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Abstract

We investigate compressible nonisothermal diffuse interface fluid models also termed capillary fluids. Such fluid models involve van der Waals' gradient energy, Korteweg's tensor, Dunn and Serrin's heat flux as well as diffusive fluxes. The density gradient is added as an extra variable and the convective and capillary fluxes of the augmented system are identified by using the Legendre transform of entropy. The augmented system of equations is recast into a normal form with symmetric hyperbolic first order terms, symmetric dissipative second order terms and antisymmetric capillary second order terms. New a priori estimates are obtained for such augmented system of equations in normal form. The time derivatives of the parabolic components are less regular than for standard hyperbolic-parabolic systems and the strongly coupling antisymmetric fluxes yields new majorizing terms. Using the augmented system in normal form and the a priori estimates, local existence of strong solutions is established in an Hilbertian framework.

1 Introduction

Diffuse interface models conceive liquid-gas interfaces as changeover zones where state variables are smoothly varying [62, 59, 1]. Diffuse interface models of second gradient type have successfully been used to describe vaporization fronts, three phase contact lines, surface diffusion as well as complex liquid-gas interfaces with topological changes [59, 1, 47, 14, 30, 2, 58]. These are strong motivations for investigating the mathematical structure and properties of the corresponding systems of partial differential equations as well as the existence of solutions.

The thermodynamics of diffuse interface models has been built by van der Waals [62] using a gradient squared term in the free energy. The associated pressure tensor has been obtained by Korteweg [51] and the heat flux by Dunn and Serrin using rational thermodynamics [23]. These equations have alternatively been derived from the kinetic theory of dense gases by Giovangigli [34, 35]. Bulk phases classical nonideal thermodynamics furthermore involve non monotone pressure laws allowing the simultaneous presence of liquid and gaseous states. The dissipative fluxes are similar to that of Navier-Stokes-Fourier fluids and the capillarity coefficient depends on temperature as deduced from the kinetic theory. The extra higher order derivative terms of capillary origin in thermodynamic functions and governing equations ensure a smooth variation of fluid properties at liquid-vapor interfaces [62, 59, 1, 47, 30, 2, 58].

In the isothermal situation, Hattori and Li [45] have first established the local existence of strong solutions to the Cauchy problem with constant capillarity and transport coefficients. Danchin and Desjardins have further obtained the existence and uniqueness of solutions in critical Besov space [18]. Kotschote has established the local existence of strong solutions in bounded domains with coefficients independent of the solution [26]. Bresch, Desjardins and Lin [8] and Bresch, Gisclon and Lacroix-Violet [9] have investigated the global existence of weak solutions in periodic or strip domains with a monotone pressure and density dependent capillarity coefficients associated with quantum fluids.

In the multi-dimensional non-isothermal situation, Haspot has investigated the well-posedness in critical spaces with a monotone pressure and a density dependent capillarity coefficient [44]. Kotschote has studied the well-posedness of strong solutions in bounded domains with very general coefficients and pressure laws [27]. Bresch et al. [9, 10], Benzoni et al. [3, 4, 5], Donatelli et al. [22] and Tzavaras et al. [7, 32] have further considered Euler-Korteweg models. Hattori and Li [46] and Kotschote [28] have studied the stability of stationary states and Nabet [57], Miranville [56], and Cancès et al. [15] have investigated related Cahn-Hilliard fluid equations.

Symmetrization is a requisite for analyzing the structure of systems of partial differential equations. Symmetrized forms have notably been used for hyperbolic systems of partial differential equations modeling fluids [42, 29, 55, 61, 17, 41, 19, 50, 6, 11] as well as for hyperbolic-parabolic systems [63, 48, 49, 36, 37, 33, 60, 21, 24, 40]. In order to symmetrize the diffuse interface fluid equations, a first step is to consider an augmented system by adding the gradient of density $\boldsymbol{w} = \boldsymbol{\nabla}\rho$ as an extra unknown following Gavrilyuk and Gouin [31], Benzoni et al. [3], Bresch et al. [10], and Kotschote [27]. In the same vein, two velocity hydrodynamics with augmented systems have been considered by Bresch et al. [12, 13]. Gavrilyuk and Gouin [31] have established that the resulting augmented system of equations can be symmetrized by using entropic variables in the special situation of inviscid fluids. The specific entropy has been used as a conserved variable—thanks to the inviscid framework—with energy playing the role of entropy [31]. We first revisit the Gavrilyuk and Gouin symmetrization method, still keeping the density gradient as an extra variable, but using the energy as a conserved unknown and restoring the natural role of entropy as is mandatory for fluids with dissipative effects. We correspondingly use of the Legendre transform of entropy—instead of energy—in order to identify the convective and capillary fluxes of the augmented system. The entropic symmetrized form is then obtained and the corresponding matrices relating the dissipative and capillary fluxes to the gradient of the entropic variable involve symmetric parts arising from dissipative effects and antisymmetric parts arising from capillarity. A major drawback of such entropic symmetrized forms, however, is that the map $\mathsf{u} \mapsto \mathsf{v}$ from the conservative variable u to the entropic variable v is not globally invertible, due to the presence of mechanical thermodynamic instabilities [3]. In addition, the dissipative effects are mixed between the entropic variable components. In order to solve these difficulties, we investigate normal variables w and normal forms for the augmented system of partial differential equations.

Normal forms for the augmented system are obtained by using for convenience the normal variable $\mathsf{w} = (\rho, \boldsymbol{w}, \boldsymbol{v}, T)^t$ where ρ denotes the density, $\boldsymbol{w}$ the density gradient, $\boldsymbol{v}$ the fluid velocity, and T the absolute temperature. The map $\mathsf{u} \mapsto \mathsf{w}$ remains globally invertible even if neither $\mathsf{u} \mapsto \mathsf{v}$ nor $\mathsf{w} \mapsto \mathsf{v}$ are globally invertible. This normal variable may be split between its hyperbolic $\mathsf{w}_{\mathrm{I}} = (\rho, \boldsymbol{w})^t$ and parabolic components $\mathsf{w}_{\mathrm{II}} = (\boldsymbol{v}, T)^t$, and the hyperbolic component w_{I} may further be split as $\mathsf{w}_{\mathrm{I}} = (\mathsf{w}_{\mathrm{I}'}, \mathsf{w}_{\mathrm{I}''})^t$ where $\mathsf{w}_{\mathrm{I}'} = \rho$ and $\mathsf{w}_{\mathrm{I}''} = \boldsymbol{w}$. The unknown w thus has $\mathsf{n} = 2d_{\mathrm{s}} + 2$ component with d_{s} denoting the space dimension wheras w_{I} and w_{II} both have $\mathsf{n}_{\mathrm{I}} = \mathsf{n}_{\mathrm{II}} = d_{\mathrm{s}} + 1$ components. The corresponding vector and matrix block structure associated with the partitioning of $\mathbb{R}^{\mathsf{n}}$ into $\mathbb{R}^{\mathsf{n}} = \mathbb{R}^{\mathsf{n}_{\mathrm{I}}} \times \mathbb{R}^{\mathsf{n}_{\mathrm{II}}}$ is used in the following. The resulting equations in normal form constitute a quasilinear symmetric-antisymmetric hyperbolic-parabolic composite system in the form

$$\overline{\mathsf{A}}_0(\mathsf{w})\partial_t \mathsf{w} \; + \sum_{i\in\mathcal{D}} \overline{\mathsf{A}}_i(\mathsf{w})\partial_i \mathsf{w} - \sum_{i,j\in\mathcal{D}} \overline{\mathsf{B}}^{\mathrm{d}}_{ij}(\mathsf{w})\partial_i\partial_j \mathsf{w} - \sum_{i,j\in\mathcal{D}} \overline{\mathsf{B}}^{\mathrm{c}}_{ij}(\mathsf{w})\partial_i\partial_j \mathsf{w} = \mathsf{h}(\mathsf{w}, \boldsymbol{\nabla}\mathsf{w}), \tag{1.1}$$

where $\overline{\mathsf{A}}_0$ is symmetric positive definite, block-diagonal $\overline{\mathsf{A}}_0 = \mathrm{diag}(\overline{\mathsf{A}}_0^{\mathrm{I},\mathrm{I}}, \overline{\mathsf{A}}_0^{\mathrm{II},\mathrm{II}})$, with $\overline{\mathsf{A}}_0^{\mathrm{II},\mathrm{II}}$ only depending on $(\mathsf{w}_{\mathrm{I}'}, \mathsf{w}_{\mathrm{II}})$. The convective matrices $\overline{\mathsf{A}}_i$, $i \in \mathcal{D}$, are symmetric and a multiple of the mass conservation equation has been added to the first equation to ensure that $\overline{\mathsf{A}}_0$ remains positive definite. The dissipation matrices satisfy the reciprocity relations $(\overline{\mathsf{B}}^{\mathrm{d}}_{ij})^t = \overline{\mathsf{B}}^{\mathrm{d}}_{ji}$, $i, j \in \mathcal{D}$, have nonzero components only into the lower right $\overline{\mathsf{B}}^{\mathrm{d}\,\mathrm{II},\mathrm{II}}_{ij}$ blocks, and $\overline{\mathsf{B}}^{\mathrm{d}\,\mathrm{II},\mathrm{II}} = \sum_{i,j\in\mathcal{D}} \overline{\mathsf{B}}^{\mathrm{d}\,\mathrm{II},\mathrm{II}}_{ij}(\mathsf{w})\xi_i\xi_j$ is positive definite for $\boldsymbol{\xi} \in \Sigma^{d_{\mathrm{s}}-1}$ where $\Sigma^{d_{\mathrm{s}}-1}$ is the sphere in d_{s} dimension. The matrices $\overline{\mathsf{B}}^{\mathrm{c}}_{ij}$ are such that $(\overline{\mathsf{B}}^{\mathrm{c}}_{ij})^t = -\overline{\mathsf{B}}^{\mathrm{c}}_{ji}$, the blocks $\overline{\mathsf{B}}^{\mathrm{c}\,\mathrm{I},\mathrm{I}}_{ij}$ vanish $\overline{\mathsf{B}}^{\mathrm{c}\,\mathrm{I},\mathrm{I}}_{ij} = 0$, and the strongly coupling blocks $\overline{\mathsf{B}}^{\mathrm{c}\,\mathrm{I},\mathrm{II}}_{ij}$ and $\overline{\mathsf{B}}^{\mathrm{c}\,\mathrm{II},\mathrm{I}}_{ij}$ only depend on $(\mathsf{w}_{\mathrm{I}'}, \mathsf{w}_{\mathrm{II}})$. The right hand side $\mathsf{h}(\mathsf{w}, \boldsymbol{\nabla}\mathsf{w})$ is in the form $\mathsf{h} = (\mathsf{h}_{\mathrm{I}}, \mathsf{h}_{\mathrm{II}})^t$ with $\mathsf{h}_{\mathrm{I}} = \mathsf{h}_{\mathrm{I}}(\mathsf{w}, \boldsymbol{\nabla}\mathsf{w}_{\mathrm{II}})$ and $\mathsf{h}_{\mathrm{II}} = \mathsf{h}_{\mathrm{I}}(\mathsf{w}, \boldsymbol{\nabla}\mathsf{w})$. We next consider a constant equilibrium state $\mathsf{w}^\star = (\mathsf{w}^\star_{\mathrm{I}}, \mathsf{w}^\star_{\mathrm{II}})^t$ with $\mathsf{w}^\star_{\mathrm{I}} = (\rho^\star, 0)^t$ and $\mathsf{w}^\star_{\mathrm{II}} = (\boldsymbol{v}^\star, T^\star)^t$ so that $\boldsymbol{w}^\star = 0$ and investigate the Cauchy problem looking for solutions such that $\mathsf{w} - \mathsf{w}^\star \in H^l(\mathbb{R}^{d_{\mathrm{s}}})$.

The structure of the system of partial differential equations may be analyzed by using its normal form (1.1). The components $\mathsf{w}_{\mathrm{I}} = (\rho, \boldsymbol{w})^t$ first form an hyperbolic variable of dimension $\mathsf{n}_{\mathrm{I}} = 1 + d_{\mathrm{s}}$ and the components $\mathsf{w}_{\mathrm{II}} = (\boldsymbol{v}, T)^t$ form a parabolic variable of dimension $\mathsf{n}_{\mathrm{II}} = d_{\mathrm{s}} + 1$ using the terminology of traditional hyperbolic-parabolic systems although the matrices $\overline{\mathsf{B}}^{\mathrm{c}}_{ij}$ introduce extra coupling. The variable $\mathsf{w}_{\mathrm{I}} = (\rho, \boldsymbol{w})^t = (\rho, \boldsymbol{\nabla}\rho)^t$ being an hyperbolic variable, the density ρ will have more regularity inherited from that of $\boldsymbol{w}$ but ρ *is not a parabolic variable*. A priori estimates given in Theorem 6.2 are obtained in the spaces

$$\mathsf{w}_{\mathrm{I}} - \mathsf{w}^\star_{\mathrm{I}} \in C^0\big([0,\bar\tau], H^l\big) \cap C^1\big([0,\bar\tau], H^{l-2}\big),$$
$$\mathsf{w}_{\mathrm{II}} - \mathsf{w}^\star_{\mathrm{II}} \in C^0\big([0,\bar\tau], H^l\big) \cap C^1\big([0,\bar\tau], H^{l-2}\big) \cap L^2\big((0,\bar\tau), H^{l+1}\big),$$

where $l \geq l_0 + 2$, $l_0 = [d_{\mathrm{s}}/2] + 1$ and $[\,]$ denotes the integer part. In particular, density estimates are in the form $\rho - \rho^\star \in C^0\big([0,\bar\tau], H^{l+1}\big)$ instead of solely $\rho \in L^2\big([0,\bar\tau], H^{l+1}\big)$ as would be the case for a

parabolic variable. More regularity is also required with $l \geq l_0 + 2$ rather than $l \geq l_0 + 1$ as for standard hyperbolic-parabolic systems [48, 33, 60]. The time derivative $\partial_t \mathsf{w}_{\text{II}}$ is indeed only in $C^0\big((0,\bar{\tau}), H^{l-2}\big)$ due to the antisymmetric coupling terms $\sum_{i,j\in\mathcal{D}} \overline{\mathsf{B}}^{\text{c}}_{ij}(\mathsf{w})\partial_i\partial_j\mathsf{w}$ involving second derivatives of hyperbolic variables.

A priori estimates and existence of solutions are established for an abstract augmented system of equations in normal form that will encompass the special situation of diffuse interface fluids. Linearized equations are initially considered and new a priori estimates are obtained for symmetric-antisymmetric linearized systems of hyperbolic-parabolic type. An important property of nonlinear as well as linearized systems is notably to maintain the natural constraint $\mathsf{w}_{\text{I}''} = \boldsymbol{\nabla}\mathsf{w}_{\text{I}'}$. In order to establish the existence of solutions for linearized equations, the higher order capillary terms $\sum_{i,j\in\mathcal{D}} \overline{\mathsf{B}}^{\text{c}}_{ij}(\mathsf{w})\partial_i\partial_j\mathsf{w}$ are first regularized using mollifiers and the hyperbolic and parabolic components are uncoupled. The limit with respect to the regularizing parameter is next performed and yields existence results for the linearized equations. Only the symmetry of the hyperbolic blocks $\overline{\mathsf{A}}^{\text{I},\text{I}}_i$, $i \in \mathcal{D}$, is required for such local existence theorems and key points are the anti-symmetry relations $(\overline{\mathsf{B}}^{\text{c}}_{ij})^t = -\overline{\mathsf{B}}^{\text{c}}_{ji}$ ensuring elimination of the capillary terms of entropy production, the vanishing of the blocks $\overline{\mathsf{B}}^{\text{c I},\text{I}}_{ij} = 0$, and the extra regularity of the coefficients $\overline{\mathsf{B}}^{\text{c I},\text{II}}_{ij}$ and $\overline{\mathsf{B}}^{\text{c II},\text{I}}_{ij}$ that only depend on more regular components of w denoted by w_{r}. The nonlinear equations are next considered and a local existence theorem of strong solutions is obtained using the symmetrized normal form as well as linearized iterates with $\mathsf{w}_{\text{r}} = (\mathsf{w}_{\text{I}'}, \mathsf{w}_{\text{II}})^t$. The functional setting is a classical Hilbertian framework that differ from previous existence theorems [44, 8, 27]. To the best of the authors' knowledge, it is the first time that symmetrized normal form is introduced for augmented system and used to establish local existence theorem of strong solutions for diffuse interface fluid models with natural general assumptions on the system coefficients.

The equations governing capillary fluids are presented in Section 2 and the augmented system is investigated in Section 3. The symmetrization of diffuse interface fluid models is addressed in Section 4. Linearized estimated and existence of solutions to the linearized equations are then studied in Section 5. Local existence of solutions to the nonlinear equations is finally obtained in Section 6.

2 Diffuse interface fluids

We present in this section the governing equations of diffuse interfaces fluids also termed Korteweg, capillary, or cohesive fluids. These equations involve van der Waals' gradient energy [62], Korteweg's tensor [51] and Dunn and Serrin's heat flux [23]. Such capillary fluids models have successfully been used for the study of phase change problems including complex liquid-gas interfaces with topological changes [1, 47, 2, 30, 58].

2.1 Van der Waals free energy

The free energy per unit volume $\mathcal{A}$ in a second gradient theory is in the form

$$\mathcal{A} = \mathcal{A}^{\text{cl}} + \tfrac{1}{2}\varkappa|\boldsymbol{\nabla}\rho|^2, \tag{2.1}$$

where $\mathcal{A}^{\text{cl}}$ denotes the classical free energy per unit volume, ρ the mass density, $\boldsymbol{\nabla} = (\partial_1, \dots, \partial_{d_{\text{s}}})^t$ the differential operator in the physical space $\mathbb{R}^{d_{\text{s}}}$, d_{s} the space dimension, and $\varkappa$ the diffuse interface or capillarity coefficient. The superscript $^{\text{cl}}$ is used to denote classic or bulk thermodynamic properties that do not involve gradients. The classic free energy $\mathcal{A}^{\text{cl}}$ only depends on the densities ρ and the absolute temperature T whereas the gradient squared term $\frac{1}{2}\varkappa|\boldsymbol{\nabla}\rho|^2$ in $\mathcal{A}$ represents an excess free energy of the interfacial region [62, 51, 59, 1]. As deduced from the kinetic theory of dense gases [34, 35] and from experimental measurements [53], the diffuse interface coefficient $\varkappa$ is assumed to only depend on temperature $\varkappa = \varkappa(T)$, at variance with the case of quantum fluids.

Using the thermodynamic relations $d\mathcal{A}^{\text{cl}} = -\mathcal{S}^{\text{cl}}dT + g^{\text{cl}}d\rho$, $\partial_T\mathcal{A} = -\mathcal{S}$, and $\partial_\rho\mathcal{A} = g$, the entropy per unit volume $\mathcal{S}$ and the Gibbs function per unit mass g are found in the form

$$\mathcal{S} = \mathcal{S}^{\text{cl}} - \tfrac{1}{2}\partial_T\varkappa|\boldsymbol{\nabla}\rho|^2, \qquad g = g^{\text{cl}}. \tag{2.2}$$

The equality of the entropy $\mathcal{S}$ with the classic entropy $\mathcal{S}^{\text{cl}}$ when $\varkappa$ is independent of temperature is in agreement with van der Waals theory [62, 51, 59] and the Gibbs function per unit volume $\mathcal{G}$ is $\mathcal{G} = \rho g = \mathcal{G}^{\text{cl}}$. The energy per unit volume $\mathcal{E} = \mathcal{A} + T\mathcal{S}$ and pressure $p = \mathcal{G} - \mathcal{A}$ are further obtained as

$$\mathcal{E} = \mathcal{E}^{\text{cl}} + \tfrac{1}{2}(\varkappa - T\partial_T\varkappa)|\boldsymbol{\nabla}\rho|^2, \qquad p = p^{\text{cl}} - \tfrac{1}{2}\varkappa|\boldsymbol{\nabla}\rho|^2, \tag{2.3}$$

and the fluid enthalpy per unit volume $\mathcal{H} = \mathcal{E} + p$ reads $\mathcal{H} = \mathcal{H}^{\rm cl} - \frac{1}{2}T\partial_T \varkappa |\boldsymbol{\nabla}\rho|^2$. The generalized volumetric Gibbs relation is finally in the form

$$T d\mathcal{S} = d\mathcal{E} - g d\rho - \varkappa \boldsymbol{\nabla}\rho \cdot d\boldsymbol{\nabla}\rho, \tag{2.4}$$

where d denotes the differentiation operator. We also introduce for later use the thermodynamic functions per unit mass $s = \mathcal{S}/\rho$, $e = \mathcal{E}/\rho$, and $h = \mathcal{H}/\rho$.

2.2 Equations for capillary fluids

The equations governing capillary fluids may be written in the form [1, 47, 34, 35]

$$\partial_t \rho + \boldsymbol{\nabla}\cdot(\rho \boldsymbol{v}) = 0, \tag{2.5}$$

$$\partial_t(\rho \boldsymbol{v}) + \boldsymbol{\nabla}\cdot(\rho \boldsymbol{v}\otimes\boldsymbol{v}) + \boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{P}} = 0, \tag{2.6}$$

$$\partial_t\big(\mathcal{E} + \tfrac{1}{2}\rho|\boldsymbol{v}|^2\big) + \boldsymbol{\nabla}\cdot\big(\boldsymbol{v}(\mathcal{E} + \tfrac{1}{2}\rho|\boldsymbol{v}|^2)\big) + \boldsymbol{\nabla}\cdot(\boldsymbol{\mathcal{Q}} + \boldsymbol{\mathcal{P}}\cdot\boldsymbol{v}) = 0, \tag{2.7}$$

where ∂_t denotes the time derivative operator, $\boldsymbol{\nabla}$ the spatial differential operator in physical space $\mathbb{R}^{d_{\rm s}}$, $\boldsymbol{v}$ the fluid velocity, $\boldsymbol{\mathcal{P}}$ the total pressure tensor, $\boldsymbol{\mathcal{Q}}$ the total heat flux, and where the pressure p and the energy $\mathcal{E}$ are given by (2.3). Vectors and tensors of physical origin in $\mathbb{R}^{d_{\rm s}}$ or $\mathbb{R}^{d_{\rm s},d_{\rm s}}$ are denoted by using bold symbols. The transport fluxes $\boldsymbol{\mathcal{P}}$ and $\boldsymbol{\mathcal{Q}}$ contain capillary as well as dissipative contributions

$$\boldsymbol{\mathcal{P}} = p\boldsymbol{I} + \varkappa\boldsymbol{\nabla}\rho\otimes\boldsymbol{\nabla}\rho - \rho\boldsymbol{\nabla}\cdot(\varkappa\boldsymbol{\nabla}\rho)\boldsymbol{I} + \boldsymbol{\mathcal{P}}^{\rm d}, \tag{2.8}$$

$$\boldsymbol{\mathcal{Q}} = \varkappa\rho\boldsymbol{\nabla}\cdot\boldsymbol{v}\,\boldsymbol{\nabla}\rho + \boldsymbol{\mathcal{Q}}^{\rm d}, \tag{2.9}$$

with $\boldsymbol{I}$ denoting the $d_{\rm s}$-dimensional unit tensor, $\boldsymbol{\mathcal{P}}^{\rm d}$ the viscous pressure tensor, and $\boldsymbol{\mathcal{Q}}^{\rm d}$ the Fourier heat flux [1, 47]. The dissipative fluxes $\boldsymbol{\mathcal{P}}^{\rm d}$ and $\boldsymbol{\mathcal{Q}}^{\rm d}$ are classically of Navier-Stokes-Fourier type [16, 25, 33]

$$\boldsymbol{\mathcal{P}}^{\rm d} = -\mathfrak{v}\boldsymbol{\nabla}\cdot\boldsymbol{v}\,\boldsymbol{I} - \eta\big(\boldsymbol{\nabla}\boldsymbol{v} + \boldsymbol{\nabla}\boldsymbol{v}^t - \tfrac{2}{d_{\rm s}}\boldsymbol{\nabla}\cdot\boldsymbol{v}\,\boldsymbol{I}\big), \tag{2.10}$$

$$\boldsymbol{\mathcal{Q}}^{\rm d} = -\lambda\boldsymbol{\nabla}T, \tag{2.11}$$

where $\mathfrak{v}$ denotes the volume viscosity, η the shear viscosity and λ the thermal conductivity. The transport coefficients $\mathfrak{v}$, η, and λ may be obtained from the kinetic theories of dense or dilute gases and only depend on ρ and T.

Remark 2.1. *The dissipative pressure tensor $\boldsymbol{\mathcal{P}}^{\rm d}$ is normally obtained in the form*

$$\boldsymbol{\mathcal{P}}^{\rm d} = -\mathfrak{v}'\boldsymbol{\nabla}\cdot\boldsymbol{v}\,\boldsymbol{I} - \eta\big(\boldsymbol{\nabla}\boldsymbol{v} + \boldsymbol{\nabla}\boldsymbol{v}^t - \tfrac{2}{3}\boldsymbol{\nabla}\cdot\boldsymbol{v}\,\boldsymbol{I}\big), \tag{2.12}$$

where $\mathfrak{v}'$ denotes the physical volume viscosity. The original pressure tensor is indeed a matrix of dimension three with a coefficient $2/3$ instead of the coefficient $2/d_{\rm s}$. However, the spatial dimension of the model $d_{\rm s}$ may possibly be reduced and the equations considered in $\mathbb{R}^{d_{\rm s}}$ with $1 \le d_{\rm s} \le 3$. Using then for convenience a coefficient $2/d_{\rm s}$ instead of $2/3$ in (2.12) is equivalent to increasing the physical volume viscosity $\mathfrak{v}'$ by the amount $\eta\big(\frac{2}{d_{\rm s}} - \frac{2}{3}\big)$. The effective volume viscosity with $d_{\rm s}$ spatial dimensions is then $\mathfrak{v} = \mathfrak{v}' + \eta\big(\frac{2}{d_{\rm s}} - \frac{2}{3}\big)$ and we note that $\mathfrak{v} + \eta(1 - \frac{2}{d_{\rm s}}) = \mathfrak{v}' + \frac{\eta}{3}$ remains positive.

Using Gibbs' relation (2.4), the governing equations (2.5)–(2.7), and the expression of transport fluxes, the entropy balance equation may also be written after some algebra as

$$\partial_t\mathcal{S} + \boldsymbol{\nabla}\cdot(\boldsymbol{v}\mathcal{S}) + \boldsymbol{\nabla}\cdot\Big(\frac{\boldsymbol{\mathcal{Q}}}{T} - \frac{\varkappa\rho\boldsymbol{\nabla}\cdot\boldsymbol{v}\boldsymbol{\nabla}\rho}{T}\Big) = \frac{\lambda}{T^2}|\boldsymbol{\nabla}T|^2 + \frac{\mathfrak{v}}{T}(\boldsymbol{\nabla}\cdot\boldsymbol{v})^2 + \frac{\eta}{2T}\big|\boldsymbol{\nabla}\boldsymbol{v} + \boldsymbol{\nabla}\boldsymbol{v}^t - \tfrac{2}{d_{\rm s}}\boldsymbol{\nabla}\cdot\boldsymbol{v}\,\boldsymbol{I}\big|^2, \tag{2.13}$$

and there is no entropy production associated with capillary phenomena. The word *capillary* is traditionally used to denote the extra gradient terms in the energy, the pressure tensor, and the heat flux. A better terminology, however, is that of *diffuse interface* since the higher order derivative terms lead to thin transition zones between phases with smooth variation of fluid properties. Another very good terminology is that of *cohesive* fluids since the extra gradient terms are due to cohesive forces.

2.3 Classical thermodynamics

The mathematical properties of the classical thermodynamic functions $\mathcal{E}^{\rm cl}$, $p^{\rm cl}$, and $\mathcal{S}^{\rm cl}$ are presented in this section [43, 39, 40]. The properties of the extended thermodynamic functions $\mathcal{E}$, p, $\mathcal{S}$, the capillarity coefficient $\varkappa$, and the transport coefficients $\mathfrak{v}$, η, λ will be addressed in the fluid section using augmented variables. We denote for convenience by $\mathsf{z}^{\rm cl}$ the variable $\mathsf{z}^{\rm cl} = (\rho, T)^t$ and by $\mathsf{u}^{\rm cl}$ the variable $\mathsf{u}^{\rm cl} = (\rho, \mathcal{E}^{\rm cl})^t$. The superscript $^{\rm cl}$ is generally used to denote quantities associated with classical thermodynamics. We may commit the traditional abuse of notation of using the same symbol for a given quantity as function of different state variables. We denote in the following by ∂ the derivation operator with respect to the variable $\mathsf{z}^{\rm cl}$. The first property $(\mathsf{H}_1^{\rm cl})$ is associated with the smoothness of thermodynamic functions and the regularity class γ denotes an integer such that $\gamma \geq 3$.

$(\mathsf{H}_1^{\rm cl})$ *The thermodynamic functions $\mathcal{E}^{\rm cl}$, $p^{\rm cl}$, and $\mathcal{S}^{\rm cl}$ are C^γ functions of the variable $\mathsf{z}^{\rm cl} = (\rho, T)^t$ defined over a simply connected nonempty open set $\mathcal{O}_{\mathsf{z}^{\rm cl}} \subset (0, \infty)^2$.*

The second property $(\mathsf{H}_2^{\rm cl})$ concerns Gibbs relation between the differentials of $\mathcal{E}^{\rm cl}$ and $\mathcal{S}^{\rm cl}$. There is also a natural constraint associated with $\mathcal{G}^{\rm cl}$ since we only consider a volumetric Gibbs differential relation [39].

$(\mathsf{H}_2^{\rm cl})$ *Assuming that $(\mathsf{H}_1^{\rm cl})$ holds and defining $\mathcal{G}^{\rm cl} = \mathcal{E}^{\rm cl} + p^{\rm cl} - T\mathcal{S}^{\rm cl}$ and $g^{\rm cl} = \mathcal{G}^{\rm cl}/\rho$, we have the volumetric Gibbs' relation between the differentials $d\mathcal{S}^{\rm cl}$ and $d\mathcal{E}^{\rm cl}$*

$$T d\mathcal{S}^{\rm cl} = d\mathcal{E}^{\rm cl} - g^{\rm cl} d\rho. \tag{2.14}$$

Defining $s^{\rm cl} = \mathcal{S}^{\rm cl}/\rho$ and $e^{\rm cl} = \mathcal{E}^{\rm cl}/\rho$ we have $g^{\rm cl} = e^{\rm cl} + p^{\rm cl}/\rho - Ts^{\rm cl}$ and it is obtained from (2.14) that $Tds^{\rm cl} = de^{\rm cl} - (p^{\rm cl}/\rho^2)d\rho$. A fundamental question associated with nonideal thermodynamics is that of thermodynamic stability. According to the second principle of thermodynamics, the evolution of an isolated system tends to maximize its entropy. The entropy of a stable isolated system should thus be a concave function of its volume and internal energy [43]. Whenever it is not the case, the system may loose its homogeneity and split between several phases in order to reach equilibrium. This notably arises with mechanically unstable fluids that split between vapor and liquid phases [43]. From a mathematical point of view, these unstable points are associated with a loss of definiteness of entropy Hessian matrices [43, 38, 39].

Proposition 2.2. *Assume that $(\mathsf{H}_1^{\rm cl})$ and $(\mathsf{H}_2^{\rm cl})$ hold and that $\mathsf{z}^{\rm cl} \mapsto \mathsf{u}^{\rm cl}$ is locally invertible. Denoting by $\widetilde{\partial}$ the derivation operator with respect to $\mathsf{u}^{\rm cl}$, the following statements are equivalent :*

(i) $\widetilde{\partial}^2_{\mathsf{u}^{\rm cl}\mathsf{u}^{\rm cl}} \mathcal{S}^{\rm cl}$ *is negative definite.*

(ii) $\partial_T \mathcal{E}^{\rm cl} > 0$ *and* $\partial_\rho p^{\rm cl} > 0$.

Proof. From Gibbs' relation (2.14) it is obtained that $\widetilde{\partial}_{\mathcal{E}^{\rm cl}} \mathcal{S}^{\rm cl} = \frac{1}{T}$ and $\widetilde{\partial}_\rho \mathcal{S}^{\rm cl} = -\frac{g^{\rm cl}}{T}$ and this implies the compatibility relation $\widetilde{\partial}_{\mathcal{E}^{\rm cl}}\big(\frac{-g^{\rm cl}}{T}\big) = \widetilde{\partial}_\rho\big(\frac{1}{T}\big)$. Moreover, for any function ϕ we have the differential relations

$$\widetilde{\partial}_{\mathcal{E}^{\rm cl}} \phi = \partial_T \phi \, \widetilde{\partial}_{\mathcal{E}^{\rm cl}} T, \qquad \partial_T \phi = \widetilde{\partial}_{\mathcal{E}^{\rm cl}} \phi \, \partial_T \mathcal{E}^{\rm cl}, \tag{2.15}$$

$$\widetilde{\partial}_\rho \phi = \partial_\rho \phi + \partial_T \phi \, \widetilde{\partial}_\rho T, \qquad \partial_\rho \phi = \widetilde{\partial}_\rho \phi + \widetilde{\partial}_{\mathcal{E}^{\rm cl}} \phi \, \partial_\rho \mathcal{E}^{\rm cl}. \tag{2.16}$$

We then note that $\widetilde{\partial}^2_{\mathcal{E}^{\rm cl}\mathcal{E}^{\rm cl}} \mathcal{S}^{\rm cl} = \widetilde{\partial}_{\mathcal{E}^{\rm cl}}(\frac{1}{T})$ so that $\widetilde{\partial}^2_{\mathcal{E}^{\rm cl}\mathcal{E}^{\rm cl}} \mathcal{S}^{\rm cl} = -\widetilde{\partial}_{\mathcal{E}^{\rm cl}} T/T^2$. Similarly, we have $\widetilde{\partial}^2_{\mathcal{E}^{\rm cl}\rho} \mathcal{S}^{\rm cl} = \widetilde{\partial}_\rho(\frac{1}{T})$ so that $\widetilde{\partial}^2_{\mathcal{E}^{\rm cl}\rho} \mathcal{S}^{\rm cl} = -\widetilde{\partial}_\rho T/T^2$. Upon letting $\phi = T$ in (2.16) we obtain that $\widetilde{\partial}_\rho T = -\widetilde{\partial}_{\mathcal{E}^{\rm cl}} T \, \partial_\rho \mathcal{E}^{\rm cl}$ and we have established that $\widetilde{\partial}^2_{\mathcal{E}^{\rm cl}\rho^{\rm cl}} \mathcal{S}^{\rm cl} = \widetilde{\partial}_{\mathcal{E}^{\rm cl}} T \, \partial_\rho \mathcal{E}^{\rm cl}/T^2$. Combining $\widetilde{\partial}_\rho T = -\widetilde{\partial}_{\mathcal{E}^{\rm cl}} T \, \partial_\rho \mathcal{E}^{\rm cl}$ with the compatibility relation $\widetilde{\partial}_{\mathcal{E}^{\rm cl}}\big(\frac{-g^{\rm cl}}{T}\big) = \widetilde{\partial}_\rho\big(\frac{1}{T}\big)$ and (2.15), we also deduce that $\partial_T(\frac{g^{\rm cl}}{T}) = -\frac{\partial_\rho \mathcal{E}^{\rm cl}}{T^2}$. In addition, from $\widetilde{\partial}^2_{\rho\rho} \mathcal{S}^{\rm cl} = -\widetilde{\partial}_\rho(\frac{g^{\rm cl}}{T})$ and from (2.16) we deduce that $\widetilde{\partial}^2_{\rho\rho} \mathcal{S}^{\rm cl} = -\partial_\rho(\frac{g^{\rm cl}}{T}) - \partial_T(\frac{g^{\rm cl}}{T}) \, \widetilde{\partial}_\rho T$ so that $\widetilde{\partial}^2_{\rho\rho} \mathcal{S}^{\rm cl} = -\frac{\partial_\rho g^{\rm cl}}{T} - \frac{\widetilde{\partial}_{\mathcal{E}^{\rm cl}} T \, (\partial_\rho \mathcal{E}^{\rm cl})^2}{T^2}$. We have thus established that $T^2 \widetilde{\partial}^2_{\mathcal{E}^{\rm cl}\mathcal{E}^{\rm cl}} \mathcal{S}^{\rm cl} = -\widetilde{\partial}_{\mathcal{E}^{\rm cl}} T$, $T^2 \widetilde{\partial}^2_{\mathcal{E}^{\rm cl}\rho} \mathcal{S}^{\rm cl} = \widetilde{\partial}_{\mathcal{E}^{\rm cl}} T \, \partial_\rho \mathcal{E}^{\rm cl}$ and $T^2 \widetilde{\partial}^2_{\rho\rho} \mathcal{S}^{\rm cl} = -T\partial_\rho g^{\rm cl} - \widetilde{\partial}_{\mathcal{E}^{\rm cl}} T \, (\partial_\rho \mathcal{E}^{\rm cl})^2$. We have already derived that $Tds^{\rm cl} = de^{\rm cl} - (p^{\rm cl}/\rho^2)d\rho$ so that $dg^{\rm cl} = -s^{\rm cl} dT + (1/\rho) dp^{\rm cl}$ and $\partial_\rho g^{\rm cl} = \partial_\rho p^{\rm cl}/\rho$, and letting $\phi = T$ in (2.15) yields $\partial_T \mathcal{E}^{\rm cl} \, \widetilde{\partial}_{\mathcal{E}^{\rm cl}} T = 1$. Combining these relations, we have established that for any test vector $\mathsf{x}^{\rm cl} = (\mathsf{x}_\rho, \mathsf{x}_\mathcal{E})^t \in \mathbb{R}^2$

$$\langle \widetilde{\partial}^2_{\mathsf{u}^{\rm cl}\mathsf{u}^{\rm cl}} \mathcal{S}^{\rm cl} \mathsf{x}^{\rm cl}, \mathsf{x}^{\rm cl} \rangle = -\frac{1}{T^2 \partial_T \mathcal{E}^{\rm cl}} (\mathsf{x}_\mathcal{E} - \partial_\rho \mathcal{E}^{\rm cl} \mathsf{x}_\rho)^2 - \frac{1}{\rho T} \partial_\rho p^{\rm cl} \, \mathsf{x}_\rho^2, \tag{2.17}$$

in such a way that thermodynamic stability is equivalent to $\partial_T \mathcal{E}^{\rm cl} > 0$ and $\partial_\rho p^{\rm cl} > 0$. □

The inequality $\partial_T \mathcal{E}^{\rm cl} > 0$ is the thermal stability condition and the inequality $\partial_\rho p^{\rm cl} > 0$ the mechanical stability condition [43, 39]. In order to integrate with respect to the variable T in the open set $\mathcal{O}_{\mathsf{z}^{\rm cl}}$ it is further required that the open set $\mathcal{O}_{\mathsf{z}^{\rm cl}}$ is increasing with respect to temperature.

Definition 2.3. *A nonempty open set $\mathcal{O}_{\mathsf{z}^{\rm cl}} \subset (0,\infty)^2$ is said to be increasing with respect to temperature if for any $\mathsf{z}^{\rm cl} = (\rho, T)^t \in \mathcal{O}_{\mathsf{z}^{\rm cl}}$ we have $\{\rho\}\times[T,\infty) \subset \mathcal{O}_{\mathsf{z}^{\rm cl}}$.*

The construction of thermodynamics from state laws also requires the open set $\mathcal{O}_{\mathsf{z}^{\rm cl}}$ to be decreasing with respect to density [39] but such a property is not needed for an existence theorem [40]. We now assume that the fluid is thermally stable and this will allow the use of the $\mathsf{u}^{\rm cl}$ variable.

