# Conditioning diffusion processes with respect to the local time at the origin 

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#### Abstract

When the unconditioned process is a diffusion process $X(t)$ of drift $\mu(x)$ and of diffusion coefficient $D=1 / 2$, the local time $A(t)=\int_{0}^{t} d \tau \delta(X(\tau))$ at the origin $x=0$ is one of the most important timeadditive observable. We construct various conditioned processes $\left[X^{*}(t), A^{*}(t)\right]$ involving the local time $A^{*}(T)$ at the time horizon $T$. When the horizon $T$ is finite, we consider the conditioning towards the final position $X^{*}(T)$ and towards the final local time $A^{*}(T)$, as well as the conditioning towards the final local time $A^{*}(T)$ alone without any condition on the final position $X^{*}(T)$. In the limit of the infinite time horizon $T \rightarrow+\infty$, we consider the conditioning towards the finite asymptotic local time $A_{\infty}^{*}<+\infty$, as well as the conditioning towards the intensive local time $a^{*}$ corresponding to the extensive behavior $A_{T} \simeq T a^{*}$, that can be compared with the appropriate 'canonical conditioning' based on the generating function of the local time in the regime of large deviations. This general construction is then applied to generate various constrained stochastic trajectories for three unconditioned diffusions with different recurrence/transience properties : (i) the simplest example of transient diffusion corresponds to the uniform strictly positive drift $\mu(x)=$ $\mu>0$; (ii) the simplest example of diffusion converging towards an equilibrium is given by the drift $\mu(x)=-\mu \operatorname{sgn}(x)$ of parameter $\mu>0$; (iii) the simplest example of recurrent diffusion that does not converge towards an equilibrium is the Brownian motion without drift $\mu=0$.


## I. INTRODUCTION

## A. Conditioning diffusion processes with respect to time-additive observables of the stochastic trajectories

Since its introduction by Doob [1, 2, the conditioning of stochastic processes (see the mathematical books [3-5] and the physics recent review [6]) have found many applications in various fields like ecology [7], finance [8] or nuclear engineering [9, 10]. Among the different conditioned diffusions that have been constructed besides the basic example of the Brownian Bridge, one can cite the Brownian excursion [11, 12], the Brownian meander [13], the taboo processes [14-19], or non-intersecting Brownian bridges [20]. Let us also mention the conditioning in the presence of killing rates [3, 21, 28] or when the killing occurs only via an absorbing boundary condition [29-32]. Note that stochastic bridges have been studied for many other Markov processes, including various diffusions processes [33-35], discretetime random walks and Lévy flights [36-38, continuous-time Markov jump processes [38, run-and-tumble trajectories [39], or processes with resetting 40].

A recent important generalization concerns the conditioning with respect to global dynamical constraints involving time-additive observables of the stochastic trajectories. In particular, the conditioning on the area has been studied via various methods for Brownian processes or bridges 41 and for Ornstein-Uhlenbeck bridges 42. The conditioning on the area and on other time-additive observables has been then analyzed both for the Brownian motion and for discretetime random walks [43]. This approach has been generalized 44] to various types of discrete-time or continuous-time Markov processes, while the time-additive observable can involve both the time spent in each configuration and the increments of the Markov process. This general reformulation of the 'microcanonical conditioning', where the time-additive observable is constrained to reach a given value after the finite time window $T$, allows to make the link 44 with the 'canonical conditioning' based on generating functions of additive observables that has been much studied recently in the field of dynamical large deviations of Markov processes over a large time-window $T$ [45 90]. The equivalence between the 'microcanonical conditioning' and the canonical conditioning' at the level of the large deviations for large time $T$ is explained in detail in the two complementary papers [68, 69] and in the HDR thesis 70.
B. Simplest examples of time-additives for a diffusion process $X(t)$ : the occupation time and the local time

For a one-dimensional diffusion process $X(t)$, two basic examples of time-additive observables are :
(i) the occupation time $O_{[a, b]}(t)$ of the space interval $[a, b]$ during the time window $[0, t]$

$$
\begin{equation*}
O_{[a, b]}(t)=\int_{0}^{t} d \tau \theta(a \leq X(\tau) \leq b) \tag{1}
\end{equation*}
$$

belongs to the interval $0 \leq O_{[a, b]}(t) \leq t$. The conditioning with respect to the occupation time of the interval $[a=0, b=+\infty$ [ has been studied recently for the Brownian motion without drift [43], while the canonical conditioning with respect to the occupation time has been analyzed for various settings [73, 74].
(ii) the local time $A_{x}(t)$ at the position $x$ during the time window $[0, t]$ (see the mathematical review 91 and references therein)

$$
\begin{equation*}
A_{x}(t)=\int_{0}^{t} d \tau \delta(X(\tau)-x) \tag{2}
\end{equation*}
$$

has for physical dimension $\frac{\text { Time }}{\text { Length }}$ so that it is actually not a 'time' despite its standard name. However, it is directly related to the occupation time as follows. On the one hand, the local time $A_{x}(t)$ of Eq. 2 can be constructed from the occupation time $O_{[x-\epsilon, x+\epsilon]}(t)$ of the space interval $[x-\epsilon, x+\epsilon]$ of size $(2 \epsilon)>0$ centered at the position $x$ in the limit $\epsilon \rightarrow 0^{+}$

$$
\begin{equation*}
A_{x}(t)=\int_{0}^{t} d \tau \lim _{\epsilon^{+} \rightarrow 0}\left(\frac{\theta(x-\epsilon \leq X(\tau) \leq x+\epsilon)}{2 \epsilon}\right)=\lim _{\epsilon^{+} \rightarrow 0}\left(\frac{O_{[x-\epsilon, x+\epsilon]}(t)}{2 \epsilon}\right) \tag{3}
\end{equation*}
$$

As a consequence, the local time $A_{x}(t)$ belongs to $[0,+\infty[$ with no upper bound. On the other hand, the occupation time $O_{[a, b]}(t)$ can be reconstructed from the local time $A_{x}(t)$ for all the internal positions $x \in[a, b]$

$$
\begin{equation*}
O_{[a, b]}(t)=\int_{0}^{t} d \tau \int_{a}^{b} d x \delta(X(\tau)-x)=\int_{a}^{b} d x A_{x}(t) \tag{4}
\end{equation*}
$$

## C. Goals of the present work

At first sight, the delta function that enters the definition of the local time in Eq. 2 might appear as very singular for the purpose of conditioning. However, as in quantum mechanics where delta impurities are well-known to be much simpler than smoother potentials, the delta function in Eq. 2 is actually a huge technical simplification compared to the case of conditioning with respect to an arbitrary general additive observable. Indeed, the exact Dyson equation associated to a single delta impurity allows to analyze the conditioning with respect to the local time for $A_{x=0}(t)=A(t)$ at the origin $x=0$ in terms of the properties of the propagator $G\left(x, t \mid x_{0}, t_{0}\right)$ of the unconditioned process $X(t)$ alone. In the present paper, it will be thus interesting to consider that the unconditioned process is a diffusion process $X(t)$ of diffusion coefficient $D=1 / 2$ with an arbitrary position-dependent drift $\mu(x)$ in order to derive the general properties before the application to various illustrative examples of drifts. Our goal is to construct various conditioned joint processes $\left[X^{*}(t), A^{*}(t)\right]$ satisfying certain conditions involving the local time $A^{*}(T)$ either at the finite time horizon $T$, or in the limit of the infinite time horizon $T \rightarrow+\infty$. For instance, the two basic cases that will be considered for the finite horizon $T$ can be summarized as follows.
(i) The conditioning towards the position $x_{T}^{*}$ and the local time $A_{T}^{*}$ at the time horizon $T$ involves the conditioned drift

$$
\mu_{T}^{\left[x_{T}^{*}, A_{T}^{*}\right]}(x, A, t)=\mu(x)+\partial_{x} \ln P\left(x_{T}^{*}, A_{T}^{*}, T \mid x, A, t\right)
$$

where $P\left(x_{T}^{*}, A_{T}^{*}, T \mid x, A, t\right)$ represents the joint propagator of the unconditioned diffusion.
(ii) The conditioning towards the local time $A_{T}^{*}$ at time horizon $T$, without any condition on the final position $x_{T}$ involves the conditioned drift

$$
\mu_{T}^{\left[A_{T}^{*}\right]}(x, A, t)=\mu(x)+\partial_{x} \ln \Pi\left(A_{T}^{*}, T \mid x, A, t\right)
$$

in terms of $\Pi\left(A_{T}^{*}, T \mid x, A, t\right)=\int d x_{T} P\left(x_{T}, A_{T}^{*}, T \mid x, A, t\right)$ for the unconditioned diffusion.
When the unconditioned diffusion is the Brownian motion of uniform drift $\mu \geq 0$ or the stochastic process with drift $\mu(x)=-\mu \operatorname{sgn}(x)$ with $\mu>0$, some examples of these conditioned drifts that will be studied are given in the two following tables.

|  | Finite time horizon $T<\infty$ |  | Infinite time horizon $T=\infty$ |
| :---: | :---: | :---: | :---: |
|  | Conditioned drift $\mu_{T}^{\left[x_{T}^{*}, A_{T}^{*}\right]}(x, A, t)$ towards the position $x_{T}^{*}$ and the local time $A_{T}^{*}$ | Conditioned drift $\mu_{T}^{\left[A_{T}^{*}\right]}(x, A, t)$ towards the local time $A_{T}^{*}$ | Conditioned drift $\mu_{\infty}^{\left[A_{\infty}^{*}\right]}(x, A)$ towards the local time $A_{\infty}^{*}$ |
| $\begin{aligned} & \text { Region } \\ & 0<A<A_{T}^{*} \end{aligned}$ | $\operatorname{sgn}(x)\left[\frac{1}{\left\|x_{T}^{*}\right\|+\|x\|+A_{T}^{*}-A}-\frac{\left\|x_{T}^{*}\right\|+\|x\|+A_{T}^{*}-A}{T-t}\right]$ | $-\operatorname{sgn}(x) \frac{\|x\|+A_{T}^{*}-A}{T-t}$ | $-\mu \operatorname{sgn}(x)$ |
| Region $A=A_{T}^{*}$ | $\frac{\left(\frac{x_{T}^{*}-x}{T-t}\right) e^{-\frac{\left(x_{T}^{*}-x\right)^{2}}{2(T-t)}}+\left(\frac{x+\operatorname{sgn}(x)\left\|x x_{T}^{*}\right\|}{T-t}\right) e^{-\frac{\left(\left\|x_{T}^{*}\right\|+\|x\|\right)^{2}}{2(T-t)}}}{e^{-\frac{\left(x_{T}^{*}-x\right)^{2}}{2(T-t)}}-e^{-\frac{\left(\left\|x_{T}^{*}\right\|+\|x\|\right)^{2}}{2(T-t)}}}$ | $\sqrt{\frac{2}{\pi(T-t)}} \frac{e^{-\frac{x^{2}}{2(T-t)}}}{\operatorname{erf}\left(\frac{\|x\|}{\sqrt{2(T-t)}}\right)} \operatorname{sgn}(x)$ | $\mu \operatorname{coth}(\mu x)$ |

Table I. Some examples of conditioned drifts $\mu_{T}^{*}(x, A, t)$ for the Brownian motion of uniform drift $\mu(x)=\mu>0$.

|  | Finite time horizon $T<\infty$ |  |
| :---: | :---: | :---: |
|  | Conditioned drift $\mu_{T}^{\left[x_{T}^{*}, A_{T}^{*}\right]}(x, A, t)$ <br> towards the position $x_{T}^{*}$ and the local time $A_{T}^{*}$ | Conditioned drift $\mu_{T}^{\left[A_{T}^{*}\right]}(x, A, t)$ towards the local time $A_{T}^{*}$ |
| $\begin{aligned} & \text { Region } \\ & 0<A<A_{T}^{*} \end{aligned}$ | $\operatorname{sgn}(x)\left[\frac{1}{\left\|x_{T}^{*}\right\|+\|x\|+A_{T}^{*}-A}-\frac{\left\|x_{T}^{*}\right\|+\|x\|+A_{T}^{*}-A}{T-t}\right]$ | $\mu \operatorname{sgn}(x)-\frac{2}{(T-t)} \frac{\left(\|x\|+A_{T}^{*}-A\right) \operatorname{sgn}(x)}{2-\mu \sqrt{2 \pi(T-t) e} \frac{\left(\|x\|+\mu(T-t)+A_{T}^{*}-A\right)^{2}}{2(T-t)} \operatorname{erfc}\left(\frac{\|x\|+\mu(T-t)+A_{T}^{*}-A}{\sqrt{2(T-t)}}\right)}$ |
| Region $A=A_{T}^{*}$ | $\frac{\left(\frac{x_{T}^{*}-x}{T-t}\right) e^{-\frac{\left(x_{T}^{*}-x\right)^{2}}{2(T-t)}}+\left(\frac{x+\operatorname{sgn}(x)\left\|x x_{T}^{*}\right\|}{T-t}\right) e^{-\frac{\left(\left\|x_{T}^{*}\right\|+\|x\|\right)^{2}}{2(T-t)}}}{e^{-\frac{\left(x_{T}^{*}-x\right)^{2}}{2(T-t)}}-e^{-\frac{\left(\left\|x_{T}^{*}\right\|+\|x\|\right)^{2}}{2(T-t)}}}$ | $2 \sqrt{\frac{2}{\tau}} \frac{\operatorname{sgn}(x)\left(\frac{1}{\sqrt{\pi}}-e^{\frac{(\|x\|+\mu \tau)^{2}}{2 \tau}} \mathcal{F}\left(\frac{\|x\|+\mu \tau}{\sqrt{2 \tau}}\right)\right)+e^{\frac{(x-\mu \tau)^{2}}{2 \tau}}\left(e^{2 \mu x} \mathcal{F}\left(\frac{x+\mu \tau}{\sqrt{2 \tau}}\right)-\mathcal{F}\left(\frac{\mu \tau-x}{\sqrt{2 \tau}}\right)\right)}{e^{\frac{(x+\mu \tau)^{2}}{2 \tau}} \operatorname{erfc}\left(\frac{x+\mu \tau}{\sqrt{2 \tau}}\right)+e^{\frac{(x-\mu \tau)^{2}}{2 \tau}} \operatorname{erfc}\left(\frac{\mu \tau-x}{\sqrt{2 \tau}}\right)-2 e^{\frac{(\|x\|+\mu \tau)^{2}}{2 \tau}} \operatorname{erfc}\left(\frac{\|x\|+\mu \tau}{\sqrt{2 \tau}}\right)}$ |

Table II. Some examples of conditioned drifts $\mu_{T}^{*}(x, A, t)$ for the stochastic process with drift, $\mu(x)=-\mu \operatorname{sgn}(x)$ with $\mu>0$. Observe that the conditioned drifts are the same as those of the Brownian motion in the case of conditioning towards the position $x_{T}^{*}$ and the local time $A_{T}^{*}$ at the finite time horizon $T$. We use the notation $\mathcal{F}(x)=x \operatorname{erfc}(x)$ where $\operatorname{erfc}(x)$ is the complementary Error function $\operatorname{erfc}(x)=1-\operatorname{erf}(x)$.

## D. Organization of the paper

The paper is organized as follows. The properties of the unconditioned diffusion process $X(t)$ with drift $\mu(x)$ are recalled in section II. We then analyze the properties of the joint propagator $P\left(x, A, t \mid x_{0}, A_{0}, t_{0}\right)$ for the position $x$ and the local time $A$ in section III, as well as the probability $\Pi\left(A, t \mid x_{0}, A_{0}, t_{0}\right)=\int d x P\left(x, A, t \mid x_{0}, A_{0}, t_{0}\right)$ in section IV, The statistical properties of the local time increment $\left[A(t)-A\left(t_{0}\right)\right]$ in the limit of the large time interval $\left(t-t_{0}\right) \rightarrow+\infty$ are discussed in section V as a function of the recurrence/transience properties of the diffusion process $X(t)$ induced by the drift $\mu(x)$. In section VI, we construct various conditioned processes $\left[X^{*}(t), A^{*}(t)\right]$ that involve the local time $A^{*}(T)$ at the finite time horizon $T$ or in the limit of the infinite time horizon $T \rightarrow+\infty$. This general framework is applied to the case of the uniform drift $\mu(x)=\mu$ with $\mu=0$ or $\mu>0$ in section VII and to the case $\mu(x)=-\mu \operatorname{sgn}(x)$ of parameter $\mu>0$ in section VIII, in order to generate stochastic trajectories of various conditioned processes with respect to the local time. Monte Carlo simulations illustrate our findings. Our conclusions are summarized in section IX. The three appendices $A, B$ and $C$ are devoted to the canonical conditioned processes $X_{p}^{*}(t)$ of parameter $p$, in order to compare with the microcanonical conditioning described in the main text.

## II. PROPERTIES OF THE UNCONDITIONED DIFFUSION PROCESS $X(t)$ WITH DRIFT $\mu(x)$

In this paper, we consider that the unconditioned process $X(t)$ is a diffusion process on the whole line $]-\infty,+\infty[$ generated by the Stochastic Differential Equation

$$
\begin{equation*}
d X(t)=\mu(X(t)) d t+d B(t) \tag{5}
\end{equation*}
$$

where $B(t)$ is a standard Brownian motion and where the position-dependent drift $\mu(x)$ is the only parameter of the model. In this section, we recall the recurrence/transience properties that will be useful to analyze the statistics of its local time $A(t)$ at the origin in the three next sections III, IV and V.

## A. Propagator $G\left(x, t \mid x_{0}, t_{0}\right)$ for the diffusion process $X(t)$

The propagator $G\left(x, t \mid x_{0}, t_{0}\right)$ for the diffusion process $X(t)$ generated by Eq. 5 satisfies the Fokker-Planck dynamics

$$
\begin{equation*}
\partial_{t} G\left(x, t \mid x_{0}, t_{0}\right)=-\partial_{x}\left[\mu(x) G\left(x, t \mid x_{0}, t_{0}\right)\right]+\frac{1}{2} \partial_{x}^{2} G\left(x, t \mid x_{0}, t_{0}\right) \tag{6}
\end{equation*}
$$

Its Laplace transform with respect to the time interval $\left(t-t_{0}\right)$

$$
\begin{equation*}
\hat{G}_{s}\left(x \mid x_{0}\right) \equiv \int_{t_{0}}^{+\infty} d t e^{-s\left(t-t_{0}\right)} G\left(x, t \mid x_{0}, t_{0}\right) \tag{7}
\end{equation*}
$$

then satisfies

$$
\begin{equation*}
-\delta\left(x-x_{0}\right)+s \hat{G}_{s}\left(x \mid x_{0}\right)=-\partial_{x}\left[\mu(x) \hat{G}_{s}\left(x \mid x_{0}\right)\right]+\frac{1}{2} \partial_{x}^{2} \hat{G}_{s}\left(x \mid x_{0}\right) \tag{8}
\end{equation*}
$$

## B. Similarity transformation towards an euclidean quantum propagator $\psi\left(x, t \mid x_{0}, t_{0}\right)$

As is well-known [92, the potential $U(x)$ defined via the following integration of the drift $\mu(y)$

$$
\begin{equation*}
U(x) \equiv-2 \int_{0}^{x} d y \mu(y) \tag{9}
\end{equation*}
$$

can be used to make the similarity transformation

$$
\begin{equation*}
G\left(x, t \mid x_{0}, t_{0}\right)=e^{-\frac{U(x)}{2}} \psi\left(x, t \mid x_{0}, t_{0}\right) e^{\frac{U\left(x_{0}\right)}{2}}=e^{\frac{U\left(x_{0}\right)-U(x)}{2}} \psi\left(x, t \mid x_{0}, t_{0}\right)=e^{\int_{x_{0}}^{x} d y \mu(y)} \psi\left(x, t \mid x_{0}, t_{0}\right) \tag{10}
\end{equation*}
$$

The Fokker-Planck Eq. 6 for the propagator $G\left(x, t \mid x_{0}, t_{0}\right)$ is then transformed into an Euclidean Schrödinger Equation for $\psi\left(x, t \mid x_{0}, t_{0}\right)$

$$
\begin{equation*}
-\partial_{t} \psi\left(x, t \mid x_{0}, t_{0}\right)=H \psi\left(x, t \mid x_{0}, t_{0}\right) \tag{11}
\end{equation*}
$$

The corresponding hermitian quantum Hamiltonian

$$
\begin{equation*}
H=-\frac{1}{2} \partial_{x}^{2}+V(x) \tag{12}
\end{equation*}
$$

involves the quantum potential

$$
\begin{equation*}
V(x) \equiv \frac{\mu^{2}(x)}{2}+\frac{\mu^{\prime}(x)}{2} \tag{13}
\end{equation*}
$$

This very specific structure of $V(x)$ in terms of the drift $\mu(x)$ allows to factorize the Hamiltonian of Eq. 12 into the supersymmetric form (see the review on supersymmetric quantum mechanics [93] and references therein)

$$
\begin{equation*}
H \equiv \frac{1}{2} Q^{\dagger} Q \tag{14}
\end{equation*}
$$

involving the first-order operator

$$
\begin{equation*}
Q \equiv-\partial_{x}+\mu(x) \tag{15}
\end{equation*}
$$

and its adjoint

$$
\begin{equation*}
Q^{\dagger} \equiv \partial_{x}+\mu(x) \tag{16}
\end{equation*}
$$

This quantum mapping allows to use all the knowledge on one-dimensional quantum Hamiltonians in general and on supersymmetric quantum Hamiltonians in particular to characterize the energy spectrum as follows.

## C. Analysis of the spectrum of the quantum supersymmetric Hamiltonian $H$

## 1. Analysis of the continuous spectrum $] V_{\infty},+\infty[$ of $H$ when it exists

The minimum of the two limiting values of the quantum potential $V(x)$ of Eq. 13 as $x \rightarrow \pm \infty$

$$
\begin{equation*}
V_{\infty}=\min [V(x \rightarrow+\infty) ; V(x \rightarrow-\infty)] \tag{17}
\end{equation*}
$$

determines the lower boundary of the continuous spectrum when it exists.
The discussion is thus as follows :
(i) if $V_{\infty}<+\infty$ is finite, then the continuous spectrum of $H$ is given by $] V_{\infty},+\infty[$. The physical interpretation is that, in the infinity region where the asymptotic value of the potential $V(x)$ is $V_{\infty}$, an eigenstate of energy $E \in] V_{\infty},+\infty\left[\right.$ behaves asymptotically like a linear combination of the plane waves $e^{ \pm i k x}$, where the relation between the wave-number $k$ and the energy $E$ is given by the corresponding eigenvalue equation for $H e^{ \pm i k x}=E e^{ \pm i k x}$ in the infinity region where the potential is $V_{\infty}$

$$
\begin{equation*}
E=\frac{k^{2}}{2}+V_{\infty} \tag{18}
\end{equation*}
$$

i.e. the wave-number $k=\sqrt{2\left(E-V_{\infty}\right)}$ is real for any energy $\left.E \in\right] V_{\infty},+\infty[$.

The simplest example is the case of the uniform drift $\mu(x)=\mu$, where the quantum potential of Eq. 13 reduces to the constant

$$
\begin{equation*}
V(x)=\frac{\mu^{2}}{2} \quad \text { for } \quad \mu(x)=\mu \tag{19}
\end{equation*}
$$

so that the continuous spectrum is $] \frac{\mu^{2}}{2},+\infty[$.
(ii) if $V_{\infty}=+\infty$ is infinite, then there is no continuous spectrum and $H$ has only an infinity of bound states.

The simplest example is the case of the Ornstein-Uhlenbeck drift $\mu(x)=-k x$ with $k>0$, where the quantum potential of Eq. 13 corresponds to the harmonic oscillator

$$
\begin{equation*}
V(x)=\frac{k^{2}}{2} x^{2}-\frac{k}{2} \quad \text { for } \quad \mu(x)=-k x \tag{20}
\end{equation*}
$$

with its well-known infinite series of discrete levels.

## 2. Analysis of the normalizable zero-energy ground-state $\phi^{G S}(x)$ of $H$ when it exists

The factorization of Eq. 14 shows that the spectrum of $H$ is positive. Let us now discuss whether $E=0$ is the ground state energy of $H$. The wavefunction $\phi^{E=0}(x)$ that is annihilated by the operator $Q$ of Eq. 15

$$
\begin{equation*}
0=Q \phi^{[E=0]}(x)=-\partial_{x} \phi^{[E=0]}(x)+\mu(x) \phi^{[E=0]}(x) \tag{21}
\end{equation*}
$$

reads in terms of the potential $U(x)$ of Eq. 9

$$
\begin{equation*}
\phi^{[E=0]}(x)=\phi^{[E=0]}(0) e^{\int_{0}^{x} d y \mu(y)}=\phi^{[E=0]}(0) e^{-\frac{U(x)}{2}} \tag{22}
\end{equation*}
$$

This wavefunction can be normalized on $x \in]-\infty,+\infty$ [ if

$$
\begin{equation*}
1=\left\langle\phi^{[E=0]} \mid \phi^{[E=0]}\right\rangle=\int_{-\infty}^{+\infty} d x\left[\phi^{[E=0]}(x)\right]^{2}=\left[\phi^{[E=0]}(0)\right]^{2} \int_{-\infty}^{+\infty} d x e^{-U(x)} \tag{23}
\end{equation*}
$$

The discussion is thus as follows :
(i) if the integral involving the potential $U(x)$ converges

$$
\begin{equation*}
\int_{-\infty}^{+\infty} d x e^{-U(x)}<+\infty \tag{24}
\end{equation*}
$$

then $H$ has the following normalizable ground state at zero-energy $E=0$

$$
\begin{equation*}
\phi^{G S}(x)=\frac{e^{-\frac{U(x)}{2}}}{\sqrt{\int_{-\infty}^{+\infty} d y e^{-U(y)}}} \tag{25}
\end{equation*}
$$

(ii) if the integral of Eq. 24 diverges

$$
\begin{equation*}
\int_{-\infty}^{+\infty} d x e^{-U(x)}=+\infty \tag{26}
\end{equation*}
$$

then the zero-energy wavefunction of Eq. 21 cannot be normalized, and the Hamiltonian $H$ has no bound state, but only the continuous spectrum discussed in the previous subsection II C1.

## D. Consequences for the Fokker-Planck propagator $G\left(x, t \mid x_{0}, t_{0}\right)$ at large time interval $\left(t-t_{0}\right)$

1. When $H$ has the zero-energy ground-state $\phi^{G S}(x): G\left(x, t \mid x_{0}, t_{0}\right)$ converges towards an equilibrium state $G_{\text {eq }}(x)$

When $H$ has the normalizable zero-energy ground-state $\phi^{G S}(x)$ of Eq. 25 then the quantum propagator $\psi\left(x, t \mid x_{0}, t_{0}\right)$ of Eq. 11 displays the long-time behavior

$$
\begin{equation*}
\psi\left(x, t \mid x_{0}, t_{0}\right) \underset{\left(t-t_{0}\right) \rightarrow+\infty}{\simeq} \phi^{G S}(x) \phi^{G S}\left(x_{0}\right)=\frac{e^{-\frac{U(x)}{2}-\frac{U\left(x_{0}\right)}{2}}}{\int_{-\infty}^{+\infty} d y e^{-U(y)}} \tag{27}
\end{equation*}
$$

So the Fokker-Planck propagator $G\left(x, t \mid x_{0}, t_{0}\right)$ obtained via the similarity transformation of Eq. 10

$$
\begin{equation*}
G\left(x, t \mid x_{0}, t_{0}\right)=e^{-\frac{U(x)}{2}} \psi\left(x, t \mid x_{0}, t_{0}\right) e^{\frac{U\left(x_{0}\right)}{2}} \underset{\left(t-t_{0}\right) \rightarrow+\infty}{\simeq} \frac{e^{-U(x)}}{\int_{-\infty}^{+\infty} d y e^{-U(y)}} \equiv G_{e q}(x) \tag{28}
\end{equation*}
$$

converges towards the Boltzmann equilibrium $G_{e q}(x)$ in the potential $U(x)$. The equilibrium state $G_{e q}(x)$ is the steady state of the Fokker-Planck dynamics of Eq. 6with no steady current

$$
\begin{equation*}
0=\mu(x) G_{e q}(x)-\frac{1}{2} \partial_{x} G_{e q}(x) \tag{29}
\end{equation*}
$$

For the Laplace transform of Eq. 7, the convergence of Eq. 28 means that $\hat{G}_{s}\left(x \mid x_{0}\right)$ is defined for $\left.s \in\right] 0,+\infty[$ with the following singularity for $s \rightarrow 0^{+}$

$$
\begin{equation*}
\hat{G}_{s}\left(x \mid x_{0}\right) \underset{s \rightarrow 0^{+}}{\simeq} \frac{G_{e q}(x)}{s} \tag{30}
\end{equation*}
$$

The simplest example of diffusion converging towards an equilibrium state is the drift $\mu(x)=-\mu \operatorname{sgn}(x)$ of parameter $\mu>0$ that will be discussed in section VIII.
2. When $H$ has only the continuum $] V_{\infty},+\infty\left[\right.$ with $V_{\infty}>0: V_{\infty}$ governs the exponential time decay of $G\left(x_{0}, t \mid x_{0}, t_{0}\right)$

When $H$ has only the continuous spectrum $] V_{\infty},+\infty\left[\right.$, where the lower boundary $V_{\infty}$ of Eq. 17 is strictly positive $V_{\infty}>0$, then the Fokker-Planck propagator $G\left(x, t \mid x_{0}, t_{0}\right)$ and the quantum propagator $\psi\left(x, t \mid x_{0}, t_{0}\right)$ are dominated by the leading exponential time decay involving $V_{\infty}$

$$
\begin{equation*}
G\left(x_{0}, t \mid x_{0}, t_{0}\right)=e^{\frac{U\left(x_{0}\right)-U(x)}{2}} \psi\left(x_{0}, t \mid x_{0}, t_{0}\right) \underset{\left(t-t_{0}\right) \rightarrow+\infty}{\propto} e^{-V_{\infty}\left(t-t_{0}\right)} \tag{31}
\end{equation*}
$$

The physical interpretation is that the diffusion process is transient and flows towards infinity. For the Laplace transform of Eq. 7. Eq. 31 means that $\hat{G}_{s}\left(x \mid x_{0}\right)$ is defined for $\left.s \in\right]-V_{\infty},+\infty[$. In particular, it remains finite for $s=0$ in contrast to the previous case of Eq. 30

$$
\begin{equation*}
\hat{G}_{s=0}\left(x_{0} \mid x_{0}\right)<+\infty \tag{32}
\end{equation*}
$$

The simplest example of transient diffusion is the uniform strictly positive drift $\mu(x)=\mu>0$ that will be discussed in section VII.
3. When $H$ has only the continuous spectrum $] V_{\infty}=0,+\infty\left[\right.$ with the vanishing lower boundary $V_{\infty}=0$

When $H$ has only the continuous spectrum $] V_{\infty}=0,+\infty\left[\right.$ with the vanishing lower boundary $V_{\infty}=0$, then the Fokker-Planck propagator $G\left(x, t \mid x_{0}, t_{0}\right)$ decays in time, but less rapidly than the exponential decay of Eq. 31 .

The simplest example of recurrent diffusion that does not converge towards an equilibrium state is of course the pure Brownian motion without drift $\mu=0$, that will be discussed in section VII.

