P^\diamond}\,.$$

Step 2: Properties of λ. First we notice that for any $N \in \mathbb{Z}^p$ we also have the equality $\mathrm{mod}_{\underline{\mathcal{F}}_{\mathbf{U}}^\diamond} \circ (\alpha(N) + \lambda, d) = \mathrm{mod}_{\underline{\mathcal{G}}_P^\diamond}$. Consequently we can assume that $\lambda(t_0) = (\tilde{z}, d)$. On the other hand, the global families $\underline{\mathcal{G}}_P^\diamond|_{V_i}$ and $(\lambda_i, d)^*\underline{\mathcal{F}}_{\mathbf{U}}^\diamond$ being $\mathcal{C}^{\mathrm{ex}}$-conjugated, the local $\mathcal{C}^{\mathrm{ex}}$-conjugacy between $\underline{\mathcal{G}}_P^\diamond$ and $\lambda^*\underline{\mathcal{F}}_{\mathbf{U}}^\diamond$ results from the lemma below. □

Lemma 3.6. *If $\mu : (Q, u_0) \to (\mathbb{C}^\tau, z_0)$ is a holomorphic map germ and $N \in \mathbb{Z}^p$, then the germs at u_0 of the marked families $(\mu, d)^*\underline{\mathcal{F}}_{\mathbf{U}}^\diamond$ and $(\alpha(N) + \mu, d)^*\underline{\mathcal{F}}_{\mathbf{U}}^\diamond$ are $\mathcal{C}^{\mathrm{ex}}$-conjugated.*

Proof. Let us denote by $\underline{\mathcal{F}}_{\mathbf{U},(z,d)}^\diamond$ the germ of $\underline{\mathcal{F}}_{\mathbf{U}}^\diamond$ at (z, d) considered as a deformation of the foliation $\mathcal{F}_{\mathbf{U}}^\diamond(z, d)$, the embedding map being the inclusion $M_0 \times \{(z, d)\} \hookrightarrow M_0 \times \mathbf{U}$.

As $(\alpha(N) + \mu, d)^*\underline{\mathcal{F}}_{\mathbf{U}}^\diamond = (\mu, d)^*(\Delta^*\underline{\mathcal{F}}_{\mathbf{U}}^\diamond)$, with

$$\Delta : (\mathbf{U}, (z_0, d)) \longrightarrow (\mathbf{U}, (\alpha(N) + z_0, d))\,, \quad \Delta(z, d) := (\alpha(N) + z, d)\,,$$

it suffices to see that $\underline{\mathcal{F}}_{\mathbf{U},(z_0,d)}^\diamond$ is $\mathcal{C}^{\mathrm{ex}}$-conjugated to $\Delta^*(\underline{\mathcal{F}}_{\mathbf{U},(\alpha(N)+z_0,d)}^\diamond)$ as a family. To lighten the text let us write

$$\mathcal{F}_0^\diamond := \mathcal{F}_{\mathbf{U}}^\diamond(z_0, d) \quad \text{and} \quad \mathcal{F}_N^\diamond := \mathcal{F}_{\mathbf{U}}^\diamond(\alpha(N) + z_0, d) = (\Delta^*\mathcal{F}_{\mathbf{U}}^\diamond)(z_0, d)\,.$$

There exists a $\mathcal{C}^{\mathrm{ex}}$-homeomorphism ϕ such that $\phi(\mathcal{F}_N^\diamond) = \mathcal{F}_0^\diamond$. Let $\underline{\mathcal{K}}_{\mathbf{U},(z_0,d)}^\diamond$ be a deformation of $\mathcal{F}_N^\diamond$ over the germ of manifold $(\mathbf{U}, (z_0, d))$ that belongs to the class $\phi^*([\underline{\mathcal{F}}_{\mathbf{U},(z_0,d)}^\diamond])$. According to Theorem 2.9 the deformations $\underline{\mathcal{F}}_{\mathbf{U},(z_0,d)}^\diamond$ and $\underline{\mathcal{F}}_{\mathbf{U},(\alpha(N)+z_0,d)}^\diamond$ are $\mathcal{C}^{\mathrm{ex}}$-universal; it follows from Theorem 2.4 and Remark 2.3 that the deformation $\underline{\mathcal{K}}_{\mathbf{U},(z_0,d)}^\diamond$ of $\mathcal{F}_N^\diamond$ is $\mathcal{C}^{\mathrm{ex}}$-universal. On the other hand, since Δ is a biholomorphism, the deformation $\Delta^*(\underline{\mathcal{F}}_{\mathbf{U},(\alpha(N)+z_0,d)}^\diamond)$ of $\mathcal{F}_N^\diamond$ is also $\mathcal{C}^{\mathrm{ex}}$-universal over the same parameter space $(\mathbf{U}, (z_0, d))$, again by Remark 2.3. By uniqueness of $\mathcal{C}^{\mathrm{ex}}$-universal deformations, $\underline{\mathcal{K}}_{\mathbf{U},(z_0,d)}^\diamond$ and $\Delta^*(\underline{\mathcal{F}}_{\mathbf{U},(\alpha(N)+z_0,d)}^\diamond)$ are $\mathcal{C}^{\mathrm{ex}}$-conjugated deformations of $\mathcal{F}_N^\diamond$. We end the proof by noting that by definition of $\phi^*([\underline{\mathcal{F}}_{\mathbf{U},(z_0,d)}^\diamond])$, the families $\underline{\mathcal{K}}_{\mathbf{U},(z_0,d)}^\diamond$ and $\underline{\mathcal{F}}_{\mathbf{U},(z_0,d)}^\diamond$ are $\mathcal{C}^{\mathrm{ex}}$-conjugated. □

Now, we consider a weaker notion of conjugacy requiring the equality of moduli maps, in other words, the $\mathcal{C}^{\mathrm{ex}}$-conjugacy fiber by fiber for each value of the parameter.

Theorem 3.7. *Let $\mathcal{F}^\diamond$ be a marked by $\mathcal{E}^\diamond$ foliation of finite type which is a generalized curve and let $\underline{\mathcal{F}}^\diamond_{\mathbf{U}}$ be the marked equisingular global family given by Theorem 2.9. If P is a connected manifold such that $H_1(P,\mathbb{Z})=0$, then:*

(1) the moduli map of any marked global family $\underline{\mathcal{G}}^\diamond_P \in \mathrm{SL}_P(\mathcal{F}^\diamond)$ factorizes through the moduli map of $\underline{\mathcal{F}}^\diamond_{\mathbf{U}}$, i.e. there exists a holomorphic map $\lambda : P \to \mathbf{U}$ such that $\mathrm{mod}_{\underline{\mathcal{G}}^\diamond_P} = \mathrm{mod}_{\underline{\mathcal{F}}^\diamond_{\mathbf{U}}} \circ \lambda$,

(2) given $t_0 \in P$ and $(\tilde{z},d) \in \mathbf{U}$ such that $\mathcal{G}^\diamond_P(t_0)$ is $\mathcal{C}^{\mathrm{ex}}$-conjugated to $\mathcal{F}^\diamond_{\mathbf{U}}(\tilde{z},d)$, there exists a unique $\lambda : P \to \mathbf{U}$ satisfying $\lambda(t_0) = (\tilde{z},d)$ and $\mathrm{mod}_{\underline{\mathcal{G}}^\diamond_P} = \mathrm{mod}_{\underline{\mathcal{F}}^\diamond_{\mathbf{U}}} \circ \lambda$,

(3) there exist $p \in \mathbb{N}$, a holomorphic action $*$ of $\mathbb{Z}^p$ on $\mathbf{U}$ and an action $\star$ of a finite product $\mathbf{I}$ of braid groups on the quotient $(\mathbf{U}/*\mathbb{Z}^p)$, such that $\mathcal{F}_{\mathbf{U}}(u_1)$ and $\mathcal{F}_{\mathbf{U}}(u_2)$ are $\mathcal{C}^{\mathrm{ex}}$-conjugated if and only if there exists $g \in \mathbf{I}$ such that $g \star (\mathbb{Z}^p * u_1) = \mathbb{Z}^p * u_2$.