$(\mathsf{H}_3^{\rm cl})$ *Assuming that $(\mathsf{H}_1^{\rm cl})$ holds, the open set $\mathcal{O}_{\mathsf{z}^{\rm cl}}$ is increasing with respect to temperature and the volumetric specific heat is positive $\partial_T \mathcal{E}^{\rm cl} > 0$ over $\mathcal{O}_{\mathsf{z}^{\rm cl}}$.*

Under such a property $(\mathsf{H}_3^{\rm cl})$ we may change of variable from $\mathsf{z}^{\rm cl} = (\rho, T)^t$ to $\mathsf{u}^{\rm cl} = (\rho, \mathcal{E}^{\rm cl})^t$ as investigated in the next lemma.

Lemma 2.4. *Assuming that $(\mathsf{H}_1^{\rm cl})$ and $(\mathsf{H}_3^{\rm cl})$ hold, the map $\mathsf{z}^{\rm cl} \mapsto \mathsf{u}^{\rm cl}$ is a C^γ diffeomorphism from $\mathcal{O}_{\mathsf{z}^{\rm cl}}$ onto an open set $\mathcal{O}_{\mathsf{u}^{\rm cl}}$.*

Proof. The map $\mathsf{z}^{\rm cl} \mapsto \mathsf{u}^{\rm cl}$ is C^γ from $(\mathsf{H}_1^{\rm cl})$ and the jacobian matrix $\partial_{\mathsf{z}^{\rm cl}} \mathsf{u}^{\rm cl}$ has determinant $\partial_T \mathcal{E}^{\rm cl}$. This jacobian $\partial_{\mathsf{z}^{\rm cl}} \mathsf{u}^{\rm cl}$ is thus nonsingular from $(\mathsf{H}_3^{\rm cl})$ so that $\mathsf{z}^{\rm cl} \mapsto \mathsf{u}^{\rm cl}$ is locally invertible and the image $\mathcal{O}_{\mathsf{u}^{\rm cl}}$ of $\mathcal{O}_{\mathsf{z}^{\rm cl}}$ is an open set. Assuming next that $\big(\rho^\flat, \mathcal{E}^{\rm cl}(\rho^\flat, T^\flat)\big) = \big(\rho^\sharp, \mathcal{E}^{\rm cl}(\rho^\sharp, T^\sharp)\big)$ and letting $\rho = \rho^\flat = \rho^\sharp$, we deduce from $\mathcal{E}^{\rm cl}(\rho, T^\flat) = \mathcal{E}^{\rm cl}(\rho, T^\sharp)$ that $\int_{T^\flat}^{T^\sharp} \partial_T \mathcal{E}^{\rm cl}(\rho, s)\, ds = 0$ keeping in mind that $\mathcal{O}_{\mathsf{z}^{\rm cl}}$ is increasing with respect to temperature. Since the specific heat $\partial_T \mathcal{E}^{\rm cl}$ remains positive over the integration segment $\{\rho\}\times[T^\flat, T^\sharp]$ we obtain that $T^\flat = T^\sharp$. The map $\mathsf{z}^{\rm cl} \mapsto \mathsf{u}^{\rm cl}$ is thus one-to-one and is therefore a global C^γ diffeomorphism from $\mathcal{O}_{\mathsf{z}^{\rm cl}}$ onto $\mathcal{O}_{\mathsf{u}^{\rm cl}}$. □

The fluid thermodynamic functions are further compatible with that of perfect gases when the density goes to zero $\rho \to 0$ [39]. This yields boundary conditions at zero densities for the construction of nonideal fluid thermodynamics from equations of state [39]. However, such a property is not required in order to investigate well posedness of diffuse interface fluid models. We neither assume that the Hessian matrix $\widetilde{\partial}^2_{\mathsf{u}^{\rm cl}\mathsf{u}^{\rm cl}} \mathcal{S}^{\rm cl}$ is negative definite since we anticipate the presence of mechanical instabilities where $\partial_\rho p^{\rm cl} < 0$ and only thermal stability is assumed with $(\mathsf{H}_3^{\rm cl})$. We will still need an instability indicator $\mathsf{m}^{\rm cl}$ in order to control negative values of the derivative $\partial_\rho p^{\rm cl}$ as investigated in the next lemma.

Lemma 2.5. *Assuming that $(\mathsf{H}_1^{\rm cl})$ holds and that $\delta > 0$ is given there exists a $C^{\gamma-1}(\mathcal{O}_{\mathsf{z}^{\rm cl}})$ function $\mathsf{m}^{\rm cl}$ such that $\mathsf{m}^{\rm cl} \geq 0$, $\mathsf{m}^{\rm cl} + \partial_\rho p^{\rm cl}/\rho T > 0$ and $\mathsf{m}^{\rm cl} = 0$ if $\partial_\rho p^{\rm cl}/\rho T \geq \delta$.*

Proof. Let $\phi \in C^\infty(\mathbb{R})$ be such that $\phi(x) = 1$ if $x \leq 1/2$, ϕ is decreasing, $\phi(x) = 0$ if $x \geq 1$, and define

$$\mathsf{m}^{\rm cl}(\rho, T) = \Big(\delta - \frac{\partial_\rho p^{\rm cl}}{\rho T}\Big)\phi\Big(\frac{\partial_\rho p^{\rm cl}}{\delta \rho T}\Big).$$

It is then easily checked that $\mathsf{m}^{\rm cl} \geq 0$, that $\mathsf{m}^{\rm cl} + \partial_\rho p^{\rm cl}/\rho T$ remains greater than $\delta/2$ that $\mathsf{m}^{\rm cl}(\rho, T) = 0$ whenever $\partial_\rho p^{\rm cl}/\rho T \geq \delta$ and m is $C^{\gamma-1}(\mathcal{O}_{\mathsf{z}^{\rm cl}})$ from $(\mathsf{H}_1^{\rm cl})$. □

Whenever stability is lost, we will add $\mathsf{m}^{\rm cl}$ to $\partial_\rho p^{\rm cl}/\rho T$ in order to obtain a positive quantity. When a thermodynamical state $\mathsf{z}^{\rm cl}$ is stable in the sense that $\partial_\rho p^{\rm cl}/\rho T > \delta$ the instability indicator $\mathsf{m}^{\rm cl}$ also vanishes in the neighborhood of $\mathsf{z}^{\rm cl}$.

Remark 2.6. *The Van der Waals pressure law is in the form [43]*

$$p^{\rm cl} = \frac{\rho r T}{1 - \rho \mathsf{b}_{\rm vw}} - \mathsf{a}_{\rm vw}(T)\rho^2,$$

where r is the gas constant per unit mass, $\mathsf{b}_{\rm vw}$ a positive constant—the covolume—and $\mathsf{a}_{\rm vw}(T)$ a non-negative C^γ function of temperature arising from attractive potential forces [43, 34]. The mass density ρ is such that $1 - \rho \mathsf{b}_{\rm vw} >$ and the open set $\mathcal{O}_{\mathsf{z}^{\rm cl}}$ is then given by $\mathcal{O}_{\mathsf{z}^{\rm cl}} = (0, 1/\mathsf{b}_{\rm vw}) \times (T_{\rm min}, \infty)$ where $T_{\rm min}$ is a positive minimum temperature. It may then be established that $e = e^{\rm pg} - (\mathsf{a}_{\rm vw} - T\partial_T \mathsf{a}_{\rm vw})\rho$ and $s = s^{\rm pg} + \partial_T \mathsf{a}_{\rm vw}\rho + r\log(1 - \mathsf{b}_{\rm vw}\rho)$ where $e^{\rm pg}$ and $s^{\rm pg}$ are the energy and entropy per unit mass of a perfect gas, respectively [39], as well as $-2\mathsf{a}_{\rm vw}/T_{\rm min} \leq \partial_\rho p^{\rm cl}/(\rho T)$ so that $\mathsf{m}^{\rm cl}$ is also uniformly bounded for these fluids. Similar properties also hold for modified van der Waals state laws like the SRK equation of state that has been found to be accurate [38, 39].

The mathematical structure of classical thermodynamics and their construction from state laws has been investigated [39]. From a physical point of view, classical thermodynamic functions for fluids may first be obtained by extending the thermostatic framework to fluids using the idea of local states [43]. A more satisfactory derivation is that from the kinetic theory of gases [16, 25, 33] since it is does take into account the presence of *macroscopic gradients*. The idealized vision that there exists a local state is replaced by the more satisfactory argument that the gas distribution function is a Maxwellian at zeroth order of Enskog expansion. Extended thermodynamics further valid in the presence of *microscopic gradients* and steep interfaces, that may also be derived from the kinetic theory [34, 35], are discussed in the next section using augmented variables.

3 The augmented system

We recast in this section the equation governing cohesive fluids as a *quasilinear second order* system of partial differential equations. A first step is to add the gradient of density as an extra unknown [31, 3, 10, 27]. A second step is to identify the proper convective, capillary and dissipative fluxes [31].

3.1 Augmented variables

An important step in order to restructure the system of partial differential equations governing diffuse interface fluids is to introduce the extra unknown vector [31, 3, 10, 27]

$$\boldsymbol{w} = \boldsymbol{\nabla}\rho. \tag{3.1}$$

The vector $\boldsymbol{w}$ is in $\mathbb{R}^{d_s}$, where d_s denotes the space dimension, and the $\boldsymbol{w}$ governing equation is obtained by applying the differential operator $\boldsymbol{\nabla}$ to the mass conservation equation (2.5). The resulting equation is written in the form

$$\partial_t \boldsymbol{w} + \sum_{i\in\mathcal{D}} \partial_i\big(\boldsymbol{w}\, v_i + \rho\boldsymbol{\nabla} v_i\big), \tag{3.2}$$

where we have denoted by $\mathcal{D} = \{1, \dots, d_s\}$ the indexing set of spatial directions, v_i the velocity in the ith spatial direction and ∂_i the derivative in the ith spatial direction so that $\boldsymbol{v} = (v_1, \dots, v_{d_s})^t$ and $\boldsymbol{\nabla} = (\partial_1, \dots, \partial_{d_s})^t$. The *augmented* conservative unknown u is then in the form

$$\mathsf{u} = \big(\rho, \boldsymbol{w}, \rho\boldsymbol{v}, \mathcal{E}^{\rm tot}\big)^t, \tag{3.3}$$

and thus includes $\boldsymbol{w}$ as well as the density ρ, the fluid momentum $\rho\boldsymbol{v}$ and the total energy per unit volume $\mathcal{E}^{\rm tot} = \mathcal{E} + \frac{1}{2}\rho|\boldsymbol{v}|^2$. Note that the transposition operation is made by block for column vectors like u for the sake of notational simplicity. The augmented natural variable $\mathsf{z} \in \mathbb{R}^{\mathsf{n}}$ is correspondingly defined as

$$\mathsf{z} = \big(\rho, \boldsymbol{w}, \boldsymbol{v}, T\big)^t, \tag{3.4}$$

and will be convenient for deriving differential identities. Both u and z are in $\mathbb{R}^{\mathsf{n}}$ where $\mathsf{n} = 2d_s + 2$ is the augmented number of unknowns.

We may then express the thermodynamic functions $\mathcal{E}$, p, and $\mathcal{S}$ in terms of the augmented natural variable z

$$\mathcal{E} = \mathcal{E}^{\rm cl} + \tfrac{1}{2}(\varkappa - T\partial_T\varkappa)|\boldsymbol{w}|^2, \qquad \mathcal{S} = \mathcal{S}^{\rm cl} - \tfrac{1}{2}\partial_T\varkappa|\boldsymbol{w}|^2, \qquad p = p^{\rm cl} - \tfrac{1}{2}\varkappa|\boldsymbol{w}|^2, \tag{3.5}$$

as well as $\mathcal{H} = \mathcal{H}^{\rm cl} - \frac{1}{2}T\partial_T\varkappa|\boldsymbol{w}|^2$ and $g = g^{\rm cl}$. We may also introduce fluid thermodynamic quantities taking into account the kinetic energy as functions of z

$$\mathcal{E}^{\rm tot} = \mathcal{E} + \tfrac{1}{2}\rho|\boldsymbol{v}|^2, \qquad \mathcal{H}^{\rm tot} = \mathcal{H} + \tfrac{1}{2}\rho|\boldsymbol{v}|^2, \qquad g^{\rm tot} = g - \tfrac{1}{2}|\boldsymbol{v}|^2, \tag{3.6}$$

as well as the densities per unit mass $s = \mathcal{S}/\rho$, $e = \mathcal{E}/\rho$, $h = \mathcal{H}/\rho$, $e^{\rm tot} = \mathcal{E}^{\rm tot}/\rho$, and $h^{\rm tot} = \mathcal{H}^{\rm tot}/\rho$. The mathematical properties of the thermodynamic functions as functions of z or u are investigated in the next section.

3.2 Fluid extended thermodynamics

The investigate in this section the mathematical properties of the thermodynamic functions $\mathcal{E}$, p, and $\mathcal{S}$, given by (3.5), as functions of z or u and this next yields the properties of $\mathcal{E}^{\rm tot}$ and $\mathcal{H}^{\rm tot}$ given by (3.6). We simply denote by ∂ the derivation operator with respect to the variable z. The first assumption $(\mathsf{H_1})$ is a natural extension of $(\mathsf{H}_1^{\rm cl})$ and the regularity class γ denotes an integer with $\gamma \geq 3$.

$(\mathsf{H_1})$ *The thermodynamic functions $\mathcal{E}$, p, and $\mathcal{S}$ are C^γ functions of the variable z defined over a simply connected nonempty open set $\mathcal{O}_\mathsf{z} \subset (0,\infty) \times \mathbb{R}^{d_{\rm s}} \times \mathbb{R}^{d_{\rm s}} \times (0,\infty)$ and the capillarity coefficient $\varkappa$ is a $C^{\gamma+1}$ function of temperature over $\mathcal{O}_\mathsf{z}$. For any $(\rho,T)^t \in \mathcal{O}_{\mathsf{z}^{\rm cl}}$ we have $(\rho,0,0,T)^t \in \mathcal{O}_\mathsf{z}$ and for any $(\rho,\boldsymbol{w},\boldsymbol{v},T)^t \in \mathcal{O}_\mathsf{z}$ we have $(\rho,T)^t \in \mathcal{O}_{\mathsf{z}^{\rm cl}}$.*

The capillarity coefficient $\varkappa$ needs to be $C^{\gamma+1}$ in order to obtain C^γ extended thermodynamic functions $\mathcal{E}$, p and $\mathcal{S}$ from (3.5) and the classical thermodynamic functions $\mathcal{E}^{\rm cl}$, $p^{\rm cl}$, and $\mathcal{S}^{\rm cl}$ are naturally defined over $\mathcal{O}_\mathsf{z}$. The capillarity coefficient $\varkappa$ only depends on temperature following the kinetic theory of dense gases [34, 35]. The next property $(\mathsf{H_2})$ concerns Gibbs relation between the differentials of $\mathcal{E}$ and $\mathcal{S}$ with a natural constraint for $\mathcal{G}$ since we consider volumetric quantities [39].

$(\mathsf{H_2})$ *Assuming that $(\mathsf{H_1})$ holds and defining $\mathcal{G} = \mathcal{E} + p - T\mathcal{S}$ and $g = \mathcal{G}/\rho$, we have the volumetric Gibbs' relation between the differentials $d\mathcal{S}$ and $d\mathcal{E}$ over $\mathcal{O}_\mathsf{z}$*

$$T d\mathcal{S} = d\mathcal{E} - g d\rho - \varkappa \boldsymbol{w}\cdot d\boldsymbol{w}. \tag{3.7}$$

When $(\mathsf{H_1})$ and $(\mathsf{H_2})$ hold, the following Gibbs' relation $T d\mathcal{S} = d\mathcal{E}^{\rm tot} - g^{\rm tot} d\rho - \varkappa \boldsymbol{w}\cdot d\boldsymbol{w} - \rho \boldsymbol{v}\cdot d\boldsymbol{v}$ is also established in terms of $\mathcal{S}$, $\mathcal{E}^{\rm tot}$ and $g^{\rm tot}$. Concerning thermodynamic stability, a result similar to that of Proposition 2.2 is established in the next proposition taking into account the $\boldsymbol{w}$ and $\boldsymbol{v}$ variables.

Proposition 3.1. *Assume that $(\mathsf{H_1})$ and $(\mathsf{H_2})$ are satisfied and that the map $\mathsf{z} \mapsto \mathsf{u}$ is locally invertible. Denoting by $\widetilde{\partial}$ the derivation operator with respect to u, the following statements are equivalent :*

(i) $\widetilde{\partial}^2_{\mathsf{uu}} \mathcal{S}$ is negative definite.

(ii) $\partial_T \mathcal{E} > 0$, $\partial_\rho p > 0$, and $\varkappa > 0$.

Proof. The proof is similar to that of Proposition 2.2 and is based on the following expression of the quadratic form associated with the entropy Hessian matrix $\widetilde{\partial}^2_{\mathsf{uu}} \mathcal{S}$

$$\begin{aligned} \langle \widetilde{\partial}^2_{\mathsf{uu}} \mathcal{S}\, \mathsf{x}, \mathsf{x} \rangle = &- \frac{1}{T^2 \partial_T \mathcal{E}} \Big(\mathsf{x}_\mathcal{E} - \partial_\rho \mathcal{E}^{\rm cl} \mathsf{x}_\rho - \boldsymbol{v}\cdot\mathsf{x}_{\boldsymbol{v}} - (\varkappa - T\partial_T \varkappa) \boldsymbol{w}\cdot\mathsf{x}_{\boldsymbol{w}} \Big)^2 \\ &- \frac{\partial_\rho p}{\rho T} \mathsf{x}_\rho^2 - \frac{\varkappa}{T} \left|\mathsf{x}_{\boldsymbol{w}}\right|^2 - \frac{1}{\rho T} \left|\mathsf{x}_{\boldsymbol{v}} - \boldsymbol{v}\mathsf{x}_\rho\right|^2, \end{aligned} \tag{3.8}$$

where $\mathsf{x} = (\mathsf{x}_T, \mathsf{x}_{\boldsymbol{w}}, \mathsf{x}_{\boldsymbol{v}}, \mathsf{x}_\mathcal{E})^t$ denotes an arbitrary vector of $\mathbb{R}^{\mathsf{n}}$. This expression then directly yields that stability is equivalent to $\partial_T \mathcal{E} > 0$, $\partial_\rho p > 0$ and $\varkappa > 0$, keeping in mind that T and ρ are positive. □

Note that the same mechanical instabilities are obtained in both Proposition 3.1 and Proposition 2.2 since $\partial_\rho p = \partial_\rho p^{\rm cl}$. This is a natural result since vaporization fronts are stabilized by higher order density derivatives and not by first order density gradients $\boldsymbol{w}$. We now generalize Definition 2.3 and Lemma 2.4 to the situation of augmented variables.

Definition 3.2. *A nonempty open set $\mathcal{O}_\mathsf{z} \subset (0,\infty) \times \mathbb{R}^{2d_{\rm s}} \times (0,\infty)$ is said to be increasing with respect to temperature if for any $\mathsf{z} = (\rho,\boldsymbol{w},\boldsymbol{v},T)^t \in \mathcal{O}_\mathsf{z}$ we have $\{\rho,\boldsymbol{w},\boldsymbol{v}\}\times[T,\infty) \subset \mathcal{O}_\mathsf{z}$.*

The strengthened assumption $(\mathsf{H_3})$ now includes the former assumption $(\mathsf{H}_3^{\rm cl})$ of Section 2.3 and assumption $(\mathsf{H_4})$ naturally assumes that the capillarity coefficient is positive.

$(\mathsf{H_3})$ *Assuming that $(\mathsf{H_1})$ holds, the open set $\mathcal{O}_\mathsf{z}$ is increasing with respect to temperature and the volumetric specific heat is positive $\partial_T \mathcal{E} = \partial_T \mathcal{E}^{\rm tot} > 0$ over $\mathcal{O}_\mathsf{z}$.*

$(\mathsf{H_4})$ *The capillarity coefficient is positive $\varkappa > 0$ over $\mathcal{O}_\mathsf{z}$.*

With Proposition 3.1 and under assumptions $(\mathsf{H_3})$ and $(\mathsf{H_4})$ only mechanical instabilities may arise. Under assumption $(\mathsf{H_3})$ we may also change of variable from z to u as investigated in the next lemma. This notably allows to consider all thermodynamic functions as functions of the conservative variable u.

Lemma 3.3. *Assume that* (H_1)-(H_3) *hold. Then the map* $\mathsf{z} \mapsto \mathsf{u}$ *is a* C^γ *diffeomorphism from the open set* $\mathcal{O}_\mathsf{z}$ *onto an open set* $\mathcal{O}_\mathsf{u}$.

Proof. The proof is similar to the situation without the $\boldsymbol{v}$ and $\boldsymbol{w}$ variables. Under the assumptions (H_1)-(H_2) the map $\mathsf{z} \mapsto \mathsf{u}$ is C^γ and its Jacobian may be evaluated in the form

$$\partial_\mathsf{z}\mathsf{u} = \begin{pmatrix} 1 & 0_{1,d_\mathrm{s}} & 0_{1,d_\mathrm{s}} & 0 \\ 0_{d_\mathrm{s},1} & \boldsymbol{I} & 0_{d_\mathrm{s},d_\mathrm{s}} & 0_{d_\mathrm{s},1} \\ \boldsymbol{v} & 0_{d_\mathrm{s},d_\mathrm{s}} & \rho\boldsymbol{I} & 0_{d_\mathrm{s},1} \\ \partial_\rho\mathcal{E}^{\mathrm{tot}} & (\varkappa - T\partial_T\varkappa)\boldsymbol{w}^t & \rho\boldsymbol{v}^t & \partial_T\mathcal{E} \end{pmatrix}, \tag{3.9}$$

where $\partial_\rho\mathcal{E}^{\mathrm{tot}} = \partial_\rho\mathcal{E}^{\mathrm{cl}} + \frac{1}{2}|\boldsymbol{v}|^2$, $\partial_T\mathcal{E} = \partial_T\mathcal{E}^{\mathrm{cl}} - \frac{1}{2}T\partial_T^2\varkappa|\boldsymbol{w}|^2$ and $0_{i,j}$ denotes a zero block with i lines and j columns. Since ρ and $\partial_T\mathcal{E}$ are positive over $\mathcal{O}_\mathsf{z}$ the Jacobian matrix $\partial_\mathsf{z}\mathsf{u}$ is nonsingular and the local inversion theorem may be used. Moreover, proceeding as in Lemma 2.4, using again the positivity of ρ and $\partial_T\mathcal{E}$ and the increasing property of the open set $\mathcal{O}_\mathsf{z}$ with respect to temperature, the map $(\rho, \boldsymbol{w}, \boldsymbol{v}, T) \mapsto (\rho, \boldsymbol{w}, \rho\boldsymbol{v}, \mathcal{E}^{\mathrm{tot}})$ is into and we conclude that $\mathsf{z} \mapsto \mathsf{u}$ is a C^γ diffeomorphism. The inverse Jacobian matrix may further be obtained after some algebra in the form

$$\partial_\mathsf{u}\mathsf{z} = \begin{pmatrix} 1 & 0_{1,d_\mathrm{s}} & 0_{1,d_\mathrm{s}} & 0 \\ 0_{d_\mathrm{s},1} & \boldsymbol{I} & 0_{d_\mathrm{s},d_\mathrm{s}} & 0_{d_\mathrm{s},1} \\ -\frac{\boldsymbol{v}}{\rho} & 0_{d_\mathrm{s},d_\mathrm{s}} & \frac{1}{\rho}\boldsymbol{I} & 0_{d_\mathrm{s},1} \\ -\frac{\partial_\rho\mathcal{E}^{\mathrm{tot}} - |\boldsymbol{v}|^2}{\partial_T\mathcal{E}} & -\frac{(\varkappa - T\partial_T\varkappa)\boldsymbol{w}^t}{\partial_T\mathcal{E}} & -\frac{\boldsymbol{v}^t}{\partial_T\mathcal{E}} & \frac{1}{\partial_T\mathcal{E}} \end{pmatrix}, \tag{3.10}$$

and these matrices $\partial_\mathsf{z}\mathsf{u}$ and $\partial_\mathsf{u}\mathsf{z}$ will later be useful in order to establish various differential identities. □

Remark 3.4. *As a typical exemple of open set* $\mathcal{O}_\mathsf{z}$, *let us assume that* $\mathcal{O}_{\mathsf{z}^{\mathrm{cl}}} \subset (0,\infty) \times (T_{\min}, \infty)$ *where* $T_{\min} > 0$ *is positive and that* $\varkappa$ *as well as* $\mathcal{E}^{\mathrm{cl}}$ *are smooth up to* $T_{\min}$. *We may then introduce*

$$\mathcal{C}(\rho) = \inf_{T\in[T_{\min},\infty)} \partial_T\mathcal{E}^{\mathrm{cl}}(\rho, T), \qquad \mathcal{K}(\rho) = \sup_{T\in[T_{\min},\infty)} |T\partial_T^2\varkappa|,$$

and assuming that $\mathcal{C}(\rho)$ *is positive and* $\mathcal{K}(\rho)$ *is finite, we may then define*

$$\mathcal{O}_\mathsf{z} = \{\ \mathsf{z} \in (0,\infty) \times \mathbb{R}^{d_\mathrm{s}} \times \mathbb{R}^{d_\mathrm{s}} \times (T_{\min},\infty);\ \ (\rho, T)^t \in \mathcal{O}_{\mathsf{z}^{\mathrm{cl}}},\ \ \mathcal{K}(\rho)\,|\boldsymbol{w}|^2 < \mathcal{C}(\rho)\ \}, \tag{3.11}$$

and Property (H_3) *is then easily established.*

The thermal stability condition $\partial_T\mathcal{E} = \partial_T\mathcal{E}^{\mathrm{cl}} - \frac{1}{2}T\partial_T^2\varkappa|\boldsymbol{w}|^2 > 0$ naturally introduce limitations on the norm of $\boldsymbol{w}$. However, the order of magnitude of $\varkappa$ shows that such conditions may hold even when $\boldsymbol{w}$ is of the order of the inverse of the range of interaction potential [35]. We do not assume that the Hessian matrix $\widetilde{\partial}^2_{\mathsf{uu}}\mathcal{S}$ is negative definite since we anticipate the presence of mechanical thermodynamic instabilities. We will also need an instability indicator m in order to control negative values of the derivative $\partial_\rho p = \partial_\rho p^{\mathrm{cl}}$ and that may be taken to be the instability indicator m^{cl} of Lemma 2.5.

Lemma 3.5. *Assuming that* (H_1) *holds, that* $\delta > 0$ *is given, and letting* $\mathsf{m}(\mathsf{z}) = \mathsf{m}^{\mathrm{cl}}(\rho, T)$, *then* m *is* $C^{\gamma-1}$ *over* $\mathcal{O}_\mathsf{z}$, *and such that* $\mathsf{m} \geq 0$, $\mathsf{m} + \partial_\rho p/\rho T > 0$ *and* $\mathsf{m} = 0$ *if* $\partial_\rho p/\rho T \geq \delta$.

We finally need to introduce the mathematical assumptions concerning the transport coefficients that may be obtained from the kinetic theory of gases, the theory of moderately dense gases or from empirical correlations [16, 25, 33, 38]. The property that $\mathfrak{v} + \eta(1 - \frac{2}{d_\mathrm{s}}) > 0$ is deduced from the relation involving the physical volume viscosity $\mathfrak{v}'$ since $\mathfrak{v} + \eta(1 - \frac{2}{d_\mathrm{s}}) = \mathfrak{v}' + \frac{\eta}{3}$. The transport coefficients only depend on (ρ, T) but including a dependence on $\boldsymbol{w}$ introduce no difficulty.

(H_5) *The coefficients* $\mathfrak{v}$, η *and* λ *are* C^γ *functions over* $\mathcal{O}_\mathsf{z}$. *The coefficients* η *and* λ *are positive, the coefficient* $\mathfrak{v}$ *is nonnegative and such that* $\mathfrak{v} + \eta(1 - \frac{2}{d_\mathrm{s}})$ *is positive over* $\mathcal{O}_\mathsf{z}$.

Assumptions (H_1)-(H_5) are assumed to hold in the remaining part of the paper whenever the diffuse interface fluid equations are considered. Since the map $\mathsf{z} \mapsto \mathsf{u}$ is a C^γ diffeomorphism, all the thermodynamic functions and transport coefficients are also C^γ functions of the conservative variable u and defined over the corresponding open set $\mathcal{O}_\mathsf{u}$.

3.3 Augmented entropic variable

The mathematical entropy σ is taken in the form

$$\sigma = -\mathcal{S} = -\mathcal{S}^{\rm cl} + \tfrac{1}{2}\partial_T \varkappa\, |\boldsymbol{w}|^2, \tag{3.12}$$

and includes capillary contributions when $\varkappa$ depends on temperature. The mathematical entropy σ is a C^γ function of z or u from assumptions (H_1)-(H_5). Differentiating σ with respect to z, we obtain from Gibb's relation (3.7) that

$$\partial_{\mathsf{z}}\sigma = \Big(\frac{g}{T} - \frac{\partial_\rho \mathcal{E}}{T}, \partial_T \varkappa\, \boldsymbol{w}^t, 0_{1,d_{\rm s}}, -\frac{\partial_T \mathcal{E}}{T}\Big).$$

Multiplying next by $\partial_{\mathsf{u}}\mathsf{z}$ on the right, and using $\partial_{\mathsf{u}}\sigma = \partial_{\mathsf{z}}\sigma\, \partial_{\mathsf{u}}\mathsf{z}$, we obtain the expression of the augmented entropic variable $\mathsf{v} = (\partial_{\mathsf{u}}\sigma)^t$

$$\mathsf{v} = \frac{1}{T}\Big(g - \tfrac{1}{2}|\boldsymbol{v}|^2, \varkappa\, \boldsymbol{w}, \boldsymbol{v}, -1\Big)^t. \tag{3.13}$$

The augmented entropic variable v is formally similar to that of compressible non capillary gases [48, 49, 33] with the addition of the extra vector $\varkappa \boldsymbol{w}$ associated with the $\boldsymbol{w}$ component of u. We may then evaluate the useful matrix $\partial_{\mathsf{z}}\mathsf{v}$ in the form

$$\partial_{\mathsf{z}}\mathsf{v} = \begin{pmatrix} \frac{\partial_\rho p}{\rho T} & 0_{1,d_{\rm s}} & -\frac{1}{T}\boldsymbol{v}^t & -\frac{\partial_\rho \mathcal{E}^{\rm tot} - |\boldsymbol{v}|^2}{T^2} \\ 0_{d_{\rm s},1} & \frac{\varkappa}{T}\boldsymbol{I} & 0_{d_{\rm s},d_{\rm s}} & -\frac{\varkappa - T\partial_T \varkappa}{T^2}\boldsymbol{w} \\ 0 & 0_{d_{\rm s},d_{\rm s}} & \frac{1}{T}\boldsymbol{I} & -\frac{1}{T^2}\boldsymbol{v} \\ 0 & 0_{1,d_{\rm s}} & 0_{1,d_{\rm s}} & \frac{1}{T^2} \end{pmatrix}. \tag{3.14}$$

In the situation of mechanical stability we have $\partial_\rho p^{\rm cl} = \partial_\rho p > 0$ so that $\partial_{\mathsf{z}}\mathsf{v}$ is nonsingular and the inverse Jacobian matrix is given by

$$\partial_{\mathsf{v}}\mathsf{z} = \begin{pmatrix} \frac{\rho T}{\partial_\rho p} & 0_{1,d_{\rm s}} & \frac{\rho T}{\partial_\rho p}\boldsymbol{v}^t & \frac{\rho T \partial_\rho \mathcal{E}^{\rm tot}}{\partial_\rho p} \\ 0 & \frac{T}{\varkappa}\boldsymbol{I} & 0_{d_{\rm s},d_{\rm s}} & \frac{T(\varkappa - T\partial_T \varkappa)}{\varkappa}\boldsymbol{w} \\ 0 & 0_{d_{\rm s},d_{\rm s}} & T\boldsymbol{I} & T\boldsymbol{v} \\ 0 & 0_{1,d_{\rm s}} & 0_{1,d_{\rm s}} & T^2 \end{pmatrix}. \tag{3.15}$$

The Jacobian matrix $\partial_{\mathsf{z}}\mathsf{v}$ has determinant $\varkappa \partial_\rho p/\rho T^5$ and the map $\mathsf{z} \mapsto \mathsf{v}$ is not globally invertible in the presence of mechanical instabilities when $\partial_\rho p = \partial_\rho p^{\rm cl}$ is changing sign. This is notably the case in liquid-vapor flows and denoting by l and g the indices of a liquid and its vapor at equilibrium, the classical equilibrium relations yield $T_{\rm l} = T_{\rm g}$, $p_{\rm l}^{\rm cl} = p_{\rm g}^{\rm cl}$, and $g_{\rm l} = g_{\rm g}$. These equilibrium conditions between a liquid and its vapor, completed with the conditions $\boldsymbol{w}_{\rm l} = \boldsymbol{w}_{\rm g} = 0$ and $\boldsymbol{v}_{\rm l} = \boldsymbol{v}_{\rm g} = 0$, mean that the entropic variables $\mathsf{v}_{\rm l}$ and $\mathsf{v}_{\rm g}$ coincide even though the corresponding natural variables $\mathsf{z}_{\rm l}$ and $\mathsf{z}_{\rm g}$ differ. We denote in the following by $\mathcal{O}_{\mathsf{z}}^{\rm st} \subset \mathcal{O}_{\mathsf{z}}$ the subset of stable states in $\mathcal{O}_{\mathsf{z}}$

$$\mathcal{O}_{\mathsf{z}}^{\rm st} = \{\, \mathsf{z} \in \mathcal{O}_{\mathsf{z}} \;\big|\; \partial_\rho p > 0 \,\}, \tag{3.16}$$

with similar definition for $\mathcal{O}_{\mathsf{u}}^{\rm st}$. Then for any $\mathsf{z} \in \mathcal{O}_{\mathsf{z}}^{\rm st}$, the map $\mathsf{z} \mapsto \mathsf{v}$ is *locally* a $C^{\gamma-1}$ diffeomorphism and thermodynamic functions may locally be considered as $C^{\gamma-1}$ functions of v. In this situation, the map $\mathsf{u} \mapsto \mathsf{v}$ is also a *local* $C^{\gamma-1}$ diffeomorphism [40] whereas in the situation of perfect gases $\mathsf{z} \mapsto \mathsf{v}$ and $\mathsf{u} \mapsto \mathsf{v}$ are always global diffeomorphisms [33].

3.4 Augmented fluxes

The governing equations of the augmented system are written in vector form

$$\partial_t \mathsf{u} + \sum_{i \in \mathcal{D}} \partial_i \big(\mathsf{F}_i + \mathsf{F}_i^{\rm c} + \mathsf{F}_i^{\rm d}\big) = 0, \tag{3.17}$$

where F_i denotes an augmented convective flux, $\mathsf{F}_i^{\rm c}$ an augmented capillary flux, $\mathsf{F}_i^{\rm d}$ an augmented dissipative flux in the ith direction and $\mathcal{D}$ the spatial direction indexing set. Both augmented convective

F_i and capillary $\mathsf{F}_i^{\rm c}$ fluxes have to be identified whereas the augmented dissipative fluxes $\mathsf{F}_i^{\rm d}$ are naturally given by [48, 49, 33]

$$\mathsf{F}_i^{\rm d} = \Big(0, 0_{d_{\rm s},1}, \boldsymbol{\mathcal{P}}_i^{\rm d}, \mathcal{Q}_i^{\rm d} + \sum_{j\in\mathcal{D}} \mathcal{P}_{ij}^{\rm d} v_j\Big)^t, \qquad i \in \mathcal{D}, \tag{3.18}$$

where $\boldsymbol{\mathcal{P}}_i^{\rm d} = (\mathcal{P}_{i1}^{\rm d}, \dots, \mathcal{P}_{id_{\rm s}}^{\rm d})^t$ and are simply obtained by adding null components for the extra hyperbolic $\boldsymbol{w}$ variable. In order to identify the proper convective F_i and capillary $\mathsf{F}_i^{\rm c}$ fluxes we adapt the method of Gavrilyuk and Gouin [31] using the Legendre transform of entropy instead of energy, although it is also possible to use thermodynamic arguments [52].

We first assume to be in the neighborhood of a stable point $\mathsf{z} \in \mathcal{O}_{\mathsf{z}}^{\rm st}$ in such a way that $\mathsf{u} \mapsto \mathsf{v}$ is locally invertible. The Legendre transform $\mathcal{L}$ of entropy is given by $\mathcal{L}(\mathsf{v}) = \langle \mathsf{v}, \mathsf{u}\rangle - \sigma$ since $\partial_{\mathsf{u}}\sigma = \mathsf{v}^t$ and may locally be defined as a function of v. In this situation, differentiating $\mathcal{L}(\mathsf{v})$ with respect to v, it is obtained that $\mathsf{u}^t = \partial_{\mathsf{v}}\mathcal{L}$. We may evaluate $\mathcal{L}$ from the expressions (3.12) of σ and (3.13) of v and it is obtained that

$$\mathcal{L} = \langle \mathsf{u}, \mathsf{v}\rangle - \sigma = \frac{1}{T}\big(p^{\rm cl} + \tfrac{1}{2}\varkappa|\boldsymbol{w}|^2\big) = \frac{1}{T}\big(p + \varkappa|\boldsymbol{w}|^2\big). \tag{3.19}$$

The Legendre transform of $\sigma(\mathsf{u})$ is then $\mathcal{L}(\mathsf{v})$ and conversely. This expression for $\mathcal{L}$ differs from that of classic compressible fluids by the extra capillary term $\frac{1}{2}\varkappa|\boldsymbol{w}|^2$. It also differs from the Legendre transform of energy by the $1/T$ factor. Adapting Gravilyuk and Gouin procedure [31] and following the structure of classical compressible fluids, *it is then enforced that* $\mathcal{L}_i = \mathcal{L} v_i$ and $\mathsf{F}_i = \big(\partial_{\mathsf{v}}(\mathcal{L} v_i)\big)^t$. The fluxes F_i may then be evaluated by using the identity $\partial_{\mathsf{v}}(\mathcal{L} v_i) = \mathsf{u} v_i + \mathcal{L}\partial_{\mathsf{v}} v_i$. From the expression of the entropic variable v it is obtained after some algebra that the convective flux F_i is given by

$$\mathsf{F}_i = \Big(\rho v_i, \boldsymbol{w} v_i, \rho \boldsymbol{v} v_i + (p + \varkappa|\boldsymbol{w}|^2)\mathfrak{b}_i, (\mathcal{H}^{\rm tot} + \varkappa|\boldsymbol{w}|^2) v_i\Big)^t, \qquad i \in \mathcal{D}, \tag{3.20}$$

where $\mathfrak{b}_i$ is the ith basis vector in the physical space $\mathbb{R}^{d_{\rm s}}$. This flux is similar to that of classic fluid with the exceptions that we have an extra capillary component $\boldsymbol{w} v_i$ associated with the $\boldsymbol{w}$ variable and that $\varkappa|\boldsymbol{w}|^2$ has been added to the pressure. Even though these fluxes have been derived in the situation of thermodynamic stability, *these expressions are now adopted for all states* $\mathsf{z} \in \mathcal{O}_{\mathsf{z}}$.

Entropic relations are then recovered from the expressions of σ and F_i, $i \in \mathcal{D}$. Defining naturally the entropy flux in the ith direction by $\mathsf{q}_i = \sigma\, v_i$ for $i \in \mathcal{D}$, we have the traditional relations

$$\partial_{\mathsf{u}}\sigma\, \partial_{\mathsf{u}}\mathsf{F}_i = \partial_{\mathsf{u}}\mathsf{q}_i, \qquad i \in \mathcal{D}. \tag{3.21}$$

It is indeed first obtained by differentiating (3.20) and using (3.13) that $\partial_{\mathsf{u}}\sigma\, \partial_{\mathsf{z}}\mathsf{F}_i = \partial_{\mathsf{z}}\mathsf{q}_i$ and multiplying on the right by $\partial_{\mathsf{u}}\mathsf{z}$ then yields (3.21). We thus have $\mathsf{q}_i = \sigma\, v_i$, $\mathcal{L}_i = \mathcal{L}\, v_i$, and the Legendre transform of $\mathsf{q}_i(\mathsf{F}_i)$ is $\mathcal{L}_i(\mathsf{v})$ when stability holds and when $\mathsf{z} \mapsto \mathsf{F}_i$ is invertible as for instance when v_i is large enough.