## III. JOINT PROPERTIES OF THE DIFFUSION PROCESS $X(t)$ AND ITS LOCAL TIME $A(t)$

In this section, we focus on the unconditioned joint process $[X(t), A(t)]:$ the position $X(t)$ and its local time $A(t)$ at the origin satisfy the Ito Stochastic Differential System

$$
\begin{align*}
d X(t) & =\mu(X(t)) d t+d B(t) \\
d A(t) & =\delta(X(t)) d t \tag{33}
\end{align*}
$$

The joint propagator $P\left(x, A, t \mid x_{0}, A_{0}, t_{0}\right)$ for the position $x$ and the local time $A$ that satisfies the Fokker-Planck dynamics

$$
\begin{equation*}
\partial_{t} P\left(x, A, t \mid x_{0}, A_{0}, t_{0}\right)=-\delta(x) \partial_{A} P\left(x, A, t \mid x_{0}, A_{0}, t_{0}\right)-\partial_{x}\left[\mu(x) P\left(x, A, t \mid x_{0}, A_{0}, t_{0}\right)\right]+\frac{1}{2} \partial_{x}^{2} P\left(x, A, t \mid x_{0}, A_{0}, t_{0}\right) \tag{34}
\end{equation*}
$$

will be useful to construct conditioned bridges involving both the final position and the final local time, as will be described in the subsection VIA.

## A. Laplace transform $\tilde{P}_{p}\left(x, t \mid x_{0}, t_{0}\right)$ with respect to the local time $\left(A-A_{0}\right) \geq 0$ : Feynman-Kac formula

For the Laplace transform $\tilde{P}_{p}\left(x, t \mid x_{0}, t_{0}\right)$ of the joint propagator $P\left(x, A, t \mid x_{0}, A_{0}, t_{0}\right)$ with respect to the local time increment $\left(A-A_{0}\right) \geq 0$

$$
\begin{equation*}
\tilde{P}_{p}\left(x, t \mid x_{0}, t_{0}\right) \equiv \int_{A_{0}}^{+\infty} d A e^{-p\left(A-A_{0}\right)} P\left(x, A, t \mid x_{0}, A_{0}, t_{0}\right) \tag{35}
\end{equation*}
$$

Eq. 34 translates into

$$
\begin{equation*}
\partial_{t} \tilde{P}_{p}\left(x, t \mid x_{0}, t_{0}\right)=-p \delta(x) \tilde{P}_{p}\left(0, t \mid x_{0}, t_{0}\right)-\partial_{x}\left[\mu(x) \tilde{P}_{p}\left(x, t \mid x_{0}, t_{0}\right)\right]+\frac{1}{2} \partial_{x}^{2} \tilde{P}_{p}\left(x, t \mid x_{0}, t_{0}\right) \tag{36}
\end{equation*}
$$

This is a standard example of the Feynman-Kac formula, where the initial Fokker-Planck dynamics of Eq. 6 is now supplemented by the additional term in $p \delta(x)$.
B. Explicit double Laplace transform $\hat{\tilde{P}}_{s, p}\left(x \mid x_{0}\right)$ of the joint propagator $P\left(x, A, t \mid x_{0}, A_{0}, t_{0}\right)$ via the Dyson Eq.

The further Laplace transform of Eq. 35 with respect to the time $\left(t-t_{0}\right)$

$$
\begin{equation*}
\hat{\tilde{P}}_{s, p}\left(x \mid x_{0}\right) \equiv \int_{t_{0}}^{+\infty} d t e^{-s\left(t-t_{0}\right)} \tilde{P}_{p}\left(x, t \mid x_{0}, t_{0}\right)=\int_{t_{0}}^{+\infty} d t e^{-s\left(t-t_{0}\right)} \int_{A_{0}}^{+\infty} d A e^{-p\left(A-A_{0}\right)} P\left(x, A, t \mid x_{0}, A_{0}, t_{0}\right) \tag{37}
\end{equation*}
$$

satisfies

$$
\begin{equation*}
-\delta\left(x-x_{0}\right)+s \hat{\tilde{P}}_{s, p}\left(x \mid x_{0}\right)=-p \delta(x) \hat{\tilde{P}}_{s, p}\left(0 \mid x_{0}\right)-\partial_{x}\left[\mu(x) \hat{\tilde{P}}_{s, p}\right]+\frac{1}{2} \partial_{x}^{2} \hat{\tilde{P}}_{s, p} \tag{38}
\end{equation*}
$$

For $p=0$, Eq. 35 coincides with the propagator $G\left(x, t \mid x_{0}, t_{0}\right)$ of the position alone described in the previous section III

$$
\begin{equation*}
\tilde{P}_{p=0}\left(x, t \mid x_{0}, t_{0}\right) \equiv \int_{A_{0}}^{+\infty} d A P\left(x, A, t \mid x_{0}, A_{0}, t_{0}\right)=G\left(x, t \mid x_{0}, t_{0}\right) \tag{39}
\end{equation*}
$$

As a consequence, Eq. 37 becomes

$$
\begin{equation*}
\hat{\tilde{P}}_{s, p=0}\left(x \mid x_{0}\right)=\hat{G}_{s}\left(x \mid x_{0}\right) \tag{40}
\end{equation*}
$$

For any $p \neq 0$, the solution $\hat{\tilde{P}}_{s, p}\left(x \mid x_{0}\right)$ of Eq. 38 satisfies the Dyson equation

$$
\begin{equation*}
\hat{\tilde{P}}_{s, p}\left(x \mid x_{0}\right)=\hat{G}_{s}\left(x \mid x_{0}\right)-p \hat{G}_{s}(x \mid 0) \hat{\tilde{P}}_{s, p}\left(0 \mid x_{0}\right) \tag{41}
\end{equation*}
$$

The self-consistency for $x=0$

$$
\begin{equation*}
\hat{\tilde{P}}_{s, p}\left(0 \mid x_{0}\right)=\hat{G}_{s}\left(0 \mid x_{0}\right)-p \hat{G}_{s}(0 \mid 0) \hat{\tilde{P}}_{s, p}\left(0 \mid x_{0}\right) \tag{42}
\end{equation*}
$$

yields

$$
\begin{equation*}
\hat{\tilde{P}}_{s, p}\left(0 \mid x_{0}\right)=\frac{\hat{G}_{s}\left(0 \mid x_{0}\right)}{1+p \hat{G}_{s}(0 \mid 0)} \tag{43}
\end{equation*}
$$

Plugging this result into Eq. 41 yields the final expression of $\hat{\tilde{P}}_{s, p}\left(x \mid x_{0}\right)$ in terms of $\hat{G}_{s}(. \mid$.

$$
\begin{equation*}
\hat{\tilde{P}}_{s, p}\left(x \mid x_{0}\right)=\hat{G}_{s}\left(x \mid x_{0}\right)-p \frac{\hat{G}_{s}(x \mid 0) \hat{G}_{s}\left(0 \mid x_{0}\right)}{1+p \hat{G}_{s}(0 \mid 0)} \tag{44}
\end{equation*}
$$

C. Explicit time Laplace transform $\hat{P}_{s}\left(x, A \mid x_{0}, A_{0}\right)$ of the joint propagator $P\left(x, A, t \mid x_{0}, A_{0}, t_{0}\right)$

The dependence with respect to the parameter $p$ in Eq. 44 can be rewritten in terms of a simple pole as

$$
\begin{align*}
\hat{\tilde{P}}_{s, p}\left(x \mid x_{0}\right) & =\hat{G}_{s}\left(x \mid x_{0}\right)-\frac{\hat{G}_{s}(x \mid 0) \hat{G}_{s}\left(0 \mid x_{0}\right)}{\hat{G}_{s}(0 \mid 0)}\left(1-\frac{1}{1+p \hat{G}_{s}(0 \mid 0)}\right) \\
& =\left[\hat{G}_{s}\left(x \mid x_{0}\right)-\frac{\hat{G}_{s}(x \mid 0) \hat{G}_{s}\left(0 \mid x_{0}\right)}{\hat{G}_{s}(0 \mid 0)}\right]+\left[\frac{\hat{G}_{s}(x \mid 0) \hat{G}_{s}\left(0 \mid x_{0}\right)}{\hat{G}_{s}^{2}(0 \mid 0)}\right] \frac{1}{p+\frac{1}{\hat{G}_{s}(0 \mid 0)}} \tag{45}
\end{align*}
$$

So the inverse Laplace transform with respect to $p$ yields that the time Laplace transform $\hat{P}_{s}\left(x, A \mid x_{0}, A_{0}\right)$ of the joint propagator $P\left(x, A, t \mid x_{0}, A_{0}, t_{0}\right)$ reads

$$
\begin{align*}
\hat{P}_{s}\left(x, A \mid x_{0}, A_{0}\right) & \equiv \int_{t_{0}}^{+\infty} d t e^{-s\left(t-t_{0}\right)} P\left(x, A, t \mid x_{0}, A_{0}, t_{0}\right) \\
& =\delta\left(A-A_{0}\right)\left[\hat{G}_{s}\left(x \mid x_{0}\right)-\frac{\hat{G}_{s}(x \mid 0) \hat{G}_{s}\left(0 \mid x_{0}\right)}{\hat{G}_{s}(0 \mid 0)}\right]+\theta\left(A>A_{0}\right)\left[\frac{\hat{G}_{s}(x \mid 0) \hat{G}_{s}\left(0 \mid x_{0}\right)}{\hat{G}_{s}^{2}(0 \mid 0)}\right] e^{-\frac{\left(A-A_{0}\right)}{\hat{G}_{s}(0 \mid 0)}} \tag{46}
\end{align*}
$$

The normalization over $A$ corresponding to $p=0$ in Eq. 40 is given by $\hat{G}_{s}\left(x \mid x_{0}\right)$

$$
\begin{equation*}
\int_{A_{0}}^{+\infty} d A \hat{P}_{s}\left(x, A \mid x_{0}, A_{0}\right)=\hat{G}_{s}\left(x \mid x_{0}\right)=\int_{t_{0}}^{+\infty} d t e^{-s\left(t-t_{0}\right)} G\left(x, t \mid x_{0}, t_{0}\right) \tag{47}
\end{equation*}
$$

Let us now explain the physical meaning of the formula of Eq. 46 in the following subsections.

1. Interpretation of the singular contribution in $\delta\left(A-A_{0}\right)$ of $P\left(x, A, t \mid x_{0}, A_{0}, t_{0}\right)$ in terms of the propagator $G^{a b s}\left(x, t \mid x_{0}, t_{0}\right)$

In Eq. 46, the singular contribution involving the delta function $\delta\left(A-A_{0}\right)$

$$
\begin{equation*}
\hat{P}_{s}^{\text {Singular }}\left(x, A \mid x_{0}, A_{0}\right)=\delta\left(A-A_{0}\right)\left[\hat{G}_{s}\left(x \mid x_{0}\right)-\frac{\hat{G}_{s}(x \mid 0) \hat{G}_{s}\left(0 \mid x_{0}\right)}{\hat{G}_{s}(0 \mid 0)}\right] \tag{48}
\end{equation*}
$$

means that the local time $A$ has kept its initial value $A_{0}$, i.e. the diffusion process has not been able to visit the origin $x=0$. As a consequence, the weight in factor of the delta function $\delta\left(A-A_{0}\right)$ should correspond to the Laplace transform $\hat{G}_{s}^{a b s}\left(x \mid x_{0}\right)$ of the propagator $G^{a b s}\left(x, t \mid x_{0}, t_{0}\right)$ in the presence of an absorbing boundary condition at the origin $x=0$

$$
\begin{equation*}
\left[\hat{G}_{s}\left(x \mid x_{0}\right)-\frac{\hat{G}_{s}(x \mid 0) \hat{G}_{s}\left(0 \mid x_{0}\right)}{\hat{G}_{s}(0 \mid 0)}\right]=\hat{G}_{s}^{a b s}\left(x \mid x_{0}\right) \equiv \int_{t_{0}}^{+\infty} d t e^{-s\left(t-t_{0}\right)} G^{a b s}\left(x, t \mid x_{0}, t_{0}\right) \tag{49}
\end{equation*}
$$

This interpretation can be also recovered by considering the limit $p \rightarrow+\infty$ in Eq. 45

$$
\begin{equation*}
\hat{\tilde{P}}_{s, p=+\infty}\left(x \mid x_{0}\right)=\left[\hat{G}_{s}\left(x \mid x_{0}\right)-\frac{\hat{G}_{s}(x \mid 0) \hat{G}_{s}\left(0 \mid x_{0}\right)}{\hat{G}_{s}(0 \mid 0)}\right] \tag{50}
\end{equation*}
$$

Indeed, in the Feynman-Kac formula of Eq. 36, the limit of $p \rightarrow+\infty$ amounts to impose the vanishing of the solution $\tilde{P}_{p=+\infty}\left(x=0, t \mid x_{0}, t_{0}\right)$ at the origin $x=0$

$$
\begin{equation*}
\tilde{P}_{p=+\infty}\left(x=0, t \mid x_{0}, t_{0}\right)=0 \tag{51}
\end{equation*}
$$

i.e. amounts to impose that the origin $x=0$ is an absorbing boundary condition.

In summary, the singular contribution of Eq. 48 can be rewritten as

$$
\begin{equation*}
\hat{P}_{s}^{\text {Singular }}\left(x, A \mid x_{0}, A_{0}\right)=\delta\left(A-A_{0}\right) \hat{G}_{s}^{a b s}\left(x \mid x_{0}\right) \tag{52}
\end{equation*}
$$

and its Laplace inversion with respect to $s$ involves the propagator $G^{a b s}\left(x, t \mid x_{0}, t_{0}\right)$ in the presence of an absorbing boundary condition at the origin $x=0$

$$
\begin{equation*}
P^{\text {Singular }}\left(x, A, t \mid x_{0}, A_{0}, t_{0}\right)=\delta\left(A-A_{0}\right) G^{a b s}\left(x, t \mid x_{0}, t_{0}\right) \tag{53}
\end{equation*}
$$

$$
\text { 2. Corresponding survival probability } S^{a b s}\left(t \mid x_{0}, t_{0}\right) \text { and absorption rate } \gamma^{a b s}\left(t \mid x_{0}, t_{0}\right)
$$

The survival probability $S^{a b s}\left(t \mid x_{0}, t_{0}\right)$ at time $t$ in the presence of an absorbing boundary at the origin $x=0$ if one starts at the position $x_{0}$ at time $t_{0}$, can be obtained from the integration of the propagator $G^{a b s}\left(x, t \mid x_{0}, t_{0}\right)$ over the final position $x$

$$
\begin{equation*}
S^{a b s}\left(t \mid x_{0}, t_{0}\right) \equiv \int_{-\infty}^{+\infty} d x G^{a b s}\left(x, t \mid x_{0}, t_{0}\right) \tag{54}
\end{equation*}
$$

The conservation of probability for the full propagator $G\left(x, t \mid x_{0}, t_{0}\right)$

$$
\begin{equation*}
\int_{-\infty}^{+\infty} d x G\left(x, t \mid x_{0}, t_{0}\right)=1 \tag{55}
\end{equation*}
$$

translates for its Laplace transform into

$$
\begin{equation*}
\int_{-\infty}^{+\infty} d x \hat{G}_{s}\left(x \mid x_{0}\right)=\int_{t_{0}}^{+\infty} d t e^{-s\left(t-t_{0}\right)} \int_{-\infty}^{+\infty} d x G\left(x, t \mid x_{0}, t_{0}\right)=\int_{t_{0}}^{+\infty} d t e^{-s\left(t-t_{0}\right)}=\frac{1}{s} \tag{56}
\end{equation*}
$$

So the time Laplace transform of Eq. 54 reads via the integration of Eq. 49 over $x$

$$
\begin{equation*}
\hat{S}_{s}^{a b s}\left(x_{0}\right) \equiv \int_{t_{0}}^{+\infty} d t e^{-s\left(t-t_{0}\right)} S^{a b s}\left(t \mid x_{0}, t_{0}\right)=\int_{-\infty}^{+\infty} d x \hat{G}_{s}^{a b s}\left(x \mid x_{0}\right)=\frac{1}{s}\left[1-\frac{\hat{G}_{s}\left(0 \mid x_{0}\right)}{\hat{G}_{s}(0 \mid 0)}\right] \tag{57}
\end{equation*}
$$

It is now useful to introduce the absorption rate $\gamma^{a b s}\left(t \mid x_{0}, t_{0}\right)$ at time $t$ when starting at the position $x_{0}$ at time $t_{0}$

$$
\begin{equation*}
\gamma^{a b s}\left(t \mid x_{0}, t_{0}\right) \equiv-\partial_{t} S^{a b s}\left(t \mid x_{0}, t_{0}\right) \tag{58}
\end{equation*}
$$

Its time Laplace transform is simple, as shown via the following integration by parts

$$
\begin{align*}
\hat{\gamma}_{s}^{a b s}\left(x_{0}\right) & =-\int_{t_{0}}^{+\infty} d t e^{-s\left(t-t_{0}\right)} \partial_{t} S^{a b s}\left(t \mid x_{0}, t_{0}\right)=-\left[e^{-s\left(t-t_{0}\right)} S^{a b s}\left(t \mid x_{0}, t_{0}\right)\right]_{t_{0}}^{+\infty}-s \hat{S}_{s}^{a b s}\left(x_{0}\right)=1-s \hat{S}_{s}^{a b s}\left(x_{0}\right) \\
& =\frac{\hat{G}_{s}\left(0 \mid x_{0}\right)}{\hat{G}_{s}(0 \mid 0)} \tag{59}
\end{align*}
$$

## 3. Interpretation of the regular contribution in $\theta\left(A>A_{0}\right)$ for $P\left(x, A, t \mid x_{0}, A_{0}, t_{0}\right)$

For the special case where the initial and the final positions are at the origin $x=0=x_{0}$, Eq. 46 reduces to

$$
\begin{equation*}
\hat{P}_{s}\left(x=0, A \mid x_{0}=0, A_{0}\right)=\theta\left(A>A_{0}\right) e^{-\frac{\left(A-A_{0}\right)}{\hat{G}_{s}(0 \mid 0)}} \tag{60}
\end{equation*}
$$

Its normalization over $A$

$$
\begin{equation*}
\int d A \hat{P}_{s}\left(x=0, A \mid x_{0}=0, A_{0}\right)=\hat{G}_{s}(0 \mid 0) \equiv \int_{t_{0}}^{+\infty} d t e^{-s\left(t-t_{0}\right)} G\left(0, t \mid 0, t_{0}\right) \tag{61}
\end{equation*}
$$

involves the propagator $G\left(0, t \mid 0, t_{0}\right)$ from the origin to the origin.
The regular contribution of Eq. 46

$$
\begin{equation*}
\hat{P}_{s}^{\text {Regular }}\left(x, A \mid x_{0}, A_{0}\right)=\theta\left(A>A_{0}\right)\left[\frac{\hat{G}_{s}(x \mid 0) \hat{G}_{s}\left(0 \mid x_{0}\right)}{\hat{G}_{s}^{2}(0 \mid 0)}\right] e^{-\frac{\left(A-A_{0}\right)}{\hat{G}_{s}(0 \mid 0)}} \tag{62}
\end{equation*}
$$

can be thus rewritten as the product of the three following functions using Eqs 59 and 60

$$
\begin{equation*}
\hat{P}_{s}^{\text {Regular }}\left(x, A \mid x_{0}, A_{0}\right)=\hat{\gamma}_{s}^{a b s}(x) \hat{P}_{s}\left(0, A \mid 0, A_{0}\right) \hat{\gamma}_{s}^{a b s}\left(x_{0}\right) \tag{63}
\end{equation*}
$$

Its Laplace inversion involves the time-convolution of the three functions

$$
\begin{equation*}
P^{\text {Regular }}\left(x, A, t \mid x_{0}, A_{0}, t_{0}\right)=\int_{t_{0}}^{t} d t_{1} \int_{t_{1}}^{t} d t_{2} \gamma^{a b s}\left(t \mid x, t_{2}\right) P\left(0, A, t_{2} \mid 0, A_{0}, t_{1}\right) \gamma^{a b s}\left(t_{1} \mid x_{0}, t_{0}\right) \tag{64}
\end{equation*}
$$

with the following physical meaning.
(i) The time $t_{1}$ is the first passage time at the origin if one starts at the initial position $x_{0}$ at time $t_{0}$, whose statistics is governed by the absorption rate $\gamma^{a b s}\left(t_{1} \mid x_{0}, t_{0}\right)$.
(ii) The time $t_{2}$ is the last passage time at the origin before reaching the final point $x$ at time $t$, where the statistics of the time interval $\left(t-t_{2}\right)$ is governed by absorption rate $\gamma^{a b s}\left(t \mid x, t_{2}\right)$ of the alternative problem when one starts at position $x$ at time $t_{2}$.
(iii) Between the first-passage-time $t_{1}$ and the last-passage-time $t_{2}$ at the origin, the statistics of the local time increment $\left(A-A_{0}\right)$ is governed by the probability $P\left(0, A, t_{2} \mid 0, A_{0}, t_{1}\right)$.

## IV. PROBABILITY DISTRIBUTION $\Pi\left(A, t \mid x_{0}, A_{0}, t_{0}\right)$ OF THE LOCAL TIME $A$ AT TIME $t$

In this section, we focus on the the probability $\Pi\left(A, t \mid x_{0}, A_{0}, t_{0}\right)$ to see the local time $A$ at time $t$ if one starts at position $x_{0}$ with the local time $A_{0}$ at time $t_{0}$. It can be obtained from the integration over the final position $x$ of the joint propagator $P\left(x, A, t \mid x_{0}, A_{0}, t_{0}\right)$ studied in the previous section and it can be thus decomposed into a singular contribution in $\delta\left(A-A_{0}\right)$ and a regular contribution in $\theta\left(A>A_{0}\right)$

$$
\begin{align*}
\Pi\left(A, t \mid x_{0}, A_{0}, t_{0}\right) & \equiv \int_{-\infty}^{+\infty} d x P\left(x, A, t \mid x_{0}, A_{0}, t_{0}\right) \\
& =\Pi^{\text {Singular }}\left(A, t \mid x_{0}, A_{0}, t_{0}\right)+\Pi^{\text {Regular }}\left(A, t \mid x_{0}, A_{0}, t_{0}\right) \tag{65}
\end{align*}
$$

This probability of Eq. 65 will be useful to construct conditioned bridges involving the local time, as will be described in the subsection VIB.

## A. Explicit Laplace transform $\hat{\Pi}_{s}\left(A \mid x_{0}, A_{0}\right)$ of the probability $\Pi\left(A, t \mid x_{0}, A_{0}, t_{0}\right)$

The Laplace transform of Eq. 65 with respect to the time interval $\left(t-t_{0}\right)$

$$
\begin{equation*}
\hat{\Pi}_{s}\left(A \mid x_{0}, A_{0}\right) \equiv \int_{t_{0}}^{+\infty} d t e^{-s\left(t-t_{0}\right)} \Pi\left(A, t \mid x_{0}, A_{0}, t_{0}\right)=\int_{-\infty}^{+\infty} d x \hat{P}_{s}\left(x, A \mid x_{0}, A_{0}\right) \tag{66}
\end{equation*}
$$

can be obtained via the integration over $x$ of $\hat{P}_{s}\left(x, A \mid x_{0}, A_{0}\right)$ given by Eq. 46 using Eq. 56

$$
\begin{align*}
& \hat{\Pi}_{s}\left(A \mid x_{0}, A_{0}\right)=\Pi_{s}^{\text {Singular }}\left(A \mid x_{0}, t_{0}\right)+\Pi_{s}^{\text {Regular }}\left(A \mid x_{0}, A_{0}\right) \\
& \Pi_{s}^{\text {Singular }}\left(A \mid x_{0}, t_{0}\right)=\delta\left(A-A_{0}\right) \frac{1}{s}\left[1-\frac{\hat{G}_{s}\left(0 \mid x_{0}\right)}{\hat{G}_{s}(0 \mid 0)}\right] \\
& \Pi_{s}^{\text {Regular }}\left(A \mid x_{0}, A_{0}\right)=\theta\left(A>A_{0}\right)\left[\frac{\hat{G}_{s}\left(0 \mid x_{0}\right)}{s \hat{G}_{s}^{2}(0 \mid 0)}\right] e^{-\frac{\left(A-A_{0}\right)}{\hat{G}_{s}(0 \mid 0)}} \tag{67}
\end{align*}
$$

## 1. Interpretation of the singular contribution in $\delta\left(A-A_{0}\right)$

The singular contribution $\Pi_{s}^{\text {Singular }}\left(A \mid x_{0}, t_{0}\right)$ of Eq. 67 involves the time Laplace transform $\hat{S}_{s}^{a b s}\left(x_{0}\right)$ of Eq. 57

$$
\begin{equation*}
\hat{\Pi}_{s}^{S i n g u l a r}\left(A \mid x_{0}, A_{0}\right)=\delta\left(A-A_{0}\right) \frac{1}{s}\left[1-\frac{\hat{G}_{s}\left(0 \mid x_{0}\right)}{\hat{G}_{s}(0 \mid 0)}\right]=\delta\left(A-A_{0}\right) \hat{S}_{s}^{a b s}\left(x_{0}\right) \tag{68}
\end{equation*}
$$

and its Laplace inversion involves the survival probability $S^{a b s}\left(t \mid x_{0}, t_{0}\right)$ of Eq. 54

$$
\begin{equation*}
\Pi^{\text {Singular }}\left(A, t \mid x_{0}, A_{0}, t_{0}\right)=\delta\left(A-A_{0}\right) S^{a b s}\left(t \mid x_{0}, t_{0}\right) \tag{69}
\end{equation*}
$$

## 2. Interpretation of the regular contribution in $\theta\left(A>A_{0}\right)$

For the special case where the initial position vanishes $x_{0}=0$, Eq. 67 reduces to

$$
\begin{equation*}
\hat{\Pi}_{s}\left(A \mid x_{0}=0, A_{0}\right)=\hat{\Pi}_{s}^{\text {Regular }}\left(A \mid x_{0}=0, A_{0}\right)=\frac{\theta\left(A>A_{0}\right)}{s \hat{G}_{s}(0 \mid 0)} e^{-\frac{\left(A-A_{0}\right)}{\hat{G}_{s}(0 \mid 0)}} \tag{70}
\end{equation*}
$$

As a consequence, the regular contribution $\hat{\Pi}_{s}^{\text {Regular }}\left(A \mid x_{0}, A_{0}\right)$ of Eq. 67 can be rewritten as the product of two functions using Eq. 59 and 70

$$
\begin{equation*}
\hat{\Pi}_{s}^{\text {Regular }}\left(A \mid x_{0}, A_{0}\right)=\theta\left(A>A_{0}\right) \frac{\hat{G}_{s}\left(0 \mid x_{0}\right)}{s \hat{G}_{s}^{2}(0 \mid 0)} e^{-\frac{\left(A-A_{0}\right)}{\hat{G}_{s}(0 \mid 0)}}=\hat{\Pi}_{s}\left(A \mid x_{0}=0, A_{0}\right) \hat{\gamma}_{s}^{a b s}\left(x_{0}\right) \tag{71}
\end{equation*}
$$

Its Laplace inversion with respect to $s$ can be written as the time-convolution of two functions

$$
\begin{equation*}
\Pi^{\text {Regular }}\left(A, t \mid x_{0}, A_{0}, t_{0}\right)=\int_{t_{0}}^{t} d t_{1} \Pi\left(A, t \mid 0, A_{0}, t_{1}\right) \gamma^{a b s}\left(t_{1} \mid x_{0}, t_{0}\right) \tag{72}
\end{equation*}
$$

with the following physical meaning :
(i) as in Eq. 64, the time $t_{1}$ is the first passage time at the origin if one starts at the initial position $x_{0}$ at time $t_{0}$, whose statistics is governed by the absorption rate $\gamma^{a b s}\left(t_{1} \mid x_{0}, t_{0}\right)$.
(ii) for the remaining time interval $\left(t-t_{1}\right), \Pi\left(A, t \mid 0, A_{0}, t_{1}\right)$ represents the probability to see the local time increment $\left(A-A_{0}\right)$ when starting at the origin $x_{0}=0$.
B. Moments $m^{[k]}\left(t \mid x_{0}, t_{0}\right)$ of order $k=1,2, .$. of the local time increment $\left[A(t)-A\left(t_{0}\right)\right]$

1. Computation of the moments $m^{[k]}\left(t \mid x_{0}, t_{0}\right)$ from the probability distribution $\Pi\left(A, t \mid x_{0}, A_{0}, t_{0}\right)$

The moments of order $k=1,2, \ldots$ of the local time increment $\left[A(t)-A\left(t_{0}\right)\right]$ only involve the regular part $\Pi^{\text {Regular }}\left(A, t \mid x_{0}, A_{0}, t_{0}\right)$ of the probability distribution $\Pi\left(A, t \mid x_{0}, A_{0}, t_{0}\right)$ of Eq. 65

$$
\begin{align*}
m^{[k]}\left(t \mid x_{0}, t_{0}\right) & \equiv \int_{A_{0}}^{+\infty} d A\left[A-A_{0}\right]^{k} \Pi\left(A, t \mid x_{0}, A_{0}, t_{0}\right) \\
& =\int_{A_{0}}^{+\infty} d A\left[A-A_{0}\right]^{k} \Pi^{\text {Regular }}\left(A, t \mid x_{0}, A_{0}, t_{0}\right) \tag{73}
\end{align*}
$$

Their Laplace transforms with respect to the time interval $\left(t-t_{0}\right)$ can be obtained from $\hat{\Pi}_{s}^{\text {Regular }}\left(A \mid x_{0}, A_{0}\right)$ of Eq. 71

$$
\begin{align*}
m_{s}^{[k]}\left(x_{0}\right) & \equiv \int_{t_{0}}^{+\infty} d t e^{-s\left(t-t_{0}\right)} m^{[k]}\left(t \mid x_{0}, t_{0}\right)=\int_{A_{0}}^{+\infty} d A\left[A-A_{0}\right]^{k} \hat{\Pi}_{s}^{\text {Regular }}\left(A \mid x_{0}, A_{0}\right) \\
& =\frac{\hat{G}_{s}\left(0 \mid x_{0}\right)}{s \hat{G}_{s}^{2}(0 \mid 0)} \int_{A_{0}}^{+\infty} d A\left[A-A_{0}\right]^{k} e^{-\frac{\left(A-A_{0}\right)}{\hat{G}_{s}(0 \mid 0)}}=\frac{k!\left[\hat{G}_{s}(0 \mid 0)\right]^{k-1} \hat{G}_{s}\left(0 \mid x_{0}\right)}{s} \tag{74}
\end{align*}
$$

2. Alternative computation of the moments $m^{[k]}\left(t \mid x_{0}, t_{0}\right)$ from the definition of Eq. 2

Alternatively, one can use the definition of Eq. 2

$$
\begin{equation*}
A(t)-A\left(t_{0}\right)=\int_{t_{0}}^{t} d \tau \delta(X(\tau)) \tag{75}
\end{equation*}
$$

to compute the moment of order $k$ for trajectories starting at $X\left(t_{0}\right)=x_{0}$ in terms of the propagator $G\left(x, t \mid y, t^{\prime}\right)$

$$
\begin{align*}
& m^{[k]}\left(t \mid x_{0}, t_{0}\right)=\int_{t_{0}}^{t} d t_{k} \int_{t_{0}}^{t} d t_{k-1} \ldots \int_{t_{0}}^{t} d t_{2} \int_{t_{0}}^{t} d t_{1}\left\langle\delta\left(X\left(t_{k}\right)\right) \delta\left(X\left(t_{k-1}\right)\right) \ldots \delta\left(X\left(t_{2}\right)\right) \delta\left(X\left(t_{1}\right)\right) \delta\left(X\left(t_{0}\right)-x_{0}\right)\right\rangle \\
& =k!\int_{t_{0}}^{t} d t_{k} \int_{t_{0}}^{t_{k}} d t_{k-1} \ldots \int_{t_{0}}^{t_{3}} d t_{2} \int_{t_{0}}^{t_{2}} d t_{1}\left\langle\delta\left(X\left(t_{k}\right)\right) \delta\left(X\left(t_{k-1}\right)\right) \ldots \delta\left(X\left(t_{2}\right)\right) \delta\left(X\left(t_{1}\right)\right) \delta\left(X\left(t_{0}\right)-x_{0}\right)\right\rangle \\
& =k!\int_{t_{0}}^{t} d t_{k} \int_{t_{0}}^{t_{k}} d t_{k-1} \ldots \int_{t_{0}}^{t_{3}} d t_{2} \int_{t_{0}}^{t_{2}} d t_{1} G\left(0, t_{k} \mid 0, t_{k-1}\right) G\left(0, t_{k-1} \mid 0, t_{k-2}\right) \ldots G\left(0, t_{2} \mid 0, t_{1}\right) G\left(0, t_{1} \mid x_{0}, t_{0}\right) \tag{76}
\end{align*}
$$

so that its Laplace transform

$$
\begin{align*}
& m_{s}^{[k]}\left(x_{0}\right)=\int_{t_{0}}^{+\infty} d t e^{-s\left(t-t_{0}\right)} m^{[k]}\left(t \mid x_{0}, t_{0}\right) \\
& =k!\int_{t_{0}}^{+\infty} d t e^{-s\left(t-t_{k}\right)-s\left(t_{k}-t_{k-1}\right)-\ldots-s\left(t_{2}-t_{1}\right)-s\left(t_{1}-t_{0}\right)} \\
& \int_{t_{0}}^{t} d t_{k} \int_{t_{0}}^{t_{k}} d t_{k-1} \ldots \int_{t_{0}}^{t_{3}} d t_{2} \int_{t_{0}}^{t_{2}} d t_{1} G\left(0, t_{k} \mid 0, t_{k-1}\right) G\left(0, t_{k-1} \mid 0, t_{k-2}\right) \ldots G\left(0, t_{2} \mid x_{0}, t_{1}\right) G\left(0, t_{1} \mid x_{0}, t_{0}\right) \\
& =k!\int_{t_{0}}^{+\infty} d t_{1} e^{-s\left(t_{1}-t_{0}\right)} G\left(0, t_{1} \mid x_{0}, t_{0}\right) \ldots \int_{t_{0}}^{+\infty} d t_{k} e^{-s\left(t_{k}-t_{k-1}\right)} G\left(0, t_{k} \mid 0, t_{k-1}\right) \int_{t_{k}}^{+\infty} d t e^{-s\left(t-t_{k}\right)} \\
& =\frac{k!\left[\hat{G}_{s}(0 \mid 0)\right]^{k-1} \hat{G}_{s}\left(0 \mid x_{0}\right)}{s} \tag{77}
\end{align*}
$$

coincides with Eq. 74 as it should.

$$
\text { 3. Example of the two first moments for } k=1 \text { and } k=2
$$

The first moment $k=1$ of Eq. 76 reduces to the single time integral

$$
\begin{equation*}
m^{[k=1]}\left(t \mid x_{0}, t_{0}\right)=\int_{t_{0}}^{t} d t_{1} G\left(0, t_{1} \mid x_{0}, t_{0}\right) \tag{78}
\end{equation*}
$$

Its growth is thus directly governed by the propagator $G\left(0, t \mid x_{0}, t_{0}\right)$

$$
\begin{equation*}
\partial_{t} m^{[k=1]}\left(t \mid x_{0}, t_{0}\right)=G\left(0, t \mid x_{0}, t_{0}\right)>0 \tag{79}
\end{equation*}
$$

The Laplace transform of Eq. 74 reads

$$
\begin{equation*}
m_{s}^{[k=1]}\left(x_{0}\right)=\frac{\hat{G}_{s}\left(0 \mid x_{0}\right)}{s} \tag{80}
\end{equation*}
$$

The second moment $k=2$ of Eq. 76 reads

$$
\begin{equation*}
m^{[k=2]}\left(t \mid x_{0}, t_{0}\right)=2 \int_{t_{0}}^{t} d t_{2} \int_{t_{0}}^{t_{2}} d t_{1} G\left(0, t_{2} \mid 0, t_{1}\right) G\left(0, t_{1} \mid x_{0}, t_{0}\right) \tag{81}
\end{equation*}
$$

with its Laplace transform of Eq. 74

$$
\begin{equation*}
m_{s}^{[k=2]}\left(x_{0}\right)=\frac{2 \hat{G}_{s}(0 \mid 0) \hat{G}_{s}\left(0 \mid x_{0}\right)}{s} \tag{82}
\end{equation*}
$$

## V. STATISTICS OF THE LOCAL TIME INCREMENT $\left(A-A_{0}\right)$ FOR LARGE TIME INTERVAL $\left(t-t_{0}\right)$

In this section, we describe how the recurrence/transience properties of the diffusion process $X(t)$ induced by the drift $\mu(x)$ produce very different behaviors for the scaling of the local time increment $\left(A-A_{0}\right)$ with respect to the large time interval $\left(t-t_{0}\right)$.