Proof. Assertion (1) follows from Theorem 3.5. Notice that the uniqueness of λ under conjugacies for each value of the parameter, stated in assertion (2), is a stronger property than that given by Theorem 3.5. In fact, the existence of the factorization λ in (2) follows again from Theorem 3.5 and the uniqueness from Lemma 3.2.

To see assertion (3) we define the action

$$* : \mathbb{Z}^p \times \mathbf{U} \to \mathbf{U}, \quad (N,(z,d)) \mapsto (z+\alpha(N),d),$$

where α is given by the exact sequence (2). Notice that the map $\mathbf{U} \to \mathrm{Mod}([\mathcal{F}^\diamond])$ defined by $(z,d) \mapsto \Lambda(z)\cdot\zeta(d) = [\mathcal{F}^\diamond_{\mathbf{U}}(z,d)]$ is surjective and $\Lambda(z)\cdot\zeta(d) = \Lambda(z')\cdot\zeta(d')$ if and only if there is $N \in \mathbb{Z}^p$ such that $N*(z,d) = (z',d')$. Thus $\mathbf{U}/*\mathbb{Z}^p$ is identified to $\mathrm{Mod}([\mathcal{F}^\diamond]) = \mathrm{SL}(\mathcal{F}^\diamond)/\sim_{\mathcal{C}^{\mathrm{ex}}}$. Let us denote by $\mathrm{SL}(\mathcal{F})$ the set of foliations having the same SL-type than $\mathcal{F}$ as was defined in the introduction by means of properties (SL1)-(SL3). The fibers of the canonical map $\mathrm{Mod}([\mathcal{F}^\diamond]) \to \mathrm{SL}(\mathcal{F})/\sim_{\mathcal{C}^{\mathrm{ex}}}$ are the orbits of the action

$$\star : \mathbf{I} \times \mathrm{Mod}([\mathcal{F}^\diamond]) \to \mathrm{Mod}([\mathcal{F}^\diamond]), \quad \varphi \star [\mathcal{G},g] = [\mathcal{G}, g\circ\varphi^{-1}],$$

where $\mathbf{I} := \mathrm{MCG}(\mathcal{E}^\diamond_{\mathcal{F}})$ is the **mapping class group** of the marked divisor $\mathcal{E}^\diamond_{\mathcal{F}}$ consisting of all the isotopy classes of homeomorphisms of $\mathcal{E}_{\mathcal{F}}$ leaving invariant the symmetric map $\cdot$ and the set $\mathrm{Sing}(\mathcal{F}^\sharp)$. It is well known that $\mathbf{I}$ is isomorphic to a finite product of braid groups. □

Corollary 3.8. *If P is a connected compact manifold such that $H_1(P,\mathbb{Z})=0$ then any marked global family $\mathcal{G}^\diamond_P \in \mathrm{SL}_P(\mathcal{F}^\diamond)$ is locally $\mathcal{C}^{\mathrm{ex}}$-trivial, and a fortiori the topological class of $\mathcal{G}^\diamond_P(t)$, $t \in P$, is constant.*

Proof of Theorem C. Assertion (0) corresponds to properties (i) and (ii) of the exact sequence (2) stated in Section 2.3. Property (1) of Theorem 2.9 implies assertion (1) of Theorem C, while assertion (2) of Theorem C is stated in Theorem 3.5. □

4. Topological equivalences for families and deformations

We will compare for global families and for germs of deformations the $\mathcal{C}^{\mathrm{ex}}$-conjugacy relation to a weaker conjugacy relation defined as the topological conjugacy before reduction, on each fiber of the family, without requiring the continuous dependence on the parameters of the conjugating homeomorphisms.

4.1. **Tame foliations.** Until now the only hypothesis that we have made on the germs of generalized curve foliations is that of finite type. Under this hypothesis, which is Krull generic [10], we have obtained, for the equivalence relation $\mathcal{C}^{\mathrm{ex}}$, complete families whose modular map is surjective. In order to obtain the same result for the equivalence relation $\mathcal{C}^0$ we must add a combinatorial assumption on the exceptional divisor $\mathcal{E}_{\mathcal{F}}$ and a dynamical assumption on the transverse structure of the foliation $\mathcal{F}$. For that let us denote by $\mathcal{E}^d_{\mathcal{F}}$ the union of irreducible components of the exceptional divisor $\mathcal{E}_{\mathcal{F}}$ which are dicritical and by $\mathcal{NC}_{\mathcal{F}}$ the set of singular points of $\mathcal{E}_{\mathcal{F}}$, called **nodal corners**, where the Camacho-Sad index of $\mathcal{F}^\sharp$ is a strictly positive real number. Let us consider the following two conditions:

(NC) *No Chain*: the closure of each connected component of $\mathcal{E}_{\mathcal{F}} \setminus \mathcal{E}^d_{\mathcal{F}}$ contains an irreducible component D with $\mathrm{card}(D \cap \mathrm{Sing}(\mathcal{F}^\sharp)) \neq 2$, i.e. there is no connected component of $\overline{\mathcal{E}_{\mathcal{F}} \setminus \mathcal{E}^d_{\mathcal{F}}}$ as in Figure 1.

(TR) *Transverse Rigidity*: if the closure of a connected component of $\mathcal{E}_{\mathcal{F}} \setminus (\mathcal{E}^d_{\mathcal{F}} \cup \mathcal{NC}_{\mathcal{F}})$ contains an irreducible component with at least 3 singular points of $\mathcal{F}^\sharp$, it also contains an irreducible component whose holonomy group for the foliation $\mathcal{F}^\sharp$ is topologically rigid, for instance unsolvable, cf. [11, 13].

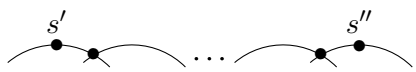

FIGURE 1. The only situation excluded by Condition (NC). Every divisor is non-dicritical and the elements of $\mathrm{Sing}(\mathcal{F}^\sharp)$ are s', s'' and the intersection points of the divisors; dicritical components may intersect any component.