On the other hand, the capillary fluxes $\mathsf{F}_i^{\rm c}$ include all higher order terms of the momentum and energy equation and will lead to an antisymmetric structure

$$\mathsf{F}_i^{\rm c} = \Big(0, \rho\boldsymbol{\nabla} v_i, -\rho\boldsymbol{\nabla}(\varkappa w_i), \rho\varkappa\boldsymbol{w}\cdot\boldsymbol{\nabla} v_i - \rho\boldsymbol{v}\cdot\boldsymbol{\nabla}(\varkappa w_i)\Big)^t, \qquad i \in \mathcal{D}. \tag{3.22}$$

It is then necessary to check that the governing equations associated with the fluxes F_i and $\mathsf{F}_i^{\rm c}$ for $i \in \mathcal{D}$ coincide with the original equations for capillary fluids (2.6) and (2.7). This is obtained by using the following differential identities for the momentum (2.6) and energy (2.7) equations

$$\boldsymbol{\nabla}\cdot\Big(\varkappa\boldsymbol{\nabla}\rho\otimes\boldsymbol{\nabla}\rho - \rho\boldsymbol{\nabla}\cdot(\varkappa\boldsymbol{\nabla}\rho)\boldsymbol{I} - \varkappa|\boldsymbol{\nabla}\rho|^2\boldsymbol{I}\Big) = -\sum_{i\in\mathcal{D}} \partial_i\big(\rho\boldsymbol{\nabla}(\varkappa\boldsymbol{w}_i)\big),$$

$$\boldsymbol{\nabla}\cdot\Big(\varkappa\rho\boldsymbol{\nabla}\cdot\boldsymbol{v}\boldsymbol{\nabla}\rho + \varkappa\boldsymbol{\nabla}\rho\,\boldsymbol{\nabla}\rho\cdot\boldsymbol{v} - \rho\boldsymbol{\nabla}\cdot(\varkappa\boldsymbol{\nabla}\rho)\,\boldsymbol{v} - \varkappa|\boldsymbol{\nabla}\rho|^2\,\boldsymbol{v}\Big) = \sum_{i\in\mathcal{D}} \partial_i\big(\rho\varkappa\boldsymbol{w}\cdot\boldsymbol{\nabla}\boldsymbol{v}_i - \rho\boldsymbol{v}\cdot\boldsymbol{\nabla}(\varkappa\boldsymbol{w}_i)\big).$$

3.5 Quasilinear form

We have obtained in Section 3.4 the augmented convective F_i, capillary $\mathsf{F}_i^{\rm c}$, and dissipative fluxes $\mathsf{F}_i^{\rm d}$ in the ith spatial direction. From the expressions (3.22) and (3.18), the capillary $\mathsf{F}_i^{\rm c}$ and dissipative $\mathsf{F}_i^{\rm d}$

fluxes may also be written in the form $\mathsf{F}_i^{\rm d} = -\sum_{j\in\mathcal{D}} \widehat{\mathsf{B}}_{ij}^{\rm d}\partial_j\mathsf{z}$ and $\mathsf{F}_i^{\rm c} = -\sum_{j\in\mathcal{D}} \widehat{\mathsf{B}}_{ij}^{\rm c}\partial_j\mathsf{z}$ where $\widehat{\mathsf{B}}_{ij}^{\rm d}$ and $\widehat{\mathsf{B}}_{ij}^{\rm c}$ are uniquely defined matrices in $\mathbb{R}^{\mathsf{n},\mathsf{n}}$. We may thus write $\mathsf{F}_i^{\rm d}$ and $\mathsf{F}_i^{\rm c}$ in the form

$$\mathsf{F}_i^{\rm d} = -\sum_{j\in\mathcal{D}} \mathsf{B}_{ij}^{\rm d}\partial_j\mathsf{u}, \qquad \mathsf{F}_i^{\rm c} = -\sum_{j\in\mathcal{D}} \mathsf{B}_{ij}^{\rm c}\partial_j\mathsf{u}, \qquad i\in\mathcal{D}, \tag{3.23}$$

where $\mathsf{B}_{ij}^{\rm d}$ and $\mathsf{B}_{ij}^{\rm c}$ are given by $\mathsf{B}_{ij}^{\rm d} = \widehat{\mathsf{B}}_{ij}^{\rm d}\partial_{\mathsf{u}}\mathsf{z}$ and $\mathsf{B}_{ij}^{\rm c} = \widehat{\mathsf{B}}_{ij}^{\rm c}\partial_{\mathsf{u}}\mathsf{z}$. The dissipation matrix $\mathsf{B}_{ij}^{\rm d}$ then relates the dissipative flux in the ith direction $\mathsf{F}_i^{\rm d}$ to the gradient of the conservative variable in the jth direction $\partial_j\mathsf{u}$ and the cohesive matrix $\mathsf{B}_{ij}^{\rm c}$ relates the capillary flux in the ith direction $\mathsf{F}_i^{\rm c}$ to the gradient of the conservative variable in the jth direction $\partial_j\mathsf{u}$.

We may next introduce the jacobian $\mathsf{A}_i = \partial_{\mathsf{u}}\mathsf{F}_i$ of the convective flux F_i with respect to the conservative variable u and the governing equations (3.17) may finally be written in the form of an augmented quasilinear second order system of partial differential equations

$$\partial_t\mathsf{u} + \sum_{i\in\mathcal{D}} \mathsf{A}_i(\mathsf{u})\partial_i\mathsf{u} - \sum_{i,j\in\mathcal{D}} \partial_i\big(\mathsf{B}_{ij}^{\rm d}(\mathsf{u})\partial_j\mathsf{u}\big) - \sum_{i,j\in\mathcal{D}} \partial_i\big(\mathsf{B}_{ij}^{\rm c}(\mathsf{u})\partial_j\mathsf{u}\big) = 0. \tag{3.24}$$

The coefficient matrices $\mathsf{A}_i = \partial_{\mathsf{u}}\mathsf{F}_i$, $\mathsf{B}_{ij}^{\rm d}$, and $\mathsf{B}_{ij}^{\rm c}$, for $i,j\in\mathcal{D}$, have at least regularity $C^{\gamma-1}$ over the open set $\mathcal{O}_{\mathsf{u}}$. It is then remarquable that the original system (2.5)–(2.7), that involves third order derivatives of density in the momentum equation, has been rewritten in the form of a quasilinear second order system. Symmetrization of the corresponding augmented system of partial differential equation is investigated in the next section.

4 Normal form of the augmented system

We investigate in this section symmetrized forms of the augmented system of conservation equations (3.24) as well as the properties of convective, dissipative and capillary matrices.

4.1 Entropic symmetrization

An entropic symmetrized form of the quasilinear system in conservative form (3.24) may be obtained by using the entropic variable v around a stable state. Assuming that z is a stable state $\mathsf{z}\in\mathcal{O}_{\mathsf{z}}^{\rm st}$ with $\partial_\rho p > 0$, the map $\mathsf{z}\mapsto\mathsf{v}$ is locally invertible so that $\mathsf{u}\mapsto\mathsf{v}$ is also locally invertible from Lemma 3.3. Proceeding as for classic fluids [48, 49, 33], the symmetrized entropic form is obtained in the form

$$\widetilde{\mathsf{A}}_0(\mathsf{w})\partial_t\mathsf{v} + \sum_{i\in\mathcal{D}} \widetilde{\mathsf{A}}_i(\mathsf{v})\partial_i\mathsf{v} - \sum_{i,j\in\mathcal{D}} \partial_i\big(\widetilde{\mathsf{B}}_{ij}^{\rm d}(\mathsf{v})\partial_j\mathsf{v}\big) - \sum_{i,j\in\mathcal{D}} \partial_i\big(\widetilde{\mathsf{B}}_{ij}^{\rm c}(\mathsf{v})\partial_j\mathsf{v}\big) = 0, \tag{4.1}$$

with $\widetilde{\mathsf{A}}_0 = \partial_{\mathsf{v}}\mathsf{u}$, $\widetilde{\mathsf{A}}_i = \mathsf{A}_i\partial_{\mathsf{v}}\mathsf{u}$, $\widetilde{\mathsf{B}}_{ij}^{\rm d} = \mathsf{B}_{ij}^{\rm d}\partial_{\mathsf{v}}\mathsf{u}$, and $\widetilde{\mathsf{B}}_{ij}^{\rm c} = \mathsf{B}_{ij}^{\rm c}\partial_{\mathsf{v}}\mathsf{u}$, for $i,j\in\mathcal{D}$.

The matrix $\widetilde{\mathsf{A}}_0 = \partial_{\mathsf{v}}\mathsf{u}$ may be evaluated as $\widetilde{\mathsf{A}}_0 = \partial_{\mathsf{z}}\mathsf{u}\,\partial_{\mathsf{z}}\mathsf{v}$ and is found to be

$$\widetilde{\mathsf{A}}_0 = \begin{pmatrix} \frac{\rho T}{\partial_\rho p} & 0_{1,d_{\rm s}} & \frac{\rho T}{\partial_\rho p}\boldsymbol{v}^t & \frac{\rho T}{\partial_\rho p}\partial_\rho\mathcal{E}^{\rm tot} \\ 0_{d_{\rm s},1} & \frac{T}{\varkappa}\boldsymbol{I} & 0_{d_{\rm s},d_{\rm s}} & \frac{T}{\varkappa}(\varkappa - T\partial_T\varkappa)\boldsymbol{w} \\ \frac{\rho T}{\partial_\rho p}\boldsymbol{v} & 0_{d_{\rm s},d_{\rm s}} & \frac{\rho}{T}\boldsymbol{I} + \frac{\rho T}{\partial_\rho p}\boldsymbol{v}\otimes\boldsymbol{v} & \frac{\rho T}{\partial_\rho p}\partial_\rho\mathcal{H}^{\rm tot}\boldsymbol{v} \\ \frac{\rho T}{\partial_\rho p}\partial_\rho\mathcal{E}^{\rm tot} & \frac{T}{\varkappa}(\varkappa - T\partial_T\varkappa)\boldsymbol{w}^t & \frac{\rho T}{\partial_\rho p}\partial_\rho\mathcal{H}^{\rm tot}\boldsymbol{v}^t & \Upsilon \end{pmatrix}, \tag{4.2}$$

where

$$\Upsilon = \frac{\rho T}{\partial_\rho p}(\partial_\rho\mathcal{E}^{\rm tot})^2 + \frac{T}{\varkappa}(\varkappa - T\partial_T\varkappa)^2|\boldsymbol{w}|^2 + \rho T|\boldsymbol{v}|^2 + T^2\partial_T\mathcal{E}.$$

The matrix $\widetilde{\mathsf{A}}_0$ is symmetric $\widetilde{\mathsf{A}}_0^t = \widetilde{\mathsf{A}}_0$ and it is easily established that $\widetilde{\mathsf{A}}_0$ is positive definite since z is assumed to be a stable state with $\partial_\rho p > 0$. Its determinant, directly obtained from $\partial_{\mathsf{v}}\mathsf{u} = \partial_{\mathsf{z}}\mathsf{u}\partial_{\mathsf{v}}\mathsf{z}$, is given by $\det\widetilde{\mathsf{A}}_0 = \rho^2T^5\partial_T\mathcal{E}/\varkappa\partial_\rho p$.

Denoting by $\boldsymbol{\xi}$ an arbitrary vector of $\mathbb{R}^{d_s}$, the convective matrices $\widetilde{\mathsf{A}}_i = \partial_{\mathsf{u}}\mathsf{F}_i\,\partial_{\mathsf{v}}\mathsf{u}$, $i \in \mathcal{D}$, that may conveniently be evaluated as $\widetilde{\mathsf{A}}_i = \partial_{\mathsf{z}}\mathsf{F}_i\,\partial_{\mathsf{v}}\mathsf{z}$, $i \in \mathcal{D}$, are given by

$$\sum_{i\in\mathcal{D}} \xi_i \widetilde{\mathsf{A}}_i = \boldsymbol{v}\cdot\boldsymbol{\xi}\,\widetilde{\mathsf{A}}_0 + T \begin{pmatrix} 0 & 0_{1,d_s} & \rho\boldsymbol{\xi}^t & \rho\boldsymbol{v}\cdot\boldsymbol{\xi} \\ 0_{d_s,1} & 0_{d_s,d_s} & \boldsymbol{w}\otimes\boldsymbol{\xi} & \boldsymbol{v}\cdot\boldsymbol{\xi}\boldsymbol{w} \\ \rho\boldsymbol{\xi} & \boldsymbol{\xi}\otimes\boldsymbol{w} & \rho(\boldsymbol{v}\otimes\boldsymbol{\xi}+\boldsymbol{\xi}\otimes\boldsymbol{v}) & (\mathcal{H}^{\rm tot}+\varkappa|\boldsymbol{w}|^2)\boldsymbol{\xi}+\rho\boldsymbol{v}\cdot\boldsymbol{\xi}\boldsymbol{v} \\ \rho\boldsymbol{v}\cdot\boldsymbol{\xi} & \boldsymbol{v}\cdot\boldsymbol{\xi}\boldsymbol{w}^t & (\mathcal{H}^{\rm tot}+\varkappa|\boldsymbol{w}|^2)\boldsymbol{\xi}^t+\rho\boldsymbol{v}\cdot\boldsymbol{\xi}\boldsymbol{v}^t & 2(\mathcal{H}^{\rm tot}+\varkappa|\boldsymbol{w}|^2)\boldsymbol{v}\cdot\boldsymbol{\xi} \end{pmatrix}. \tag{4.3}$$

These convective matrices $\widetilde{\mathsf{A}}_i$ are symmetric $\widetilde{\mathsf{A}}_i^t = \widetilde{\mathsf{A}}_i$, $i \in \mathcal{D}$, and underline the role of the modified pressure $p + \varkappa|\boldsymbol{w}|^2$. If the lines and columns associated with the extra variable $\boldsymbol{w}$ are removed and if the capillarity coefficient $\varkappa$ is set to zero, the resulting matrices are formally similar to that obtained for compressible Navier-Stokes-Fourier equations [48, 33].

The second order dissipative flux matrices $\widetilde{\mathsf{B}}_{ij}^{\rm d}$ may further be decomposed in the form

$$\widetilde{\mathsf{B}}_{ij}^{\rm d} = \mathfrak{v}\widetilde{\mathsf{B}}_{ij}^{\mathfrak{v}} + \eta\widetilde{\mathsf{B}}_{ij}^{\eta} + \lambda\widetilde{\mathsf{B}}_{ij}^{\lambda}, \tag{4.4}$$

where $\mathfrak{v}$ denotes the volume viscosity, η the shear viscosity and λ the thermal conductivity. Denoting by $\boldsymbol{\xi}$ and $\boldsymbol{\zeta}$ arbitrary vectors of $\mathbb{R}^{d_s}$, the matrices $\widetilde{\mathsf{B}}_{ij}^{\mathfrak{v}}$, $\widetilde{\mathsf{B}}_{ij}^{\eta}$ and $\widetilde{\mathsf{B}}_{ij}^{\lambda}$ are found in the form

$$\sum_{i,j\in\mathcal{D}} \xi_i\zeta_j \widetilde{\mathsf{B}}_{ij}^{\mathfrak{v}} = T \begin{pmatrix} 0 & 0_{1,d_s} & 0_{1,d_s} & 0 \\ 0_{d_s,1} & 0_{d_s,d_s} & 0_{d_s,d_s} & 0_{d_s,1} \\ 0_{d_s,1} & 0_{d_s,d_s} & \boldsymbol{\xi}\otimes\boldsymbol{\zeta} & \boldsymbol{\xi}\,\boldsymbol{\zeta}\cdot\boldsymbol{v} \\ 0 & 0_{1,d_s} & \boldsymbol{\zeta}^t\boldsymbol{\xi}\cdot\boldsymbol{v} & \boldsymbol{\xi}\cdot\boldsymbol{v}\,\boldsymbol{\zeta}\cdot\boldsymbol{v} \end{pmatrix}, \tag{4.5}$$

$$\sum_{i,j\in\mathcal{D}} \xi_i\zeta_j \widetilde{\mathsf{B}}_{ij}^{\eta} = T \begin{pmatrix} 0 & 0_{1,d_s} & 0_{1,d_s} & 0 \\ 0_{d_s,1} & 0_{d_s,d_s} & 0_{d_s,d_s} & 0_{d_s,1} \\ 0_{d_s,1} & 0_{d_s,d_s} & \boldsymbol{\xi}\cdot\boldsymbol{\zeta}\boldsymbol{I} + \boldsymbol{\zeta}\otimes\boldsymbol{\xi} - \frac{2}{d_s}\boldsymbol{\xi}\otimes\boldsymbol{\zeta} & \boldsymbol{\xi}\cdot\boldsymbol{\zeta}\boldsymbol{v} + \boldsymbol{\zeta}\,\boldsymbol{\xi}\cdot\boldsymbol{v} - \frac{2}{d_s}\boldsymbol{\xi}\,\boldsymbol{\zeta}\cdot\boldsymbol{v} \\ 0 & 0_{1,d_s} & \boldsymbol{\xi}\cdot\boldsymbol{\zeta}\boldsymbol{v}^t + \boldsymbol{\zeta}\cdot\boldsymbol{v}\,\boldsymbol{\xi}^t - \frac{2}{d_s}\boldsymbol{\xi}\cdot\boldsymbol{v}\boldsymbol{\zeta}^t & \boldsymbol{\xi}\cdot\boldsymbol{\zeta}\boldsymbol{v}\cdot\boldsymbol{v} + (1-\frac{2}{d_s})\boldsymbol{\xi}\cdot\boldsymbol{v}\,\boldsymbol{\zeta}\cdot\boldsymbol{v} \end{pmatrix}, \tag{4.6}$$

$$\sum_{i,j\in\mathcal{D}} \xi_i\zeta_j \widetilde{\mathsf{B}}_{ij}^{\lambda} = T^2\boldsymbol{\xi}\cdot\boldsymbol{\zeta} \begin{pmatrix} 0 & 0_{1,d_s} & 0_{1,d_s} & 0 \\ 0_{d_s,1} & 0_{d_s,d_s} & 0_{d_s,d_s} & 0_{d_s,1} \\ 0_{d_s,1} & 0_{d_s,d_s} & 0_{d_s,d_s} & 0_{d_s,1} \\ 0 & 0_{1,d_s} & 0_{1,d_s} & 1 \end{pmatrix}. \tag{4.7}$$

These dissipative second order matrices $\widetilde{\mathsf{B}}_{ij}^{\rm d}$, $\widetilde{\mathsf{B}}_{ij}^{\mathfrak{v}}$, $\widetilde{\mathsf{B}}_{ij}^{\eta}$, and $\widetilde{\mathsf{B}}_{ij}^{\lambda}$, $i, j \in \mathcal{D}$, are similar to those for compressible fluids except that there are extra line and columns of zeros associated with the $\boldsymbol{w}$ variable [49, 33]. We have the reciprocity relations $(\widetilde{\mathsf{B}}_{ij}^{\rm d})^t = \widetilde{\mathsf{B}}_{ji}^{\rm d}$, $i, j \in \mathcal{D}$, and the blocks $\sum_{i,j\in\mathcal{D}} \xi_i\xi_j \widetilde{\mathsf{B}}_{ij}^{\rm d\,II,II}$ is positive definite for any $\xi \in \Sigma^{d_s-1}$.

On the other hand, the cohesive flux matrices are found in the form

$$\sum_{i,j\in\mathcal{D}} \xi_i\zeta_j \widetilde{\mathsf{B}}_{ij}^{\rm c} = \rho T \begin{pmatrix} 0 & 0_{1,d_s} & 0_{1,d_s} & 0 \\ 0_{d_s,1} & 0_{d_s,d_s} & -\boldsymbol{\zeta}\otimes\boldsymbol{\xi} & -\boldsymbol{\zeta}\,\boldsymbol{\xi}\cdot\boldsymbol{v} \\ 0_{d_s,1} & \boldsymbol{\zeta}\otimes\boldsymbol{\xi} & 0_{d_s,d_s} & \varkappa\,\boldsymbol{\zeta}\,\boldsymbol{\xi}\cdot\boldsymbol{w} \\ 0 & \boldsymbol{\xi}^t\,\boldsymbol{\zeta}\cdot\boldsymbol{v} & -\varkappa\,\boldsymbol{\xi}^t\,\boldsymbol{\zeta}\cdot\boldsymbol{w} & \varkappa(\boldsymbol{\xi}\cdot\boldsymbol{w}\,\boldsymbol{\zeta}\cdot\boldsymbol{v} - \boldsymbol{\xi}\cdot\boldsymbol{v}\,\boldsymbol{\zeta}\cdot\boldsymbol{w}) \end{pmatrix}. \tag{4.8}$$

The cohesive matrices $\widetilde{\mathsf{B}}_{ij}^{\rm c}$ are such that $(\widetilde{\mathsf{B}}_{ij}^{\rm c})^t = -\widetilde{\mathsf{B}}_{ji}^{\rm c}$, $i, j \in \mathcal{D}$, and *introduce extra coupling between the hyperbolic and parabolic variables*. The blocks $\widetilde{\mathsf{B}}_{ij}^{\rm c,I,I}$ vanish $\widetilde{\mathsf{B}}_{ij}^{\rm c,I,I} = 0$ and the blocks $\widetilde{\mathsf{B}}_{ij}^{\rm c,I,II}$ and $\widetilde{\mathsf{B}}_{ij}^{\rm c,II,I}$ only depend on $(\rho, \boldsymbol{v}, T)$. We also note that the matrices $\widetilde{\mathsf{A}}_0$, $\widetilde{\mathsf{A}}_i$, $i \in \mathcal{D}$, $\widetilde{\mathsf{B}}_{ij}^{\rm d}$ and $\widetilde{\mathsf{B}}_{ij}^{\rm c}$, $i, j \in \mathcal{D}$ have at least regularity $C^{\gamma-2}$ over $\mathcal{O}_{\mathsf{v}}$.

The entropic symmetrized form introduced by Gravilyuk and Gouin was concerned with inviscid fluids with capillary effects [31]. The corresponding governing equations may be obtained from (2.5)–(2.9) by letting to zero the dissipative fluxes $\boldsymbol{\mathcal{P}}^{\rm d}=0$ and $\boldsymbol{\mathcal{Q}}^{\rm d}=0$. Gravilyuk and Gouin have used the conservative variable $(\rho,\boldsymbol{w},\rho\boldsymbol{v},\mathcal{S})^t$ with the entropy equation $\partial_t\mathcal{S}+\boldsymbol{\nabla\cdot}(\mathcal{S}\boldsymbol{v})=0$ and the energy $\mathcal{E}$ playing the role of entropy. This procedure, however, is only valid for *inviscid* fluids and this is why we have restored the natural role of entropy. On the other hand, *a major difficulty* with the entropic variable v, notably identified by Benzoni et al [3], is that the map $\mathsf{u}\mapsto\mathsf{v}$ is generally not globally invertible. The matrix $\widetilde{\mathsf{A}}_0$ also becomes singular at points where $\partial_\rho p=0$ and is not anymore positive definite in the presence of mechanical instabilities when $\partial_\rho p$ becomes negative. This is why it is mandatory to use normal variables w such that $\mathsf{u}\mapsto\mathsf{w}$ remains invertible as investigated in the next section.

4.2 Normal symmetrization

We investigate in this section a normal form of the augmented system of equations using the augmented natural variable

$$\mathsf{w}=\big(\rho,\boldsymbol{w},\boldsymbol{v},T\big)^t. \tag{4.9}$$

In the particular situation of the compressible Navier-Stokes equations without capillarity effects, it has been established that the variable $(\rho,\boldsymbol{v},T)^t$ may be used as a normal variable [49, 33]. It is thus natural to seek if the particular augmented variable $\mathsf{w}=\big(\rho,\boldsymbol{w},\boldsymbol{v},T\big)^t$ that coincide with z also leads to a normal symmetrized form of the augmented system. We use in the following the vector and matrix block structure induced by the partitioning of $\mathbb{R}^{\mathsf{n}}$ into $\mathbb{R}^{\mathsf{n}}=\mathbb{R}^{\mathsf{n}_{\rm I}}\times\mathbb{R}^{\mathsf{n}_{\rm II}}$ with $\mathsf{n}=\mathsf{n}_{\rm I}+\mathsf{n}_{\rm II}$, $\mathsf{n}_{\rm I}=\mathsf{n}_{\rm II}=d_{\rm s}+1$, and $\mathsf{w}=(\mathsf{w}_{\rm I},\mathsf{w}_{\rm II})^t$ with $\mathsf{w}_{\rm I}=(\rho,\boldsymbol{w})^t$ and $\mathsf{w}_{\rm II}=(\boldsymbol{v},T)^t$. It will also be convenient in the following to split the hyperbolic variable $\mathsf{w}_{\rm I}$ as $\mathsf{w}_{\rm I}=(\mathsf{w}_{\rm I'},\mathsf{w}_{\rm I''})^t$ where $\mathsf{w}_{\rm I'}=\rho$ and $\mathsf{w}_{\rm I''}=\boldsymbol{w}$.

The normal form of the system is written for convenience in nonconservative form

$$\overline{\mathsf{A}}_0(\mathsf{w})\partial_t\mathsf{w}+\sum_{i\in\mathcal{D}}\overline{\mathsf{A}}_i(\mathsf{w})\partial_i\mathsf{w}-\sum_{i,j\in\mathcal{D}}\overline{\mathsf{B}}^{\rm d}_{ij}(\mathsf{w})\partial_i\partial_j\mathsf{w}-\sum_{i,j\in\mathcal{D}}\overline{\mathsf{B}}^{\rm c}_{ij}(\mathsf{w})\partial_i\partial_j\mathsf{w}=\mathsf{h}(\mathsf{w},\boldsymbol{\nabla}\mathsf{w}), \tag{4.10}$$

where h has the structure

$$\mathsf{h}=\big(\mathsf{h}_{\rm I}(\mathsf{w},\boldsymbol{\nabla}\mathsf{w}_{\rm II}),\mathsf{h}_{\rm II}(\mathsf{w},\boldsymbol{\nabla}\mathsf{w})\big)^t. \tag{4.11}$$

The standard method in order to derive normal forms for systems of conservation laws is to substitute $\mathsf{v}=\mathsf{v}(\mathsf{w})$ in the entropic symmetrized form and to multiply on the left by $(\partial_{\mathsf{w}}\mathsf{v})^t$ in order to maintain the symmetry properties of convective and dissipative matrices [49, 33]. The matrix in front of $\partial_t\mathsf{w}$ would then be $(\partial_{\mathsf{z}}\mathsf{v})^t\widetilde{\mathsf{A}}_0\partial_{\mathsf{z}}\mathsf{v}$, that may be rewritten $(\partial_{\mathsf{z}}\mathsf{v})^t\partial_{\mathsf{z}}\mathsf{u}$, and the matrix in front of $\partial_i\mathsf{w}$ would be $(\partial_{\mathsf{z}}\mathsf{v})^t\widetilde{\mathsf{A}}_i\partial_{\mathsf{z}}\mathsf{v}$, that may also be rewritten $(\partial_{\mathsf{z}}\mathsf{v})^t\partial_{\mathsf{z}}\mathsf{F}_i$. This standard procedure, however, must be modified in order to take into account the apparition of singularities due to thermodynamic mechanical instabilities arising when $\partial_\rho p$ vanishes and becomes negative. To this aim, after multiplication on the left by $(\partial_{\mathsf{z}}\mathsf{v})^t$, *we add the total mass conservation equation multiplied by the instability indicator* m of Lemma 3.5 to the first equation.

The matrices $\overline{\mathsf{A}}_0$ and $\overline{\mathsf{A}}_i$ are thus taken in the form

$$\overline{\mathsf{A}}_0=(\partial_{\mathsf{w}}\mathsf{v})^t\partial_{\mathsf{w}}\mathsf{u}+\mathsf{m}\,\mathsf{e}_1\otimes\mathsf{e}_1,\qquad\qquad\overline{\mathsf{A}}_i=(\partial_{\mathsf{w}}\mathsf{v})^t\partial_{\mathsf{w}}\mathsf{F}_i+\mathsf{m}v_i\,\mathsf{e}_1\otimes\mathsf{e}_1,\quad i\in\mathcal{D}, \tag{4.12}$$

where e_i, $1\le i\le\mathsf{n}$, denotes the canonical basis of $\mathbb{R}^{\mathsf{n}}$. There is also a cancelation of singularities when rewriting $(\partial_{\mathsf{w}}\mathsf{v})^t\partial_{\mathsf{v}}\mathsf{u}\,\partial_{\mathsf{w}}\mathsf{v}$ as $(\partial_{\mathsf{w}}\mathsf{v})^t\partial_{\mathsf{w}}\mathsf{u}$ and $(\partial_{\mathsf{w}}\mathsf{v})^t\partial_{\mathsf{v}}\mathsf{F}_i\,\partial_{\mathsf{w}}\mathsf{v}$ as $(\partial_{\mathsf{w}}\mathsf{v})^t\partial_{\mathsf{w}}\mathsf{F}_i$. The right hand side is correspondingly in the form

$$\mathsf{h}=\sum_{i,j\in\mathcal{D}}(\partial_{\mathsf{w}}\mathsf{v})^t\partial_i\widehat{\mathsf{B}}^{\rm d}_{ij}\partial_j\mathsf{w}+\sum_{i,j\in\mathcal{D}}(\partial_{\mathsf{w}}\mathsf{v})^t\partial_i\widehat{\mathsf{B}}^{\rm c}_{ij}\partial_j\mathsf{w}-\mathsf{m}\rho\boldsymbol{\nabla\cdot}\boldsymbol{v}\,\mathsf{e}_1, \tag{4.13}$$

where we have used $\mathsf{F}^{\rm d}_i=-\sum_{j\in\mathcal{D}}\widehat{\mathsf{B}}^{\rm d}_{ij}\partial_j\mathsf{w}$ and $\mathsf{F}^{\rm c}_i=-\sum_{j\in\mathcal{D}}\widehat{\mathsf{B}}^{\rm c}_{ij}\partial_j\mathsf{w}$ keeping in mind that $\mathsf{B}^{\rm d}_{ij}=\widehat{\mathsf{B}}^{\rm d}_{ij}\partial_{\mathsf{u}}\mathsf{w}$ and $\mathsf{B}^{\rm c}_{ij}=\widehat{\mathsf{B}}^{\rm c}_{ij}\partial_{\mathsf{u}}\mathsf{w}$ with $\mathsf{w}=\mathsf{z}$. The second order matrices, that are initially in the form $\overline{\mathsf{B}}^{\rm d}_{ij}=(\partial_{\mathsf{w}}\mathsf{v})^t\widetilde{\mathsf{B}}^{\rm d}_{ij}\partial_{\mathsf{w}}\mathsf{v}$ and $\overline{\mathsf{B}}^{\rm c}_{ij}=(\partial_{\mathsf{w}}\mathsf{v})^t\widetilde{\mathsf{B}}^{\rm c}_{ij}\partial_{\mathsf{w}}\mathsf{v}$, are also simplified by using $\widehat{\mathsf{B}}^{\rm d}_{ij}$ and $\widehat{\mathsf{B}}^{\rm c}_{ij}$ in the form

$$\overline{\mathsf{B}}^{\rm d}_{ij}=(\partial_{\mathsf{w}}\mathsf{v})^t\widehat{\mathsf{B}}^{\rm d}_{ij},\qquad\qquad\overline{\mathsf{B}}^{\rm c}_{ij}=(\partial_{\mathsf{w}}\mathsf{v})^t\widehat{\mathsf{B}}^{\rm c}_{ij},\quad i\in\mathcal{D}, \tag{4.14}$$

We may now obtain the matrices of the normal form (4.10) using previous expressions for $\partial_{\mathsf{w}}\mathsf{v}$ and $\partial_{\mathsf{w}}\mathsf{u}$ in (3.14) and (3.9) and using the matrices $\partial_{\mathsf{w}}\mathsf{F}_i$, $\widehat{\mathsf{B}}^{\mathrm{d}}_{ij}$, and $\widehat{\mathsf{B}}^{\mathrm{c}}_{ij}$ that are easily evaluated. The matrix $\overline{\mathsf{A}}_0$ is found in the form

$$\overline{\mathsf{A}}_0 = \begin{pmatrix} \mathsf{m} + \frac{\partial_\rho p}{\rho T} & 0_{1,d_{\mathrm{s}}} & 0_{1,d_{\mathrm{s}}} & 0 \\ 0_{d_{\mathrm{s}},1} & \frac{\varkappa}{T}\boldsymbol{I} & 0_{d_{\mathrm{s}},d_{\mathrm{s}}} & 0_{d_{\mathrm{s}},1} \\ 0_{d_{\mathrm{s}},1} & 0_{d_{\mathrm{s}},d_{\mathrm{s}}} & \frac{\rho}{T}\boldsymbol{I} & 0_{d_{\mathrm{s}},1} \\ 0 & 0_{1,d_{\mathrm{s}}} & 0_{1,d_{\mathrm{s}}} & \frac{\partial_T \mathcal{E}}{T^2} \end{pmatrix}, \tag{4.15}$$

and is thus symmetric $\overline{\mathsf{A}}_0 = \overline{\mathsf{A}}_0^t$ and block diagonal $\overline{\mathsf{A}}_0 = \mathrm{diag}(\overline{\mathsf{A}}_0^{\mathrm{I},\mathrm{I}}, \overline{\mathsf{A}}_0^{\mathrm{II},\mathrm{II}})$. The matrix $\overline{\mathsf{A}}_0$ positive definite since $\partial_T\mathcal{E} > 0$ from (H_3) and $\mathsf{m} + \frac{\partial_\rho p}{\rho T} > 0$ from Lemma 3.5 and $\overline{\mathsf{A}}_0^{\mathrm{II},\mathrm{II}}$ only depends on $(\rho, \boldsymbol{v}, T)$, that is, on $(\mathsf{w}_{\mathrm{I}'}, \mathsf{w}_{\mathrm{II}})$. The corresponding quadratic form may be written as

$$\langle \overline{\mathsf{A}}_0 \mathsf{x}, \mathsf{x}\rangle = \Big(\mathsf{m} + \frac{\partial_\rho p}{\rho T}\Big)\mathsf{x}_\rho^2 + \frac{\varkappa}{T}|\mathsf{x}_{\boldsymbol{w}}|^2 + \frac{\rho}{T}|\mathsf{x}_{\boldsymbol{v}}|^2 + \frac{\partial_T\mathcal{E}}{T^2}\mathsf{x}_T^2, \tag{4.16}$$

where the vector $\mathsf{x} \in \mathbb{R}^{\mathsf{n}}$ has been decomposed in the form $\mathsf{x} = (\mathsf{x}_\rho, \mathsf{x}_{\boldsymbol{w}}, \mathsf{x}_{\boldsymbol{v}}, \mathsf{x}_T)^t$. Denoting by $\boldsymbol{\xi}$ an arbitrary vector of $\mathbb{R}^{d_{\mathrm{s}}}$, the matrix $\sum_{i\in\mathcal{D}} \xi_i \overline{\mathsf{A}}_i$ is found to be

$$\sum_{i\in\mathcal{D}} \xi_i \overline{\mathsf{A}}_i = \boldsymbol{v}\cdot\boldsymbol{\xi}\,\overline{\mathsf{A}}_0 + \frac{1}{T}\begin{pmatrix} 0 & 0_{1,d_{\mathrm{s}}} & \partial_\rho p\,\boldsymbol{\xi}^t & 0 \\ 0_{d_{\mathrm{s}},1} & 0_{d_{\mathrm{s}},d_{\mathrm{s}}} & \varkappa \boldsymbol{w}\otimes\boldsymbol{\xi} & 0_{d_{\mathrm{s}},1} \\ \partial_\rho p\,\boldsymbol{\xi} & \varkappa\boldsymbol{\xi}\otimes\boldsymbol{w} & 0_{d_{\mathrm{s}},d_{\mathrm{s}}} & (\partial_T p + \partial_T\varkappa|\boldsymbol{w}|^2)\boldsymbol{\xi} \\ 0 & 0_{1,d_{\mathrm{s}}} & (\partial_T p + \partial_T\varkappa|\boldsymbol{w}|^2)\boldsymbol{\xi}^t & 0 \end{pmatrix}, \tag{4.17}$$

and all the matrices $\overline{\mathsf{A}}_i$ are symmetric $\overline{\mathsf{A}}_i = \overline{\mathsf{A}}_i^t$ for $i \in \mathcal{D}$.

The second order flux matrices are split in the form

$$\overline{\mathsf{B}}^{\mathrm{d}}_{ij} = \mathfrak{v}\overline{\mathsf{B}}^{\mathfrak{v}}_{ij} + \eta\overline{\mathsf{B}}^{\eta}_{ij} + \lambda\overline{\mathsf{B}}^{\lambda}_{ij}, \tag{4.18}$$

and, denoting by $\boldsymbol{\xi}$ and $\boldsymbol{\zeta}$ arbitrary vectors of $\mathbb{R}^{d_{\mathrm{s}}}$, it is found that

$$\sum_{i,j\in\mathcal{D}} \xi_i\zeta_j \overline{\mathsf{B}}^{\mathfrak{v}}_{ij} = \frac{1}{T}\begin{pmatrix} 0 & 0_{1,d_{\mathrm{s}}} & 0_{1,d_{\mathrm{s}}} & 0 \\ 0_{d_{\mathrm{s}},1} & 0_{d_{\mathrm{s}},d_{\mathrm{s}}} & 0_{d_{\mathrm{s}},d_{\mathrm{s}}} & 0_{d_{\mathrm{s}},1} \\ 0_{d_{\mathrm{s}},1} & 0_{d_{\mathrm{s}},d_{\mathrm{s}}} & \boldsymbol{\xi}\otimes\boldsymbol{\zeta} & 0_{d_{\mathrm{s}},1} \\ 0 & 0_{1,d_{\mathrm{s}}} & 0_{1,d_{\mathrm{s}}} & 0 \end{pmatrix}. \tag{4.19}$$

$$\sum_{i,j\in\mathcal{D}} \xi_i\zeta_j \overline{\mathsf{B}}^{\eta}_{ij} = \frac{1}{T}\begin{pmatrix} 0 & 0_{1,d_{\mathrm{s}}} & 0_{1,d_{\mathrm{s}}} & 0 \\ 0_{d_{\mathrm{s}},1} & 0_{d_{\mathrm{s}},d_{\mathrm{s}}} & 0_{d_{\mathrm{s}},d_{\mathrm{s}}} & 0_{d_{\mathrm{s}},1} \\ 0_{d_{\mathrm{s}},1} & 0_{d_{\mathrm{s}},d_{\mathrm{s}}} & \boldsymbol{\xi}\cdot\boldsymbol{\zeta}\,\boldsymbol{I} + \boldsymbol{\zeta}\otimes\boldsymbol{\xi} - \frac{2}{d_{\mathrm{s}}}\boldsymbol{\xi}\otimes\boldsymbol{\zeta} & 0_{d_{\mathrm{s}},1} \\ 0 & 0_{1,d_{\mathrm{s}}} & 0_{1,d_{\mathrm{s}}} & 0 \end{pmatrix}. \tag{4.20}$$

$$\sum_{i,j\in\mathcal{D}} \xi_i\zeta_j \overline{\mathsf{B}}^{\lambda}_{ij} = \frac{1}{T^2}\begin{pmatrix} 0 & 0_{1,d_{\mathrm{s}}} & 0_{1,d_{\mathrm{s}}} & 0 \\ 0_{d_{\mathrm{s}},1} & 0_{d_{\mathrm{s}},d_{\mathrm{s}}} & 0_{d_{\mathrm{s}},d_{\mathrm{s}}} & 0_{d_{\mathrm{s}},1} \\ 0_{d_{\mathrm{s}},1} & 0_{d_{\mathrm{s}},d_{\mathrm{s}}} & 0_{d_{\mathrm{s}},d_{\mathrm{s}}} & 0_{d_{\mathrm{s}},1} \\ 0 & 0_{1,d_{\mathrm{s}}} & 0_{1,d_{\mathrm{s}}} & 1 \end{pmatrix}. \tag{4.21}$$

The matrices $\overline{\mathsf{B}}^{\mathrm{d}}_{ij}$, $\overline{\mathsf{B}}^{\mathfrak{v}}_{ij}$, $\overline{\mathsf{B}}^{\eta}_{ij}$, and $\overline{\mathsf{B}}^{\lambda}_{ij}$, $i, j \in \mathcal{D}$, are similar to those of compressible fluids except that there are extra lines and columns of zeros associated with the $\boldsymbol{w}$ variable [49, 33] and we have the reciprocity relations $(\overline{\mathsf{B}}^{\mathrm{d}}_{ij})^t = \overline{\mathsf{B}}^{\mathrm{d}}_{ji}$, $i, j \in \mathcal{D}$. Denoting by $\overline{\mathsf{B}}^{\mathrm{d}\,\mathrm{II},\mathrm{II}}_{ij}$ the lower right bloc of size $d_{\mathrm{s}}+1$ of $\overline{\mathsf{B}}^{\mathrm{d}}_{ij}$ and by $\overline{\mathsf{B}}^{\mathrm{d}\,\mathrm{II},\mathrm{II}}$ the matrix $\overline{\mathsf{B}}^{\mathrm{d}\,\mathrm{II},\mathrm{II}} = \sum_{i,j\in\mathcal{D}} \xi_i\xi_j \overline{\mathsf{B}}^{\mathrm{d}\,\mathrm{II},\mathrm{II}}_{ij}$ where $\boldsymbol{\xi} \in \Sigma^{d_{\mathrm{s}}-1}$ we have

$$T^2\langle \overline{\mathsf{B}}^{\mathrm{d}\,\mathrm{II},\mathrm{II}}\mathsf{x}_{\mathrm{II}}, \mathsf{x}_{\mathrm{II}}\rangle = T\Big(\mathfrak{v} + \eta\big(1 - \tfrac{2}{d_{\mathrm{s}}}\big)\Big)(\boldsymbol{\xi}\cdot\mathsf{x}_{\boldsymbol{v}})^2 + T\eta\,|\mathsf{x}_{\boldsymbol{v}}|^2 + \lambda\,\mathsf{x}_T^2, \tag{4.22}$$

for any test vector $\mathsf{x}_{\text{II}} = (\mathsf{x}_{\boldsymbol{v}}, \mathsf{x}_{\mathcal{E}})^t$ so that $\overline{\mathsf{B}}^{\text{d II,II}}$ is positive definite under assumption (H_5).

