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Conditioning diffusion processes with respect to the local time at the origin

Alain Mazzolo

Université Paris-Saclay, CEA, Service d'Études des Réacteurs et de Mathématiques Appliquées, 91191, Gif-sur-Yvette, France

Cécile Monthus

Université Paris-Saclay, CNRS, CEA, Institut de Physique Théorique, 91191 Gif-sur-Yvette, France

When the unconditioned process is a diffusion process $X(t)$ of drift $\mu(x)$ and of diffusion coefficient $D = 1/2$, the local time $A(t) = \int_0^t d\tau \delta(X(\tau))$ at the origin $x = 0$ is one of the most important time-additive observable. We construct various conditioned processes $[X^*(t), A^*(t)]$ involving the local time $A^*(T)$ at the time horizon T . When the horizon T is finite, we consider the conditioning towards the final position $X^*(T)$ and towards the final local time $A^*(T)$, as well as the conditioning towards the final local time $A^*(T)$ alone without any condition on the final position $X^*(T)$. In the limit of the infinite time horizon $T \rightarrow +\infty$, we consider the conditioning towards the finite asymptotic local time $A_\infty^* < +\infty$, as well as the conditioning towards the intensive local time a^* corresponding to the extensive behavior $A_T \simeq Ta^*$, that can be compared with the appropriate 'canonical conditioning' based on the generating function of the local time in the regime of large deviations. This general construction is then applied to generate various constrained stochastic trajectories for three unconditioned diffusions with different recurrence/transience properties : (i) the simplest example of transient diffusion corresponds to the uniform strictly positive drift $\mu(x) = \mu > 0$; (ii) the simplest example of diffusion converging towards an equilibrium is given by the drift $\mu(x) = -\mu \operatorname{sgn}(x)$ of parameter $\mu > 0$; (iii) the simplest example of recurrent diffusion that does not converge towards an equilibrium is the Brownian motion without drift $\mu = 0$.

I. INTRODUCTION

A. Conditioning diffusion processes with respect to time-additive observables of the stochastic trajectories

Since its introduction by Doob [1, 2], the conditioning of stochastic processes (see the mathematical books [3–5] and the physics recent review [6]) have found many applications in various fields like ecology [7], finance [8] or nuclear engineering [9, 10]. Among the different conditioned diffusions that have been constructed besides the basic example of the Brownian Bridge, one can cite the Brownian excursion [11, 12], the Brownian meander [13], the taboo processes [14–19], or non-intersecting Brownian bridges [20]. Let us also mention the conditioning in the presence of killing rates [3, 21–28] or when the killing occurs only via an absorbing boundary condition [29–32]. Note that stochastic bridges have been studied for many other Markov processes, including various diffusions processes [33–35], discrete-time random walks and Lévy flights [36–38], continuous-time Markov jump processes [38], run-and-tumble trajectories [39], or processes with resetting [40].

A recent important generalization concerns the conditioning with respect to global dynamical constraints involving time-additive observables of the stochastic trajectories. In particular, the conditioning on the area has been studied via various methods for Brownian processes or bridges [41] and for Ornstein-Uhlenbeck bridges [42]. The conditioning on the area and on other time-additive observables has been then analyzed both for the Brownian motion and for discrete-time random walks [43]. This approach has been generalized [44] to various types of discrete-time or continuous-time Markov processes, while the time-additive observable can involve both the time spent in each configuration and the increments of the Markov process. This general reformulation of the 'microcanonical conditioning', where the time-additive observable is constrained to reach a given value after the finite time window T , allows to make the link [44] with the 'canonical conditioning' based on generating functions of additive observables that has been much studied recently in the field of dynamical large deviations of Markov processes over a large time-window T [45–90]. The equivalence between the 'microcanonical conditioning' and the canonical conditioning' at the level of the large deviations for large time T is explained in detail in the two complementary papers [68, 69] and in the HDR thesis [70].

B. Simplest examples of time-additives for a diffusion process $X(t)$: the occupation time and the local time

For a one-dimensional diffusion process $X(t)$, two basic examples of time-additive observables are :

(i) the occupation time $O_{[a,b]}(t)$ of the space interval $[a, b]$ during the time window $[0, t]$

$$O_{[a,b]}(t) = \int_0^t d\tau \theta(a \leq X(\tau) \leq b) \quad (1)$$

belongs to the interval $0 \leq O_{[a,b]}(t) \leq t$. The conditioning with respect to the occupation time of the interval $[a = 0, b = +\infty[$ has been studied recently for the Brownian motion without drift [43], while the canonical conditioning with respect to the occupation time has been analyzed for various settings [73, 74].

(ii) the local time $A_x(t)$ at the position x during the time window $[0, t]$ (see the mathematical review [91] and references therein)

$$A_x(t) = \int_0^t d\tau \delta(X(\tau) - x) \quad (2)$$

has for physical dimension $\frac{\text{Time}}{\text{Length}}$ so that it is actually not a 'time' despite its standard name. However, it is directly related to the occupation time as follows. On the one hand, the local time $A_x(t)$ of Eq. 2 can be constructed from the occupation time $O_{[x-\epsilon, x+\epsilon]}(t)$ of the space interval $[x - \epsilon, x + \epsilon]$ of size $(2\epsilon) > 0$ centered at the position x in the limit $\epsilon \rightarrow 0^+$

$$A_x(t) = \int_0^t d\tau \lim_{\epsilon \rightarrow 0^+} \left(\frac{\theta(x - \epsilon \leq X(\tau) \leq x + \epsilon)}{2\epsilon} \right) = \lim_{\epsilon \rightarrow 0^+} \left(\frac{O_{[x-\epsilon, x+\epsilon]}(t)}{2\epsilon} \right) \quad (3)$$

As a consequence, the local time $A_x(t)$ belongs to $[0, +\infty[$ with no upper bound. On the other hand, the occupation time $O_{[a,b]}(t)$ can be reconstructed from the local time $A_x(t)$ for all the internal positions $x \in [a, b]$

$$O_{[a,b]}(t) = \int_0^t d\tau \int_a^b dx \delta(X(\tau) - x) = \int_a^b dx A_x(t) \quad (4)$$

C. Goals of the present work

At first sight, the delta function that enters the definition of the local time in Eq. 2 might appear as very singular for the purpose of conditioning. However, as in quantum mechanics where delta impurities are well-known to be much simpler than smoother potentials, the delta function in Eq. 2 is actually a huge technical simplification compared to the case of conditioning with respect to an arbitrary general additive observable. Indeed, the exact Dyson equation associated to a single delta impurity allows to analyze the conditioning with respect to the local time for $A_{x=0}(t) = A(t)$ at the origin $x = 0$ in terms of the properties of the propagator $G(x, t|x_0, t_0)$ of the unconditioned process $X(t)$ alone. In the present paper, it will be thus interesting to consider that the unconditioned process is a diffusion process $X(t)$ of diffusion coefficient $D = 1/2$ with an arbitrary position-dependent drift $\mu(x)$ in order to derive the general properties before the application to various illustrative examples of drifts. Our goal is to construct various conditioned joint processes $[X^*(t), A^*(t)]$ satisfying certain conditions involving the local time $A^*(T)$ either at the finite time horizon T , or in the limit of the infinite time horizon $T \rightarrow +\infty$. For instance, the two basic cases that will be considered for the finite horizon T can be summarized as follows.

(i) The conditioning towards the position x_T^* and the local time A_T^* at the time horizon T involves the conditioned drift

$$\mu_T^{[x_T^*, A_T^*]}(x, A, t) = \mu(x) + \partial_x \ln P(x_T^*, A_T^*, T|x, A, t)$$

where $P(x_T^*, A_T^*, T|x, A, t)$ represents the joint propagator of the unconditioned diffusion.

(ii) The conditioning towards the local time A_T^* at time horizon T , without any condition on the final position x_T involves the conditioned drift

$$\mu_T^{[A_T^*]}(x, A, t) = \mu(x) + \partial_x \ln \Pi(A_T^*, T|x, A, t)$$

in terms of $\Pi(A_T^*, T|x, A, t) = \int dx_T P(x_T, A_T^*, T|x, A, t)$ for the unconditioned diffusion.

When the unconditioned diffusion is the Brownian motion of uniform drift $\mu \geq 0$ or the stochastic process with drift $\mu(x) = -\mu \operatorname{sgn}(x)$ with $\mu > 0$, some examples of these conditioned drifts that will be studied are given in the two following tables.

	Finite time horizon $T < \infty$		Infinite time horizon $T = \infty$
	Conditioned drift $\mu_T^{[x_T^*, A_T^*]}(x, A, t)$ towards the position x_T^* and the local time A_T^*	Conditioned drift $\mu_T^{[A_T^*]}(x, A, t)$ towards the local time A_T^*	Conditioned drift $\mu_\infty^{[A_\infty^*]}(x, A)$ towards the local time A_∞^*
Region $0 < A < A_T^*$	$\text{sgn}(x) \left[\frac{1}{ x_T^* + x + A_T^* - A} - \frac{ x_T^* + x + A_T^* - A}{T - t} \right]$	$-\text{sgn}(x) \frac{ x + A_T^* - A}{T - t}$	$-\mu \text{sgn}(x)$
Region $A = A_T^*$	$\frac{\left(\frac{x_T^* - x}{T - t}\right) e^{-\frac{(x_T^* - x)^2}{2(T - t)}} + \left(\frac{x + \text{sgn}(x) x_T^* }{T - t}\right) e^{-\frac{(x_T^* + x)^2}{2(T - t)}}}{e^{-\frac{(x_T^* - x)^2}{2(T - t)}} - e^{-\frac{(x_T^* + x)^2}{2(T - t)}}}$	$\sqrt{\frac{2}{\pi(T - t)}} \frac{e^{-\frac{x^2}{2(T - t)}}}{\text{erf}\left(\frac{ x }{\sqrt{2(T - t)}}\right)} \text{sgn}(x)$	$\mu \coth(\mu x)$

Table I. Some examples of conditioned drifts $\mu_T^*(x, A, t)$ for the Brownian motion of uniform drift $\mu(x) = \mu > 0$.

	Finite time horizon $T < \infty$	
	Conditioned drift $\mu_T^{[x_T^*, A_T^*]}(x, A, t)$ towards the position x_T^* and the local time A_T^*	Conditioned drift $\mu_T^{[A_T^*]}(x, A, t)$ towards the local time A_T^*
Region $0 < A < A_T^*$	$\text{sgn}(x) \left[\frac{1}{ x_T^* + x + A_T^* - A} - \frac{ x_T^* + x + A_T^* - A}{T - t} \right]$	$\mu \text{sgn}(x) - \frac{2}{(T - t)} \frac{(x + A_T^* - A) \text{sgn}(x)}{2 - \mu \sqrt{2\pi(T - t)} e^{\frac{(x + \mu(T - t) + A_T^* - A)^2}{2(T - t)}} \text{erfc}\left(\frac{ x + \mu(T - t) + A_T^* - A}{\sqrt{2(T - t)}}\right)}$
Region $A = A_T^*$	$\frac{\left(\frac{x_T^* - x}{T - t}\right) e^{-\frac{(x_T^* - x)^2}{2(T - t)}} + \left(\frac{x + \text{sgn}(x) x_T^* }{T - t}\right) e^{-\frac{(x_T^* + x)^2}{2(T - t)}}}{e^{-\frac{(x_T^* - x)^2}{2(T - t)}} - e^{-\frac{(x_T^* + x)^2}{2(T - t)}}}$	$2\sqrt{\frac{2}{\pi}} \frac{\text{sgn}(x) \left(\frac{1}{\sqrt{\pi}} e^{-\frac{(x + \mu\tau)^2}{2\tau}} \mathcal{F}\left(\frac{ x + \mu\tau}{\sqrt{2\tau}}\right) \right) + e^{\frac{(x - \mu\tau)^2}{2\tau}} \left(e^{2\mu x \mathcal{F}\left(\frac{x + \mu\tau}{\sqrt{2\tau}}\right)} - \mathcal{F}\left(\frac{\mu\tau - x}{\sqrt{2\tau}}\right) \right)}{e^{\frac{(x + \mu\tau)^2}{2\tau}} \text{erfc}\left(\frac{x + \mu\tau}{\sqrt{2\tau}}\right) + e^{\frac{(x - \mu\tau)^2}{2\tau}} \text{erfc}\left(\frac{\mu\tau - x}{\sqrt{2\tau}}\right) - 2e^{\frac{(x + \mu\tau)^2}{2\tau}} \text{erfc}\left(\frac{ x + \mu\tau}{\sqrt{2\tau}}\right)}$

Table II. Some examples of conditioned drifts $\mu_T^*(x, A, t)$ for the stochastic process with drift, $\mu(x) = -\mu \text{sgn}(x)$ with $\mu > 0$. Observe that the conditioned drifts are the same as those of the Brownian motion in the case of conditioning towards the position x_T^* and the local time A_T^* at the finite time horizon T . We use the notation $\mathcal{F}(x) = x \text{erfc}(x)$ where $\text{erfc}(x)$ is the complementary Error function $\text{erfc}(x) = 1 - \text{erf}(x)$.

D. Organization of the paper

The paper is organized as follows. The properties of the unconditioned diffusion process $X(t)$ with drift $\mu(x)$ are recalled in section II. We then analyze the properties of the joint propagator $P(x, A, t | x_0, A_0, t_0)$ for the position x and the local time A in section III, as well as the probability $\Pi(A, t | x_0, A_0, t_0) = \int dx P(x, A, t | x_0, A_0, t_0)$ in section IV. The statistical properties of the local time increment $[A(t) - A(t_0)]$ in the limit of the large time interval $(t - t_0) \rightarrow +\infty$ are discussed in section V as a function of the recurrence/transience properties of the diffusion process $X(t)$ induced by the drift $\mu(x)$. In section VI, we construct various conditioned processes $[X^*(t), A^*(t)]$ that involve the local time $A^*(T)$ at the finite time horizon T or in the limit of the infinite time horizon $T \rightarrow +\infty$. This general framework is applied to the case of the uniform drift $\mu(x) = \mu$ with $\mu = 0$ or $\mu > 0$ in section VII, and to the case $\mu(x) = -\mu \text{sgn}(x)$ of parameter $\mu > 0$ in section VIII, in order to generate stochastic trajectories of various conditioned processes with respect to the local time. Monte Carlo simulations illustrate our findings. Our conclusions are summarized in section IX. The three appendices A, B and C are devoted to the canonical conditioned processes $X_p^*(t)$ of parameter p , in order to compare with the microcanonical conditioning described in the main text.

II. PROPERTIES OF THE UNCONDITIONED DIFFUSION PROCESS $X(t)$ WITH DRIFT $\mu(x)$

In this paper, we consider that the unconditioned process $X(t)$ is a diffusion process on the whole line $]-\infty, +\infty[$ generated by the Stochastic Differential Equation

$$dX(t) = \mu(X(t))dt + dB(t) \quad (5)$$

where $B(t)$ is a standard Brownian motion and where the position-dependent drift $\mu(x)$ is the only parameter of the model. In this section, we recall the recurrence/transience properties that will be useful to analyze the statistics of its local time $A(t)$ at the origin in the three next sections III, IV and V.

A. Propagator $G(x, t|x_0, t_0)$ for the diffusion process $X(t)$

The propagator $G(x, t|x_0, t_0)$ for the diffusion process $X(t)$ generated by Eq. 5 satisfies the Fokker-Planck dynamics

$$\partial_t G(x, t|x_0, t_0) = -\partial_x [\mu(x)G(x, t|x_0, t_0)] + \frac{1}{2}\partial_x^2 G(x, t|x_0, t_0) \quad (6)$$

Its Laplace transform with respect to the time interval $(t - t_0)$

$$\hat{G}_s(x|x_0) \equiv \int_{t_0}^{+\infty} dt e^{-s(t-t_0)} G(x, t|x_0, t_0) \quad (7)$$

then satisfies

$$-\delta(x - x_0) + s\hat{G}_s(x|x_0) = -\partial_x [\mu(x)\hat{G}_s(x|x_0)] + \frac{1}{2}\partial_x^2 \hat{G}_s(x|x_0) \quad (8)$$

B. Similarity transformation towards an euclidean quantum propagator $\psi(x, t|x_0, t_0)$

As is well-known [92], the potential $U(x)$ defined via the following integration of the drift $\mu(y)$

$$U(x) \equiv -2 \int_0^x dy \mu(y) \quad (9)$$

can be used to make the similarity transformation

$$G(x, t|x_0, t_0) = e^{-\frac{U(x)}{2}} \psi(x, t|x_0, t_0) e^{\frac{U(x_0)}{2}} = e^{\frac{U(x_0) - U(x)}{2}} \psi(x, t|x_0, t_0) = e^{\int_{x_0}^x dy \mu(y)} \psi(x, t|x_0, t_0) \quad (10)$$

The Fokker-Planck Eq. 6 for the propagator $G(x, t|x_0, t_0)$ is then transformed into an Euclidean Schrödinger Equation for $\psi(x, t|x_0, t_0)$

$$-\partial_t \psi(x, t|x_0, t_0) = H \psi(x, t|x_0, t_0) \quad (11)$$

The corresponding hermitian quantum Hamiltonian

$$H = -\frac{1}{2}\partial_x^2 + V(x) \quad (12)$$

involves the quantum potential

$$V(x) \equiv \frac{\mu^2(x)}{2} + \frac{\mu'(x)}{2} \quad (13)$$

This very specific structure of $V(x)$ in terms of the drift $\mu(x)$ allows to factorize the Hamiltonian of Eq. 12 into the supersymmetric form (see the review on supersymmetric quantum mechanics [93] and references therein)

$$H \equiv \frac{1}{2} Q^\dagger Q \quad (14)$$

involving the first-order operator

$$Q \equiv -\partial_x + \mu(x) \quad (15)$$

and its adjoint

$$Q^\dagger \equiv \partial_x + \mu(x) \quad (16)$$

This quantum mapping allows to use all the knowledge on one-dimensional quantum Hamiltonians in general and on supersymmetric quantum Hamiltonians in particular to characterize the energy spectrum as follows.