Condition (NC) is technical and, as the generalized curve condition, only depends on a finite order jet of the differential form defining $\mathcal{F}$. In the presence of chains $\mathcal{C}^0$-classification must be approached differently and it will depend on open questions about the topology of Cremer biholomorphisms in one complex variable. Property (TR) is satisfied for a dense open set for the Krull topology of differential 1-forms fulfilling condition (NC), cf. [3]. The following theorem, first proved in [6] with additional assumptions, then generalized in [8, Theorem 11.4] using results of [14], justifies these two hypothesis:

Theorem 4.1 ([8, Theorem A])**.** *Two germs of generalized curves foliations $\mathcal{F}$ and $\mathcal{G}$ satisfying (NC) and (TR) are $\mathcal{C}^0$-conjugated if and only if they are $\mathcal{C}^{\mathrm{ex}}$-conjugated.*

Definition 4.2. *A germ of singular foliation is called* ***tame*** *if it is a generalized curve of finite type satisfying conditions (NC) and (TR).*

Remark 4.3. For a global equisingular family over a connected manifold properties (NC), (TR) and being of finite type are satisfied by any fiber as soon as they are satisfied by one fiber. □

Proof of Theorem A. We mark $\mathcal{F}$ by $\mathcal{E}_{\mathcal{F}}$ using the identity map; to obtain $\underline{\mathcal{F}}_{\mathbf{U}}$ we apply Theorem C to $\mathcal{F}^\diamond = (\mathcal{F}, \mathrm{id}_{\mathcal{E}_{\mathcal{F}}})$, that also provides a marking $(f_u)_{u\in\mathbf{U}}$ on $\underline{\mathcal{F}}_{\mathbf{U}}$.

We begin by proving assertion (1). Since $\mathcal{G}$ has the same SL-type as $\mathcal{F}$ there exists a homeomorphism $\varphi : \mathcal{E}_{\mathcal{F}} \to \mathcal{E}_{\mathcal{G}}$ satisfying properties (SL1)-(SL3). We consider the

marked by $\mathcal{E}^\diamond_{\mathcal{F}} = (\mathcal{E}_{\mathcal{F}}, \mathrm{Sing}(\mathcal{F}^\sharp), \cdot)$ foliation $\mathcal{G}^\diamond = (\mathcal{G}, \varphi)$ which has the same marked SL-type as $\mathcal{F}^\diamond$. By assertion (1) of Theorem C there exists $u_0 \in \mathbf{U}$ such that $\mathcal{G}^\diamond$ is $\mathcal{C}^{\mathrm{ex}}$-conjugated to $\mathcal{F}^\diamond_{\mathbf{U}}(u_0)$. A fortiori, $\mathcal{G}$ is $\mathcal{C}^0$-conjugated to $\mathcal{F}_{\mathbf{U}}(u_0)$.

Let us prove now assertion (2). As $\mathcal{G}_P(t_0)$ is $\mathcal{C}^0$-conjugated to $\mathcal{F}_{\mathbf{U}}(u_0)$, by Theorem 4.1 there is a $\mathcal{C}^{\mathrm{ex}}$-conjugacy $\phi : \mathcal{F}_{\mathbf{U}}(u_0) \to \mathcal{G}_P(t_0)$. The composition $g_{t_0} := \phi^\sharp \circ f_{u_0} : \mathcal{E}_{\mathcal{F}} \to \mathcal{E}_{\mathcal{G}_P(t_0)}$ of the lifting of ϕ through the reduction maps and the marking of $\mathcal{F}_{\mathbf{U}}(u_0)$ defines a marking of $\mathcal{G}_P(t_0)$ such that $\mathcal{G}^\diamond_P(t_0)$ is $\mathcal{C}^{\mathrm{ex}}$-conjugated to $\mathcal{F}^\diamond_{\mathbf{U}}(u_0)$ by ϕ^{-1}. Since P is simply connected, by assertion (b) of Remark 2.5, the marking g_{t_0} extends to a marking $(g_t)_{t\in P}$ of the global family $\underline{\mathcal{G}}_P$. We apply assertion (2) of Theorem C to $\underline{\mathcal{G}}^\diamond_P = (\underline{\mathcal{G}}_P, (g_t)_{t\in P})$ and we obtain a (unique) holomorphic map $\lambda : P \to \mathbf{U}$ such that $\lambda(t_0) = u_0$ and for any $t \in P$ the germs of marked families $\underline{\mathcal{G}}^\diamond_{P,t}$ and $\lambda^*\underline{\mathcal{F}}^\diamond_{\mathbf{U},\lambda(t)}$ over the germ of manifold (P, t) are $\mathcal{C}^{\mathrm{ex}}$-conjugated. A fortiori, the germs of families $\underline{\mathcal{G}}_{P,t}$ and $\lambda^*\underline{\mathcal{F}}_{\mathbf{U},\lambda(t)}$ are $\mathcal{C}^0$-conjugated.

Redundancy property (3) in Theorem A follows from assertion (3) in Theorem 3.7 taking into account that $\mathcal{C}^{\mathrm{ex}}$-conjugacy and $\mathcal{C}^0$-conjugacy are equivalent for tame foliations, see Theorem 4.1. □

4.2. **Weak and strong conjugacies of families.** In this section we will prove Theorem B of the introduction. Before that we state a marked version of that result in which the hypothesis are weaker.

Theorem 4.4. *Let $\underline{\mathcal{F}}^\diamond_Q$ and $\underline{\mathcal{G}}^\diamond_Q$ be marked by $\mathcal{E}^\diamond$ equisingular global families of foliations over a complex manifold Q, whose fibers are generalized curves of finite type. The following properties are equivalent*

(1) for any $u \in Q$ the marked foliations $\mathcal{F}^\diamond_Q(u)$ and $\mathcal{G}^\diamond_Q(u)$ are $\mathcal{C}^{\mathrm{ex}}$-conjugated,
(2) the marked global families $\underline{\mathcal{F}}^\diamond_Q$ and $\underline{\mathcal{G}}^\diamond_Q$ are locally $\mathcal{C}^{\mathrm{ex}}$-conjugated.

Proof. The implication (2) ⇒ (1) is trivial. To prove the converse we can assume that Q is connected and simply connected. Let us fix a fiber $\mathcal{F}^\diamond := \mathcal{F}^\diamond_Q(\tilde{u})$, $\tilde{u} \in Q$. According to the connectedness of Q and Remark 2.7, each $\mathcal{G}^\diamond_Q(u)$, $u \in Q$, belongs to $\mathrm{SL}(\mathcal{F}^\diamond)$, see Definition 2.6. By assertion (1) we have the equality

$$\mathrm{mod}_{\underline{\mathcal{G}}^\diamond_Q} = \mathrm{mod}_{\underline{\mathcal{F}}^\diamond_Q} : Q \longrightarrow \mathrm{Mod}([\mathcal{F}^\diamond]) .$$