The cohesive matrices $\overline{\mathsf{B}}^{\text{c}}_{ij}$ are also found in the form

$$\sum_{i,j\in\mathcal{D}} \xi_i\zeta_j \overline{\mathsf{B}}^{\text{c}}_{ij} = \frac{\rho}{T}\begin{pmatrix} 0 & 0_{1,d_{\text{s}}} & 0_{1,d_{\text{s}}} & 0 \\ 0_{d_{\text{s}},1} & 0_{d_{\text{s}},d_{\text{s}}} & -\varkappa\boldsymbol{\zeta}\otimes\boldsymbol{\xi} & 0_{d_{\text{s}},1} \\ 0_{d_{\text{s}},1} & \varkappa\boldsymbol{\zeta}\otimes\boldsymbol{\xi} & 0_{d_{\text{s}},d_{\text{s}}} & \partial_T\varkappa\,\boldsymbol{w}\cdot\boldsymbol{\xi}\boldsymbol{\zeta} \\ 0 & 0_{1,d_{\text{s}}} & -\partial_T\varkappa\,\boldsymbol{w}\cdot\boldsymbol{\zeta}\boldsymbol{\xi}^t & 0 \end{pmatrix}. \tag{4.23}$$

The (I, I) upper left block vanishes $\overline{\mathsf{B}}^{\text{c I,I}}_{ij} = 0$ and the coupling (I, II) and (II, I) blocks $\overline{\mathsf{B}}^{\text{c I,II}}_{ij}$ and $\overline{\mathsf{B}}^{\text{c II,I}}_{ij}$ only depend on $(\rho, \boldsymbol{v}, T)$, that is, on $(\mathsf{w}_{\text{I}'}, \mathsf{w}_{\text{II}})$, so that they will have more regularity. The antisymmetric reciprocity relations $(\widehat{\mathsf{B}}^{\text{c}}_{ij})^t = -\widehat{\mathsf{B}}^{\text{c}}_{ji}$, $i, j \in \mathcal{D}$, also hold and are associated physically with the fact that capillarity does not produce entropy.

After some calculus, the hyperbolic part h_{I} of the right hand side in (4.10) is found to be

$$\mathsf{h}_{\text{I}} = \Big(-\mathsf{m}\rho\boldsymbol{\nabla}\cdot\boldsymbol{v}, -\frac{\varkappa}{T}\sum_{i\in\mathcal{D}} w_i\boldsymbol{\nabla}v_i\Big)^t, \tag{4.24}$$

where $\boldsymbol{w} = (w_1, \dots, w_{d_{\text{s}}})^t$. The derivatives of density appearing in $\mathsf{h}_{\text{I}''}$ of h_{I} have notably been written as components of $\boldsymbol{w}$ to ensure that the gradient constraint holds for linearized equations as investigated in the next section. On the other hand, the parabolic part is in the general form

$$\mathsf{h}_{\text{II}} = \mathsf{h}_{\text{II}}(\mathsf{w}, \boldsymbol{\nabla}\mathsf{w}), \tag{4.25}$$

and depends on the complete gradient $\boldsymbol{\nabla}\mathsf{w}$. The term $\mathsf{h}_{\text{I}'}$ is linear in the gradient $\boldsymbol{\nabla}\mathsf{w}_{\text{II}}$ and arises from the stabilization procedure wheras $\mathsf{h}_{\text{I}''}$ has been rewritten formally as linear in $\boldsymbol{\nabla}\mathsf{w}_{\text{II}}$ for convenience by using $\boldsymbol{w} = \boldsymbol{\nabla}\rho$ but is intrinsically quadratic. The term h_{II} is quadratic in terms of the gradient $\boldsymbol{\nabla}\mathsf{w}$ with coefficients depending on w that have at least regularity $C^{\gamma-2}$. We finally note that the matrices $\overline{\mathsf{A}}_0$, $\overline{\mathsf{A}}_i$, $i \in \mathcal{D}$, $\overline{\mathsf{B}}^{\text{d}}_{ij}$ and $\overline{\mathsf{B}}^{\text{c}}_{ij}$, $i, j \in \mathcal{D}$, as well as the coefficients in front of the gradients of the right hand sides h_{I} and h_{II} have at least regularity $C^{\gamma-2}$ over $\mathcal{O}_{\mathsf{w}}$.

The general structure of the system of partial differential equations (3.17) may then be discussed with the normal form (4.10). The variable $\mathsf{w}_{\text{I}} = (\rho, \boldsymbol{w})^t$ with $\boldsymbol{w} = \boldsymbol{\nabla}\rho$ is first an hyperbolic variable, i.e., when w_{II} is given, w_{I} is governed by a symmetric hyperbolic system of conservation laws. Similarly, $\mathsf{w}_{\text{II}} = (\boldsymbol{v}, T)^t$ is a parabolic variable, i.e., when w_{I} is given, w_{II} is governed by a strongly parabolic system of conservation laws. Incidentally, for such symmetric system, Petrovsky parabolicity is equivalent to strong parabolicity [40]. As a consequence, the variable ρ is *not* a parabolic variable but rather the couple $(\rho, \boldsymbol{\nabla}\rho)^t$ is an hyperbolic variable. We have also obtained a new type of composite hyperbolic-parabolic system involving matrices $\overline{\mathsf{B}}^{\text{d}}_{ij}$ with symmetry properties and matrices $\overline{\mathsf{B}}^{\text{c}}_{ij}$ with antisymmetry properties. The matrices $\overline{\mathsf{B}}^{\text{c}}_{ij}$ introduce extra couplings between the hyperbolic and parabolic variables and these coupling blocks $\overline{\mathsf{B}}^{\text{c I,II}}_{ij}$ and $\overline{\mathsf{B}}^{\text{c II,I}}_{ij}$ will require more regularity. Last but not least, it will be important in practice to maintain the constraint $\boldsymbol{\nabla}\rho = \boldsymbol{w}$ that is discussed in the next section.

4.3 Gradient constraint and linearization

We investigate in this section how the natural gradient constraint $\boldsymbol{w} = \boldsymbol{\nabla}\rho$ is a consequence of the governing equations (4.10) with the hyperbolic part of the right hand side in the form (4.24). This *natural gradient constraint* will be a key point in order to establish that the density ρ *has more regularity*. The nonlinear equations in normal form are first considered and then the more complex situation of linearized equations.

Considering the nonlinear equations in normal form (4.10) we note that, after some calculus, the first equation governing the density ρ is in the form

$$(\mathsf{m} + \frac{1}{\rho T}\partial_\rho p)\big(\partial_t\rho + \boldsymbol{\nabla}\cdot(\rho\boldsymbol{v})\big) = 0.$$

Since $\mathsf{m} + \frac{1}{\rho T}\partial_\rho p > 0$ from Lemma 3.5 we deduce that $\partial_t\rho + \boldsymbol{\nabla}\cdot(\rho\boldsymbol{v}) = 0$ and (2.5) has been recovered. Note the importance of the stabilizing terms in (4.12) involving the instability indicator m of Lemma 3.5.

Taking the gradient of this relation we obtain that $\partial_t(\boldsymbol{\nabla}\rho) + \boldsymbol{\nabla}\cdot\big(\boldsymbol{\nabla}\rho\otimes\boldsymbol{v} + \rho(\boldsymbol{\nabla}\boldsymbol{v})^t\big) = 0$. On the other hand, after some algebra, the second equation of (4.10) governing $\boldsymbol{w}$ reads

$$\frac{\varkappa}{T}\big(\partial_t\boldsymbol{w} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\boldsymbol{w} + \boldsymbol{w}\boldsymbol{\nabla}\cdot\boldsymbol{v} + \rho\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{v}) + (\boldsymbol{\nabla}\boldsymbol{v})^t\cdot\boldsymbol{w}\big) = 0.$$

Dividing by $\varkappa/T$ and subtracting the $\boldsymbol{\nabla}\rho$ equation then yields

$$\partial_t(\boldsymbol{w} - \boldsymbol{\nabla}\rho) + \boldsymbol{v}\cdot\boldsymbol{\nabla}(\boldsymbol{w} - \boldsymbol{\nabla}\rho) + (\boldsymbol{w} - \boldsymbol{\nabla}\rho)\boldsymbol{\nabla}\cdot\boldsymbol{v} + (\boldsymbol{\nabla}\boldsymbol{v})^t\cdot(\boldsymbol{w} - \boldsymbol{\nabla}\rho) = 0, \tag{4.26}$$

and this Friedrichs system with respect to $\boldsymbol{w} - \boldsymbol{\nabla}\rho$ implies that $\boldsymbol{w} - \boldsymbol{\nabla}\rho = 0$ provided that the solution w is smooth enough, that the constraint $\boldsymbol{w}_0 - \boldsymbol{\nabla}\rho_0 = 0$ holds at the initial time $t = 0$, with $\mathsf{w}(0,\cdot)$ written in the form $\mathsf{w}(0,\boldsymbol{x}) = \big(\rho_0(\boldsymbol{x}), \boldsymbol{w}_0(\boldsymbol{x}), \boldsymbol{v}_0(\boldsymbol{x}), T_0(\boldsymbol{x})\big)^t$, and that the state at infinity $\mathsf{w}^\star$ is naturally such that $\boldsymbol{w}^\star = 0$ [6].

In order to establish existence results for the nonlinear system (4.10), we will need to use linearized equations. In is then mandatory to ensure that the natural gradient constraint $\boldsymbol{w} = \boldsymbol{\nabla}\rho$ *also holds for solutions of some properly selected linearized equations*. To this aim, the linearized unknown $\widetilde{\mathsf{w}}$ is decomposed in the form $\widetilde{\mathsf{w}} = (\widetilde{\mathsf{w}}_{\mathrm{I}}, \widetilde{\mathsf{w}}_{\mathrm{II}})^t = \big(\widetilde{\rho}, \widetilde{\boldsymbol{w}}, \widetilde{\boldsymbol{v}}, \widetilde{T}\big)^t$ and the linearized system taken in the form

$$\overline{\mathsf{A}}_0(\mathsf{w})\partial_t\widetilde{\mathsf{w}} + \sum_{i\in\mathcal{D}}\overline{\mathsf{A}}_i(\mathsf{w})\partial_i\widetilde{\mathsf{w}} - \sum_{i,j\in\mathcal{D}}\overline{\mathsf{B}}^{\mathrm{d}}_{ij}(\mathsf{w})\partial_i\partial_j\widetilde{\mathsf{w}} - \sum_{i,j\in\mathcal{D}}\overline{\mathsf{B}}^{\mathrm{c}}_{ij}(\mathsf{w})\partial_i\partial_j\widetilde{\mathsf{w}} = \big(\widetilde{\mathsf{h}}_{\mathrm{I}}, \mathsf{h}_{\mathrm{II}}(\mathsf{w}, \boldsymbol{\nabla}\mathsf{w})\big)^t, \tag{4.27}$$

where the hyperbolic part of the right hand side $\widetilde{\mathsf{h}}_{\mathrm{I}}$ is taken to be

$$\widetilde{\mathsf{h}}_{\mathrm{I}} = \Big(-\mathsf{m}\,\rho\,\boldsymbol{\nabla}\cdot\widetilde{\boldsymbol{v}}, -\sum_{i\in\mathcal{D}}\frac{\varkappa}{T}\widetilde{w}_i\boldsymbol{\nabla}v_i\Big)^t. \tag{4.28}$$

The linearized equations may thus be rewritten as

$$\overline{\mathsf{A}}_0(\mathsf{w})\partial_t\widetilde{\mathsf{w}} + \sum_{i\in\mathcal{D}}\overline{\mathsf{A}}'_i(\mathsf{w})\partial_i\widetilde{\mathsf{w}} - \sum_{i,j\in\mathcal{D}}\overline{\mathsf{B}}^{\mathrm{d}}_{ij}(\mathsf{w})\partial_i\partial_j\widetilde{\mathsf{w}} - \sum_{i,j\in\mathcal{D}}\overline{\mathsf{B}}^{\mathrm{c}}_{ij}(\mathsf{w})\partial_i\partial_j\widetilde{\mathsf{w}} + \overline{\mathsf{L}}(\mathsf{w}, \boldsymbol{\nabla}\mathsf{w}_{\mathrm{II}})\widetilde{\mathsf{w}} = \mathsf{h}'(\mathsf{w}, \boldsymbol{\nabla}\mathsf{w}), \tag{4.29}$$

where

$$\mathsf{h}' = \big(0_{\mathsf{n}_{\mathrm{I}},1}, \mathsf{h}_{\mathrm{II}}(\mathsf{w}, \boldsymbol{\nabla}\mathsf{w})\big)^t, \tag{4.30}$$

$$\overline{\mathsf{A}}'_i(\mathsf{w}) = \overline{\mathsf{A}}_i(\mathsf{w}) + \mathsf{m}\rho\mathsf{e}_1\otimes\mathsf{e}_{d+1+i}, \qquad \overline{\mathsf{L}}(\mathsf{w}, \boldsymbol{\nabla}\mathsf{w}_{\mathrm{II}}) = \sum_{i\in\mathcal{D}}\frac{\varkappa}{T}(0, \boldsymbol{\nabla}v_i, 0_{1,\mathsf{n}_{\mathrm{I}}}, 0)^t\otimes\mathsf{e}_{i+1}, \tag{4.31}$$

with $\mathsf{e}_1, \ldots, \mathsf{e}_{\mathsf{n}}$ denoting the canonical basis of $\mathbb{R}^{\mathsf{n}} = \mathbb{R}^{2d_{\mathrm{s}}+2}$. We have in particular that

$$\sum_{i\in\mathcal{D}}\big(\overline{\mathsf{A}}'_i(\mathsf{w}) - \overline{\mathsf{A}}_i(\mathsf{w})\big)\partial_i\widetilde{\mathsf{w}} = \big(\mathsf{m}\,\rho\,\boldsymbol{\nabla}\cdot\widetilde{\boldsymbol{v}}, 0_{d_{\mathrm{s}},1}, 0_{d_{\mathrm{s}},1}, 0\big)^t, \qquad \overline{\mathsf{L}}(\mathsf{w}, \boldsymbol{\nabla}\mathsf{w}_{\mathrm{II}})\widetilde{\mathsf{w}} = \Big(0, \sum_{i\in\mathcal{D}}\frac{\varkappa}{T}\widetilde{w}_i\boldsymbol{\nabla}v_i, 0_{d_{\mathrm{s}},1}, 0\Big)^t, \tag{4.32}$$

and $\overline{\mathsf{L}}$ is a block diagonal matrix $\overline{\mathsf{L}} = \mathrm{diag}(\,\overline{\mathsf{L}}^{\mathrm{I},\mathrm{I}}, \overline{\mathsf{L}}^{\mathrm{II},\mathrm{II}}\,)$ with $\overline{\mathsf{L}}^{\mathrm{II},\mathrm{II}} = \mathrm{diag}(0_{d_{\mathrm{s}},d_{\mathrm{s}}}, 0)$ and

$$\overline{\mathsf{L}}^{\mathrm{I},\mathrm{I}} = \mathrm{diag}\Big(0, \sum_{i\in\mathcal{D}}\frac{\varkappa}{T}\boldsymbol{\nabla}v_i\otimes\mathfrak{b}_i\Big).$$

A consequence of (4.31) is notably that the hyperbolic blocks $\overline{\mathsf{A}}'^{\mathrm{I},\mathrm{I}}_i$ and $\overline{\mathsf{A}}^{\mathrm{I},\mathrm{I}}_i$ coincide.

We then deduce that the first equation of (4.29) governing $\widetilde{\rho}$ is in the form

$$(\mathsf{m} + \frac{1}{\rho T}\partial_\rho p)(\partial_t\widetilde{\rho} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\widetilde{\rho} + \rho\boldsymbol{\nabla}\cdot\widetilde{\boldsymbol{v}}) = 0,$$

so that $\partial_t\widetilde{\rho} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\widetilde{\rho} + \rho\boldsymbol{\nabla}\cdot\widetilde{\boldsymbol{v}} = 0$ and the equation governing $\boldsymbol{\nabla}\widetilde{\rho}$ is next obtained by applying the $\boldsymbol{\nabla}$ operator. The second equation of (4.29) governing $\widetilde{\boldsymbol{w}}$, after simplification by the $\varkappa/T$ factor, is obtained in the form

$$\partial_t\widetilde{\boldsymbol{w}} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\widetilde{\boldsymbol{w}} + \boldsymbol{w}\boldsymbol{\nabla}\cdot\widetilde{\boldsymbol{v}} + \rho\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\widetilde{\boldsymbol{v}}) + \boldsymbol{\nabla}\boldsymbol{v}^t\cdot\widetilde{\boldsymbol{w}} = 0.$$

Subtracting the $\boldsymbol{\nabla}\widetilde{\rho}$ equation from the $\widetilde{\boldsymbol{w}}$ equation then yields

$$\partial_t(\widetilde{\boldsymbol{w}} - \boldsymbol{\nabla}\widetilde{\rho}) + \boldsymbol{v}\cdot\boldsymbol{\nabla}(\widetilde{\boldsymbol{w}} - \boldsymbol{\nabla}\widetilde{\rho}) + (\boldsymbol{w} - \boldsymbol{\nabla}\rho)\,\boldsymbol{\nabla}\cdot\widetilde{\boldsymbol{v}} + \boldsymbol{\nabla}\boldsymbol{v}^t\cdot(\widetilde{\boldsymbol{w}} - \boldsymbol{\nabla}\widetilde{\rho}) = 0, \tag{4.33}$$

and this equation was the motivation for choosing (4.28). The Friedrichs system (4.33) with respect to $\widetilde{\boldsymbol{w}} - \boldsymbol{\nabla}\widetilde{\rho}$ thus implies that $\widetilde{\boldsymbol{w}} - \boldsymbol{\nabla}\widetilde{\rho} = 0$ provided that the constraint $\boldsymbol{w} - \boldsymbol{\nabla}\rho = 0$ holds, that the solutions w and $\widetilde{\mathsf{w}}$ are smooth enough, that the constraint $\widetilde{\boldsymbol{w}}_0 - \boldsymbol{\nabla}\widetilde{\rho}_0 = 0$ holds at the initial time $t = 0$, and that the state at infinity $\widetilde{\mathsf{w}}^\star$ is naturally such that $\widetilde{\boldsymbol{w}}^\star = 0$ [6]. The natural gradient constraint may thus be transmitted from w to $\widetilde{\mathsf{w}}$ by using suitable linearizations like (4.29). The right hand sides are also naturally related by $\sum_{i\in\mathcal{D}}\big(\overline{\mathsf{A}}_i'(\mathsf{w}) - \overline{\mathsf{A}}_i(\mathsf{w})\big)\partial_i\mathsf{w} + \overline{\mathsf{L}}(\mathsf{w}, \boldsymbol{\nabla}\mathsf{w}_{\rm II})\mathsf{w} + \mathsf{h}(\mathsf{w}, \boldsymbol{\nabla}\mathsf{w}) = \mathsf{h}'(\mathsf{w}, \boldsymbol{\nabla}\mathsf{w})$ for any regular function w.

5 Linearized equations

In order to establish the existence of strong solutions to the system in normal form (4.10), we need to investigate linearized equations in the form (4.29). A priori estimates and existence theorems are obtained in this section for abstract linearized systems that cover the particular situation of diffuse interface fluids (4.29).

5.1 Linearized estimates

An abstract hyperbolic-parabolic linearized system is considered in the form

$$\overline{\mathsf{A}}_0(\mathsf{w})\partial_t\widetilde{\mathsf{w}} + \sum_{i\in\mathcal{D}}\overline{\mathsf{A}}_i'(\mathsf{w})\partial_i\widetilde{\mathsf{w}} - \sum_{i,j\in\mathcal{D}}\overline{\mathsf{B}}^{\rm d}_{ij}(\mathsf{w})\partial_i\partial_j\widetilde{\mathsf{w}} - \sum_{i,j\in\mathcal{D}}\overline{\mathsf{B}}^{\rm c}_{ij}(\mathsf{w})\partial_i\partial_j\widetilde{\mathsf{w}} + \overline{\mathsf{L}}(\mathsf{w}, \nabla\mathsf{w}_{\rm r})\widetilde{\mathsf{w}} = \mathsf{f} + \mathsf{g}, \tag{5.1}$$

where t denotes the time variable, ∂_i the derivative in the ith spatial direction, w a given function, $\mathsf{w}_{\rm r}$ a subset of w components that have more regularity, and $\widetilde{\mathsf{w}}$ the linearized unknown. The variables w and $\widetilde{\mathsf{w}}$ are assumed to have $\mathsf{n} \geq 1$ components and $\mathbb{R}^{\mathsf{n}}$ is decomposed into $\mathbb{R}^{\mathsf{n}} = \mathbb{R}^{\mathsf{n}_{\rm I}} \times \mathbb{R}^{\mathsf{n}_{\rm II}}$ with $\mathsf{n} = \mathsf{n}_{\rm I} + \mathsf{n}_{\rm II}$ and the subvariable $\mathsf{w}_{\rm r}$ is composed of $\mathsf{n}_{\rm r}$ components of w with $1 \leq \mathsf{n}_{\rm r} \leq \mathsf{n}$. The bloc structure induced by the partitioning of $\mathbb{R}^{\mathsf{n}}$ is used in the following so that $\widetilde{\mathsf{w}}$ is decomposed into $\widetilde{\mathsf{w}} = (\widetilde{\mathsf{w}}_{\rm I}, \widetilde{\mathsf{w}}_{\rm II})^t$ with $\widetilde{\mathsf{w}}_{\rm I} \in \mathbb{R}^{\mathsf{n}_{\rm I}}$ and $\widetilde{\mathsf{w}}_{\rm II} \in \mathbb{R}^{\mathsf{n}_{\rm II}}$. The integer l_0 denotes the minimum integer for an embedding of the Sobolev space $H^{l_0}(\mathbb{R}^{d_{\rm s}})$ into the space of bounded continuous functions $l_0 = [d_{\rm s}/2] + 1$ and l is an integer such that $l \geq l_0 + 2$.

The following assumptions are made concerning the linearized equations (5.1). The matrix $\overline{\mathsf{A}}_0$ is symmetric, positive definite, block diagonal $\overline{\mathsf{A}}_0 = \mathrm{diag}(\overline{\mathsf{A}}_0^{\rm I,I}, \overline{\mathsf{A}}_0^{\rm II,II})$, and the block $\overline{\mathsf{A}}_0^{\rm II,II}$ only depends on the subvariable $\mathsf{w}_{\rm r}$. The matrices $\overline{\mathsf{A}}_i'(\mathsf{w}) \in \mathbb{R}^{\mathsf{n},\mathsf{n}}$, $i \in \mathcal{D}$, have their hyperbolic blocks $\overline{\mathsf{A}}_i'^{\rm I,I}$ symmetric, the dissipation matrices are such that $(\overline{\mathsf{B}}^{\rm d}_{ij})^t = \overline{\mathsf{B}}^{\rm d}_{ji}$, $i, j \in \mathcal{D}$, have nonzero components only into the right lower $\overline{\mathsf{B}}^{\rm d\,II,II}_{ij}$ blocks, and $\overline{\mathsf{B}}^{\rm d\,II,II} = \sum_{i,j\in\mathcal{D}}\overline{\mathsf{B}}^{\rm d\,II,II}_{ij}\xi_i\xi_j$ is positive definite for $\boldsymbol{\xi} \in \Sigma^{d_{\rm s}-1}$ where $\Sigma^{d_{\rm s}-1}$ is the sphere in $d_{\rm s}$ dimension. The matrices $\overline{\mathsf{B}}^{\rm c}_{ij}$ are such that $(\overline{\mathsf{B}}^{\rm c}_{ij})^t = -\overline{\mathsf{B}}^{\rm c}_{ji}$, the blocks $\overline{\mathsf{B}}^{\rm c,I,I}_{ij}$ vanish $\overline{\mathsf{B}}^{\rm c,I,I}_{ij} = 0$, and the blocks $\overline{\mathsf{B}}^{\rm c,I,II}_{ij}$ and $\overline{\mathsf{B}}^{\rm c,II,I}_{ij}$ only depend on the subvariable $\mathsf{w}_{\rm r}$. The matrix $\overline{\mathsf{L}} = \overline{\mathsf{L}}(\mathsf{w}, \mathfrak{p}_{\rm r})$ is block diagonal $\overline{\mathsf{L}} = \mathrm{diag}(\overline{\mathsf{L}}^{\rm I,I}, \overline{\mathsf{L}}^{\rm II,II})$ and is a linear function of $\mathfrak{p}_{\rm r} \in \mathbb{R}^{\mathsf{n}_{\rm r},d_{\rm s}}$ so that $\overline{\mathsf{L}}^{\rm I,I} = \mathfrak{L}^{\rm I,I}(\mathsf{w})\mathfrak{p}_{\rm r}$ and $\overline{\mathsf{L}}^{\rm II,II} = \mathfrak{L}^{\rm II,II}(\mathsf{w})\mathfrak{p}_{\rm r}$ where $\mathfrak{L}^{\rm I,I} \in \mathbb{R}^{\mathsf{n}_{\rm I},\mathsf{n}_{\rm I},\mathsf{n}_{\rm r},d_{\rm s}}$ and $\mathfrak{L}^{\rm II,II} \in \mathbb{R}^{\mathsf{n}_{\rm II},\mathsf{n}_{\rm II},\mathsf{n}_{\rm r},d_{\rm s}}$ depend on w. The system coefficients $\overline{\mathsf{A}}_0$, $\overline{\mathsf{A}}_i'$ $\overline{\mathsf{B}}^{\rm d}_{ij}$, $\overline{\mathsf{B}}^{\rm c}_{ij}$, $i, j \in \mathcal{D}$, $\mathfrak{L}^{\rm I,I}$, and $\mathfrak{L}^{\rm II,II}$ are smooth functions of $\mathsf{w} \subset \mathbb{R}^{\mathsf{n}}$ defined over an open set $\mathcal{O}_{\mathsf{w}}$ with at least C^{l+2} regularity. The right hand sides f and g as well as w are given functions of time and space defined over $[0, \bar{\tau}] \times \mathbb{R}^{d_{\rm s}}$ for some given positive time $\bar{\tau} > 0$.

The norm in the Sobolev space $H^l = H^l(\mathbb{R}^d)$ is denoted by $|\bullet|_l$ and otherwise by $|\bullet|_A$ in the functional space A. Similarly, $\langle\ ,\ \rangle$ denotes the scalar product in $\mathbb{R}^{\mathsf{n}}$, $|\bullet|$ the Euclidean norm in $\mathbb{R}$ or $\mathbb{R}^{\mathsf{n}}$, the Frobenius norm in $\mathbb{R}^{\mathsf{n},\mathsf{n}}$, and the Euclidean distance between $\mathfrak{w} \in \mathcal{O}_{\mathsf{w}}$ and the boundary $\partial\mathcal{O}_{\mathsf{w}}$ is denoted by $\mathrm{dist}(\mathfrak{w}, \partial\mathcal{O}_{\mathsf{w}})$. If $\alpha = (\alpha_1, \ldots, \alpha_{d_{\rm s}}) \in \mathbb{N}^{d_{\rm s}}$ is a multiindex, ∂^α denotes the differential operator $\partial_1^{\alpha_1}\cdots\partial_{d_{\rm s}}^{\alpha_{d_{\rm s}}}$ and $|\alpha|$ the order $|\alpha| = \alpha_1 + \cdots + \alpha_{d_{\rm s}}$. The square of $k^{\rm th}$ derivatives of scalar functions ϕ, like T, ρ, or v_i, $1 \leq i \leq d_{\rm s}$, is defined by

$$|\partial^k\phi|^2 = \sum_{|\alpha|=k}\frac{|\alpha|!}{\alpha!}(\partial^\alpha\phi)^2 = \sum_{1\leq i_1,\ldots,i_k\leq d_{\rm s}}(\partial_{i_1}\cdots\partial_{i_k}\phi)^2, \tag{5.2}$$

where $|\alpha|!/\alpha!$ are the multinomial coefficients and similarly for a vector function like v the norm $|\partial^k v|^2$ stands for $|\partial^k v|^2 = \sum_{1\leq i\leq d_{\rm s}}|\partial^k v_i|^2$. Finally, for any map $\phi : [0, \bar{\tau}]\times\mathbb{R}^{d_{\rm s}} \mapsto \mathbb{R}^{\mathsf{n}}$, where $\bar{\tau} > 0$ is positive, $\phi(\tau)$ denotes the partial map $\boldsymbol{x} \mapsto \phi(\tau, \boldsymbol{x})$ from $\mathbb{R}^{d_{\rm s}}$ to $\mathbb{R}^{\mathsf{n}}$ and for $\tau \in [0, \bar{\tau}]$.

Denoting by $\mathsf{w}^\star \in \mathcal{O}_\mathsf{w}$ a constant state in the w variable, it is assumed that w is such that

$$\begin{cases} \mathsf{w}_{\rm I} - \mathsf{w}_{\rm I}^\star \in C^0\big([0,\bar\tau],H^l\big) \cap C^1\big([0,\bar\tau],H^{l-2}\big), \\ \mathsf{w}_{\rm II} - \mathsf{w}_{\rm II}^\star \in C^0\big([0,\bar\tau],H^l\big) \cap C^1\big([0,\bar\tau],H^{l-2}\big) \cap L^2\big((0,\bar\tau),H^{l+1}\big), \end{cases} \tag{5.3}$$

and the quantities M, $M_{\rm t}$ and $M_{\rm r}$ are defined by

$$M^2 = \sup_{0\le\tau\le\bar\tau} |\mathsf{w}(\tau)-\mathsf{w}^\star|_l^2, \qquad M_{\rm t}^2 = \int_0^{\bar\tau} |\partial_t \mathsf{w}(\tau)|_{l-2}^2 \, d\tau, \qquad M_{\rm r}^2 = \int_0^{\bar\tau} |\nabla \mathsf{w}_{\rm r}(\tau)|_l^2 \, d\tau. \tag{5.4}$$

The quantities M and $M_{\rm t}$ are traditionally used to estimate solutions of hyperbolic-parabolic linearized systems. The nonstandard use of $M_{\rm r}$ is required due to the strong couplings arising from the antisymmetric blocks $\overline{\mathsf{B}}^{\rm c,I,II}(\mathsf{w}_{\rm r})_{ij}$ and $\overline{\mathsf{B}}^{\rm c,II,I}(\mathsf{w}_{\rm r})_{ij}$ and from the matrix $\overline{\mathsf{L}} = \overline{\mathsf{L}}(\mathsf{w},\nabla\mathsf{w}_{\rm r})$. It is assumed that $\mathcal{O}_0$ is an open set such that $\mathcal{O}_0 \subset \overline{\mathcal{O}}_0 \subset \mathcal{O}_\mathsf{w}$, that a_1 is such that $0 < a_1 < \operatorname{dist}(\overline{\mathcal{O}}_0, \partial\mathcal{O}_\mathsf{w})$, and the open set $\mathcal{O}_1$ is defined by

$$\mathcal{O}_1 = \{\, \mathsf{w} \in \mathcal{O}_\mathsf{w};\ \operatorname{dist}(\mathsf{w},\overline{\mathcal{O}}_0) < a_1 \,\}. \tag{5.5}$$

It is further assumed that $\mathsf{w}_0(\boldsymbol{x}) = \mathsf{w}(0,\boldsymbol{x}) \in \mathcal{O}_0$, and that $\mathsf{w}(t,\boldsymbol{x}) \in \mathcal{O}_1$, for $(t,\boldsymbol{x}) \in [0,\bar\tau]\times\mathbb{R}^{d_{\rm s}}$ so that the values of w are controlled with the open set $\mathcal{O}_1$.

Let now l' be such that $1 \le l' \le l$ and assume that the right hand sides $\mathsf{f} \in \mathbb{R}^{\mathsf{n}}$ and $\mathsf{g} \in \mathbb{R}^{\mathsf{n}}$ are such that

$$\mathsf{f} \in C^0\big([0,\bar\tau],H^{l'-1}\big) \cap L^1\big((0,\bar\tau),H^{l'}\big), \tag{5.6}$$

$$\mathsf{g} \in C^0\big([0,\bar\tau],H^{l'-1}\big), \qquad \mathsf{g}_{\rm I} = 0. \tag{5.7}$$

Denoting by $\widetilde{\mathsf{w}}^\star$ a constant state in $\mathbb{R}^{\mathsf{n}}$ such that $\overline{\mathsf{L}}(\mathsf{w},\mathfrak{p}_{\rm r})\widetilde{\mathsf{w}}^\star = 0$ for any $\mathsf{w} \in \mathcal{O}_\mathsf{w}$ and $\mathfrak{p}_{\rm r} \in \mathbb{R}^{\mathsf{n}_{\rm r},d_{\rm s}}$, a priori estimates of $\widetilde{\mathsf{w}}-\widetilde{\mathsf{w}}^\star$ are obtained in the following theorem for solutions $\widetilde{\mathsf{w}}$ to the linearized equations (5.1).

Theorem 5.1. *Assume that the solution $\widetilde{\mathsf{w}}$ of the linearized system* (5.1) *is such that*

$$\begin{aligned} &\widetilde{\mathsf{w}}_{\rm I} - \widetilde{\mathsf{w}}_{\rm I}^\star \in C^0\big([0,\bar\tau],H^{l'}\big) \cap C^1\big([0,\bar\tau],H^{l'-2}\big), \\ &\widetilde{\mathsf{w}}_{\rm II} - \widetilde{\mathsf{w}}_{\rm II}^\star \in C^0\big([0,\bar\tau],H^{l'}\big) \cap C^1\big([0,\bar\tau],H^{l'-2}\big) \cap L^2\big((0,\bar\tau),H^{l'+1}\big), \end{aligned} \tag{5.8}$$

where $1 \le l' \le l$ and denote by $\widetilde{\mathsf{w}}_0$ the initial state $\widetilde{\mathsf{w}}_0(\boldsymbol{x}) = \widetilde{\mathsf{w}}(0,\boldsymbol{x})$. Then there exists constants $\mathsf{c}_1(\mathcal{O}_1) \ge 1$ and $\mathsf{c}_2(\mathcal{O}_1,M) \ge 1$, with $\mathsf{c}_2(\mathcal{O}_1,M)$ increasing with M, such that for any $t \in [0,\bar\tau]$

$$\begin{aligned} \sup_{0\le\tau\le t} |\widetilde{\mathsf{w}}(\tau)-\widetilde{\mathsf{w}}^\star|_{l'}^2 + \int_0^t |\widetilde{\mathsf{w}}_{\rm II}(\tau)-\widetilde{\mathsf{w}}_{\rm II}^\star|_{l'+1}^2 \, d\tau \le\ & \mathsf{c}_1^2 \exp\big(\mathsf{c}_2\big(t + M_{\rm t}\sqrt{t} + M_{\rm r}\sqrt{t}\,\big)\big)\Big(|\widetilde{\mathsf{w}}_0-\widetilde{\mathsf{w}}^\star|_{l'}^2 \\ &+ \mathsf{c}_2\Big\{\int_0^t |\mathsf{f}|_{l'}\,d\tau\Big\}^2 + \mathsf{c}_2\int_0^t |\mathsf{g}_{\rm II}|_{l'-1}^2\,d\tau\Big), \end{aligned} \tag{5.9}$$

These estimates differ from classical estimates for hyperbolic-parabolic linearized systems in several points. The Sobolev order l associated with w is first larger than l_0+2, rather than l_0+1, since the time derivative $\partial_t\mathsf{w}_{\rm II}$ is only in H^{l-2} due to the coupling blocks $\overline{\mathsf{B}}_{ij}^{\rm c,II,I}$ and the apparition of second derivatives of the hyperbolic variable. The extra coupling terms $\overline{\mathsf{B}}^{\rm c,I,II}(\mathsf{w}_{\rm r})_{ij}$ and $\overline{\mathsf{B}}^{\rm c,II,I}(\mathsf{w}_{\rm r})_{ij}$ also require more regularity by solely depending on the subvariable $\mathsf{w}_{\rm r}$. There is also a new contribution $M_{\rm r}\sqrt{t}$ in the exponential term of the right hand side of (5.9).

Proof. The proof of the estimates (5.9) is divided in several steps, i.e., preliminaries, zeroth order estimates and l'th order estimates.

Step 0. Preliminaries. In order to alleviate notation in the proof $\delta\widetilde{\mathsf{w}}$ denotes for short $\delta\widetilde{\mathsf{w}} = \widetilde{\mathsf{w}}-\widetilde{\mathsf{w}}^\star$ and since $\overline{\mathsf{L}}(\mathsf{w},\mathfrak{p}_{\rm r})\widetilde{\mathsf{w}}^\star = 0$ for $\mathsf{w} \in \mathcal{O}_\mathsf{w}$ and $\mathfrak{p}_{\rm r} \in \mathbb{R}^{\mathsf{n}_{\rm r},d_{\rm s}}$ we may replace $\widetilde{\mathsf{w}}$ by $\delta\widetilde{\mathsf{w}}$ in the linearized equations. In the following c_0 denotes a generic constant independent of both $\mathcal{O}_1$ and M, $\delta_1 = \delta(\mathcal{O}_1) \le 1$ a generic small constant only depending on $\mathcal{O}_1$, $\mathsf{c}_1 = \mathsf{c}_1(\mathcal{O}_1) \ge 1$ a generic large constant only depending on $\mathcal{O}_1$, and $\mathsf{c}_2 = \mathsf{c}_2(\mathcal{O}_1,M) \ge 1$ a generic large constant depending on $\mathcal{O}_1$ and M. The various occurrences of these constants may be distinguished and the minimum of all δ_1 and the maxima of all c_1 and c_2 may be taken at the end of the proof so that only single constants ultimately remain. The dependence on $d_{\rm s}$, l, n, as well as on the system coefficients of these estimating constants is left implicit. For $k \ge 0$ and $\phi \in H^k$ the norm $E_k^2(\phi)$ is defined by

$$E_k^2(\phi) = \sum_{0\le|\alpha|\le k} \frac{|\alpha|!}{\alpha!} \int_{\mathbb{R}^{d_{\rm s}}} \big\langle \overline{\mathsf{A}}_0(\mathsf{w})\partial^\alpha\phi, \partial^\alpha\phi\big\rangle \, d\boldsymbol{x}. \tag{5.10}$$

In order to establish the a priori estimates, it is sufficient to consider the situation where $\widetilde{\mathsf{w}}$ is smooth since one may use mollifiers [48].