## A. When $X(t)$ is transient : the local time increment $\left(A-A_{0}\right)$ remains finite for $\left(t-t_{0}\right) \rightarrow+\infty$

Among transient diffusions, the simplest example is the uniform strictly positive drift $\mu(x)=\mu>0$ that will be discussed in section VII.

When the diffusion process $X(t)$ is transient with the exponential time decay of Eq. 31 for the propagator $G\left(x, t \mid x_{0}, t\right)$, then the first moment $m^{[k=1]}\left(t \mid x_{0}, t_{0}\right)$ of the local time increment $\left[A(t)-A\left(t_{0}\right)\right]$ of Eq. 78 converges towards the finite value $m^{[k=1]}\left(\infty \mid x_{0}\right)$ for $\left(t-t_{0}\right) \rightarrow+\infty$

$$
\begin{equation*}
m^{[k=1]}\left(t \mid x_{0}, t_{0}\right) \underset{\left(t-t_{0}\right) \rightarrow+\infty}{\simeq} m^{[k=1]}\left(\infty \mid x_{0}\right)=\int_{0}^{+\infty} d t_{1} G\left(0, t_{1} \mid x_{0}, 0\right)<+\infty \tag{83}
\end{equation*}
$$

More generally, the local time increment $\left(A-A_{0}\right)$ will remain a finite random variable for $\left(t-t_{0}\right) \rightarrow+\infty$ with the following notation for the limit of the distribution $\Pi\left(A, t \mid x_{0}, A_{0}, t_{0}\right)$ of Eq. 65

$$
\begin{align*}
\Pi\left(A, \infty \mid x_{0}, A_{0}\right) & =\lim _{\left(t-t_{0}\right) \rightarrow+\infty} \Pi\left(A, t \mid x_{0}, A_{0}, t_{0}\right) \\
& =\Pi^{\text {Singular }}\left(A, \infty \mid x_{0}, A_{0}\right)+\Pi^{\text {Regular }}\left(A, \infty \mid x_{0}, A_{0}\right) \tag{84}
\end{align*}
$$

Let us now discuss its singular and regular contributions.
(i) The singular contribution $\Pi^{\text {Singular }}\left(A, \infty \mid x_{0}, A_{0}\right)$ involves the infinite-time limit of Eq. 69

$$
\begin{equation*}
\Pi^{\text {Singular }}\left(A, \infty \mid x_{0}, A_{0}, t_{0}\right)=\delta\left(A-A_{0}\right) S^{a b s}\left(\infty \mid x_{0}\right) \tag{85}
\end{equation*}
$$

that involves the probability to survive forever

$$
\begin{equation*}
S^{a b s}\left(\infty \mid x_{0}\right)=\lim _{\left(t-t_{0}\right) \rightarrow+\infty} S^{a b s}\left(t \mid x_{0}, t_{0}\right) \tag{86}
\end{equation*}
$$

This probability to escape towards infinity without visiting the origin $x=0$ can be obtained from the Laplace transform $\hat{S}_{s}^{a b s}\left(x_{0}\right)$ of Eq. 57 by considering the limit $s \rightarrow 0$ of

$$
\begin{equation*}
S^{a b s}\left(\infty \mid x_{0}\right)=\lim _{s \rightarrow 0}\left[s \hat{S}_{s}^{a b s}\left(x_{0}\right)\right]=\lim _{s \rightarrow 0}\left[1-\frac{\hat{G}_{s}\left(0 \mid x_{0}\right)}{\hat{G}_{s}(0 \mid 0)}\right]=1-\frac{\hat{G}_{0}\left(0 \mid x_{0}\right)}{\hat{G}_{0}(0 \mid 0)} \tag{87}
\end{equation*}
$$

(ii) The Regular contribution $\Pi^{\text {Regular }}\left(A, \infty \mid x_{0}, A_{0}\right)$ can be obtained from the Laplace transform $\hat{\Pi}_{s}^{\text {Regular }}\left(A \mid x_{0}, A_{0}\right)$ of Eq. 71 by considering the limit $s \rightarrow 0$ of

$$
\begin{align*}
\Pi^{\text {Regular }}\left(A, \infty \mid x_{0}, A_{0}\right) & =\lim _{s \rightarrow 0}\left[s \hat{\Pi}_{s}^{\text {Regular }}\left(A \mid x_{0}, A_{0}\right)\right]=\lim _{s \rightarrow 0}\left[\theta\left(A>A_{0}\right) \frac{\hat{G}_{s}\left(0 \mid x_{0}\right)}{\hat{G}_{s}^{2}(0 \mid 0)} e^{-\frac{\left(A-A_{0}\right)}{\hat{G}_{s}(0 \mid 0)}}\right] \\
& =\theta\left(A>A_{0}\right) \frac{\hat{G}_{0}\left(0 \mid x_{0}\right)}{\hat{G}_{0}^{2}(0 \mid 0)} e^{-\frac{\left(A-A_{0}\right)}{G_{0}(0 \mid 0)}} \\
& =\theta\left(A>A_{0}\right)\left[1-S^{a b s}\left(\infty \mid x_{0}\right)\right] \frac{1}{\hat{G}_{0}(0 \mid 0)} e^{-\frac{\left(A-A_{0}\right)}{\hat{G}_{0}(0 \mid 0)}} \tag{88}
\end{align*}
$$

Its physical meaning can be understood as follows : with the complementary probability $\left[1-S^{a b s}\left(\infty \mid x_{0}\right)\right]$ with respect to Eq. 87, the diffusion process $X(t)$ visits the origin before escaping towards infinity, and then the local time $\left(A-A_{0}\right)$ is an exponential random variable with the finite scale

$$
\begin{equation*}
\hat{G}_{s=0}(0 \mid 0) \equiv \int_{0}^{+\infty} d t G(0, t \mid 0,0) \tag{89}
\end{equation*}
$$

B. When $X(t)$ converges towards an equilibrium : the increment $\left(A-A_{0}\right)$ grows extensively in $\left(t-t_{0}\right)$

Among diffusions converging towards equilibrium, one simple example is is the drift $\mu(x)=-\mu \operatorname{sgn}(x)$ of parameter $\mu>0$ that will be discussed in section VIII.

When the diffusion process $X(t)$ converges towards the Boltzmann equilibrium state of Eq. 28

$$
\begin{equation*}
G_{e q}(x)=\frac{e^{-U(x)}}{\int_{-\infty}^{+\infty} d y e^{-U(y)}}=\left[\phi_{0}^{G S}(x)\right]^{2} \tag{90}
\end{equation*}
$$

then the first moment $m^{[k=1]}\left(t \mid x_{0}, t_{0}\right)$ of the local time increment $\left[A(t)-A\left(t_{0}\right)\right]$ discussed in EqS 78,79 is extensive with respect to the time interval $\left(t-t_{0}\right)$

$$
\begin{equation*}
m^{[k=1]}\left(t \mid x_{0}, t_{0}\right) \underset{\left(t-t_{0}\right) \rightarrow+\infty}{\simeq}\left(t-t_{0}\right) G_{e q}(x=0) \tag{91}
\end{equation*}
$$

The corresponding intensive local time

$$
\begin{equation*}
a \equiv \frac{A-A_{0}}{t-t_{0}} \tag{92}
\end{equation*}
$$

then converges in the thermodynamic limit $\left(t-t_{0}\right)=+\infty$ towards its equilibrium value

$$
\begin{equation*}
a_{e q}=G_{e q}(0)=\frac{e^{-U(0)}}{\int_{-\infty}^{+\infty} d y e^{-U(y)}}=\left[\phi_{0}^{G S}(0)\right]^{2} \tag{93}
\end{equation*}
$$

For large but finite $\left(t-t_{0}\right)$, it is thus interesting to analyze its large deviations properties.

$$
\text { 1. Large deviations properties of the intensive local time } a=\frac{A-A_{0}}{t-t_{0}}
$$

The probability $\Pi\left(A=A_{0}+\left(t-t_{0}\right) a, t \mid x_{0}, A_{0}, t_{0}\right)$ to see the intensive local time $a$ different from its equilibrium value $a_{e q}$ will display the large deviation form with respect to $\left(t-t_{0}\right)$

$$
\begin{equation*}
\Pi\left(A=A_{0}+\left(t-t_{0}\right) a, t \mid x_{0}, A_{0}, t_{0}\right) \underset{\left(t-t_{0}\right) \rightarrow+\infty}{\propto} e^{-\left(t-t_{0}\right) I(a)} \tag{94}
\end{equation*}
$$

The positive rate function $I(a) \geq 0$ is defined for $a \in\left[0,+\infty\left[\right.\right.$ and vanishes only for the equilibrium value $a_{e q}$ of Eq. 93 where it is minimum

$$
\begin{equation*}
I\left(a_{e q}\right)=0=I^{\prime}\left(a_{e q}\right) \tag{95}
\end{equation*}
$$

The Central Limit theorem governing the small Gaussian fluctuations around $a_{e q}$ can be recovered via the Taylor expansion at second order of the rate function $I(a)$ around $a_{e q}$

$$
\begin{equation*}
I(a)=\frac{\left(a-a_{e q}\right)^{2}}{2} I^{\prime \prime}\left(a_{e q}\right)+o\left(\left(a-a_{e q}\right)^{2}\right) \tag{96}
\end{equation*}
$$

The link with the singular and regular contributions of $\Pi\left(A, t \mid x_{0}, A_{0}, t_{0}\right)$ can be understood as follows.
(i) The survival probability $S^{a b s}\left(t \mid x_{0}, t_{0}\right)$ representing the weight of the singular contribution $\Pi^{\text {Singular }}\left(A, t \mid x_{0}, A_{0}, t_{0}\right)$ of Eq. 69 corresponds to the value $a=0$ and will thus display the following exponential decay with respect to $\left(t-t_{0}\right)$ that involves the boundary value $I(a=0)$

$$
\begin{equation*}
S^{a b s}\left(t \mid x_{0}, t_{0}\right) \underset{\left(t-t_{0}\right) \rightarrow+\infty}{\propto} e^{-\left(t-t_{0}\right) I(a=0)} \tag{97}
\end{equation*}
$$

(ii) For the regular contribution, the compatibility at leading order in the exponentials between the large deviation form of Eq. 94 and the Laplace transform $\hat{\Pi}_{s}^{\text {Regular }}\left(A \mid x_{0}, A_{0}\right)$ of Eq. 71 yields 95 ]

$$
\begin{equation*}
e^{-\frac{\left(A-A_{0}\right)}{G_{s}(0 \mid 0)}} \underset{\left(A-A_{0}\right) \rightarrow+\infty}{\simeq} \int_{t_{0}}^{+\infty} d t e^{-s\left(t-t_{0}\right)} e^{-\left(t-t_{0}\right) I\left(\frac{A-A_{0}}{t-t_{0}}\right)} \tag{98}
\end{equation*}
$$

It is thus convenient to make a change of variable in the integral from the time $t$ towards the intensive local time $a=\frac{A-A_{0}}{t-t_{0}}$

$$
\begin{equation*}
e^{-\frac{\left(A-A_{0}\right)}{\hat{G}_{s}(0 \mid 0)}} \underset{\left(A-A_{0}\right) \rightarrow+\infty}{\simeq} \int_{0}^{+\infty} \frac{d a}{a^{2}}\left(A-A_{0}\right) e^{-\left(A-A_{0}\right) \frac{s+I(a)}{a}} \tag{99}
\end{equation*}
$$

For large increment $\left(A-A_{0}\right) \rightarrow+\infty$, the evaluation of this integral via the saddle-point method allows to obtain the following link between $\hat{G}_{s}(0 \mid 0)$ and the rate function $I(a)$ 95]

$$
\begin{align*}
\frac{1}{\hat{G}_{s}(0 \mid 0)} & =\frac{s+I(a)}{a} \\
0 & =\partial_{a}\left[\frac{s+I(a)}{a}\right]=\frac{s+I^{\prime}(a)}{a}-\frac{s+I(a)}{a^{2}} \tag{100}
\end{align*}
$$

This quasi-Legendre transform can be written in reciprocal form in order to compute the rate function $I(a)$ from the knowledge of $\hat{G}_{s}(0 \mid 0)$, as discussed in detail in the next subsection.

## 2. Evaluation of the leading order of $\Pi^{\text {Regular }}\left(A=A_{0}+\left(t-t_{0}\right) a, t \mid x_{0}, A_{0}, t_{0}\right)$ with the prefactors

For the Doob conditioned processes that will be discussed in section VI, one needs to compute the dependence with respect to the initial position $x_{0}$, so that one needs to include the prefactors in the reciprocal calculation concerning the Laplace inverse of $\hat{\Pi}_{s}^{\text {Regular }}\left(A \mid x_{0}, A_{0}\right)$ of Eq. 71

$$
\begin{align*}
\Pi^{\text {Regular }}\left(A, t \mid x_{0}, A_{0}, t_{0}\right) & =\int_{c-i \infty}^{c+i \infty} \frac{d s}{2 i \pi} e^{s\left(t-t_{0}\right)} \hat{\Pi}_{s}^{\text {Regular }}\left(A \mid x_{0}, A_{0}\right) \\
& =\int_{c-i \infty}^{c+i \infty} \frac{d s}{2 i \pi} e^{s\left(t-t_{0}\right)} \frac{\hat{G}_{s}\left(0 \mid x_{0}\right)}{s \hat{G}_{s}^{2}(0 \mid 0)} e^{-\frac{\left(A-A_{0}\right)}{\hat{G}_{s}(0 \mid 0)}} \tag{101}
\end{align*}
$$

Here the goal is to evaluate this regular contribution for $A=A_{0}+\left(t-t_{0}\right) a$

$$
\begin{equation*}
\Pi^{\text {Regular }}\left(A=A_{0}+\left(t-t_{0}\right) a, t \mid x_{0}, A_{0}, t_{0}\right)=\int_{c-i \infty}^{c+i \infty} \frac{d s}{2 i \pi} \frac{\hat{G}_{s}\left(0 \mid x_{0}\right)}{s \hat{G}_{s}^{2}(0 \mid 0)} e^{-\left(t-t_{0}\right)\left[\frac{a}{\hat{G}_{s}(0 \mid 0)}-s\right]} \tag{102}
\end{equation*}
$$

For large $\left(t-t_{0}\right)$, the saddle-point evaluation of this integral will be governed by the solution $s_{a}$ of the following equation in $s$ as a function of the parameter $a$

$$
\begin{equation*}
0=\partial_{s}\left[\frac{a}{\hat{G}_{s}(0 \mid 0)}-s\right]=a \partial_{s}\left[\frac{1}{\hat{G}_{s}(0 \mid 0)}\right]-1 \quad \text { with solution } s=s_{a} \tag{103}
\end{equation*}
$$

In the integral of Eq. 102 one then needs to make the change of variable around this saddle-point value $s_{a}$

$$
\begin{equation*}
s=s_{a}+i \omega \tag{104}
\end{equation*}
$$

The Taylor expansion at second order in $\omega$ of the function in the exponential

$$
\begin{equation*}
\left[\frac{a}{\hat{G}_{s}(0 \mid 0)}-s\right]_{s=s_{a}+i \omega}=I(a)+0+\frac{\omega^{2}}{2} K(a)+o\left(\omega^{2}\right) \tag{105}
\end{equation*}
$$

involves the two functions

$$
\begin{align*}
I(a) & =\left.\left(\frac{a}{\hat{G}_{s}(0 \mid 0)}-s\right)\right|_{s=s_{a}} \\
K(a) & =-\left.\left(\partial_{s}^{2}\left[\frac{a}{\hat{G}_{s}(0 \mid 0)}-s\right]\right)\right|_{s=s_{a}} \tag{106}
\end{align*}
$$

In particular, the rate function $I(a)$ can be computed from the knowledge of $\hat{G}_{s}(0 \mid 0)$ via Eq. 106 using the saddlepoint value $s_{a}$ determined by Eq. 103 . As it should for consistency, Eqs 106 and 103 correspond to the reciprocal quasi-Legendre transform of Eq. 100. Simple examples will be given in Eqs 189 191 192 193, as well as in Eqs 228 247|251 252.

Putting everything together, one obtains the final result for the leading order of Eq. 102 based on the remaining Gaussian integral over $\omega$

$$
\begin{align*}
\Pi^{\text {Regular }}\left(A=A_{0}+\left(t-t_{0}\right) a, t \mid x_{0}, A_{0}, t_{0}\right) & \underset{\left(t-t_{0}\right) \rightarrow+\infty}{\simeq} \frac{\hat{G}_{s_{a}}\left(0 \mid x_{0}\right)}{s_{a} \hat{G}_{s_{a}}^{2}(0 \mid 0)} e^{-\left(t-t_{0}\right) I(a)} \int_{\infty}^{\infty} \frac{d \omega}{2 \pi} e^{-\left(t-t_{0}\right) \frac{K(a)}{2} \omega^{2}} \\
& \simeq \frac{\hat{G}_{s_{a}}\left(0 \mid x_{0}\right)}{\simeq} \frac{e^{-\left(t-t_{0}\right) I(a)}}{\sqrt{2 \pi\left(t-t_{0}\right) K(a)}} \tag{107}
\end{align*}
$$

Note that the dependence with respect to the initial position $x_{0}$ is only in the function $\hat{G}_{s_{a}}\left(0 \mid x_{0}\right)$ evaluated for the saddle-point value $s_{a}$ determined by Eq. 103 .
3. Evaluation of the leading order of $P^{\text {Regular }}\left(x, A=A_{0}+\left(t-t_{0}\right) a, t \mid x_{0}, A_{0}, t_{0}\right)$ with the prefactors

Similarly, let us now consider the Laplace inverse of $P_{s}^{\text {Regular }}\left(x, A \mid x_{0}, A_{0}\right)$ of Eq. 62

$$
\begin{align*}
P^{\text {Regular }}\left(x, A, t \mid x_{0}, A_{0}, t_{0}\right) & =\int_{c-i \infty}^{c+i \infty} \frac{d s}{2 i \pi} e^{s\left(t-t_{0}\right)} \hat{P}_{s}^{\text {Regular }}\left(x, A \mid x_{0}, A_{0}\right) \\
& =\int_{c-i \infty}^{c+i \infty} \frac{d s}{2 i \pi} e^{s\left(t-t_{0}\right)}\left[\frac{\hat{G}_{s}(x \mid 0) \hat{G}_{s}\left(0 \mid x_{0}\right)}{\hat{G}_{s}^{2}(0 \mid 0)}\right] e^{-\frac{\left(A-A_{0}\right)}{\hat{G}_{s}(0 \mid 0)}} \tag{108}
\end{align*}
$$

Again we are interested into the value $A=A_{0}+\left(t-t_{0}\right) a$

$$
\begin{equation*}
P^{\text {Regular }}\left(x, A=A_{0}+\left(t-t_{0}\right) a, t \mid x_{0}, A_{0}, t_{0}\right)=\int_{c-i \infty}^{c+i \infty} \frac{d s}{2 i \pi}\left[\frac{\hat{G}_{s}(x \mid 0) \hat{G}_{s}\left(0 \mid x_{0}\right)}{\hat{G}_{s}^{2}(0 \mid 0)}\right] e^{-\left(t-t_{0}\right)\left[\frac{a}{\hat{G}_{s}(0 \mid 0)}-s\right]} \tag{109}
\end{equation*}
$$

So we can use the same saddle-point method described in the previous subsection to obtain the final result analog to Eq. 107

$$
\begin{equation*}
P^{\text {Regular }}\left(x, A=A_{0}+\left(t-t_{0}\right) a, t \mid x_{0}, A_{0}, t_{0}\right) \underset{\left(t-t_{0}\right) \rightarrow+\infty}{\simeq} \frac{\hat{G}_{s_{a}}(x \mid 0) \hat{G}_{s_{a}}\left(0 \mid x_{0}\right)}{\hat{G}_{s_{a}}^{2}(0 \mid 0)} \frac{e^{-\left(t-t_{0}\right) I(a)}}{\sqrt{2 \pi\left(t-t_{0}\right) K(a)}} \tag{110}
\end{equation*}
$$

Note that the dependence with respect to the initial position $x_{0}$ and to the final position $x$ are only in the functions $\hat{G}_{s_{a}}\left(0 \mid x_{0}\right)$ and $\hat{G}_{s_{a}}(x \mid 0)$ evaluated for the saddle-point value $s_{a}$ determined by Eq. 103 .

## C. When $X(t)$ is recurrent but does not converge towards an equilibrium state

Among recurrent diffusions that do not converge towards an equilibrium state, the simplest case is the pure Brownian motion without drift $\mu=0$, that will be discussed in section VII

When the diffusion process $X(t)$ is recurrent but does not converge towards an equilibrium state, then the first moment $m^{[k=1]}\left(t \mid x_{0}, t_{0}\right)$ of the local time increment $\left[A(t)-A\left(t_{0}\right)\right]$ of Eq. 78 will diverge for $\left(t-t_{0}\right) \rightarrow+\infty$ in contrast to Eq. 83

$$
\begin{equation*}
m^{[k=1]}\left(t \mid x_{0}, t_{0}\right) \underset{\left(t-t_{0}\right) \rightarrow+\infty}{\simeq}+\infty \tag{111}
\end{equation*}
$$

However this divergence will be weaker than the extensive behavior of Eq. 91

$$
\begin{equation*}
\frac{m^{[k=1]}\left(t \mid x_{0}, t_{0}\right)}{\left(t-t_{0}\right)} \underset{\left(t-t_{0}\right) \rightarrow+\infty}{\simeq} 0 \tag{112}
\end{equation*}
$$

Nevertheless, the saddle-point evaluations of Eq. 102 can still be performed to obtain as in Eq. 107 the leading behavior

$$
\begin{equation*}
\Pi^{\text {Regular }}\left(A=A_{0}+\left(t-t_{0}\right) a, t \mid x_{0}, A_{0}, t_{0}\right) \underset{\left(t-t_{0}\right) \rightarrow+\infty}{\propto} \frac{\hat{G}_{s_{a}}\left(0 \mid x_{0}\right)}{s_{a} \hat{G}_{s_{a}}^{2}(0 \mid 0)} \frac{e^{-\left(t-t_{0}\right) I(a)}}{\sqrt{2 \pi\left(t-t_{0}\right) K(a)}} \tag{113}
\end{equation*}
$$

The only important difference is that the rate function $I(a)$ defined for $a \in[0,+\infty[$ will now vanish at the boundary $a=0$ where it is minimum

$$
\begin{equation*}
I(a=0)=0=I^{\prime}\left(a=0^{+}\right) \tag{114}
\end{equation*}
$$

instead of the finite value $a_{e q}>0$ of Eq. 93 discussed in the previous subsection.

## VI. CONSTRUCTION OF CONDITIONED PROCESSES INVOLVING THE LOCAL TIME

In this section, the goal is to construct various conditioned joint processes $\left[X^{*}(t), A^{*}(t)\right]$ satisfying certain conditions involving the local time.

## A. Conditioned Bridge towards the position $x_{T}^{*}$ and the local time $A_{T}^{*}$ at the time horizon $T$

When the initial position is $x_{0}$ with $A_{0}=0$ at time $t=0$, the conditioning towards the final position $x_{T}^{*}$ and the final local time $A_{T}^{*}$ at the time horizon $T$ leads to the following conditioned probability $P_{T}^{\left[x_{T}^{*}, A_{T}^{*}\right]}(x, A, t)$ for the position $x$ and the local time $A$ at some interior time $t \in] 0, T$ [ in terms of the unconditioned joint propagator $P\left(x_{2}, A_{2}, t_{2} \mid x_{1}, A_{1}, t_{1}\right)$ described in section III

$$
\begin{equation*}
P_{T}^{\left[x_{T}^{*}, A_{T}^{*}\right]}(x, A, t)=\frac{P\left(x_{T}^{*}, A_{T}^{*}, T \mid x, A, t\right) P\left(x, A, t \mid x_{0}, A_{0}=0,0\right)}{P\left(x_{T}^{*}, A_{T}^{*}, T \mid x_{0}, A_{0}=0,0\right)} \tag{115}
\end{equation*}
$$

As described in detail in 43, 44, the corresponding conditioned process $\left[X^{*}(t), A^{*}(t)\right]$ then satisfies the Ito system analog to Eq. 33

$$
\begin{align*}
d X^{*}(t) & =\mu_{T}^{\left[x_{T}^{*}, A_{T}^{*}\right]}\left(X^{*}(t), A^{*}(t), t\right) d t+d B(t) \\
d A^{*}(t) & =\delta\left(X^{*}(t)\right) d t \tag{116}
\end{align*}
$$

where the conditioned drift $\mu_{T}^{\left[x_{T}^{*}, A_{T}^{*}\right]}(x, A, t)$ involves the unconditioned drift $\mu(x)$ and the logarithmic derivative of the unconditioned propagator $P\left(x_{T}^{*}, A_{T}^{*}, T \mid x, A, t\right)$ with respect to $x$

$$
\begin{equation*}
\mu_{T}^{\left[x_{T}^{*}, A_{T}^{*}\right]}(x, A, t)=\mu(x)+\partial_{x} \ln P\left(x_{T}^{*}, A_{T}^{*}, T \mid x, A, t\right) \tag{117}
\end{equation*}
$$

The decomposition of the unconditioned joint propagator $P\left(x_{T}^{*}, A_{T}^{*}, T \mid x, A, t\right)$ into its singular contribution of Eq. 53 corresponding to $A_{T}^{*}=A$ and its regular contribution $P^{\text {Regular }}\left(x_{T}^{*}, A_{T}^{*}, T \mid x, A, t\right)$ of Eq. 64 corresponding to $A_{T}^{*}>A$

$$
\begin{equation*}
P\left(x_{T}^{*}, A_{T}^{*}, T \mid x, A, t\right)=\delta\left(A_{T}^{*}-A\right) G^{a b s}\left(x_{T}^{*}, T \mid x, t\right)+\theta\left(A_{T}^{*}>A\right) P^{\text {Regular }}\left(x_{T}^{*}, A_{T}^{*}, T \mid x, A, t\right) \tag{118}
\end{equation*}
$$

yields that the conditioned dynamics can be decomposed into the two following regions.
(i) In the region $A_{0}=0 \leq A<A_{T}^{*}$ where the local time $A$ has not yet reached its conditioned final value $A_{T}^{*}$, the conditioned drift of Eq. 117 involves the regular contribution of the propagator

$$
\begin{equation*}
\mu_{T}^{\left[x_{T}^{*}, A_{T}^{*}\right]}\left(x, A<A_{T}^{*}, t\right)=\mu(x)+\partial_{x} \ln P^{\text {Regular }}\left(x_{T}^{*}, A_{T}^{*}, T \mid x, A, t\right) \tag{119}
\end{equation*}
$$

(ii) In the region $A=A_{T}^{*}$ where the local time $A$ has already reached its conditioned final value $A_{T}^{*}$, and where the position $x$ cannot visit the origin $x=0$ anymore, the conditioned drift of Eq. 117 involves the singular contribution of the propagator

$$
\begin{equation*}
\mu_{T}^{\left[x_{T}^{*}, A_{T}^{*}\right]}\left(x, A=A_{T}^{*}, t\right)=\mu(x)+\partial_{x} \ln G^{a b s}\left(x_{T}^{*}, T \mid x, t\right) \tag{120}
\end{equation*}
$$

As a consequence, it only depends on the propagator $G^{a b s}\left(x_{T}^{*}, T \mid x, t\right)$ in the presence of an absorbing boundary at the origin so that one recovers the standard problem of a diffusion conditioned to avoid the origin. Note that for the special case where the final conditioned position is at the origin $x_{T}^{*}=0$, this region (ii) does not exist, and the local time $A$ should reach its final conditioned value $A_{T}^{*}$ only at the final time $T$.