C. Analysis of the spectrum of the quantum supersymmetric Hamiltonian H

1. Analysis of the continuous spectrum $]V_\infty, +\infty[$ of H when it exists

The minimum of the two limiting values of the quantum potential $V(x)$ of Eq. 13 as $x \rightarrow \pm\infty$

$$V_\infty = \min[V(x \rightarrow +\infty); V(x \rightarrow -\infty)] \quad (17)$$

determines the lower boundary of the continuous spectrum when it exists.

The discussion is thus as follows :

(i) if $V_\infty < +\infty$ is finite, then the continuous spectrum of H is given by $]V_\infty, +\infty[$. The physical interpretation is that, in the infinity region where the asymptotic value of the potential $V(x)$ is V_∞ , an eigenstate of energy $E \in]V_\infty, +\infty[$ behaves asymptotically like a linear combination of the plane waves $e^{\pm ikx}$, where the relation between the wave-number k and the energy E is given by the corresponding eigenvalue equation for $He^{\pm ikx} = Ee^{\pm ikx}$ in the infinity region where the potential is V_∞

$$E = \frac{k^2}{2} + V_\infty \quad (18)$$

i.e. the wave-number $k = \sqrt{2(E - V_\infty)}$ is real for any energy $E \in]V_\infty, +\infty[$.

The simplest example is the case of the uniform drift $\mu(x) = \mu$, where the quantum potential of Eq. 13 reduces to the constant

$$V(x) = \frac{\mu^2}{2} \quad \text{for } \mu(x) = \mu \quad (19)$$

so that the continuous spectrum is $]\frac{\mu^2}{2}, +\infty[$.

(ii) if $V_\infty = +\infty$ is infinite, then there is no continuous spectrum and H has only an infinity of bound states.

The simplest example is the case of the Ornstein-Uhlenbeck drift $\mu(x) = -kx$ with $k > 0$, where the quantum potential of Eq. 13 corresponds to the harmonic oscillator

$$V(x) = \frac{k^2}{2}x^2 - \frac{k}{2} \quad \text{for } \mu(x) = -kx \quad (20)$$

with its well-known infinite series of discrete levels.

2. Analysis of the normalizable zero-energy ground-state $\phi^{GS}(x)$ of H when it exists

The factorization of Eq. 14 shows that the spectrum of H is positive. Let us now discuss whether $E = 0$ is the ground state energy of H . The wavefunction $\phi^{E=0}(x)$ that is annihilated by the operator Q of Eq. 15

$$0 = Q\phi^{[E=0]}(x) = -\partial_x \phi^{[E=0]}(x) + \mu(x)\phi^{[E=0]}(x) \quad (21)$$

reads in terms of the potential $U(x)$ of Eq. 9

$$\phi^{[E=0]}(x) = \phi^{[E=0]}(0)e^{\int_0^x dy \mu(y)} = \phi^{[E=0]}(0)e^{-\frac{U(x)}{2}} \quad (22)$$

This wavefunction can be normalized on $x \in]-\infty, +\infty[$ if

$$1 = \langle \phi^{[E=0]} | \phi^{[E=0]} \rangle = \int_{-\infty}^{+\infty} dx \left[\phi^{[E=0]}(x) \right]^2 = \left[\phi^{[E=0]}(0) \right]^2 \int_{-\infty}^{+\infty} dx e^{-U(x)} \quad (23)$$

The discussion is thus as follows :

(i) if the integral involving the potential $U(x)$ converges

$$\int_{-\infty}^{+\infty} dx e^{-U(x)} < +\infty \quad (24)$$

then H has the following normalizable ground state at zero-energy $E = 0$

$$\phi^{GS}(x) = \frac{e^{-\frac{U(x)}{2}}}{\sqrt{\int_{-\infty}^{+\infty} dy e^{-U(y)}}} \quad (25)$$

(ii) if the integral of Eq. 24 diverges

$$\int_{-\infty}^{+\infty} dx e^{-U(x)} = +\infty \quad (26)$$

then the zero-energy wavefunction of Eq. 21 cannot be normalized, and the Hamiltonian H has no bound state, but only the continuous spectrum discussed in the previous subsection II C 1.

D. Consequences for the Fokker-Planck propagator $G(x, t|x_0, t_0)$ at large time interval $(t - t_0)$

1. When H has the zero-energy ground-state $\phi^{GS}(x) : G(x, t|x_0, t_0)$ converges towards an equilibrium state $G_{eq}(x)$

When H has the normalizable zero-energy ground-state $\phi^{GS}(x)$ of Eq. 25, then the quantum propagator $\psi(x, t|x_0, t_0)$ of Eq. 11 displays the long-time behavior

$$\psi(x, t|x_0, t_0) \underset{(t-t_0) \rightarrow +\infty}{\simeq} \phi^{GS}(x) \phi^{GS}(x_0) = \frac{e^{-\frac{U(x)}{2} - \frac{U(x_0)}{2}}}{\int_{-\infty}^{+\infty} dy e^{-U(y)}} \quad (27)$$

So the Fokker-Planck propagator $G(x, t|x_0, t_0)$ obtained via the similarity transformation of Eq. 10

$$G(x, t|x_0, t_0) = e^{-\frac{U(x)}{2}} \psi(x, t|x_0, t_0) e^{\frac{U(x_0)}{2}} \underset{(t-t_0) \rightarrow +\infty}{\simeq} \frac{e^{-U(x)}}{\int_{-\infty}^{+\infty} dy e^{-U(y)}} \equiv G_{eq}(x) \quad (28)$$

converges towards the Boltzmann equilibrium $G_{eq}(x)$ in the potential $U(x)$. The equilibrium state $G_{eq}(x)$ is the steady state of the Fokker-Planck dynamics of Eq. 6 with no steady current

$$0 = \mu(x) G_{eq}(x) - \frac{1}{2} \partial_x G_{eq}(x) \quad (29)$$

For the Laplace transform of Eq. 7, the convergence of Eq. 28 means that $\hat{G}_s(x|x_0)$ is defined for $s \in]0, +\infty[$ with the following singularity for $s \rightarrow 0^+$

$$\hat{G}_s(x|x_0) \underset{s \rightarrow 0^+}{\simeq} \frac{G_{eq}(x)}{s} \quad (30)$$

The simplest example of diffusion converging towards an equilibrium state is the drift $\mu(x) = -\mu \operatorname{sgn}(x)$ of parameter $\mu > 0$ that will be discussed in section VIII.

2. When H has only the continuum $]V_\infty, +\infty[$ with $V_\infty > 0$: V_∞ governs the exponential time decay of $G(x_0, t|x_0, t_0)$

When H has only the continuous spectrum $]V_\infty, +\infty[$, where the lower boundary V_∞ of Eq. 17 is strictly positive $V_\infty > 0$, then the Fokker-Planck propagator $G(x, t|x_0, t_0)$ and the quantum propagator $\psi(x, t|x_0, t_0)$ are dominated by the leading exponential time decay involving V_∞

$$G(x_0, t|x_0, t_0) = e^{\frac{U(x_0) - U(x)}{2}} \psi(x_0, t|x_0, t_0) \underset{(t-t_0) \rightarrow +\infty}{\propto} e^{-V_\infty(t-t_0)} \quad (31)$$

The physical interpretation is that the diffusion process is transient and flows towards infinity. For the Laplace transform of Eq. 7, Eq. 31 means that $\hat{G}_s(x|x_0)$ is defined for $s \in]-V_\infty, +\infty[$. In particular, it remains finite for $s = 0$ in contrast to the previous case of Eq. 30

$$\hat{G}_{s=0}(x_0|x_0) < +\infty \quad (32)$$

The simplest example of transient diffusion is the uniform strictly positive drift $\mu(x) = \mu > 0$ that will be discussed in section VII.

3. When H has only the continuous spectrum $]V_\infty = 0, +\infty[$ with the vanishing lower boundary $V_\infty = 0$

When H has only the continuous spectrum $]V_\infty = 0, +\infty[$ with the vanishing lower boundary $V_\infty = 0$, then the Fokker-Planck propagator $G(x, t|x_0, t_0)$ decays in time, but less rapidly than the exponential decay of Eq. 31.

The simplest example of recurrent diffusion that does not converge towards an equilibrium state is of course the pure Brownian motion without drift $\mu = 0$, that will be discussed in section VII.

III. JOINT PROPERTIES OF THE DIFFUSION PROCESS $X(t)$ AND ITS LOCAL TIME $A(t)$

In this section, we focus on the unconditioned joint process $[X(t), A(t)]$: the position $X(t)$ and its local time $A(t)$ at the origin satisfy the Ito Stochastic Differential System

$$\begin{aligned} dX(t) &= \mu(X(t))dt + dB(t) \\ dA(t) &= \delta(X(t))dt \end{aligned} \quad (33)$$

The joint propagator $P(x, A, t|x_0, A_0, t_0)$ for the position x and the local time A that satisfies the Fokker-Planck dynamics

$$\partial_t P(x, A, t|x_0, A_0, t_0) = -\delta(x)\partial_A P(x, A, t|x_0, A_0, t_0) - \partial_x [\mu(x)P(x, A, t|x_0, A_0, t_0)] + \frac{1}{2}\partial_x^2 P(x, A, t|x_0, A_0, t_0) \quad (34)$$

will be useful to construct conditioned bridges involving both the final position and the final local time, as will be described in the subsection VIA.

A. Laplace transform $\tilde{P}_p(x, t|x_0, t_0)$ with respect to the local time $(A - A_0) \geq 0$: Feynman-Kac formula

For the Laplace transform $\tilde{P}_p(x, t|x_0, t_0)$ of the joint propagator $P(x, A, t|x_0, A_0, t_0)$ with respect to the local time increment $(A - A_0) \geq 0$

$$\tilde{P}_p(x, t|x_0, t_0) \equiv \int_{A_0}^{+\infty} dA e^{-p(A-A_0)} P(x, A, t|x_0, A_0, t_0) \quad (35)$$

Eq. 34 translates into

$$\partial_t \tilde{P}_p(x, t|x_0, t_0) = -p\delta(x)\tilde{P}_p(0, t|x_0, t_0) - \partial_x [\mu(x)\tilde{P}_p(x, t|x_0, t_0)] + \frac{1}{2}\partial_x^2 \tilde{P}_p(x, t|x_0, t_0) \quad (36)$$

This is a standard example of the Feynman-Kac formula, where the initial Fokker-Planck dynamics of Eq. 6 is now supplemented by the additional term in $p\delta(x)$.

B. Explicit double Laplace transform $\hat{\tilde{P}}_{s,p}(x|x_0)$ of the joint propagator $P(x, A, t|x_0, A_0, t_0)$ via the Dyson Eq.

The further Laplace transform of Eq. 35 with respect to the time $(t - t_0)$

$$\hat{\tilde{P}}_{s,p}(x|x_0) \equiv \int_{t_0}^{+\infty} dt e^{-s(t-t_0)} \tilde{P}_p(x, t|x_0, t_0) = \int_{t_0}^{+\infty} dt e^{-s(t-t_0)} \int_{A_0}^{+\infty} dA e^{-p(A-A_0)} P(x, A, t|x_0, A_0, t_0) \quad (37)$$

satisfies

$$-\delta(x - x_0) + s\hat{\tilde{P}}_{s,p}(x|x_0) = -p\delta(x)\hat{\tilde{P}}_{s,p}(0|x_0) - \partial_x [\mu(x)\hat{\tilde{P}}_{s,p}] + \frac{1}{2}\partial_x^2 \hat{\tilde{P}}_{s,p} \quad (38)$$

For $p = 0$, Eq. 35 coincides with the propagator $G(x, t|x_0, t_0)$ of the position alone described in the previous section II

$$\tilde{P}_{p=0}(x, t|x_0, t_0) \equiv \int_{A_0}^{+\infty} dA P(x, A, t|x_0, A_0, t_0) = G(x, t|x_0, t_0) \quad (39)$$

As a consequence, Eq. 37 becomes

$$\hat{\tilde{P}}_{s,p=0}(x|x_0) = \hat{G}_s(x|x_0) \quad (40)$$

For any $p \neq 0$, the solution $\hat{\tilde{P}}_{s,p}(x|x_0)$ of Eq. 38 satisfies the Dyson equation

$$\hat{\tilde{P}}_{s,p}(x|x_0) = \hat{G}_s(x|x_0) - p\hat{G}_s(x|0)\hat{\tilde{P}}_{s,p}(0|x_0) \quad (41)$$

The self-consistency for $x = 0$

$$\hat{\tilde{P}}_{s,p}(0|x_0) = \hat{G}_s(0|x_0) - p\hat{G}_s(0|0)\hat{\tilde{P}}_{s,p}(0|x_0) \quad (42)$$

yields

$$\hat{P}_{s,p}(0|x_0) = \frac{\hat{G}_s(0|x_0)}{1 + p\hat{G}_s(0|0)} \quad (43)$$

Plugging this result into Eq. 41 yields the final expression of $\hat{P}_{s,p}(x|x_0)$ in terms of $\hat{G}_s(\cdot|\cdot)$

$$\hat{P}_{s,p}(x|x_0) = \hat{G}_s(x|x_0) - p \frac{\hat{G}_s(x|0)\hat{G}_s(0|x_0)}{1 + p\hat{G}_s(0|0)} \quad (44)$$

C. Explicit time Laplace transform $\hat{P}_s(x, A|x_0, A_0)$ of the joint propagator $P(x, A, t|x_0, A_0, t_0)$

The dependence with respect to the parameter p in Eq. 44 can be rewritten in terms of a simple pole as

$$\begin{aligned} \hat{P}_{s,p}(x|x_0) &= \hat{G}_s(x|x_0) - \frac{\hat{G}_s(x|0)\hat{G}_s(0|x_0)}{\hat{G}_s(0|0)} \left(1 - \frac{1}{1 + p\hat{G}_s(0|0)} \right) \\ &= \left[\hat{G}_s(x|x_0) - \frac{\hat{G}_s(x|0)\hat{G}_s(0|x_0)}{\hat{G}_s(0|0)} \right] + \left[\frac{\hat{G}_s(x|0)\hat{G}_s(0|x_0)}{\hat{G}_s^2(0|0)} \right] \frac{1}{p + \frac{1}{\hat{G}_s(0|0)}} \end{aligned} \quad (45)$$

So the inverse Laplace transform with respect to p yields that the time Laplace transform $\hat{P}_s(x, A|x_0, A_0)$ of the joint propagator $P(x, A, t|x_0, A_0, t_0)$ reads

$$\begin{aligned} \hat{P}_s(x, A|x_0, A_0) &\equiv \int_{t_0}^{+\infty} dt e^{-s(t-t_0)} P(x, A, t|x_0, A_0, t_0) \\ &= \delta(A - A_0) \left[\hat{G}_s(x|x_0) - \frac{\hat{G}_s(x|0)\hat{G}_s(0|x_0)}{\hat{G}_s(0|0)} \right] + \theta(A > A_0) \left[\frac{\hat{G}_s(x|0)\hat{G}_s(0|x_0)}{\hat{G}_s^2(0|0)} \right] e^{-\frac{(A-A_0)}{\hat{G}_s(0|0)}} \end{aligned} \quad (46)$$

The normalization over A corresponding to $p = 0$ in Eq. 40 is given by $\hat{G}_s(x|x_0)$

$$\int_{A_0}^{+\infty} dA \hat{P}_s(x, A|x_0, A_0) = \hat{G}_s(x|x_0) = \int_{t_0}^{+\infty} dt e^{-s(t-t_0)} G(x, t|x_0, t_0) \quad (47)$$

Let us now explain the physical meaning of the formula of Eq. 46 in the following subsections.