We will use the classical nonlinear estimate

$$|f(\phi)-f(0)|_k \le \mathsf{c}_0\|f\|_{\mathcal{C}^k(\mathcal{O}_\phi)}(1+\|\phi\|_{L^\infty})^{k-1}\,|\phi|_k, \qquad k\ge 1, \tag{5.11}$$

valid for $\phi\in H^k(\mathbb{R}^{d_{\mathrm{s}}})$, $\mathcal{O}_\phi$ an open ball containing the range of ϕ, and f a C^k function over $\mathcal{O}_\phi$. For any $u\in H^{\overline{l}}$ and $v\in H^k$ with $0\le k\le\overline{l}$ and $\overline{l}\ge l_0=[d_{\mathrm{s}}/2]+1$ we also have

$$|uv|_k^2 \le \mathsf{c}_0|u|_{\overline{l}}^2|v|_k^2, \qquad 0\le k\le\overline{l}. \tag{5.12}$$

We further have the Sobolev type inequality valid for any $\overline{l}\ge l_0$

$$\|\phi\|_{L^\infty}\le \mathsf{c}_0|\phi|_{\overline{l}}. \tag{5.13}$$

Denoting by $[\partial^\alpha,u]v=\partial^\alpha(uv)-u\partial^\alpha v$ the commutator between ∂^α and u, and assuming that $1\le l'\le\overline{l}$ and $\overline{l}\ge l_0+1$ we have the commutator estimate

$$\sum_{0\le|\alpha|\le l'}\big|[\partial^\alpha,u]v\big|_0\le \mathsf{c}_0|\nabla u|_{\overline{l}-1}|v|_{l'-1}, \tag{5.14}$$

valid for any u and v with $\nabla u\in H^{l-1}$ and $v\in H^{l'-1}$. Finally, we also have the Garding inequality [65, 64]

$$\delta_1|\phi_{\mathrm{II}}|_1^2\le\sum_{i,j\in\mathcal{D}}\int_{\mathbb{R}^{d_{\mathrm{s}}}}\langle\overline{\mathsf{B}}_{ij}^{\mathrm{d}\,\mathrm{II},\mathrm{II}}(\mathsf{w})\partial_j\phi_{\mathrm{II}},\partial_i\phi_{\mathrm{II}}\rangle\,d\boldsymbol{x}+\mathsf{c}_2|\phi_{\mathrm{II}}|_0^2, \tag{5.15}$$

valid for $\phi_{\mathrm{II}}:\mathbb{R}^{d_{\mathrm{s}}}\mapsto\mathbb{R}^{\mathsf{n}_{\mathrm{II}}}$ vector valued function in $H^1(\mathbb{R}^{d_{\mathrm{s}}})$.

Step 1. The zeroth order inequality. Multiplying (5.1) by $\delta\widetilde{\mathsf{w}}=\widetilde{\mathsf{w}}-\widetilde{\mathsf{w}}^\star$ and integrating over $\mathbb{R}^{d_{\mathrm{s}}}$, we obtain that

$$\begin{aligned}&\int_{\mathbb{R}^{d_{\mathrm{s}}}}\langle\delta\widetilde{\mathsf{w}},\overline{\mathsf{A}}_0(\mathsf{w})\partial_t\delta\widetilde{\mathsf{w}}\rangle\,d\boldsymbol{x}+\sum_{i\in\mathcal{D}}\int_{\mathbb{R}^{d_{\mathrm{s}}}}\langle\delta\widetilde{\mathsf{w}},\overline{\mathsf{A}}_i'(\mathsf{w})\partial_i\delta\widetilde{\mathsf{w}}\rangle\,d\boldsymbol{x}-\sum_{i,j\in\mathcal{D}}\int_{\mathbb{R}^{d_{\mathrm{s}}}}\langle\delta\widetilde{\mathsf{w}},\overline{\mathsf{B}}_{ij}^{\mathrm{d}}(\mathsf{w})\partial_i\partial_j\delta\widetilde{\mathsf{w}}\rangle\,d\boldsymbol{x}\\&\quad-\sum_{i,j\in\mathcal{D}}\int_{\mathbb{R}^{d_{\mathrm{s}}}}\langle\delta\widetilde{\mathsf{w}},\overline{\mathsf{B}}_{ij}^{\mathrm{c}}(\mathsf{w})\partial_i\partial_j\delta\widetilde{\mathsf{w}}\rangle\,d\boldsymbol{x}+\int_{\mathbb{R}^{d_{\mathrm{s}}}}\langle\delta\widetilde{\mathsf{w}},\overline{\mathsf{L}}(\mathsf{w},\nabla\mathsf{w}_{\mathrm{r}})\delta\widetilde{\mathsf{w}}\rangle\,d\boldsymbol{x}=\int_{\mathbb{R}^{d_{\mathrm{s}}}}\langle\delta\widetilde{\mathsf{w}},\mathsf{f}\rangle\,d\boldsymbol{x}+\int_{\mathbb{R}^{d_{\mathrm{s}}}}\langle\delta\widetilde{\mathsf{w}},\mathsf{g}\rangle\,d\boldsymbol{x}.\end{aligned} \tag{5.16}$$

The time derivative term in (5.16) is rewritten by using the symmetry of $\overline{\mathsf{A}}_0$

$$\langle\delta\widetilde{\mathsf{w}},\overline{\mathsf{A}}_0(\mathsf{w})\partial_t\delta\widetilde{\mathsf{w}}\rangle=\tfrac12\partial_t\langle\delta\widetilde{\mathsf{w}},\overline{\mathsf{A}}_0(\mathsf{w})\delta\widetilde{\mathsf{w}}\rangle-\tfrac12\langle\delta\widetilde{\mathsf{w}},\partial_t\overline{\mathsf{A}}_0(\mathsf{w})\delta\widetilde{\mathsf{w}}\rangle,$$

and $\partial_t\overline{\mathsf{A}}_0(\mathsf{w})=\partial_{\mathsf{w}}\overline{\mathsf{A}}_0\,\partial_t\mathsf{w}$ is estimated with $|\partial_t\overline{\mathsf{A}}_0|_{L^\infty}\le\mathsf{c}_0|\partial_t\overline{\mathsf{A}}_0|_{l-2}\le\mathsf{c}_1|\partial_t\mathsf{w}|_{l-2}$.

For the convective terms in (5.16) the products $\langle\delta\widetilde{\mathsf{w}},\overline{\mathsf{A}}_i'(\mathsf{w})\partial_i\delta\widetilde{\mathsf{w}}\rangle$ are evaluated by blocks and the (I,I) terms are rewritten by using the symmetry of $\overline{\mathsf{A}}_i'^{\mathrm{I},\mathrm{I}}$

$$\langle\delta\widetilde{\mathsf{w}}_{\mathrm{I}},\overline{\mathsf{A}}_i'^{\mathrm{I},\mathrm{I}}(\mathsf{w})\partial_i\delta\widetilde{\mathsf{w}}_{\mathrm{I}}\rangle=\tfrac12\partial_i\langle\delta\widetilde{\mathsf{w}}_{\mathrm{I}},\overline{\mathsf{A}}_i'^{\mathrm{I},\mathrm{I}}(\mathsf{w})\delta\widetilde{\mathsf{w}}_{\mathrm{I}}\rangle-\tfrac12\langle\delta\widetilde{\mathsf{w}}_{\mathrm{I}},\partial_i\overline{\mathsf{A}}_i'^{\mathrm{I},\mathrm{I}}(\mathsf{w})\delta\widetilde{\mathsf{w}}_{\mathrm{I}}\rangle,$$

and $\partial_i\overline{\mathsf{A}}_i'^{\mathrm{I},\mathrm{I}}(\mathsf{w})=\partial_{\mathsf{w}}\overline{\mathsf{A}}_i'^{\mathrm{I},\mathrm{I}}\partial_i\mathsf{w}$ is estimated with $|\partial_i\overline{\mathsf{A}}_i'^{\mathrm{I},\mathrm{I}}|_{L^\infty}\le\mathsf{c}_0|\partial_i\overline{\mathsf{A}}_i'^{\mathrm{I},\mathrm{I}}|_{l-2}\le\mathsf{c}_1M\le\mathsf{c}_2$. The (I,II) and $(\mathrm{II},\mathrm{II})$ convective terms are directly estimated as

$$\Big|\int_{\mathbb{R}^{d_{\mathrm{s}}}}\langle\delta\widetilde{\mathsf{w}}_{\mathrm{I}},\overline{\mathsf{A}}_i'^{\mathrm{I},\mathrm{II}}(\mathsf{w})\partial_i\delta\widetilde{\mathsf{w}}_{\mathrm{II}}\rangle\,d\boldsymbol{x}\Big|\le\mathsf{c}_1|\delta\widetilde{\mathsf{w}}_{\mathrm{I}}|_0|\delta\widetilde{\mathsf{w}}_{\mathrm{II}}|_1,$$

$$\Big|\int_{\mathbb{R}^{d_{\mathrm{s}}}}\langle\delta\widetilde{\mathsf{w}}_{\mathrm{II}},\overline{\mathsf{A}}_i'^{\mathrm{II},\mathrm{II}}(\mathsf{w})\partial_i\delta\widetilde{\mathsf{w}}_{\mathrm{II}}\rangle\,d\boldsymbol{x}\Big|\le\mathsf{c}_1|\delta\widetilde{\mathsf{w}}_{\mathrm{II}}|_0|\delta\widetilde{\mathsf{w}}_{\mathrm{II}}|_1.$$

The (II,I) terms are integrated by parts

$$\int_{\mathbb{R}^{d_{\mathrm{s}}}}\langle\delta\widetilde{\mathsf{w}}_{\mathrm{II}},\overline{\mathsf{A}}_i'^{\mathrm{II},\mathrm{I}}(\mathsf{w})\partial_i\delta\widetilde{\mathsf{w}}_{\mathrm{I}}\rangle\,d\boldsymbol{x}=-\int_{\mathbb{R}^{d_{\mathrm{s}}}}\langle\partial_i\delta\widetilde{\mathsf{w}}_{\mathrm{II}},\overline{\mathsf{A}}_i'^{\mathrm{II},\mathrm{I}}(\mathsf{w})\delta\widetilde{\mathsf{w}}_{\mathrm{I}}\rangle\,d\boldsymbol{x}-\int_{\mathbb{R}^{d_{\mathrm{s}}}}\langle\delta\widetilde{\mathsf{w}}_{\mathrm{II}},\partial_i\overline{\mathsf{A}}_i'^{\mathrm{II},\mathrm{I}}(\mathsf{w})\delta\widetilde{\mathsf{w}}_{\mathrm{I}}\rangle\,d\boldsymbol{x},$$

so that

$$\Big|\int_{\mathbb{R}^{d_s}} \langle \delta\widetilde{\mathsf{w}}_{\mathrm{II}}, \overline{\mathsf{A}}_i'^{\mathrm{II},\mathrm{I}}(\mathsf{w})\partial_i\delta\widetilde{\mathsf{w}}_{\mathrm{I}}\rangle\, d\boldsymbol{x}\Big| \le \mathsf{c}_1|\delta\widetilde{\mathsf{w}}_{\mathrm{I}}|_0|\delta\widetilde{\mathsf{w}}_{\mathrm{II}}|_1 + \mathsf{c}_2|\delta\widetilde{\mathsf{w}}_{\mathrm{I}}|_0|\delta\widetilde{\mathsf{w}}_{\mathrm{II}}|_0 \le \mathsf{c}_2|\delta\widetilde{\mathsf{w}}_{\mathrm{I}}|_0|\delta\widetilde{\mathsf{w}}_{\mathrm{II}}|_1,$$

using $|\overline{\mathsf{A}}_i|_{L^\infty} \le \mathsf{c}_1$ and $|\partial_i\overline{\mathsf{A}}_i|_{L^\infty} \le \mathsf{c}_2$ since $\partial_i\overline{\mathsf{A}}_i = \partial_{\mathsf{w}}\overline{\mathsf{A}}_i\,\partial_i\mathsf{w}$.

The dissipative terms in (5.16) only involve II components and are integrated by parts

$$\begin{aligned}-\sum_{i,j\in\mathcal{D}}\int_{\mathbb{R}^{d_s}} \langle \delta\widetilde{\mathsf{w}}, \overline{\mathsf{B}}^{\mathrm{d}}_{ij}(\mathsf{w})\partial_i\partial_j\delta\widetilde{\mathsf{w}}\rangle\, d\boldsymbol{x} &= \sum_{i,j\in\mathcal{D}}\int_{\mathbb{R}^{d_s}} \langle \partial_i\delta\widetilde{\mathsf{w}}_{\mathrm{II}}, \overline{\mathsf{B}}^{\mathrm{d}\,\mathrm{II},\mathrm{II}}_{ij}(\mathsf{w})\partial_j\delta\widetilde{\mathsf{w}}_{\mathrm{II}}\rangle\, d\boldsymbol{x}\\ &\quad+\sum_{i,j\in\mathcal{D}}\int_{\mathbb{R}^{d_s}} \langle \delta\widetilde{\mathsf{w}}_{\mathrm{II}}, \partial_i\overline{\mathsf{B}}^{\mathrm{d}\,\mathrm{II},\mathrm{II}}_{ij}(\mathsf{w})\partial_j\delta\widetilde{\mathsf{w}}_{\mathrm{II}}\rangle\, d\boldsymbol{x}.\end{aligned}$$

The first sum is estimated by using Garding inequality whereas the remaining terms are estimated by using $|\partial_i\overline{\mathsf{B}}^{\mathrm{d}}_{ij}(\mathsf{w})|_{L^\infty} \le \mathsf{c}_2$.

For the antisymmetric terms, we first integrate by parts to get that

$$\begin{aligned}-\sum_{i,j\in\mathcal{D}}\int_{\mathbb{R}^{d_s}} \langle \delta\widetilde{\mathsf{w}}, \overline{\mathsf{B}}^{\mathrm{c}}_{ij}(\mathsf{w})\partial_i\partial_j\delta\widetilde{\mathsf{w}}\rangle\, d\boldsymbol{x} &= \sum_{i,j\in\mathcal{D}}\int_{\mathbb{R}^{d_s}} \langle \partial_i\delta\widetilde{\mathsf{w}}, \overline{\mathsf{B}}^{\mathrm{c}}_{ij}(\mathsf{w})\partial_j\delta\widetilde{\mathsf{w}}\rangle\, d\boldsymbol{x}\\ &\quad+\sum_{i,j\in\mathcal{D}}\int_{\mathbb{R}^{d_s}} \langle \delta\widetilde{\mathsf{w}}, \partial_i\overline{\mathsf{B}}^{\mathrm{c}}_{ij}(\mathsf{w})\partial_j\delta\widetilde{\mathsf{w}}\rangle\, d\boldsymbol{x}.\end{aligned}$$

The first sum vanishes

$$\sum_{i,j\in\mathcal{D}}\int_{\mathbb{R}^{d_s}} \langle \partial_i\delta\widetilde{\mathsf{w}}, \overline{\mathsf{B}}^{\mathrm{c}}_{ij}(\mathsf{w})\partial_j\delta\widetilde{\mathsf{w}}\rangle\, d\boldsymbol{x} = 0,$$

from the antisymmetric properties $(\overline{\mathsf{B}}^{\mathrm{c}}_{ij})^t = -\overline{\mathsf{B}}^{\mathrm{c}}_{ji}$ of the $\overline{\mathsf{B}}^{\mathrm{c}}_{ij}$ matrices and the remaining terms are decomposed by using the underlying block structure. The (I,I) contributions vanish since $\overline{\mathsf{B}}^{\mathrm{c}\,\mathrm{I},\mathrm{I}}_{ij} = 0$ whereas both (I,II) and $(\mathrm{II},\mathrm{II})$ contributions are estimated as

$$\sum_{i,j\in\mathcal{D}}\Big|\int_{\mathbb{R}^{d_s}} \langle \delta\widetilde{\mathsf{w}}_{\mathrm{I}}, \partial_i\overline{\mathsf{B}}^{\mathrm{c}\,\mathrm{I},\mathrm{II}}_{ij}(\mathsf{w})\partial_j\delta\widetilde{\mathsf{w}}_{\mathrm{II}}\rangle\, d\boldsymbol{x}\Big| \le \mathsf{c}_2|\delta\widetilde{\mathsf{w}}_{\mathrm{I}}|_0|\delta\widetilde{\mathsf{w}}_{\mathrm{II}}|_1,$$

$$\sum_{i,j\in\mathcal{D}}\Big|\int_{\mathbb{R}^{d_s}} \langle \delta\widetilde{\mathsf{w}}_{\mathrm{II}}, \partial_i\overline{\mathsf{B}}^{\mathrm{c}\,\mathrm{II},\mathrm{II}}_{ij}(\mathsf{w})\partial_j\delta\widetilde{\mathsf{w}}_{\mathrm{II}}\rangle\, d\boldsymbol{x}\Big| \le \mathsf{c}_2|\delta\widetilde{\mathsf{w}}_{\mathrm{II}}|_0|\delta\widetilde{\mathsf{w}}_{\mathrm{II}}|_1.$$

The more difficult (II,I) terms are first integrated by parts

$$\begin{aligned}\sum_{i,j\in\mathcal{D}}\int_{\mathbb{R}^{d_s}} \langle \delta\widetilde{\mathsf{w}}_{\mathrm{II}}, \partial_i\overline{\mathsf{B}}^{\mathrm{c}\,\mathrm{II},\mathrm{I}}_{ij}(\mathsf{w})\partial_j\delta\widetilde{\mathsf{w}}_{\mathrm{I}}\rangle\, d\boldsymbol{x} &= -\sum_{i,j\in\mathcal{D}}\int_{\mathbb{R}^{d_s}} \langle \partial_j\delta\widetilde{\mathsf{w}}_{\mathrm{II}}, \partial_i\overline{\mathsf{B}}^{\mathrm{c}\,\mathrm{II},\mathrm{I}}_{ij}(\mathsf{w})\delta\widetilde{\mathsf{w}}_{\mathrm{I}}\rangle\, d\boldsymbol{x}\\ &\quad-\sum_{i,j\in\mathcal{D}}\int_{\mathbb{R}^{d_s}} \langle \delta\widetilde{\mathsf{w}}_{\mathrm{II}}, \partial_i\partial_j\overline{\mathsf{B}}^{\mathrm{c}\,\mathrm{II},\mathrm{I}}_{ij}(\mathsf{w})\delta\widetilde{\mathsf{w}}_{\mathrm{I}}\rangle\, d\boldsymbol{x}.\end{aligned}$$

The terms of the first sum are estimated as the (I,II) terms whereas the terms in the second sum are estimated by using $|\partial_i\partial_j\overline{\mathsf{B}}^{\mathrm{c}\,\mathrm{II},\mathrm{I}}_{ij}| \le \mathsf{c}_2$ since $l \ge l_0+2$ and $\partial_i\partial_j\overline{\mathsf{B}}^{\mathrm{c}\,\mathrm{II},\mathrm{I}}_{ij} = \partial^2_{\mathsf{ww}}\overline{\mathsf{B}}^{\mathrm{c}\,\mathrm{II},\mathrm{I}}_{ij}\,\partial_i\mathsf{w}\partial_j\mathsf{w} + \partial_{\mathsf{w}}\overline{\mathsf{B}}^{\mathrm{c}\,\mathrm{II},\mathrm{I}}_{ij}\,\partial_i\partial_j\mathsf{w}$ so that the (II,I) terms are also majorized by $\mathsf{c}_2|\delta\widetilde{\mathsf{w}}_{\mathrm{I}}|_0|\delta\widetilde{\mathsf{w}}_{\mathrm{II}}|_1$.

The $\overline{\mathsf{L}}$ terms are finally evaluated by using $\big|\overline{\mathsf{L}}(\mathsf{w},\nabla\mathsf{w}_{\mathrm{r}})\big|_{L^\infty} \le \mathsf{c}_2$ so that

$$\int_{\mathbb{R}^{d_s}} \langle \delta\widetilde{\mathsf{w}}, \overline{\mathsf{L}}(\mathsf{w},\nabla\mathsf{w}_{\mathrm{r}})\delta\widetilde{\mathsf{w}}\rangle\, d\boldsymbol{x} \le \mathsf{c}_2|\delta\widetilde{\mathsf{w}}|_0^2,$$

and the right hand side terms with

$$\int_{\mathbb{R}^{d_s}} \langle \delta\widetilde{\mathsf{w}}, \mathsf{f}\rangle\, d\boldsymbol{x} \le \mathsf{c}_1|\delta\widetilde{\mathsf{w}}|_0|\mathsf{f}|_0, \qquad \int_{\mathbb{R}^{d_s}} \langle \delta\widetilde{\mathsf{w}}, \mathsf{g}\rangle\, d\boldsymbol{x} \le |\delta\widetilde{\mathsf{w}}|_0|\mathsf{g}|_0.$$

Combining the previous inequalities and using $|\delta\widetilde{\mathsf{w}}|_0 \le \mathsf{c}_1 E_0(\delta\widetilde{\mathsf{w}})$ it has been established that

$$\partial_t E_0^2(\delta\widetilde{\mathsf{w}}) + \delta_1|\delta\widetilde{\mathsf{w}}_{\mathrm{II}}|_1^2 \le \mathsf{c}_1|\mathsf{f}|_0|\delta\widetilde{\mathsf{w}}|_0 + \mathsf{c}_1|\mathsf{g}_{\mathrm{II}}|_0^2 + \mathsf{c}_2(1+|\partial_t\mathsf{w}|_{l-2})E_0^2(\delta\widetilde{\mathsf{w}}). \tag{5.17}$$

Step 2. The l'th order inequality. Multiplying (5.1) by $\overline{\mathsf{A}}_0^{-1}$, applying the αth spatial derivative operator ∂^α, and multiplying again by $\overline{\mathsf{A}}_0$ first yields

$$\overline{\mathsf{A}}_0(\mathsf{w})\partial_t\partial^\alpha\widetilde{\mathsf{w}}+\sum_{i\in\mathcal{D}}\overline{\mathsf{A}}_i'(\mathsf{w})\partial_i\partial^\alpha\widetilde{\mathsf{w}}-\sum_{i,j\in\mathcal{D}}\overline{\mathsf{B}}_{ij}^{\mathrm{d}}(\mathsf{w})\partial_i\partial_i\partial^\alpha\widetilde{\mathsf{w}}-\sum_{i,j\in\mathcal{D}}\overline{\mathsf{B}}_{ij}^{\mathrm{c}}(\mathsf{w})\partial_i\partial_i\partial^\alpha\widetilde{\mathsf{w}}+\overline{\mathsf{L}}(\mathsf{w},\nabla\mathsf{w})\partial^\alpha\widetilde{\mathsf{w}}=\mathsf{h}^\alpha, \quad (5.18)$$

where

$$\begin{aligned}\mathsf{h}^\alpha=&\,\overline{\mathsf{A}}_0\partial^\alpha\big(\overline{\mathsf{A}}_0^{-1}\mathsf{f}\big)+\overline{\mathsf{A}}_0\partial^\alpha\big(\overline{\mathsf{A}}_0^{-1}\mathsf{g}\big)-\sum_{i\in\mathcal{D}}\overline{\mathsf{A}}_0\big[\partial^\alpha,\overline{\mathsf{A}}_0^{-1}\,\overline{\mathsf{A}}_i'\big]\partial_i\widetilde{\mathsf{w}}\\&-\overline{\mathsf{A}}_0\big[\partial^\alpha,\overline{\mathsf{A}}_0^{-1}\,\overline{\mathsf{L}}\big]\widetilde{\mathsf{w}}+\sum_{i,j\in\mathcal{D}}\overline{\mathsf{A}}_0\big[\partial^\alpha,\overline{\mathsf{A}}_0^{-1}\,\overline{\mathsf{B}}_{ij}^{\mathrm{d}}\big]\partial_i\partial_j\widetilde{\mathsf{w}}+\sum_{i,j\in\mathcal{D}}\overline{\mathsf{A}}_0\big[\partial^\alpha,\overline{\mathsf{A}}_0^{-1}\,\overline{\mathsf{B}}_{ij}^{\mathrm{c}}\big]\partial_i\partial_j\widetilde{\mathsf{w}}. \quad (5.19)\end{aligned}$$

We next multiply scalarly (5.18) by $\partial^\alpha\delta\widetilde{\mathsf{w}}$, myltiply by $|\alpha|!/\alpha!$, integrate over $\mathbb{R}^d$, sum over $1\le|\alpha|\le l'$, and add the zeroth order estimate. We then proceed exactly as for the zeroth order estimate (5.17)—with $\partial^\alpha\delta\widetilde{\mathsf{w}}$ in place of $\delta\widetilde{\mathsf{w}}$—in order to first rewrite the terms arising from the left hand side of (5.18). Further using $|\delta\widetilde{\mathsf{w}}|_{l'}\le\mathsf{c}_1E_{l'}(\delta\widetilde{\mathsf{w}})$, it is obtained that

$$\partial_tE_{l'}^2(\delta\widetilde{\mathsf{w}})+\delta_1|\delta\widetilde{\mathsf{w}}_{\mathrm{II}}|_{l'+1}^2\le\mathsf{c}_2(1+|\partial_t\mathsf{w}|_{l-2})E_{l'}^2(\delta\widetilde{\mathsf{w}})\;+\sum_{0\le|\alpha|\le l'}\frac{|\alpha|!}{\alpha!}\int_{\mathbb{R}^{d_{\mathrm{s}}}}\langle\mathsf{h}^\alpha,\partial^\alpha\delta\widetilde{\mathsf{w}}\rangle\,d\boldsymbol{x},$$

and we now need to estimate all terms $\int_{\mathbb{R}^{d_{\mathrm{s}}}}\langle\mathsf{h}^\alpha,\partial^\alpha\delta\widetilde{\mathsf{w}}\rangle\,d\boldsymbol{x}$ arising from the right hand side h^α (5.19). Since the zeroth order terms with $\alpha=0$ in the residuals $\int_{\mathbb{R}^{d_{\mathrm{s}}}}\langle\mathsf{h}^\alpha,\partial^\alpha\delta\widetilde{\mathsf{w}}\rangle\,d\boldsymbol{x}$ have already been examined with (5.17), it is sufficient to analyze the terms such that $1\le|\alpha|\le l'$ and we note that for such multiindex α we have $\partial^\alpha\widetilde{\mathsf{w}}=\partial^\alpha\delta\widetilde{\mathsf{w}}$.

The right hand side contributions involving f may be directly be estimated as

$$\Big|\int_{\mathbb{R}^{d_{\mathrm{s}}}}\big\langle\overline{\mathsf{A}}_0\partial^\alpha\big(\overline{\mathsf{A}}_0^{-1}\mathsf{f}\big),\partial^\alpha\delta\widetilde{\mathsf{w}}\big\rangle\,d\boldsymbol{x}\Big|\le|\overline{\mathsf{A}}_0|_\infty\,|\overline{\mathsf{A}}_0^{-1}\mathsf{f}|_{l'}\,|\delta\widetilde{\mathsf{w}}|_{l'}\le\mathsf{c}_2|\mathsf{f}|_{l'}\,|\delta\widetilde{\mathsf{w}}|_{l'},$$

by using

$$|\overline{\mathsf{A}}_0^{-1}\mathsf{f}|_{l'}\le\mathsf{c}_1\big(1+|\overline{\mathsf{A}}_0^{-1}(\mathsf{w})-\overline{\mathsf{A}}_0^{-1}(\mathsf{w}^\star)|_l\big)\,|\mathsf{f}|_{l'}\le\mathsf{c}_2|\mathsf{f}|_{l'}.$$

For the right hand side terms involving g, the cases where $|\alpha|\le l'-1$ and $|\alpha|=l'$ must be distinguished. In the situation where $|\alpha|\le l'-1$ we may proceed as for the f term, keeping in mind that $\mathsf{g}=(0,\mathsf{g}_{\mathrm{II}})^t$ and that $\overline{\mathsf{A}}_0$ is block diagonal, to get that

$$\Big|\int_{\mathbb{R}^{d_{\mathrm{s}}}}\big\langle\overline{\mathsf{A}}_0\partial^\alpha\big(\overline{\mathsf{A}}_0^{-1}\mathsf{g}\big),\partial^\alpha\delta\widetilde{\mathsf{w}}\big\rangle\,d\boldsymbol{x}\Big|\le|\overline{\mathsf{A}}_0|_\infty\,|\overline{\mathsf{A}}_0^{-1}\mathsf{g}|_{l'-1}\,|\delta\widetilde{\mathsf{w}}_{\mathrm{II}}|_{l'-1}\le\mathsf{c}_2|\mathsf{g}|_{l'-1}\,|\delta\widetilde{\mathsf{w}}_{\mathrm{II}}|_{l'}.$$

When $|\alpha|=l'$, we may select any spatial direction i_0 such that $\alpha=\alpha'+\mathfrak{b}_{i_0}$, $|\alpha'|=l'-1$ and $\partial^\alpha=\partial^{\alpha'}\partial_{i_0}$, and integrate by parts the i_0th derivative with

$$\int_{\mathbb{R}^{d_{\mathrm{s}}}}\big\langle\overline{\mathsf{A}}_0\partial^\alpha\big(\overline{\mathsf{A}}_0^{-1}\mathsf{g}\big),\partial^\alpha\delta\widetilde{\mathsf{w}}\big\rangle\,d\boldsymbol{x}=-\int_{\mathbb{R}^{d_{\mathrm{s}}}}\big\langle\partial^{\alpha'}\big((\overline{\mathsf{A}}_0^{\mathrm{II},\mathrm{II}})^{-1}\mathsf{g}_{\mathrm{II}}\big),\partial_{i_0}(\overline{\mathsf{A}}_0^{\mathrm{II},\mathrm{II}}\partial^\alpha\delta\widetilde{\mathsf{w}}_{\mathrm{II}})\big\rangle\,d\boldsymbol{x}.$$

Using then $\big|(\overline{\mathsf{A}}_0^{\mathrm{II},\mathrm{II}})^{-1}\mathsf{g}_{\mathrm{II}}\big|_{l'-1}\le\mathsf{c}_2|\mathsf{g}_{\mathrm{II}}|_{l'-1}$ and $|\partial_{i_0}(\overline{\mathsf{A}}_0^{\mathrm{II},\mathrm{II}}\partial^\alpha\delta\widetilde{\mathsf{w}}_{\mathrm{II}})|_{l'}\le\mathsf{c}_2|\delta\widetilde{\mathsf{w}}_{\mathrm{II}}|_{l'+1}$ we obtain that

$$\Big|\int_{\mathbb{R}^{d_{\mathrm{s}}}}\big\langle\overline{\mathsf{A}}_0\partial^\alpha\big(\overline{\mathsf{A}}_0^{-1}\mathsf{g}\big),\partial^\alpha\delta\widetilde{\mathsf{w}}\big\rangle\,d\boldsymbol{x}\Big|\le\mathsf{c}_2|\mathsf{g}_{\mathrm{II}}|_{l'-1}\,|\delta\widetilde{\mathsf{w}}_{\mathrm{II}}|_{l'+1}.$$

The convective and dissipative contributions may next be estimated by using commutator estimates (5.14) and this yields

$$\Big|\int_{\mathbb{R}^{d_{\mathrm{s}}}}\big\langle\overline{\mathsf{A}}_0\big[\partial^\alpha,\overline{\mathsf{A}}_0^{-1}\,\overline{\mathsf{A}}_i'\big]\partial_i\widetilde{\mathsf{w}},\partial^\alpha\delta\widetilde{\mathsf{w}}\big\rangle\,d\boldsymbol{x}\Big|\le\mathsf{c}_2|\delta\widetilde{\mathsf{w}}|_{l'}^2,$$

$$\Big|\int_{\mathbb{R}^{d_{\mathrm{s}}}}\big\langle\overline{\mathsf{A}}_0\big[\partial^\alpha,\overline{\mathsf{A}}_0^{-1}\,\overline{\mathsf{B}}_{ij}^{\mathrm{d}}\big]\partial_i\partial_j\widetilde{\mathsf{w}},\partial^\alpha\delta\widetilde{\mathsf{w}}\big\rangle\,d\boldsymbol{x}\Big|\le\mathsf{c}_2\,|\delta\widetilde{\mathsf{w}}_{\mathrm{II}}|_{l'+1}\,|\delta\widetilde{\mathsf{w}}_{\mathrm{II}}|_{l'},$$

using the block structure of the dissipation matrices and of $\overline{\mathsf{A}}_0$.

For the antisymmetric terms

$$\int_{\mathbb{R}^{d_s}} \langle \overline{\mathsf{A}}_0 \big[\partial^\alpha, \overline{\mathsf{A}}_0^{-1}\, \overline{\mathsf{B}}^{\mathrm{c}}_{ij}\big] \partial_i\partial_j \widetilde{\mathsf{w}}, \partial^\alpha \delta\widetilde{\mathsf{w}}\rangle\, d\boldsymbol{x},$$

we must again use the underlying block structure. The (I,I) terms first vanish and the (I,II) and $(\mathrm{II},\mathrm{II})$ terms are estimated using the commutator inequality as for the dissipative terms. The difficult terms are the (II,I) terms that are integrated by parts using the commutator identity

$$[\partial^\alpha, \mathfrak{V}]\partial_i\phi = \partial_i([\partial^\alpha, \mathfrak{V}]\phi) - [\partial^\alpha, \partial_i\mathfrak{V}]\phi, \tag{5.20}$$

with $\mathfrak{V} = (\overline{\mathsf{A}}_0^{\mathrm{II},\mathrm{II}})^{-1}\, \overline{\mathsf{B}}^{\mathrm{c}\,\mathrm{II},\mathrm{I}}_{ij}$ and $\phi = \partial_j\delta\widetilde{\mathsf{w}}_{\mathrm{I}}$ and this yields

$$\begin{aligned}
-\sum_{i,j\in\mathcal{D}} \int_{\mathbb{R}^{d_s}} \langle \overline{\mathsf{A}}_0^{\mathrm{II},\mathrm{II}} \big[\partial^\alpha, (\overline{\mathsf{A}}_0^{\mathrm{II},\mathrm{II}})^{-1}\, \overline{\mathsf{B}}^{\mathrm{c}\,\mathrm{II},\mathrm{I}}_{ij}\big] \partial_i\partial_j\delta\widetilde{\mathsf{w}}_{\mathrm{I}}, \partial^\alpha\delta\widetilde{\mathsf{w}}_{\mathrm{II}}\rangle\, d\boldsymbol{x} = \\
\sum_{i,j\in\mathcal{D}} \int_{\mathbb{R}^{d_s}} \big\langle \big[\partial^\alpha, (\overline{\mathsf{A}}_0^{\mathrm{II},\mathrm{II}})^{-1}\, \overline{\mathsf{B}}^{\mathrm{c}\,\mathrm{II},\mathrm{I}}_{ij}\big] \partial_j\delta\widetilde{\mathsf{w}}_{\mathrm{I}}, \partial_i(\overline{\mathsf{A}}_0^{\mathrm{II},\mathrm{II}}\partial^\alpha\delta\widetilde{\mathsf{w}}_{\mathrm{II}})\big\rangle\, d\boldsymbol{x} \\
+\sum_{i,j\in\mathcal{D}} \int_{\mathbb{R}^{d_s}} \big\langle \overline{\mathsf{A}}_0^{\mathrm{II},\mathrm{II}} \big[\partial^\alpha, \partial_i\big((\overline{\mathsf{A}}_0^{\mathrm{II},\mathrm{II}})^{-1}\, \overline{\mathsf{B}}^{\mathrm{c}\,\mathrm{II},\mathrm{I}}_{ij}\big)\big] \partial_j\delta\widetilde{\mathsf{w}}_{\mathrm{I}}, \partial^\alpha\delta\widetilde{\mathsf{w}}_{\mathrm{II}}\big\rangle\, d\boldsymbol{x},
\end{aligned}$$

where we have used the symmetry of $\overline{\mathsf{A}}_0^{\mathrm{II},\mathrm{II}}$ and integrated by parts all terms of the first sum. We may then use the commutator estimates for both sums and that $(\overline{\mathsf{A}}_0^{\mathrm{II},\mathrm{II}})^{-1}\, \overline{\mathsf{B}}^{\mathrm{c}\,\mathrm{II},\mathrm{I}}_{ij}$ only depends on w_{r}, for the terms of the second sum, to get an upper bound in the form

$$\mathsf{c}_2|\delta\widetilde{\mathsf{w}}_{\mathrm{I}}|_{l'}\, |\delta\widetilde{\mathsf{w}}_{\mathrm{II}}|_{l'+1} + \mathsf{c}_2|\nabla\mathsf{w}_{\mathrm{r}}|_l|\delta\widetilde{\mathsf{w}}_{\mathrm{I}}|_{l'}\, |\delta\widetilde{\mathsf{w}}_{\mathrm{II}}|_{l'} \le \mathsf{c}_2|\delta\widetilde{\mathsf{w}}|_{l'}\, |\delta\widetilde{\mathsf{w}}_{\mathrm{II}}|_{l'+1} + \mathsf{c}_2|\nabla\mathsf{w}_{\mathrm{r}}|_l|\delta\widetilde{\mathsf{w}}|^2_{l'}.$$

The term associated with $\overline{\mathsf{A}}_0\big[\partial^\alpha, \overline{\mathsf{A}}_0^{-1}\, \overline{\mathsf{L}}\,\big]\widetilde{\mathsf{w}}$ is also easily controlled in terms of $|\nabla\mathsf{w}_{\mathrm{r}}|_l$ and $|\delta\widetilde{\mathsf{w}}|_l$ with

$$\Big|\int_{\mathbb{R}^{d_s}} \langle \overline{\mathsf{A}}_0\big[\partial^\alpha, \overline{\mathsf{A}}_0^{-1}\, \overline{\mathsf{L}}\,\big]\widetilde{\mathsf{w}}, \partial^\alpha\delta\widetilde{\mathsf{w}}\rangle\, d\boldsymbol{x}\Big| \le \mathsf{c}_2|\nabla\mathsf{w}_{\mathrm{r}}|_l|\delta\widetilde{\mathsf{w}}|^2_{l'},$$

since $\overline{\mathsf{L}} = \mathrm{diag}(\overline{\mathsf{L}}^{\mathrm{I},\mathrm{I}}, \overline{\mathsf{L}}^{\mathrm{II},\mathrm{II}})$ is a linear function of $\nabla\mathsf{w}_{\mathrm{r}}$.

Collecting all contributions and using $|\delta\widetilde{\mathsf{w}}|_{l'} \le \mathsf{c}_1 E_{l'}(\delta\widetilde{\mathsf{w}})$ it has been established that

$$\partial_t E^2_{l'}(\delta\widetilde{\mathsf{w}}) + \delta_1|\delta\widetilde{\mathsf{w}}_{\mathrm{II}}|^2_{l'+1} \le \mathsf{c}_2\big(1 + |\partial_t\mathsf{w}|_{l-2} + |\nabla\mathsf{w}_{\mathrm{r}}|_l\big)E^2_{l'}(\delta\widetilde{\mathsf{w}}) + \mathsf{c}_2|\mathsf{f}|_{l'}E_{l'}(\delta\widetilde{\mathsf{w}}) + \mathsf{c}_2|\mathsf{g}_{\mathrm{II}}|^2_{l'-1}. \tag{5.21}$$

We may then apply Gronwall lemma in order to deduce the required a priori estimates. □

5.2 Existence of solutions to linearized equations

We investigate in this section existence of solutions to the linearized equations (5.1). Existence of solutions is obtained by first regularizing the antisymmetric strongly coupling terms $\overline{\mathsf{B}}^{\mathrm{c}}_{ij}\partial_i\partial_j\delta\widetilde{\mathsf{w}}$ and next by letting the regularizing parameter ϵ to go to zero. The regularization allows in particular to uncouple the hyperbolic and parabolic problems.