Examples of conditioned bridges towards the position $x_{T}^{*}$ and the local time $A_{T}^{*}$ at the time horizon $T$ will be given in subsections VIIE and VIIIE

Generalization : conditioning towards some joint distribution $P_{T}^{*}\left(x_{T}, A_{T}\right)$ of the position $x_{T}$ and of the local time $A_{T}$
If instead of the bridge described above, one wishes to impose some joint distribution $P_{T}^{*}\left(x_{T}, A_{T}\right)$ of the final position $x_{T}$ and of the final local time $A_{T}$ at the time horizon $T$, the conditioned probability $P^{*}(x, A, t)$ for the position and the local time $A$ at some interior time $t \in] 0, T$ reads

$$
\begin{equation*}
P_{T}^{*}(x, A, t)=Q_{T}(x, A, t) P\left(x, A, t \mid x_{0}, A_{0}=0,0\right) \tag{121}
\end{equation*}
$$

where the function $Q_{T}(x, A, t)$ reads in terms of the final distribution $P_{T}^{*}\left(x_{T}, A_{T}\right)$ that one wishes to impose

$$
\begin{equation*}
Q_{T}(x, A, t) \equiv \int_{-\infty}^{+\infty} d x_{T} \int_{0}^{+\infty} d A_{T} P^{*}\left(x_{T}, A_{T}, T\right) \frac{P\left(x_{T}, A_{T}, T \mid x, A, t\right)}{P\left(x_{T}, A_{T}, T \mid x_{0}, A_{0}=0,0\right)} \tag{122}
\end{equation*}
$$

As a consequence, the conditioned drift $\mu^{*}(x, A, t)$ now involves the logarithmic derivative of the function $Q(x, A, t)$ of Eq. 122 with respect to $x$

$$
\begin{equation*}
\mu_{T}^{*}(x, A, t)=\mu(x)+\partial_{x} \ln Q_{T}(x, A, t) \tag{123}
\end{equation*}
$$

## B. Conditioned Bridge towards the local time $A_{T}^{*}$ at the time horizon $T$

If the conditioning is towards the local time $A_{T}^{*}$ at time horizon $T$, without any condition on the final position $x_{T}$, the conditioned probability for the position $x$ and the local time $A$ at some interior time $\left.t \in\right] 0, T[$ involves the unconditioned probability $\Pi\left(A_{2}, t_{2} \mid x_{1}, A_{1}, t_{1}\right)$ described in section IV

$$
\begin{equation*}
P_{T}^{\left[A_{T}^{*}\right]}(x, A, t)=\frac{\Pi\left(A_{T}, T \mid x, A, t\right) P\left(x, A, t \mid x_{0}, A_{0}=0,0\right)}{\Pi\left(A_{T}, T \mid x_{0}, A_{0}=0,0\right)} \tag{124}
\end{equation*}
$$

The corresponding conditioned drift then involves the logarithmic derivative of the unconditioned probability $\Pi\left(A_{T}^{*}, T \mid x, A, t\right)$ with respect to $x$

$$
\begin{equation*}
\mu_{T}^{\left[A_{T}^{*}\right]}(x, A, t)=\mu(x)+\partial_{x} \ln \Pi\left(A_{T}^{*}, T \mid x, A, t\right) \tag{125}
\end{equation*}
$$

Again, the decomposition of $\Pi\left(A_{T}^{*}, T \mid x, A, t\right)$ into its singular contribution of Eq. 69 corresponding to $A_{T}^{*}=A$ and its regular contribution $\Pi^{\text {Regular }}\left(A_{T}^{*}, T \mid x, A, t\right)$ of Eq. 71 corresponding to $A_{T}^{*}>A$

$$
\begin{equation*}
\Pi\left(A_{T}^{*}, T \mid x, A, t\right)=\delta\left(A_{T}^{*}-A\right) S^{a b s}(T \mid x, t)+\theta\left(A_{T}^{*}>A\right) \Pi^{\text {Regular }}\left(A_{T}^{*}, T \mid x, A, t\right) \tag{126}
\end{equation*}
$$

yields that the conditioned dynamics can be decomposed into the two following regions.
(i) In the region $A_{0}=0 \leq A<A_{T}^{*}$ where the local time $A$ has not yet reached its conditioned final value $A_{T}^{*}$, the conditioned drift of Eq. 125 involves the regular contribution

$$
\begin{equation*}
\mu_{T}^{\left[A_{T}^{*}\right]}\left(x, A<A_{T}^{*}, t\right)=\mu(x)+\partial_{x} \ln \Pi^{\text {Regular }}\left(A_{T}^{*}, T \mid x, A, t\right) \tag{127}
\end{equation*}
$$

(ii) In the region $A=A_{T}^{*}$ where the local time $A$ has already reached its conditioned final value $A_{T}^{*}$, and where the position $x$ cannot visit the origin $x=0$ anymore, the conditioned drift of Eq. 125 involves the singular contribution

$$
\begin{equation*}
\mu_{T}^{\left[A_{T}^{*}\right]}\left(x, A=A_{T}^{*}, t\right)=\mu(x)+\partial_{x} \ln \Pi^{\text {Singular }}\left(A_{T}^{*}, T \mid x, A, t\right)=\mu(x)+\partial_{x} \ln S^{a b s}(T \mid x, t) \tag{128}
\end{equation*}
$$

It depends only on the survival probability $S^{a b s}\left(x_{T}^{*}, T \mid x, t\right)$ in the presence of an absorbing boundary at the origin, so that one recovers the standard problem of a diffusion conditioned to survive up to time $T$.

Example of the conditioned bridge towards the local time $A_{T}^{*}$ at the time horizon $T$ will be given in subsections VIIF and VIIIF.

It is now interesting to consider two possibilities in the limit of the infinite horizon $T \rightarrow+\infty$, as described in the two next subsections.

## 1. Conditioning towards the finite asymptotic local time $A_{\infty}<+\infty$ for the infinite horizon $T \rightarrow+\infty$

If one wishes to impose the finite asymptotic local time $A_{\infty}<+\infty$ for the infinite horizon $T \rightarrow+\infty$, one needs to analyze the limit of the infinite horizon $T \rightarrow+\infty$ for the conditioned drift of Eq. 127

$$
\begin{equation*}
\mu_{\infty}^{\left[A_{\infty}^{*}\right]}\left(x, A<A_{\infty}^{*}, t\right)=\mu(x)+\lim _{T \rightarrow+\infty}\left[\partial_{x} \ln \Pi^{\text {Regular }}\left(A_{\infty}^{*}, T \mid x, A, t\right)\right] \tag{129}
\end{equation*}
$$

and for the conditioned drift of Eq. 128

$$
\begin{equation*}
\mu_{\infty}^{\left[A_{\infty}^{*}\right]}\left(x, A=A_{\infty}^{*}, t\right)=\mu(x)+\lim _{T \rightarrow+\infty}\left[\partial_{x} \ln \Pi^{\text {Singular }}\left(A_{\infty}^{*}, T \mid x, A, t\right)\right]=\mu(x)+\lim _{T \rightarrow+\infty}\left[\partial_{x} \ln S^{a b s}(T \mid x, t)\right] \tag{130}
\end{equation*}
$$

An example of the conditioning towards the finite asymptotic local time $A_{\infty}<+\infty$ for the infinite horizon $T \rightarrow+\infty$ will be given in subsection VIIH.
2. Conditioning towards the intensive local time $a^{*}=\frac{A_{T}^{*}}{T}$ for large time horizon $T \rightarrow+\infty$

If one wishes to impose instead the fixed intensive local time $a^{*}$ for large time horizon $T \rightarrow+\infty$, one needs to plug the value

$$
\begin{equation*}
A_{T}^{*}=T a^{*} \tag{131}
\end{equation*}
$$

into the conditioned drift of Eq. 127

$$
\begin{equation*}
\mu_{T}^{\left[T a^{*}\right]}\left(x, A<T a^{*}, t\right)=\mu(x)+\partial_{x} \ln \Pi^{\text {Regular }}\left(T a^{*}, T \mid x, A, t\right) \tag{132}
\end{equation*}
$$

The asymptotic form of Eq. 107 for the propagator $\Pi^{\text {Regular }}\left(T a^{*}, T \mid x, A, t\right)$ in the region $t \ll T$

$$
\begin{equation*}
\Pi^{\text {Regular }}\left(T a^{*}, T \mid x, A, t\right) \underset{(T-t) \rightarrow+\infty}{\simeq} \frac{\hat{G}_{s_{a_{t}}}(0 \mid x)}{s_{a_{t}} \hat{G}_{s_{a_{t}}}^{2}(0 \mid 0)} \frac{e^{-(T-t) I\left(a_{t}\right)}}{\sqrt{2 \pi(T-t) K\left(a_{t}\right)}} \tag{133}
\end{equation*}
$$

involves the corresponding intensive local time $a_{t}$ on the time interval $(T-t)$

$$
\begin{equation*}
a_{t} \equiv \frac{T a^{*}-A}{T-t} \underset{T \rightarrow+\infty}{\simeq} a^{*} \tag{134}
\end{equation*}
$$

that reduces to $a^{*}$ at leading order when $T \rightarrow+\infty$. So at leading order for the large time horizon $T \rightarrow+\infty$, the conditioned drift of Eq. 132 reduces

$$
\begin{equation*}
\mu_{T}^{\left[T a^{*}\right]}\left(x, A<T a^{*}, t\right) \underset{T \rightarrow+\infty}{\simeq} \mu(x)+\partial_{x} \ln \hat{G}_{s_{a^{*}}}(0 \mid x) \equiv \mu_{\infty}^{\left[a^{*}\right]}(x) \tag{135}
\end{equation*}
$$

to the time-independent drift $\mu_{\infty}^{\left[a^{*}\right]}(x)$ where $s_{a^{*}}$ should be computed as the solution of the saddle-point Equation 103

$$
\begin{equation*}
0=a^{*} \partial_{s}\left[\frac{1}{\hat{G}_{s}(0 \mid 0)}\right]-1 \tag{136}
\end{equation*}
$$

Using the similarity transformation of Eq. 10 for time-Laplace transforms

$$
\begin{equation*}
\hat{G}_{s}(0 \mid x)=e^{-\int_{0}^{x} d y \mu(y)} \hat{\psi}_{s}(0 \mid x) \tag{137}
\end{equation*}
$$

one obtains that the conditioned drift of Eq. 135 only involves the logarithmic derivation with respect to $x$ of the time Laplace transform $\hat{\psi}_{a^{*}}(0 \mid x)$ of the quantum propagator

$$
\begin{equation*}
\mu_{\infty}^{\left[a^{*}\right]}(x) \equiv \mu(x)+\partial_{x} \ln \hat{G}_{s_{a^{*}}}(0 \mid x)=\partial_{x} \ln \hat{\psi}_{s_{a^{*}}}(0 \mid x) \tag{138}
\end{equation*}
$$

while Eq. 136 for $s_{a}^{*}$ becomes

$$
\begin{equation*}
0=a^{*} \partial_{s}\left[\frac{1}{\hat{\psi}_{s}(0 \mid 0)}\right]-1 \tag{139}
\end{equation*}
$$

The conditioned potential $U_{\infty}^{\left[a^{*}\right]}(x)$ associated to the conditioned drift $\mu_{\infty}^{\left[a^{*}\right]}(x)$ of Eq. 138 via Eq. 9

$$
\begin{equation*}
U^{\left[a^{*}\right]}(x) \equiv-2 \int_{0}^{x} d y \mu_{\infty}^{\left[a^{*}\right]}(x)=\ln \left(\frac{\hat{\psi}_{s_{a^{*}}}(0 \mid 0)}{\hat{\psi}_{s_{a^{*}}}(0 \mid x)}\right)^{2} \tag{140}
\end{equation*}
$$

corresponds to the conditioned Boltzmann equilibrium of Eq. 28

$$
\begin{equation*}
G_{e q}^{\left[a^{*}\right]}(x)=\frac{e^{-U^{\left[a^{*}\right]}(x)}}{\int_{-\infty}^{+\infty} d y e^{-U^{\left[a^{*}\right]}(y)}}=\frac{\left[\hat{\psi}_{s_{a^{*}}}(0 \mid x)\right]^{2}}{\int_{-\infty}^{+\infty} d y\left[\hat{\psi}_{s_{a^{*}}}(0 \mid y)\right]^{2}} \tag{141}
\end{equation*}
$$

Example of the conditioning towards the intensive local time $a^{*}=\frac{A_{T}^{*}}{T}$ for large time horizon $T \rightarrow+\infty$ will be given in subsections VII G and VIII G.

## 3. Generalization : conditioning towards the distribution $\Pi_{T}^{*}\left(A_{T}\right)$ of the local time $A_{T}$ at the time horizon $T$

If instead of the bridge corresponding to the single value $A_{T}^{*}$, one wishes to impose some distribution $\Pi_{T}^{*}\left(A_{T}\right)$ of the local time $A_{T}$ at the time horizon $T$, the conditioned probability for the position $x$ and the local time $A$ at some interior time $t \in] 0, T[$ is given by

$$
\begin{equation*}
P_{T}^{*}(x, A, t)=Q_{T}(x, A, t) P\left(x, A, t \mid x_{0}, A_{0}=0,0\right) \tag{142}
\end{equation*}
$$

where the function $Q_{T}(x, A, t)$ involves the final distribution $\Pi_{T}^{*}\left(A_{T}\right)$ that one wishes to impose

$$
\begin{equation*}
Q_{T}(x, A, t) \equiv \int_{0}^{+\infty} d A_{T} \Pi^{*}\left(A_{T}, T\right) \frac{\Pi\left(A_{T}, T \mid x, A, t\right)}{\Pi\left(A_{T}, T \mid x_{0}, A_{0}=0,0\right)} \tag{143}
\end{equation*}
$$

Its logarithmic derivative with respect to $x$ allows to compute the corresponding conditioned drift

$$
\begin{equation*}
\mu_{T}^{*}(x, A, t)=\mu(x)+\partial_{x} \ln Q_{T}(x, A, t) \tag{144}
\end{equation*}
$$

## VII. APPLICATION TO THE UNIFORM DRIFT $\mu \geq 0$

In this section, the unconditioned process is the Brownian motion with uniform drift $\mu(x)=\mu \geq 0$, so the Ito system of Eq. 33 reads

$$
\begin{align*}
d X(t) & =\mu d t+d B(t) \\
d A(t) & =\delta(X(t)) d t \tag{145}
\end{align*}
$$

Note that in the transient cases $\mu>0$, the local time increment $\left(A-A_{0}\right)$ of this unconditioned process remains finite for $\left(t-t_{0}\right) \rightarrow+\infty$ as described in subsection V A, while $\mu=0$ corresponds to the case of recurrent diffusion that does not converge towards an equilibrium state discussed in subsection VC.

## A. Properties of the unconditioned diffusion process $X(t)$ alone

## 1. Propagator $G\left(x, t \mid x_{0}, t_{0}\right)$ for the position alone

The propagator $G\left(x, t \mid x_{0}, t_{0}\right)$ discussed in section II is Gaussian

$$
\begin{equation*}
G\left(x, t \mid x_{0}, t_{0}\right)=\frac{1}{\sqrt{2 \pi\left(t-t_{0}\right)}} e^{-\frac{\left[x-x_{0}-\mu\left(t-t_{0}\right)\right]^{2}}{2\left(t-t_{0}\right)}}=\frac{1}{\sqrt{2 \pi\left(t-t_{0}\right)}} e^{-\frac{\left(x-x_{0}\right)^{2}}{2\left(t-t_{0}\right)}+\mu\left(x-x_{0}\right)-\frac{\mu^{2}}{2}\left(t-t_{0}\right)} \tag{146}
\end{equation*}
$$

and its time Laplace transform reads

$$
\begin{align*}
\tilde{G}_{s}\left(x \mid x_{0}\right) & \equiv \int_{t_{0}}^{+\infty} d t e^{-s\left(t-t_{0}\right)} G\left(x, t \mid x_{0}, t_{0}\right)=\frac{e^{\mu\left(x-x_{0}\right)}}{\sqrt{2 \pi}} \int_{0}^{+\infty} d \tau \tau^{-\frac{1}{2}} e^{-\left(s+\frac{\mu^{2}}{2}\right) \tau} e^{-\frac{\left(x-x_{0}\right)^{2}}{2 \tau}} \\
& =\frac{e^{\mu\left(x-x_{0}\right)-\sqrt{\mu^{2}+2 s}\left|x-x_{0}\right|}}{\sqrt{\mu^{2}+2 s}} \tag{147}
\end{align*}
$$

## 2. Properties in the presence of an absorbing boundary at the origin $x=0$

The Laplace transform $\hat{G}_{s}^{a b s}\left(x \mid x_{0}\right)$ of Eq. 49 reads using Eq. 147

$$
\begin{align*}
\hat{G}_{s}^{a b s}\left(x \mid x_{0}\right) & =\hat{G}_{s}\left(x \mid x_{0}\right)-\frac{\hat{G}_{s}(x \mid 0) \hat{G}_{s}\left(0 \mid x_{0}\right)}{\hat{G}_{s}(0 \mid 0)}=\frac{e^{\mu\left(x-x_{0}\right)-\sqrt{\mu^{2}+2 s}\left|x-x_{0}\right|}}{\sqrt{\mu^{2}+2 s}}-\frac{e^{\mu\left(x-x_{0}\right)-\sqrt{\mu^{2}+2 s}\left(|x|+\left|x_{0}\right|\right)}}{\sqrt{\mu^{2}+2 s}} \\
& =\frac{e^{\mu\left(x-x_{0}\right)}}{\sqrt{\mu^{2}+2 s}}\left[e^{-\sqrt{\mu^{2}+2 s}\left|x-x_{0}\right|}-e^{-\sqrt{\mu^{2}+2 s}\left(|x|+\left|x_{0}\right|\right)}\right] \tag{148}
\end{align*}
$$

Its Laplace inversion with respect to $s$ yields the propagator $G^{a b s}\left(x, t \mid x_{0}, t_{0}\right)$ in the presence of an absorbing boundary at the origin $x=0$

$$
\begin{equation*}
G^{a b s}\left(x, t \mid x_{0}, t_{0}\right)=\frac{e^{\mu\left(x-x_{0}\right)-\frac{\mu^{2}}{2}\left(t-t_{0}\right)}}{\sqrt{2 \pi\left(t-t_{0}\right)}}\left[e^{-\frac{\left(x-x_{0}\right)^{2}}{2\left(t-t_{0}\right)}}-e^{-\frac{\left(|x|+\left.\left|x_{0}\right|\right|^{2}\right.}{2\left(t-t_{0}\right)}}\right] \tag{149}
\end{equation*}
$$

in agreement with the method of images.
The Laplace transform $\hat{\gamma}_{s}^{a b s}\left(x_{0}\right)$ of Eq. 59 reads using Eq. 147

$$
\begin{equation*}
\hat{\gamma}_{s}^{a b s}\left(x_{0}\right)=\frac{\hat{G}_{s}\left(0 \mid x_{0}\right)}{\hat{G}_{s}(0 \mid 0)}=e^{-\mu x_{0}-\sqrt{\mu^{2}+2 s}\left|x_{0}\right|} \tag{150}
\end{equation*}
$$

The rewriting of the Laplace transform of Eq. 147 in terms of the parameter $\alpha>0$

$$
\begin{equation*}
\frac{e^{-\alpha \sqrt{\mu^{2}+2 s}}}{\sqrt{\mu^{2}+2 s}}=\int_{0}^{+\infty} \frac{d \tau}{\sqrt{2 \pi \tau}} e^{-\left(s+\frac{\mu^{2}}{2}\right) \tau} e^{-\frac{\alpha^{2}}{2 \tau}} \tag{151}
\end{equation*}
$$

allows to obtain via the derivation with respect to $\alpha$

$$
\begin{align*}
e^{-\alpha \sqrt{\mu^{2}+2 s}} & =-\partial_{\alpha}\left(\frac{e^{-\alpha \sqrt{\mu^{2}+2 s}}}{\sqrt{\mu^{2}+2 s}}\right)=-\int_{0}^{+\infty} \frac{d \tau}{\sqrt{2 \pi \tau}} e^{-\left(s+\frac{\mu^{2}}{2}\right) \tau} \partial_{\alpha} e^{-\frac{\alpha^{2}}{2 \tau}} \\
& =\int_{0}^{+\infty} d \tau e^{-\left(s+\frac{\mu^{2}}{2}\right) \tau}\left(\frac{\alpha}{\sqrt{2 \pi} \tau^{\frac{3}{2}}}\right) e^{-\frac{\alpha^{2}}{2 \tau}} \tag{152}
\end{align*}
$$

The Laplace inversion of Eq. 150 thus yields the absorption rate $\gamma^{a b s}\left(t \mid x_{0}, t_{0}\right)$ at time $t$ if one starts at position $x_{0}$ at time $t_{0}$

$$
\begin{equation*}
\gamma^{a b s}\left(t \mid x_{0}, t_{0}\right)=\frac{\left|x_{0}\right|}{\sqrt{2 \pi}\left(t-t_{0}\right)^{\frac{3}{2}}} e^{-\mu x_{0}-\frac{\mu^{2}}{2}\left(t-t_{0}\right)-\frac{x_{0}^{2}}{2\left(t-t_{0}\right)}} \tag{153}
\end{equation*}
$$

Finally the survival probability $S^{a b s}\left(t \mid x_{0}, t_{0}\right)$ of Eq. 54 can be obtained from the integral over the final position $x$ of the propagator $G^{a b s}\left(x, t \mid x_{0}, t_{0}\right)$ of Eq. 149

$$
\begin{equation*}
S^{a b s}\left(t \mid x_{0}, t_{0}\right) \equiv \int_{-\infty}^{+\infty} d x G^{a b s}\left(x, t \mid x_{0}, t_{0}\right)=\int_{-\infty}^{+\infty} d x \frac{e^{\mu\left(x-x_{0}\right)-\frac{\mu^{2}}{2}\left(t-t_{0}\right)}}{\sqrt{2 \pi\left(t-t_{0}\right)}}\left[e^{-\frac{\left(x-x_{0}\right)^{2}}{2\left(t-t_{0}\right)}}-e^{-\frac{\left(|x|+\left.\left|x_{0}\right|\right|^{2}\right.}{2\left(t-t_{0}\right)}}\right] \tag{154}
\end{equation*}
$$

Its Laplace transform of Eq. 57 reads using Eq. 150

$$
\begin{equation*}
\hat{S}_{s}^{a b s}\left(x_{0}\right)=\frac{1}{s}\left[1-\frac{\hat{G}_{s}\left(0 \mid x_{0}\right)}{\hat{G}_{s}(0 \mid 0)}\right]=\frac{1}{s}\left[1-e^{-\mu x_{0}-\sqrt{\mu^{2}+2 s}\left|x_{0}\right|}\right] \tag{155}
\end{equation*}
$$

For $\mu>0$, the forever-survival probability $S^{a b s[\mu>0]}\left(\infty \mid x_{0}\right)$ at infinite time $\left(t-t_{0}\right) \rightarrow+\infty$ when starting at the position $x_{0}$ can be recovered via the limit $s \rightarrow 0$ of

$$
S^{a b s[\mu>0]}\left(\infty \mid x_{0}\right)=\lim _{s \rightarrow 0}\left[s \hat{S}_{s}^{a b s}\left(x_{0}\right)\right]=1-e^{-\mu\left(x_{0}+\left|x_{0}\right|\right)}= \begin{cases}0 & \text { if } x_{0} \leq 0  \tag{156}\\ 1-e^{-2 \mu x_{0}} & \text { if } x_{0}>0\end{cases}
$$

It remains finite for $x_{0}>0$, since the particle can escape towards $(+\infty)$ without touching the origin $x=0$. The finite-time survival probability $S^{a b s}\left(t \mid x_{0}, t_{0}\right)$ given by Eq. 154 can be expressed in terms of the Error function $\operatorname{erf}(x)$ and and in terms of the complementary Error function $\operatorname{erfc}(x)=1-\operatorname{erf}(x)$ as

$$
S^{a b s}\left(t \mid x_{0}, t_{0}\right)= \begin{cases}\frac{1}{2}\left[1+\operatorname{erf}\left(\frac{x_{0}+\mu\left(t-t_{0}\right)}{\sqrt{2\left(t-t_{0}\right)}}\right)-e^{-2 \mu x_{0}} \operatorname{erfc}\left(\frac{x_{0}-\mu\left(t-t_{0}\right)}{\sqrt{2\left(t-t_{0}\right)}}\right)\right] & \text { if } x_{0}>0  \tag{157}\\ \frac{1}{2}\left[e^{-2 \mu x_{0}}\left(-2+\operatorname{erfc}\left(\frac{x_{0}-\mu\left(t-t_{0}\right)}{\sqrt{2\left(t-t_{0}\right)}}\right)\right)+\operatorname{erfc}\left(\frac{x_{0}+\mu\left(t-t_{0}\right)}{\sqrt{2\left(t-t_{0}\right)}}\right)\right] & \text { if } x_{0}<0\end{cases}
$$

In the region $x_{0} \leq 0$ where the forever-survival $S^{a b s[\mu>0]}\left(\infty \mid x_{0}\right)$ vanishes, using the asymptotic behavior of the complementary Error function

$$
\operatorname{erfc}(x) \simeq \begin{cases}e^{-x^{2}}\left(\frac{1}{\sqrt{\pi} x}-\frac{1}{2 \sqrt{\pi} x^{3}}\right) & \text { when } x \rightarrow \infty  \tag{158}\\ 2+e^{-x^{2}}\left(\frac{1}{\sqrt{\pi} x}-\frac{1}{2 \sqrt{\pi} x^{3}}\right) & \text { when } x \rightarrow-\infty\end{cases}
$$

the asymptotic behavior of $S^{a b s}\left(t \mid x_{0}, t_{0}\right)$ for large time $\left(t-t_{0}\right)$ and fixed $x_{0}$ reads

$$
\begin{equation*}
S^{a b s[\mu>0]}\left(t \mid x_{0}<0, t_{0}\right) \underset{\left(t-t_{0}\right) \rightarrow+\infty}{\simeq} \sqrt{\frac{2}{\pi}} \frac{\left|x_{0}\right| e^{\mu\left|x_{0}\right|-\frac{\mu^{2}}{2}\left(t-t_{0}\right)}}{\mu^{2}\left(t-t_{0}\right)^{\frac{3}{2}}} \tag{159}
\end{equation*}
$$

For $\mu=0$, the forever-survival probability $S^{a b s[\mu=0]}\left(\infty \mid x_{0}\right)$ of Eq. 156 vanishes for any $x_{0}$. The leading singularity of Eq. 155 for $s \rightarrow 0^{+}$

$$
\begin{equation*}
\hat{S}_{s}^{a b s[\mu=0]}\left(x_{0}\right)=\frac{1}{s}\left[1-e^{-\sqrt{2 s}\left|x_{0}\right|}\right] \underset{s \rightarrow 0^{+}}{\simeq}\left|x_{0}\right| \sqrt{\frac{2}{s}} \tag{160}
\end{equation*}
$$

allows to recover the dominant asymptotic behavior for large time

$$
\begin{equation*}
S^{a b s[\mu=0]}\left(t \mid x_{0}, t_{0}\right) \underset{\left(t-t_{0}\right) \rightarrow+\infty}{\simeq}\left|x_{0}\right| \sqrt{\frac{2}{\pi\left(t-t_{0}\right)}} \tag{161}
\end{equation*}
$$

B. Joint propagator $P\left(x, A, t \mid x_{0}, A_{0}, t_{0}\right)$ for the unconditioned joint process $[X(t), A(t)]$

The singular contribution of Eq. 53 involves the propagator $G^{a b s}\left(x, t \mid x_{0}, t_{0}\right)$ of Eq. 149

$$
\begin{align*}
P^{\text {Singular }}\left(x, A, t \mid x_{0}, A_{0}, t_{0}\right) & =\delta\left(A-A_{0}\right) G^{a b s}\left(x, t \mid x_{0}, t_{0}\right) \\
& =\delta\left(A-A_{0}\right) \frac{e^{\mu\left(x-x_{0}\right)-\frac{\mu^{2}}{2}\left(t-t_{0}\right)}}{\sqrt{2 \pi\left(t-t_{0}\right)}}\left[e^{-\frac{\left(x-x_{0}\right)^{2}}{2\left(t-t_{0}\right)}}-e^{-\frac{\left(|x|+\left|x_{0}\right|\right)^{2}}{2\left(t-t_{0}\right)}}\right] \tag{162}
\end{align*}
$$

The Laplace transform $\hat{P}_{s}^{\text {Regular }}\left(x, A \mid x_{0}, A_{0}\right)$ of Eq. 62 reads using Eq. 147

$$
\begin{align*}
\hat{P}_{s}^{\text {Regular }}\left(x, A \mid x_{0}, A_{0}\right) & =\theta\left(A>A_{0}\right)\left[\frac{\hat{G}_{s}(x \mid 0) \hat{G}_{s}\left(0 \mid x_{0}\right)}{\hat{G}_{s}^{2}(0 \mid 0)}\right] e^{-\frac{\left(A-A_{0}\right)}{\hat{G}_{s}(0 \mid 0)}} \\
& =\theta\left(A>A_{0}\right) e^{\mu\left(x-x_{0}\right)-\sqrt{\mu^{2}+2 s}\left(|x|+\left|x_{0}\right|+A-A_{0}\right)} \tag{163}
\end{align*}
$$

Its Laplace inversion using Eq. 152 yields

$$
\begin{equation*}
P^{\text {Regular }}\left(x, A, t \mid x_{0}, A_{0}, t_{0}\right)=\theta\left(A>A_{0}\right) e^{\mu\left(x-x_{0}\right)-\frac{\mu^{2}}{2}\left(t-t_{0}\right)}\left(\frac{|x|+\left|x_{0}\right|+A-A_{0}}{\sqrt{2 \pi}\left(t-t_{0}\right)^{\frac{3}{2}}}\right) e^{-\frac{\left(|x|+\left|x_{0}\right|+A-A_{0}\right)^{2}}{2\left(t-t_{0}\right)}} \tag{164}
\end{equation*}
$$

In summary, the joint propagator $P\left(x, A, t \mid x_{0}, A_{0}, t_{0}\right)$ involving the two contributions of Eq. 162 and Eq. 164 reads

$$
\begin{align*}
P\left(x, A, t \mid x_{0}, A_{0}, t_{0}\right) & =P^{\text {Singular }}\left(x, A, t \mid x_{0}, A_{0}, t_{0}\right)+P^{\text {Regular }}\left(x, A, t \mid x_{0}, A_{0}, t_{0}\right) \\
& =\delta\left(A-A_{0}\right) \frac{e^{\mu\left(x-x_{0}\right)-\frac{\mu^{2}}{2}\left(t-t_{0}\right)}}{\sqrt{2 \pi\left(t-t_{0}\right)}}\left[e^{-\frac{\left(x-x_{0}\right)^{2}}{2\left(t-t_{0}\right)}}-e^{-\frac{\left(|x|+\mid x_{0}\right)^{2}}{2\left(t-t_{0}\right)}}\right] \\
& +\theta\left(A>A_{0}\right) e^{\mu\left(x-x_{0}\right)-\frac{\mu^{2}}{2}\left(t-t_{0}\right)}\left(\frac{|x|+\left|x_{0}\right|+A-A_{0}}{\sqrt{2 \pi}\left(t-t_{0}\right)^{\frac{3}{2}}}\right) e^{-\frac{\left(|x|+\left|x_{0}\right|+A-A_{0}\right)^{2}}{2\left(t-t_{0}\right)}} \tag{165}
\end{align*}
$$

For $\mu=0$ and $x_{0}=0$ (i.e. a standard Brownian motion), the regular part of the propagator $P\left(x, A, t \mid x_{0}, A_{0}, t_{0}\right)$ reduces to

$$
\begin{equation*}
P^{\text {Regular }}(x, A, t \mid 0,0,0)=\frac{|x|+A}{\sqrt{2 \pi t^{3}}} e^{-\frac{(|x|+A)^{2}}{2 t}} \tag{166}
\end{equation*}
$$

a result that can be found in the mathematical literature 94 .