1. Interpretation of the singular contribution in $\delta(A - A_0)$ of $P(x, A, t|x_0, A_0, t_0)$ in terms of the propagator $G^{abs}(x, t|x_0, t_0)$

In Eq. 46, the singular contribution involving the delta function $\delta(A - A_0)$

$$\hat{P}_s^{Singular}(x, A|x_0, A_0) = \delta(A - A_0) \left[\hat{G}_s(x|x_0) - \frac{\hat{G}_s(x|0)\hat{G}_s(0|x_0)}{\hat{G}_s(0|0)} \right] \quad (48)$$

means that the local time A has kept its initial value A_0 , i.e. the diffusion process has not been able to visit the origin $x = 0$. As a consequence, the weight in factor of the delta function $\delta(A - A_0)$ should correspond to the Laplace transform $\hat{G}_s^{abs}(x|x_0)$ of the propagator $G^{abs}(x, t|x_0, t_0)$ in the presence of an absorbing boundary condition at the origin $x = 0$

$$\left[\hat{G}_s(x|x_0) - \frac{\hat{G}_s(x|0)\hat{G}_s(0|x_0)}{\hat{G}_s(0|0)} \right] = \hat{G}_s^{abs}(x|x_0) \equiv \int_{t_0}^{+\infty} dt e^{-s(t-t_0)} G^{abs}(x, t|x_0, t_0) \quad (49)$$

This interpretation can be also recovered by considering the limit $p \rightarrow +\infty$ in Eq. 45

$$\hat{P}_{s,p \rightarrow +\infty}(x|x_0) = \left[\hat{G}_s(x|x_0) - \frac{\hat{G}_s(x|0)\hat{G}_s(0|x_0)}{\hat{G}_s(0|0)} \right] \quad (50)$$

Indeed, in the Feynman-Kac formula of Eq. 36, the limit of $p \rightarrow +\infty$ amounts to impose the vanishing of the solution $\tilde{P}_{p=+\infty}(x=0, t|x_0, t_0)$ at the origin $x=0$

$$\tilde{P}_{p=+\infty}(x=0, t|x_0, t_0) = 0 \quad (51)$$

i.e. amounts to impose that the origin $x=0$ is an absorbing boundary condition.

In summary, the singular contribution of Eq. 48 can be rewritten as

$$\hat{P}_s^{Singular}(x, A|x_0, A_0) = \delta(A - A_0) \hat{G}_s^{abs}(x|x_0) \quad (52)$$

and its Laplace inversion with respect to s involves the propagator $G^{abs}(x, t|x_0, t_0)$ in the presence of an absorbing boundary condition at the origin $x=0$

$$P^{Singular}(x, A, t|x_0, A_0, t_0) = \delta(A - A_0) G^{abs}(x, t|x_0, t_0) \quad (53)$$

2. Corresponding survival probability $S^{abs}(t|x_0, t_0)$ and absorption rate $\gamma^{abs}(t|x_0, t_0)$

The survival probability $S^{abs}(t|x_0, t_0)$ at time t in the presence of an absorbing boundary at the origin $x=0$ if one starts at the position x_0 at time t_0 , can be obtained from the integration of the propagator $G^{abs}(x, t|x_0, t_0)$ over the final position x

$$S^{abs}(t|x_0, t_0) \equiv \int_{-\infty}^{+\infty} dx G^{abs}(x, t|x_0, t_0) \quad (54)$$

The conservation of probability for the full propagator $G(x, t|x_0, t_0)$

$$\int_{-\infty}^{+\infty} dx G(x, t|x_0, t_0) = 1 \quad (55)$$

translates for its Laplace transform into

$$\int_{-\infty}^{+\infty} dx \hat{G}_s(x|x_0) = \int_{t_0}^{+\infty} dt e^{-s(t-t_0)} \int_{-\infty}^{+\infty} dx G(x, t|x_0, t_0) = \int_{t_0}^{+\infty} dt e^{-s(t-t_0)} = \frac{1}{s} \quad (56)$$

So the time Laplace transform of Eq. 54 reads via the integration of Eq. 49 over x

$$\hat{S}_s^{abs}(x_0) \equiv \int_{t_0}^{+\infty} dt e^{-s(t-t_0)} S^{abs}(t|x_0, t_0) = \int_{-\infty}^{+\infty} dx \hat{G}_s^{abs}(x|x_0) = \frac{1}{s} \left[1 - \frac{\hat{G}_s(0|x_0)}{\hat{G}_s(0|0)} \right] \quad (57)$$

It is now useful to introduce the absorption rate $\gamma^{abs}(t|x_0, t_0)$ at time t when starting at the position x_0 at time t_0

$$\gamma^{abs}(t|x_0, t_0) \equiv -\partial_t S^{abs}(t|x_0, t_0) \quad (58)$$

Its time Laplace transform is simple, as shown via the following integration by parts

$$\begin{aligned} \hat{\gamma}_s^{abs}(x_0) &= - \int_{t_0}^{+\infty} dt e^{-s(t-t_0)} \partial_t S^{abs}(t|x_0, t_0) = - \left[e^{-s(t-t_0)} S^{abs}(t|x_0, t_0) \right]_{t_0}^{+\infty} - s \hat{S}_s^{abs}(x_0) = 1 - s \hat{S}_s^{abs}(x_0) \\ &= \frac{\hat{G}_s(0|x_0)}{\hat{G}_s(0|0)} \end{aligned} \quad (59)$$

3. Interpretation of the regular contribution in $\theta(A > A_0)$ for $P(x, A, t|x_0, A_0, t_0)$

For the special case where the initial and the final positions are at the origin $x=0=x_0$, Eq. 46 reduces to

$$\hat{P}_s(x=0, A|x_0=0, A_0) = \theta(A > A_0) e^{-\frac{(A-A_0)}{\hat{G}_s(0|0)}} \quad (60)$$

Its normalization over A

$$\int dA \hat{P}_s(x=0, A|x_0=0, A_0) = \hat{G}_s(0|0) \equiv \int_{t_0}^{+\infty} dt e^{-s(t-t_0)} G(0, t|0, t_0) \quad (61)$$

involves the propagator $G(0, t|0, t_0)$ from the origin to the origin.

The regular contribution of Eq. 46

$$\hat{P}_s^{Regular}(x, A|x_0, A_0) = \theta(A > A_0) \left[\frac{\hat{G}_s(x|0)\hat{G}_s(0|x_0)}{\hat{G}_s^2(0|0)} \right] e^{-\frac{(A-A_0)}{\hat{G}_s(0|0)}} \quad (62)$$

can be thus rewritten as the product of the three following functions using Eqs 59 and 60

$$\hat{P}_s^{Regular}(x, A|x_0, A_0) = \hat{\gamma}_s^{abs}(x) \hat{P}_s(0, A|0, A_0) \hat{\gamma}_s^{abs}(x_0) \quad (63)$$

Its Laplace inversion involves the time-convolution of the three functions

$$P^{Regular}(x, A, t|x_0, A_0, t_0) = \int_{t_0}^t dt_1 \int_{t_1}^t dt_2 \gamma^{abs}(t|x, t_2) P(0, A, t_2|0, A_0, t_1) \gamma^{abs}(t_1|x_0, t_0) \quad (64)$$

with the following physical meaning.

(i) The time t_1 is the first passage time at the origin if one starts at the initial position x_0 at time t_0 , whose statistics is governed by the absorption rate $\gamma^{abs}(t_1|x_0, t_0)$.

(ii) The time t_2 is the last passage time at the origin before reaching the final point x at time t , where the statistics of the time interval $(t - t_2)$ is governed by absorption rate $\gamma^{abs}(t|x, t_2)$ of the alternative problem when one starts at position x at time t_2 .

(iii) Between the first-passage-time t_1 and the last-passage-time t_2 at the origin, the statistics of the local time increment $(A - A_0)$ is governed by the probability $P(0, A, t_2|0, A_0, t_1)$.

IV. PROBABILITY DISTRIBUTION $\Pi(A, t|x_0, A_0, t_0)$ OF THE LOCAL TIME A AT TIME t

In this section, we focus on the the probability $\Pi(A, t|x_0, A_0, t_0)$ to see the local time A at time t if one starts at position x_0 with the local time A_0 at time t_0 . It can be obtained from the integration over the final position x of the joint propagator $P(x, A, t|x_0, A_0, t_0)$ studied in the previous section and it can be thus decomposed into a singular contribution in $\delta(A - A_0)$ and a regular contribution in $\theta(A > A_0)$

$$\begin{aligned} \Pi(A, t|x_0, A_0, t_0) &\equiv \int_{-\infty}^{+\infty} dx P(x, A, t|x_0, A_0, t_0) \\ &= \Pi^{Singular}(A, t|x_0, A_0, t_0) + \Pi^{Regular}(A, t|x_0, A_0, t_0) \end{aligned} \quad (65)$$

This probability of Eq. 65 will be useful to construct conditioned bridges involving the local time, as will be described in the subsection VI B.

A. Explicit Laplace transform $\hat{\Pi}_s(A|x_0, A_0)$ of the probability $\Pi(A, t|x_0, A_0, t_0)$

The Laplace transform of Eq. 65 with respect to the time interval $(t - t_0)$

$$\hat{\Pi}_s(A|x_0, A_0) \equiv \int_{t_0}^{+\infty} dt e^{-s(t-t_0)} \Pi(A, t|x_0, A_0, t_0) = \int_{-\infty}^{+\infty} dx \hat{P}_s(x, A|x_0, A_0) \quad (66)$$

can be obtained via the integration over x of $\hat{P}_s(x, A|x_0, A_0)$ given by Eq. 46 using Eq. 56

$$\begin{aligned} \hat{\Pi}_s(A|x_0, A_0) &= \Pi_s^{Singular}(A|x_0, A_0) + \Pi_s^{Regular}(A|x_0, A_0) \\ \Pi_s^{Singular}(A|x_0, A_0) &= \delta(A - A_0) \frac{1}{s} \left[1 - \frac{\hat{G}_s(0|x_0)}{\hat{G}_s(0|0)} \right] \\ \Pi_s^{Regular}(A|x_0, A_0) &= \theta(A > A_0) \left[\frac{\hat{G}_s(0|x_0)}{s \hat{G}_s^2(0|0)} \right] e^{-\frac{(A-A_0)}{\hat{G}_s(0|0)}} \end{aligned} \quad (67)$$

1. Interpretation of the singular contribution in $\delta(A - A_0)$

The singular contribution $\Pi_s^{Singular}(A|x_0, t_0)$ of Eq. 67 involves the time Laplace transform $\hat{S}_s^{abs}(x_0)$ of Eq. 57

$$\hat{\Pi}_s^{Singular}(A|x_0, A_0) = \delta(A - A_0) \frac{1}{s} \left[1 - \frac{\hat{G}_s(0|x_0)}{\hat{G}_s(0|0)} \right] = \delta(A - A_0) \hat{S}_s^{abs}(x_0) \quad (68)$$

and its Laplace inversion involves the survival probability $S^{abs}(t|x_0, t_0)$ of Eq. 54

$$\Pi^{Singular}(A, t|x_0, A_0, t_0) = \delta(A - A_0) S^{abs}(t|x_0, t_0) \quad (69)$$

2. Interpretation of the regular contribution in $\theta(A > A_0)$

For the special case where the initial position vanishes $x_0 = 0$, Eq. 67 reduces to

$$\hat{\Pi}_s(A|x_0 = 0, A_0) = \hat{\Pi}_s^{Regular}(A|x_0 = 0, A_0) = \frac{\theta(A > A_0)}{s \hat{G}_s(0|0)} e^{-\frac{(A-A_0)}{\hat{G}_s(0|0)}} \quad (70)$$

As a consequence, the regular contribution $\hat{\Pi}_s^{Regular}(A|x_0, A_0)$ of Eq. 67 can be rewritten as the product of two functions using Eq. 59 and 70

$$\hat{\Pi}_s^{Regular}(A|x_0, A_0) = \theta(A > A_0) \frac{\hat{G}_s(0|x_0)}{s \hat{G}_s^2(0|0)} e^{-\frac{(A-A_0)}{\hat{G}_s(0|0)}} = \hat{\Pi}_s(A|x_0 = 0, A_0) \hat{\gamma}_s^{abs}(x_0) \quad (71)$$

Its Laplace inversion with respect to s can be written as the time-convolution of two functions

$$\Pi^{Regular}(A, t|x_0, A_0, t_0) = \int_{t_0}^t dt_1 \Pi(A, t|0, A_0, t_1) \gamma^{abs}(t_1|x_0, t_0) \quad (72)$$

with the following physical meaning :

- (i) as in Eq. 64, the time t_1 is the first passage time at the origin if one starts at the initial position x_0 at time t_0 , whose statistics is governed by the absorption rate $\gamma^{abs}(t_1|x_0, t_0)$.
- (ii) for the remaining time interval $(t - t_1)$, $\Pi(A, t|0, A_0, t_1)$ represents the probability to see the local time increment $(A - A_0)$ when starting at the origin $x_0 = 0$.

B. Moments $m^{[k]}(t|x_0, t_0)$ of order $k = 1, 2, \dots$ of the local time increment $[A(t) - A(t_0)]$

1. Computation of the moments $m^{[k]}(t|x_0, t_0)$ from the probability distribution $\Pi(A, t|x_0, A_0, t_0)$

The moments of order $k = 1, 2, \dots$ of the local time increment $[A(t) - A(t_0)]$ only involve the regular part $\Pi^{Regular}(A, t|x_0, A_0, t_0)$ of the probability distribution $\Pi(A, t|x_0, A_0, t_0)$ of Eq. 65

$$\begin{aligned} m^{[k]}(t|x_0, t_0) &\equiv \int_{A_0}^{+\infty} dA [A - A_0]^k \Pi(A, t|x_0, A_0, t_0) \\ &= \int_{A_0}^{+\infty} dA [A - A_0]^k \Pi^{Regular}(A, t|x_0, A_0, t_0) \end{aligned} \quad (73)$$

Their Laplace transforms with respect to the time interval $(t - t_0)$ can be obtained from $\hat{\Pi}_s^{Regular}(A|x_0, A_0)$ of Eq. 71

$$\begin{aligned} m_s^{[k]}(x_0) &\equiv \int_{t_0}^{+\infty} dt e^{-s(t-t_0)} m^{[k]}(t|x_0, t_0) = \int_{A_0}^{+\infty} dA [A - A_0]^k \hat{\Pi}_s^{Regular}(A|x_0, A_0) \\ &= \frac{\hat{G}_s(0|x_0)}{s \hat{G}_s^2(0|0)} \int_{A_0}^{+\infty} dA [A - A_0]^k e^{-\frac{(A-A_0)}{\hat{G}_s(0|0)}} = \frac{k! \left[\hat{G}_s(0|0) \right]^{k-1} \hat{G}_s(0|x_0)}{s} \end{aligned} \quad (74)$$

2. *Alternative computation of the moments $m^{[k]}(t|x_0, t_0)$ from the definition of Eq. 2*

Alternatively, one can use the definition of Eq. 2

$$A(t) - A(t_0) = \int_{t_0}^t d\tau \delta(X(\tau)) \quad (75)$$

to compute the moment of order k for trajectories starting at $X(t_0) = x_0$ in terms of the propagator $G(x, t|y, t')$

$$\begin{aligned} m^{[k]}(t|x_0, t_0) &= \int_{t_0}^t dt_k \int_{t_0}^{t_k} dt_{k-1} \dots \int_{t_0}^{t_2} dt_2 \int_{t_0}^{t_1} dt_1 \langle \delta(X(t_k)) \delta(X(t_{k-1})) \dots \delta(X(t_2)) \delta(X(t_1)) \delta(X(t_0) - x_0) \rangle \\ &= k! \int_{t_0}^t dt_k \int_{t_0}^{t_k} dt_{k-1} \dots \int_{t_0}^{t_3} dt_2 \int_{t_0}^{t_2} dt_1 \langle \delta(X(t_k)) \delta(X(t_{k-1})) \dots \delta(X(t_2)) \delta(X(t_1)) \delta(X(t_0) - x_0) \rangle \\ &= k! \int_{t_0}^t dt_k \int_{t_0}^{t_k} dt_{k-1} \dots \int_{t_0}^{t_3} dt_2 \int_{t_0}^{t_2} dt_1 G(0, t_k|0, t_{k-1}) G(0, t_{k-1}|0, t_{k-2}) \dots G(0, t_2|0, t_1) G(0, t_1|x_0, t_0) \end{aligned} \quad (76)$$

so that its Laplace transform

$$\begin{aligned} m_s^{[k]}(x_0) &= \int_{t_0}^{+\infty} dt e^{-s(t-t_0)} m^{[k]}(t|x_0, t_0) \\ &= k! \int_{t_0}^{+\infty} dt e^{-s(t-t_k) - s(t_k-t_{k-1}) - \dots - s(t_2-t_1) - s(t_1-t_0)} \\ &\quad \int_{t_0}^t dt_k \int_{t_0}^{t_k} dt_{k-1} \dots \int_{t_0}^{t_3} dt_2 \int_{t_0}^{t_2} dt_1 G(0, t_k|0, t_{k-1}) G(0, t_{k-1}|0, t_{k-2}) \dots G(0, t_2|x_0, t_1) G(0, t_1|x_0, t_0) \\ &= k! \int_{t_0}^{+\infty} dt_1 e^{-s(t_1-t_0)} G(0, t_1|x_0, t_0) \dots \int_{t_0}^{+\infty} dt_k e^{-s(t_k-t_{k-1})} G(0, t_k|0, t_{k-1}) \int_{t_k}^{+\infty} dt e^{-s(t-t_k)} \\ &= \frac{k! \left[\hat{G}_s(0|0) \right]^{k-1} \hat{G}_s(0|x_0)}{s} \end{aligned} \quad (77)$$

coincides with Eq. 74 as it should.