Let $\underline{\mathcal{F}}^\diamond_{\mathbf{U}}$ be the marked global family given by Theorem 2.9. Let us consider $(\tilde{z}, d) \in \mathbf{U}$ such that $\mathcal{F}^\diamond_Q(\tilde{u})$ is $\mathcal{C}^{\mathrm{ex}}$-conjugated to $\mathcal{F}^\diamond_{\mathbf{U}}(\tilde{z}, d)$. Since $\mathcal{F}^\diamond_Q(\tilde{u})$ and $\mathcal{G}^\diamond_Q(\tilde{u})$ are $\mathcal{C}^{\mathrm{ex}}$-conjugated, Theorem 3.5 provides holomorphic maps $\lambda, \mu : Q \to \mathbb{C}^\tau \times \{d\} \subset \mathbf{U}$ satisfying $\lambda(\tilde{u}) = \mu(\tilde{u}) = (\tilde{z}, d)$, such that $\underline{\mathcal{F}}^\diamond_Q$ is locally $\mathcal{C}^{\mathrm{ex}}$-conjugated to $\lambda^*\underline{\mathcal{F}}^\diamond_{\mathbf{U}}$ and $\underline{\mathcal{G}}^\diamond_Q$ is locally $\mathcal{C}^{\mathrm{ex}}$-conjugated to $\mu^*\underline{\mathcal{F}}^\diamond_{\mathbf{U}}$. We thus have:

$$\mathrm{mod}_{\underline{\mathcal{F}}^\diamond_{\mathbf{U}}} \circ \lambda = \mathrm{mod}_{\underline{\mathcal{F}}^\diamond_Q} = \mathrm{mod}_{\underline{\mathcal{G}}^\diamond_Q} = \mathrm{mod}_{\underline{\mathcal{F}}^\diamond_{\mathbf{U}}} \circ \mu .$$

Consequently λ and μ are two liftings of the map $\mathrm{mod}_{\underline{\mathcal{G}}^\diamond_Q} = \mathrm{mod}_{\underline{\mathcal{F}}^\diamond_Q}$ through the map $\mathrm{mod}_{\underline{\mathcal{F}}^\diamond_{\mathbf{U}}}$, which coincide at the point $\tilde{u}$. It follows from assertion (2) of Theorem 3.7 that $\lambda = \mu$. Therefore $\underline{\mathcal{F}}^\diamond_Q$ and $\underline{\mathcal{G}}^\diamond_Q$ are locally $\mathcal{C}^{\mathrm{ex}}$-conjugated, since they are both locally $\mathcal{C}^{\mathrm{ex}}$-conjugated to $\lambda^*\underline{\mathcal{F}}^\diamond_{\mathbf{U}} = \mu^*\underline{\mathcal{F}}^\diamond_{\mathbf{U}}$. □

Now we will use Theorem 4.4 to prove Theorem B of the introduction.

Proof of Theorem B. Thanks to Theorem 4.1, assertions (1) and (2) are equivalent. The implication (3) ⟹ (1) is trivial. To prove (2) ⟹ (3) let us fix a point $\tilde{u}$ in Q and a marking $f_{\tilde{u}} : \mathcal{E} \xrightarrow{\sim} \mathcal{E}_{\mathcal{F}_Q(\tilde{u})}$ of the fiber $\mathcal{F}_Q(\tilde{u})$ by an appropriate marked divisor $\mathcal{E}^\diamond$. By restricting both families to a suitable neighborhood of $\tilde{u}$ we may

assume that Q is connected and simply connected. Thanks to (b) in Remark 2.5, $f_{\tilde{u}}$ extends to a marking $(f_u)_{u\in Q}$ of the global family $\underline{\mathcal{F}}_Q$ and we will write:

$$\underline{\mathcal{F}}_Q^{\diamond} := (\underline{\mathcal{F}}_Q, (f_u)_{u\in Q}) \quad \text{and} \quad \mathcal{F}^{\diamond} := (\mathcal{F}_Q(\tilde{u}), f_{\tilde{u}})\,.$$

According to Theorem 4.4, in order to obtain assertion (3) it only remains to prove the existence of a marking $(G_u)_{u\in Q}$ of $\underline{\mathcal{G}}_Q$ such that for each $u \in Q$ the marked foliation $\mathcal{G}_Q^{\diamond}(u) := (\mathcal{G}_Q(u), G_u)$ is $\mathcal{C}^{\text{ex}}$-conjugated to $\mathcal{F}_Q^{\diamond}(u)$:

$$[(\mathcal{G}_Q(u), G_u)] = [(\mathcal{F}_Q(u), f_u)] \in \text{Mod}([\mathcal{F}^{\diamond}])\,, \quad u \in Q\,. \tag{11}$$

For this, we choose for each $u \in Q$ a $\mathcal{C}^{\text{ex}}$-conjugacy

$$\varphi_u : (M(u), \theta(u)) \xrightarrow{\sim} (N(u), \vartheta(u))\,, \quad \varphi_u(\mathcal{F}_Q(u)) = \mathcal{G}_Q(u)\,,$$

and we denote by $\varphi_u^{\sharp} : (M_{\mathcal{F}_Q(u)}, \mathcal{E}_{\mathcal{F}_Q(u)}) \to (M_{\mathcal{G}_Q(u)}, \mathcal{E}_{\mathcal{G}_Q(u)})$ the germ of homeomorphism obtained by lifting it through the reduction of singularities of $\mathcal{F}_Q(u)$ and $\mathcal{G}_Q(u)$. We endow $\underline{\mathcal{G}}_Q$ with a marking by $\mathcal{E}^{\diamond}$

$$g_u : \mathcal{E} \to \mathcal{E}_{\mathcal{G}_Q(u)}\,, \quad u \in Q\,,$$

obtained thanks to Remark 2.5 by extending the marking $\varphi_{\tilde{u}}^{\sharp} \circ f_{\tilde{u}}$ of $\mathcal{G}_Q(\tilde{u})$. We also consider the following pre-marking of $\underline{\mathcal{G}}_Q$:

$$\varphi_u^{\sharp} \circ f_u : \mathcal{E} \to \mathcal{E}_{\mathcal{G}_Q(u)}\,, \quad u \in Q\,.$$

Since φ_u is a $\mathcal{C}^{\text{ex}}$-conjugacy from $\mathcal{F}_Q^{\diamond}(u)$ to $(\mathcal{G}_Q(u),\, \varphi_u^{\sharp} \circ f_u)$ and $\mathcal{F}_Q^{\diamond}(u)$ belongs to $\text{SL}(\mathcal{F}^{\diamond})$, this pre-marking satisfies

$$(\mathcal{G}_Q(u),\, \varphi_u^{\sharp} \circ f_u) \in \text{SL}(\mathcal{F}^{\diamond})$$

for each $u \in Q$.