## C. Probability $\Pi\left(A, t \mid x_{0}, A_{0}, t_{0}\right)$ to see the local time $A$ at time $t$

The probability $\Pi\left(A, t \mid x_{0}, A_{0}, t_{0}\right)$ of Eq. 65 can be obtained via the integration of the joint propagator $P\left(x, A, t \mid x_{0}, A_{0}, t_{0}\right)$ of Eq. 165 over the final position $x$

$$
\begin{equation*}
\Pi\left(A, t \mid x_{0}, A_{0}, t_{0}\right) \equiv \int_{-\infty}^{+\infty} d x P\left(A, t \mid x_{0}, A_{0}, t_{0}\right) \tag{167}
\end{equation*}
$$

Its singular contribution of Eq. 69 involves the survival probability $S^{a b s}\left(t \mid x_{0}, t_{0}\right)$ of Eq. 154

$$
\begin{equation*}
\Pi^{\text {Singular }}\left(A, t \mid x_{0}, A_{0}, t_{0}\right)=\delta\left(A-A_{0}\right) S^{a b s}\left(t \mid x_{0}, t_{0}\right) \tag{168}
\end{equation*}
$$

The Laplace transform of Eq. 71 reads using Eq. 147

$$
\begin{equation*}
\hat{\Pi}_{s}^{\text {Regular }}\left(A \mid x_{0}, A_{0}\right)=\theta\left(A>A_{0}\right) \frac{\hat{G}_{s}\left(0 \mid x_{0}\right)}{s \hat{G}_{s}^{2}(0 \mid 0)} e^{-\frac{\left(A-A_{0}\right)}{G_{s}(0 \mid 0)}}=\theta\left(A>A_{0}\right) \frac{\sqrt{\mu^{2}+2 s}}{s} e^{-\mu x_{0}-\sqrt{\mu^{2}+2 s}\left(\left|x_{0}\right|+A-A_{0}\right)} \tag{169}
\end{equation*}
$$

$$
\text { 1. } \text { Case } \mu=0
$$

For the case $\mu=0$, Eq. 169 reduces to

$$
\begin{equation*}
\hat{\Pi}_{s}^{\text {Regular }[\mu=0]}\left(A \mid x_{0}, A_{0}\right)=\theta\left(A>A_{0}\right) \sqrt{\frac{2}{s}} e^{-\sqrt{2 s}\left(\left|x_{0}\right|+A-A_{0}\right)} \tag{170}
\end{equation*}
$$

so that Eq. 151 for $\mu=0$ can be used to obtain the Laplace inversion of Eq. 170

$$
\begin{equation*}
\Pi^{\text {Regular }[\mu=0]}\left(A, t \mid x_{0}, A_{0}, t_{0}\right)=\theta\left(A>A_{0}\right) \sqrt{\frac{2}{\pi\left(t-t_{0}\right)}} e^{-\frac{\left(\left|x_{0}\right|+A-A_{0}\right)^{2}}{2\left(t-t_{0}\right)}} \tag{171}
\end{equation*}
$$

The singular contribution of Eq. 168 involves the survival probability of Eq. 154

$$
\begin{align*}
\Pi^{\text {Singular }[\mu=0]}\left(A, t \mid x_{0}, A_{0}, t_{0}\right) & =\delta\left(A-A_{0}\right) S^{a b s[\mu=0]}\left(t \mid x_{0}, t_{0}\right) \\
& =\delta\left(A-A_{0}\right) \int_{-\infty}^{+\infty} d x \frac{e^{-\frac{x^{2}+x_{0}^{2}}{2\left(t-t_{0}\right)}}}{\sqrt{2 \pi\left(t-t_{0}\right)}}\left[e^{\frac{x x_{0}}{\left(t-t_{0}\right)}}-e^{-\frac{\left|x x_{0}\right|}{\left(t-t_{0}\right)}}\right] \\
& =\delta\left(A-A_{0}\right) e^{-\frac{x_{0}^{2}}{2\left(t-t_{0}\right)}} \int_{-\infty}^{+\infty} d z \frac{e^{-\frac{z^{2}}{2}}}{\sqrt{2 \pi}}\left[e^{\frac{z x_{0}}{\sqrt{t-t_{0}}}}-e^{-\frac{\left|z x_{0}\right|}{\sqrt{t-t_{0}}}}\right] \\
& =\delta\left(A-A_{0}\right) \operatorname{erf}\left(\frac{\left|x_{0}\right|}{\sqrt{2\left(t-t_{0}\right)}}\right) \tag{172}
\end{align*}
$$

Putting together the two contributions, one obtains that $\Pi^{[\mu=0]}\left(A, t \mid x_{0}, A_{0}, t_{0}\right)$ reads

$$
\begin{align*}
\Pi^{[\mu=0]}\left(A, t \mid x_{0}, A_{0}, t_{0}\right) & =\delta\left(A-A_{0}\right) S^{a b s[\mu=0]}\left(t \mid x_{0}, t_{0}\right)+\Pi^{\text {Regular }[\mu=0]}\left(A, t \mid x_{0}, A_{0}, t_{0}\right) \\
& =\delta\left(A-A_{0}\right) \operatorname{erf}\left(\frac{\left|x_{0}\right|}{\sqrt{2\left(t-t_{0}\right)}}\right)+\theta\left(A>A_{0}\right) \sqrt{\frac{2}{\pi\left(t-t_{0}\right)}} e^{-\frac{\left(\left|x_{0}\right|+A-A_{0}\right)^{2}}{2\left(t-t_{0}\right)}} \tag{173}
\end{align*}
$$

Since $\operatorname{erf}(x) \underset{x \rightarrow 0}{\simeq} 2 x / \sqrt{\pi}$, we get the asymptotic behavior for large $\left(t-t_{0}\right)$

$$
\begin{equation*}
\Pi^{[\mu=0]}\left(A, t \mid x_{0}, A_{0}, t_{0}\right) \underset{\left(t-t_{0}\right) \rightarrow+\infty}{\simeq} \sqrt{\frac{2}{\pi\left(t-t_{0}\right)}}\left[\delta\left(A-A_{0}\right)\left|x_{0}\right|+\theta\left(A>A_{0}\right) e^{-\frac{\left(\left|x_{0}\right|+A-A_{0}\right)^{2}}{2\left(t-t_{0}\right)}}\right] \tag{174}
\end{equation*}
$$

In the case where $x_{0}=0, A_{0}=0$ and $t_{0}=0$, the propagator of Eq. 173 reduces to the half-Gaussian distribution

$$
\begin{equation*}
\Pi^{[\mu=0]}(A, t \mid 0,0,0)=\theta(A>0) \sqrt{\frac{2}{\pi t}} e^{-\frac{A^{2}}{2 t}} \tag{175}
\end{equation*}
$$

a result that can be found in [95, 96].
The first moment $m^{[k=1]}\left(t \mid x_{0}, t_{0}=0\right)$ of the local time increment can be computed via Eq. 73 or via Eq. 78

$$
\begin{align*}
m^{[k=1]}\left(t \mid x_{0}, t_{0}=0\right) & =\int_{0}^{\infty} d A A \Pi^{[\mu=0]}\left(A, t \mid x_{0}, A_{0}=0, t_{0}=0\right) \\
& =\int_{0}^{t} d t_{1} G\left(0, t_{1} \mid x_{0}, t_{0}=0\right)=\int_{0}^{t} d t_{1} \frac{e^{-\frac{x_{0}^{2}}{2 t_{1}}}}{\sqrt{2 \pi t_{1}}} \\
& =\sqrt{\frac{2 t}{\pi}} e^{-\frac{x_{0}^{2}}{2 t}}-\left|x_{0}\right| \operatorname{erfc}\left(\frac{\left|x_{0}\right|}{\sqrt{2 t}}\right) \tag{176}
\end{align*}
$$

and displays the power-law asymptotic growth independent of $x_{0}$

$$
\begin{equation*}
m^{[k=1]}\left(t \mid x_{0}, t_{0}=0\right) \underset{t \rightarrow+\infty}{\simeq} \sqrt{\frac{2 t}{\pi}} \tag{177}
\end{equation*}
$$

that is intermediate as it should between the finite case of Eq. 83 and the extensive case of Eq. 91, since the Brownian motion without drift $\mu=0$ is recurrent but does not converge towards an equilibrium distribution.

The second moment $m^{[k=2]}\left(t \mid x_{0}, t_{0}=0\right)$ of the local time increment can be computed via Eq. 73 or Eq. 81

$$
\begin{align*}
m^{[k=2]}\left(t \mid x_{0}, t_{0}=0\right) & =\int_{0}^{\infty} d A A^{2} \Pi^{[\mu=0]}\left(A, t \mid x_{0}, A_{0}=0, t_{0}=0\right)=2 \int_{0}^{t} d t_{2} \int_{0}^{t_{2}} d t_{1} G\left(0, t_{2} \mid 0, t_{1}\right) G\left(0, t_{1} \mid x_{0}, t_{0}=0\right) \\
& =\frac{1}{\pi} \int_{0}^{t} d t_{2} \int_{0}^{t_{2}} d t_{1} \frac{e^{-\frac{x_{0}^{2}}{2 t_{1}}}}{\sqrt{\left(t_{2}-t_{1}\right) t_{1}}}=\frac{1}{\pi} \int_{0}^{t} d t_{1} \frac{e^{-\frac{x_{0}^{2}}{2 t_{1}}}}{\sqrt{t_{1}}} \int_{t_{1}}^{t} \frac{d t_{2}}{\sqrt{t_{2}-t_{1}}}=\frac{2}{\pi} \int_{0}^{t} d t_{1} e^{-\frac{x_{0}^{2}}{2 t_{1}}} \frac{\sqrt{t-t_{1}}}{\sqrt{t_{1}}} \\
& =\left(t+x_{0}^{2}\right) \operatorname{erfc}\left(\frac{\left|x_{0}\right|}{\sqrt{2 t}}\right)-\left|x_{0}\right| \sqrt{\frac{2 t}{\pi}} e^{-\frac{x_{0}^{2}}{2 t}} \tag{178}
\end{align*}
$$

with the following asymptotic growth independent of $x_{0}$

$$
\begin{equation*}
m^{[k=2]}\left(t \mid x_{0}, t_{0}=0\right) \underset{t \rightarrow+\infty}{\simeq} t \tag{179}
\end{equation*}
$$

## 2. Case $\mu>0$

For the case $\mu>0$, integrating the singular and regular part of the joint propagator $P\left(x, A, t \mid x_{0}, A_{0}, t_{0}\right)$ of Eq. 165 with respect to the final position $x$, gives respectively

$$
\begin{align*}
\Pi^{\text {Singular }}\left(A, t \mid x_{0}, A_{0}, t_{0}\right) & =\delta\left(A-A_{0}\right) \int_{-\infty}^{+\infty} d x \frac{e^{\mu\left(x-x_{0}\right)-\frac{\mu^{2}}{2}\left(t-t_{0}\right)}}{\sqrt{2 \pi\left(t-t_{0}\right)}}\left[e^{-\frac{\left(x-x_{0}\right)^{2}}{2\left(t-t_{0}\right)}}-e^{-\frac{\left(|x|+\left|x_{0}\right|\right)^{2}}{2\left(t-t_{0}\right)}}\right]  \tag{180}\\
& =\delta\left(A-A_{0}\right)\left[1-\frac{1}{2} e^{-\mu x_{0}}\left(e^{-\mu\left|x_{0}\right|} \operatorname{erfc}\left(\frac{\left|x_{0}\right|-\mu\left(t-t_{0}\right)}{\sqrt{2\left(t-t_{0}\right)}}\right)+e^{\mu\left|x_{0}\right|} \operatorname{erfc}\left(\frac{\left|x_{0}\right|+\mu\left(t-t_{0}\right)}{\sqrt{2\left(t-t_{0}\right)}}\right)\right)\right]
\end{align*}
$$

and

$$
\begin{align*}
& \Pi^{\text {Regular }}\left(A, t \mid x_{0}, A_{0}, t_{0}\right)=\theta\left(A>A_{0}\right) \int_{-\infty}^{+\infty} d x e^{\mu\left(x-x_{0}\right)-\frac{\mu^{2}}{2}\left(t-t_{0}\right)}\left(\frac{|x|+\left|x_{0}\right|+A-A_{0}}{\sqrt{2 \pi}\left(t-t_{0}\right)^{\frac{3}{2}}}\right) e^{-\frac{\left(|x|+\left|x_{0}\right|+A-A_{0}\right)^{2}}{2\left(t-t_{0}\right)}} \\
& =\theta\left(A>A_{0}\right)\left[\sqrt{\frac{2}{\pi\left(t-t_{0}\right)}} e^{-\mu x_{0}} e^{-\frac{\mu^{2}}{2}\left(t-t_{0}\right)} e^{-\frac{\left(\left|x_{0}\right|+A-A_{0}\right)^{2}}{2\left(t-t_{0}\right)}}\right.  \tag{181}\\
& \left.+\frac{1}{2} \mu e^{-\mu x_{0}}\left(e^{-\mu\left(\left|x_{0}\right|+A-A_{0}\right)} \operatorname{erfc}\left(\frac{\left|x_{0}\right|-\mu\left(t-t_{0}\right)+A-A_{0}}{\sqrt{2\left(t-t_{0}\right)}}\right)-e^{\mu\left(\left|x_{0}\right|+A-A_{0}\right)} \operatorname{erfc}\left(\frac{\left|x_{0}\right|+\mu\left(t-t_{0}\right)+A-A_{0}}{\sqrt{2\left(t-t_{0}\right)}}\right)\right)\right]
\end{align*}
$$

The limit of the infinite time interval $\left(t-t_{0}\right) \rightarrow+\infty$ yields

$$
\begin{align*}
& \Pi\left(A, \infty \mid x_{0}, A_{0}\right)=\Pi^{\text {Singular }}\left(A, \infty \mid x_{0}, A_{0}\right)+\Pi^{\text {Regular }}\left(A, \infty \mid x_{0}, A_{0}\right) \\
& =\delta\left(A-A_{0}\right)\left[1-e^{-\mu\left(x_{0}+\left|x_{0}\right|\right)}\right]+\theta\left(A>A_{0}\right) \mu e^{-\mu\left(x_{0}+\left|x_{0}\right|+A-A_{0}\right)} \\
& = \begin{cases}\theta\left(A>A_{0}\right) \mu e^{-\mu\left(A-A_{0}\right)} & \text { if } x_{0}<0 \\
\delta\left(A-A_{0}\right)\left[1-e^{-2 \mu x_{0}}\right]+e^{-2 \mu x_{0}} \theta\left(A>A_{0}\right) \mu e^{-\mu\left(A-A_{0}\right)} & \text { if } x_{0}>0\end{cases} \tag{182}
\end{align*}
$$

In the region $x_{0}<0$ where the limit of the singular contribution vanishes $\Pi^{\operatorname{Singular}}\left(A, \infty \mid x_{0}<0, A_{0}\right)=0$, one can use the asymptotic behaviors of the erfc function given by Eq. 158 to obtain that the leading contribution to Eq. 180 reads for large time $\left(t-t_{0}\right)$

$$
\begin{equation*}
\Pi^{\text {Singular }}\left(A, t \mid x_{0}<, A_{0}, t_{0}\right) \underset{\left(t-t_{0}\right) \rightarrow+\infty}{\simeq} \delta\left(A-A_{0}\right)\left|x_{0}\right| \sqrt{\frac{2}{\pi}} \frac{e^{-\frac{\left(\left|x_{0}\right|+\mu\left(t-t_{0}\right)\right)^{2}}{2\left(t-t_{0}\right)}}}{\mu^{2}\left(t-t_{0}\right)^{3 / 2}} \tag{183}
\end{equation*}
$$



Figure 1. Plain curve: mean local time, dash curve: derivative of the mean local time. Blue lines correspond to $x_{0}=-1$, Red lines correspond to $x_{0}=1$. The constant drift $\mu$ is equal to 1 .

The first moment $m^{[k=1]}\left(t \mid x_{0}, t_{0}=0\right)$ of the local time increment of Eqs 73 and 78

$$
\begin{align*}
m^{[k=1]}\left(t \mid x_{0}, t_{0}=0\right) & =\int_{0}^{\infty} d A A \Pi^{[\mu]}\left(A, t \mid x_{0}, A_{0}=0, t_{0}=0\right)=\int_{0}^{t} d t_{1} G\left(0, t_{1} \mid x_{0}, t_{0}=0\right) \\
& =\int_{0}^{t} d t_{1} \frac{e^{-\frac{\left(x_{0}-\mu t_{1}\right)^{2}}{2 t_{1}}}}{\sqrt{2 \pi t_{1}}} \\
& =\frac{e^{-\mu\left(\left|x_{0}\right|+x_{0}\right)} \operatorname{erfc}\left(\frac{\left|x_{0}\right|-\mu t}{\sqrt{2 t}}\right)-e^{\mu\left(\left|x_{0}\right|-x_{0}\right)} \operatorname{erfc}\left(\frac{\left|x_{0}\right|+\mu t}{\sqrt{2 t}}\right)}{2 \mu} \tag{184}
\end{align*}
$$

converges to a finite asymptotic value which depends strongly on the sign of the initial position $x_{0}$ of the process

$$
\lim _{t \rightarrow \infty} m^{[k=1]}\left(t \mid x_{0}, t_{0}=0\right)=\frac{e^{-\mu\left(x_{0}-\left|x_{0}\right|\right)}}{\mu}= \begin{cases}\frac{1}{\mu} & \text { if } x_{0}<0  \tag{185}\\ \frac{e^{-2 \mu x_{0}}}{\mu} & \text { if } x_{0}>0\end{cases}
$$

In the region $x_{0}>0$ where the process has a finite probability to escape towards $(+\infty)$ without visiting the origin, the mean local time is of course smaller than in the region $x_{0}<0$ where the process is certain to cross the origin. In the latter case, as expected, its average asymptotic value is the same as if the process started from 0 .

The two first time-derivatives of the first moments

$$
\begin{align*}
& \partial_{t} m^{[k=1]}\left(t \mid x_{0}, t_{0}=0\right)=\frac{e^{-\frac{\left(x_{0}+\mu t\right)^{2}}{2 t}}}{\sqrt{2 \pi t}} \\
& \partial_{t}^{2} m^{[k=1]}\left(t \mid x_{0}, t_{0}=0\right)=\frac{e^{-\frac{\left(x_{0}+\mu t\right)^{2}}{2 t}}}{2 \sqrt{2 \pi t^{5}}}\left(x_{0}^{2}-t\left(1+t \mu^{2}\right)\right) \tag{186}
\end{align*}
$$

shows that the mean local time increases the most at a time $T=\frac{\sqrt{1+4 x_{0}^{2} \mu^{2}}-1}{2 \mu^{2}}$ which is independent of the sign of $x_{0}$. The behavior of the mean local time and its derivative is shown in Fig. 1 .

The second moment $m^{[k=2]}\left(t \mid x_{0}, t_{0}=0\right)$ of the local time increment of Eqs 73 and 81

$$
\begin{align*}
& m^{[k=2]}\left(t \mid x_{0}, t_{0}=0\right)=\int_{0}^{\infty} d A A^{2} \Pi^{[\mu=0]}\left(A, t \mid x_{0}, A_{0}=0, t_{0}=0\right)=2 \int_{0}^{t} d t_{2} \int_{0}^{t_{2}} d t_{1} G\left(0, t_{2} \mid 0, t_{1}\right) G\left(0, t_{1} \mid x_{0}, t_{0}=0\right) \\
& = \begin{cases}\frac{1}{\mu^{2}} \operatorname{erfc}\left(\frac{x_{0}+\mu t}{\sqrt{2 t}}\right)+\frac{1}{\mu^{2}} e^{-\frac{\mu}{2}\left(6 x_{0}+\mu t\right)}\left[-2 e^{2 \mu x_{0}} \operatorname{erfc}\left(\frac{x_{0}}{\sqrt{2 t}}\right)+e^{\mu x_{0}+\frac{\mu^{2}}{2} t} \operatorname{erfc}\left(\frac{x_{0}-\mu t}{\sqrt{2 t}}\right)\right] & \text { if } x_{0}>0 \\
\frac{1}{\mu^{2}} \operatorname{erfc}\left(\frac{-x_{0}-\mu t}{\sqrt{2 t}}\right)+\frac{1}{\mu^{2}} e^{-\frac{\mu}{2}\left(6 x_{0}+\mu t\right)}\left[-2 e^{2 \mu x_{0}} \operatorname{erfc}\left(\frac{-x_{0}}{\sqrt{2 t}}\right)+e^{\mu x_{0}+\frac{\mu^{2}}{2} t} \operatorname{erfc}\left(\frac{x_{0}-\mu t}{\sqrt{2 t}}\right)\right] & \text { if } x_{0}<0\end{cases} \tag{187}
\end{align*}
$$

## D. Large deviations of the intensive local time $a$ for the Brownian motion without drift $\mu=0$

$$
\text { 1. Rate function } I(a) \text { for the intensive local time } a=\frac{A-A_{0}}{t-t_{0}} \in[0,+\infty[
$$

The probability distribution $\Pi^{[\mu=0]}\left(A, t \mid x_{0}, A_{0}, t_{0}\right)$ of Eq. 173 allows to evaluate the probability to see $A=A_{0}+$ $a\left(t-t_{0}\right)$

$$
\begin{align*}
\Pi^{[\mu=0]}\left(A=A_{0}+a\left(t-t_{0}\right), t \mid x_{0}, A_{0}, t_{0}\right) & =\delta\left(a\left(t-t_{0}\right)\right) e^{-\frac{x_{0}^{2}}{2\left(t-t_{0}\right)}} \int_{-\infty}^{+\infty} d z \frac{e^{-\frac{z^{2}}{2}}}{\sqrt{2 \pi}}\left[e^{\frac{z x_{0}}{\sqrt{t-t_{0}}}}-e^{-\frac{\left|z x_{0}\right|}{\sqrt{t-t_{0}}}}\right] \\
& +\theta(a>0) \sqrt{\frac{2}{\pi\left(t-t_{0}\right)}} e^{-\frac{x_{0}^{2}}{2\left(t-t_{0}\right)}-\left|x_{0}\right| a-\frac{a^{2}}{2}\left(t-t_{0}\right)} \tag{188}
\end{align*}
$$

So the large deviations of the intensive local time $a$ are governed by the simple rate function 95 ]

$$
\begin{equation*}
I(a)=\frac{a^{2}}{2} \text { for } a \in[0,+\infty[ \tag{189}
\end{equation*}
$$

that vanishes and is minimum at its boundary value $a=0$ in agreement with Eq. 114 .
If one includes the prefactors, the leading order of the regular contribution of Eq. 188 reads

$$
\begin{equation*}
\Pi^{\text {Regular }[\mu=0]}\left(A=A_{0}+a\left(t-t_{0}\right), t \mid x_{0}, A_{0}, t_{0}\right) \underset{\left(t-t_{0}\right) \rightarrow+\infty}{\simeq} \sqrt{\frac{2}{\pi\left(t-t_{0}\right)}} e^{-\left|x_{0}\right| a-\left(t-t_{0}\right) I(a)} \tag{190}
\end{equation*}
$$

The agreement with the general formula of Eq. 113 can be checked using Eq. 147 for $\mu=0$

$$
\begin{equation*}
\tilde{G}_{s}^{[\mu=0]}\left(x \mid x_{0}\right)=\frac{e^{-\sqrt{2 s}\left|x-x_{0}\right|}}{\sqrt{2 s}} \tag{191}
\end{equation*}
$$

and Eq. 103

$$
\begin{equation*}
0=a \partial_{s}[\sqrt{2 s}]-1=\frac{a}{\sqrt{2 s}}-1 \tag{192}
\end{equation*}
$$

that leads to the saddle-point

$$
\begin{equation*}
s_{a}=\frac{a^{2}}{2} \tag{193}
\end{equation*}
$$

2. Rate function $I(a, v)$ for the intensive local time $a=\frac{A-A_{0}}{t-t_{0}}$ and the intensive displacement $v=\frac{x-x_{0}}{t-t_{0}}$

The joint propagator of Eq. 165 for the case $\mu=0$

$$
\begin{align*}
P^{[\mu=0]}\left(x, A, t \mid x_{0}, A_{0}, t_{0}\right) & =\delta\left(A-A_{0}\right) \frac{e^{-\frac{x^{2}+x_{0}^{2}}{2\left(t-t_{0}\right)}}}{\sqrt{2 \pi\left(t-t_{0}\right)}}\left[e^{\frac{x x_{0}}{2\left(t-t_{0}\right)}}-e^{-\frac{\left|x x_{0}\right|}{\left(t-t_{0}\right)}}\right] \\
& +\theta\left(A>A_{0}\right)\left(\frac{|x|+\left|x_{0}\right|+A-A_{0}}{\sqrt{2 \pi}\left(t-t_{0}\right)^{\frac{3}{2}}}\right) e^{-\frac{x^{2}+x_{0}^{2}+\left(A-A_{0}\right)^{2}}{2\left(t-t_{0}\right)}} e^{-\frac{\left|x x_{0}\right|+\left(|x|+\left|x_{0}\right|\right)\left(A-A_{0}\right)}{\left(t-t_{0}\right)}} \tag{194}
\end{align*}
$$

allows to evaluate the joint probability to see $x=x_{0}+v\left(t-t_{0}\right)$ and $A=A_{0}+a\left(t-t_{0}\right)$

$$
\begin{align*}
& P^{[\mu=0]}\left(x=x_{0}+v\left(t-t_{0}\right), A=A_{0}+a\left(t-t_{0}\right), t \mid x_{0}, A_{0}, t_{0}\right)=\delta\left(a\left(t-t_{0}\right)\right) \frac{e^{-\frac{\left[x_{0}+v\left(t-t_{0}\right)\right]^{2}+x_{0}^{2}}{2\left(t-t_{0}\right)}}}{\sqrt{2 \pi\left(t-t_{0}\right)}}\left[e^{\frac{\left[x_{0}+v\left(t-t_{0}\right)\right] x_{0}}{2\left(t-t_{0}\right)}}-e^{-\frac{\left|\left[x_{0}+v\left(t-t_{0}\right)\right] x_{0}\right|}{\left(t-t_{0}\right)}}\right] \\
& +\theta(a>0)\left(\frac{\left|x_{0}+v\left(t-t_{0}\right)\right|+\left|x_{0}\right|+a\left(t-t_{0}\right)}{\sqrt{2 \pi}\left(t-t_{0}\right)^{\frac{3}{2}}}\right) e^{-\frac{\left[x_{0}+v\left(t-t_{0}\right)\right]^{2}+x_{0}^{2}+a^{2}\left(t-t_{0}\right)^{2}}{2\left(t-t_{0}\right)}} e^{-\frac{\left|\left[x_{0}+v\left(t-t_{0}\right)\right] x_{0}\right|+\left(\left|x_{0}+v\left(t-t_{0}\right)\right|+\left|x_{0}\right|\right) a\left(t-t_{0}\right)}{\left(t-t_{0}\right)}} \\
& =\delta\left(a\left(t-t_{0}\right)\right) \frac{e^{-\frac{v^{2}}{2}\left(t-t_{0}\right)-x_{0} v-\frac{x_{0}^{2}}{\left(t-t_{0}\right)}}}{\sqrt{2 \pi\left(t-t_{0}\right)}}\left[e^{\frac{v x_{0}}{2}+\frac{x_{0}^{2}}{2\left(t-t_{0}\right)}}-e^{-\frac{\left|x_{0}^{2}+v x_{0}\left(t-t_{0}\right)\right|}{\left(t-t_{0}\right)}}\right] \\
& +\theta(a>0)\left(\frac{\left|x_{0}+v\left(t-t_{0}\right)\right|+\left|x_{0}\right|+a\left(t-t_{0}\right)}{\sqrt{2 \pi}\left(t-t_{0}\right)^{\frac{3}{2}}}\right) e^{-\frac{v^{2}+a^{2}}{2}\left(t-t_{0}\right)-x_{0} v-\frac{x_{0}^{2}}{\left(t-t_{0}\right)}-\frac{\left|x_{0}^{2}+x_{0} v\left(t-t_{0}\right)\right|}{\left(t-t_{0}\right)}-\left(\left|x_{0}+v\left(t-t_{0}\right)\right|+\left|x_{0}\right|\right) a} \tag{195}
\end{align*}
$$

So the large deviations for the joint probability of the intensive local time $a=\frac{A-A_{0}}{t-t_{0}} \in[0,+\infty[$ and of the intensive displacement $\left.v=\frac{x-x_{0}}{t-t_{0}} \in\right]-\infty,+\infty[$

$$
\begin{equation*}
P^{[\mu=0]}\left(x=x_{0}+v\left(t-t_{0}\right), A=A_{0}+a\left(t-t_{0}\right), t \mid x_{0}, A_{0}, t_{0}\right) \underset{\left(t-t_{0}\right) \rightarrow+\infty}{\propto} e^{-\left(t-t_{0}\right) I(a, v)} \tag{196}
\end{equation*}
$$

are governed by the rate function

$$
\begin{equation*}
I(a, v)=\frac{v^{2}+a^{2}}{2}+|v| a=\frac{(|v|+a)^{2}}{2} \quad \text { for } \quad a \in[0,+\infty[\quad \text { and } \quad v \in]-\infty,+\infty[ \tag{197}
\end{equation*}
$$

For any $a \in[0,+\infty[$, the joint rate function $I(a, v)$ is minimum for $v=0$ where one recovers $I(a)$ of Eq. 189 .
As a final remark, let us stress that the joint rate function $I(a, v)$ only occurs for diffusion processes that are recurrent but do not converge towards an equilibrium for the following reasons.
(i) For transient processes, the local time $\left(A_{T}-A_{0}\right)$ remains a finite random variable for $T \rightarrow+\infty$, while the large deviations properties of the intensive displacement $v=\frac{x-x_{0}}{t-t_{0}}$ are governed by some rate function $I(v)$.
(ii) For processes converging towards an equilibrium, the total displacement $\left(x-x_{0}\right)$ remains a finite random variable for $T \rightarrow+\infty$, while the large deviations properties of the intensive local time $a=\frac{A-A_{0}}{t-t_{0}}$ are governed by some rate function $I(a)$ as described in subsection V B 1 .