We introduce a mollifying sequence of functions $\mathfrak{a}_\epsilon = \epsilon^{-d_s}\mathfrak{a}(\mathbf{r}/\epsilon)$ where $0 < \epsilon \le 1$ and $\mathfrak{a}$ is a smooth function positive in the ball $|\mathbf{r}| < 1$ of $\mathbb{R}^{d_s}$ and zero otherwise with its integral equal to unity $\int\mathfrak{a}\, d\mathbf{r} = 1$. The corresponding mollifying convolution operators are denoted by R_ϵ

$$\mathsf{R}_\epsilon\phi(\mathbf{r}) = \int \mathfrak{a}_\epsilon(\mathbf{r}-\hat{\mathbf{r}})\phi(\hat{\mathbf{r}})\, d\hat{\mathbf{r}}, \qquad \phi \in L^1(\mathbb{R}^{d_s}) \qquad 0 < \epsilon \le 1. \tag{5.22}$$

The regularizing operators R_ϵ, $0 < \epsilon \le 1$, are symmetric, commute with differentiation operators, and for any integers $k \ge 0$ and $j \ge 0$ and any $\phi \in H^k$ we have [63, 48]

$$|\mathsf{R}_\epsilon\phi|_k \le \mathsf{c}_0|\phi|_k, \tag{5.23}$$

$$\lim_{\epsilon\to 0} |\mathsf{R}_\epsilon\phi - \phi|_k = 0, \tag{5.24}$$

$$|\mathsf{R}_\epsilon\phi|_{k+j} \le \frac{\mathsf{c}_0}{\epsilon^j}|\phi|_k. \tag{5.25}$$

Denoting by $[\mathsf{R}_\epsilon, \phi]$ the commutator defined by $[\mathsf{R}_\epsilon, \phi]\psi = \mathsf{R}_\epsilon(\phi\,\psi) - \phi\,\mathsf{R}_\epsilon(\psi)$ we also have the estimates [48, 20]

$$\big|[\mathsf{R}_\epsilon, \phi]\nabla\psi\big|_k \le \mathsf{c}_0(|\phi|_{L^\infty} + |\nabla\phi|_{l-1})|\psi|_k, \tag{5.26}$$

as well as the limit [48, 20]

$$\lim_{\epsilon\to 0}[\mathsf{R}_\epsilon, \phi]\nabla\psi = 0. \tag{5.27}$$

The antisymmetric strongly coupling terms are regularized in the form

$$-\sum_{i,j\in\mathcal{D}} \mathsf{R}_\epsilon \overline{\mathsf{B}}^{\mathrm{c}}_{ij}(\mathsf{w})\mathsf{R}_\epsilon\partial_i\partial_j\widetilde{\mathsf{w}} = -\sum_{i,j\in\mathcal{D}} \mathsf{R}_\epsilon \overline{\mathsf{B}}^{\mathrm{c}}_{ij}(\mathsf{w})\partial_i\partial_j\mathsf{R}_\epsilon\widetilde{\mathsf{w}},$$

and the regularized equations read

$$\begin{aligned}\overline{\mathsf{A}}_0(\mathsf{w})\partial_t\widetilde{\mathsf{w}} \; &+\sum_{i\in\mathcal{D}} \overline{\mathsf{A}}'_i(\mathsf{w})\partial_i\widetilde{\mathsf{w}} - \sum_{i,j\in\mathcal{D}} \overline{\mathsf{B}}^{\mathrm{d}}_{ij}(\mathsf{w})\partial_i\partial_j\widetilde{\mathsf{w}} \\ &- \sum_{i,j\in\mathcal{D}} \mathsf{R}_\epsilon \overline{\mathsf{B}}^{\mathrm{c}}_{ij}(\mathsf{w})\mathsf{R}_\epsilon\partial_i\partial_j\widetilde{\mathsf{w}} + \overline{\mathsf{L}}(\mathsf{w}, \nabla\mathsf{w}_{\mathrm{r}})\widetilde{\mathsf{w}} = \mathsf{f} + \mathsf{g}.\end{aligned} \tag{5.28}$$

We first establish an existence result for the regularized equations (5.28) with ϵ fixed and then let the regularizing parametrer ϵ to go to zero. The assumptions on the system coefficients $\overline{\mathsf{A}}_0$, $\overline{\mathsf{A}}'_i$, $\overline{\mathsf{B}}^{\mathrm{d}}_{ij}$, $\overline{\mathsf{B}}^{\mathrm{c}}_{ij}$, $i, j \in \mathcal{D}$ $\overline{\mathsf{L}}$ and $\mathfrak{L} = (\mathfrak{L}^{\mathrm{I},\mathrm{I}}, \mathfrak{L}^{\mathrm{II},\mathrm{II}})$ of Section 5.1 are kept as well as the assumptions (5.3) on w. We remind that $\overline{\mathsf{L}} = \overline{\mathsf{L}}(\mathsf{w}, \mathfrak{p}_{\mathrm{r}})$ is block diagonal $\overline{\mathsf{L}} = \mathrm{diag}(\overline{\mathsf{L}}^{\mathrm{I},\mathrm{I}}, \overline{\mathsf{L}}^{\mathrm{II},\mathrm{II}})$ and is a linear function of $\nabla\mathsf{w}_{\mathrm{r}}$ with $\overline{\mathsf{L}}^{\mathrm{I},\mathrm{I}} = \mathfrak{L}^{\mathrm{I},\mathrm{I}}(\mathsf{w})\nabla\mathsf{w}_{\mathrm{r}}$ and $\overline{\mathsf{L}}^{\mathrm{II},\mathrm{II}} = \mathfrak{L}^{\mathrm{II},\mathrm{II}}(\mathsf{w})\nabla\mathsf{w}_{\mathrm{r}}$. The system coefficients are thus of class at least C^{l+2} over $\mathcal{O}_{\mathsf{w}}$ and the right hand sides are assumed to be such that

$$\mathsf{f} \in C^0\big([0,\bar\tau], H^{l-1}\big) \cap L^1\big((0,\bar\tau), H^l\big), \tag{5.29}$$

$$\mathsf{g} \in C^0\big([0,\bar\tau], H^{l-1}\big), \qquad \mathsf{g}_{\mathrm{I}} = 0. \tag{5.30}$$

We also denote by $\widetilde{\mathsf{w}}^\star$ a constant state in $\mathbb{R}^{\mathsf{n}}$ such that $\overline{\mathsf{L}}(\mathsf{w}, \mathfrak{p}_{\mathrm{r}})\widetilde{\mathsf{w}}^\star = 0$ for any $\mathsf{w} \in \mathcal{O}_{\mathsf{w}}$ and any $\mathfrak{p}_{\mathrm{r}} \in \mathbb{R}^{\mathsf{n}_{\mathrm{r}}, d_{\mathrm{s}}}$.

Proposition 5.2. *Assuming that $\widetilde{\mathsf{w}}_0$ is such that $\widetilde{\mathsf{w}}_0 - \widetilde{\mathsf{w}}^\star \in H^l$, there exists a unique solution $\widetilde{\mathsf{w}}^\epsilon$ to the regularized linearized equations* (5.28) *with initial condition $\widetilde{\mathsf{w}}_0$ and regularity* (5.8).

Proof. Solutions of the linearized *coupled* system (5.28) are obtained as fixed points $\widetilde{\widetilde{\mathsf{w}}} = \widetilde{\mathsf{w}}$ of the following linearized *uncoupled* system in the unknown $\widetilde{\widetilde{\mathsf{w}}} = (\widetilde{\widetilde{\mathsf{w}}}_{\mathrm{I}}, \widetilde{\widetilde{\mathsf{w}}}_{\mathrm{II}})$

$$\overline{\mathsf{A}}_0^{\mathrm{I},\mathrm{I}}(\mathsf{w})\partial_t\widetilde{\widetilde{\mathsf{w}}}_{\mathrm{I}} + \sum_{i\in\mathcal{D}} \overline{\mathsf{A}}_i'^{\mathrm{I},\mathrm{I}}(\mathsf{w})\partial_i\widetilde{\widetilde{\mathsf{w}}}_{\mathrm{I}} + \overline{\mathsf{L}}^{\mathrm{I},\mathrm{I}}(\mathsf{w}, \nabla\mathsf{w}_{\mathrm{r}})\widetilde{\widetilde{\mathsf{w}}}_{\mathrm{I}} = \widetilde{\mathsf{f}}_{\mathrm{I}}(\epsilon, \mathsf{w}, \widetilde{\mathsf{w}}, \nabla\widetilde{\widetilde{\mathsf{w}}}_{\mathrm{II}}), \tag{5.31}$$

$$\overline{\mathsf{A}}_0^{\mathrm{II},\mathrm{II}}(\mathsf{w})\partial_t\widetilde{\widetilde{\mathsf{w}}}_{\mathrm{II}} - \sum_{i,j\in\mathcal{D}} \overline{\mathsf{B}}_{ij}^{\mathrm{d}\,\mathrm{II},\mathrm{II}}(\mathsf{w})\partial_i\partial_j\widetilde{\widetilde{\mathsf{w}}}_{\mathrm{II}} + \overline{\mathsf{L}}^{\mathrm{II},\mathrm{II}}(\mathsf{w}, \nabla\mathsf{w}_{\mathrm{r}})\widetilde{\widetilde{\mathsf{w}}}_{\mathrm{II}} = \widetilde{\mathsf{f}}_{\mathrm{II}}(\epsilon, \mathsf{w}, \widetilde{\mathsf{w}}, \nabla\widetilde{\mathsf{w}}), \tag{5.32}$$

with the initial condition $\widetilde{\widetilde{\mathsf{w}}}(0, \boldsymbol{x}) = \widetilde{\mathsf{w}}_0(\boldsymbol{x})$. In these equations we have denoted

$$\widetilde{\mathsf{f}}_{\mathrm{I}} = \mathsf{f}_{\mathrm{I}} - \sum_{i\in\mathcal{D}} \overline{\mathsf{A}}_i'^{\mathrm{I},\mathrm{II}}(\mathsf{w})\partial_i\widetilde{\widetilde{\mathsf{w}}}_{\mathrm{II}} + \sum_{i,j\in\mathcal{D}} \mathsf{R}_\epsilon\overline{\mathsf{B}}_{ij}^{\mathrm{c}\,\mathrm{I},\mathrm{II}}(\mathsf{w}_{\mathrm{r}})\mathsf{R}_\epsilon\partial_i\partial_j\widetilde{\widetilde{\mathsf{w}}}_{\mathrm{II}} \tag{5.33}$$

$$\begin{aligned}\widetilde{\mathsf{f}}_{\mathrm{II}} = \mathsf{f}_{\mathrm{II}} + \mathsf{g}_{\mathrm{II}} &- \sum_{i\in\mathcal{D}} \big(\overline{\mathsf{A}}_i'^{\mathrm{II},\mathrm{I}}(\mathsf{w})\partial_i\widetilde{\mathsf{w}}_{\mathrm{I}} + \overline{\mathsf{A}}_i'^{\mathrm{II},\mathrm{II}}(\mathsf{w})\partial_i\widetilde{\mathsf{w}}_{\mathrm{II}}\big) \\ &+ \sum_{i,j\in\mathcal{D}} \big(\mathsf{R}_\epsilon\overline{\mathsf{B}}_{ij}^{\mathrm{c}\,\mathrm{II},\mathrm{I}}(\mathsf{w}_{\mathrm{r}})\mathsf{R}_\epsilon\partial_i\partial_j\widetilde{\mathsf{w}}_{\mathrm{I}} + \mathsf{R}_\epsilon\overline{\mathsf{B}}_{ij}^{\mathrm{c}\,\mathrm{II},\mathrm{II}}(\mathsf{w})\mathsf{R}_\epsilon\partial_i\partial_j\widetilde{\mathsf{w}}_{\mathrm{II}}\big),\end{aligned} \tag{5.34}$$

keeping in mind that the coupling blocks $\overline{\mathsf{B}}_{ij}^{\mathrm{c}\,\mathrm{II},\mathrm{I}}$ and $\overline{\mathsf{B}}_{ij}^{\mathrm{c}\,\mathrm{I},\mathrm{II}}$ only depends on w_{r}.

The system defining $\widetilde{\widetilde{\mathsf{w}}}_{\mathrm{II}}$ is symmetric strongly parabolic and classical existence theorems [48] warrants the existence of $\widetilde{\widetilde{\mathsf{w}}}_{\mathrm{II}}$ such that

$$\widetilde{\widetilde{\mathsf{w}}}_{\mathrm{II}} - \widetilde{\mathsf{w}}_{\mathrm{II}}^\star \in C^0\big([0,\bar\tau], H^l\big) \cap C^1\big([0,\bar\tau], H^{l-1}\big) \cap L^2\big((0,\bar\tau), H^{l+1}\big),$$

with the following estimates for $0 \le t \le \bar\tau$

$$\sup_{0\le\tau\le t} |\widetilde{\widetilde{\mathsf{w}}}_{\mathrm{II}} - \widetilde{\mathsf{w}}^\star_{\mathrm{II}}|^2_l + \int_0^t |\widetilde{\widetilde{\mathsf{w}}}_{\mathrm{II}} - \widetilde{\mathsf{w}}^\star_{\mathrm{II}}|^2_{l+1}\, d\tau \le$$

$$\widetilde{\mathsf{c}}_1^2 \exp\bigl(\widetilde{\mathsf{c}}_2(t + M_{\mathrm{t}}\sqrt{t} + M_{\mathrm{r}}\sqrt{t}\,)\bigr)\Bigl(|\widetilde{\mathsf{w}}_{0,\mathrm{II}} - \widetilde{\mathsf{w}}^\star_{\mathrm{II}}|^2_l + \widetilde{\mathsf{c}}_2 \int_0^t |\widetilde{\mathsf{f}}_{\mathrm{II}}|^2_{l-1}\, d\tau\Bigr),$$

where the constants $\widetilde{\mathsf{c}}_1$ and $\widetilde{\mathsf{c}}_2$ *depend on* ϵ, that is, $\widetilde{\mathsf{c}}_1 = \widetilde{\mathsf{c}}_1(\epsilon, \mathcal{O}_1)$ and $\widetilde{\mathsf{c}}_2 = \widetilde{\mathsf{c}}_2(\epsilon, \mathcal{O}_1, M)$. Similarly, the system defining $\widetilde{\widetilde{\mathsf{w}}}_{\mathrm{I}}$ is symmetric hyperbolic and classical existence theorems [48] warrants the existence of $\widetilde{\widetilde{\mathsf{w}}}_{\mathrm{I}}$ such that

$$\widetilde{\widetilde{\mathsf{w}}}_{\mathrm{I}} - \widetilde{\mathsf{w}}^\star_{\mathrm{I}} \in C^0\bigl([0,\bar\tau], H^l\bigr) \cap C^1\bigl([0,\bar\tau], H^{l-1}\bigr),$$

with the estimates for $0 \le t \le \bar\tau$

$$\sup_{0\le\tau\le t} |\widetilde{\widetilde{\mathsf{w}}}_{\mathrm{I}} - \widetilde{\mathsf{w}}^\star_{\mathrm{I}}|^2_l \le \widetilde{\mathsf{c}}_1^2 \exp\bigl(\widetilde{\mathsf{c}}_2(t + M_{\mathrm{t}}\sqrt{t} + M_{\mathrm{r}}\sqrt{t}\,)\bigr)\Bigl(|\widetilde{\mathsf{w}}_{0,\mathrm{I}} - \widetilde{\mathsf{w}}^\star_{\mathrm{I}}|^2_l + \widetilde{\mathsf{c}}_2\, t\int_0^t |\widetilde{\mathsf{f}}_{\mathrm{I}}|^2_l\, d\tau\Bigr).$$

We may now define the successive approximation sequence $\{\widetilde{\mathsf{w}}^k\}_{k\ge0}$ with $\widetilde{\mathsf{w}}^0 = \widetilde{\mathsf{w}}^\star$ and $\widetilde{\mathsf{w}}^{k+1} = \widetilde{\widetilde{\mathsf{w}}}^k$ for $k \ge 0$ and letting for short $\delta^{k+1}\widetilde{\mathsf{w}} = \widetilde{\mathsf{w}}^{k+1} - \widetilde{\mathsf{w}}^k$ we have to estimate $\delta^{k+1}\widetilde{\mathsf{w}}$ in order to establish that $(\widetilde{\mathsf{w}}^k)_{k\ge0}$ is a Cauchy sequence. Forming the difference between two iterations, letting $\delta^{k+1}\widetilde{\mathsf{f}}_{\mathrm{I}} = \widetilde{\mathsf{f}}_{\mathrm{I}}^{k+1} - \widetilde{\mathsf{f}}_{\mathrm{I}}^k$ and $\delta^{k+1}\widetilde{\mathsf{f}}_{\mathrm{II}} = \widetilde{\mathsf{f}}_{\mathrm{II}}^{k+1} - \widetilde{\mathsf{f}}_{\mathrm{II}}^k$, and using similar estimates for linear symmetric hyperbolic systems and linear symmetric strongly parabolic systems, it is obtained that

$$\sup_{0\le\tau\le t} |\delta^{k+1}\widetilde{\mathsf{w}}_{\mathrm{II}}(\tau)|^2_l + \int_0^t |\delta^{k+1}\widetilde{\mathsf{w}}_{\mathrm{II}}|^2_{l+1}\, d\tau \le \widetilde{\mathsf{c}}_1^2 \exp\bigl(\widetilde{\mathsf{c}}_2(t + M_{\mathrm{t}}\sqrt{t} + M_{\mathrm{r}}\sqrt{t}\,)\bigr)\Bigl(\widetilde{\mathsf{c}}_2 \int_0^t |\delta^{k+1}\widetilde{\mathsf{f}}_{\mathrm{II}}|^2_{l-1}\, d\tau\Bigr),$$

as well as

$$\sup_{0\le\tau\le t} |\delta^{k+1}\widetilde{\mathsf{w}}_{\mathrm{I}}(\tau)|^2_l \le \widetilde{\mathsf{c}}_1^2 \exp\bigl(\widetilde{\mathsf{c}}_2(t + M_{\mathrm{t}}\sqrt{t} + M_{\mathrm{r}}\sqrt{t}\,)\bigr)\Bigl(\widetilde{\mathsf{c}}_2\, t\int_0^t |\delta^{k+1}\widetilde{\mathsf{f}}_{\mathrm{I}}|^2_l\, d\tau\Bigr).$$

Defining then

$$\gamma^k(t) = \sup_{0\le\tau\le t} |\delta^k\widetilde{\mathsf{w}}(\tau)|^2_l,$$

and using (5.23) and (5.25), we obtain that

$$|\delta^{k+1}\widetilde{\mathsf{f}}_{\mathrm{I}}|_l \le \widetilde{\mathsf{c}}_2\bigl(|\delta^k\widetilde{\mathsf{w}}|_l + |\delta^{k+1}\widetilde{\mathsf{w}}_{\mathrm{II}}|_{l+1}\bigr), \qquad |\delta^{k+1}\widetilde{\mathsf{f}}_{\mathrm{II}}|_{l-1} \le \widetilde{\mathsf{c}}_2|\delta^k\widetilde{\mathsf{w}}|_l,$$

so that

$$\gamma^{k+1}(t) \le C\int_0^t \gamma^k(\tau)\, d\tau,$$

where $C = C(\epsilon, \mathcal{O}_1, M, M_{\mathrm{t}}, \bar\tau)$ depends on ϵ, $\mathcal{O}_1$, M, M_{t} and $\bar\tau$. Since γ^1 is bounded over $[0,\bar\tau]$, say $\gamma^1(t) \le K$, it is thus obtained that

$$\gamma^{k+1}(t) \le K\frac{C^k t^k}{k!}, \qquad 0 \le t \le \bar\tau, \qquad k \ge 0,$$

so that $(\widetilde{\mathsf{w}}^k)_{k\ge0}$ is a Cauchy sequence. The limit of this Cauchy sequence is then a fixed point of the iteration system and thus a solution of the linearized system (5.1) in $C^0\bigl([0,\bar\tau], H^l\bigr)$. It is further obtained from the estimates and standard functional analysis argument that $\int_0^t |\widetilde{\mathsf{w}}_{\mathrm{II}} - \widetilde{\mathsf{w}}^\star_{\mathrm{II}}|^2_{l+1}\, d\tau$ is finite so that the fixed point has the required regularity properties. □

We now investigate new a priori estimates independent of ϵ for the solutions $\widetilde{\mathsf{w}}^\epsilon$ of the regularized equations (5.28).

Proposition 5.3. *Keep the assumptions of Proposition 5.2 and denote by* $\widetilde{\mathsf{w}}^\epsilon$ *the solution of* (5.28) *with regularity* (5.8). *There exists constants* $\mathsf{c}_1(\mathcal{O}_1) \ge 1$ *and* $\mathsf{c}_2(\mathcal{O}_1, M) \ge 1$ *increasing with* M *and independent of* ϵ, *such that for any* $t \in [0,\bar\tau]$ *and any* $1 \le l' \le l$

$$\sup_{0\le\tau\le t} |\widetilde{\mathsf{w}}^\epsilon(\tau) - \widetilde{\mathsf{w}}^\star|^2_{l'} + \int_0^t |\widetilde{\mathsf{w}}^\epsilon_{\mathrm{II}}(\tau) - \widetilde{\mathsf{w}}^\star_{\mathrm{II}}|^2_{l'+1}\, d\tau \le \mathsf{c}_1^2 \exp\bigl(\mathsf{c}_2\bigl(t + M_{\mathrm{t}}\sqrt{t} + M_{\mathrm{r}}\sqrt{t}\,\bigr)\bigr)\Bigl(|\widetilde{\mathsf{w}}^\epsilon_0 - \widetilde{\mathsf{w}}^\star|^2_{l'}$$
$$+ \mathsf{c}_2\Bigl\{\int_0^t |\mathsf{f}|_{l'}\, d\tau\Bigr\}^2 + \mathsf{c}_2\int_0^t |\mathsf{g}_{\mathrm{II}}|^2_{l'-1}\, d\tau\Bigr). \tag{5.35}$$

Proof. The proof is similar to that of Theorem 5.1 and only the differences involving the second order regularized matrices $\mathsf{R}_\epsilon \overline{\mathsf{B}}^{\mathrm{c}}_{ij} \mathsf{R}_\epsilon$ need to be analyzed. We denote for short $\delta\widetilde{\mathsf{w}}^\epsilon = \widetilde{\mathsf{w}}^\epsilon - \widetilde{\mathsf{w}}^\star$ and the main idea is to avoid the use of (5.25) and to only use (5.23) in all estimates. For the zeroth order estimates, integrating by parts and using that R_ϵ are symmetric operators, it is obtained that

$$-\sum_{i,j\in\mathcal{D}} \int_{\mathbb{R}^{d_{\mathrm{s}}}} \langle \delta\widetilde{\mathsf{w}}^\epsilon, \mathsf{R}_\epsilon \overline{\mathsf{B}}^{\mathrm{c}}_{ij}(\mathsf{w}) \mathsf{R}_\epsilon \partial_i \partial_j \delta\widetilde{\mathsf{w}}^\epsilon \rangle \, d\boldsymbol{x} = \sum_{i,j\in\mathcal{D}} \int_{\mathbb{R}^{d_{\mathrm{s}}}} \langle \mathsf{R}_\epsilon \partial_i \delta\widetilde{\mathsf{w}}^\epsilon, \overline{\mathsf{B}}^{\mathrm{c}}_{ij}(\mathsf{w}) \mathsf{R}_\epsilon \partial_j \delta\widetilde{\mathsf{w}}^\epsilon \rangle \, d\boldsymbol{x}$$
$$+ \sum_{i,j\in\mathcal{D}} \int_{\mathbb{R}^{d_{\mathrm{s}}}} \langle \delta\widetilde{\mathsf{w}}^\epsilon, \partial_i \mathsf{R}_\epsilon \overline{\mathsf{B}}^{\mathrm{c}}_{ij}(\mathsf{w}) \mathsf{R}_\epsilon \partial_j \delta\widetilde{\mathsf{w}}^\epsilon \rangle \, d\boldsymbol{x}.$$

The first sum vanishes

$$\sum_{i,j\in\mathcal{D}} \int_{\mathbb{R}^{d_{\mathrm{s}}}} \langle \mathsf{R}_\epsilon \partial_i \delta\widetilde{\mathsf{w}}^\epsilon, \overline{\mathsf{B}}^{\mathrm{c}}_{ij}(\mathsf{w}) \mathsf{R}_\epsilon \partial_j \delta\widetilde{\mathsf{w}}^\epsilon \rangle \, d\boldsymbol{x} = 0,$$

from the symmetry properties of R_ϵ and the antisymmetry properties of the $\overline{\mathsf{B}}^{\mathrm{c}}_{ij}$ matrices and the remaining terms of the second sum are then decomposed by using the underlying block structure. The (I,I) contributions also vanish since $\overline{\mathsf{B}}^{\mathrm{c}\,\mathrm{I},\mathrm{I}}_{ij} = 0$, both (I,II) and $(\mathrm{II},\mathrm{II})$ contributions are estimated as in the proof of Theorem 5.1 by using (5.23) so that

$$\sum_{i,j\in\mathcal{D}} \Big| \int_{\mathbb{R}^{d_{\mathrm{s}}}} \langle \delta\widetilde{\mathsf{w}}^\epsilon_{\mathrm{I}}, \mathsf{R}_\epsilon \partial_i \overline{\mathsf{B}}^{\mathrm{c}\,\mathrm{I},\mathrm{II}}_{ij}(\mathsf{w}) \mathsf{R}_\epsilon \partial_j \delta\widetilde{\mathsf{w}}^\epsilon_{\mathrm{II}} \rangle \, d\boldsymbol{x} \Big| \le \mathsf{c}_2 |\delta\widetilde{\mathsf{w}}^\epsilon_{\mathrm{I}}|_0 |\delta\widetilde{\mathsf{w}}^\epsilon_{\mathrm{II}}|_1,$$

$$\sum_{i,j\in\mathcal{D}} \Big| \int_{\mathbb{R}^{d_{\mathrm{s}}}} \langle \delta\widetilde{\mathsf{w}}^\epsilon_{\mathrm{II}}, \mathsf{R}_\epsilon \partial_i \overline{\mathsf{B}}^{\mathrm{c}\,\mathrm{II},\mathrm{II}}_{ij}(\mathsf{w}) \mathsf{R}_\epsilon \partial_j \delta\widetilde{\mathsf{w}}^\epsilon_{\mathrm{II}} \rangle \, d\boldsymbol{x} \Big| \le \mathsf{c}_2 |\delta\widetilde{\mathsf{w}}^\epsilon_{\mathrm{II}}|_0 |\delta\widetilde{\mathsf{w}}^\epsilon_{\mathrm{II}}|_1.$$

The last terms (II,I) are again integrated by parts in the form

$$\sum_{i,j\in\mathcal{D}} \int_{\mathbb{R}^{d_{\mathrm{s}}}} \langle \delta\widetilde{\mathsf{w}}^\epsilon_{\mathrm{II}}, \mathsf{R}_\epsilon \partial_i \overline{\mathsf{B}}^{\mathrm{c}\,\mathrm{II},\mathrm{I}}_{ij}(\mathsf{w}) \mathsf{R}_\epsilon \partial_j \delta\widetilde{\mathsf{w}}^\epsilon_{\mathrm{I}} \rangle \, d\boldsymbol{x} = -\sum_{i,j\in\mathcal{D}} \int_{\mathbb{R}^{d_{\mathrm{s}}}} \langle \partial_j \delta\widetilde{\mathsf{w}}^\epsilon_{\mathrm{II}}, \mathsf{R}_\epsilon \partial_i \overline{\mathsf{B}}^{\mathrm{c}\,\mathrm{II},\mathrm{I}}_{ij}(\mathsf{w}) \mathsf{R}_\epsilon \delta\widetilde{\mathsf{w}}^\epsilon_{\mathrm{I}} \rangle \, d\boldsymbol{x}$$
$$- \sum_{i,j\in\mathcal{D}} \int_{\mathbb{R}^{d_{\mathrm{s}}}} \langle \delta\widetilde{\mathsf{w}}^\epsilon_{\mathrm{II}}, \mathsf{R}_\epsilon \partial_i \partial_j \overline{\mathsf{B}}^{\mathrm{c}\,\mathrm{II},\mathrm{I}}_{ij}(\mathsf{w}) \mathsf{R}_\epsilon \delta\widetilde{\mathsf{w}}^\epsilon_{\mathrm{I}} \rangle . \, d\boldsymbol{x}.$$

The terms of the first sum are estimated as the (I,II) terms whereas fo the second sum we may use that $|\partial_i \partial_j \overline{\mathsf{B}}^{\mathrm{c}\,\mathrm{II},\mathrm{I}}_{ij}| \le \mathsf{c}_2$ since $l \ge l_0 + 2$ so that the (I,II) terms are also majorized by $\mathsf{c}_2 |\delta\widetilde{\mathsf{w}}_{\mathrm{I}}|_0 |\delta\widetilde{\mathsf{w}}_{\mathrm{II}}|_1$. We have thus established an inequality in the form (5.17) for the solutions of the regularized system.

For the l'th estimates, the antisymmetric terms are again integrated by parts and from the symmetry properties of R_ϵ and the antisymmetry properties of the $\overline{\mathsf{B}}^{\mathrm{c}}_{ij}$ matrices we have for any $1 \le |\alpha| \le l'$

$$\sum_{i,j\in\mathcal{D}} \int_{\mathbb{R}^{d_{\mathrm{s}}}} \langle \mathsf{R}_\epsilon \partial_i \partial^\alpha \widetilde{\mathsf{w}}^\epsilon, \overline{\mathsf{B}}^{\mathrm{c}}_{ij}(\mathsf{w}) \mathsf{R}_\epsilon \partial_j \partial^\alpha \widetilde{\mathsf{w}}^\epsilon \rangle \, d\boldsymbol{x} = 0.$$

The remaining terms are treated as for the zeroth order estimates and it is obtained that

$$\sum_{i,j\in\mathcal{D}} \Big| \int_{\mathbb{R}^{d_{\mathrm{s}}}} \langle \partial^\alpha \widetilde{\mathsf{w}}^\epsilon_{\mathrm{I}}, \mathsf{R}_\epsilon \partial_i \overline{\mathsf{B}}^{\mathrm{c}\,\mathrm{I},\mathrm{II}}_{ij}(\mathsf{w}) \mathsf{R}_\epsilon \partial_j \partial^\alpha \widetilde{\mathsf{w}}^\epsilon_{\mathrm{II}} \rangle \, d\boldsymbol{x} \Big| \le \mathsf{c}_2 |\delta\widetilde{\mathsf{w}}^\epsilon_{\mathrm{I}}|_{l'} |\delta\widetilde{\mathsf{w}}^\epsilon_{\mathrm{II}}|_{l'+1},$$

$$\sum_{i,j\in\mathcal{D}} \Big| \int_{\mathbb{R}^{d_{\mathrm{s}}}} \langle \partial^\alpha \widetilde{\mathsf{w}}^\epsilon_{\mathrm{II}}, \mathsf{R}_\epsilon \partial_i \overline{\mathsf{B}}^{\mathrm{c}\,\mathrm{II},\mathrm{II}}_{ij}(\mathsf{w}) \mathsf{R}_\epsilon \partial_j \partial^\alpha \widetilde{\mathsf{w}}^\epsilon_{\mathrm{II}} \rangle \, d\boldsymbol{x} \Big| \le \mathsf{c}_2 |\delta\widetilde{\mathsf{w}}^\epsilon_{\mathrm{II}}|_{l'} |\delta\widetilde{\mathsf{w}}^\epsilon_{\mathrm{II}}|_{l'+1}.$$

The last terms (II,I) are integrated by parts in the form

$$\sum_{i,j\in\mathcal{D}} \int_{\mathbb{R}^{d_{\mathrm{s}}}} \langle \partial^\alpha \widetilde{\mathsf{w}}^\epsilon_{\mathrm{II}}, \mathsf{R}_\epsilon \partial_i \overline{\mathsf{B}}^{\mathrm{c}\,\mathrm{II},\mathrm{I}}_{ij}(\mathsf{w}) \mathsf{R}_\epsilon \partial_j \partial^\alpha \widetilde{\mathsf{w}}^\epsilon_{\mathrm{I}} \rangle \, d\boldsymbol{x} = -\sum_{i,j\in\mathcal{D}} \int_{\mathbb{R}^{d_{\mathrm{s}}}} \langle \partial_j \partial^\alpha \widetilde{\mathsf{w}}^\epsilon_{\mathrm{II}}, \mathsf{R}_\epsilon \partial_i \overline{\mathsf{B}}^{\mathrm{c}\,\mathrm{II},\mathrm{I}}_{ij}(\mathsf{w}) \mathsf{R}_\epsilon \partial^\alpha \widetilde{\mathsf{w}}^\epsilon_{\mathrm{I}} \rangle \, d\boldsymbol{x}$$
$$- \sum_{i,j\in\mathcal{D}} \int_{\mathbb{R}^{d_{\mathrm{s}}}} \langle \partial^\alpha \widetilde{\mathsf{w}}^\epsilon_{\mathrm{II}}, \mathsf{R}_\epsilon \partial_i \partial_j \overline{\mathsf{B}}^{\mathrm{c}\,\mathrm{II},\mathrm{I}}_{ij}(\mathsf{w}) \mathsf{R}_\epsilon \partial^\alpha \widetilde{\mathsf{w}}^\epsilon_{\mathrm{I}} \rangle \, d\boldsymbol{x}.$$

The terms in the first sum is then estimated as the (I,II) terms and the terms in the the second sum may be estimated by using $|\partial_i \partial_j \mathsf{R}_\epsilon \overline{\mathsf{B}}^{\mathrm{c}\,\mathrm{II},\mathrm{I}}_{ij}| \le \mathsf{c}_2$.

It now only remains to analyze the antisymmetric commutator terms in the form

$$\int_{\mathbb{R}^{d_s}} \langle \overline{\mathsf{A}}_0 \big[\partial^\alpha, \overline{\mathsf{A}}_0^{-1}\, \mathsf{R}_\epsilon \overline{\mathsf{B}}^{\mathrm{c}}_{ij}\big] \mathsf{R}_\epsilon \partial_i \partial_j \widetilde{\mathsf{w}}^\epsilon, \partial^\alpha \delta\widetilde{\mathsf{w}} \rangle \, d\boldsymbol{x},$$

and we again use the underlying block structure with a treatment similar to the proof of Theorem 5.1. The (I,I) terms first vanish and the (I,II) and $(\mathrm{II},\mathrm{II})$ terms are estimated using the commutator and convolution operators inequalities. The difficult terms are only the (II,I) terms that are integrated by parts using (5.20) so that

$$\begin{aligned} -\int_{\mathbb{R}^{d_s}} \langle \overline{\mathsf{A}}_0 \big[\partial^\alpha, \overline{\mathsf{A}}_0^{-1}\, \mathsf{R}_\epsilon \overline{\mathsf{B}}^{\mathrm{c}}_{ij}\big] \mathsf{R}_\epsilon \partial_i \partial_j \widetilde{\mathsf{w}}^\epsilon, \partial^\alpha \delta\widetilde{\mathsf{w}} \rangle \, d\boldsymbol{x} = & \int_{\mathbb{R}^{d_s}} \big\langle \big[\partial^\alpha, \overline{\mathsf{A}}_0^{-1}\, \mathsf{R}_\epsilon \overline{\mathsf{B}}^{\mathrm{c}}_{ij}\big] \mathsf{R}_\epsilon \partial_j \widetilde{\mathsf{w}}^\epsilon, \partial_i (\overline{\mathsf{A}}_0 \partial^\alpha \delta\widetilde{\mathsf{w}}) \big\rangle \, d\boldsymbol{x} \\ & + \int_{\mathbb{R}^{d_s}} \langle \overline{\mathsf{A}}_0 \big[\partial^\alpha, \partial_i (\overline{\mathsf{A}}_0^{-1}\, \mathsf{R}_\epsilon \overline{\mathsf{B}}^{\mathrm{c}}_{ij})\big] \mathsf{R}_\epsilon \partial_i \partial_j \widetilde{\mathsf{w}}^\epsilon, \partial^\alpha \delta\widetilde{\mathsf{w}} \rangle \, d\boldsymbol{x}. \end{aligned}$$

We then use that $(\overline{\mathsf{A}}_0^{\mathrm{II},\mathrm{II}})^{-1}\, \mathsf{R}_\epsilon \overline{\mathsf{B}}^{\mathrm{c}\,\mathrm{II},\mathrm{I}}_{ij}$ only depends on w_{r} so that an inequality similar to (5.21) is then obtained and using Gronwall lemma completes the proof. □

Existence of solutions to the linearized equations (5.1) in finally obtained by letting the regularizing parameter ϵ to go to zero.

Theorem 5.4. *Keep the assumption on the system coefficients of Theorem 5.1. Denote by $\widetilde{\mathsf{w}}_0$ an initial state with $\widetilde{\mathsf{w}}_0 - \widetilde{\mathsf{w}}^\star \in H^l$ and assume that the right hand sides $\mathsf{f} \in \mathbb{R}^{\mathsf{n}}$ and $\mathsf{g} \in \mathbb{R}^{\mathsf{n}}$ are such that* (5.29) *and* (5.30) *hold. There exists a solution $\widetilde{\mathsf{w}}$ to the linearized equations* (5.1) *with regularity* (5.8) *and such that the estimates* (5.9) *hold.*

Proof. We first establish that the family of solutions $(\widetilde{\mathsf{w}}^\epsilon)_{0<\epsilon\leq 1}$ obtained in Proposition 5.2 and estimated in Proposition 5.3 form a Cauchy sequence in $C^0\big([0,\bar\tau], H^{l-1}\big)$.