Let us denote by $\text{MCG}(\mathcal{E}^{\diamond})$ the mapping class group of $\mathcal{E}^{\diamond} = (\mathcal{E}, \Sigma, \cdot)$, that is the group of isotopy classes $[\phi]$ of homeomorphisms $\phi : \mathcal{E} \xrightarrow{\sim} \mathcal{E}$ leaving invariant the symmetric map $\cdot$ and the set Σ. For each $[\phi] \in \text{MCG}(\mathcal{E}^{\diamond})$ let us consider the set

$$K_{[\phi]} := \left\{ u \in Q\ ;\ [g_u^{-1} \circ \varphi_u^{\sharp} \circ f_u] = [\phi] \right\}.$$

Since $\text{MCG}(\mathcal{E}^{\diamond})$ is countable and

$$\bigcup_{[\phi]\in \text{MCG}(\mathcal{E}^{\diamond})} K_{[\phi]} = Q\,,$$

there exists an element $[\Phi] \in \text{MCG}(\mathcal{E}^{\diamond})$ such that $K_{[\Phi]}$ is not contained in any countable union of proper closed analytic subsets of Q. Let us consider the marked by $\mathcal{E}^{\diamond}$ equisingular global family

$$\underline{\mathcal{G}}_Q^{\diamond} := \left(\underline{\mathcal{G}}_Q,\, (G_u)_{u\in Q}\right)\,, \quad G_u := g_u \circ \Phi\,.$$

We highlight that

$$\varphi_u(\mathcal{F}_Q^{\diamond}(u)) = \mathcal{G}_Q^{\diamond}(u)\,, \quad \text{if} \quad u \in K_{[\Phi]}\,, \tag{12}$$

as in this case $g_u \circ \Phi$ is isotopic to $\varphi_u^{\sharp} \circ f_u$. Therefore $\mathcal{G}_Q^{\diamond}(u)$ belongs to $\text{SL}(\mathcal{F}^{\diamond})$ when $u \in K_{[\Phi]}$. It follows from Remark 2.7 that $\underline{\mathcal{G}}_Q^{\diamond}$ belongs to $\text{SL}_Q(\mathcal{F}^{\diamond})$ and we can consider the map $Q \ni u \mapsto [\mathcal{G}_Q^{\diamond}(u)] \in \text{Mod}([\mathcal{F}^{\diamond}])$.

Let us now consider the map $\beta : \text{Mod}([\mathcal{F}^{\diamond}]) \to \mathbf{D}$ in the exact sequence (2). By Corollary 3.3 there is $d \in \mathbf{D}$ such that $\beta([\mathcal{F}_Q^{\diamond}(u)]) = d$, for every $u \in Q$. From (12)

and Corollary 3.3 we also have $\beta([\mathcal{G}_Q^\diamond(u)]) = d$, for every $u \in Q$. Let us fix $u_1 \in K_{[\Phi]}$ and $z_1 \in \mathbb{C}^\tau$ satisfying

$$[\mathcal{F}_Q^\diamond(u_1)] = [\mathcal{G}_Q^\diamond(u_1)] = \Lambda(z_1) \cdot \zeta(d)$$

By Theorem 3.5 there exist two holomorphic maps

$$\lambda : Q \longrightarrow \mathbb{C}^\tau\,, \qquad \lambda' : Q \longrightarrow \mathbb{C}^\tau\,,$$

satisfying $\lambda(u_1) = \lambda'(u_1) = z_1$ and

$$\Lambda(\lambda(u)) \cdot \zeta(d) = [\mathcal{F}_Q^\diamond(u)]\,, \quad \Lambda(\lambda'(u)) \cdot \zeta(d) = [\mathcal{G}_Q^\diamond(u)]\,, \quad u \in Q\,, \tag{13}$$

where $\Lambda : \mathbb{C}^\tau \to \mathrm{Mod}([\mathcal{F}^\diamond])$ is the map in (2). From (12) for $u \in K_{[\Phi]}$ we have:

$$\Lambda(\lambda(u)) \cdot \zeta(d) = \Lambda(\lambda'(u)) \cdot \zeta(d) \quad \text{if} \quad u \in K_{[\Phi]}$$

and $\lambda(u) - \lambda'(u)$ belongs to $\ker(\Lambda)$ for $u \in K_{[\Phi]}$. Let us choose $N_u \in \mathbb{Z}^p$ such that

$$\lambda(u) = \alpha(N_u) + \lambda'(u)\,, \quad u \in K_{[\Phi]}\,,$$

and for each $N \in \mathbb{Z}^p$ let us consider the following sets:

$$L_N := \{u \in K_{[\Phi]}\ ;\ N_u = N\} \subset L'_N = \{u \in Q\ ;\ \lambda(u) - \lambda'(u) = \alpha(N)\} \subset Q\,.$$

Clearly we have:

$$K_{[\Phi]} \ = \bigcup_{N \in \mathbb{Z}^p} L_N \subset \bigcup_{N \in \mathbb{Z}^p} L'_N\,.$$

Since each L'_N is a closed analytic subset of Q and $K_{[\Phi]}$ is not contained in a countable union of proper such sets, there exists $\widetilde{N} \in \mathbb{Z}^p$ such that $L'_{\widetilde{N}} = Q$. Consequently:

$$\lambda(u) = \alpha(\widetilde{N}) + \lambda'(u) \quad \text{for every} \quad u \in Q\,.$$

Then equalities (13) give the required equalities (11); that ends the proof. □

4.3. **Conjugacies of families versus conjugacies of deformations.** According to Remark 2.5, any deformation of a marked foliation may be canonically endowed with a marking. We will see that under finite type assumptions, the notion of conjugacy of deformations is equivalent to that of the conjugacy of their marked associated families.

Theorem 4.5. *Let us consider a finite type foliation $\mathcal{F}$ which is a generalized curve, $f : \mathcal{E} \to \mathcal{E}_\mathcal{F}$ a marking of $\mathcal{F}$, $(\underline{\mathcal{F}}_{Q^\cdot}, \iota)$ and $(\underline{\mathcal{G}}_{Q^\cdot}, \delta)$ two equisingular deformations of $\mathcal{F}$ over a germ of manifold $Q^\cdot := (Q, \tilde{u})$. Let us denote by $\underline{\mathcal{F}}_{Q^\cdot}^\diamond$ and $\underline{\mathcal{G}}_{Q^\cdot}^\diamond$ the families $\underline{\mathcal{F}}_{Q^\cdot}$ resp. $\underline{\mathcal{G}}_{Q^\cdot}$ endowed with the markings induced by the markings $\iota^\sharp \circ f$ and $\delta^\sharp \circ f$ of their special fibers. Then the following properties are equivalent:*

(1) there is a $\mathcal{C}^{\mathrm{ex}}$-conjugacy ϕ between the germs of families of $\underline{\mathcal{F}}_{Q^\cdot}$ and $\underline{\mathcal{G}}_{Q^\cdot}$ such that the lifting of $\delta^{-1} \circ \phi \circ \iota$ through the reduction of singularities of $\mathcal{F}$ leaves invariant each irreducible component of $\mathcal{E}_\mathcal{F}$,
(2) the marked families $\underline{\mathcal{F}}_{Q^\cdot}^\diamond$ and $\underline{\mathcal{G}}_{Q^\cdot}^\diamond$ are $\mathcal{C}^{\mathrm{ex}}$-conjugated,
(3) the deformations $(\underline{\mathcal{F}}_{Q^\cdot}, \iota)$ and $(\underline{\mathcal{G}}_{Q^\cdot}, \delta)$ are $\mathcal{C}^{\mathrm{ex}}$-conjugated.

This theorem will be easily obtained using the following extension result.