## E. Conditioning towards the position $x_{T}^{*}$ and the local time $A_{T}^{*}$ at the finite time horizon $T$

Let us now apply the framework described in the subsection VIA. Using the explicit joint propagator of Eq. 165

$$
\begin{align*}
& \ln P\left(x_{T}, A_{T}, T \mid x, A, t\right)=\mu\left(x_{T}-x\right)-\frac{\mu^{2}}{2}(T-t)-\ln (\sqrt{2 \pi(T-t)}) \\
& +\ln \left[\delta\left(A_{T}-A\right)\left(e^{-\frac{\left(x_{T}-x\right)^{2}}{2(T-t)}}-e^{-\frac{\left(\left|x_{T}\right|+|x|\right)^{2}}{2(T-t)}}\right)+\theta\left(A_{T}>A\right) \frac{\left(\left|x_{T}\right|+|x|+A_{T}-A\right)}{(T-t)} e^{-\frac{\left(\left|x_{T}\right|+|x|+A_{T}-A\right)^{2}}{2(T-t)}}\right] \tag{198}
\end{align*}
$$

one obtains that the conditioned drift of Eq. 117

$$
\begin{align*}
& \mu_{T}^{\left[x_{T}^{*}, A_{T}^{*}\right]}(x, A, t)=\mu+\partial_{x} \ln P\left(x_{T}^{*}, A_{T}^{*}, T \mid x, A, t\right) \\
& =\partial_{x} \ln \left[\delta\left(A_{T}^{*}-A\right)\left(e^{-\frac{\left(x_{T}^{*}-x\right)^{2}}{2(T-t)}}-e^{-\frac{\left(\left|x_{T}^{*}\right|+|x|\right)^{2}}{2(T-t)}}\right)+\theta\left(A_{T}^{*}>A\right) \frac{\left(\left|x_{T}^{*}\right|+|x|+A_{T}^{*}-A\right)}{(T-t)} e^{-\frac{\left(\left|x_{T}^{*}\right|+|x|+A_{T}^{*}-A\right)^{2}}{2(T-t)}}\right] \tag{199}
\end{align*}
$$

does not depend on the initial unconditioned drift $\mu$ anymore and can be decomposed into the two following regions for $x_{T}^{*} \neq 0$.
(i) In the region $A_{0}=0 \leq A<A_{T}^{*}$ where the local time $A$ has not yet reached its conditioned final value $A_{T}^{*}$, Eq. 199 reduces to

$$
\begin{equation*}
\mu_{T}^{\left[x_{T}^{*}, A_{T}^{*}\right]}\left(x, A<A_{T}^{*}, t\right)=\operatorname{sgn}(x)\left[\frac{1}{\left|x_{T}^{*}\right|+|x|+A_{T}^{*}-A}-\frac{\left|x_{T}^{*}\right|+|x|+A_{T}^{*}-A}{T-t}\right] \tag{200}
\end{equation*}
$$

(ii) In the region $A=A_{T}^{*}$ where the local time $A$ has already reached its conditioned final value $A_{T}^{*}$, and where the position $x$ cannot visit the origin $x=0$ anymore, the drift of Eq. 199 reduces to

$$
\begin{equation*}
\mu_{T}^{\left[x_{T}^{*}, A_{T}^{*}\right]}\left(x, A=A_{T}^{*}, t\right)=\frac{\left(\frac{x_{T}^{*}-x}{T-t}\right) e^{-\frac{\left(x_{T}^{*}-x\right)^{2}}{2(T-t)}}+\left(\frac{x+\operatorname{sgn}(x)\left|x_{T}^{*}\right|}{T-t}\right) e^{-\frac{\left(\left|x_{T}^{*}\right|+|x|\right)^{2}}{2(T-t)}}}{e^{-\frac{\left(x_{T}^{*}-x\right)^{2}}{2(T-t)}}-e^{-\frac{\left(\left|x_{T}^{*}\right|+|x|\right)^{2}}{2(T-t)}}} \tag{201}
\end{equation*}
$$

The fact that the initial unconditioned drift $\mu$ does not appear in the conditioned drift of Eqs 200 and 201 is actually an immediate consequence of a more general result stating that the constraints can be imposed one after the other [32, 41. By first imposing the final position of the process, one obtains a Brownian bridge which does not depend on the original drift. Then, imposing additional constraints on this process (whatever they are: local time, area under the curve, etc...) will not change this result. In addition, in the particular case where $A_{T}^{*}=0$ (in other word the process cannot cross the origin) then only the singular part of the propagator contributes to the conditioned drift. Assume moreover that $x_{T}^{*}$ and $x$ are positive, then Eq. 201 reduces to

$$
\begin{equation*}
\mu_{T}^{\left[x_{T}^{*}, A_{T}^{*}\right]}\left(x, A=A_{T}^{*}=0, t\right)=\frac{\left(\frac{x_{T}^{*}-x}{T-t}\right) e^{-\frac{\left(x_{T}^{*}-x\right)^{2}}{2(T-t)}}+\left(\frac{x_{T}^{*}+x}{T-t}\right) e^{-\frac{\left(x_{T}^{*}+x\right)^{2}}{2(T-t)}}}{e^{-\frac{\left(x_{T}^{*}-x\right)^{2}}{2(T-t)}}-e^{-\frac{\left(x_{T}^{*}+x\right)^{2}}{2(T-t)}}} \tag{202}
\end{equation*}
$$

which is the drift of a Brownian bridge conditioned to stay positive, as it should be. This equation can be found in (4, 6].


Figure 2. Examples of realizations of the Brownian bridge conditioned to end at the final position $x_{T}^{*}=0.5$ and to have the final local time $A_{T}^{*}=1$ at the finite time horizon $T=1$ (see the conditioned drift of Eqs. 200 and 201). For each trajectory, the associated local time $A(t)$ is shown as a function of the time $t \in[0, T]$. Top figure: the process begins at the position $x_{0}=2$. Bottom figure: the process begins at position $x_{0}=-0.5$. The encapsulated graphs show the convergence of both processes to the desired final value $x_{T}^{*}=0.5$. The time step used in the discretization is $d t=10^{-5}$.

## F. Case $\mu=0$ : Conditioning towards the local time $A_{T}^{*}$ at the finite time horizon $T$

Let us now apply the framework described in the subsection VIB. Using the explicit probability of Eq. 173

$$
\begin{equation*}
\Pi^{[\mu=0]}\left(A_{T}^{*}, T \mid x, A, t\right)=\delta\left(A_{T}^{*}-A\right) \operatorname{erf}\left(\frac{|x|}{\sqrt{2(T-t)}}\right)+\theta\left(A_{T}^{*}>A\right) \sqrt{\frac{2}{\pi(T-t)}} e^{-\frac{\left(|x|+A_{T}^{*}-A\right)^{2}}{2(T-t)}} \tag{203}
\end{equation*}
$$

one obtains that the conditioned drift of Eq. 125

$$
\begin{equation*}
\mu_{T}^{\left[A_{T}^{*}\right]}(x, A, t)=\partial_{x} \ln \Pi^{[\mu=0]}\left(A_{T}^{*}, T \mid x, A, t\right) \tag{204}
\end{equation*}
$$

can be decomposed into the two following regions.
(i) In the region $A_{0}=0 \leq A<A_{T}^{*}$ where the local time $A$ has not yet reached its conditioned final value $A_{T}^{*}$, the conditioned drift of Eq. 204 reduces to

$$
\begin{align*}
\mu_{T}^{\left[A_{T}^{*}\right]}\left(x, A<A_{T}^{*}, t\right) & =\partial_{x} \ln \left[\sqrt{\frac{2}{\pi(T-t)}} e^{-\frac{\left(|x|+A_{T}^{*}-A\right)^{2}}{2(T-t)}}\right] \\
& =-\operatorname{sgn}(x) \frac{|x|+A_{T}^{*}-A}{T-t} \tag{205}
\end{align*}
$$

(ii) In the region $A=A_{T}^{*}$ where the local time $A$ has already reached its conditioned final value $A_{T}^{*}$, and where the position $x$ cannot visit the origin $x=0$ anymore, the conditioned drift of Eq. 204 reads

$$
\begin{align*}
\mu_{T}^{\left[A_{T}^{*}\right]}\left(x, A=A_{T}^{*}, t\right) & =\partial_{x} \ln \left[\operatorname{erf}\left(\frac{|x|}{\sqrt{2(T-t)}}\right)\right] \\
& =\sqrt{\frac{2}{\pi(T-t)}} \frac{e^{-\frac{x^{2}}{2(T-t)}}}{\operatorname{erf}\left(\frac{|x|}{\sqrt{2(T-t)}}\right)} \operatorname{sgn}(x) \tag{206}
\end{align*}
$$

The asymptotic behavior near the origin $x \rightarrow 0$ is given by

$$
\begin{equation*}
\mu_{T}^{\left[A_{T}^{*}\right]}\left[x, A=A_{T}^{*}, t\right] \underset{x \rightarrow 0}{\simeq} \frac{1}{x}-\frac{x}{3(T-t)} \tag{207}
\end{equation*}
$$

Due to the $1 / x$ term, the origin $x=0$ is an entrance boundary that the process cannot cross, therefore in the second region the local time can no longer increase, as wished.
G. Case $\mu=0$ : Conditioning towards the intensive local time $a^{*}=\frac{A_{T}^{*}}{T}$ in the limit $T \rightarrow+\infty$

In order to impose the intensive local time $a^{*}=\frac{A_{T}^{*}}{T}$ in the limit $T \rightarrow+\infty$, one can plug $A_{T}^{*}=T a^{*}$ into the conditioned drift of Eq. 205 to obtain at leading order for $T \rightarrow+\infty$ while $t$ remains finite

$$
\begin{equation*}
\mu_{T}^{\left[T a^{*}\right]}\left(x, A<T a^{*}, t\right)=-\operatorname{sgn}(x) \frac{|x|+\left(T a^{*}-A\right)}{T-t} \underset{T \rightarrow+\infty}{\simeq}-\operatorname{sgn}(x) a^{*} \equiv \mu_{\infty}^{\left[a^{*}\right]}(x) \tag{208}
\end{equation*}
$$

The agreement with the general formula of Eq. 135 for the drift $\mu_{\infty}^{\left[a_{\infty}^{*}\right]}(x)$ can be checked using Eq. 147

$$
\begin{equation*}
\tilde{G}_{s}^{[\mu=0]}(0 \mid x)=\frac{e^{-\sqrt{2 s}|x|}}{\sqrt{2 s}} \tag{209}
\end{equation*}
$$

and the saddle-point value $s_{a^{*}}=\frac{\left[a^{*}\right]^{2}}{2}$ of Eq. 193 to obtain

$$
\begin{equation*}
\mu_{\infty}^{\left[a^{*}\right]}(x)=\partial_{x} \ln \hat{G}_{s_{a^{*}}}(0 \mid x)=\partial_{x}\left(-\sqrt{2 s_{a^{*}}}|x|-\ln \left(\sqrt{2 s_{a^{*}}}\right)\right)=-\sqrt{2 s_{a^{*}}} \operatorname{sgn}(x)=-\operatorname{sgn}(x) a^{*} \tag{210}
\end{equation*}
$$

As explained in the Appendices, this result can be also recovered via the appropriate canonical conditioning leading to Eq. C13.
H. Case $\mu>0$ : Conditioning towards the finite local time $A_{\infty}^{*}<+\infty$ at the infinite time horizon $T=+\infty$

The framework described in the subsection VIB 1 can be applied as follows.
(i) In the region $A_{0}=0 \leq A<A_{\infty}^{*}$ where the local time $A$ has not yet reached its conditioned asymptotic value $A_{T}^{*}$, the conditioned drift $\mu_{A_{\infty}^{*}}^{*}\left(x, A<A_{\infty}^{*}\right)$ of Eq. 129 can be obtained from $\Pi^{\text {Regular }}\left(A, \infty \mid x_{0}, A_{0}\right)$ given in Eq. 182

$$
\begin{align*}
\mu_{A_{\infty}^{*}}^{*}\left(x, A<A_{\infty}^{*}\right) & =\mu+\partial_{x} \ln \Pi^{\text {Regular }}\left(A_{\infty}^{*}, \infty \mid x, A\right)=\mu+\partial_{x} \ln \left[\mu e^{-\mu\left(x+|x|+A_{\infty}^{*}-A\right)}\right] \\
& =-\mu \operatorname{sgn}(x) \tag{211}
\end{align*}
$$



Figure 3. Examples of realizations of the Brownian process conditioned to have the final local time $A_{T}^{*}=1$ at the finite time horizon $T=1$ (see the conditioned drift of Eqs. 205 and 206). For each trajectory, the associated local time $A(t)$ is shown as a function of the time $t \in[0, T]$. The process can end at any final position $x_{T}$, while the initial position is $x_{0}=-0.5$ here. The time step used in the discretization is $d t=10^{-5}$.

So in the region $x<0$, the conditioned drift coincide with the initial unconditioned drift $\mu$, while in the region $x>0$, the conditioned drift is opposite to the initial unconditioned drift $\mu$ in order to visit again the origin and to increase the local time.
(ii) In the region $A=A_{\infty}^{*}$ where the local time $A$ has already reached its conditioned asymptotic value $A_{T}^{*}$, the conditioned drift of Eq. 130 can be obtained from $\Pi^{\text {Singular }}\left(A_{\infty}^{*}, \infty \mid x, A\right)$ given in Eq. 182 for $x>0$

$$
\begin{align*}
\mu_{A_{\infty}^{*}}^{*}\left(x>0, A_{\infty}^{*}\right) & =\mu+\partial_{x} \ln \Pi^{\text {Singular }}\left(A_{\infty}^{*}, \infty \mid x, A\right)=\mu+\partial_{x} \ln \left[1-e^{-\mu 2 x}\right] \\
& =\mu\left(\frac{e^{2 \mu x}+1}{e^{2 \mu x}-1}\right)=\mu \operatorname{coth}(\mu x) \tag{212}
\end{align*}
$$

For $x<0$, the conditioned drift of Eq. 130 should be computed using the leading asymptotic form of $\Pi^{\text {Singular }}\left(A_{\infty}^{*}, T \mid x<\right.$ $0, A, t)$ for large time interval $(T-t)$ given in Eq. 183

$$
\begin{align*}
\mu_{A_{\infty}^{*}}^{*}\left(x<0, A=A_{\infty}^{*}\right) & =\mu+\lim _{T \rightarrow+\infty}\left(\partial_{x} \ln \Pi^{\text {Singular }}\left(A_{\infty}^{*}, T \mid x<0, A, t\right)\right)=\mu+\lim _{T \rightarrow+\infty}\left(\partial_{x} \ln \left[(-x) \sqrt{\frac{2}{\pi}} \frac{e^{-\frac{(-x+\mu T)^{2}}{2 T}}}{\mu^{2} T^{3 / 2}}\right]\right) \\
& =\lim _{T \rightarrow+\infty}\left(\frac{1}{x}-\frac{x}{T}\right)=\frac{1}{x} \tag{213}
\end{align*}
$$

In both cases, whether $x>0$ or $x<0$, when $x \rightarrow 0$, the conditioned drift behaves as $\mu_{A_{\infty}^{*}}^{*}\left(x \rightarrow 0, A=A_{\infty}^{*}\right) \simeq 1 / x$. The origin $x=0$ is thus an entrance boundary that the process cannot cross, therefore in the second region (ii), the local time cannot increase any further, as expected.


Figure 4. Examples of realization of the Brownian process with drift $\mu>0$ where the local time is conditioned towards the finite asymptotic value $A_{\infty}^{*}=1$ at the infinite time horizon $T=+\infty$ (see the conditioned drift of Eqs. 211] and 212). For each trajectory, the associated local time $A(t)$ is shown as a function of the time $t$. The dashed vertical lines indicate the time when the local time reaches the finite asymptotic value $A_{\infty}^{*}=1$. The time step used in the discretization is $d t=10^{-5}$.

## VIII. APPLICATION TO THE DRIFT $\mu(x)=-\mu \operatorname{sgn}(x)$ OF PARAMETER $\mu>0$

In this section, we consider the drift directed towards the origin $x=0$ of amplitude $\mu>0$

$$
\begin{equation*}
\mu(x)=-\mu \operatorname{sgn}(x) \tag{214}
\end{equation*}
$$

This process is sometimes called Brownian motion with alternating drift (or bang-bang process [97, 98]) and was originally introduced by de Gennes to study dry friction 99.
The associated potential $U(x)$ of Eq. 9

$$
\begin{equation*}
U(x)=2 \mu \int_{0}^{x} d y \operatorname{sgn}(y)=2 \mu|x| \tag{215}
\end{equation*}
$$

corresponds to the normalizable equilibrium Boltzmann distribution of Eq. 28

$$
\begin{equation*}
G_{e q}(x)=\frac{e^{-U(x)}}{\int_{-\infty}^{+\infty} d y e^{-U(y)}}=\mu e^{-2 \mu|x|} \tag{216}
\end{equation*}
$$

So the unconditioned dynamics converges towards this equilibrium distribution for $t \rightarrow+\infty$, and the local time increment $\left(A-A_{0}\right)$ grows extensively in $\left(t-t_{0}\right)$ as discussed in subsection VB.
A. Properties of the unconditioned diffusion process $X(t)$ alone

## 1. Propagator $G\left(x, t \mid x_{0}, t_{0}\right)$ for the position alone

Via the similarity transformation of Eq. 10 based on the potential $U(x)$ of Eq. 215

$$
\begin{equation*}
G\left(x, t \mid x_{0}, t_{0}\right)=e^{\mu\left(\left|x_{0}\right|-|x|\right)} \psi\left(x, t \mid x_{0}, t_{0}\right) \tag{217}
\end{equation*}
$$

the Fokker-Planck Eq. 6 for the propagator $G\left(x, t \mid x_{0}, t_{0}\right)$ becomes the Schrödinger Equation of Eq. 11 for $\psi\left(x, t \mid x_{0}, t_{0}\right)$, where the quantum Hamiltonian of Eq. 12 involves the potential of Eq. 13 that reads for the drift of Eq. 214

$$
\begin{equation*}
V(x)=\frac{\mu^{2}}{2}-\mu \delta(x) \tag{218}
\end{equation*}
$$

As a consequence, the Hamiltonian of Eq. 12 can be decomposed

$$
\begin{equation*}
H=H_{0}+H_{1} \tag{219}
\end{equation*}
$$

into the two contributions

$$
\begin{align*}
& H_{0} \equiv-\frac{1}{2} \partial_{x}^{2}+\frac{\mu^{2}}{2} \\
& H_{1} \equiv-\mu \delta(x) \tag{220}
\end{align*}
$$

When the contribution $H_{1}$ is absent, the Schrödinger propagator $\psi^{[0]}\left(x, t \mid x_{0}, t_{0}\right)$ associated to the Hamiltonian $H_{0}$ whose potential reduces to the constant $\frac{\mu^{2}}{2}$ is given by

$$
\begin{equation*}
\psi^{[0]}\left(x, t \mid x_{0}, t_{0}\right)=\frac{1}{\sqrt{2 \pi\left(t-t_{0}\right)}} e^{-\frac{\left(x-x_{0}\right)^{2}}{2\left(t-t_{0}\right)}-\frac{\mu^{2}}{2}\left(t-t_{0}\right)} \tag{221}
\end{equation*}
$$

while its time Laplace transform reads

$$
\begin{equation*}
\tilde{\psi}_{s}^{[0]}\left(x \mid x_{0}\right) \equiv \int_{t_{0}}^{+\infty} d t e^{-s\left(t-t_{0}\right)} \psi^{[0]}\left(x, t \mid x_{0}, t_{0}\right)=\frac{1}{\sqrt{2 \pi}} \int_{0}^{+\infty} d \tau \tau^{-\frac{1}{2}} e^{-\left(s+\frac{\mu^{2}}{2}\right) \tau} e^{-\frac{\left(x-x_{0}\right)^{2}}{2 \tau}}=\frac{e^{-\sqrt{\mu^{2}+2 s}\left|x-x_{0}\right|}}{\sqrt{\mu^{2}+2 s}} \tag{222}
\end{equation*}
$$

When the contribution $H_{1}$ is present, the Laplace transform $\tilde{\psi}_{s}\left(x \mid x_{0}\right)$ of the Schrödinger propagator $\psi\left(x, t \mid x_{0}, t_{0}\right)$ for the full Hamiltonian $H$ can be computed from $\tilde{\psi}_{s}^{[0]}\left(x \mid x_{0}\right)$ of Eq. 222 via the Dyson formula analog to Eq. 44 to obtain

$$
\begin{align*}
\tilde{\psi}_{s}\left(x \mid x_{0}\right) & =\tilde{\psi}_{s}^{[0]}\left(x \mid x_{0}\right)+\tilde{\psi}_{s}^{[0]}(x \mid 0) \frac{\mu}{1-\mu \tilde{\psi}_{s}^{[0]}(0 \mid 0)} \tilde{\psi}_{s}^{[0]}\left(0 \mid x_{0}\right) \\
& =\frac{e^{-\sqrt{\mu^{2}+2 s}\left|x-x_{0}\right|}}{\sqrt{\mu^{2}+2 s}}+\frac{e^{-\sqrt{\mu^{2}+2 s}|x|}}{\sqrt{\mu^{2}+2 s}}\left(\frac{\mu}{1-\frac{\mu}{\sqrt{\mu^{2}+2 s}}}\right) \frac{e^{-\sqrt{\mu^{2}+2 s}\left|x_{0}\right|}}{\sqrt{\mu^{2}+2 s}} \\
& =\frac{1}{\sqrt{\mu^{2}+2 s}}\left[e^{-\sqrt{\mu^{2}+2 s}\left|x-x_{0}\right|}+\frac{\mu}{\sqrt{\mu^{2}+2 s}-\mu} e^{-\sqrt{\mu^{2}+2 s}\left(|x|+\left|x_{0}\right|\right)}\right] \tag{223}
\end{align*}
$$

So the Laplace transform $\tilde{G}_{s}\left(x \mid x_{0}\right)$ of the Fokker-Planck propagator $G\left(x, t \mid x_{0}, t_{0}\right)$ of Eq. 217 reads

$$
\begin{align*}
\tilde{G}_{s}\left(x \mid x_{0}\right) & \equiv \int_{t_{0}}^{+\infty} d t e^{-s\left(t-t_{0}\right)} G\left(x, t \mid x_{0}, t_{0}\right)=\int_{t_{0}}^{+\infty} d t e^{-s\left(t-t_{0}\right)} e^{\mu\left(\left|x_{0}\right|-|x|\right)} \psi\left(x, t \mid x_{0}, t_{0}\right)=e^{\mu\left(\left|x_{0}\right|-|x|\right)} \tilde{\psi}_{s}\left(x \mid x_{0}\right) \\
& =\frac{e^{\mu\left(\left|x_{0}\right|-|x|\right)}}{\sqrt{\mu^{2}+2 s}}\left[e^{-\sqrt{\mu^{2}+2 s}\left|x-x_{0}\right|}+\frac{\mu}{\sqrt{\mu^{2}+2 s}-\mu} e^{-\sqrt{\mu^{2}+2 s}\left(|x|+\left|x_{0}\right|\right)}\right] \tag{224}
\end{align*}
$$

The limit $s \rightarrow 0$ of

$$
\begin{align*}
\lim _{s \rightarrow 0}\left[s \tilde{G}_{s}\left(x \mid x_{0}\right)\right] & =\lim _{s \rightarrow 0}\left(s \frac{e^{\mu\left(\left|x_{0}\right|-|x|\right)}}{\sqrt{\mu^{2}+2 s}}\left[e^{-\sqrt{\mu^{2}+2 s}\left|x-x_{0}\right|}+\frac{\mu\left(\sqrt{\mu^{2}+2 s}+\mu\right)}{2 s} e^{-\sqrt{\mu^{2}+2 s}\left(|x|+\left|x_{0}\right|\right)}\right]\right) \\
& =\mu e^{-2 \mu|x|}=G_{e q}(x) \tag{225}
\end{align*}
$$

allows to recover the equilibrium distribution $G_{e q}(x)$ of Eq. 216 as it should.

## 2. Properties in the presence of an absorbing boundary at the origin $x=0$

In the presence of an absorbing boundary at the origin $x=0$, the present model $\mu(x)=-\mu \operatorname{sgn}(x)$ is of course very similar to the previous section concerning the model $\mu(x)=\mu$ : the two models coincide for $x_{0}<0$, and the region $x_{0}>0$ could be obtained by symmetry for the present model. However, one can also use the general formula as follows.

The evaluation of Eq. 224 for the special case $x=0$

$$
\begin{equation*}
\tilde{G}_{s}\left(0 \mid x_{0}\right)=\frac{e^{\mu\left|x_{0}\right|}}{\sqrt{\mu^{2}+2 s}}\left[e^{-\sqrt{\mu^{2}+2 s}\left|x_{0}\right|}+\frac{\mu}{\sqrt{\mu^{2}+2 s}-\mu} e^{-\sqrt{\mu^{2}+2 s}\left|x_{0}\right|}\right]=\frac{e^{\left(\mu-\sqrt{\mu^{2}+2 s}\right)\left|x_{0}\right|}}{\sqrt{\mu^{2}+2 s}-\mu} \tag{226}
\end{equation*}
$$

for the special case $x_{0}=0$

$$
\begin{equation*}
\tilde{G}_{s}(x \mid 0)=\frac{e^{-\mu|x|}}{\sqrt{\mu^{2}+2 s}}\left[e^{-\sqrt{\mu^{2}+2 s}|x|}+\frac{\mu}{\sqrt{\mu^{2}+2 s}-\mu} e^{-\sqrt{\mu^{2}+2 s}|x|}\right]=\frac{e^{-\left(\mu+\sqrt{\mu^{2}+2 s}\right)|x|}}{\sqrt{\mu^{2}+2 s}-\mu} \tag{227}
\end{equation*}
$$

and for the special case $x=0=x_{0}$

$$
\begin{equation*}
\tilde{G}_{s}(0 \mid 0)=\frac{1}{\sqrt{\mu^{2}+2 s}-\mu} \tag{228}
\end{equation*}
$$

allows to compute the Laplace transform $\hat{G}_{s}^{a b s}\left(x \mid x_{0}\right)$ via Eq. 49

$$
\begin{equation*}
\hat{G}_{s}^{a b s}\left(x \mid x_{0}\right) \equiv \hat{G}_{s}\left(x \mid x_{0}\right)-\frac{\hat{G}_{s}(x \mid 0) \hat{G}_{s}\left(0 \mid x_{0}\right)}{\hat{G}_{s}(0 \mid 0)}=\frac{e^{\mu\left(\left|x_{0}\right|-|x|\right)}}{\sqrt{\mu^{2}+2 s}}\left[e^{-\sqrt{\mu^{2}+2 s}\left|x-x_{0}\right|}-e^{-\sqrt{\mu^{2}+2 s}\left(|x|+\left|x_{0}\right|\right)}\right] \tag{229}
\end{equation*}
$$

The Laplace inversion using Eq. 151 yields the propagator $G^{a b s}\left(x, t \mid x_{0}, t_{0}\right)$ in the presence of an absorbing boundary at the origin $x=0$

$$
\begin{align*}
G^{a b s}\left(x, A, t \mid x_{0}, A_{0}, t_{0}\right) & =e^{\mu\left(\left|x_{0}\right|-|x|\right)} \frac{e^{-\frac{\mu^{2}}{2}\left(t-t_{0}\right)}}{\sqrt{2 \pi\left(t-t_{0}\right)}}\left[e^{-\frac{\left(x-x_{0}\right)^{2}}{2\left(t-t_{0}\right)}}-e^{-\frac{\left(|x|+\left|x_{0}\right|\right)^{2}}{2\left(t-t_{0}\right)}}\right] \\
& =e^{\mu\left(\left|x_{0}\right|-|x|\right)} \frac{e^{-\frac{\mu^{2}}{2}\left(t-t_{0}\right)-\frac{x^{2}+x_{0}^{2}}{2\left(t-t_{0}\right)}}}{\sqrt{2 \pi\left(t-t_{0}\right)}}\left[e^{\frac{x x_{0}}{\left(t-t_{0}\right)}}-e^{-\frac{\left|x x_{0}\right|}{\left(t-t_{0}\right)}}\right] \tag{230}
\end{align*}
$$

in agreement with the method of images.
The Laplace transform $\hat{\gamma}_{s}^{a b s}\left(x_{0}\right)$ of Eq. 59 reads

$$
\begin{equation*}
\hat{\gamma}_{s}^{a b s}\left(x_{0}\right)=\frac{\hat{G}_{s}\left(0 \mid x_{0}\right)}{\hat{G}_{s}(0 \mid 0)}=e^{\left(\mu-\sqrt{\mu^{2}+2 s}\right)\left|x_{0}\right|} \tag{231}
\end{equation*}
$$

Its Laplace inversion yields the absorption rate $\gamma^{a b s}\left(t \mid x_{0}, t_{0}\right)$

$$
\begin{equation*}
\gamma^{a b s}\left(t \mid x_{0}, t_{0}\right)=\frac{\left|x_{0}\right|}{\sqrt{2 \pi}\left(t-t_{0}\right)^{\frac{3}{2}}} e^{\mu\left|x_{0}\right|-\frac{\mu^{2}}{2}\left(t-t_{0}\right)-\frac{x_{0}^{2}}{2\left(t-t_{0}\right)}} \tag{232}
\end{equation*}
$$

One can check the normalization to unity for any starting point $x_{0}$

$$
\begin{equation*}
\int_{t_{0}}^{+\infty} d t \gamma^{a b s}\left(t \mid x_{0}, t_{0}\right)=\hat{\gamma}_{s=0}^{a b s}\left(x_{0}\right)=1 \tag{233}
\end{equation*}
$$

The survival probability $S^{a b s}\left(t \mid x_{0}, t_{0}\right)$ of Eq. 54 can be obtained from the integral over the final position $x$ of the propagator $G^{a b s}\left(x, t \mid x_{0}, t_{0}\right)$ of Eq. 230

$$
\begin{equation*}
S^{a b s}\left(t \mid x_{0}, t_{0}\right) \equiv \int_{-\infty}^{+\infty} d x G^{a b s}\left(x, t \mid x_{0}, t_{0}\right)=\frac{e^{-\frac{\mu^{2}}{2}\left(t-t_{0}\right)+\mu\left|x_{0}\right|-\frac{x_{0}^{2}}{2\left(t-t_{0}\right)}}}{\sqrt{2 \pi\left(t-t_{0}\right)}} \int_{-\infty}^{+\infty} d x e^{-\mu|x|-\frac{x^{2}}{2\left(t-t_{0}\right)}}\left[e^{\frac{x x_{0}}{\left(t-t_{0}\right)}}-e^{-\frac{\left|x x_{0}\right|}{\left(t-t_{0}\right)}}\right] \tag{234}
\end{equation*}
$$

Its asymptotic decay for large time $\left(t-t_{0}\right)$ is given by

$$
\begin{equation*}
S^{a b s}\left(t \mid x_{0}, t_{0}\right) \underset{\left(t-t_{0}\right) \rightarrow+\infty}{\simeq} \frac{e^{-\frac{\mu^{2}}{2}\left(t-t_{0}\right)+\mu\left|x_{0}\right|}}{\sqrt{2 \pi\left(t-t_{0}\right)}} \int_{-\infty}^{+\infty} d x e^{-\mu|x|}\left[\frac{x x_{0}+\left|x x_{0}\right|}{t-t_{0}}\right]=\sqrt{\frac{2}{\pi}} \frac{\left|x_{0}\right| e^{\mu\left|x_{0}\right|-\frac{\mu^{2}}{2}\left(t-t_{0}\right)}}{\mu^{2}\left(t-t_{0}\right)^{\frac{3}{2}}} \tag{235}
\end{equation*}
$$

## B. Joint propagator $P\left(x, A, t \mid x_{0}, A_{0}, t_{0}\right)$ for the unconditioned joint process $[X(t), A(t)]$

The singular contribution of Eq. 53 involves the propagator $G^{a b s}\left(x, t \mid x_{0}, t_{0}\right)$ of Eq. 230

$$
\begin{align*}
P^{\text {Singular }}\left(x, A, t \mid x_{0}, A_{0}, t_{0}\right) & =\delta\left(A-A_{0}\right) G^{a b s}\left(x, t \mid x_{0}, t_{0}\right) \\
& =\delta\left(A-A_{0}\right) e^{\mu\left(\left|x_{0}\right|-|x|\right)} \frac{e^{-\frac{\mu^{2}}{2}\left(t-t_{0}\right)}}{\sqrt{2 \pi\left(t-t_{0}\right)}}\left[e^{-\frac{\left(x-x_{0}\right)^{2}}{2\left(t-t_{0}\right)}}-e^{-\frac{\left(|x|+\left|x_{0}\right|\right)^{2}}{2\left(t-t_{0}\right)}}\right] \tag{236}
\end{align*}
$$