3. *Example of the two first moments for $k = 1$ and $k = 2$*

The first moment $k = 1$ of Eq. 76 reduces to the single time integral

$$m^{[k=1]}(t|x_0, t_0) = \int_{t_0}^t dt_1 G(0, t_1|x_0, t_0) \quad (78)$$

Its growth is thus directly governed by the propagator $G(0, t|x_0, t_0)$

$$\partial_t m^{[k=1]}(t|x_0, t_0) = G(0, t|x_0, t_0) > 0 \quad (79)$$

The Laplace transform of Eq. 74 reads

$$m_s^{[k=1]}(x_0) = \frac{\hat{G}_s(0|x_0)}{s} \quad (80)$$

The second moment $k = 2$ of Eq. 76 reads

$$m^{[k=2]}(t|x_0, t_0) = 2 \int_{t_0}^t dt_2 \int_{t_0}^{t_2} dt_1 G(0, t_2|0, t_1) G(0, t_1|x_0, t_0) \quad (81)$$

with its Laplace transform of Eq. 74

$$m_s^{[k=2]}(x_0) = \frac{2\hat{G}_s(0|0)\hat{G}_s(0|x_0)}{s} \quad (82)$$

V. STATISTICS OF THE LOCAL TIME INCREMENT $(A - A_0)$ FOR LARGE TIME INTERVAL $(t - t_0)$

In this section, we describe how the recurrence/transience properties of the diffusion process $X(t)$ induced by the drift $\mu(x)$ produce very different behaviors for the scaling of the local time increment $(A - A_0)$ with respect to the large time interval $(t - t_0)$.

A. When $X(t)$ is transient : the local time increment $(A - A_0)$ remains finite for $(t - t_0) \rightarrow +\infty$

Among transient diffusions, the simplest example is the uniform strictly positive drift $\mu(x) = \mu > 0$ that will be discussed in section VII.

When the diffusion process $X(t)$ is transient with the exponential time decay of Eq. 31 for the propagator $G(x, t|x_0, t)$, then the first moment $m^{[k=1]}(t|x_0, t_0)$ of the local time increment $[A(t) - A(t_0)]$ of Eq. 78 converges towards the finite value $m^{[k=1]}(\infty|x_0)$ for $(t - t_0) \rightarrow +\infty$

$$m^{[k=1]}(t|x_0, t_0) \underset{(t-t_0) \rightarrow +\infty}{\simeq} m^{[k=1]}(\infty|x_0) = \int_0^{+\infty} dt_1 G(0, t_1|x_0, 0) < +\infty \quad (83)$$

More generally, the local time increment $(A - A_0)$ will remain a finite random variable for $(t - t_0) \rightarrow +\infty$ with the following notation for the limit of the distribution $\Pi(A, t|x_0, A_0, t_0)$ of Eq. 65

$$\begin{aligned} \Pi(A, \infty|x_0, A_0) &= \lim_{(t-t_0) \rightarrow +\infty} \Pi(A, t|x_0, A_0, t_0) \\ &= \Pi^{Singular}(A, \infty|x_0, A_0) + \Pi^{Regular}(A, \infty|x_0, A_0) \end{aligned} \quad (84)$$

Let us now discuss its singular and regular contributions.

(i) The singular contribution $\Pi^{Singular}(A, \infty|x_0, A_0)$ involves the infinite-time limit of Eq. 69

$$\Pi^{Singular}(A, \infty|x_0, A_0, t_0) = \delta(A - A_0) S^{abs}(\infty|x_0) \quad (85)$$

that involves the probability to survive forever

$$S^{abs}(\infty|x_0) = \lim_{(t-t_0) \rightarrow +\infty} S^{abs}(t|x_0, t_0) \quad (86)$$

This probability to escape towards infinity without visiting the origin $x = 0$ can be obtained from the Laplace transform $\hat{S}_s^{abs}(x_0)$ of Eq. 57 by considering the limit $s \rightarrow 0$ of

$$S^{abs}(\infty|x_0) = \lim_{s \rightarrow 0} [s \hat{S}_s^{abs}(x_0)] = \lim_{s \rightarrow 0} \left[1 - \frac{\hat{G}_s(0|x_0)}{\hat{G}_s(0|0)} \right] = 1 - \frac{\hat{G}_0(0|x_0)}{\hat{G}_0(0|0)} \quad (87)$$

(ii) The Regular contribution $\Pi^{Regular}(A, \infty|x_0, A_0)$ can be obtained from the Laplace transform $\hat{\Pi}_s^{Regular}(A|x_0, A_0)$ of Eq. 71 by considering the limit $s \rightarrow 0$ of

$$\begin{aligned} \Pi^{Regular}(A, \infty|x_0, A_0) &= \lim_{s \rightarrow 0} [s \hat{\Pi}_s^{Regular}(A|x_0, A_0)] = \lim_{s \rightarrow 0} \left[\theta(A > A_0) \frac{\hat{G}_s(0|x_0)}{\hat{G}_s^2(0|0)} e^{-\frac{(A-A_0)}{\hat{G}_s(0|0)}} \right] \\ &= \theta(A > A_0) \frac{\hat{G}_0(0|x_0)}{\hat{G}_0^2(0|0)} e^{-\frac{(A-A_0)}{\hat{G}_0(0|0)}} \\ &= \theta(A > A_0) [1 - S^{abs}(\infty|x_0)] \frac{1}{\hat{G}_0(0|0)} e^{-\frac{(A-A_0)}{\hat{G}_0(0|0)}} \end{aligned} \quad (88)$$

Its physical meaning can be understood as follows : with the complementary probability $[1 - S^{abs}(\infty|x_0)]$ with respect to Eq. 87, the diffusion process $X(t)$ visits the origin before escaping towards infinity, and then the local time $(A - A_0)$ is an exponential random variable with the finite scale

$$\hat{G}_{s=0}(0|0) \equiv \int_0^{+\infty} dt G(0, t|0, 0) \quad (89)$$

B. When $X(t)$ converges towards an equilibrium : the increment $(A - A_0)$ grows extensively in $(t - t_0)$

Among diffusions converging towards equilibrium, one simple example is the drift $\mu(x) = -\mu \operatorname{sgn}(x)$ of parameter $\mu > 0$ that will be discussed in section VIII.

When the diffusion process $X(t)$ converges towards the Boltzmann equilibrium state of Eq. 28

$$G_{eq}(x) = \frac{e^{-U(x)}}{\int_{-\infty}^{+\infty} dy e^{-U(y)}} = [\phi_0^{GS}(x)]^2 \quad (90)$$

then the first moment $m^{[k=1]}(t|x_0, t_0)$ of the local time increment $[A(t) - A(t_0)]$ discussed in Eqs 78, 79 is extensive with respect to the time interval $(t - t_0)$

$$m^{[k=1]}(t|x_0, t_0) \underset{(t-t_0) \rightarrow +\infty}{\simeq} (t - t_0) G_{eq}(x = 0) \quad (91)$$

The corresponding intensive local time

$$a \equiv \frac{A - A_0}{t - t_0} \quad (92)$$

then converges in the thermodynamic limit $(t - t_0) = +\infty$ towards its equilibrium value

$$a_{eq} = G_{eq}(0) = \frac{e^{-U(0)}}{\int_{-\infty}^{+\infty} dy e^{-U(y)}} = [\phi_0^{GS}(0)]^2 \quad (93)$$

For large but finite $(t - t_0)$, it is thus interesting to analyze its large deviations properties.

1. Large deviations properties of the intensive local time $a = \frac{A - A_0}{t - t_0}$

The probability $\Pi(A = A_0 + (t - t_0)a, t|x_0, A_0, t_0)$ to see the intensive local time a different from its equilibrium value a_{eq} will display the large deviation form with respect to $(t - t_0)$

$$\Pi(A = A_0 + (t - t_0)a, t|x_0, A_0, t_0) \underset{(t-t_0) \rightarrow +\infty}{\propto} e^{-(t-t_0)I(a)} \quad (94)$$

The positive rate function $I(a) \geq 0$ is defined for $a \in [0, +\infty[$ and vanishes only for the equilibrium value a_{eq} of Eq. 93 where it is minimum

$$I(a_{eq}) = 0 = I'(a_{eq}) \quad (95)$$

The Central Limit theorem governing the small Gaussian fluctuations around a_{eq} can be recovered via the Taylor expansion at second order of the rate function $I(a)$ around a_{eq}

$$I(a) = \frac{(a - a_{eq})^2}{2} I''(a_{eq}) + o((a - a_{eq})^2) \quad (96)$$

The link with the singular and regular contributions of $\Pi(A, t|x_0, A_0, t_0)$ can be understood as follows.

(i) The survival probability $S^{abs}(t|x_0, t_0)$ representing the weight of the singular contribution $\Pi^{Singular}(A, t|x_0, A_0, t_0)$ of Eq. 69 corresponds to the value $a = 0$ and will thus display the following exponential decay with respect to $(t - t_0)$ that involves the boundary value $I(a = 0)$

$$S^{abs}(t|x_0, t_0) \underset{(t-t_0) \rightarrow +\infty}{\propto} e^{-(t-t_0)I(a=0)} \quad (97)$$

(ii) For the regular contribution, the compatibility at leading order in the exponentials between the large deviation form of Eq. 94 and the Laplace transform $\hat{\Pi}_s^{Regular}(A|x_0, A_0)$ of Eq. 71 yields [95]

$$e^{-\frac{(A-A_0)}{\hat{G}_s(0|0)}} \underset{(A-A_0) \rightarrow +\infty}{\simeq} \int_{t_0}^{+\infty} dt e^{-s(t-t_0)} e^{-(t-t_0)I\left(\frac{A-A_0}{t-t_0}\right)} \quad (98)$$

It is thus convenient to make a change of variable in the integral from the time t towards the intensive local time $a = \frac{A-A_0}{t-t_0}$

$$e^{-\frac{(A-A_0)}{\hat{G}_s(0|0)}} \underset{(A-A_0) \rightarrow +\infty}{\simeq} \int_0^{+\infty} \frac{da}{a^2} (A - A_0) e^{-(A-A_0)\frac{s+I(a)}{a}} \quad (99)$$

For large increment $(A - A_0) \rightarrow +\infty$, the evaluation of this integral via the saddle-point method allows to obtain the following link between $\hat{G}_s(0|0)$ and the rate function $I(a)$ [95]

$$\begin{aligned} \frac{1}{\hat{G}_s(0|0)} &= \frac{s + I(a)}{a} \\ 0 &= \partial_a \left[\frac{s + I(a)}{a} \right] = \frac{s + I'(a)}{a} - \frac{s + I(a)}{a^2} \end{aligned} \quad (100)$$

This quasi-Legendre transform can be written in reciprocal form in order to compute the rate function $I(a)$ from the knowledge of $\hat{G}_s(0|0)$, as discussed in detail in the next subsection.

2. Evaluation of the leading order of $\Pi^{Regular}(A = A_0 + (t - t_0)a, t|x_0, A_0, t_0)$ with the prefactors

For the Doob conditioned processes that will be discussed in section VI, one needs to compute the dependence with respect to the initial position x_0 , so that one needs to include the prefactors in the reciprocal calculation concerning the Laplace inverse of $\hat{\Pi}_s^{Regular}(A|x_0, A_0)$ of Eq. 71

$$\begin{aligned} \Pi^{Regular}(A, t|x_0, A_0, t_0) &= \int_{c-i\infty}^{c+i\infty} \frac{ds}{2i\pi} e^{s(t-t_0)} \hat{\Pi}_s^{Regular}(A|x_0, A_0) \\ &= \int_{c-i\infty}^{c+i\infty} \frac{ds}{2i\pi} e^{s(t-t_0)} \frac{\hat{G}_s(0|x_0)}{s\hat{G}_s^2(0|0)} e^{-\frac{(A-A_0)}{\hat{G}_s(0|0)}} \end{aligned} \quad (101)$$

Here the goal is to evaluate this regular contribution for $A = A_0 + (t - t_0)a$

$$\Pi^{Regular}(A = A_0 + (t - t_0)a, t|x_0, A_0, t_0) = \int_{c-i\infty}^{c+i\infty} \frac{ds}{2i\pi} \frac{\hat{G}_s(0|x_0)}{s\hat{G}_s^2(0|0)} e^{-(t-t_0)\left[\frac{a}{\hat{G}_s(0|0)} - s\right]} \quad (102)$$

For large $(t - t_0)$, the saddle-point evaluation of this integral will be governed by the solution s_a of the following equation in s as a function of the parameter a

$$0 = \partial_s \left[\frac{a}{\hat{G}_s(0|0)} - s \right] = a \partial_s \left[\frac{1}{\hat{G}_s(0|0)} \right] - 1 \quad \text{with solution } s = s_a \quad (103)$$

In the integral of Eq. 102, one then needs to make the change of variable around this saddle-point value s_a

$$s = s_a + i\omega \quad (104)$$

The Taylor expansion at second order in ω of the function in the exponential

$$\left[\frac{a}{\hat{G}_s(0|0)} - s \right]_{s=s_a+i\omega} = I(a) + 0 + \frac{\omega^2}{2} K(a) + o(\omega^2) \quad (105)$$

involves the two functions

$$\begin{aligned} I(a) &= \left(\frac{a}{\hat{G}_s(0|0)} - s \right) \Big|_{s=s_a} \\ K(a) &= - \left(\partial_s^2 \left[\frac{a}{\hat{G}_s(0|0)} - s \right] \right) \Big|_{s=s_a} \end{aligned} \quad (106)$$

In particular, the rate function $I(a)$ can be computed from the knowledge of $\hat{G}_s(0|0)$ via Eq. 106 using the saddle-point value s_a determined by Eq. 103. As it should for consistency, Eqs 106 and 103 correspond to the reciprocal quasi-Legendre transform of Eq. 100. Simple examples will be given in Eqs 189 191 192 193, as well as in Eqs 228 247 251 252.

Putting everything together, one obtains the final result for the leading order of Eq. 102 based on the remaining Gaussian integral over ω

$$\begin{aligned} \Pi^{Regular}(A = A_0 + (t - t_0)a, t|x_0, A_0, t_0) &\underset{(t-t_0) \rightarrow +\infty}{\simeq} \frac{\hat{G}_{s_a}(0|x_0)}{s_a \hat{G}_{s_a}^2(0|0)} e^{-(t-t_0)I(a)} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-(t-t_0)\frac{K(a)}{2}\omega^2} \\ &\underset{(t-t_0) \rightarrow +\infty}{\simeq} \frac{\hat{G}_{s_a}(0|x_0)}{s_a \hat{G}_{s_a}^2(0|0)} \frac{e^{-(t-t_0)I(a)}}{\sqrt{2\pi(t-t_0)K(a)}} \end{aligned} \quad (107)$$

Note that the dependence with respect to the initial position x_0 is only in the function $\hat{G}_{s_a}(0|x_0)$ evaluated for the saddle-point value s_a determined by Eq. 103.

3. *Evaluation of the leading order of $P^{Regular}(x, A = A_0 + (t - t_0)a, t|x_0, A_0, t_0)$ with the prefactors*

Similarly, let us now consider the Laplace inverse of $P_s^{Regular}(x, A|x_0, A_0)$ of Eq. 62

$$\begin{aligned} P^{Regular}(x, A, t|x_0, A_0, t_0) &= \int_{c-i\infty}^{c+i\infty} \frac{ds}{2i\pi} e^{s(t-t_0)} \hat{P}_s^{Regular}(x, A|x_0, A_0) \\ &= \int_{c-i\infty}^{c+i\infty} \frac{ds}{2i\pi} e^{s(t-t_0)} \left[\frac{\hat{G}_s(x|0)\hat{G}_s(0|x_0)}{\hat{G}_s^2(0|0)} \right] e^{-\frac{(A-A_0)}{\hat{G}_s(0|0)}} \end{aligned} \quad (108)$$

Again we are interested into the value $A = A_0 + (t - t_0)a$

$$P^{Regular}(x, A = A_0 + (t - t_0)a, t|x_0, A_0, t_0) = \int_{c-i\infty}^{c+i\infty} \frac{ds}{2i\pi} \left[\frac{\hat{G}_s(x|0)\hat{G}_s(0|x_0)}{\hat{G}_s^2(0|0)} \right] e^{-(t-t_0) \left[\frac{a}{\hat{G}_s(0|0)} - s \right]} \quad (109)$$

So we can use the same saddle-point method described in the previous subsection to obtain the final result analog to Eq. 107

$$P^{Regular}(x, A = A_0 + (t - t_0)a, t|x_0, A_0, t_0) \underset{(t-t_0) \rightarrow +\infty}{\simeq} \frac{\hat{G}_{s_a}(x|0)\hat{G}_{s_a}(0|x_0)}{\hat{G}_{s_a}^2(0|0)} \frac{e^{-(t-t_0)I(a)}}{\sqrt{2\pi(t-t_0)K(a)}} \quad (110)$$

Note that the dependence with respect to the initial position x_0 and to the final position x are only in the functions $\hat{G}_{s_a}(0|x_0)$ and $\hat{G}_{s_a}(x|0)$ evaluated for the saddle-point value s_a determined by Eq. 103.

C. When $X(t)$ is recurrent but does not converge towards an equilibrium state

Among recurrent diffusions that do not converge towards an equilibrium state, the simplest case is the pure Brownian motion without drift $\mu = 0$, that will be discussed in section VII.