Theorem 4.6. *Let $\underline{\mathcal{F}}_{Q^\cdot}$ be an equisingular family over a germ of manifold $Q^\cdot = (Q, \tilde{u})$ whose fibers are finite type generalized curves. Let ϕ be a $\mathcal{C}^{\mathrm{ex}}$-automorphism of the special fiber $\mathcal{F}_{Q^\cdot}(\tilde{u})$ and $\phi^\sharp : M_{\mathcal{F}_{Q^\cdot}(\tilde{u})} \xrightarrow{\sim} M_{\mathcal{F}_{Q^\cdot}(\tilde{u})}$ the lifting of ϕ through the reduction of singularities of $\mathcal{F}_{Q^\cdot}(\tilde{u})$. If $\phi^\sharp$ leaves invariant each irreducible component of the exceptional divisor $\mathcal{E}_{\mathcal{F}_{Q^\cdot}(\tilde{u})}$, then ϕ extends to a $\mathcal{C}^{\mathrm{ex}}$-automorphism of the family $\underline{\mathcal{F}}_{Q^\cdot}$.*

Proof of Theorem 4.5 . The implications (3) $\Rightarrow$ (2) $\Rightarrow$ (1) are trivial. To see that (1) $\Rightarrow$ (3), we denote by $\phi_{\tilde{u}}$ the restriction of ϕ to the fibers over $\tilde{u}$ and we consider the $\mathcal{C}^{\mathrm{ex}}$-homeomorphism $\psi := \iota \circ \delta^{-1} \circ \phi_{\tilde{u}}$ which is an automorphism of $\mathcal{F}_{Q^\cdot}(\tilde{u})$. Notice that ψ is conjugated by ι to $\delta^{-1} \circ \phi \circ \iota$ and consequently its lifting $\psi^\sharp$ through the reduction of singularities of $\mathcal{F}_{Q^\cdot}(\tilde{u})$ leaves invariant each irreducible component of the exceptional divisor $\mathcal{E}_{\mathcal{F}_{Q^\cdot}(\tilde{u})}$. By Theorem 4.6 ψ extends to a $\mathcal{C}^{\mathrm{ex}}$-automorphism Ψ of the family $\underline{\mathcal{F}}_{Q^\cdot}$. Then the $\mathcal{C}^{\mathrm{ex}}$-homeomorphism $\Phi := \phi \circ \Psi^{-1}$ satisfies $\Phi(\underline{\mathcal{F}}_{Q^\cdot}) = \underline{\mathcal{F}}_{Q^\cdot}$ and

$$\Phi \circ \iota = \Phi_{\tilde{u}} \circ \iota = \phi_{\tilde{u}} \circ \Psi_{\tilde{u}}^{-1} \circ \iota = \phi_{\tilde{u}} \circ \psi^{-1} \circ \iota = \phi_{\tilde{u}} \circ \phi_{\tilde{u}}^{-1} \circ \delta \circ \iota^{-1} \circ \iota = \delta.$$

□

The proof of Theorem 4.6 is based on the following property of the pull-back map ϕ^* introduced in Theorem 2.4.

Lemma 4.7. *Let $\mathcal{F}$ be a germ of foliation and let ϕ be a $\mathcal{C}^{\mathrm{ex}}$-homeomorphism that conjugates $\mathcal{F}$ to itself. Assume that $\mathcal{F}$ is a finite type generalized curve and that the lifting $\phi^\sharp$ of ϕ through the reduction of singularities of $\mathcal{F}$ leaves invariant each irreducible component of the exceptional divisor $\mathcal{E}_{\mathcal{F}}$. Then for any pointed manifold $P^\cdot$ the pull-back map $\phi^* : \mathrm{Def}_{\mathcal{F}}^{P^\cdot} \xrightarrow{\sim} \mathrm{Def}_{\mathcal{F}}^{P^\cdot}$ is the identity map.*

Proof of Theorem 4.6. Let ι be the canonical embedding of the special fiber $\mathcal{F}_{Q^\cdot}(\tilde{u})$ in the family $\underline{\mathcal{F}}_{Q^\cdot}$, so that $(\underline{\mathcal{F}}_{Q^\cdot}, \iota)$ is a deformation of $\mathcal{F}_{Q^\cdot}(\tilde{u})$. By definition, there is an element $(\underline{\mathcal{G}}_{Q^\cdot}, \delta)$ of $\phi^*([\underline{\mathcal{F}}_{Q^\cdot}, \iota])$ that is conjugated to $(\underline{\mathcal{F}}_{Q^\cdot}, \iota)$ by a $\mathcal{C}^{\mathrm{ex}}$-homeomorphism germ $\Phi : \underline{\mathcal{G}}_{Q^\cdot} \to \underline{\mathcal{F}}_{Q^\cdot}$ such that $\Phi \circ \delta = \iota \circ \phi$. By Lemma 4.7 we have $\phi^*([\underline{\mathcal{F}}_{Q^\cdot}, \iota]) = [\underline{\mathcal{F}}_{Q^\cdot}, \iota]$. This means that $(\underline{\mathcal{G}}_{Q^\cdot}, \delta) \sim_{\mathcal{C}^{\mathrm{ex}}} (\underline{\mathcal{F}}_{Q^\cdot}, \iota)$, and there is a $\mathcal{C}^{\mathrm{ex}}$-homeomorphism germ Ψ such that $\Psi(\underline{\mathcal{F}}_{Q^\cdot}) = \underline{\mathcal{G}}_{Q^\cdot}$ and $\Psi \circ \iota = \delta$. Hence $\Phi \circ \Psi \circ \iota = \iota \circ \phi$, i.e. $\Phi \circ \Psi$ extends ϕ. □

It remains to prove Lemma 4.7 but before we will recall some representation results proved in [9]. Let **Fol** be the category of germs of generalized curves on $(\mathbb{C}^2, 0)$ and excellent conjugacies. Let $\mathbf{Fol_{ft}} \subset \mathbf{Fol}$ be the full subcategory consisting in finite type foliations. Let $\mathbf{Man}^\cdot \subset \mathbf{Set}^\cdot$ be the categories of pointed complex manifolds and sets. In [9, §5.3 and §1.2] we have introduced the contravariant functor $\mathbf{Fol_{ft}} \to \mathbf{Man}^\cdot$, $\mathcal{F} \mapsto H^1(\mathsf{A}_{\mathcal{F}}, \mathcal{T}_{\mathcal{F}})$. Any excellent conjugacy $\phi : \mathcal{G} \to \mathcal{F}$ induces the graph morphism $\mathsf{A}_\phi : \mathsf{A}_{\mathcal{G}} \to \mathsf{A}_{\mathcal{F}}$, $\star \mapsto \phi^\sharp(\star)$, which allows to define a morphism $[\phi^*] : H^1(\mathsf{A}_{\mathcal{F}}, \mathcal{T}_{\mathcal{F}}) \to H^1(\mathsf{A}_{\mathcal{G}}, \mathcal{T}_{\mathcal{G}})$ in the following way:

$$[\phi^*]([X_{D,\langle D,D'\rangle}]) = [Y_{D,\langle D,D'\rangle}], \quad Y_{D,\langle D,D'\rangle} = (\phi^\sharp)^* X_{\mathsf{A}_\phi(D,\langle D,D'\rangle)}\,.$$