The Laplace transform $\hat{P}_{s}^{\text {Regular }}\left(x, A \mid x_{0}, A_{0}\right)$ of Eq. 62 reads using Eqs 226,227 and 228

$$
\begin{align*}
\hat{P}_{s}^{\text {Regular }}\left(x, A \mid x_{0}, A_{0}\right) & =\theta\left(A>A_{0}\right)\left[\frac{\hat{G}_{s}(x \mid 0) \hat{G}_{s}\left(0 \mid x_{0}\right)}{\hat{G}_{s}^{2}(0 \mid 0)}\right] e^{-\frac{\left(A-A_{0}\right)}{\hat{G}_{s}(0 \mid 0)}} \\
& =\theta\left(A>A_{0}\right) e^{\mu\left(\left|x_{0}\right|-|x|+A-A_{0}\right)} e^{-\sqrt{\mu^{2}+2 s}\left(\left|x_{0}\right|+|x|+A-A_{0}\right)} \tag{237}
\end{align*}
$$

Equation 152 allows to compute the Laplace inversion

$$
\begin{equation*}
P^{\text {Regular }}\left(x, A, t \mid x_{0}, A_{0}, t_{0}\right)=\theta\left(A>A_{0}\right) e^{\mu\left(\left|x_{0}\right|-|x|+A-A_{0}\right)-\frac{\mu^{2}}{2}\left(t-t_{0}\right)}\left(\frac{\left|x_{0}\right|+|x|+A-A_{0}}{\sqrt{2 \pi}\left(t-t_{0}\right)^{\frac{3}{2}}}\right) e^{-\frac{\left(\left|x_{0}\right|+|x|+A-A_{0}\right)^{2}}{2\left(t-t_{0}\right)}} \tag{238}
\end{equation*}
$$

So the joint propagator $P\left(x, A, t \mid x_{0}, A_{0}, t_{0}\right)$ involving the two contributions of Eq. 236 and Eq. 238 reads

$$
\begin{align*}
P\left(x, A, t \mid x_{0}, A_{0}, t_{0}\right) & =\delta\left(A-A_{0}\right) e^{\mu\left(\left|x_{0}\right|-|x|\right)} \frac{e^{-\frac{\mu^{2}}{2}\left(t-t_{0}\right)}}{\sqrt{2 \pi\left(t-t_{0}\right)}}\left[e^{-\frac{\left(x-x_{0}\right)^{2}}{2\left(t-t_{0}\right)}}-e^{-\frac{\left(|x|+\left|x_{0}\right|\right)^{2}}{2\left(t-t_{0}\right)}}\right] \\
& +\theta\left(A>A_{0}\right) e^{\mu\left(\left|x_{0}\right|-|x|+A-A_{0}\right)-\frac{\mu^{2}}{2}\left(t-t_{0}\right)}\left(\frac{\left|x_{0}\right|+|x|+A-A_{0}}{\sqrt{2 \pi}\left(t-t_{0}\right)^{\frac{3}{2}}}\right) e^{-\frac{\left(\left|x_{0}\right|+|x|+A-A_{0}\right)^{2}}{2\left(t-t_{0}\right)}} \tag{239}
\end{align*}
$$

The integration of this joint propagator $P\left(x, A, t \mid x_{0}, A_{0}, t_{0}\right)$ of Eq. 239 over the local time $A$

$$
\begin{align*}
\int_{0}^{+\infty} d A P\left(x, A, t \mid x_{0}, A_{0}, t_{0}\right) & =e^{\mu\left(\left|x_{0}\right|-|x|\right)} \frac{e^{-\frac{\mu^{2}}{2}\left(t-t_{0}\right)}}{\sqrt{2 \pi\left(t-t_{0}\right)}}\left[e^{-\frac{\left(x-x_{0}\right)^{2}}{2\left(t-t_{0}\right)}}-e^{-\frac{\left(|x|+\left|x_{0}\right|\right)^{2}}{2\left(t-t_{0}\right)}}\right]  \tag{240}\\
& +\int_{A_{0}}^{+\infty} d A e^{\mu\left(\left|x_{0}\right|-|x|+A-A_{0}\right)-\frac{\mu^{2}}{2}\left(t-t_{0}\right)}\left(\frac{\left|x_{0}\right|+|x|+A-A_{0}}{\sqrt{2 \pi\left(t-t_{0}\right)^{\frac{3}{2}}}}\right) e^{-\frac{\left(\left|x_{0}\right|+|x|+A-A_{0}\right)^{2}}{2\left(t-t_{0}\right)}} \\
& =e^{-2 \mu|x|}\left[\frac{1}{\sqrt{2 \pi\left(t-t_{0}\right)}} e^{\mu\left(\left|x_{0}\right|+|x|\right)-\frac{\mu^{2}}{2}\left(t-t_{0}\right)-\frac{\left(x-x_{0}\right)^{2}}{2\left(t-t_{0}\right)}}+\frac{\mu}{2} \operatorname{erfc}\left(\frac{\left|x_{0}\right|+|x|-\mu\left(t-t_{0}\right)}{\sqrt{2\left(t-t_{0}\right)}}\right)\right]
\end{align*}
$$

allows to recover the free propagator of the Brownian motion with alternating drift [28] as it should.

## C. Probability $\Pi\left(A, t \mid x_{0}, A_{0}, t_{0}\right)$ to see the local time $A$ at time $t$

The probability $\Pi\left(A, t \mid x_{0}, A_{0}, t_{0}\right)$ of Eq. 65 can be obtained via the integration of the joint propagator $P\left(x, A, t \mid x_{0}, A_{0}, t_{0}\right)$ of Eq. 239 over the final position $x$

$$
\begin{equation*}
\Pi\left(A, t \mid x_{0}, A_{0}, t_{0}\right) \equiv \int_{-\infty}^{+\infty} d x P\left(A, t \mid x_{0}, A_{0}, t_{0}\right) \tag{241}
\end{equation*}
$$

Its singular contribution of Eq. 69 involves the survival probability $S^{a b s}\left(t \mid x_{0}, t_{0}\right)$ of Eq. 234 with the asymptotic behavior of Eq. 235

$$
\begin{align*}
\Pi^{\text {Singular }}\left(A, t \mid x_{0}, A_{0}, t_{0}\right) & =\delta\left(A-A_{0}\right) S^{a b s}\left(t \mid x_{0}, t_{0}\right) \\
& \simeq\left(t-t_{0}\right) \rightarrow+\infty  \tag{242}\\
\simeq & \delta\left(A-A_{0}\right) \sqrt{\frac{2}{\pi}} \frac{\left|x_{0}\right| e^{\mu\left|x_{0}\right|-\frac{\mu^{2}}{2}\left(t-t_{0}\right)}}{\mu^{2}\left(t-t_{0}\right)^{\frac{3}{2}}}
\end{align*}
$$

Its regular contribution can be obtained from the integration over $x$ of Eq. 238

$$
\begin{align*}
& \Pi^{\text {Regular }}\left(A, t \mid x_{0}, A_{0}, t_{0}\right) \equiv \int_{-\infty}^{+\infty} d x P^{\text {Regular }}\left(A, t \mid x_{0}, A_{0}, t_{0}\right) \\
& =\theta\left(A>A_{0}\right) \frac{e^{-\frac{\mu^{2}}{2}\left(t-t_{0}\right)+\mu\left(\left|x_{0}\right|+A-A_{0}\right)}}{\sqrt{2 \pi}\left(t-t_{0}\right)^{\frac{3}{2}}} \int_{-\infty}^{+\infty} d x e^{-\mu|x|}\left(|x|+\left|x_{0}\right|+A-A_{0}\right) e^{-\frac{\left(|x|+\left|x_{0}\right|+A-A_{0}\right)^{2}}{2\left(t-t_{0}\right)}} \\
& =\theta\left(A>A_{0}\right) \frac{e^{-\frac{\mu^{2}}{2}\left(t-t_{0}\right)+\mu\left(\left|x_{0}\right|+A-A_{0}\right)}}{\sqrt{2 \pi\left(t-t_{0}\right)}} 2 \int_{0}^{+\infty} d x e^{-\mu x} \frac{\left(x+\left|x_{0}\right|+A-A_{0}\right)}{t-t_{0}} e^{-\frac{\left(x+\left|x_{0}\right|+A-A_{0}\right)^{2}}{2\left(t-t_{0}\right)}} \\
& =-\theta\left(A>A_{0}\right) \sqrt{\frac{2}{\pi\left(t-t_{0}\right)}} e^{-\frac{\mu^{2}}{2}\left(t-t_{0}\right)+\mu\left(\left|x_{0}\right|+A-A_{0}\right)}\left(\left[e^{-\mu x} e^{-\frac{\left(x+\left|x_{0}\right|+A-A_{0}\right)^{2}}{2\left(t-t_{0}\right)}}\right]_{x=0}^{x=+\infty}+\mu \int_{0}^{+\infty} d x e^{-\mu x-\frac{\left(x+\left|x_{0}\right|+A-A_{0}\right)^{2}}{2\left(t-t_{0}\right)}}\right) \\
& =\theta\left(A>A_{0}\right) \sqrt{\frac{2}{\pi\left(t-t_{0}\right)}} e^{-\frac{\mu^{2}}{2}\left(t-t_{0}\right)+\mu\left(\left|x_{0}\right|+A-A_{0}\right)}\left(e^{-\frac{\left(\left|x_{0}\right|+A-A_{0}\right)^{2}}{2\left(t-t_{0}\right)}}-\mu e^{-\frac{\left(\left|x_{0}\right|+A-A_{0}\right)^{2}}{2\left(t-t_{0}\right)}} \int_{0}^{+\infty} d x e^{-\frac{x^{2}}{2\left(t-t_{0}\right)}-x\left[\mu+\frac{\left(\left|x_{0}\right|+A-A_{0}\right)}{\left(t-t_{0}\right)}\right]}\right) \\
& =\theta\left(A>A_{0}\right) \sqrt{\frac{2}{\pi\left(t-t_{0}\right)}} e^{-\frac{\left(\left|x_{0}\right|+A-A_{0}-\mu\left(t-t_{0}\right)\right)^{2}}{2\left(t-t_{0}\right)}}\left(1-\mu \int_{0}^{+\infty} d x e^{-\frac{x^{2}}{2\left(t-t_{0}\right)}-x\left[\mu+\frac{\left(\left|x_{0}\right|+A-A_{0}\right)}{\left(t-t_{0}\right)}\right]}\right) \\
& =\theta\left(A>A_{0}\right) e^{-\frac{\left(\left|x_{0}\right|+A-A_{0}-\mu\left(t-t_{0}\right)\right)^{2}}{2\left(t-t_{0}\right)}}\left[\sqrt{\frac{2}{\pi\left(t-t_{0}\right)}}-\mu e^{\frac{\left(\left|x_{0}\right|+A-A_{0}+\mu\left(t-t_{0}\right)\right)^{2}}{2\left(t-t_{0}\right)}} \operatorname{erfc}\left(\frac{\left|x_{0}\right|+\mu\left(t-t_{0}\right)+A-A_{0}}{\sqrt{2\left(t-t_{0}\right)}}\right)\right] \tag{243}
\end{align*}
$$

For large time interval $\left(t-t_{0}\right)$, the leading behavior is given by

$$
\begin{align*}
& \Pi^{\text {Regular }}\left(A, t \mid x_{0}, A_{0}, t_{0}\right) \underset{\left(t-t_{0}\right) \rightarrow+\infty}{\simeq} \theta\left(A>A_{0}\right) \sqrt{\frac{2}{\pi\left(t-t_{0}\right)}} e^{-\frac{\left(\left|x_{0}\right|+A-A_{0}-\mu\left(t-t_{0}\right)\right)^{2}}{2\left(t-t_{0}\right)}}\left(1-\mu \int_{0}^{+\infty} d x e^{-x\left[\mu+\frac{\left(\left|x_{0}\right|+A-A_{0}\right)}{\left(t-t_{0}\right)}\right]}\right) \\
& \left(t-t_{0}\right) \rightarrow+\infty \\
& \simeq \tag{244}
\end{align*}\left(A>A_{0}\right) \sqrt{\frac{2}{\pi\left(t-t_{0}\right)}} e^{-\frac{\left(\left|x_{0}\right|+A-A_{0}-\mu\left(t-t_{0}\right)\right)^{2}}{2\left(t-t_{0}\right)}}\left(1-\frac{\mu}{\mu+\frac{\left(\left|x_{0}\right|+A-A_{0}\right)}{\left(t-t_{0}\right)}}\right) .
$$

Note that for $\mu=0$, we recover the expression $\Pi^{\text {Regular }}\left(A, t \mid x_{0}, A_{0}, t_{0}\right)$ of the standard Brownian motion Eq. 171, as expected.

## D. Large deviations properties of the intensive local time $a=\frac{A-A_{0}}{t-t_{0}} \in[0,+\infty[$

The probability to see $A=A_{0}+\left(t-t_{0}\right) a$ in Eq. 239 reads

$$
\begin{align*}
& P\left(x, A=A_{0}+\left(t-t_{0}\right) a, t \mid x_{0}, A_{0}, t_{0}\right)=\delta\left(\left(t-t_{0}\right) a\right) e^{\mu\left(\left|x_{0}\right|-|x|\right)} \frac{e^{-\frac{\mu^{2}}{2}\left(t-t_{0}\right)}}{\sqrt{2 \pi\left(t-t_{0}\right)}}\left[e^{-\frac{\left(x-x_{0}\right)^{2}}{2\left(t-t_{0}\right)}}-e^{-\frac{\left(|x|+\left|x_{0}\right|\right)^{2}}{2\left(t-t_{0}\right)}}\right] \\
& +\theta(a>0) e^{\mu\left(\left|x_{0}\right|-|x|+\left(t-t_{0}\right) a\right)-\frac{\mu^{2}}{2}\left(t-t_{0}\right)}\left(\frac{\left|x_{0}\right|+|x|+\left(t-t_{0}\right) a}{\sqrt{2 \pi}\left(t-t_{0}\right)^{\frac{3}{2}}}\right) e^{-\frac{\left.\left(\left|x_{0}\right|+|x|+t-t_{0}\right) a\right)^{2}}{2\left(t-t_{0}\right)}} \tag{245}
\end{align*}
$$

The large deviations of the intensive local time $a=\frac{A-A_{0}}{t-t_{0}} \in[0,+\infty[$

$$
\begin{equation*}
P\left(x, A=A_{0}+\left(t-t_{0}\right) a, t \mid x_{0}, A_{0}, t_{0}\right) \underset{\left(t-t_{0}\right) \rightarrow+\infty}{\propto} e^{-\left(t-t_{0}\right) I(a)} \tag{246}
\end{equation*}
$$

thus involve the rate function 95 ]

$$
\begin{equation*}
I(a)=-\mu a+\frac{\mu^{2}}{2}+\frac{a^{2}}{2}=\frac{(a-\mu)^{2}}{2} \quad \text { for } \quad a \in[0,+\infty[ \tag{247}
\end{equation*}
$$

The boundary value $I(a=0)$ at the origin $a=0$

$$
\begin{equation*}
I(a=0)=\frac{\mu^{2}}{2} \tag{248}
\end{equation*}
$$

governs the decay of the survival probability of Eq. 97 .
The equilibrium value $a_{e q}$ of Eq. 95 where the rate function $I(a)$ of Eq. 247 and its first derivative $I^{\prime}(a)$ vanish (Eq. 95 is simply $a_{e q}=\mu$. It coincides with the value $G_{e q}(0)$ of the equilibrium distribution of $G_{e q}(x)$ of Eq. 216 at the origin $x=0$ as it should

$$
\begin{equation*}
a_{e q}=\mu=G_{e q}(0) \tag{249}
\end{equation*}
$$

If one includes the prefactors, the leading order of the regular contribution of Eq. 245 reads

$$
\begin{equation*}
P^{\text {Regular }}\left(x, A=A_{0}+\left(t-t_{0}\right) a, t \mid x_{0}, A_{0}, t_{0}\right) \underset{\left(t-t_{0}\right) \rightarrow+\infty}{\simeq} \frac{a}{\sqrt{2 \pi\left(t-t_{0}\right)}} e^{-(\mu+a)|x|+(\mu-a)\left|x_{0}\right|} e^{-\left(t-t_{0}\right) I(a)} \tag{250}
\end{equation*}
$$

The agreement with the general formula of Eq. 110 can be checked using Eqs 226, 227, 228 as well as Eq. 103

$$
\begin{equation*}
0=a \partial_{s}\left[\sqrt{\mu^{2}+2 s}-\mu\right]-1=\frac{a}{\sqrt{\mu^{2}+2 s}}-1 \tag{251}
\end{equation*}
$$

that leads to the saddle-point

$$
\begin{equation*}
s_{a}=\frac{a^{2}-\mu^{2}}{2} \tag{252}
\end{equation*}
$$

## E. Conditioning towards the position $x_{T}^{*}$ and the local time $A_{T}^{*}$ at the finite time horizon $T$

Let us now apply the framework described in the subsection VIA. Using the explicit joint propagator of Eq. 239

$$
\begin{align*}
& \ln P\left(x_{T}, A_{T}, T \mid x, A, t\right)=\mu\left(|x|-\left|x_{T}\right|\right)-\frac{\mu^{2}}{2}(T-t)-\ln (\sqrt{2 \pi(T-t)}) \\
& +\ln \left(\delta\left(A_{T}-A\right)\left[e^{-\frac{\left(x_{T}-x\right)^{2}}{2(T-t)}}-e^{-\frac{\left(\left|x_{T}\right|+|x|\right)^{2}}{2(T-t)}}\right]+\theta\left(A_{T}>A\right)\left(\frac{|x|+\left|x_{T}\right|+A_{T}-A}{T-t}\right) e^{\mu\left(A_{T}-A\right)-\frac{\left(|x|+\left|x_{T}\right|+A_{T}-A\right)^{2}}{2(T-t)}}\right) \tag{253}
\end{align*}
$$

one obtains the conditioned drift of Eq. 117

$$
\begin{align*}
& \mu_{T}^{\left[x_{T}^{*}, A_{T}^{*}\right]}(x, A, t)=-\mu \operatorname{sgn}(x)+\partial_{x} \ln P\left(x_{T}^{*}, A_{T}^{*}, T \mid x, A, t\right) \\
& =\partial_{x} \ln \left(\delta\left(A_{T}^{*}-A\right)\left[e^{-\frac{\left(x_{T}^{*}-x\right)^{2}}{2(T-t)}}-e^{-\frac{\left(\left|x_{T}^{*}\right|+|x|\right)^{2}}{2(T-t)}}\right]+\theta\left(A_{T}^{*}>A\right)\left(\frac{|x|+\left|x_{T}^{*}\right|+A_{T}^{*}-A}{T-t}\right) e^{\mu\left(A_{T}^{*}-A\right)-\frac{\left(|x|+\left|x_{T}^{*}\right|+A_{T}^{*}-A\right)^{2}}{2(T-t)}}\right) \tag{254}
\end{align*}
$$

It actually coincides with the conditioned drift of Eq. 199 with its two regions of Eqs 200 and 201

$$
\begin{align*}
& \mu_{T}^{\left[x_{T}^{*}, A_{T}^{*}\right]}\left(x, A<A_{T}^{*}, t\right)=\operatorname{sgn}(x)\left[\frac{1}{\left|x_{T}^{*}\right|+|x|+A_{T}^{*}-A}-\frac{\left|x_{T}^{*}\right|+|x|+A_{T}^{*}-A}{T-t}\right] \\
& \mu_{T}^{\left[x_{T}^{*}, A_{T}^{*}\right]}\left(x, A=A_{T}^{*}, t\right)=\frac{\frac{x_{T}^{*}-x}{T-t} e^{-\frac{\left(x_{T}^{*}-x\right)^{2}}{2(T-t)}}+\frac{x+\operatorname{sgn}(x)\left|x_{T}^{*}\right|}{T-t} e^{-\frac{\left(\left|x_{T}^{*}\right|+|x|\right)^{2}}{2(T-t)}}}{e^{-\frac{\left(x_{T}^{*}-x\right)^{2}}{2(T-t)}}-e^{-\frac{\left(\left|x_{T}^{*}\right|+|x|\right)^{2}}{2(T-t)}}} \tag{255}
\end{align*}
$$

Corresponding stochastic trajectories have already been shown on figure 2

## F. Conditioning towards the local time $A_{T}^{*}$ at the finite time horizon $T$

Let us now apply the framework described in the subsection VIB.
(i) In the region $A_{0}=0 \leq A<A_{T}^{*}$ where the local time $A$ has not yet reached its conditioned final value $A_{T}^{*}$, the conditioned drift of Eq. 127 involves the regular contribution $\Pi^{\text {Regular }}\left(A_{T}^{*}, T \mid x, A, t\right)$ of Eq. 243

$$
\begin{align*}
& \mu_{T}^{\left[A_{T}^{*}\right]}\left(x, A<A_{T}^{*}, t\right)=-\mu \operatorname{sgn}(x)+\partial_{x} \ln \Pi^{\text {Regular }}\left(A_{T}^{*}, T \mid x, A, t\right) \\
& =-\mu \operatorname{sgn}(x)+\partial_{x} \ln \left[e^{-\frac{\left(|x|+A_{T}^{*}-A-\mu(T-t)\right)^{2}}{2(T-t)}}\left(\sqrt{\frac{2}{\pi(T-t)}}-\mu e^{\frac{\left(|x|+A_{T}^{*}-A+\mu(T-t)\right)^{2}}{2(T-t)}} \operatorname{erfc}\left(\frac{|x|+\mu(T-t)+A_{T}^{*}-A}{\sqrt{2(T-t)}}\right)\right)\right] \\
& =-\operatorname{sgn}(x) \frac{|x|+A_{T}^{*}-A}{T-t}+\partial_{x} \ln \left(1-\mu \int_{0}^{+\infty} d y e^{-\frac{y^{2}}{2(T-t)}-y\left[\mu+\frac{\left(|x|+A_{T}^{*}-A\right)}{(T-t)}\right]}\right) \\
& =\mu \operatorname{sgn}(x)-\frac{2}{(T-t)} \frac{\left(|x|+A_{T}^{*}-A\right)}{2-\mu \sqrt{2 \pi(T-t)} e^{\frac{\left(|x|+\mu(T-t)+A_{T}^{*}-A\right)^{2}}{2(T-t)}} \operatorname{erfc}\left(\frac{|x|+\mu(T-t)+A_{T}^{*}-A}{\sqrt{2(T-t)}}\right)} \operatorname{sgn}(x) \tag{256}
\end{align*}
$$

(ii) In the region $A=A_{T}^{*}$ where the local time $A$ has already reached its conditioned final value $A_{T}^{*}$, and where the position $x$ cannot visit the origin $x=0$ anymore, the conditioned drift of Eq. 128 involves the survival probability $S^{a b s}(T \mid x, t)$ of Eq. 234

$$
\begin{align*}
\mu_{T}^{\left[A_{T}^{*}\right]}\left(x, A=A_{T}^{*}, t\right) & =-\mu \operatorname{sgn}(x)+\partial_{x} \ln S^{a b s}(T \mid x, t) \\
& =-\mu \operatorname{sgn}(x)+\partial_{x} \ln \left(\frac{e^{-\frac{\mu^{2}}{2}(T-t)+\mu|x|-\frac{x^{2}}{2(T-t)}}}{\sqrt{2 \pi(T-t)}} \int_{-\infty}^{+\infty} d y e^{-\mu|y|-\frac{y^{2}}{2(T-t)}}\left[e^{\frac{y x}{(T-t)}}-e^{-\frac{|y x|}{(T-t)}}\right]\right) \\
& =\partial_{x} \ln \left(\int_{-\infty}^{+\infty} d y e^{-\mu|y|-\frac{y^{2}}{2(T-t)}}\left[e^{\frac{y x}{(T-t)}}-e^{-\frac{|y x|}{(T-t)}}\right]\right) \\
& =2 \sqrt{\frac{2}{\tau}} \frac{\operatorname{sgn}(x)\left(\frac{1}{\sqrt{\pi}}-e^{\frac{(|x|+\mu \tau)^{2}}{2 \tau}} \mathcal{F}\left(\frac{|x|+\mu \tau}{\sqrt{2 \tau}}\right)\right)+e^{\frac{(x-\mu \tau)^{2}}{2 \tau}}\left(e^{2 \mu x} \mathcal{F}\left(\frac{x+\mu \tau}{\sqrt{2 \tau}}\right)-\mathcal{F}\left(\frac{\mu \tau-x}{\sqrt{2 \tau}}\right)\right)}{e^{\frac{(x+\mu)^{2}}{2 \tau}} \operatorname{erfc}\left(\frac{x+\mu \tau}{\sqrt{2 \tau}}\right)+e^{\frac{(x-\mu \tau)^{2}}{2 \tau}} \operatorname{erfc}\left(\frac{\mu \tau-x}{\sqrt{2 \tau}}\right)-2 e^{\frac{(|x|+\mu \tau)^{2}}{2 \tau}} \operatorname{erfc}\left(\frac{|x|+\mu \tau}{\sqrt{2 \tau}}\right)} \tag{257}
\end{align*}
$$

where $\mathcal{F}(x)=x \operatorname{erfc}(x)$ and $\tau=T-t$.
In the second region, the asymptotic behavior near the origin $x \rightarrow 0$ is

$$
\begin{equation*}
\mu_{T}^{\left[A_{T}^{*}\right]}\left(x, A=A_{T}^{*}, t\right) \underset{x \rightarrow 0}{\simeq} \frac{1}{x}+x\left(\frac{1}{T-t}+\frac{\mu^{2}}{3}+\frac{1}{-3(T-t)+3 \mu \sqrt{\frac{\pi(T-t)^{3}}{2}} e^{\frac{(T-t) \mu^{2}}{2}} \operatorname{erfc}\left(\mu \sqrt{\frac{T-t}{2}}\right)}\right) \tag{258}
\end{equation*}
$$

Again, the $1 / x$ term prevents the process from crossing the origin and the local time cannot increase anymore.

## G. Case $\mu=0$ : Conditioning towards the intensive value $a^{*}=\frac{A_{T}^{*}}{T}$ in the limit $T \rightarrow+\infty$

A direct consequence of Eq. 255 is that the conditioning towards the intensive value $a^{*}=\frac{A_{T}^{*}}{T}$ in the limit $T \rightarrow+\infty$ will give exactly the same conditioned drift of Eq. 208

$$
\begin{equation*}
\mu_{T}^{\left[T a^{*}\right]}\left(x, A<T a^{*}, t\right) \underset{T \rightarrow+\infty}{\simeq}-\operatorname{sgn}(x) a^{*} \equiv \mu_{\infty}^{\left[a^{*}\right]}(x) \tag{259}
\end{equation*}
$$

The agreement with the general formula of Eq. 135 for the conditioned drift $\mu_{\infty}^{\left[a^{*}\right]}(x)$ can be checked by Eq. 226

$$
\begin{equation*}
\tilde{G}_{s}(0 \mid x)=\frac{e^{\left(\mu-\sqrt{\mu^{2}+2 s}\right)|x|}}{\sqrt{\mu^{2}+2 s}-\mu} \tag{260}
\end{equation*}
$$

and the saddle-point value $s_{a^{*}}=\frac{\left(a^{*}\right)^{2}-\mu^{2}}{2}$ of Eq. 252 to obtain

$$
\begin{align*}
\mu_{\text {Bridge }\left[a^{*}\right]}^{*}(x) & =-\mu \operatorname{sgn}(x)+\partial_{x} \ln \hat{G}_{s_{a^{*}}}(0 \mid x)=-\mu \operatorname{sgn}(x)+\partial_{x}\left(\left(\mu-\sqrt{\mu^{2}+2 s_{a^{*}}}\right)|x|-\ln \left(\sqrt{\mu^{2}+2 s_{a^{*}}}-\mu\right)\right) \\
& =-\sqrt{\mu^{2}+2 s_{a^{*}}} \operatorname{sgn}(x)=-\operatorname{sgn}(x) a^{*} \tag{261}
\end{align*}
$$

As explained in the Appendices, this result can be also recovered via the appropriate canonical conditioning leading to Eq. B20.


Figure 5. Examples of realization of the Brownian process with alternating drift conditioned to have the final local time $A_{T}^{*}=0.2$ at the finite time horizon $T=1$ (see the conditioned drift of Eqs. 256 and 257). For each trajectory, the associated local time $A(t)$ is shown as a function of the time $t \in[0, T]$. The process can end at any final position $x_{T}$, while the initial position is $x_{0}=-0.5$ here. Observe that the blue realization reaches the desired local time value only at the very end of the time-window (as shown in the encapsulated plot). The time step used in the discretization is $d t=10^{-5}$.

## IX. CONCLUSION

In the present paper, we have analyzed the conditioning of a diffusion process $X(t)$ of drift $\mu(x)$ and of diffusion coefficient $D=1 / 2$ with respect to its local time $A_{x=0}(t)=A(t)$ at the origin $x=0$. Our goal was to construct various conditioned joint processes $\left[X^{*}(t), A^{*}(t)\right]$ satisfying certain conditions involving the local time $A^{*}(T)$ at the finite time horizon $T$ or in the limit of the infinite time horizon $T \rightarrow+\infty$.
For the case of the finite time horizon $T$, we have studied :
(a) the conditioning towards the final position $X^{*}(T)$ and towards the final local time $A^{*}(T)$. In other words, this case corresponds to conditioning a generalized Brownian bridge with respect to its local time at the final time.
(b) the conditioning towards the final local time $A^{*}(T)$ alone without any condition on the final position $X^{*}(T)$. In the limit of the infinite time horizon $T \rightarrow+\infty$, we have analyzed :
(1) the conditioning towards the finite asymptotic local time $A_{\infty}^{*}<+\infty$
(2) the conditioning towards the intensive local time $a^{*}$ corresponding to the extensive behavior $A_{T} \simeq T a^{*}$, that we have compared in the Appendices to the appropriate 'canonical conditioning' based on the generating function of the local time in the regime of large deviations.
This general construction has been applied to generate various constrained stochastic trajectories for three unconditioned diffusions with different recurrence/transience properties :
(i) as simplest example of a transient diffusion, we have considered the uniform strictly positive drift $\mu(x)=\mu>0$
(ii) as simplest example of a diffusion converging towards an equilibrium, we have chosen the drift $\mu(x)=-\mu \operatorname{sgn}(x)$ of parameter $\mu>0$
(iii) as simplest example of a recurrent diffusion that does not converge towards an equilibrium, we have focused on the Brownian motion without drift $\mu=0$.

The generalization of the present work to analyze the conditioning with respect to two local times is described in [100].

## Appendix A: Notion of canonical conditioned process $X_{p}^{*}(t)$ of parameter $p$ conjugated to the local time

As recalled in the Introduction, the 'canonical conditioning' based on generating functions of time-additive observables for Markov processes over a large time-window $T$ has recently been used extensively in the field of non-equilibrium statistical physics 45-90. Its physical meaning comes from the equivalence at the level of the large deviations for large time $T$ between the 'canonical conditioning' and the 'microcanonical conditioning' (see the two detailed papers [68, 69 and the HDR thesis [70] with references therein). In this Appendix, it is thus interesting to analyze the 'canonical conditioning' of parameter $p$ conjugated to the local time increment $\left[A(t)-A\left(t_{0}\right)\right]$ and to compare with the 'microcanonical conditioning' described in the main text.

## 1. Canonical conditioned process $X_{p}^{*}(t)$ of parameter $p$ based on the Laplace transform $\tilde{P}_{p}\left(x, t \mid x_{0}, t_{0}\right)$

The canonical conditioning is based on the Laplace transform $\tilde{P}_{p}\left(x, t \mid x_{0}, t_{0}\right)$ of Eq. 35 with respect to the local time increment $\left(A-A_{0}\right)$, where the Laplace parameter $p$ conjugated to the local time increment $\left(A-A_{0}\right)$ is fixed.