When the diffusion process $X(t)$ is recurrent but does not converge towards an equilibrium state, then the first moment $m^{[k=1]}(t|x_0, t_0)$ of the local time increment $[A(t) - A(t_0)]$ of Eq. 78 will diverge for $(t - t_0) \rightarrow +\infty$ in contrast to Eq. 83

$$m^{[k=1]}(t|x_0, t_0) \underset{(t-t_0) \rightarrow +\infty}{\simeq} +\infty \quad (111)$$

However this divergence will be weaker than the extensive behavior of Eq. 91

$$\frac{m^{[k=1]}(t|x_0, t_0)}{(t - t_0)} \underset{(t-t_0) \rightarrow +\infty}{\simeq} 0 \quad (112)$$

Nevertheless, the saddle-point evaluations of Eq. 102 can still be performed to obtain as in Eq. 107 the leading behavior

$$\Pi^{Regular}(A = A_0 + (t - t_0)a, t|x_0, A_0, t_0) \underset{(t-t_0) \rightarrow +\infty}{\propto} \frac{\hat{G}_{s_a}(0|x_0)}{s_a \hat{G}_{s_a}^2(0|0)} \frac{e^{-(t-t_0)I(a)}}{\sqrt{2\pi(t-t_0)K(a)}} \quad (113)$$

The only important difference is that the rate function $I(a)$ defined for $a \in [0, +\infty[$ will now vanish at the boundary $a = 0$ where it is minimum

$$I(a = 0) = 0 = I'(a = 0^+) \quad (114)$$

instead of the finite value $a_{eq} > 0$ of Eq. 93 discussed in the previous subsection.

VI. CONSTRUCTION OF CONDITIONED PROCESSES INVOLVING THE LOCAL TIME

In this section, the goal is to construct various conditioned joint processes $[X^*(t), A^*(t)]$ satisfying certain conditions involving the local time.

A. Conditioned Bridge towards the position x_T^* and the local time A_T^* at the time horizon T

When the initial position is x_0 with $A_0 = 0$ at time $t = 0$, the conditioning towards the final position x_T^* and the final local time A_T^* at the time horizon T leads to the following conditioned probability $P_T^{[x_T^*, A_T^*]}(x, A, t)$ for the position x and the local time A at some interior time $t \in]0, T[$ in terms of the unconditioned joint propagator $P(x_2, A_2, t_2 | x_1, A_1, t_1)$ described in section III

$$P_T^{[x_T^*, A_T^*]}(x, A, t) = \frac{P(x_T^*, A_T^*, T | x, A, t) P(x, A, t | x_0, A_0 = 0, 0)}{P(x_T^*, A_T^*, T | x_0, A_0 = 0, 0)} \quad (115)$$

As described in detail in [43, 44], the corresponding conditioned process $[X^*(t), A^*(t)]$ then satisfies the Ito system analog to Eq. 33

$$\begin{aligned} dX^*(t) &= \mu_T^{[x_T^*, A_T^*]}(X^*(t), A^*(t), t) dt + dB(t) \\ dA^*(t) &= \delta(X^*(t)) dt \end{aligned} \quad (116)$$

where the conditioned drift $\mu_T^{[x_T^*, A_T^*]}(x, A, t)$ involves the unconditioned drift $\mu(x)$ and the logarithmic derivative of the unconditioned propagator $P(x_T^*, A_T^*, T | x, A, t)$ with respect to x

$$\mu_T^{[x_T^*, A_T^*]}(x, A, t) = \mu(x) + \partial_x \ln P(x_T^*, A_T^*, T | x, A, t) \quad (117)$$

The decomposition of the unconditioned joint propagator $P(x_T^*, A_T^*, T | x, A, t)$ into its singular contribution of Eq. 53 corresponding to $A_T^* = A$ and its regular contribution $P^{Regular}(x_T^*, A_T^*, T | x, A, t)$ of Eq. 64 corresponding to $A_T^* > A$

$$P(x_T^*, A_T^*, T | x, A, t) = \delta(A_T^* - A) G^{abs}(x_T^*, T | x, t) + \theta(A_T^* > A) P^{Regular}(x_T^*, A_T^*, T | x, A, t) \quad (118)$$

yields that the conditioned dynamics can be decomposed into the two following regions.

(i) In the region $A_0 = 0 \leq A < A_T^*$ where the local time A has not yet reached its conditioned final value A_T^* , the conditioned drift of Eq. 117 involves the regular contribution of the propagator

$$\mu_T^{[x_T^*, A_T^*]}(x, A < A_T^*, t) = \mu(x) + \partial_x \ln P^{Regular}(x_T^*, A_T^*, T | x, A, t) \quad (119)$$

(ii) In the region $A = A_T^*$ where the local time A has already reached its conditioned final value A_T^* , and where the position x cannot visit the origin $x = 0$ anymore, the conditioned drift of Eq. 117 involves the singular contribution of the propagator

$$\mu_T^{[x_T^*, A_T^*]}(x, A = A_T^*, t) = \mu(x) + \partial_x \ln G^{abs}(x_T^*, T | x, t) \quad (120)$$

As a consequence, it only depends on the propagator $G^{abs}(x_T^*, T | x, t)$ in the presence of an absorbing boundary at the origin so that one recovers the standard problem of a diffusion conditioned to avoid the origin. Note that for the special case where the final conditioned position is at the origin $x_T^* = 0$, this region (ii) does not exist, and the local time A should reach its final conditioned value A_T^* only at the final time T .

Examples of conditioned bridges towards the position x_T^* and the local time A_T^* at the time horizon T will be given in subsections VII E and VIII E.

Generalization : conditioning towards some joint distribution $P_T^(x_T, A_T)$ of the position x_T and of the local time A_T*

If instead of the bridge described above, one wishes to impose some joint distribution $P_T^*(x_T, A_T)$ of the final position x_T and of the final local time A_T at the time horizon T , the conditioned probability $P^*(x, A, t)$ for the position and the local time A at some interior time $t \in]0, T[$ reads

$$P_T^*(x, A, t) = Q_T(x, A, t) P(x, A, t | x_0, A_0 = 0, 0) \quad (121)$$

where the function $Q_T(x, A, t)$ reads in terms of the final distribution $P_T^*(x_T, A_T)$ that one wishes to impose

$$Q_T(x, A, t) \equiv \int_{-\infty}^{+\infty} dx_T \int_0^{+\infty} dA_T P^*(x_T, A_T, T) \frac{P(x_T, A_T, T | x, A, t)}{P(x_T, A_T, T | x_0, A_0 = 0, 0)} \quad (122)$$

As a consequence, the conditioned drift $\mu^*(x, A, t)$ now involves the logarithmic derivative of the function $Q(x, A, t)$ of Eq. 122 with respect to x

$$\mu_T^*(x, A, t) = \mu(x) + \partial_x \ln Q_T(x, A, t) \quad (123)$$

B. Conditioned Bridge towards the local time A_T^* at the time horizon T

If the conditioning is towards the local time A_T^* at time horizon T , without any condition on the final position x_T , the conditioned probability for the position x and the local time A at some interior time $t \in]0, T[$ involves the unconditioned probability $\Pi(A_2, t_2 | x_1, A_1, t_1)$ described in section IV

$$P_T^{[A_T^*]}(x, A, t) = \frac{\Pi(A_T, T | x, A, t) P(x, A, t | x_0, A_0 = 0, 0)}{\Pi(A_T, T | x_0, A_0 = 0, 0)} \quad (124)$$

The corresponding conditioned drift then involves the logarithmic derivative of the unconditioned probability $\Pi(A_T^*, T | x, A, t)$ with respect to x

$$\mu_T^{[A_T^*]}(x, A, t) = \mu(x) + \partial_x \ln \Pi(A_T^*, T | x, A, t) \quad (125)$$

Again, the decomposition of $\Pi(A_T^*, T | x, A, t)$ into its singular contribution of Eq. 69 corresponding to $A_T^* = A$ and its regular contribution $\Pi^{Regular}(A_T^*, T | x, A, t)$ of Eq. 71 corresponding to $A_T^* > A$

$$\Pi(A_T^*, T | x, A, t) = \delta(A_T^* - A) S^{abs}(T | x, t) + \theta(A_T^* > A) \Pi^{Regular}(A_T^*, T | x, A, t) \quad (126)$$

yields that the conditioned dynamics can be decomposed into the two following regions.

(i) In the region $A_0 = 0 \leq A < A_T^*$ where the local time A has not yet reached its conditioned final value A_T^* , the conditioned drift of Eq. 125 involves the regular contribution

$$\mu_T^{[A_T^*]}(x, A < A_T^*, t) = \mu(x) + \partial_x \ln \Pi^{Regular}(A_T^*, T | x, A, t) \quad (127)$$

(ii) In the region $A = A_T^*$ where the local time A has already reached its conditioned final value A_T^* , and where the position x cannot visit the origin $x = 0$ anymore, the conditioned drift of Eq. 125 involves the singular contribution

$$\mu_T^{[A_T^*]}(x, A = A_T^*, t) = \mu(x) + \partial_x \ln \Pi^{Singular}(A_T^*, T | x, A, t) = \mu(x) + \partial_x \ln S^{abs}(T | x, t) \quad (128)$$

It depends only on the survival probability $S^{abs}(x_T^*, T | x, t)$ in the presence of an absorbing boundary at the origin, so that one recovers the standard problem of a diffusion conditioned to survive up to time T .

Example of the conditioned bridge towards the local time A_T^* at the time horizon T will be given in subsections VII F and VIII F.

It is now interesting to consider two possibilities in the limit of the infinite horizon $T \rightarrow +\infty$, as described in the two next subsections.

1. Conditioning towards the finite asymptotic local time $A_\infty < +\infty$ for the infinite horizon $T \rightarrow +\infty$

If one wishes to impose the finite asymptotic local time $A_\infty < +\infty$ for the infinite horizon $T \rightarrow +\infty$, one needs to analyze the limit of the infinite horizon $T \rightarrow +\infty$ for the conditioned drift of Eq. 127

$$\mu_\infty^{[A_\infty^*]}(x, A < A_\infty^*, t) = \mu(x) + \lim_{T \rightarrow +\infty} [\partial_x \ln \Pi^{Regular}(A_\infty^*, T | x, A, t)] \quad (129)$$

and for the conditioned drift of Eq. 128

$$\mu_\infty^{[A_\infty^*]}(x, A = A_\infty^*, t) = \mu(x) + \lim_{T \rightarrow +\infty} [\partial_x \ln \Pi^{Singular}(A_\infty^*, T | x, A, t)] = \mu(x) + \lim_{T \rightarrow +\infty} [\partial_x \ln S^{abs}(T | x, t)] \quad (130)$$

An example of the conditioning towards the finite asymptotic local time $A_\infty < +\infty$ for the infinite horizon $T \rightarrow +\infty$ will be given in subsection VII H.

2. Conditioning towards the intensive local time $a^* = \frac{A_T^*}{T}$ for large time horizon $T \rightarrow +\infty$

If one wishes to impose instead the fixed intensive local time a^* for large time horizon $T \rightarrow +\infty$, one needs to plug the value

$$A_T^* = T a^* \quad (131)$$

into the conditioned drift of Eq. 127

$$\mu_T^{[Ta^*]}(x, A < Ta^*, t) = \mu(x) + \partial_x \ln \Pi^{Regular}(Ta^*, T|x, A, t) \quad (132)$$

The asymptotic form of Eq. 107 for the propagator $\Pi^{Regular}(Ta^*, T|x, A, t)$ in the region $t \ll T$

$$\Pi^{Regular}(Ta^*, T|x, A, t) \underset{(T-t) \rightarrow +\infty}{\simeq} \frac{\hat{G}_{s_{a_t}}(0|x)}{s_{a_t} \hat{G}_{s_{a_t}}^2(0|0)} \frac{e^{-(T-t)I(a_t)}}{\sqrt{2\pi(T-t)K(a_t)}} \quad (133)$$

involves the corresponding intensive local time a_t on the time interval $(T-t)$

$$a_t \equiv \frac{Ta^* - A}{T-t} \underset{T \rightarrow +\infty}{\simeq} a^* \quad (134)$$

that reduces to a^* at leading order when $T \rightarrow +\infty$. So at leading order for the large time horizon $T \rightarrow +\infty$, the conditioned drift of Eq. 132 reduces

$$\mu_T^{[Ta^*]}(x, A < Ta^*, t) \underset{T \rightarrow +\infty}{\simeq} \mu(x) + \partial_x \ln \hat{G}_{s_{a^*}}(0|x) \equiv \mu_\infty^{[a^*]}(x) \quad (135)$$

to the time-independent drift $\mu_\infty^{[a^*]}(x)$ where s_{a^*} should be computed as the solution of the saddle-point Equation 103

$$0 = a^* \partial_s \left[\frac{1}{\hat{G}_s(0|0)} \right] - 1 \quad (136)$$

Using the similarity transformation of Eq. 10 for time-Laplace transforms

$$\hat{G}_s(0|x) = e^{-\int_0^x dy \mu(y)} \hat{\psi}_s(0|x) \quad (137)$$

one obtains that the conditioned drift of Eq. 135 only involves the logarithmic derivation with respect to x of the time Laplace transform $\hat{\psi}_{s_{a^*}}(0|x)$ of the quantum propagator

$$\mu_\infty^{[a^*]}(x) \equiv \mu(x) + \partial_x \ln \hat{G}_{s_{a^*}}(0|x) = \partial_x \ln \hat{\psi}_{s_{a^*}}(0|x) \quad (138)$$

while Eq. 136 for s_a^* becomes

$$0 = a^* \partial_s \left[\frac{1}{\hat{\psi}_s(0|0)} \right] - 1 \quad (139)$$

The conditioned potential $U_\infty^{[a^*]}(x)$ associated to the conditioned drift $\mu_\infty^{[a^*]}(x)$ of Eq. 138 via Eq. 9

$$U^{[a^*]}(x) \equiv -2 \int_0^x dy \mu_\infty^{[a^*]}(y) = \ln \left(\frac{\hat{\psi}_{s_{a^*}}(0|0)}{\hat{\psi}_{s_{a^*}}(0|x)} \right)^2 \quad (140)$$

corresponds to the conditioned Boltzmann equilibrium of Eq. 28

$$G_{eq}^{[a^*]}(x) = \frac{e^{-U^{[a^*]}(x)}}{\int_{-\infty}^{+\infty} dy e^{-U^{[a^*]}(y)}} = \frac{[\hat{\psi}_{s_{a^*}}(0|x)]^2}{\int_{-\infty}^{+\infty} dy [\hat{\psi}_{s_{a^*}}(0|y)]^2} \quad (141)$$

Example of the conditioning towards the intensive local time $a^* = \frac{A_T^*}{T}$ for large time horizon $T \rightarrow +\infty$ will be given in subsections VII G and VIII G.

3. Generalization : conditioning towards the distribution $\Pi_T^*(A_T)$ of the local time A_T at the time horizon T

If instead of the bridge corresponding to the single value A_T^* , one wishes to impose some distribution $\Pi_T^*(A_T)$ of the local time A_T at the time horizon T , the conditioned probability for the position x and the local time A at some interior time $t \in]0, T[$ is given by

$$P_T^*(x, A, t) = Q_T(x, A, t) P(x, A, t|x_0, A_0 = 0, 0) \quad (142)$$

where the function $Q_T(x, A, t)$ involves the final distribution $\Pi_T^*(A_T)$ that one wishes to impose

$$Q_T(x, A, t) \equiv \int_0^{+\infty} dA_T \Pi^*(A_T, T) \frac{\Pi(A_T, T|x, A, t)}{\Pi(A_T, T|x_0, A_0 = 0, 0)} \quad (143)$$

Its logarithmic derivative with respect to x allows to compute the corresponding conditioned drift

$$\mu_T^*(x, A, t) = \mu(x) + \partial_x \ln Q_T(x, A, t) \quad (144)$$

VII. APPLICATION TO THE UNIFORM DRIFT $\mu \geq 0$

In this section, the unconditioned process is the Brownian motion with uniform drift $\mu(x) = \mu \geq 0$, so the Ito system of Eq. 33 reads

$$\begin{aligned} dX(t) &= \mu dt + dB(t) \\ dA(t) &= \delta(X(t))dt \end{aligned} \quad (145)$$

Note that in the transient cases $\mu > 0$, the local time increment $(A - A_0)$ of this unconditioned process remains finite for $(t - t_0) \rightarrow +\infty$ as described in subsection V A, while $\mu = 0$ corresponds to the case of recurrent diffusion that does not converge towards an equilibrium state discussed in subsection V C.