We also considered the contravariant functor $\mathrm{Fac} : \mathbf{Man}^\cdot \times \mathbf{Fol_{ft}} \to \mathbf{Set}^\cdot$ defined by

$$\mathrm{Fac} : (P^\cdot, \mathcal{F}) \mapsto \mathcal{O}(P^\cdot, H^1(\mathsf{A}_{\mathcal{F}}, \mathcal{T}_{\mathcal{F}})^\cdot)\,,$$

and if $\mu : Q^\cdot \to P^\cdot$ and $\phi : \mathcal{G} \to \mathcal{F}$ are morphisms in $\mathbf{Man}^\cdot$ and $\mathbf{Fol_{ft}}$ respectively, then Fac_ϕ^μ sends $\lambda \in \mathcal{O}(P^\cdot, H^1(\mathsf{A}_{\mathcal{F}}, \mathcal{T}_{\mathcal{F}})^\cdot)$ into $[\phi^*] \circ \lambda \circ \mu \in \mathcal{O}(Q^\cdot, H^1(\mathsf{A}_{\mathcal{G}}, \mathcal{T}_{\mathcal{G}})^\cdot)$.

On the other hand, Theorem 2.4 allows us to consider the correspondence

$$\mathrm{Def} : (P^{\cdot}, \mathcal{F}) \mapsto \mathrm{Def}_{\mathcal{F}}^{P^{\cdot}} := \{\text{equisingular deformations of } \mathcal{F} \text{ over } P^{\cdot}\}/\sim_{\mathcal{C}^{\mathrm{ex}}}$$

together with the morphisms $\mathrm{Def}_{\phi}^{\mu} = \phi^* \circ \mu^* = \mu^* \circ \phi^*$. According to [9, Theorem 3.11] $\mathrm{Def} : \mathbf{Man}^{\cdot} \times \mathbf{Fol}_{\mathbf{ft}} \to \mathbf{Set}^{\cdot}$ is a contravariant functor.

Theorem 4.8 ([9, Theorem 6.3])**.** *There is an isomorphism of functors* $\xi : \mathrm{Def} \xrightarrow{\sim} \mathrm{Fac}$.

Proof of Lemma 4.7. Thanks to Theorem 4.8 it suffices to prove that the morphisms

$$\mathrm{Fac}_{\phi}^{\mathrm{id}_{P^{\cdot}}} : \mathcal{O}(P^{\cdot}, H^1(\mathsf{A}_{\mathcal{F}}, \mathcal{T}_{\mathcal{F}})^{\cdot}) \to \mathcal{O}(P^{\cdot}, H^1(\mathsf{A}_{\mathcal{F}}, \mathcal{T}_{\mathcal{F}})^{\cdot}), \quad \lambda \mapsto [\phi^*] \circ \lambda,$$

are the identity maps. Indeed the naturality of ξ gives the following commutative diagrams

$$\begin{array}{ccc} \mathrm{Def}_{\mathcal{F}}^{P^{\cdot}} & \xrightarrow{\phi^*} & \mathrm{Def}_{\mathcal{F}}^{P^{\cdot}} \\ \xi_{\mathcal{F}}^{P^{\cdot}} \downarrow \wr & & \wr \downarrow \xi_{\mathcal{F}}^{P^{\cdot}} \\ \mathcal{O}(P^{\cdot}, H^1(\mathsf{A}_{\mathcal{F}}, \mathcal{T}_{\mathcal{F}})^{\cdot}) & \xrightarrow[\mathrm{Fac}_{\phi}^{\mathrm{id}_{P^{\cdot}}}]{} & \mathcal{O}(P^{\cdot}, H^1(\mathsf{A}_{\mathcal{F}}, \mathcal{T}_{\mathcal{F}})^{\cdot}). \end{array}$$

The map $\mathrm{Fac}_{\phi}^{\mathrm{id}_{P^{\cdot}}}$ is the identity if and only if the pull-back map $[\phi^*] : H^1(\mathsf{A}_{\mathcal{F}}, \mathcal{T}_{\mathcal{F}}) \longrightarrow H^1(\mathsf{A}_{\mathcal{F}}, \mathcal{T}_{\mathcal{F}})$ is the identity. Since each irreducible component $D \in \mathsf{Ve}_{\mathsf{A}_{\mathcal{F}}}$ is fixed by $\phi^\sharp$, the induced graph morphism $\mathsf{A}_{\phi} : \mathsf{A}_{\mathcal{F}} \to \mathsf{A}_{\mathcal{F}}$ is the identity map and

$$[\phi^*] : H^1(\mathsf{A}_{\mathcal{F}}, \mathcal{T}_{\mathcal{F}}) \longrightarrow H^1(\mathsf{A}_{\mathcal{F}}, \mathcal{T}_{\mathcal{F}}), \quad [(X_{D,\mathsf{e}})_{D \in \mathsf{e} \in \mathsf{Ed}_{\mathsf{A}_{\mathcal{F}}}}] \mapsto \left[(\phi^{\sharp *}(X_{D,\mathsf{e}}))_{D \in \mathsf{e} \in \mathsf{Ed}_{\mathsf{A}_{\mathcal{F}}}}\right].$$

Thanks to Theorem 2.11, it suffices to see that $\phi^{\sharp *} : \mathcal{T}_{\mathcal{F}}(\mathsf{e}) \to \mathcal{T}_{\mathcal{F}}(\mathsf{e})$ is the identity for each $\mathsf{e} = \langle D, D' \rangle \in \mathsf{Ed}_{\mathsf{A}_{\mathcal{F}}}$ such that $\mathcal{T}_{\mathcal{F}}(\mathsf{e})$ is one-dimensional. The germ at $\{m\} = D \cap D'$ of the foliation $\mathcal{F}^\sharp$ is either linearizable non-resonant or resonant normalizable and non-linearizable. Let us fix a local chart $(z_1, z_2) : \Omega \xrightarrow{\sim} \overline{\mathbb{D}}_r \times \overline{\mathbb{D}}_r$ centered at the point m, satisfying

$$r > 1, \quad z(m) = (0,0), \quad D \cup D' = \{z_1 z_2 = 0\}$$

such that the foliation $\mathcal{F}^\sharp$ is defined on Ω either by the 1-form $\omega = \omega_L$ or by $\omega = \omega_N$, with

- $\omega_L = a z_1 dz_2 + b z_2 dz_1$, with a, $b, \in \mathbb{C}$, $b/a \notin \mathbb{Q}$,
- $\omega_N = b z_1 (1 + \zeta (z_1^a z_2^b)^k) dz_2 + a z_1 (1 + (\zeta - 1)(z_1^a z_2^b)^k) dz_2$, $a, b, k \in \mathbb{N}^*$, $\zeta \in \mathbb{C}$.