For the bridge conditioned to end at the position $x_{T}^{*}$ at the time horizon $T$, the conditioned probability for the position $x$ at an interior time $t \in[0, T]$ reads

$$
\begin{equation*}
P_{T}^{\left[x_{T}^{*} ; p\right]}(x, t)=\frac{\tilde{P}_{p}\left(x_{T}, T \mid x, t\right) \tilde{P}_{p}\left(x, t \mid x_{0}, 0\right)}{\tilde{P}_{p}\left(x_{T}, T \mid x_{0}, 0\right)} \tag{A1}
\end{equation*}
$$

The corresponding Ito dynamics for the conditioned process $X_{p}^{*}(t)$ of parameter $p$

$$
\begin{equation*}
d X_{p}^{*}(t)=\mu_{p}^{*}\left(X_{p}^{*}(t), t\right) d t+d B(t) \tag{A2}
\end{equation*}
$$

involves the conditioned drift

$$
\begin{equation*}
\mu_{T}^{\left[x_{T}^{*} ; p\right]}(x, t)=\mu(x)+\partial_{x} \ln \tilde{P}_{p}\left(x_{T}, T \mid x, t\right) \tag{A3}
\end{equation*}
$$

## 2. Properties of the $p$-deformed propagator $\tilde{P}_{p}\left(x, t \mid x_{0}, t_{0}\right)$

The forward dynamics of the propagator $\tilde{P}_{p}\left(x, t \mid x_{0}, t_{0}\right)$ given by the Feynman-Kac formula of Eq. 36

$$
\begin{equation*}
\partial_{t} \tilde{P}_{p}\left(x, t \mid x_{0}, t_{0}\right)=-p \delta(x) \tilde{P}_{p}\left(x, t \mid x_{0}, t_{0}\right)-\partial_{x}\left[\mu(x) \tilde{P}_{p}\left(x, t \mid x_{0}, t_{0}\right)\right]+\frac{1}{2} \partial_{x}^{2} \tilde{P}_{p}\left(x, t \mid x_{0}, t_{0}\right) \equiv \mathcal{F}_{p} \tilde{P}_{p}\left(x, t \mid x_{0}, t_{0}\right) \tag{A4}
\end{equation*}
$$

involves the generator

$$
\begin{equation*}
\mathcal{F}_{p} \equiv-p \delta(x)-\partial_{x} \mu(x)+\frac{1}{2} \partial_{x}^{2} \tag{A5}
\end{equation*}
$$

Its adjoint

$$
\begin{equation*}
\mathcal{F}_{p}^{\dagger}=-p \delta(x)+\mu(x) \partial_{x}+\frac{1}{2} \partial_{x}^{2} \tag{A6}
\end{equation*}
$$

governs the backward dynamics of the propagator $\tilde{P}_{p}\left(x_{T}, T \mid x, t\right)$

$$
\begin{equation*}
\partial_{t} \tilde{P}_{p}\left(x_{T}, T \mid x, t\right)=\mathcal{F}_{p}^{\dagger} \tilde{P}_{p}\left(x_{T}, T \mid x, t\right)=-p \delta(x) \tilde{P}_{p}\left(x_{T}, T \mid x, t\right)+\mu(x) \partial_{x} \tilde{P}_{p}\left(x_{T}, T \mid x, t\right)+\frac{1}{2} \partial_{x}^{2} \tilde{P}_{p}\left(x_{T}, T \mid x, t\right) \tag{A7}
\end{equation*}
$$

a. Physical meaning of the $p$-deformed Fokker-Planck dynamics

With respect to the dynamics of Eq. 6 corresponding to $p=0$, the additional term in the forward dynamics of Eq. A4 corresponds to the killing rate of amplitude $p>0$ localized at the origin $x=0$

$$
\begin{equation*}
k(x)=p \delta(x) \tag{A8}
\end{equation*}
$$

It is however also interesting to consider the case $p<0$ in the Laplace transform of Eq. 35: then the additional term in the Feynman-Kac formula of Eq. 36 corresponds instead to the reproducing rate of amplitude $(-p)>0$ localized at the origin $x=0$

$$
\begin{equation*}
r(x)=(-p) \delta(x) \tag{A9}
\end{equation*}
$$

## b. Physical meaning of the associated p-deformed quantum Hamiltonian $H_{p}$

Via the similarity transformation analogous to Eq. 10 that involves the potential $U(x)$ of Eq. 9

$$
\begin{equation*}
\tilde{P}_{p}\left(x, t \mid x_{0}, t_{0}\right)=e^{-\frac{U(x)}{2}} \psi\left(x, t \mid x_{0}, t_{0}\right) e^{\frac{U\left(x_{0}\right)}{2}}=e^{\int_{x_{0}}^{x} d y \mu(y)} \psi_{p}\left(x, t \mid x_{0}, t_{0}\right) \tag{A10}
\end{equation*}
$$

the forward dynamics of Eq. A4 for $\tilde{P}_{p}\left(x, t \mid x_{0}, t_{0}\right)$ translates into the Euclidean Schrödinger Equation for $\psi_{p}\left(x, t \mid x_{0}, t_{0}\right)$

$$
\begin{equation*}
-\partial_{t} \psi_{p}\left(x, t \mid x_{0}, t_{0}\right)=H_{p} \psi_{p}\left(x, t \mid x_{0}, t_{0}\right) \tag{A11}
\end{equation*}
$$

With respect to the Hamiltonian $H$ of Eq. 12 involving the potential $V(x)$ of Eq. 13 , the quantum Hamiltonian $H_{p}$ contains an additional delta potential of amplitude $p$ localized at the origin $x=0$

$$
\begin{equation*}
H_{p}=H+p \delta(x)=-\frac{1}{2} \partial_{x}^{2}+V(x)+p \delta(x)=-\frac{1}{2} \partial_{x}^{2}+\frac{\mu^{2}(x)}{2}+\frac{\mu^{\prime}(x)}{2}+p \delta(x) \tag{A12}
\end{equation*}
$$

So the case $p>0$ associated to the killing rate of Eq. A8 corresponds to an additional repulsive delta potential, while the case $p<0$ associated to the reproducing rate of Eq. A9 corresponds to an additional attractive delta potential.

## 3. Canonical conditioning for large horizon $T$ when the Hamiltonian $H_{p}$ has a normalizable ground-state

a. Propagator $\tilde{P}_{p}\left(x, t \mid x_{0}, t_{0}\right)$ for large time $\left(t-t_{0}\right)$ when the $p$-deformed Hamiltonian $H_{p}$ has a normalizable ground-state

When the $p$-deformed quantum Hamiltonian $H_{p}$ has a normalizable ground-state $\phi_{p}^{G S}(x)$ of energy $E_{p}$

$$
\begin{equation*}
H_{p} \phi_{p}^{G S}(x)=E_{p} \phi_{p}^{G S}(x) \tag{A13}
\end{equation*}
$$

the ground state can be chosen real and positive $\phi_{p}^{G S}(x) \geq 0$ with the normalization

$$
\begin{equation*}
\left\langle\phi_{p}^{G S} \mid \phi_{p}^{G S}\right\rangle=\int_{-\infty}^{+\infty} d x\left[\phi_{p}^{G S}(x)\right]^{2}=1 \tag{A14}
\end{equation*}
$$

This ground-state $\phi_{p}^{G S}(x)$ and its energy $E_{p}$ determine the leading asymptotic behavior of the quantum propagator

$$
\begin{equation*}
\psi_{p}\left(x, t \mid x_{0}, t_{0}\right) \underset{\left(t-t_{0}\right) \rightarrow+\infty}{\simeq} e^{-\left(t-t_{0}\right) E_{p}} \phi_{p}^{G S}(x) \phi_{p}^{G S}\left(x_{0}\right) \tag{A15}
\end{equation*}
$$

The corresponding asymptotic behavior of the propagator $\tilde{P}_{p}\left(x, t \mid x_{0}, t_{0}\right)$ given by the similarity transformation of Eq. A10 reads

$$
\begin{align*}
\tilde{P}_{p}\left(x, t \mid x_{0}, t_{0}\right) & =e^{-\frac{U(x)}{2}} \psi_{p}\left(x, t \mid x_{0}, t_{0}\right) e^{\frac{U\left(x_{0}\right)}{2}} \\
& \simeq  \tag{A16}\\
& \simeq\left(t-t_{0}\right) \rightarrow+\infty
\end{align*} e^{-\left(t-t_{0}\right) E_{p}}\left[e^{-\frac{U(x)}{2}} \phi_{p}^{G S}(x)\right]\left[e^{\frac{U\left(x_{0}\right)}{2}} \phi_{p}^{G S}\left(x_{0}\right)\right] \equiv e^{-\left(t-t_{0}\right) E_{p}} r_{p}(x) l_{p}\left(x_{0}\right), ~ l
$$

where

$$
\begin{align*}
& r_{p}(x) \equiv e^{-\frac{U(x)}{2}} \phi_{p}^{G S}(x)=e^{\int_{0}^{x} d y \mu(y)} \phi_{p}^{G S}(x) \\
& l_{p}\left(x_{0}\right) \equiv e^{\frac{U\left(x_{0}\right)}{2}} \phi_{p}^{G S}\left(x_{0}\right)=e^{-\int_{0}^{x_{0}} d y \mu(y)} \phi_{p}^{G S}\left(x_{0}\right) \tag{A17}
\end{align*}
$$

correspond to the positive right and left eigenvectors of the generator of Eq. A5 associated to the eigenvalue $\left(-E_{p}\right)$

$$
\begin{align*}
& -E_{p} r_{p}(x)=\mathcal{F}_{p} r_{p}(x)=-p \delta(x) r_{p}(x)-\partial_{x}\left[\mu(x) r_{p}(x)\right]+\frac{1}{2} \partial_{x}^{2} r_{p}(x) \\
& -E_{p} l_{p}(x)=\mathcal{F}_{p}^{\dagger} l_{p}(x)=-p \delta(x) l_{p}(x)+\mu(x) \partial_{x} l_{p}(x)+\frac{1}{2} \partial_{x}^{2} l_{p}(x) \tag{A18}
\end{align*}
$$

The normalization inherited from Eq. A14 reads

$$
\begin{equation*}
\left\langle l_{p} \mid r_{p}\right\rangle=\int_{-\infty}^{+\infty} d x r_{k}(x) l_{k}(x)=\int_{-\infty}^{+\infty} d x\left[\phi_{p}^{G S}(x)\right]^{2}=1 \tag{A19}
\end{equation*}
$$

For the double Laplace transform $\hat{\tilde{P}}_{s, p}\left(x \mid x_{0}\right)$ of Eq. 37 , the asymptotic behavior of Eq. A16 for large $\left(t-t_{0}\right)$ means that $\hat{\tilde{P}}_{s, p}\left(x \mid x_{0}\right)$ exists for $\left.s \in\right]-E_{p},+\infty\left[\right.$ with the following pole singularity for $s \rightarrow\left(-E_{p}\right)^{+}$

$$
\begin{equation*}
\hat{\tilde{P}}_{s, p}\left(x \mid x_{0}\right) \underset{s \rightarrow\left(-E_{p}\right)^{+}}{\simeq} \int_{t_{0}}^{+\infty} d t e^{-\left(s+E_{p}\right)\left(t-t_{0}\right)} r_{p}(x) l_{p}\left(x_{0}\right)=\frac{r_{p}(x) l_{p}\left(x_{0}\right)}{s+E_{p}} \tag{A20}
\end{equation*}
$$

b. Simplifications for the canonical conditioning for large horizon $T$ when $H_{p}$ has a normalizable ground-state

When the $p$-deformed quantum Hamiltonian $H_{p}$ has a normalizable ground-state $\phi_{p}^{G S}(x)$, the asymptotic behavior of Eq. A16 can be plugged into the three propagators of Eq. A1 to obtain that the conditioned density at any interior time $0 \ll t \ll T$

$$
\begin{equation*}
P_{T}^{\left[x_{T}^{*} ; p\right]}(x, t) \underset{0 \ll t \ll T}{\sim} \frac{e^{-E_{p}(T-t)} r_{p}\left(x_{T}\right) l_{p}(x) e^{-E_{p} t} r_{p}(x) l_{p}\left(x_{0}\right)}{e^{-E_{p} T} r_{p}\left(x_{T}\right) l_{p}\left(x_{0}\right)}=l_{p}(x) r_{p}(x) \equiv P_{p}^{*}(x) \tag{A21}
\end{equation*}
$$

does not depend on the interior time $t$ anymore. This steady conditioned density $P_{p}^{*}(x)$ only involves the product of the left and right eigenvectors of Eq. A17 and can be thus rewritten as the square of the ground-state $\phi_{p}^{G S}(x)$ of the quantum Hamiltonian $H_{p}$ alone

$$
\begin{equation*}
P_{p}^{*}(x)=l_{p}(x) r_{p}(x)=\left[\phi_{p}^{G S}(x)\right]^{2} \tag{A22}
\end{equation*}
$$

The corresponding conditioned drift of Eq. A3 is also independent of the interior time $t$

$$
\begin{equation*}
\mu_{T}^{\left[x_{T}^{*} ; p\right]}(x, t) \underset{0 \ll t \ll T}{\simeq} \mu(x)+\partial_{x} \ln \left[e^{-E_{p}(T-t)} r_{p}\left(x_{T}\right) l_{p}(x)\right]=\mu(x)+\partial_{x} \ln \left[l_{p}(x)\right] \equiv \mu_{p}^{*}(x) \tag{A23}
\end{equation*}
$$

Since $\mu_{p}^{*}(x)$ involves the initial drift $\mu(x)$ and the logarithmic derivative of the left eigenvector $l_{p}(x)$ of Eq. A17, it can be rewritten in terms of the logarithmic derivative of the ground-state $\phi_{p}^{G S}(x)$ of the quantum Hamiltonian $H_{p}$ alone

$$
\begin{equation*}
\mu_{p}^{*}(x)=\mu(x)+\partial_{x} \ln \left[e^{-\int_{0}^{x} d y \mu(y)} \phi_{p}^{G S}(x)\right]=\partial_{x} \ln \left[\phi_{p}^{G S}(x)\right] \tag{A24}
\end{equation*}
$$

## c. Conclusion

In summary, when the $p$-deformed quantum Hamiltonian $H_{p}$ has a normalizable ground-state $\phi_{p}^{G S}(x)$, then the canonical conditioned process $X_{p}^{*}(t)$ becomes simple for large time horizon $T \rightarrow+\infty$ in the region of interior times $0 \ll t \ll T$ : its steady density $P_{p}^{*}(x)$ of Eq. A22 and the conditioned drift $\mu_{p}^{*}(x)$ of Eq. A24 are time-independent and involve only the normalizable ground-state $\phi_{p}^{G S}(x)$ of $H_{p}$.

The physical meaning of this conditioned process $X_{p}^{*}(t)$ depends on whether $H=H_{p=0}$ has also a normalizable ground-state or not :
(i) the case where $H=H_{p=0}$ has also a normalizable ground-state $\phi_{p=0}^{G S}(x)$, i.e. where the unconditioned process $X(t)$ converges towards an equilibrium state $G_{e q}(x)$ is discussed in Appendix B
(ii) the case where $H=H_{p=0}$ has no normalizable ground-state is discussed in Appendix $\mathbb{C}$.

## Appendix B: Canonical conditioning when the unconditioned process $X(t)$ has an equilibrium state

In this Appendix, we consider the case where the unconditioned process $X(t)$ converges towards the equilibrium state $G_{e q}(x)$ of Eq. 28, so that the quantum Hamiltonian $H$ of Eq. 12 has a normalizable ground-state $\phi^{G S}(x)$ given by Eq. 25. Then the ground-state $\phi_{p}^{G S}(x)$ of the Hamiltonian $H_{p}$ of Eq. A12 can be interpreted as a deformation of this unperturbed ground-state $\phi^{G S}(x)=\phi_{0}^{G S}(x)$, with the following consequences for the physical meaning of the canonical conditioning of parameter $p$

1. Link between $E_{p}$ and the rate function $I(a)$ governing the large deviations of the intensive local time $a$

The ground-state energy $E_{p}$ of the $p$-deformed Hamiltonian $H_{p}$ governs the asymptotic behavior of Eq. A16 of the Laplace transform $\tilde{P}_{p}\left(x, T \mid x_{0}, 0\right)$ of the local time $A_{T}$ introduced in Eq. 35

$$
\begin{align*}
\tilde{P}_{p}\left(x, T \mid x_{0}, 0\right) & \equiv \int_{0}^{+\infty} d A e^{-p A} P\left(x, A, T \mid x_{0}, A_{0}, 0\right) \\
& \underset{T \rightarrow+\infty}{\sim} e^{-T E_{p}}\left[e^{-\frac{U(x)}{2}} \phi_{p}^{G S}(x)\right]\left[e^{\frac{U\left(x_{0}\right)}{2}} \phi_{p}^{G S}\left(x_{0}\right)\right] \equiv e^{-T E_{p}} r_{p}(x) l_{p}\left(x_{0}\right) \tag{B1}
\end{align*}
$$

The large deviations properties of Eq. 94 can also be used to evaluate the generating function of $\left(A_{T}-A_{0}\right)=T a$ via the saddle-point method for large $T$

$$
\begin{equation*}
\left\langle e^{-p\left(A_{T}-A_{0}\right)}\right\rangle=\left\langle e^{-p T a}\right\rangle=\int_{0}^{+\infty} d a e^{-p T a} P_{T}(a) \underset{T \rightarrow+\infty}{\simeq} \int_{0}^{+\infty} d a e^{-T[p a+I(a)]} \underset{T \rightarrow+\infty}{\simeq} e^{-T E_{p}} \tag{B2}
\end{equation*}
$$

So the energy $E_{p}$ governing the asymptotic behavior of Eq. B1 for the propagator $\tilde{P}_{p}\left(x, t \mid x_{0}, t_{0}\right)$ is the Legendre transform of the rate function $I(a)$

$$
\begin{align*}
p a+I(a) & =E_{p} \\
p+I^{\prime}(a) & =0 \tag{B3}
\end{align*}
$$

while the reciprocal Legendre transform reads

$$
\begin{align*}
I(a) & =E_{p}-p a \\
a & =\frac{d E_{p}}{d p} \tag{B4}
\end{align*}
$$

As a consequence, the canonical conditioning of parameter $p$ discussed in section A 3b can be considered as asymptotically equivalent for large $T$ to the microcanonical conditioning of subsection VIB 2 towards the intensive local time $a_{p}^{*}=\frac{d E_{p}}{d p}$ corresponding to the Legendre value of Eq. B4. Note that this relation $a_{p}^{*}=\frac{d E_{p}}{d p}$ has a very simple interpretation via the first-order perturbation theory for the energy $E_{p}$ of the ground state $\phi_{p}^{G S}(x)$ in quantum mechanics when the parameter $p$ is changed into $(p+\epsilon)$

$$
\begin{equation*}
a_{p}^{*}=\frac{d E_{p}}{d p}=\lim _{\epsilon \rightarrow 0}\left(\frac{E_{p+\epsilon}-E_{p}}{\epsilon}\right)=\left\langle\phi_{p}^{G S}\right| \delta(x)\left|\phi_{p}^{G S}\right\rangle=\left[\phi_{p}^{G S}(x=0)\right]^{2}=P_{p}^{*}(x=0) \tag{B5}
\end{equation*}
$$

that corresponds to the conditioned steady state $P_{p}^{*}(x)$ of Eq. A22 at the origin $x=0$.

## Relation between the Laplace parameter $p$ and the time-Laplace parameter $s$ in the large deviations analysis

The comparison between
(i) the Legendre transform of Eqs. B3, B4 between $I(a)$ and $E_{p}$
(ii) the quasi-Legendre transform of Eqs. 100,103 and 106 between $I(a)$ and $\frac{1}{\hat{G}_{s}(0 \mid 0)}$
allows to eliminate the variable $a$ to obtain the following relations between the Laplace parameter $p$ and the time-Laplace parameter $s$

$$
\begin{align*}
& s=-E_{p} \\
& p=-\frac{1}{\hat{G}_{s}(0 \mid 0)} \tag{B6}
\end{align*}
$$

2. Example of the drift $\mu(x)=-\mu \operatorname{sgn}(x)$ with $\mu>0$
a. Computation of the energy $E_{p}$ via the Legendre transform of the rate function $I(a)$

For the explicit rate function $I(a)$ of Eq. 247, the Legendre transform of Eq. B3 yields the following properties.
(i) The microcanonical conditioning to the intensive local time $a^{*}$ is asymptotically equivalent in the thermodynamic limit $T \rightarrow+\infty$ to the canonical conditioning of parameter

$$
\begin{equation*}
p=-I^{\prime}\left(a^{*}\right)=\mu-a^{*} \tag{B7}
\end{equation*}
$$

so that the domain $a^{*} \in[0,+\infty[$ of definition for the intensive local time corresponds to the following domain for the Laplace variable $p$

$$
\begin{equation*}
p \in]-\infty, \mu] \tag{B8}
\end{equation*}
$$

Reciprocally, the canonical conditioning of parameter $p$ is asymptotically equivalent in the thermodynamic limit $T \rightarrow+\infty$ to the microcanonical conditioning of the intensive local time

$$
\begin{equation*}
a_{p}^{*}=\mu-p \tag{B9}
\end{equation*}
$$

(ii) The energy $E_{p}$ of Eq. B3 reads using Eq. 247 and Eq. B7

$$
\begin{equation*}
E_{p}=p a+I(a)=p a+\frac{(a-\mu)^{2}}{2}=p(\mu-p)+\frac{p^{2}}{2}=p \mu-\frac{p^{2}}{2} \tag{B10}
\end{equation*}
$$

## b. Direct analysis of the ground-state of the p-deformed Hamiltonian $H_{p}$

For the drift $\mu(x)=-\mu \operatorname{sgn}(x)$ with $\mu>0$, the Hamiltonian $H=H^{[\mu]}$ of Eq. 220 of parameter $\mu>0$

$$
\begin{equation*}
H^{[\mu]}=-\frac{1}{2} \partial_{x}^{2}+\frac{\mu^{2}}{2}-\mu \delta(x) \tag{B11}
\end{equation*}
$$

has the zero-energy normalized ground-state of Eq. 25 using Eq. 216

$$
\begin{equation*}
\phi^{G S[\mu]}(x)=\sqrt{G_{e q}(x)}=\sqrt{\mu} e^{-\mu|x|} \tag{B12}
\end{equation*}
$$

The $p$-deformed Hamiltonian $H_{p}=H_{p}^{[\mu]}$ of Eq. A12

$$
\begin{align*}
H_{p}^{[\mu]} & =H^{[\mu]}+p \delta(x)=-\frac{1}{2} \partial_{x}^{2}+\frac{\mu^{2}}{2}-(\mu-p) \delta(x) \\
& =-\frac{1}{2} \partial_{x}^{2}+\frac{(\mu-p)^{2}}{2}-(\mu-p) \delta(x)+\frac{\mu^{2}-(\mu-p)^{2}}{2} \\
& \equiv H^{[\mu-p]}+E_{p} \tag{B13}
\end{align*}
$$

can be thus interpreted in the domain $p \in]-\infty, \mu[$ of Eq. $\overline{B 8} 8$ as the sum of :
(i) the Hamiltonian $H^{[\mu-p]}$ of effective drift $(\mu-p)>0$ in Eq. B11, with its zero-energy normalized ground-state of Eq. B12

$$
\begin{equation*}
\phi^{G S[\mu-p]}(x)=\sqrt{\mu-p} e^{-(\mu-p)|x|} \tag{B14}
\end{equation*}
$$

(ii) the remaining constant in Eq. B13

$$
\begin{equation*}
E_{p}=\frac{\mu^{2}-(\mu-p)^{2}}{2}=p \mu-\frac{p^{2}}{2} \tag{B15}
\end{equation*}
$$

that directly represents the ground-state energy of $H_{p}^{[\mu]}$ and that coincides with the alternative analysis of Eq. B15 as it should.

$$
\text { c. Canonical conditioned process } X_{p}^{*}(t) \text { of parameter } p
$$

Since Eq. B 14 is the ground-state of the $p$-deformed Hamiltonian $H_{p}^{[\mu]}$

$$
\begin{equation*}
\phi_{p}^{G S[\mu]}(x)=\phi^{G S[\mu-p]}(x)=\sqrt{\mu-p} e^{-(\mu-p)|x|} \tag{B16}
\end{equation*}
$$

one obtains that the conditioned drift of Eq. A24 reduces to

$$
\begin{equation*}
\mu_{p}^{*}(x)=\partial_{x} \ln \left[\phi_{p}^{G S[\mu]}(x)\right]=-(\mu-p) \operatorname{sgn}(x) \tag{B17}
\end{equation*}
$$

So the canonical conditioning of parameter $p \in]-\infty, \mu[$ simply amounts to change the amplitude $\mu$ of the unconditioned drift $\mu(x)=-\mu \operatorname{sgn}(x)$ into the amplitude $(\mu-p)$. As a consequence, the corresponding conditioned equilibrium state

$$
\begin{equation*}
G_{p}^{e q[\mu]}(x)=G^{e q[\mu-p]}(x)=(\mu-p) e^{-2(\mu-p)|x|} \tag{B18}
\end{equation*}
$$

will produce the following equilibrium value for the intensive local time

$$
\begin{equation*}
a_{p}^{e q[\mu]}=G_{p}^{e q[\mu]}(x=0)=\mu-p \tag{B19}
\end{equation*}
$$

in agreement with the Legendre correspondence of Eq. B7.
Finally, the conditioned drift of Eq. B17 can be rewritten in terms of $a_{p}^{*}=\mu-p$ of Eq. B9 as

$$
\begin{equation*}
\mu_{p}^{*}(x)=-a_{p}^{*} \operatorname{sgn}(x) \tag{B20}
\end{equation*}
$$

in agreement with the microcanonical conditioning of Eq. 261 in the main text.

## Appendix C: Canonical conditioning when the unconditioned process $X(t)$ has no equilibrium state

In this Appendix, we consider the case where the unconditioned process $X(t)$ has no equilibrium state, so that the quantum Hamiltonian $H$ of Eq. 12 has no bound state. For the Hamiltonian $H_{p}$ of Eq. A12 the presence of a bound state depends on the sign of $p$ as follows.
(i) The case $p>0$ corresponds to an additional repulsive delta potential at the origin $x=0$ and will not change the range $] V_{\infty},+\infty\left[\right.$ of the continuous spectrum of $H$, since $H$ and $H_{p}$ have the same potential at $x \rightarrow \pm \infty$ in Eq. 17 .
(ii) The case $p<0$ corresponds to an additional attractive delta potential at the origin $x=0$ that produces a normalizable ground state for $H_{p}$ as we now describe.

## 1. Emergence of a bound state in the attractive case $p<0$

For the double Laplace transform $\hat{\tilde{P}}_{s, p}\left(x \mid x_{0}\right)$ of Eq. 37 , the result of Eq. 45

$$
\begin{equation*}
\hat{\tilde{P}}_{s, p}\left(x \mid x_{0}\right)=\left[\hat{G}_{s}\left(x \mid x_{0}\right)-\frac{\hat{G}_{s}(x \mid 0) \hat{G}_{s}\left(0 \mid x_{0}\right)}{\hat{G}_{s}(0 \mid 0)}\right]+\left[\frac{\hat{G}_{s}(x \mid 0) \hat{G}_{s}\left(0 \mid x_{0}\right)}{\hat{G}_{s}^{2}(0 \mid 0)}\right] \frac{1}{p+\frac{1}{\hat{G}_{s}(0 \mid 0)}} \tag{C1}
\end{equation*}
$$

shows that for $p<0$, a new singularity will appear in $\hat{\tilde{P}}_{s, p}\left(x \mid x_{0}\right)$ with respect to $\hat{G}_{s}\left(x \mid x_{0}\right)$ when the variable $s$ make the denominator vanish

$$
\begin{equation*}
0=p+\frac{1}{\hat{G}_{s}(0 \mid 0)} \tag{C2}
\end{equation*}
$$

The comparison with the pole in Eq. A20 shows that the value of $s$ satisfying Eq. C2 is directly related to the ground-state energy $E_{p}$ of $H_{p}$

$$
\begin{equation*}
s=-E_{p} \tag{C3}
\end{equation*}
$$

i.e. the relations between $p$ and $s$ of Eqs C2 and C3 are the same as in Eq. B6

## 2. Example of the uniform drift $\mu \geq 0$

a. Computation of the energy $E_{p}$ via the pole analysis of Eqs C2 C 3

For the case of the uniform drift $\mu \geq 0$, the Laplace transform $\hat{G}_{s}(0 \mid 0)$ of Eq. 147 yields that Eq. C2 reads

$$
\begin{equation*}
0=p+\sqrt{\mu^{2}+2 s} \tag{C4}
\end{equation*}
$$

So for any $p<0$, its solution $s=-E_{p}$ of Eq. C3 leads to the energy

$$
\begin{equation*}
E_{p<0}=\frac{\mu^{2}-p^{2}}{2} \tag{C5}
\end{equation*}
$$

for the ground state of $H_{p}$ that emerges below the continuous spectrum $] \frac{\mu^{2}}{2},+\infty[$.
b. Direct analysis of the ground-state of the $p$-deformed Hamiltonian $H_{p}$ for $p<0$

For the drift $\mu(x)=\mu$, the Hamiltonian $H$ of Eq. 12

$$
\begin{equation*}
H=-\frac{1}{2} \partial_{x}^{2}+\frac{\mu^{2}}{2} \tag{C6}
\end{equation*}
$$

has no bound-state, but only a continuous spectrum $] \frac{\mu^{2}}{2},+\infty[$.
The $p$-deformed Hamiltonian $H_{p}$ of Eq. A12 reads

$$
\begin{equation*}
H_{p}^{[\mu]}=H^{[\mu]}+p \delta(x)=-\frac{1}{2} \partial_{x}^{2}+\frac{\mu^{2}}{2}+p \delta(x) \tag{C7}
\end{equation*}
$$

For the repulsive case $p>0, H_{p}$ keeps the continuous spectrum $] \frac{\mu^{2}}{2},+\infty[$.
However, for the attractive case $p<0$, a bound state emerges below the continuous spectrum $] \frac{\mu^{2}}{2},+\infty[$. It is exponentially localized around the origin

$$
\begin{equation*}
\phi_{p<0}^{G S}(x)=\sqrt{(-p)} e^{-(-p)|x|} \tag{C8}
\end{equation*}
$$

Its energy

$$
\begin{equation*}
E_{p<0}=\frac{\mu^{2}-p^{2}}{2} \tag{C9}
\end{equation*}
$$

coincides with the other derivation of Eq. C5 as it should.
c. Canonical conditioned process $X_{p}^{*}(t)$ of parameter $p<0$

The conditioned drift of Eq. A24 reads

$$
\begin{equation*}
\mu_{p<0}^{*}(x)=\partial_{x} \ln \left[\phi_{p}^{G S}(x)\right]=-(-p) \operatorname{sgn}(x) \tag{C10}
\end{equation*}
$$

The corresponding conditioned equilibrium state reads

$$
\begin{equation*}
G_{p}^{e q}(x)=\left[\phi_{p<0}^{G S}(x)\right]^{2}=(-p) e^{-2(-p)|x|} \tag{C11}
\end{equation*}
$$

So the canonical conditioning of parameter $p<0$ is asymptotically equivalent to the microcanonical conditioning towards the intensive local time

$$
\begin{equation*}
a_{p}^{*}=G_{p}^{e q}(x=0)=-p>0 \tag{C12}
\end{equation*}
$$

So the conditioned drift of Eq. C10 can be rewritten as

$$
\begin{equation*}
\mu_{p<0}^{*}(x)=-a_{p}^{*} \operatorname{sgn}(x) \tag{C13}
\end{equation*}
$$

in agreement with the microcanonical conditioning of Eq. 210 in the main text.
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