We note that the difference $\widetilde{\mathsf{w}}^\epsilon - \widetilde{\mathsf{w}}^{\epsilon'}$ is the solution of the system of partial differential equations

$$\begin{aligned} \overline{\mathsf{A}}_0(\mathsf{w})\partial_t(\widetilde{\mathsf{w}}^\epsilon - \widetilde{\mathsf{w}}^{\epsilon'}) \; & + \sum_{i\in\mathcal{D}} \overline{\mathsf{A}}'_i(\mathsf{w})\partial_i(\widetilde{\mathsf{w}}^\epsilon - \widetilde{\mathsf{w}}^{\epsilon'}) - \sum_{i,j\in\mathcal{D}} \overline{\mathsf{B}}^{\mathrm{d}}_{ij}(\mathsf{w})\partial_i\partial_j(\widetilde{\mathsf{w}}^\epsilon - \widetilde{\mathsf{w}}^{\epsilon'}) \\ & - \sum_{i,j\in\mathcal{D}} \mathsf{R}_\epsilon \overline{\mathsf{B}}^{\mathrm{c}}_{ij}(\mathsf{w})\mathsf{R}_\epsilon\partial_i\partial_j(\widetilde{\mathsf{w}}^\epsilon - \widetilde{\mathsf{w}}^{\epsilon'}) + \overline{\mathsf{L}}(\mathsf{w},\nabla\mathsf{w}_{\mathrm{r}})(\widetilde{\mathsf{w}}^\epsilon - \widetilde{\mathsf{w}}^{\epsilon'}) \\ & = \sum_{i,j\in\mathcal{D}} \big(\mathsf{R}_\epsilon \overline{\mathsf{B}}^{\mathrm{c}}_{ij}(\mathsf{w})\mathsf{R}_\epsilon - \mathsf{R}_{\epsilon'} \overline{\mathsf{B}}^{\mathrm{c}}_{ij}(\mathsf{w})\mathsf{R}_{\epsilon'}\big)\partial_i\partial_j\widetilde{\mathsf{w}}^{\epsilon'}. \end{aligned} \tag{5.36}$$

The right hand side is written in the form $\mathsf{f}+\mathsf{g}$ with $\mathsf{f} = (\mathsf{f}_{\mathrm{I}}, 0)^t$ and $\mathsf{g} = (0, \mathsf{g}_{\mathrm{II}})^t$ and from the linearized estimates for regularized systems applied to $\widetilde{\mathsf{w}}^\epsilon - \widetilde{\mathsf{w}}^{\epsilon'}$ and with $l' = l-1$, we deduce that for any $t \in [0,\bar\tau]$

$$\begin{aligned} \sup_{0\leq\tau\leq t} |\widetilde{\mathsf{w}}^\epsilon - \widetilde{\mathsf{w}}^{\epsilon'}|^2_{l-1} + \int_0^t |\widetilde{\mathsf{w}}^\epsilon_{\mathrm{II}} - \widetilde{\mathsf{w}}^{\epsilon'}_{\mathrm{II}}|^2_l \, d\tau \leq \; & \mathsf{c}_1^2 \exp\big(\mathsf{c}_2\big(t + M_{\mathrm{t}}\sqrt{t} + M_{\mathrm{r}}\sqrt{t}\,\big)\big) \\ & \times \Big(\mathsf{c}_2\, t \int_0^t \Big| \big(\mathsf{R}_\epsilon \overline{\mathsf{B}}^{\mathrm{c}\,\mathrm{I},\mathrm{II}}_{ij}(\mathsf{w})\mathsf{R}_\epsilon - \mathsf{R}_{\epsilon'} \overline{\mathsf{B}}^{\mathrm{c}\,\mathrm{I},\mathrm{II}}_{ij}(\mathsf{w})\mathsf{R}_{\epsilon'}\big)\partial_i\partial_j \widetilde{\mathsf{w}}^{\epsilon'}_{\mathrm{II}} \Big|^2_{l-1} d\tau \\ & + \mathsf{c}_2 \int_0^t \Big| \big(\mathsf{R}_\epsilon \overline{\mathsf{B}}^{\mathrm{c}\,\mathrm{II},\mathrm{I}}_{ij}(\mathsf{w})\mathsf{R}_\epsilon - \mathsf{R}_{\epsilon'} \overline{\mathsf{B}}^{\mathrm{c}\,\mathrm{II},\mathrm{I}}_{ij}(\mathsf{w})\mathsf{R}_{\epsilon'}\big)\partial_i\partial_j \widetilde{\mathsf{w}}^{\epsilon'}_{\mathrm{I}} \Big|^2_{l-2} d\tau \\ & + \mathsf{c}_2 \int_0^t \Big| \big(\mathsf{R}_\epsilon \overline{\mathsf{B}}^{\mathrm{c}\,\mathrm{II},\mathrm{II}}_{ij}(\mathsf{w})\mathsf{R}_\epsilon - \mathsf{R}_{\epsilon'} \overline{\mathsf{B}}^{\mathrm{c}\,\mathrm{II},\mathrm{II}}_{ij}(\mathsf{w})\mathsf{R}_{\epsilon'}\big)\partial_i\partial_j \widetilde{\mathsf{w}}^{\epsilon'}_{\mathrm{II}} \Big|_{l-2} d\tau \Big). \end{aligned} \tag{5.37}$$

We then note that the right hand side converge to zero as ϵ and ϵ' go to zero from (5.24) so that the family $(\widetilde{\mathsf{w}}^\epsilon)_{0<\epsilon\leq}$ form a Cauchy sequence in $C^0\big([0,\bar\tau], H^{l-1}\big)$ as $\epsilon \to 0$. We may then define

$$\widetilde{\mathsf{w}} = \lim_{\epsilon\to 0} \widetilde{\mathsf{w}}^\epsilon,$$

so that $\widetilde{\mathsf{w}} \in C^0\big([0,\bar\tau], H^{l-1}\big)$ and using standard argument from functional analysis and the estimates of Proposition 5.3 we obtain that $\widetilde{\mathsf{w}}$ is in $L^\infty\big((0,\bar\tau), H^l\big)$ and that $\int_0^t |\widetilde{\mathsf{w}}_{\mathrm{II}}(\tau) - \widetilde{\mathsf{w}}^\star_{\mathrm{II}}|^2_{l+1}\, d\tau$ is finite.

In order to establish that $\widetilde{\mathsf{w}}$ is indeed in $C^0\big([0,\bar\tau],H^l\big)$ we introduce the regularized solutions $\widetilde{\mathsf{w}}^\delta=\mathsf{R}_\delta\widetilde{\mathsf{w}}$ and establish that the family $(\widetilde{\mathsf{w}}^\delta)_{0<\delta\le1}$ form a Cauchy sequence in $C^0\big([0,\bar\tau],H^l\big)$. To this aim, we note that

$$\overline{\mathsf{A}}_0(\mathsf{w})\partial_t\widetilde{\mathsf{w}}^\delta+\sum_{i\in\mathcal{D}}\overline{\mathsf{A}}_i'(\mathsf{w})\partial_i\widetilde{\mathsf{w}}^\delta-\sum_{i,j\in\mathcal{D}}\overline{\mathsf{B}}_{ij}^{\mathrm{d}}(\mathsf{w})\partial_i\partial_j\widetilde{\mathsf{w}}^\delta-\sum_{i,j\in\mathcal{D}}\overline{\mathsf{B}}_{ij}^{\mathrm{c}}(\mathsf{w})\partial_i\partial_j\widetilde{\mathsf{w}}^\delta$$
$$+\overline{\mathsf{L}}(\mathsf{w},\nabla\mathsf{w}_{\mathrm{r}})\widetilde{\mathsf{w}}^\delta=\overline{\mathsf{A}}_0\mathsf{R}_\delta\big(\overline{\mathsf{A}}_0^{-1}\mathsf{f}\big)+\overline{\mathsf{A}}_0\mathsf{R}_\delta\big(\overline{\mathsf{A}}_0^{-1}\mathsf{g}\big)+\mathsf{h}^\delta,\tag{5.38}$$

where

$$\mathsf{h}^\delta=-\sum_{i\in\mathcal{D}}\overline{\mathsf{A}}_0\big[\mathsf{R}_\delta,\overline{\mathsf{A}}_0^{-1}\,\overline{\mathsf{A}}_i'\big]\partial_i\widetilde{\mathsf{w}}-\overline{\mathsf{A}}_0\big[\mathsf{R}_\delta,\overline{\mathsf{A}}_0^{-1}\,\overline{\mathsf{L}}\,\big]\widetilde{\mathsf{w}}$$
$$+\sum_{i,j\in\mathcal{D}}\overline{\mathsf{A}}_0\big[\mathsf{R}_\delta,\overline{\mathsf{A}}_0^{-1}\,\overline{\mathsf{B}}_{ij}^{\mathrm{d}}\big]\partial_i\partial_j\widetilde{\mathsf{w}}+\sum_{i,j\in\mathcal{D}}\overline{\mathsf{A}}_0\big[\mathsf{R}_\delta,\overline{\mathsf{A}}_0^{-1}\,\overline{\mathsf{B}}_{ij}^{\mathrm{c}}\big]\partial_i\partial_j\widetilde{\mathsf{w}}.\tag{5.39}$$

We may thus write that

$$\overline{\mathsf{A}}_0(\mathsf{w})\partial_t(\widetilde{\mathsf{w}}^\delta-\widetilde{\mathsf{w}}^{\delta'})+\sum_{i\in\mathcal{D}}\overline{\mathsf{A}}_i'(\mathsf{w})\partial_i(\widetilde{\mathsf{w}}^\delta-\widetilde{\mathsf{w}}^{\delta'})-\sum_{i,j\in\mathcal{D}}\overline{\mathsf{B}}_{ij}^{\mathrm{d}}(\mathsf{w})\partial_i\partial_j(\widetilde{\mathsf{w}}^\delta-\widetilde{\mathsf{w}}^{\delta'})$$
$$-\sum_{i,j\in\mathcal{D}}\overline{\mathsf{B}}_{ij}^{\mathrm{c}}(\mathsf{w})\partial_i\partial_j(\widetilde{\mathsf{w}}^\delta-\widetilde{\mathsf{w}}^{\delta'})+\overline{\mathsf{L}}(\mathsf{w},\nabla\mathsf{w}_{\mathrm{r}})(\widetilde{\mathsf{w}}^\delta-\widetilde{\mathsf{w}}^{\delta'})=$$
$$\overline{\mathsf{A}}_0(\mathsf{R}_\delta-\mathsf{R}_{\delta'})\big(\overline{\mathsf{A}}_0^{-1}\mathsf{f}\big)+\overline{\mathsf{A}}_0(\mathsf{R}_\delta-\mathsf{R}_{\delta'})\big(\overline{\mathsf{A}}_0^{-1}\mathsf{g}\big)+\mathsf{h}^\delta-\mathsf{h}^{\delta'}.\tag{5.40}$$

We then note that $\overline{\mathsf{A}}_0(\mathsf{R}_\delta-\mathsf{R}_{\delta'})\big(\overline{\mathsf{A}}_0^{-1}\mathsf{f}\big)$ goes to zero in H^l and $\overline{\mathsf{A}}_0(\mathsf{R}_\delta-\mathsf{R}_{\delta'})\big(\overline{\mathsf{A}}_0^{-1}\mathsf{g}\big)$ goes to zero in H^{l-1} from (5.24) and that the I components of both h^δ and $\mathsf{h}^{\delta'}$ go to zero in H^l and II components of both h^δ and $\mathsf{h}^{\delta'}$ go to zero in H^{l-1} from (5.27) so that the family $(\widetilde{\mathsf{w}}^\delta)_{0<\delta\le1}$ form a Cauchy sequence in $C^0\big([0,\bar\tau],H^l\big)$ from the linearized estimates and this completes the proof. □

6 Nonlinear equations

A local existence theorem of strong solutions is first established in an abstract framework and then applied to the particular situation of diffuse interface fluids.

6.1 Structural assumptions

An abstract symmetric-antisymmetric hyperbolic-parabolic augmented system in normal form is considered in the form

$$\overline{\mathsf{A}}_0(\mathsf{w})\partial_t\mathsf{w}+\sum_{i\in\mathcal{D}}\overline{\mathsf{A}}_i(\mathsf{w})\partial_i\mathsf{w}-\sum_{i,j\in\mathcal{D}}\overline{\mathsf{B}}_{ij}^{\mathrm{d}}(\mathsf{w})\partial_i\partial_j\mathsf{w}-\sum_{i,j\in\mathcal{D}}\overline{\mathsf{B}}_{ij}^{\mathrm{c}}(\mathsf{w})\partial_i\partial_j\mathsf{w}=\mathsf{h}(\mathsf{w},\nabla\mathsf{w}),\tag{6.1}$$

where t denotes the time variable, ∂_i the derivative in the ith spatial direction, w the normal variable, d_{s} the space dimension and $\mathcal{D}=\{1,\dots,d_{\mathrm{s}}\}$ the indexing set of spatial directions. The variables w has $\mathsf{n}\ge1$ components and $\mathbb{R}^{\mathsf{n}}$ is decomposed into $\mathbb{R}^{\mathsf{n}}=\mathbb{R}^{\mathsf{n}_{\mathrm{I}}}\times\mathbb{R}^{\mathsf{n}_{\mathrm{II}}}$ with $\mathsf{n}=\mathsf{n}_{\mathrm{I}}+\mathsf{n}_{\mathrm{II}}$ so that w is decomposed into $\mathsf{w}=(\mathsf{w}_{\mathrm{I}},\mathsf{w}_{\mathrm{II}})^t$ where $\mathsf{w}_{\mathrm{I}}\in\mathbb{R}^{\mathsf{n}_{\mathrm{I}}}$ denotes the hyperbolic components and $\mathsf{w}_{\mathrm{II}}\in\mathbb{R}^{\mathsf{n}_{\mathrm{II}}}$ the parabolic components. The system (6.1) is assumed to be an *augmented system* in the sense that w_{I} should be $\mathsf{w}_{\mathrm{I}}=(\mathsf{w}_{\mathrm{I}'},\mathsf{w}_{\mathrm{I}''})^t$ with $\mathsf{w}_{\mathrm{I}''}=\nabla\mathsf{w}_{\mathrm{I}'}$. The augmented system (6.1) has thus been built by adding the gradient $\mathsf{w}_{\mathrm{I}''}$ of a scalar hyperbolic component $\mathsf{w}_{\mathrm{I}'}$ to the unknowns. More complex situations with extra hyperbolic components or systems augmented with higher order derivatives lay out of the scope of the present work. We denote by w_{r} the subset of $\mathsf{n}_{\mathrm{r}}=1+\mathsf{n}_{\mathrm{II}}$ components of w defined by $\mathsf{w}_{\mathrm{r}}=(\mathsf{w}_{\mathrm{I}'},\mathsf{w}_{\mathrm{II}})^t$ that will have more regularity.

The following general assumptions are made concerning the system (6.1). The matrix $\overline{\mathsf{A}}_0\in\mathbb{R}^{\mathsf{n},\mathsf{n}}$ is symmetric, positive definite, block diagonal $\overline{\mathsf{A}}_0=\mathrm{diag}(\overline{\mathsf{A}}_0^{\mathrm{I},\mathrm{I}},\overline{\mathsf{A}}_0^{\mathrm{II},\mathrm{II}})$, and the block $\overline{\mathsf{A}}_0^{\mathrm{II},\mathrm{II}}$ only depends on the subvariable w_{r}. The matrices $\overline{\mathsf{A}}_i\in\mathbb{R}^{\mathsf{n},\mathsf{n}}$, $i\in\mathcal{D}$, are symmetric, the dissipation matrices are such that $(\overline{\mathsf{B}}_{ij}^{\mathrm{d}})^t=\overline{\mathsf{B}}_{ji}^{\mathrm{d}}$, $i,j\in\mathcal{D}$, have nonzero components only into the right lower $\overline{\mathsf{B}}_{ij}^{\mathrm{d}\,\mathrm{II},\mathrm{II}}$ blocks,

and $\overline{\mathsf{B}}^{\mathrm{d}\,\mathrm{II},\mathrm{II}} = \sum_{i,j\in\mathcal{D}} \overline{\mathsf{B}}^{\mathrm{d}\,\mathrm{II},\mathrm{II}}_{ij}(\mathsf{w})\xi_i\xi_j$ is positive definite for $\boldsymbol{\xi} \in \Sigma^{d_\mathrm{s}-1}$. The matrices $\overline{\mathsf{B}}^{\mathrm{c}}_{ij}$ are such that $(\overline{\mathsf{B}}^{\mathrm{c}}_{ij})^t = -\overline{\mathsf{B}}^{\mathrm{c}}_{ji}$, the blocks $\overline{\mathsf{B}}^{\mathrm{c},\mathrm{I}\mathrm{I}}_{ij}$ vanish $\overline{\mathsf{B}}^{\mathrm{c},\mathrm{I}\mathrm{I}}_{ij} = 0$, and the blocks $\overline{\mathsf{B}}^{\mathrm{c},\mathrm{I},\mathrm{II}}_{ij}$ and $\overline{\mathsf{B}}^{\mathrm{c},\mathrm{II},\mathrm{I}}_{ij}$ only depend on the subvariable w_r. The right hand side $\mathsf{h} = (\mathsf{h}_\mathrm{I}, \mathsf{h}_\mathrm{II})^t$ is in the general form

$$\mathsf{h}_\mathrm{I} = \sum_{i\in\mathcal{D}} \overline{\mathsf{M}}^{\mathrm{I}}_i(\mathsf{w})\partial_i\mathsf{w}_\mathrm{r} + \sum_{i,j\in\mathcal{D}} \overline{\mathsf{M}}^{\mathrm{I},\mathrm{I}}_{ij}(\mathsf{w})\partial_i\mathsf{w}_\mathrm{r}\partial_j\mathsf{w}_\mathrm{r}, \tag{6.2}$$

$$\mathsf{h}_\mathrm{II} = \sum_{i\in\mathcal{D}} \overline{\mathsf{M}}^{\mathrm{II}}_i(\mathsf{w})\partial_i\mathsf{w} + \sum_{i,j\in\mathcal{D}} \overline{\mathsf{M}}^{\mathrm{II},\mathrm{II}}_{ij}(\mathsf{w})\partial_i\mathsf{w}\partial_j\mathsf{w}, \tag{6.3}$$

where $\overline{\mathsf{M}}^{\mathrm{I}}_i \in \mathbb{R}^{\mathsf{n}_\mathrm{I},\mathsf{n}_\mathrm{r}}$, $\overline{\mathsf{M}}^{\mathrm{I},\mathrm{I}}_{ij}(\mathsf{w}) \in \mathbb{R}^{\mathsf{n}_\mathrm{I},\mathsf{n}_\mathrm{r},\mathsf{n}_\mathrm{r}}$, $\overline{\mathsf{M}}^{\mathrm{II}}_i \in \mathbb{R}^{\mathsf{n}_\mathrm{II},\mathsf{n}}$, and $\overline{\mathsf{M}}^{\mathrm{II},\mathrm{II}}_{ij} \in \mathbb{R}^{\mathsf{n}_\mathrm{II},\mathsf{n},\mathsf{n}}$ are linear operators. The hyperbolic components h_I thus includes linear and quadratic contributions in terms of $\nabla\mathsf{w}_\mathrm{r}$ whereas h_II includes linear and quadratic contributions in terms of $\nabla\mathsf{w}$. All the system coefficients $\overline{\mathsf{A}}_0$, $\overline{\mathsf{A}}_i$, $i \in \mathcal{D}$, $\overline{\mathsf{B}}^{\mathrm{d}}_{ij}$, $\overline{\mathsf{B}}^{\mathrm{c}}_{ij}$, $i, j \in \mathcal{D}$, $\overline{\mathsf{M}}^{\mathrm{I}}_i$, $\overline{\mathsf{M}}^{\mathrm{I},\mathrm{I}}_{ij}$, $\overline{\mathsf{M}}^{\mathrm{II}}_i$, and $\overline{\mathsf{M}}^{\mathrm{II},\mathrm{II}}_{ij}$ are assumed to have at least regularity $C^{l+2}(\mathcal{O}_\mathsf{w})$ where $\mathcal{O}_\mathsf{w}$ is a simply connected open set in $\mathbb{R}^{d_\mathrm{s}}$. We also denote by $\mathsf{w}^\star$ a constant state in the open set $\mathcal{O}_\mathsf{w}$ such that $\mathsf{w}^\star_{\mathrm{I}''} = 0$. In addition, it is assumed that whenever $\mathsf{w}_{0\mathrm{I}''} = \nabla\mathsf{w}_{0\mathrm{I}'}$, the regular solutions of (6.1) are such that $\mathsf{w}_{\mathrm{I}''} = \nabla\mathsf{w}_{\mathrm{I}'}$.

We further assume that there exists a linearized version of (6.1) in the form

$$\overline{\mathsf{A}}_0(\mathsf{w})\partial_t\widetilde{\mathsf{w}} + \sum_{i\in\mathcal{D}} \overline{\mathsf{A}}'_i(\mathsf{w})\partial_i\widetilde{\mathsf{w}} - \sum_{i,j\in\mathcal{D}} \overline{\mathsf{B}}^{\mathrm{d}}_{ij}(\mathsf{w})\partial_i\partial_j\widetilde{\mathsf{w}} - \sum_{i,j\in\mathcal{D}} \overline{\mathsf{B}}^{\mathrm{c}}_{ij}(\mathsf{w})\partial_i\partial_j\widetilde{\mathsf{w}} + \overline{\mathsf{L}}(\mathsf{w},\nabla\mathsf{w}_\mathrm{r})\widetilde{\mathsf{w}} = \mathsf{h}'(\mathsf{w},\nabla\mathsf{w}). \tag{6.4}$$

The hyperbolic blocks $\overline{\mathsf{A}}'^{\mathrm{I},\mathrm{I}}_i$ of the matrices $\overline{\mathsf{A}}'_i$ are assumed to be given by $\overline{\mathsf{A}}'^{\mathrm{I},\mathrm{I}}_i = \overline{\mathsf{A}}^{\mathrm{I},\mathrm{I}}_i$ and are thus symmetric. The matrix $\overline{\mathsf{L}} = \overline{\mathsf{L}}(\mathsf{w},\mathfrak{p}_\mathrm{r})$ is assumed to be block diagonal $\overline{\mathsf{L}} = \mathrm{diag}(\overline{\mathsf{L}}^{\mathrm{I},\mathrm{I}}, \overline{\mathsf{L}}^{\mathrm{II},\mathrm{II}})$ and to be a linear function of $\mathfrak{p}_\mathrm{r} \in \mathbb{R}^{\mathsf{n}_\mathrm{r},d_\mathrm{s}}$ so that $\overline{\mathsf{L}}^{\mathrm{I},\mathrm{I}} = \mathfrak{L}^{\mathrm{I},\mathrm{I}}(\mathsf{w})\mathfrak{p}_\mathrm{r}$ and $\overline{\mathsf{L}}^{\mathrm{II},\mathrm{II}} = \mathfrak{L}^{\mathrm{II},\mathrm{II}}(\mathsf{w})\mathfrak{p}_\mathrm{r}$ where $\mathfrak{L}^{\mathrm{I},\mathrm{I}} \in \mathbb{R}^{\mathsf{n}_\mathrm{I},\mathsf{n},\mathsf{n}_\mathrm{r},d_\mathrm{s}}$ and $\mathfrak{L}^{\mathrm{II},\mathrm{II}} \in \mathbb{R}^{\mathsf{n}_\mathrm{II},\mathsf{n},\mathsf{n}_\mathrm{r},d_\mathrm{s}}$ depend on w. The systems coefficients are naturally related by

$$\sum_{i\in\mathcal{D}} \big(\overline{\mathsf{A}}'_i(\mathsf{w}) - \overline{\mathsf{A}}_i(\mathsf{w})\big)\mathfrak{p} + \overline{\mathsf{L}}(\mathsf{w},\mathfrak{p}_\mathrm{r})\mathsf{w} + \mathsf{h}(\mathsf{w},\mathfrak{p}) = \mathsf{h}'(\mathsf{w},\mathfrak{p}) \tag{6.5}$$

for any $\mathsf{w} \in \mathcal{O}_\mathsf{w}$ and any vector $\mathfrak{p} \in \mathbb{R}^{\mathsf{n}}$. From (6.5) and the relations $\overline{\mathsf{A}}'^{\mathrm{I},\mathrm{I}}_i = \overline{\mathsf{A}}^{\mathrm{I},\mathrm{I}}_i$ we notably deduce that h' has a structure similar to that of h with corresponding matrices $\overline{\mathsf{M}}'^{\mathrm{I}}_i$, $\overline{\mathsf{M}}'^{\mathrm{I},\mathrm{I}}_{ij}$, $\overline{\mathsf{M}}'^{\mathrm{II}}_i$, and $\overline{\mathsf{M}}'^{\mathrm{II},\mathrm{II}}_{ij}$. The matrices $\overline{\mathsf{A}}'_i(\mathsf{w})$, the coefficients $\mathfrak{L}^{\mathrm{I},\mathrm{I}}$ and $\mathfrak{L}^{\mathrm{II},\mathrm{II}}$, and the coefficients $\overline{\mathsf{M}}'^{\mathrm{I}}_i$, $\overline{\mathsf{M}}'^{\mathrm{I},\mathrm{I}}_{ij}$, $\overline{\mathsf{M}}'^{\mathrm{II}}_i$, and $\overline{\mathsf{M}}'^{\mathrm{II},\mathrm{II}}_{ij}$ are assumed to have at least regularity C^{l+2} over $\mathcal{O}_\mathsf{w}$. The state $\mathsf{w}^\star$ is also such that $\overline{\mathsf{L}}(\mathsf{w},\mathfrak{p}_\mathrm{r})\mathsf{w}^\star = 0$ for any $\mathsf{w} \in \mathcal{O}_\mathsf{w}$ and $\mathfrak{p}_\mathrm{r} \in \mathbb{R}^{\mathsf{n}_\mathrm{r},d_\mathrm{s}}$. We further assume the linearized version (6.4) of the system is such that the constraint $\mathsf{w}_{\mathrm{I}''} = \nabla\mathsf{w}_{\mathrm{I}'}$ is transmitted from w to $\widetilde{\mathsf{w}}$, that is, whenever w and $\widetilde{\mathsf{w}}$ are regular, $\mathsf{w}_{\mathrm{I}''} = \nabla\mathsf{w}_{\mathrm{I}'}$, $\widetilde{\mathsf{w}}_{0\mathrm{I}''} = \nabla\widetilde{\mathsf{w}}_{0\mathrm{I}'}$, and $\widetilde{\mathsf{w}}^\star_{\mathrm{I}''} = 0$ then that $\widetilde{\mathsf{w}}_{\mathrm{I}''} = \nabla\widetilde{\mathsf{w}}_{\mathrm{I}'}$. The structural assumptions of Section 5.1 are notably satisfied by the linearized system (6.4).

6.2 Existence of solutions to nonlinear equations

We establish in this section local existence of solutions for augmented systems in the general form (6.1) under the structural assumptions listed in Section 6.1.

Theorem 6.1. *Let $d \geq 1$, $l \geq l_0 + 2$, $l_0 = [d/2] + 1$, be integers and let $b > 0$. Let $\mathcal{O}_0$ be such that $\overline{\mathcal{O}}_0 \subset \mathcal{O}_\mathsf{w}$, a_1 such that $0 < a_1 < \mathrm{dist}(\overline{\mathcal{O}}_0, \partial\mathcal{O}_\mathsf{w})$, and $\mathcal{O}_1 = \{\,\mathsf{w} \in \mathcal{O}_\mathsf{w}; \mathrm{dist}(\mathsf{w},\overline{\mathcal{O}}_0) < a_1\,\}$. There exists $\bar\tau > 0$ depending on $\mathcal{O}_1$ and b such that for any w_0 with $\mathsf{w}_0 \in \overline{\mathcal{O}}_0$, $\mathsf{w}_0 - \mathsf{w}^\star \in H^l$, $\mathsf{w}_{0\mathrm{I}''} = \nabla\mathsf{w}_{0\mathrm{I}'}$ and*

$$|\mathsf{w}_0 - \mathsf{w}^\star|^2_l < b^2, \tag{6.6}$$

there exists a unique local solution w to the system (6.1) with initial condition $\mathsf{w}(0,\boldsymbol{x}) = \mathsf{w}_0(\boldsymbol{x})$, such that $\mathsf{w}(t,\boldsymbol{x}) \in \mathcal{O}_1$ for $(t,\boldsymbol{x}) \in [0,\bar\tau]\times\mathbb{R}^{d_\mathrm{s}}$, $\mathsf{w}_{\mathrm{I}''} = \nabla\mathsf{w}_{\mathrm{I}'}$, and

$$\begin{aligned}
&\mathsf{w}_\mathrm{I} - \mathsf{w}^\star_\mathrm{I} \in C^0\big([0,\bar\tau],H^l\big) \cap C^1\big([0,\bar\tau],H^{l-2}\big),\\
&\mathsf{w}_\mathrm{II} - \mathsf{w}^\star_\mathrm{II} \in C^0\big([0,\bar\tau],H^l\big) \cap C^1\big([0,\bar\tau],H^{l-2}\big) \cap L^2\big((0,\bar\tau),H^{l+1}\big).
\end{aligned}$$

Moreover, there exists $\mathsf{c}_\mathrm{loc} \geq 1$ only depending on $\mathcal{O}_1$ and b such that

$$\sup_{0\leq\tau\leq\bar\tau} |\mathsf{w}(\tau) - \mathsf{w}^\star|^2_l + \int_0^{\bar\tau} |\mathsf{w}_\mathrm{II}(\tau) - \mathsf{w}^\star_\mathrm{II}|^2_{l+1} d\tau \leq \mathsf{c}^2_\mathrm{loc}|\mathsf{w}_0 - \mathsf{w}^\star|^2_l. \tag{6.7}$$

Proof. Solutions to the nonlinear system (6.1) are fixed points $\widetilde{\mathsf{w}} = \mathsf{w}$ of the linearized equations (6.4). Fixed points are investigated in the space $\mathsf{w} \in \mathsf{X}^l_{\bar\tau}\big(\mathcal{O}_1, \overline{M}\,\big)$, where $\overline{M}$ is a positive number that will be determined later, and defined by $\mathsf{w}-\mathsf{w}^\star \in C^0\big([0,\bar\tau], H^l\big)$, $\partial_t \mathsf{w} \in C^0\big([0,\bar\tau], H^{l-2}\big)$, $\mathsf{w}_{\rm II}-\mathsf{w}^\star_{\rm II} \in L^2\big((0,\bar\tau), H^{l+1}\big)$, $\mathsf{w}(t,\boldsymbol{x}) \in \mathcal{O}_1$, $\mathsf{w}_{\rm I}{}'' = \nabla \mathsf{w}_{\rm I}{}'$, and

$$\sup_{0\le\tau\le\bar\tau} |\mathsf{w}(\tau) - \mathsf{w}^\star|_l^2 + \int_0^{\bar\tau} |\mathsf{w}_{\rm II}(\tau) - \mathsf{w}^\star_{\rm II}|^2_{l+1}\, d\tau \le \overline{M}^2, \tag{6.8}$$

$$\int_0^{\bar\tau} |\partial_t \mathsf{w}(\tau)|^2_{l-2}\, d\tau \le \overline{M}^2, \qquad \int_0^{\bar\tau} |\nabla \mathsf{w}_{\rm r}(\tau)|^2_l\, d\tau \le \overline{M}^2. \tag{6.9}$$

We first have to establish that the space $\mathsf{X}^l_{\bar\tau}\big(\mathcal{O}_1, \overline{M}\,\big)$ is invariant by the map $\mathsf{w} \mapsto \widetilde{\mathsf{w}}$ for a suitable $\overline{M}$ and for a time $\bar\tau$ small enough, where $\widetilde{\mathsf{w}}$ denotes the solution of the linearized system (6.4) with w_0 as initial condition and with $\widetilde{\mathsf{w}}^\star = \mathsf{w}^\star$. For w in $\mathsf{X}^l_{\bar\tau}\big(\mathcal{O}_1, \overline{M}\,\big)$ the solution $\widetilde{\mathsf{w}}$ of the linearized equations (6.4) exists from Theorem 5.4, the estimates of Theorem 5.1 hold as well as the gradient constraint $\widetilde{\mathsf{w}}_{\rm I}{}'' = \nabla \widetilde{\mathsf{w}}_{\rm I}{}'$. Letting then

$$\widetilde{M}^2 = \sup_{0\le\tau\le\bar\tau} |\widetilde{\mathsf{w}}(\tau) - \mathsf{w}^\star|_l^2 + \int_0^{\bar\tau} |\widetilde{\mathsf{w}}_{\rm II}(\tau) - \mathsf{w}^\star_{\rm II}|^2_{l+1}\, d\tau,$$

$$\widetilde{M}_{\rm t}^2 = \int_0^{\bar\tau} |\partial_t \widetilde{\mathsf{w}}(\tau)|^2_{l-2}\, d\tau, \qquad \widetilde{M}_{\rm r}^2 = \int_0^{\bar\tau} |\nabla \widetilde{\mathsf{w}_{\rm r}}(\tau)|^2_l\, d\tau,$$

we have to estimate $\widetilde{M}$, $\widetilde{M}_{\rm t}$ and $\widetilde{M}_{\rm r}$.

From the estimates of Theorem 5.1 with $l' = l$, further using (6.9) and $\Big(\int_0^{\bar\tau} |\mathsf{h}'_{\rm I}|_l\, d\tau\Big)^2 \le \bar\tau \int_0^{\bar\tau} |\mathsf{h}'_{\rm I}|^2_l\, d\tau$, we may write that

$$\widetilde{M}^2 \le \mathsf{c}_1^2 \exp\big(\mathsf{c}_2\big(\bar\tau + 2\overline{M}\sqrt{\bar\tau}\big)\big)\Big(|\widetilde{\mathsf{w}}_0 - \mathsf{w}^\star|_l^2 + \mathsf{c}_2\bar\tau \int_0^{\bar\tau} |\mathsf{h}'_{\rm I}|^2_l\, d\tau + \mathsf{c}_2 \int_0^{\bar\tau} |\mathsf{h}'_{\rm II}|^2_{l-1}\, d\tau\Big), \tag{6.10}$$

with constants $\mathsf{c}_1(\mathcal{O}_1)$ and $\mathsf{c}_2(\mathcal{O}_1, \overline{M})$. On the other hand, using the structure (6.2) and (6.3) of the right hand side h, the relation (6.5) expressing h', les nonlinear estimates (5.11) and (5.12), and the definition of $\mathsf{X}^l_{\bar\tau}\big(\mathcal{O}_1, \overline{M}\,\big)$, we obtain

$$\int_0^{\bar\tau} |\mathsf{h}'_{\rm I}|^2_l\, d\tau \le \mathsf{c}_2(\mathcal{O}_1, \overline{M})\overline{M}^2,$$

$$\int_0^{\bar\tau} |\mathsf{h}'_{\rm II}|^2_{l-1}\, d\tau \le \bar\tau \mathsf{c}_2(\mathcal{O}_1, \overline{M})\overline{M}^2.$$

Moreover, from the linearized equations (6.4), we may evaluate the time derivative $\partial_t \widetilde{\mathsf{w}}$ in term of the spatial gradients in the form

$$\int_0^{\bar\tau} |\partial_t \widetilde{\mathsf{w}}(\tau)|^2_{l-2}\, d\tau \le \bar\tau \mathsf{c}'_2(\mathcal{O}_1, \overline{M})\big(\widetilde{M}^2 + \overline{M}^2\big), \tag{6.11}$$

for some constant $\mathsf{c}'_2(\mathcal{O}_1, \overline{M}) \ge 1$ increasing with $\overline{M}$. We now define $\overline{M}_b$ by

$$\overline{M}_b = 2\mathsf{c}_1(\mathcal{O}_1)b,$$

and assume that $\bar\tau \le 1$ is small enough such that

$$\exp\big(\mathsf{c}_2(\mathcal{O}_1, \overline{M}_b)(\bar\tau + \overline{M}_b\sqrt{\bar\tau})\big) \le 2, \tag{6.12}$$

$$2\mathsf{c}_2^2(\mathcal{O}_1, \overline{M}_b)\bar\tau\big(2\mathsf{c}_1(\mathcal{O}_1)\big)^2 \le 1, \tag{6.13}$$

$$2\bar\tau \mathsf{c}'_2(\mathcal{O}_1, \overline{M}_b) \le 1, \tag{6.14}$$

as well as $\mathsf{c}_0\overline{M}_b\sqrt{\bar\tau} < a_1$ where c_0 is such that $\|\phi\|_{L^\infty} \le \mathsf{c}_0|\phi|_{l-2}$.

We claim that for any w in $\mathsf{X}^l_{\bar\tau}\big(\mathcal{O}_1, \overline{M}_b\big)$, any w_0 such that $\mathsf{w}_0 - \mathsf{w}^\star \in H^l$, $\mathsf{w}_{0\rm I}{}'' = \nabla \mathsf{w}_{0\rm I}{}'$, $\mathsf{w}_0 \in \overline{\mathcal{O}}_0$, and $|\mathsf{w}_0 - \mathsf{w}^\star|_l^2 < b^2$, the solution $\widetilde{\mathsf{w}}$ to the linearized equations (6.4) with initial condition $\widetilde{\mathsf{w}}_0 = \mathsf{w}_0$ stays in the same space $\mathsf{X}^l_{\bar\tau}\big(\mathcal{O}_1, \overline{M}_b\big)$. We first obtain from (6.12) and (6.13) that

$$\widetilde{M}^2 \le 2\mathsf{c}_1^2(b^2 + 2\bar\tau\mathsf{c}_2^2\overline{M}_b^2) \le 4\mathsf{c}_1^2 b^2 = \overline{M}_b^2.$$

Moreover from the gradient constraint transmitted to $\widetilde{\mathsf{w}}$, and since $\bar\tau$ is such that $\bar\tau \le 1$, we have an estimate in the form

$$\widetilde{M}_{\mathrm{r}}^2 = \int_0^{\bar\tau} |\nabla\widetilde{\mathsf{w}_{\mathrm{r}}}(\tau)|_l^2 \, d\tau \le \widetilde{M}^2 \le \overline{M}_b^2.$$

In addition, for the time derivatives, we obtain from (6.11) and (6.14) that

$$\widetilde{M}_{\mathrm{t}}^2 \le \bar\tau \mathsf{c}_2'(\mathcal{O}_1, \overline{M}^{b\,2})\big(\widetilde{M}^2 + \overline{M}_b^2\big) \le \overline{M}_b^2,$$

and we also have $\widetilde{\mathsf{w}}_{\mathrm{I}''} = \nabla\widetilde{\mathsf{w}}_{\mathrm{I}'}$. Finally, from the time derivative estimates and $\mathsf{c}_0\overline{M}_b\sqrt{\bar\tau} < a_1$ we also obtain that $\widetilde{\mathsf{w}} \in \mathcal{O}_1$ and we have established that $\mathsf{X}_{\bar\tau}^l\big(\mathcal{O}_1, \overline{M}_b\big)$ is invariant.