According to [9, Lemma 5.4] in both cases, $\omega = \omega_L$ or $\omega = \omega_N$, there exists an explicit holomorphic vector field Z on Ω, that is tangent to $\{z_1 = 1\}$ such that $[Z]$ generates $\mathcal{T}_{\mathcal{F}}(\mathsf{e})$:

$$Z = z_2 \frac{\partial}{\partial z_2} \text{ if } \omega = \omega_L, \qquad Z = \frac{(z_1^a z_2^b)^k}{1 + \zeta (z_1^a z_2^b)^k} z_2 \frac{\partial}{\partial z_2} \text{ if } \omega = \omega_N.$$

Let us fix a point $p \in \Omega$ with coordinates $z_1 = \varepsilon$, $z_2 = 0$, where $\varepsilon \in \mathbb{R}_{>0}$ is sufficiently small so that $\phi^\sharp$ is holomorphic on $\Omega_\varepsilon := \{|z_1|, |z_2| \leq \varepsilon\} \subset \Omega$ and $\phi^\sharp(\Omega_\varepsilon) \subset \Omega$. For $q \in \Omega$ we will denote by Z_q the germ of Z at q. We must prove the equality $[\phi^{\sharp *} Z_m] = [Z_m]$ in $\mathcal{T}_{\mathcal{F}}(\mathsf{e})$, or equivalently that $Z_m - (D\phi^\sharp \cdot Z_m) \circ \phi^{\sharp -1}$ is tangent to $\mathcal{F}^\sharp$. We will use the following fact about the quotient sheaf $\underline{\mathcal{B}}_{\mathcal{F}}/\underline{\mathcal{X}}_{\mathcal{F}}$ of basic and tangent vector fields of $\mathcal{F}^\sharp$, cf. §2.3:

- *Let X be a section of the sheaf $\underline{\mathcal{B}}_{\mathcal{F}}/\underline{\mathcal{X}}_{\mathcal{F}}$ restricted to a connected open subset V of an invariant irreducible component of $\mathcal{E}_{\mathcal{F}}$. If the germ of X at some point p of V is zero, then $X = 0$.*

Indeed if p is a regular point, by local triviality, the section is zero along the whole regular part of V. The vanishing at the remaining singularities follows by analytic continuation. If p is a singular point, then the germ of X at a regular point close to p is zero and we conclude similarly.

Thanks to this property it suffices to show that at the point $\phi^\sharp(p)$ the vector field germ $Z_{\phi^\sharp(p)} - (D\phi^\sharp \cdot Z_p) \circ \phi^{\sharp\,-1}$ is tangent to $\mathcal{F}^\sharp$. Let us choose a simple path

$$\gamma : [0,1] \to \{z_1 \neq 0\,,\, z_2 = 0\}\,, \quad \gamma(0) = p\,, \quad \gamma(1) = \phi^\sharp(p)\,,$$

and a germ at p of holomorphic submersion $I_p : (\Omega, p) \to (\mathbb{C}, 0)$ constant on the leaves of $\mathcal{F}^\sharp$ whose restriction to $\{z_1 = \varepsilon\}$ is equal to z_2. Let us denote by $I_{\phi^\sharp(p)} : (\Omega, \phi^\sharp(p)) \to (\mathbb{C}, 0)$ the analytic extension of I_p along γ, which coincides with its extension as first integral of $\mathcal{F}^\sharp$. The vector field Z being basic, the germ of holomorphic vector field on $(\mathbb{C}, 0)$

$$Z^\flat = z\frac{\partial}{\partial z} \quad \text{if} \quad \omega = \omega_L\,, \qquad \text{or} \qquad Z^\flat = \frac{z^{bk}}{1+\zeta z^{bk}} z\frac{\partial}{\partial z} \quad \text{if} \quad \omega = \omega_N\,,$$

satisfies the relations

$$DI_p \cdot Z_p = Z^\flat \circ I_p \quad \text{and} \quad DI_{\phi^\sharp(p)} \cdot Z_{\phi^\sharp(p)} = Z^\flat \circ I_{\phi^\sharp(p)}\,, \tag{14}$$

the second equality resulting from the first one by analytic extension. On the other hand, the germ of $\phi^\sharp$ at p factorizes through the first integrals, inducing a biholomophism germ $\phi^\flat : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ such that

$$I_{\phi^\sharp(p)} \circ \phi^\sharp = \phi^\flat \circ I_p\,. \tag{15}$$

Using the chain rule we have:

$$\begin{aligned} DI_{\phi^\sharp(p)} \cdot \big((D\phi^\sharp \cdot Z_p) \circ \phi^{\sharp\,-1}\big) &= \big((DI_{\phi^\sharp(p)} \circ \phi^\sharp) \cdot D\phi^\sharp \cdot Z_p\big) \circ \phi^{\sharp\,-1} \\ &= \big(D(I_{\phi^\sharp(p)} \circ \phi^\sharp) \cdot Z_p\big) \circ \phi^{\sharp\,-1} \overset{(15)}{=} \big(D(\phi^\flat \circ I_p) \cdot Z_p\big) \circ \phi^{\sharp\,-1} \\ &= \big((D\phi^\flat \circ I_p) \cdot DI_p \cdot Z_p\big) \circ \phi^{\sharp\,-1} \overset{(14)}{=} \big((D\phi^\flat \circ I_p) \cdot (Z^\flat \circ I_p)\big) \circ \phi^{\sharp\,-1} \\ &= (D\phi^\flat \cdot Z^\flat) \circ I_p \circ \phi^{\sharp\,-1}\,. \end{aligned} \tag{16}$$

Since $\phi^\sharp$ is defined on a neighborhood of the singular point m, $\phi^\flat$ commutes with the biholomorphism of holonomy of $\mathcal{F}^\sharp$ along D and around m. According to [8, Proposition 6.10] (cases (L) and (R)) there is some $t_1 \in \mathbb{C}$ and a linear periodic map $\ell : \mathbb{C} \to \mathbb{C}$ such that:

$$\phi^\flat = \ell \circ \exp(Z^\flat)[t_1]\,, \quad \ell^*(Z^\flat) = Z^\flat\,;$$

therefore $\phi^\flat$ leaves $Z^\flat$ invariant. Hence the equality $D\phi^\flat \cdot Z^\flat = Z^\flat \circ \phi^\flat$ holds and using it we obtain

$$\begin{aligned} DI_{\phi^\sharp(p)} \cdot \big((D\phi^\sharp \cdot Z_p) \circ \phi^{\sharp\,-1}\big) &\overset{(16)}{=} (D\phi^\flat \cdot Z^\flat) \circ I_p \circ \phi^{\sharp\,-1} = Z^\flat \circ \phi^\flat \circ I_p \circ \phi^{\sharp\,-1} \\ &\overset{(15)}{=} Z^\flat \circ I_{\phi^\sharp(p)} \circ \phi^\sharp \circ \phi^{\sharp\,-1} \overset{(14)}{=} DI_{\phi^\sharp(p)} \cdot Z_{\phi^\sharp(p)}\,. \end{aligned}$$

We finally have the equality

$$DI_{\phi^\sharp(p)} \cdot \big(Z_{\phi^\sharp(p)} - (D\phi^\sharp \cdot Z_p) \circ \phi^{\sharp\,-1}\big) = 0\,,$$

that shows that the vector field germ $Z_{\phi^\sharp(p)} - (D\phi^\sharp \cdot Z_p) \circ \phi^{\sharp\,-1}$ is tangent to $\mathcal{F}^\sharp$. $\square$

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