A. Properties of the unconditioned diffusion process $X(t)$ alone

1. Propagator $G(x, t|x_0, t_0)$ for the position alone

The propagator $G(x, t|x_0, t_0)$ discussed in section II is Gaussian

$$G(x, t|x_0, t_0) = \frac{1}{\sqrt{2\pi(t-t_0)}} e^{-\frac{[x-x_0-\mu(t-t_0)]^2}{2(t-t_0)}} = \frac{1}{\sqrt{2\pi(t-t_0)}} e^{-\frac{(x-x_0)^2}{2(t-t_0)} + \mu(x-x_0) - \frac{\mu^2}{2}(t-t_0)} \quad (146)$$

and its time Laplace transform reads

$$\begin{aligned} \tilde{G}_s(x|x_0) &\equiv \int_{t_0}^{+\infty} dt e^{-s(t-t_0)} G(x, t|x_0, t_0) = \frac{e^{\mu(x-x_0)}}{\sqrt{2\pi}} \int_0^{+\infty} d\tau \tau^{-\frac{1}{2}} e^{-\left(s + \frac{\mu^2}{2}\right)\tau} e^{-\frac{(x-x_0)^2}{2\tau}} \\ &= \frac{e^{\mu(x-x_0) - \sqrt{\mu^2 + 2s}|x-x_0|}}{\sqrt{\mu^2 + 2s}} \end{aligned} \quad (147)$$

2. Properties in the presence of an absorbing boundary at the origin $x = 0$

The Laplace transform $\hat{G}_s^{abs}(x|x_0)$ of Eq. 49 reads using Eq. 147

$$\begin{aligned} \hat{G}_s^{abs}(x|x_0) &= \hat{G}_s(x|x_0) - \frac{\hat{G}_s(x|0)\hat{G}_s(0|x_0)}{\hat{G}_s(0|0)} = \frac{e^{\mu(x-x_0) - \sqrt{\mu^2 + 2s}|x-x_0|}}{\sqrt{\mu^2 + 2s}} - \frac{e^{\mu(x-x_0) - \sqrt{\mu^2 + 2s}(|x|+|x_0|)}}{\sqrt{\mu^2 + 2s}} \\ &= \frac{e^{\mu(x-x_0)}}{\sqrt{\mu^2 + 2s}} \left[e^{-\sqrt{\mu^2 + 2s}|x-x_0|} - e^{-\sqrt{\mu^2 + 2s}(|x|+|x_0|)} \right] \end{aligned} \quad (148)$$

Its Laplace inversion with respect to s yields the propagator $G^{abs}(x, t|x_0, t_0)$ in the presence of an absorbing boundary at the origin $x = 0$

$$G^{abs}(x, t|x_0, t_0) = \frac{e^{\mu(x-x_0) - \frac{\mu^2}{2}(t-t_0)}}{\sqrt{2\pi(t-t_0)}} \left[e^{-\frac{(x-x_0)^2}{2(t-t_0)}} - e^{-\frac{(|x|+|x_0|)^2}{2(t-t_0)}} \right] \quad (149)$$

in agreement with the method of images.

The Laplace transform $\hat{\gamma}_s^{abs}(x_0)$ of Eq. 59 reads using Eq. 147

$$\hat{\gamma}_s^{abs}(x_0) = \frac{\hat{G}_s(0|x_0)}{\hat{G}_s(0|0)} = e^{-\mu x_0 - \sqrt{\mu^2 + 2s}|x_0|} \quad (150)$$

The rewriting of the Laplace transform of Eq. 147 in terms of the parameter $\alpha > 0$

$$\frac{e^{-\alpha\sqrt{\mu^2 + 2s}}}{\sqrt{\mu^2 + 2s}} = \int_0^{+\infty} \frac{d\tau}{\sqrt{2\pi\tau}} e^{-\left(s + \frac{\mu^2}{2}\right)\tau} e^{-\frac{\alpha^2}{2\tau}} \quad (151)$$

allows to obtain via the derivation with respect to α

$$\begin{aligned} e^{-\alpha\sqrt{\mu^2+2s}} &= -\partial_\alpha \left(\frac{e^{-\alpha\sqrt{\mu^2+2s}}}{\sqrt{\mu^2+2s}} \right) = -\int_0^{+\infty} \frac{d\tau}{\sqrt{2\pi\tau}} e^{-\left(s+\frac{\mu^2}{2}\right)\tau} \partial_\alpha e^{-\frac{\alpha^2}{2\tau}} \\ &= \int_0^{+\infty} d\tau e^{-\left(s+\frac{\mu^2}{2}\right)\tau} \left(\frac{\alpha}{\sqrt{2\pi\tau^{\frac{3}{2}}}} \right) e^{-\frac{\alpha^2}{2\tau}} \end{aligned} \quad (152)$$

The Laplace inversion of Eq. 150 thus yields the absorption rate $\gamma^{abs}(t|x_0, t_0)$ at time t if one starts at position x_0 at time t_0

$$\gamma^{abs}(t|x_0, t_0) = \frac{|x_0|}{\sqrt{2\pi}(t-t_0)^{\frac{3}{2}}} e^{-\mu x_0 - \frac{\mu^2}{2}(t-t_0) - \frac{x_0^2}{2(t-t_0)}} \quad (153)$$

Finally the survival probability $S^{abs}(t|x_0, t_0)$ of Eq. 54 can be obtained from the integral over the final position x of the propagator $G^{abs}(x, t|x_0, t_0)$ of Eq. 149

$$S^{abs}(t|x_0, t_0) \equiv \int_{-\infty}^{+\infty} dx G^{abs}(x, t|x_0, t_0) = \int_{-\infty}^{+\infty} dx \frac{e^{\mu(x-x_0) - \frac{\mu^2}{2}(t-t_0)}}{\sqrt{2\pi(t-t_0)}} \left[e^{-\frac{(x-x_0)^2}{2(t-t_0)}} - e^{-\frac{(|x|+|x_0|)^2}{2(t-t_0)}} \right] \quad (154)$$

Its Laplace transform of Eq. 57 reads using Eq. 150

$$\hat{S}_s^{abs}(x_0) = \frac{1}{s} \left[1 - \frac{\hat{G}_s(0|x_0)}{\hat{G}_s(0|0)} \right] = \frac{1}{s} \left[1 - e^{-\mu x_0 - \sqrt{\mu^2+2s}|x_0|} \right] \quad (155)$$

For $\mu > 0$, the forever-survival probability $S^{abs[\mu>0]}(\infty|x_0)$ at infinite time $(t-t_0) \rightarrow +\infty$ when starting at the position x_0 can be recovered via the limit $s \rightarrow 0$ of

$$S^{abs[\mu>0]}(\infty|x_0) = \lim_{s \rightarrow 0} \left[s \hat{S}_s^{abs}(x_0) \right] = 1 - e^{-\mu(x_0+|x_0|)} = \begin{cases} 0 & \text{if } x_0 \leq 0 \\ 1 - e^{-2\mu x_0} & \text{if } x_0 > 0 \end{cases} \quad (156)$$

It remains finite for $x_0 > 0$, since the particle can escape towards $(+\infty)$ without touching the origin $x = 0$. The finite-time survival probability $S^{abs}(t|x_0, t_0)$ given by Eq. 154 can be expressed in terms of the Error function $\text{erf}(x)$ and in terms of the complementary Error function $\text{erfc}(x) = 1 - \text{erf}(x)$ as

$$S^{abs}(t|x_0, t_0) = \begin{cases} \frac{1}{2} \left[1 + \text{erf} \left(\frac{x_0 + \mu(t-t_0)}{\sqrt{2(t-t_0)}} \right) - e^{-2\mu x_0} \text{erfc} \left(\frac{x_0 - \mu(t-t_0)}{\sqrt{2(t-t_0)}} \right) \right] & \text{if } x_0 > 0 \\ \frac{1}{2} \left[e^{-2\mu x_0} \left(-2 + \text{erfc} \left(\frac{x_0 - \mu(t-t_0)}{\sqrt{2(t-t_0)}} \right) \right) + \text{erfc} \left(\frac{x_0 + \mu(t-t_0)}{\sqrt{2(t-t_0)}} \right) \right] & \text{if } x_0 < 0 \end{cases} \quad (157)$$

In the region $x_0 \leq 0$ where the forever-survival $S^{abs[\mu>0]}(\infty|x_0)$ vanishes, using the asymptotic behavior of the complementary Error function

$$\text{erfc}(x) \simeq \begin{cases} e^{-x^2} \left(\frac{1}{\sqrt{\pi}x} - \frac{1}{2\sqrt{\pi}x^3} \right) & \text{when } x \rightarrow \infty \\ 2 + e^{-x^2} \left(\frac{1}{\sqrt{\pi}x} - \frac{1}{2\sqrt{\pi}x^3} \right) & \text{when } x \rightarrow -\infty \end{cases} \quad (158)$$

the asymptotic behavior of $S^{abs}(t|x_0, t_0)$ for large time $(t-t_0)$ and fixed x_0 reads

$$S^{abs[\mu>0]}(t|x_0 < 0, t_0) \underset{(t-t_0) \rightarrow +\infty}{\simeq} \sqrt{\frac{2}{\pi}} \frac{|x_0| e^{\mu|x_0| - \frac{\mu^2}{2}(t-t_0)}}{\mu^2(t-t_0)^{\frac{3}{2}}} \quad (159)$$

For $\mu = 0$, the forever-survival probability $S^{abs[\mu=0]}(\infty|x_0)$ of Eq. 156 vanishes for any x_0 . The leading singularity of Eq. 155 for $s \rightarrow 0^+$

$$\hat{S}_s^{abs[\mu=0]}(x_0) = \frac{1}{s} \left[1 - e^{-\sqrt{2s}|x_0|} \right] \underset{s \rightarrow 0^+}{\simeq} |x_0| \sqrt{\frac{2}{s}} \quad (160)$$

allows to recover the dominant asymptotic behavior for large time

$$S^{abs[\mu=0]}(t|x_0, t_0) \underset{(t-t_0) \rightarrow +\infty}{\simeq} |x_0| \sqrt{\frac{2}{\pi(t-t_0)}} \quad (161)$$

B. Joint propagator $P(x, A, t|x_0, A_0, t_0)$ for the unconditioned joint process $[X(t), A(t)]$

The singular contribution of Eq. 53 involves the propagator $G^{abs}(x, t|x_0, t_0)$ of Eq. 149

$$\begin{aligned} P^{Singular}(x, A, t|x_0, A_0, t_0) &= \delta(A - A_0)G^{abs}(x, t|x_0, t_0) \\ &= \delta(A - A_0)\frac{e^{\mu(x-x_0)-\frac{\mu^2}{2}(t-t_0)}}{\sqrt{2\pi(t-t_0)}} \left[e^{-\frac{(x-x_0)^2}{2(t-t_0)}} - e^{-\frac{(|x|+|x_0|)^2}{2(t-t_0)}} \right] \end{aligned} \quad (162)$$

The Laplace transform $\hat{P}_s^{Regular}(x, A|x_0, A_0)$ of Eq. 62 reads using Eq. 147

$$\begin{aligned} \hat{P}_s^{Regular}(x, A|x_0, A_0) &= \theta(A > A_0) \left[\frac{\hat{G}_s(x|0)\hat{G}_s(0|x_0)}{\hat{G}_s^2(0|0)} \right] e^{-\frac{(A-A_0)}{\hat{G}_s(0|0)}} \\ &= \theta(A > A_0)e^{\mu(x-x_0)-\sqrt{\mu^2+2s}(|x|+|x_0|+A-A_0)} \end{aligned} \quad (163)$$

Its Laplace inversion using Eq. 152 yields

$$P^{Regular}(x, A, t|x_0, A_0, t_0) = \theta(A > A_0)e^{\mu(x-x_0)-\frac{\mu^2}{2}(t-t_0)} \left(\frac{|x|+|x_0|+A-A_0}{\sqrt{2\pi(t-t_0)^{\frac{3}{2}}}} \right) e^{-\frac{(|x|+|x_0|+A-A_0)^2}{2(t-t_0)}} \quad (164)$$

In summary, the joint propagator $P(x, A, t|x_0, A_0, t_0)$ involving the two contributions of Eq. 162 and Eq. 164 reads

$$\begin{aligned} P(x, A, t|x_0, A_0, t_0) &= P^{Singular}(x, A, t|x_0, A_0, t_0) + P^{Regular}(x, A, t|x_0, A_0, t_0) \\ &= \delta(A - A_0)\frac{e^{\mu(x-x_0)-\frac{\mu^2}{2}(t-t_0)}}{\sqrt{2\pi(t-t_0)}} \left[e^{-\frac{(x-x_0)^2}{2(t-t_0)}} - e^{-\frac{(|x|+|x_0|)^2}{2(t-t_0)}} \right] \\ &\quad + \theta(A > A_0)e^{\mu(x-x_0)-\frac{\mu^2}{2}(t-t_0)} \left(\frac{|x|+|x_0|+A-A_0}{\sqrt{2\pi(t-t_0)^{\frac{3}{2}}}} \right) e^{-\frac{(|x|+|x_0|+A-A_0)^2}{2(t-t_0)}} \end{aligned} \quad (165)$$

For $\mu = 0$ and $x_0 = 0$ (i.e. a standard Brownian motion), the regular part of the propagator $P(x, A, t|x_0, A_0, t_0)$ reduces to

$$P^{Regular}(x, A, t|0, 0, 0) = \frac{|x|+A}{\sqrt{2\pi t^3}} e^{-\frac{(|x|+A)^2}{2t}} \quad (166)$$

a result that can be found in the mathematical literature [94].

C. Probability $\Pi(A, t|x_0, A_0, t_0)$ to see the local time A at time t

The probability $\Pi(A, t|x_0, A_0, t_0)$ of Eq. 65 can be obtained via the integration of the joint propagator $P(x, A, t|x_0, A_0, t_0)$ of Eq. 165 over the final position x

$$\Pi(A, t|x_0, A_0, t_0) \equiv \int_{-\infty}^{+\infty} dx P(A, t|x_0, A_0, t_0) \quad (167)$$

Its singular contribution of Eq. 69 involves the survival probability $S^{abs}(t|x_0, t_0)$ of Eq. 154

$$\Pi^{Singular}(A, t|x_0, A_0, t_0) = \delta(A - A_0)S^{abs}(t|x_0, t_0) \quad (168)$$

The Laplace transform of Eq. 71 reads using Eq. 147

$$\hat{\Pi}_s^{Regular}(A|x_0, A_0) = \theta(A > A_0) \frac{\hat{G}_s(0|x_0)}{s\hat{G}_s^2(0|0)} e^{-\frac{(A-A_0)}{\hat{G}_s(0|0)}} = \theta(A > A_0) \frac{\sqrt{\mu^2+2s}}{s} e^{-\mu x_0 - \sqrt{\mu^2+2s}(|x_0|+A-A_0)} \quad (169)$$

1. Case $\mu = 0$

For the case $\mu = 0$, Eq. 169 reduces to

$$\hat{\Pi}_s^{Regular[\mu=0]}(A|x_0, A_0) = \theta(A > A_0) \sqrt{\frac{2}{s}} e^{-\sqrt{2s}(|x_0| + A - A_0)} \quad (170)$$

so that Eq. 151 for $\mu = 0$ can be used to obtain the Laplace inversion of Eq. 170

$$\Pi^{Regular[\mu=0]}(A, t|x_0, A_0, t_0) = \theta(A > A_0) \sqrt{\frac{2}{\pi(t-t_0)}} e^{-\frac{(|x_0| + A - A_0)^2}{2(t-t_0)}} \quad (171)$$

The singular contribution of Eq. 168 involves the survival probability of Eq. 154

$$\begin{aligned} \Pi^{Singular[\mu=0]}(A, t|x_0, A_0, t_0) &= \delta(A - A_0) S^{abs[\mu=0]}(t|x_0, t_0) \\ &= \delta(A - A_0) \int_{-\infty}^{+\infty} dx \frac{e^{-\frac{x^2 + x_0^2}{2(t-t_0)}}}{\sqrt{2\pi(t-t_0)}} \left[e^{\frac{xx_0}{(t-t_0)}} - e^{-\frac{|xx_0|}{(t-t_0)}} \right] \\ &= \delta(A - A_0) e^{-\frac{x_0^2}{2(t-t_0)}} \int_{-\infty}^{+\infty} dz \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} \left[e^{\frac{zx_0}{\sqrt{t-t_0}}} - e^{-\frac{|zx_0|}{\sqrt{t-t_0}}} \right] \\ &= \delta(A - A_0) \operatorname{erf} \left(\frac{|x_0|}{\sqrt{2(t-t_0)}} \right) \end{aligned} \quad (172)$$

Putting together the two contributions, one obtains that $\Pi^{[\mu=0]}(A, t|x_0, A_0, t_0)$ reads

$$\begin{aligned} \Pi^{[\mu=0]}(A, t|x_0, A_0, t_0) &= \delta(A - A_0) S^{abs[\mu=0]}(t|x_0, t_0) + \Pi^{Regular[\mu=0]}(A, t|x_0, A_0, t_0) \\ &= \delta(A - A_0) \operatorname{erf} \left(\frac{|x_0|}{\sqrt{2(t-t_0)}} \right) + \theta(A > A_0) \sqrt{\frac{2}{\pi(t-t_0)}} e^{-\frac{(|x_0| + A - A_0)^2}{2(t-t_0)}} \end{aligned} \quad (173)$$

Since $\operatorname{erf}(x) \underset{x \rightarrow 0}{\simeq} 2x/\sqrt{\pi}$, we get the asymptotic behavior for large $(t - t_0)$

$$\Pi^{[\mu=0]}(A, t|x_0, A_0, t_0) \underset{(t-t_0) \rightarrow +\infty}{\simeq} \sqrt{\frac{2}{\pi(t-t_0)}} \left[\delta(A - A_0)|x_0| + \theta(A > A_0) e^{-\frac{(|x_0| + A - A_0)^2}{2(t-t_0)}} \right] \quad (174)$$

In the case where $x_0 = 0$, $A_0 = 0$ and $t_0 = 0$, the propagator of Eq. 173 reduces to the half-Gaussian distribution

$$\Pi^{[\mu=0]}(A, t|0, 0, 0) = \theta(A > 0) \sqrt{\frac{2}{\pi t}} e^{-\frac{A^2}{2t}} \quad (175)$$

a result that can be found in [95, 96].