Noting that the constant state $\mathsf{w}^\star$ is in $\mathsf{X}_{\bar\tau}^l\big(\mathcal{O}_1, \overline{M}_b\big)$, the sequence of successive approximations $\{\mathsf{w}^k\}_{k\ge0}$ starting at $\mathsf{w}^0 = \mathsf{w}^\star$ is thus well defined with $\mathsf{w}^{k+1} = \widetilde{\mathsf{w}}^k$, i.e., with w^{k+1} is obtained as the solution $\widetilde{\mathsf{w}} = \mathsf{w}^{k+1}$ of linearized equations with $\mathsf{w} = \mathsf{w}^k$ and with the initial condition w_0. Let $\delta^k\mathsf{w}$ denotes the difference $\delta^k\mathsf{w} = \mathsf{w}^{k+1} - \mathsf{w}^k$ for $k \ge 0$. For a suitable $\overline{\tau}$ small enough, we claim that the map $\mathsf{w} \mapsto \widetilde{\mathsf{w}}$ is a contraction for a weaker norm. Consider w and $\widehat{\mathsf{w}}$ in $\mathsf{X}_{\bar\tau}^l\big(\mathcal{O}_1, \overline{M}_b\big)$, and define $\delta\mathsf{w} = \mathsf{w} - \widehat{\mathsf{w}}$ and $\delta\widetilde{\mathsf{w}} = \widetilde{\mathsf{w}} - \widetilde{\widehat{\mathsf{w}}}$ where $\widetilde{\mathsf{w}}$ and $\widetilde{\widehat{\mathsf{w}}}$ are the solutions of the corresponding linearized equations with initial condition w_0. Forming the difference between the linearized equations, one may obtain that

$$\begin{aligned}\overline{\mathsf{A}}_0(\widehat{\mathsf{w}})\partial_t\delta\widetilde{\mathsf{w}} + \sum_{i\in\mathcal{D}} \overline{\mathsf{A}}_i(\widehat{\mathsf{w}})\partial_i\delta\widetilde{\mathsf{w}} - \sum_{i,j\in\mathcal{D}} \overline{\mathsf{B}}_{ij}^{\mathrm{d}}(\widehat{\mathsf{w}})\partial_i\partial_j\delta\widetilde{\mathsf{w}} \\ - \sum_{i,j\in\mathcal{D}} \overline{\mathsf{B}}_{ij}^{\mathrm{c}}(\widehat{\mathsf{w}})\partial_i\partial_j\delta\widetilde{\mathsf{w}} + \overline{\mathsf{L}}(\widehat{\mathsf{w}}, \nabla\mathsf{w}_{\mathrm{r}})\partial_i\partial_j\delta\widetilde{\mathsf{w}} = \delta\mathsf{h}',\end{aligned} \tag{6.15}$$

where

$$\begin{aligned}\delta\mathsf{h}' = &- \sum_{i\in\mathcal{D}} \big(\overline{\mathsf{A}}_0(\widehat{\mathsf{w}})\overline{\mathsf{A}}_0^{-1}(\mathsf{w})\overline{\mathsf{A}}_i'(\mathsf{w}) - \overline{\mathsf{A}}_i'(\widehat{\mathsf{w}})\big)\partial_i\mathsf{w} \\ &+ \sum_{i,j\in\mathcal{D}} \big(\overline{\mathsf{A}}_0(\widehat{\mathsf{w}})\overline{\mathsf{A}}_0^{-1}(\mathsf{w})\overline{\mathsf{B}}_{ij}^{\mathrm{d}}(\mathsf{w}) - \overline{\mathsf{B}}_{ij}^{\mathrm{d}}(\widehat{\mathsf{w}})\big)\partial_i\partial_j\mathsf{w} \\ &+ \sum_{i,j\in\mathcal{D}} \big(\overline{\mathsf{A}}_0(\widehat{\mathsf{w}})\overline{\mathsf{A}}_0^{-1}(\mathsf{w})\overline{\mathsf{B}}_{ij}^{\mathrm{c}}(\mathsf{w}) - \overline{\mathsf{B}}_{ij}^{\mathrm{c}}(\widehat{\mathsf{w}})\big)\partial_i\partial_j\mathsf{w} \\ &+ \big(\overline{\mathsf{A}}_0(\widehat{\mathsf{w}})\overline{\mathsf{A}}_0^{-1}(\mathsf{w}) - \mathbb{I}\big)\,\mathsf{h}'(\mathsf{w}, \nabla\mathsf{w}) + \mathsf{h}'(\mathsf{w}, \nabla\mathsf{w}) - \mathsf{h}'(\widehat{\mathsf{w}}, \nabla\widehat{\mathsf{w}}).\end{aligned}$$

These relations and the structural assumptions (6.2) (6.3) and (6.5) imply that $|\delta\mathsf{h}_{\mathrm{I}}'|_{l-2} \le \mathsf{c}_2|\delta\mathsf{w}|_{l-2} + \mathsf{c}_2|\delta\mathsf{w}_{\mathrm{II}}|_{l-1}$ and $|\delta\mathsf{h}_{\mathrm{II}}'|_{l-3} \le \mathsf{c}_2|\delta\mathsf{w}|_{l-2}$. The linearized estimates with $l' = l-2$ then yield that

$$\sup_{0\le\tau\le\bar\tau} |\delta\widetilde{\mathsf{w}}(\tau)|_{l-2}^2 + \int_0^{\bar\tau} |\delta\widetilde{\mathsf{w}}_{\mathrm{II}}(\tau)|_{l-1}^2 \, d\tau \le \mathsf{c}_2\bar\tau\Big(\sup_{0\le\tau\le\bar\tau} |\delta\mathsf{w}(\tau)|_{l-2}^2 + \int_0^{\bar\tau} |\delta\mathsf{w}_{\mathrm{II}}(\tau)|_{l-1}^2 \, d\tau\Big).$$

Assuming that $\mathsf{c}_2\bar\tau \le \frac{1}{2}$, we then have

$$\sup_{0\le\tau\le\bar\tau} |\delta\widetilde{\mathsf{w}}(\tau)|_{l-2}^2 + \int_0^{\bar\tau} |\delta\widetilde{\mathsf{w}}_{\mathrm{II}}(\tau)|_{l-1}^2 \, d\tau \le \frac{\mathsf{c}_0}{2^k}, \qquad 0 \le k, \tag{6.16}$$

where c_0 depends on $\mathcal{O}_1$, b, and the data but is independent of k. The sequence of successive approximation $\{\mathsf{w}^k\}_{k\ge0}$ is thus convergent over $[0, \bar\tau]$ for the norm $\sup_{0\le\tau\le\bar\tau} |\delta\widetilde{\mathsf{w}}(\tau)|_{l-2}^2 + \int_0^{\bar\tau} |\delta\widetilde{\mathsf{w}}_{\mathrm{II}}(\tau)|_{l-1}^2 \, d\tau$ towards a fixed point $\overline{\mathsf{w}} \in C^0\big([0,\bar\tau], H^{l-2}\big)$ that is a solution of (6.1). Since the sequence $\{\mathsf{w}^k\}_{k\ge0}$ is bounded in the space $\mathsf{X}_{\bar\tau}^l\big(\mathcal{O}_1, \overline{M}_b\big)$, it follows from standard functional analysis arguments that the limit function $\overline{\mathsf{w}}$ is also in $L^\infty\big((0,\bar\tau), H^l\big)$ and such that $\overline{\mathsf{w}}_{\mathrm{II}} - \mathsf{w}_{\mathrm{II}}^\star \in L^2\big((0,\bar\tau), H^{l+1}\big)$.

In order to establish that the limit $\overline{\mathsf{w}}$ is also in the space $C^0\big((0,\bar\tau), H^l\big)$ we consider the regularized functions $\mathsf{w}^\delta = \mathsf{R}_\delta\overline{\mathsf{w}}$ and we establish that the family $(\mathsf{w}^\delta)_{0<\delta\le1}$ form a Cauchy sequence in $C^0\big([0,\bar\tau], H^l\big)$. We indeed note that

$$\begin{aligned}\overline{\mathsf{A}}_0(\overline{\mathsf{w}})\partial_t\mathsf{w}^\delta + \sum_{i\in\mathcal{D}} \overline{\mathsf{A}}_i'(\overline{\mathsf{w}})\partial_i\mathsf{w}^\delta - \sum_{i,j\in\mathcal{D}} \overline{\mathsf{B}}_{ij}^{\mathrm{d}}(\overline{\mathsf{w}})\partial_i\partial_j\mathsf{w}^\delta - \sum_{i,j\in\mathcal{D}} \overline{\mathsf{B}}_{ij}^{\mathrm{c}}(\overline{\mathsf{w}})\partial_i\partial_j\mathsf{w}^\delta \\ + \overline{\mathsf{L}}(\overline{\mathsf{w}}, \nabla\overline{\mathsf{w}}_{\mathrm{r}})\mathsf{w}^\delta = \overline{\mathsf{A}}_0\mathsf{R}_\delta\big(\overline{\mathsf{A}}_0^{-1}\bar{\mathsf{h}}'\big) + \mathsf{h}'^\delta,\end{aligned} \tag{6.17}$$

where $\bar{\mathsf{h}}' = \mathsf{h}'(\overline{\mathsf{w}}, \nabla\overline{\mathsf{w}})$ with

$$\mathsf{h}'^{\delta} = -\sum_{i\in\mathcal{D}} \overline{\mathsf{A}}_0\big[\mathsf{R}_\delta, \overline{\mathsf{A}}_0^{-1}\,\overline{\mathsf{A}}_i'\big]\partial_i\overline{\mathsf{w}} - \overline{\mathsf{A}}_0\big[\mathsf{R}_\delta, \overline{\mathsf{A}}_0^{-1}\,\overline{\mathsf{L}}\,\big]\overline{\mathsf{w}} \tag{6.18}$$

$$+ \sum_{i,j\in\mathcal{D}} \overline{\mathsf{A}}_0\big[\mathsf{R}_\delta, \overline{\mathsf{A}}_0^{-1}\,\overline{\mathsf{B}}^{\mathrm{d}}_{ij}\big]\partial_i\partial_j\overline{\mathsf{w}} + \sum_{i,j\in\mathcal{D}} \overline{\mathsf{A}}_0\big[\mathsf{R}_\delta, \overline{\mathsf{A}}_0^{-1}\,\overline{\mathsf{B}}^{\mathrm{c}}_{ij}\big]\partial_i\partial_j\overline{\mathsf{w}}. \tag{6.19}$$

We may then write that

$$\begin{aligned}\overline{\mathsf{A}}_0(\overline{\mathsf{w}})\partial_t(\widetilde{\mathsf{w}}^{\delta} - \widetilde{\mathsf{w}}^{\delta'}) + \sum_{i\in\mathcal{D}} \overline{\mathsf{A}}_i'(\overline{\mathsf{w}})\partial_i(\widetilde{\mathsf{w}}^{\delta} - \widetilde{\mathsf{w}}^{\delta'}) - \sum_{i,j\in\mathcal{D}} \overline{\mathsf{B}}^{\mathrm{d}}_{ij}(\overline{\mathsf{w}})\partial_i\partial_j(\widetilde{\mathsf{w}}^{\delta} - \widetilde{\mathsf{w}}^{\delta'})\\ - \sum_{i,j\in\mathcal{D}} \overline{\mathsf{B}}^{\mathrm{c}}_{ij}(\overline{\mathsf{w}})\partial_i\partial_j(\widetilde{\mathsf{w}}^{\delta} - \widetilde{\mathsf{w}}^{\delta'}) + \overline{\mathsf{L}}(\overline{\mathsf{w}}, \nabla\overline{\mathsf{w}}_{\mathrm{r}})(\widetilde{\mathsf{w}}^{\delta} - \widetilde{\mathsf{w}}^{\delta'}) =\\ \overline{\mathsf{A}}_0(\mathsf{R}_\delta - \mathsf{R}_{\delta'})\big(\overline{\mathsf{A}}_0^{-1}\bar{\mathsf{h}}'\big) + \mathsf{h}'^{\delta} - \mathsf{h}'^{\delta'}.\end{aligned} \tag{6.20}$$

We may then observe that the I component of $\overline{\mathsf{A}}_0(\mathsf{R}_\delta - \mathsf{R}_{\delta'})\big(\overline{\mathsf{A}}_0^{-1}\bar{\mathsf{h}}'\big)$, h'^{δ} and $\mathsf{h}'^{\delta'}$ go to zero in H^l from from (5.24) and the II component of $\overline{\mathsf{A}}_0(\mathsf{R}_\delta - \mathsf{R}_{\delta'})\big(\overline{\mathsf{A}}_0^{-1}\bar{\mathsf{h}}'\big)$, h'^{δ} and $\mathsf{h}'^{\delta'}$ go to zero in H^{l-1} from (5.27) so that the family $(\widetilde{\mathsf{w}}^{\delta})_{0<\delta\leq 1}$ form a Cauchy sequence in $C^0\big([0,\bar\tau], H^l\big)$ and this completes the proof. □

6.3 Application to diffuse interface fluids

We apply in this section Theorem 6.1 to the diffuse interface fluid equations in normal form (4.10) under the assumptions (H_1)-(H_5). The unknown vector reads $\mathsf{w} = (\rho, \boldsymbol{w}, \boldsymbol{v}, T)^t$ and has $\mathsf{n} = 2d_{\mathrm{s}} + 2$ components with $\mathsf{n}_{\mathrm{I}} = d_{\mathrm{s}} + 1$ hyperbolic components $\mathsf{w}_{\mathrm{I}} = (\rho, \boldsymbol{w})^t$ and $\mathsf{n}_{\mathrm{II}} = d_{\mathrm{s}} + 1$ parabolic components $\mathsf{w}_{\mathrm{II}} = (\boldsymbol{v}, T)^t$ and the subvariable w_{r} is given by $\mathsf{w}_{\mathrm{r}} = (\rho, \boldsymbol{v}, T)^t$ with $\mathsf{n}_{\mathrm{r}} = 1 + \mathsf{n}_{\mathrm{II}}$. Letting $\mathcal{O}_{\mathsf{w}}$ to be the open set $\mathcal{O}_{\mathsf{z}}$ of (H_1), we select $\mathsf{w}^\star = (\rho^\star, 0, \boldsymbol{v}^\star, T^\star)^t \in \mathcal{O}_{\mathsf{w}}$, an open set $\mathcal{O}_0$ with $\overline{\mathcal{O}}_0 \subset \mathcal{O}_{\mathsf{w}}$ and $\mathsf{w}^\star \in \mathcal{O}_0$, a_1 such that $0 < a_1 < \mathrm{dist}(\overline{\mathcal{O}}_0, \partial\mathcal{O}_{\mathsf{w}})$, and we define $\mathcal{O}_1 = \{\, \mathsf{w} \in \mathcal{O}_{\mathsf{w}}; \mathrm{dist}(\mathsf{w}, \overline{\mathcal{O}}_0) < a_1 \,\}$.

It is assumed that the regularity class of thermodynamic functions γ is such that $\gamma - 2 \geq l + 2 \geq l_0 + 3$ so that all the system coefficients $\overline{\mathsf{A}}_0$, $\overline{\mathsf{A}}_i$, $i \in \mathcal{D}$, $\overline{\mathsf{B}}^{\mathrm{d}}_{ij}$, $\overline{\mathsf{B}}^{\mathrm{c}}_{ij}$, $i, j \in \mathcal{D}$, have at least regularity C^{l+2} over $\mathcal{O}_{\mathsf{w}}$. Moreover there exists a linearized version in the form (6.4) that enforce that gradient constraint. The matrix $\overline{\mathsf{L}} = \overline{\mathsf{L}}(\mathsf{w}, \boldsymbol{\nabla}\mathsf{w}_{\mathrm{r}})$ is block diagonal $\overline{\mathsf{L}} = \mathrm{diag}(\overline{\mathsf{L}}^{\mathrm{I,I}}, \overline{\mathsf{L}}^{\mathrm{II,II}})$ with $\overline{\mathsf{L}}^{\mathrm{II,II}} = 0$ and $\overline{\mathsf{L}}^{\mathrm{I,I}}$ is a linear function of $\boldsymbol{\nabla}\mathsf{w}_{\mathrm{r}}$ so that $\overline{\mathsf{L}}^{\mathrm{I,I}} = \mathfrak{L}^{\mathrm{I,I}}(\mathsf{w})\boldsymbol{\nabla}\mathsf{w}_{\mathrm{r}}$ and the right hand sides are in the form (6.2) and (6.3). The systems coefficients $\mathfrak{L}^{\mathrm{I,I}}$, $\overline{\mathrm{M}}_i'^{\mathrm{I}}$, $\overline{\mathrm{M}}_{ij}'^{\mathrm{I,I}}$, $\overline{\mathrm{M}}_i'^{\mathrm{II}}$, and $\overline{\mathrm{M}}_{ij}'^{\mathrm{II,II}}$ also have at least regularity $C^{l+2}(\mathcal{O}_{\mathsf{w}})$. Application of Theorem 6.1 then yields the following existence result.

Theorem 6.2. *Let $d \geq 1$, $l \geq l_0 + 2$, and $b > 0$. There exists $\bar\tau > 0$ depending on $\mathcal{O}_1$ and b such that for any w_0 with $\mathsf{w}_0 \in \overline{\mathcal{O}}_0$, $\mathsf{w}_0 - \mathsf{w}^\star \in H^l$, $\boldsymbol{w}_0 = \boldsymbol{\nabla}\rho_0$ and $|\mathsf{w}_0 - \mathsf{w}^\star|_l^2 < b^2$ there exists a unique local solution w to the system (4.10) with initial condition $\mathsf{w}(0, \boldsymbol{x}) = \mathsf{w}_0(\boldsymbol{x})$, $\mathsf{w}(t, \boldsymbol{x}) \in \mathcal{O}_1$ for $(t, \boldsymbol{x}) \in [0, \bar\tau] \times \mathbb{R}^{d_{\mathrm{s}}}$, $\boldsymbol{w} = \boldsymbol{\nabla}\rho$ and with*

$$\rho - \rho^\star \in C^0\big([0,\bar\tau], H^{l+1}\big), \tag{6.21}$$

$$\boldsymbol{v} - \boldsymbol{v}^\star \in C^0\big([0,\bar\tau], H^l\big) \cap L^2\big((0,\bar\tau), H^{l+1}\big) \tag{6.22}$$

$$T - T^\star \in C^0\big([0,\bar\tau], H^l\big) \cap L^2\big((0,\bar\tau), H^{l+1}\big). \tag{6.23}$$

The time derivatives are also such that $\partial_t\rho \in C^0\big([0,\bar\tau], H^{l-1}\big) \cap L^2\big((0,\bar\tau), H^l\big)$, $\partial_t\boldsymbol{v} \in C^0\big([0,\bar\tau], H^{l-2}\big)$ and $\partial_t T \in C^0\big([0,\bar\tau], H^{l-2}\big)$. Moreover, there exists $\mathsf{c}_{\mathrm{loc}} \geq 1$ only depending on $\mathcal{O}_1$ and b such that

$$\sup_{0\leq\tau\leq\bar\tau} |\rho(\tau) - \rho^\star|_{l+1}^2 + \sup_{0\leq\tau\leq\bar\tau} |\boldsymbol{v}(\tau) - \boldsymbol{v}^\star|_l^2 + \sup_{0\leq\tau\leq\bar\tau} |T(\tau) - T^\star|_l^2 + \int_0^{\bar\tau} |\boldsymbol{v}(\tau) - \boldsymbol{v}^\star|_{l+1}^2 d\tau \tag{6.24}$$

$$+ \int_0^{\bar\tau} |T(\tau) - T^\star|_{l+1}^2 d\tau \leq \mathsf{c}_{\mathrm{loc}}^2\Big(|\rho_0(\tau) - \rho^\star|_{l+1}^2 + |\boldsymbol{v}_0(\tau) - \boldsymbol{v}^\star|_l^2 + |T_0(\tau) - T^\star|_l^2\Big). \tag{6.25}$$

Proof. In order to apply Theorem 6.1 we have to check that the assumptions listed in Section 6.1 are satisfied for the system in normal form (4.10).

The matrix $\overline{\mathsf{A}}_0$ given by (4.15) is symmetric positive definite, block-diagonal $\overline{\mathsf{A}}_0 = \mathrm{diag}(\overline{\mathsf{A}}_0^{\mathrm{I},\mathrm{I}}, \overline{\mathsf{A}}_0^{\mathrm{II},\mathrm{II}})$, with $\overline{\mathsf{A}}_0^{\mathrm{II},\mathrm{II}}$ only depending on $(\mathsf{w}_{\mathrm{I}'}, \mathsf{w}_{\mathrm{II}})$, and is positive definite form (4.16). The convective matrices $\overline{\mathsf{A}}_i$, $i \in \mathcal{D}$, given by (4.17) are also symmetric. The dissipation matrices have been decomposed in the form (4.18) with $\overline{\mathsf{B}}_{ij}^{\mathfrak{v}}$, $\overline{\mathsf{B}}_{ij}^{\eta}$, and $\overline{\mathsf{B}}_{ij}^{\lambda}$ given repectively by (4.19), (4.20), and (4.21). The reciprocity relations $(\overline{\mathsf{B}}_{ij}^{\mathrm{d}})^t = \overline{\mathsf{B}}_{ji}^{\mathrm{d}}$, $i, j \in \mathcal{D}$, thus hold and the matrices $\overline{\mathsf{B}}_{ij}^{\mathrm{d}}$ have nonzero components only into the lower right $\overline{\mathsf{B}}_{ij}^{\mathrm{d}\,\mathrm{II},\mathrm{II}}$ blocks. Moreover, from the expression of the quadratic form (4.22) we deduce that $\overline{\mathsf{B}}^{\mathrm{d}\,\mathrm{II},\mathrm{II}} = \sum_{i,j\in\mathcal{D}} \overline{\mathsf{B}}_{ij}^{\mathrm{d}\,\mathrm{II},\mathrm{II}}(\mathsf{w})\xi_i\xi_j$ is positive definite for $\boldsymbol{\xi} \in \Sigma^{d_{\mathrm{s}}-1}$. The matrices $\overline{\mathsf{B}}_{ij}^{\mathrm{c}}$ are given by (4.23) and are such that $(\overline{\mathsf{B}}_{ij}^{\mathrm{c}})^t = -\overline{\mathsf{B}}_{ji}^{\mathrm{c}}$, the blocks $\overline{\mathsf{B}}_{ij}^{\mathrm{c}\,\mathrm{I},\mathrm{I}}$ vanish $\overline{\mathsf{B}}_{ij}^{\mathrm{c}\,\mathrm{I},\mathrm{I}} = 0$, and the strongly coupling blocks $\overline{\mathsf{B}}_{ij}^{\mathrm{c}\,\mathrm{I},\mathrm{II}}$ and $\overline{\mathsf{B}}_{ij}^{\mathrm{c}\,\mathrm{II},\mathrm{I}}$ only depend $(\rho, \boldsymbol{v}, T)$. The right hand side $\mathsf{h}(\mathsf{w}, \boldsymbol{\nabla}\mathsf{w})$ is given by (4.24) and (4.25) that is covered by the more general situation (6.2) and (6.3). The system coefficients $\overline{\mathsf{A}}_0$, $\overline{\mathsf{A}}_i$, $i \in \mathcal{D}$, $\overline{\mathsf{B}}_{ij}^{\mathrm{d}}$, $\overline{\mathsf{B}}_{ij}^{\mathrm{c}}$, $i, j \in \mathcal{D}$, $\overline{\mathrm{M}}_i^{\mathrm{I}}$, $\overline{\mathrm{M}}_{ij}^{\mathrm{I},\mathrm{I}}$, $\overline{\mathrm{M}}_i^{\mathrm{II}}$, and $\overline{\mathrm{M}}_{ij}^{\mathrm{II},\mathrm{II}}$ obtained in Section 4.2 have at least regularity $C^{l+2}(\mathcal{O}_{\mathsf{w}})$ since the regularity class γ of thermodynamic functions is such that $\gamma - 2 \geq l + 2$.

There exists also a linearized version of (6.1) in the form (6.4) that enforce the gradient constraint. This linearized version (4.29) has been obtained in Section 4.3 and the corresponding coefficients $\overline{\mathsf{A}}_i'$, $\overline{\mathsf{L}}$, and h' satisfy all assumptions listed in Section 6.1 as well as Section 5.1. More specifically, as a consequence of (4.31), the hyperbolic blocks $\overline{\mathsf{A}}_i'^{\mathrm{I},\mathrm{I}}$ and $\overline{\mathsf{A}}_i^{\mathrm{I},\mathrm{I}}$ coincide, the matrix $\overline{\mathsf{L}} = \overline{\mathsf{L}}(\mathsf{w}, \mathfrak{p}_{\mathrm{r}})$ is block diagonal $\overline{\mathsf{L}} = \mathrm{diag}(\overline{\mathsf{L}}^{\mathrm{I},\mathrm{I}}, \overline{\mathsf{L}}^{\mathrm{II},\mathrm{II}})$ with $\overline{\mathsf{L}}^{\mathrm{II},\mathrm{II}} = 0$ and $\overline{\mathsf{L}}^{\mathrm{I},\mathrm{I}}$ is a linear function of $\mathfrak{p}_{\mathrm{r}} \in \mathbb{R}^{\mathsf{n}_{\mathrm{r}},d_{\mathrm{s}}}$ so that $\overline{\mathsf{L}}^{\mathrm{I},\mathrm{I}} = \mathfrak{L}^{\mathrm{I},\mathrm{I}}(\mathsf{w})\mathfrak{p}_{\mathrm{r}}$ where $\mathfrak{L}^{\mathrm{I},\mathrm{I}} \in \mathbb{R}^{\mathsf{n}_{\mathrm{I}},\mathsf{n},\mathsf{n}_{\mathrm{r}},d_{\mathrm{s}}}$ depend on w. The systems coefficients are naturally related by $\sum_{i\in\mathcal{D}}\big(\overline{\mathsf{A}}_i'(\mathsf{w}) - \overline{\mathsf{A}}_i(\mathsf{w})\big)\mathfrak{p} + \overline{\mathsf{L}}(\mathsf{w}, \mathfrak{p}_{\mathrm{r}})\mathsf{w} + \mathsf{h}(\mathsf{w}, \mathfrak{p}) = \mathsf{h}'(\mathsf{w}, \mathfrak{p})$ as established in Section 4.3. The matrices $\overline{\mathsf{A}}_i'(\mathsf{w})$, the coefficients $\mathfrak{L}^{\mathrm{I},\mathrm{I}}$, and the coefficients $\overline{\mathrm{M}}_i'^{\mathrm{I}}$, $\overline{\mathrm{M}}_{ij}'^{\mathrm{I},\mathrm{I}}$, $\overline{\mathrm{M}}_i'^{\mathrm{II}}$, and $\overline{\mathrm{M}}_{ij}'^{\mathrm{II},\mathrm{II}}$ also have at least regularity C^{l+2} over $\mathcal{O}_{\mathsf{w}}$ since the regularity class γ of thermodynamic functions is such that $\gamma - 2 \geq l + 2$. The state $\mathsf{w}^\star$ is finally such that $\overline{\mathsf{L}}(\mathsf{w}, \mathfrak{p}_{\mathrm{r}})\mathsf{w}^\star = 0$ for any $\mathsf{w} \in \mathcal{O}_{\mathsf{w}}$ and $\mathfrak{p}_{\mathrm{r}} \in \mathbb{R}^{\mathsf{n}_{\mathrm{r}},d_{\mathrm{s}}}$ since $\boldsymbol{w}^\star = 0$. All assumptions listed in Section 6.1 are thus satisfied and this completes the proof. □

We thus conclude that there exists local strong solutions to the system of diffuse interface fluids. The model notably takes into account the temperature dependence of the capillarity coefficient and general assumptions for the transport coefficients. The model also takes into account the presence of mechanically unstable points with a nonideal thermodynamics.

7 Conclusion

The mathematical modeling of diffuse interface fluids has been investigated. The mathematical structure of classical thermodynamics with instabilities as well as that of extended thermodynamics with temperature dependent capillarity coefficients has been presented. Normal forms have been obtained for the equations governing diffuse interface fluids with symmetric second order dissipation matrices and anti-symmetric second order capillarity matrices. The couple $(\rho, \boldsymbol{\nabla}\rho)$ has been shown to be an hyperbolic variable whereas $(\boldsymbol{v}, T)$ is the traditional parabolic variables as in the Navier-Stokes-Fourier system. The antisymmetric type second order terms of capillary origin introduce strong couplings between the hyperbolic and parabolic variables.

New linearized estimates have been obtained for augmented systems in normal form. Local existence has been obtained for diffuse interface fluids in an Hilbertian framework using a normal form. A key point has been the use of the gradient constraint.

Natural extension of high scientific interest would be to consider boundary value problems in bounded domains with proper boundary conditions. Existence of global solutions around thermodynamically stable constant states as well as around nonconstant liquid-vapor steady equilibrium profiles in an Hilbertian framework is also of high scientific interest and will require using a new Kawashima condition for augmented systems.

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Statements and Declarations

On the behalf of the authors there are no conflicts of interest with the authors.

A Entropy production

Denoting by D_t the convective derivative $D_t = \partial_t + \boldsymbol{v}\cdot\boldsymbol{\nabla}$ we obtain from Gibbs' relation (2.4) that

$$T D_t\mathcal{S} = D_t\mathcal{E} - g D_t\rho - \varkappa\boldsymbol{\nabla}\rho\cdot D_t\boldsymbol{\nabla}\rho.$$

The convective derivatives $D_t\mathcal{E}$, $D_t\rho$, and $D_t\boldsymbol{\nabla}\rho$ may then be evaluated from the governing equations (2.5)–(2.7) and after some algebra and integrations by parts it is obtained that

$$\begin{aligned}\partial_t\mathcal{S} + \boldsymbol{\nabla}\cdot(\boldsymbol{v}\mathcal{S}) + \boldsymbol{\nabla}\cdot\Big(\frac{\boldsymbol{\mathcal{Q}}}{T} - \frac{\varkappa\rho\boldsymbol{\nabla}\cdot\boldsymbol{v}\boldsymbol{\nabla}\rho}{T}\Big) = &-\frac{1}{T}\Big(\boldsymbol{\mathcal{P}} - p\boldsymbol{I} - \varkappa\boldsymbol{\nabla}\rho\otimes\boldsymbol{\nabla}\rho + \rho\boldsymbol{\nabla}\cdot(\varkappa\boldsymbol{\nabla}\rho)\boldsymbol{I}\Big)\mathbf{:}\boldsymbol{\nabla}\boldsymbol{v}\\ &- \Big(\boldsymbol{\mathcal{Q}} - \varkappa\rho\boldsymbol{\nabla}\cdot\boldsymbol{v}\boldsymbol{\nabla}\rho\Big)\cdot\frac{\boldsymbol{\nabla}T}{T^2},\end{aligned} \tag{A.1}$$

where $\mathbf{v}\mathbf{:}\mathbf{w}$ denotes the full contraction between any two tensors $\mathbf{v}$ and $\mathbf{w}$. Using the expressions of for $\boldsymbol{\mathcal{P}}$ and $\boldsymbol{\mathcal{Q}}$ as well as (A.1) we obtain the equation (2.13) and there is no entropy production associated with capillary phenomena.

References

[1] D.M. Anderson, G.B. McFadden and A.A. Wheeler, Diffuse Interface Methods in Fluid Mechanics, Ann. Rev. Fluid Mech., 30 (1998), 139–165.

[2] P. Barbante and A. Frezzotti, A comparison of models for the evaporation of a Lennard-Jones fluid, Eur. J. Mech. B. Fluids, 64 (2017), 69-80.

[3] S. Benzoni-Gavage, R. Danchin, S. Descombes, and D. Jamet, Structure of Korteweg models and stability of diffuse interfaces, Interfaces Free Bound., 7 (2005), 371–414.

[4] S. Benzoni-Gavage, R. Danchin, and S. Descombes, Well-posedness of one-dimensional Korteweg models, Elec. J. Diff. Equs., 59 (2006), 1–35.

[5] S. Benzoni-Gavage, R. Danchin, and S. Descombes, On the well-posedness of the Euler-Korteweg model in several space dimensions, Indiana J. Math., 56 (2007), 1499–1579.

[6] S. Benzoni and D. Serre, *Multi-Dimensional Hyperbolic Partial Differential Equations*. First Order Systems and Applications. Oxford Mathematical Monographs, Oxford (2007).

[7] T. Dębiec, P. Gwiazda, A. Świerczewska-Gwiazda, and A. Tzavaras, Conservation of energy for the Euler-Korteweg equations, Cal. Var., 57 (2018), 160.

[8] D. Bresch, B. Desjardins and C.K. Lin, On some compressible fluid models: Korteweg, lubrication and shallow water systems, Comm. Part. Diff. Eqs., 28 (2003), 843–868.

[9] D. Bresch, M. Gisclon and I. Lacroix-Violet, Navier-Stokes-Korteweg and Euler-Korteweg systems: Application to quantum fluids, Arch. rat. Mech. Anal., 233 (2019), 975–1025.

[10] D. Bresch, F. Couderc, P. Noble , J.-P. Vila, A generalization of the quantum Bohm identity: Hyperbolic CFL condition for Euler-Korteweg equations, C. Rendu Acad. Sci., 354 (2008), 39–43.

[11] D. Bresch and B. Desjardins, On the Existence of Global Weak Solutions to the Navier-Stokes Equations for Viscous Compressible and Heat Conducting Fluids, J. Math. Pure Appl., 87 (2007), 57–90.

[12] D. Bresch, V. Giovangigli and E. Zatorska, Two-velocity hydrodynamics in fluid mechanics: Part I Well Posedness for Zero Mach Number Systems, J. Math. Pures App., 104 (2015), 762–800.

[13] D. Bresch, B. Desjardins and E. Zatorska, Two-velocity hydrodynamics in fluid mechanics: Part II Existence of global κ-entropy solutions to the compressible Navier-Stokes system with degenerate viscosities, J. Math. Pures App., 104 (2015), 801–836.

[14] E. Bretin, S. Masnou, and E. Oudet, Phase-field approximations of the Willmore functional and flow, Numer. Math., 131 (2015), 115–171.

[15] C. Cancès, D.Matthes, F. Nabet, A two-phase two-fluxes degenerate Cahn-Hilliard model as constrained Wasserstein gradient flow, Arch. Rational Mech. Anal., 233 (2019), 837-866.

[16] S. Chapman and T.G. Cowling, *The Mathematical Theory of Non-Uniform Gases*. Cambridge University Press, Cambridge, 1970.

[17] G.Q. Chen, C.D. Levermore, and T.P. Liu, Hyperbolic conservation laws with stiff relaxation terms and entropy. Comm. Pure Appl. Math. 47 (1994), 787–830.

[18] R. Danchin and B. Desjardins, Existence of solutions for compressible fluid models of Korteweg type, Annales IHP, Anal. non linéaire, 18 (2001), 97–133.

[19] C.M. Dafermos, *Hyperbolic Conservation Laws in Continuum Physics*. Springer-Verlag, Heidelberg 2000.

[20] R.J. DiPerna and P.L. Lions, Ordinary differential equations, transport theory and Sobolev spaces, Invent. Math., 98 (1989), 511-547.

[21] D. Donatelli and P. Marcati, Convergence of singular limits for multi-d semilinear hyperbolic systems to parabolic systems, Trans. Amer. Soc., 356, (2004), 2093–2121.

[22] D. Donatelli, E. Feireisl and P. Marcati, Well/Ill posedness for the Euler-Korteweg-Poisson system and related problems, Comm. Partial Diff. Equat., 40, (2014), 1314–1335.

[23] J.E. Dunn and J. Serrin, On the thermomechanics of interstitial working, Archive Rat. Mech. Anal. 133 (1985), 95–133.

[24] E. Feireisl, *Dynamics of viscous compressible fluids*. Oxford University press, Oxford 2004.

[25] J.H. Ferziger and H.G. Kaper, Mathematical Theory of Transport Processes in Gases. North Holland Pub. Co., Amsterdam, 1972.

[26] M. Kotschote, Strong solutions for a compressible fluid model of Korteweg type, Anal. Inst. H. Poincaré, 25 (2008), 679–696.

[27] M. Kotschote, Dynamics of compressible non-isothermal fluids of new-newtonian Korteweg type, SIAM J. Math. Anal., 44 (2012), 74–101.

[28] M. Kotschote, Existence and time-asymptotics of global strong solutions to dynamic Korteweg models, Indiana J. Math., 63 (2014), 21–51.

[29] K.O. Friedrichs and P. D. Lax, Systems of conservation laws with a convex extension. Proc. Nat. Acad. Sci. USA 68 (1971), 1686–1688.

[30] P. Gaillard, V. Giovangigli and L. Matuszewski, A Diffuse Interface Lox/Hydrogen Transcritical Flame Model, Combust. Theory Model., 20 (2016), 486–520.

[31] S. Gavrilyuk and H. Gouin, Symmetric form of governing equations for capillary fluids, "Trends in applications of mathematics to mechanics" (Nice, 1998), Monogr. Surv. Pure Appl. Math., Vol. 106, Chapman & Hall/CRC, Boca Raton, FL, 306–311, (2000).

[32] J. Giesselmann and A. Tzavaras, Stability properties of the Euler-Korteweg system with nonmonotone pressures, App. Anal., 96 (2017), 1528–1546.

[33] V. Giovangigli, *Multicomponent Flow Modeling*, Birkhaüser, Boston, 1999.

[34] V. Giovangigli, Kinetic Derivation of Diffuse-Interface Fluid Models, Phys. Rev. E, 102 (2020), 012110.

[35] V. Giovangigli, Kinetic Derivation of Cahn-Hilliard Fluid Models, Phys. Rev. E, 104 (2021), 054109.

[36] V. Giovangigli and M. Massot, The local Cauchy problem for multicomponent reactive flows in Full Vibrational Nonequilibrium. Math. Meth. Appl. Sci. 21 (1998), 1415–1439.

[37] V. Giovangigli and M. Massot, Asymptotic stability of equilibrium states for multicomponent reactive flows, Math. Mod. Meth. Appl. Sci. 8 (1998), 251–297.

[38] V. Giovangigli, L. Matuszewski, and F. Dupoirieux, Detailed modeling of planar transcritical H_2-O_2-N_2 flames, Combustion Theory and Modelling, 15, (2011), pp. 141–182.

[39] V. Giovangigli and L. Matuszewski, Supercritical fluid thermodynamics from equations of state, Physica D, 241 (2012), 649–670.

[40] V. Giovangigli and L. Matuszewski, Mathematical modeling of supercritical multicomponent reactive fluids. Math. Mod. Meth. Sci. 23, 2193–2251 (2013).

[41] E. Godlewski and P.A. Raviart, *Numerical Approximation of Hyperbolic Systems of Conservation Laws* Springer-Verlag, Heidelberg (1996).

[42] S. Godunov, An interesting class of quasilinear systems, Sov. Math. Dokl, 2 (1961), 947–949.

[43] E.A. Guggenheim, *Thermodynamics*, North-Holland, Amsterdam 1967.

[44] B. Haspot, Existence of strang solutions for nonisothermal Korteweg system, Annales Math. Blaise Pascal, 16, (2009) 431–481.

[45] H. Hattori and D. Li, Solutions for two-dimensional system for materials of Korteweg type, J. Partial Diff. Equ., 9 (1996), 323–342.

[46] H. Hattori and D. Li, Global solutions of a high dimensional system for Korteweg materials, J. Math. Anal., 198 (1996), 84–97.

[47] D. Jamet, Diffuse interface models in fluid mechanics, Semantic Scholar, Corpus Id: 18437499, (2005).

[48] S. Kawashima, Systems of hyperbolic-parabolic composite type, with application to the equations of magnetohydrodynamics. Doctoral Thesis, Kyoto University, 1984.

[49] S. Kawashima and Y. Shizuta, On the normal form of the symmetric hyperbolic-parabolic systems associated with the conservation laws. Tôhoku Math. J. 40 (1988), 449-464.

[50] S. Kawashima and W.A. Yong, Dissipative structure and entropy for hyperbolic systems of conservation laws, Arch. Rat. Mech. Anal., 174 (2004), 345–364.

[51] D.J. Korteweg, Sur la Forme que Prennent les Equations du Mouvement Fluide si l'on tient Compte de Forces Capillaires Causées par les Variations de Densité Considérables mais Continues et sur la Théorie de la Capillarité dans l'Hypothèse d'une Variations Continue de la Densité, Arch. Neerl. Sci Exactes, 6 (1901), 1–20.

[52] Y. Le Calvez, *Modélisation mathématique et simulation numériques des mélanges fluides avec interfaces diffuses*. PhD thesis, (in preparation) (2023).

[53] H. Lin, Y.-Y. Duan, and Q. Min, Gradient Theory Modeling of Surface Tension for Pure Fluids and Binary Mixtures, Fluid Phase Equil., 254 (2007), pp. 75–90.

[54] T.P. Liu, Hyperbolic conservation laws with relaxation, Comm. Math. Physics, 108 (1987), 153-175.

[55] A. Majda, Compressible fluid flow and systems of conservation laws in several space variables, Springer, New York, 1984.

[56] A. Miranville, The Cahn-Hilliard equation and some of its variants, AIMS Math., 2 (2017), 479–544.

[57] F. Nabet, Convergence of a finite-volume scheme for the Cahn-Hilliard equation with dynamic boundary conditions, IMA J. Num. Anal., 36 (2016), 1898–1942.

[58] D. Nayigizente, S. Ducruix, and T. Schmitt, Development of an interface thickening method for the direct numerical simulation of compressible liquid-vapor flows in the framework of the second gradient theory, Phys. Fluids, 33 (2021), 052119.

[59] J.S. Rowlinson and B. Widom, *Molecular Theory of Capillarity*, Dover publications, Mineola, 1989 and 2002.

[60] Serre D., The structure of dissipative viscous system of conservation laws, Physica D 239 (2010), 1381–1386.

[61] T. Ruggeri, Thermodynamics and symmetric hyperbolic systems. Rend. Sem. Mat. Univ. Torino., (1988), 167–183 .

[62] J.D. van der Waals, Thermodynamische Theorie der Capillariteit in de Onderstelling van Continue Dichtheidsverandering Verhand. Kon. Akad. V Wetensch. Amst., (1893). (in Dutch; English translation in J. Stat. Phys., 20 (1979), 197–244).

[63] A.I. Vol'pert and S.I. Hudjaev, On the Cauchy Problem for Composite Systems of Nonlinear Differential Equations, Math USSR Sbornik, 16 (1972), 517–544.

[64] Z.B. Yang and W.A. Yong, Validity of the Chapman-Enskog expansion for a class of hyperbolic relaxation systems, J. Diff. Equ., 258 (2015), 2745–2766.

[65] K. Yosida, *Functional Analysis*. Springer-Verlag, Berlin, (1980).