The first moment $m^{[k=1]}(t|x_0, t_0 = 0)$ of the local time increment can be computed via Eq. 73 or via Eq. 78

$$\begin{aligned} m^{[k=1]}(t|x_0, t_0 = 0) &= \int_0^\infty dA A \Pi^{[\mu=0]}(A, t|x_0, A_0 = 0, t_0 = 0) \\ &= \int_0^t dt_1 G(0, t_1|x_0, t_0 = 0) = \int_0^t dt_1 \frac{e^{-\frac{x_0^2}{2t_1}}}{\sqrt{2\pi t_1}} \\ &= \sqrt{\frac{2t}{\pi}} e^{-\frac{x_0^2}{2t}} - |x_0| \operatorname{erfc} \left(\frac{|x_0|}{\sqrt{2t}} \right) \end{aligned} \quad (176)$$

and displays the power-law asymptotic growth independent of x_0

$$m^{[k=1]}(t|x_0, t_0 = 0) \underset{t \rightarrow +\infty}{\simeq} \sqrt{\frac{2t}{\pi}} \quad (177)$$

that is intermediate as it should be between the finite case of Eq. 83 and the extensive case of Eq. 91, since the Brownian motion without drift $\mu = 0$ is recurrent but does not converge towards an equilibrium distribution.

The second moment $m^{[k=2]}(t|x_0, t_0 = 0)$ of the local time increment can be computed via Eq. 73 or Eq. 81

$$\begin{aligned}
m^{[k=2]}(t|x_0, t_0 = 0) &= \int_0^\infty dA A^2 \Pi^{[\mu=0]}(A, t|x_0, A_0 = 0, t_0 = 0) = 2 \int_0^t dt_2 \int_0^{t_2} dt_1 G(0, t_2|0, t_1) G(0, t_1|x_0, t_0 = 0) \\
&= \frac{1}{\pi} \int_0^t dt_2 \int_0^{t_2} dt_1 \frac{e^{-\frac{x_0^2}{2t_1}}}{\sqrt{(t_2 - t_1)t_1}} = \frac{1}{\pi} \int_0^t dt_1 \frac{e^{-\frac{x_0^2}{2t_1}}}{\sqrt{t_1}} \int_{t_1}^t \frac{dt_2}{\sqrt{t_2 - t_1}} = \frac{2}{\pi} \int_0^t dt_1 e^{-\frac{x_0^2}{2t_1}} \frac{\sqrt{t - t_1}}{\sqrt{t_1}} \\
&= (t + x_0^2) \operatorname{erfc}\left(\frac{|x_0|}{\sqrt{2t}}\right) - |x_0| \sqrt{\frac{2t}{\pi}} e^{-\frac{x_0^2}{2t}}
\end{aligned} \tag{178}$$

with the following asymptotic growth independent of x_0

$$m^{[k=2]}(t|x_0, t_0 = 0) \underset{t \rightarrow +\infty}{\simeq} t \tag{179}$$

2. Case $\mu > 0$

For the case $\mu > 0$, integrating the singular and regular part of the joint propagator $P(x, A, t|x_0, A_0, t_0)$ of Eq. 165 with respect to the final position x , gives respectively

$$\begin{aligned}
\Pi^{Singular}(A, t|x_0, A_0, t_0) &= \delta(A - A_0) \int_{-\infty}^{+\infty} dx \frac{e^{\mu(x-x_0) - \frac{\mu^2}{2}(t-t_0)}}{\sqrt{2\pi(t-t_0)}} \left[e^{-\frac{(x-x_0)^2}{2(t-t_0)}} - e^{-\frac{(|x|+|x_0|)^2}{2(t-t_0)}} \right] \\
&= \delta(A - A_0) \left[1 - \frac{1}{2} e^{-\mu x_0} \left(e^{-\mu|x_0|} \operatorname{erfc}\left(\frac{|x_0| - \mu(t-t_0)}{\sqrt{2(t-t_0)}}\right) + e^{\mu|x_0|} \operatorname{erfc}\left(\frac{|x_0| + \mu(t-t_0)}{\sqrt{2(t-t_0)}}\right) \right) \right]
\end{aligned} \tag{180}$$

and

$$\begin{aligned}
\Pi^{Regular}(A, t|x_0, A_0, t_0) &= \theta(A > A_0) \int_{-\infty}^{+\infty} dx e^{\mu(x-x_0) - \frac{\mu^2}{2}(t-t_0)} \left(\frac{|x| + |x_0| + A - A_0}{\sqrt{2\pi(t-t_0)}} \right) e^{-\frac{(|x|+|x_0|+A-A_0)^2}{2(t-t_0)}} \\
&= \theta(A > A_0) \left[\sqrt{\frac{2}{\pi(t-t_0)}} e^{-\mu x_0} e^{-\frac{\mu^2}{2}(t-t_0)} e^{-\frac{(|x_0|+A-A_0)^2}{2(t-t_0)}} \right. \\
&\quad \left. + \frac{1}{2} \mu e^{-\mu x_0} \left(e^{-\mu(|x_0|+A-A_0)} \operatorname{erfc}\left(\frac{|x_0| - \mu(t-t_0) + A - A_0}{\sqrt{2(t-t_0)}}\right) - e^{\mu(|x_0|+A-A_0)} \operatorname{erfc}\left(\frac{|x_0| + \mu(t-t_0) + A - A_0}{\sqrt{2(t-t_0)}}\right) \right) \right]
\end{aligned} \tag{181}$$

The limit of the infinite time interval $(t - t_0) \rightarrow +\infty$ yields

$$\begin{aligned}
\Pi(A, \infty|x_0, A_0) &= \Pi^{Singular}(A, \infty|x_0, A_0) + \Pi^{Regular}(A, \infty|x_0, A_0) \\
&= \delta(A - A_0) \left[1 - e^{-\mu(x_0+|x_0|)} \right] + \theta(A > A_0) \mu e^{-\mu(x_0+|x_0|+A-A_0)} \\
&= \begin{cases} \theta(A > A_0) \mu e^{-\mu(A-A_0)} & \text{if } x_0 < 0 \\ \delta(A - A_0) [1 - e^{-2\mu x_0}] + e^{-2\mu x_0} \theta(A > A_0) \mu e^{-\mu(A-A_0)} & \text{if } x_0 > 0 \end{cases}
\end{aligned} \tag{182}$$

In the region $x_0 < 0$ where the limit of the singular contribution vanishes $\Pi^{Singular}(A, \infty|x_0 < 0, A_0) = 0$, one can use the asymptotic behaviors of the erfc function given by Eq. 158 to obtain that the leading contribution to Eq. 180 reads for large time $(t - t_0)$

$$\Pi^{Singular}(A, t|x_0 < 0, A_0, t_0) \underset{(t-t_0) \rightarrow +\infty}{\simeq} \delta(A - A_0) |x_0| \sqrt{\frac{2}{\pi}} \frac{e^{-\frac{(|x_0|+\mu(t-t_0))^2}{2(t-t_0)}}}{\mu^2(t-t_0)^{3/2}} \tag{183}$$

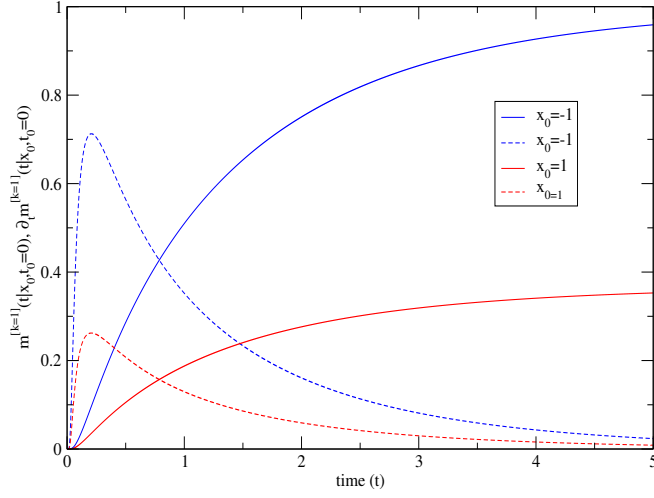


Figure 1. Plain curve: mean local time, dash curve: derivative of the mean local time. Blue lines correspond to $x_0 = -1$, Red lines correspond to $x_0 = 1$. The constant drift μ is equal to 1.

The first moment $m^{[k=1]}(t|x_0, t_0 = 0)$ of the local time increment of Eqs 73 and 78

$$\begin{aligned}
 m^{[k=1]}(t|x_0, t_0 = 0) &= \int_0^\infty dA A \Pi^{[\mu]}(A, t|x_0, A_0 = 0, t_0 = 0) = \int_0^t dt_1 G(0, t_1|x_0, t_0 = 0) \\
 &= \int_0^t dt_1 \frac{e^{-\frac{(x_0 - \mu t_1)^2}{2t_1}}}{\sqrt{2\pi t_1}} \\
 &= \frac{e^{-\mu(|x_0| + x_0)} \operatorname{erfc}\left(\frac{|x_0| - \mu t}{\sqrt{2t}}\right) - e^{\mu(|x_0| - x_0)} \operatorname{erfc}\left(\frac{|x_0| + \mu t}{\sqrt{2t}}\right)}{2\mu}
 \end{aligned} \tag{184}$$

converges to a finite asymptotic value which depends strongly on the sign of the initial position x_0 of the process

$$\lim_{t \rightarrow \infty} m^{[k=1]}(t|x_0, t_0 = 0) = \frac{e^{-\mu(x_0 - |x_0|)}}{\mu} = \begin{cases} \frac{1}{\mu} & \text{if } x_0 < 0 \\ \frac{e^{-2\mu x_0}}{\mu} & \text{if } x_0 > 0 \end{cases} \tag{185}$$

In the region $x_0 > 0$ where the process has a finite probability to escape towards $(+\infty)$ without visiting the origin, the mean local time is of course smaller than in the region $x_0 < 0$ where the process is certain to cross the origin. In the latter case, as expected, its average asymptotic value is the same as if the process started from 0.

The two first time-derivatives of the first moments

$$\begin{aligned}
 \partial_t m^{[k=1]}(t|x_0, t_0 = 0) &= \frac{e^{-\frac{(x_0 + \mu t)^2}{2t}}}{\sqrt{2\pi t}} \\
 \partial_t^2 m^{[k=1]}(t|x_0, t_0 = 0) &= \frac{e^{-\frac{(x_0 + \mu t)^2}{2t}}}{2\sqrt{2\pi t^5}} (x_0^2 - t(1 + t\mu^2))
 \end{aligned} \tag{186}$$

shows that the mean local time increases the most at a time $T = \frac{\sqrt{1+4x_0^2\mu^2}-1}{2\mu^2}$ which is independent of the sign of x_0 . The behavior of the mean local time and its derivative is shown in Fig.1.

The second moment $m^{[k=2]}(t|x_0, t_0 = 0)$ of the local time increment of Eqs 73 and 81

$$m^{[k=2]}(t|x_0, t_0 = 0) = \int_0^\infty dA A^2 \Pi^{[\mu=0]}(A, t|x_0, A_0 = 0, t_0 = 0) = 2 \int_0^t dt_2 \int_0^{t_2} dt_1 G(0, t_2|0, t_1) G(0, t_1|x_0, t_0 = 0)$$

$$= \begin{cases} \frac{1}{\mu^2} \operatorname{erfc}\left(\frac{x_0 + \mu t}{\sqrt{2t}}\right) + \frac{1}{\mu^2} e^{-\frac{\mu}{2}(6x_0 + \mu t)} \left[-2e^{2\mu x_0} \operatorname{erfc}\left(\frac{x_0}{\sqrt{2t}}\right) + e^{\mu x_0 + \frac{\mu^2}{2}t} \operatorname{erfc}\left(\frac{x_0 - \mu t}{\sqrt{2t}}\right) \right] & \text{if } x_0 > 0 \\ \frac{1}{\mu^2} \operatorname{erfc}\left(\frac{-x_0 - \mu t}{\sqrt{2t}}\right) + \frac{1}{\mu^2} e^{-\frac{\mu}{2}(6x_0 + \mu t)} \left[-2e^{2\mu x_0} \operatorname{erfc}\left(\frac{-x_0}{\sqrt{2t}}\right) + e^{\mu x_0 + \frac{\mu^2}{2}t} \operatorname{erfc}\left(\frac{x_0 - \mu t}{\sqrt{2t}}\right) \right] & \text{if } x_0 < 0 \end{cases} \quad (187)$$

D. Large deviations of the intensive local time a for the Brownian motion without drift $\mu = 0$

1. Rate function $I(a)$ for the intensive local time $a = \frac{A - A_0}{t - t_0} \in [0, +\infty[$

The probability distribution $\Pi^{[\mu=0]}(A, t|x_0, A_0, t_0)$ of Eq. 173 allows to evaluate the probability to see $A = A_0 + a(t - t_0)$

$$\Pi^{[\mu=0]}(A = A_0 + a(t - t_0), t|x_0, A_0, t_0) = \delta(a(t - t_0)) e^{-\frac{x_0^2}{2(t-t_0)}} \int_{-\infty}^{+\infty} dz \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} \left[e^{\frac{zx_0}{\sqrt{t-t_0}}} - e^{-\frac{|zx_0|}{\sqrt{t-t_0}}} \right]$$

$$+ \theta(a > 0) \sqrt{\frac{2}{\pi(t-t_0)}} e^{-\frac{x_0^2}{2(t-t_0)} - |x_0|a - \frac{a^2}{2}(t-t_0)} \quad (188)$$

So the large deviations of the intensive local time a are governed by the simple rate function [95]

$$I(a) = \frac{a^2}{2} \quad \text{for } a \in [0, +\infty[\quad (189)$$

that vanishes and is minimum at its boundary value $a = 0$ in agreement with Eq. 114.

If one includes the prefactors, the leading order of the regular contribution of Eq. 188 reads

$$\Pi^{Regular[\mu=0]}(A = A_0 + a(t - t_0), t|x_0, A_0, t_0) \underset{(t-t_0) \rightarrow +\infty}{\simeq} \sqrt{\frac{2}{\pi(t-t_0)}} e^{-|x_0|a - (t-t_0)I(a)} \quad (190)$$

The agreement with the general formula of Eq. 113 can be checked using Eq. 147 for $\mu = 0$

$$\tilde{G}_s^{[\mu=0]}(x|x_0) = \frac{e^{-\sqrt{2s}|x-x_0|}}{\sqrt{2s}} \quad (191)$$

and Eq. 103

$$0 = a\partial_s \left[\sqrt{2s} \right] - 1 = \frac{a}{\sqrt{2s}} - 1 \quad (192)$$

that leads to the saddle-point

$$s_a = \frac{a^2}{2} \quad (193)$$

2. Rate function $I(a, v)$ for the intensive local time $a = \frac{A - A_0}{t - t_0}$ and the intensive displacement $v = \frac{x - x_0}{t - t_0}$

The joint propagator of Eq. 165 for the case $\mu = 0$

$$P^{[\mu=0]}(x, A, t|x_0, A_0, t_0) = \delta(A - A_0) \frac{e^{-\frac{x^2 + x_0^2}{2(t-t_0)}}}{\sqrt{2\pi(t-t_0)}} \left[e^{\frac{xx_0}{2(t-t_0)}} - e^{-\frac{|xx_0|}{(t-t_0)}} \right]$$

$$+ \theta(A > A_0) \left(\frac{|x| + |x_0| + A - A_0}{\sqrt{2\pi(t-t_0)}^{\frac{3}{2}}} \right) e^{-\frac{x^2 + x_0^2 + (A - A_0)^2}{2(t-t_0)}} e^{-\frac{|xx_0| + (|x| + |x_0|)(A - A_0)}{(t-t_0)}} \quad (194)$$

allows to evaluate the joint probability to see $x = x_0 + v(t - t_0)$ and $A = A_0 + a(t - t_0)$

$$\begin{aligned}
P^{[\mu=0]}(x = x_0 + v(t - t_0), A = A_0 + a(t - t_0), t | x_0, A_0, t_0) &= \delta(a(t - t_0)) \frac{e^{-\frac{[x_0 + v(t - t_0)]^2 + x_0^2}{2(t - t_0)}}}{\sqrt{2\pi(t - t_0)}} \left[e^{\frac{[x_0 + v(t - t_0)]x_0}{2(t - t_0)}} - e^{-\frac{[x_0 + v(t - t_0)]x_0}{(t - t_0)}} \right] \\
&+ \theta(a > 0) \left(\frac{|x_0 + v(t - t_0)| + |x_0| + a(t - t_0)}{\sqrt{2\pi(t - t_0)}^{\frac{3}{2}}} \right) e^{-\frac{[x_0 + v(t - t_0)]^2 + x_0^2 + a^2(t - t_0)^2}{2(t - t_0)}} e^{-\frac{[x_0 + v(t - t_0)]x_0 + (|x_0 + v(t - t_0)| + |x_0|)a(t - t_0)}{(t - t_0)}} \\
&= \delta(a(t - t_0)) \frac{e^{-\frac{v^2}{2}(t - t_0) - x_0 v - \frac{x_0^2}{(t - t_0)}}}{\sqrt{2\pi(t - t_0)}} \left[e^{\frac{v x_0}{2} + \frac{x_0^2}{2(t - t_0)}} - e^{-\frac{|x_0^2 + v x_0(t - t_0)|}{(t - t_0)}} \right] \\
&+ \theta(a > 0) \left(\frac{|x_0 + v(t - t_0)| + |x_0| + a(t - t_0)}{\sqrt{2\pi(t - t_0)}^{\frac{3}{2}}} \right) e^{-\frac{v^2 + a^2}{2}(t - t_0) - x_0 v - \frac{x_0^2}{(t - t_0)} - \frac{|x_0^2 + v x_0(t - t_0)|}{(t - t_0)} - (|x_0 + v(t - t_0)| + |x_0|)a} \quad (195)
\end{aligned}$$

So the large deviations for the joint probability of the intensive local time $a = \frac{A - A_0}{t - t_0} \in [0, +\infty[$ and of the intensive displacement $v = \frac{x - x_0}{t - t_0} \in]-\infty, +\infty[$

$$P^{[\mu=0]}(x = x_0 + v(t - t_0), A = A_0 + a(t - t_0), t | x_0, A_0, t_0) \underset{(t - t_0) \rightarrow +\infty}{\propto} e^{-(t - t_0)I(a, v)} \quad (196)$$

are governed by the rate function

$$I(a, v) = \frac{v^2 + a^2}{2} + |v|a = \frac{(|v| + a)^2}{2} \quad \text{for } a \in [0, +\infty[\text{ and } v \in]-\infty, +\infty[\quad (197)$$

For any $a \in [0, +\infty[$, the joint rate function $I(a, v)$ is minimum for $v = 0$ where one recovers $I(a)$ of Eq. 189.

As a final remark, let us stress that the joint rate function $I(a, v)$ only occurs for diffusion processes that are recurrent but do not converge towards an equilibrium for the following reasons.

(i) For transient processes, the local time $(A_T - A_0)$ remains a finite random variable for $T \rightarrow +\infty$, while the large deviations properties of the intensive displacement $v = \frac{x - x_0}{t - t_0}$ are governed by some rate function $I(v)$.

(ii) For processes converging towards an equilibrium, the total displacement $(x - x_0)$ remains a finite random variable for $T \rightarrow +\infty$, while the large deviations properties of the intensive local time $a = \frac{A - A_0}{t - t_0}$ are governed by some rate function $I(a)$ as described in subsection V B 1.

E. Conditioning towards the position x_T^* and the local time A_T^* at the finite time horizon T

Let us now apply the framework described in the subsection VI A. Using the explicit joint propagator of Eq. 165

$$\begin{aligned}
\ln P(x_T, A_T, T | x, A, t) &= \mu(x_T - x) - \frac{\mu^2}{2}(T - t) - \ln \left(\sqrt{2\pi(T - t)} \right) \\
&+ \ln \left[\delta(A_T - A) \left(e^{-\frac{(x_T - x)^2}{2(T - t)}} - e^{-\frac{(|x_T| + |x|)^2}{2(T - t)}} \right) + \theta(A_T > A) \frac{(|x_T| + |x| + A_T - A)}{(T - t)} e^{-\frac{(|x_T| + |x| + A_T - A)^2}{2(T - t)}} \right] \quad (198)
\end{aligned}$$

one obtains that the conditioned drift of Eq. 117

$$\begin{aligned}
\mu_T^{[x_T^*, A_T^*]}(x, A, t) &= \mu + \partial_x \ln P(x_T^*, A_T^*, T | x, A, t) \\
&= \partial_x \ln \left[\delta(A_T^* - A) \left(e^{-\frac{(x_T^* - x)^2}{2(T - t)}} - e^{-\frac{(|x_T^*| + |x|)^2}{2(T - t)}} \right) + \theta(A_T^* > A) \frac{(|x_T^*| + |x| + A_T^* - A)}{(T - t)} e^{-\frac{(|x_T^*| + |x| + A_T^* - A)^2}{2(T - t)}} \right] \quad (199)
\end{aligned}$$

does not depend on the initial unconditioned drift μ anymore and can be decomposed into the two following regions for $x_T^* \neq 0$.

(i) In the region $A_0 = 0 \leq A < A_T^*$ where the local time A has not yet reached its conditioned final value A_T^* , Eq. 199 reduces to

$$\mu_T^{[x_T^*, A_T^*]}(x, A < A_T^*, t) = \text{sgn}(x) \left[\frac{1}{|x_T^*| + |x| + A_T^* - A} - \frac{|x_T^*| + |x| + A_T^* - A}{T - t} \right] \quad (200)$$

(ii) In the region $A = A_T^*$ where the local time A has already reached its conditioned final value A_T^* , and where the position x cannot visit the origin $x = 0$ anymore, the drift of Eq. 199 reduces to

$$\mu_T^{[x_T^*, A_T^*]}(x, A = A_T^*, t) = \frac{\left(\frac{x_T^* - x}{T - t} \right) e^{-\frac{(x_T^* - x)^2}{2(T - t)}} + \left(\frac{x + \text{sgn}(x)|x_T^*|}{T - t} \right) e^{-\frac{(|x_T^*| + |x|)^2}{2(T - t)}}}{e^{-\frac{(x_T^* - x)^2}{2(T - t)}} - e^{-\frac{(|x_T^*| + |x|)^2}{2(T - t)}}} \quad (201)$$

The fact that the initial unconditioned drift μ does not appear in the conditioned drift of Eqs 200 and 201 is actually an immediate consequence of a more general result stating that the constraints can be imposed one after the other [32, 41]. By first imposing the final position of the process, one obtains a Brownian bridge which does not depend on the original drift. Then, imposing additional constraints on this process (whatever they are: local time, area under the curve, etc...) will not change this result. In addition, in the particular case where $A_T^* = 0$ (in other word the process cannot cross the origin) then only the singular part of the propagator contributes to the conditioned drift. Assume moreover that x_T^* and x are positive, then Eq. 201 reduces to

$$\mu_T^{[x_T^*, A_T^*]}(x, A = A_T^* = 0, t) = \frac{\left(\frac{x_T^* - x}{T - t}\right) e^{-\frac{(x_T^* - x)^2}{2(T-t)}} + \left(\frac{x_T^* + x}{T - t}\right) e^{-\frac{(x_T^* + x)^2}{2(T-t)}}}{e^{-\frac{(x_T^* - x)^2}{2(T-t)}} - e^{-\frac{(x_T^* + x)^2}{2(T-t)}}} \quad (202)$$

which is the drift of a Brownian bridge conditioned to stay positive, as it should be. This equation can be found in [4, 6].

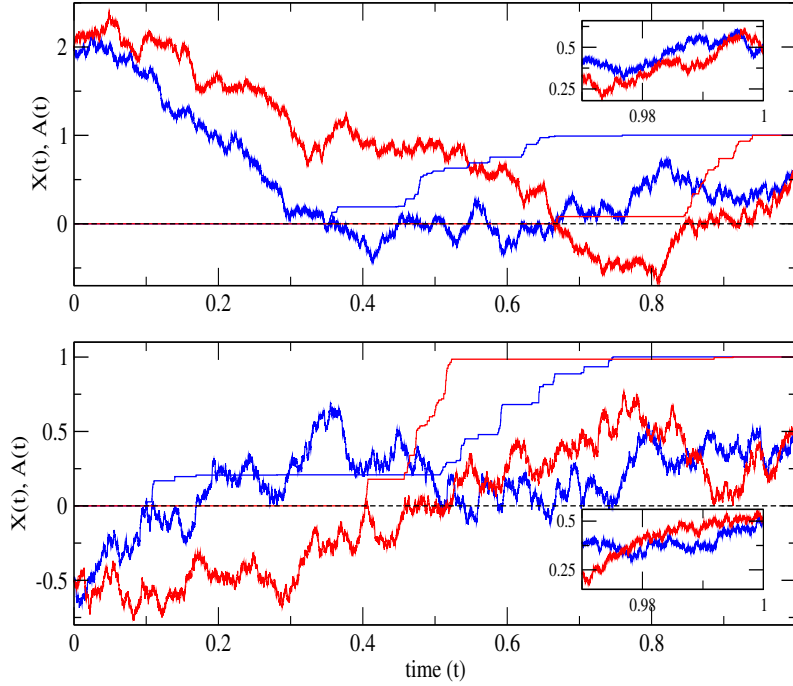


Figure 2. Examples of realizations of the Brownian bridge conditioned to end at the final position $x_T^* = 0.5$ and to have the final local time $A_T^* = 1$ at the finite time horizon $T = 1$ (see the conditioned drift of Eqs. 200 and 201). For each trajectory, the associated local time $A(t)$ is shown as a function of the time $t \in [0, T]$. Top figure: the process begins at the position $x_0 = 2$. Bottom figure: the process begins at position $x_0 = -0.5$. The encapsulated graphs show the convergence of both processes to the desired final value $x_T^* = 0.5$. The time step used in the discretization is $dt = 10^{-5}$.

F. Case $\mu = 0$: Conditioning towards the local time A_T^* at the finite time horizon T

Let us now apply the framework described in the subsection VIB. Using the explicit probability of Eq. 173

$$\Pi^{[\mu=0]}(A_T^*, T | x, A, t) = \delta(A_T^* - A) \operatorname{erf} \left(\frac{|x|}{\sqrt{2(T-t)}} \right) + \theta(A_T^* > A) \sqrt{\frac{2}{\pi(T-t)}} e^{-\frac{(|x| + A_T^* - A)^2}{2(T-t)}} \quad (203)$$

one obtains that the conditioned drift of Eq. 125

$$\mu_T^{[A_T^*]}(x, A, t) = \partial_x \ln \Pi^{[\mu=0]}(A_T^*, T|x, A, t) \quad (204)$$

can be decomposed into the two following regions.

(i) In the region $A_0 = 0 \leq A < A_T^*$ where the local time A has not yet reached its conditioned final value A_T^* , the conditioned drift of Eq. 204 reduces to

$$\begin{aligned} \mu_T^{[A_T^*]}(x, A < A_T^*, t) &= \partial_x \ln \left[\sqrt{\frac{2}{\pi(T-t)}} e^{-\frac{(|x|+A_T^*-A)^2}{2(T-t)}} \right] \\ &= -\text{sgn}(x) \frac{|x| + A_T^* - A}{T-t} \end{aligned} \quad (205)$$

(ii) In the region $A = A_T^*$ where the local time A has already reached its conditioned final value A_T^* , and where the position x cannot visit the origin $x = 0$ anymore, the conditioned drift of Eq. 204 reads

$$\begin{aligned} \mu_T^{[A_T^*]}(x, A = A_T^*, t) &= \partial_x \ln \left[\text{erf} \left(\frac{|x|}{\sqrt{2(T-t)}} \right) \right] \\ &= \sqrt{\frac{2}{\pi(T-t)}} \frac{e^{-\frac{x^2}{2(T-t)}}}{\text{erf} \left(\frac{|x|}{\sqrt{2(T-t)}} \right)} \text{sgn}(x) \end{aligned} \quad (206)$$

The asymptotic behavior near the origin $x \rightarrow 0$ is given by

$$\mu_T^{[A_T^*]}[x, A = A_T^*, t] \underset{x \rightarrow 0}{\simeq} \frac{1}{x} - \frac{x}{3(T-t)} \quad (207)$$

Due to the $1/x$ term, the origin $x = 0$ is an entrance boundary that the process cannot cross, therefore in the second region the local time can no longer increase, as wished.

G. Case $\mu = 0$: Conditioning towards the intensive local time $a^* = \frac{A_T^*}{T}$ in the limit $T \rightarrow +\infty$

In order to impose the intensive local time $a^* = \frac{A_T^*}{T}$ in the limit $T \rightarrow +\infty$, one can plug $A_T^* = Ta^*$ into the conditioned drift of Eq. 205 to obtain at leading order for $T \rightarrow +\infty$ while t remains finite

$$\mu_T^{[Ta^*]}(x, A < Ta^*, t) = -\text{sgn}(x) \frac{|x| + (Ta^* - A)}{T-t} \underset{T \rightarrow +\infty}{\simeq} -\text{sgn}(x)a^* \equiv \mu_\infty^{[a^*]}(x) \quad (208)$$

The agreement with the general formula of Eq. 135 for the drift $\mu_\infty^{[a^*]}(x)$ can be checked using Eq. 147

$$\tilde{G}_s^{[\mu=0]}(0|x) = \frac{e^{-\sqrt{2s}|x|}}{\sqrt{2s}} \quad (209)$$

and the saddle-point value $s_{a^*} = \frac{[a^*]^2}{2}$ of Eq. 193 to obtain

$$\mu_\infty^{[a^*]}(x) = \partial_x \ln \hat{G}_{s_{a^*}}(0|x) = \partial_x (-\sqrt{2s_{a^*}}|x| - \ln(\sqrt{2s_{a^*}})) = -\sqrt{2s_{a^*}} \text{sgn}(x) = -\text{sgn}(x)a^* \quad (210)$$

As explained in the Appendices, this result can be also recovered via the appropriate canonical conditioning leading to Eq. C13.

H. Case $\mu > 0$: Conditioning towards the finite local time $A_\infty^* < +\infty$ at the infinite time horizon $T = +\infty$

The framework described in the subsection VIB1 can be applied as follows.

(i) In the region $A_0 = 0 \leq A < A_\infty^*$ where the local time A has not yet reached its conditioned asymptotic value A_∞^* , the conditioned drift $\mu_{A_\infty^*}^*(x, A < A_\infty^*)$ of Eq. 129 can be obtained from $\Pi^{Regular}(A, \infty|x, A_0)$ given in Eq. 182

$$\begin{aligned} \mu_{A_\infty^*}^*(x, A < A_\infty^*) &= \mu + \partial_x \ln \Pi^{Regular}(A_\infty^*, \infty|x, A) = \mu + \partial_x \ln \left[\mu e^{-\mu(x+|x|+A_\infty^*-A)} \right] \\ &= -\mu \text{sgn}(x) \end{aligned} \quad (211)$$

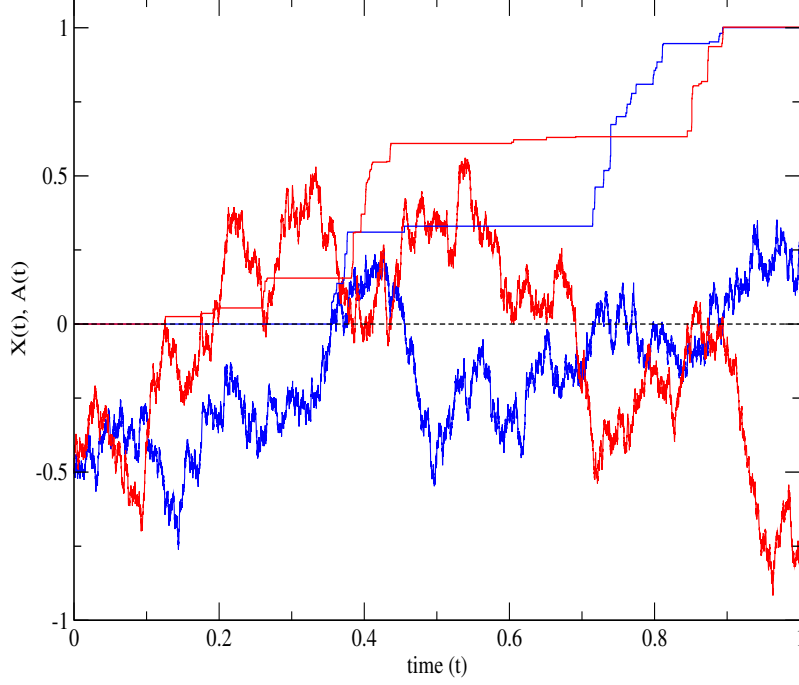


Figure 3. Examples of realizations of the Brownian process conditioned to have the final local time $A_T^* = 1$ at the finite time horizon $T = 1$ (see the conditioned drift of Eqs. 205 and 206). For each trajectory, the associated local time $A(t)$ is shown as a function of the time $t \in [0, T]$. The process can end at any final position x_T , while the initial position is $x_0 = -0.5$ here. The time step used in the discretization is $dt = 10^{-5}$.

So in the region $x < 0$, the conditioned drift coincide with the initial unconditioned drift μ , while in the region $x > 0$, the conditioned drift is opposite to the initial unconditioned drift μ in order to visit again the origin and to increase the local time.

(ii) In the region $A = A_\infty^*$ where the local time A has already reached its conditioned asymptotic value A_T^* , the conditioned drift of Eq. 130 can be obtained from $\Pi^{Singular}(A_\infty^*, \infty | x, A)$ given in Eq. 182 for $x > 0$

$$\begin{aligned} \mu_{A_\infty^*}^*(x > 0, A_\infty^*) &= \mu + \partial_x \ln \Pi^{Singular}(A_\infty^*, \infty | x, A) = \mu + \partial_x \ln [1 - e^{-\mu 2x}] \\ &= \mu \left(\frac{e^{2\mu x} + 1}{e^{2\mu x} - 1} \right) = \mu \coth(\mu x) \end{aligned} \quad (212)$$

For $x < 0$, the conditioned drift of Eq. 130 should be computed using the leading asymptotic form of $\Pi^{Singular}(A_\infty^*, T | x < 0, A, t)$ for large time interval $(T - t)$ given in Eq. 183

$$\begin{aligned} \mu_{A_\infty^*}^*(x < 0, A = A_\infty^*) &= \mu + \lim_{T \rightarrow +\infty} (\partial_x \ln \Pi^{Singular}(A_\infty^*, T | x < 0, A, t)) = \mu + \lim_{T \rightarrow +\infty} \left(\partial_x \ln \left[(-x) \sqrt{\frac{2}{\pi}} \frac{e^{-\frac{(-x+\mu T)^2}{2T}}}{\mu^2 T^{3/2}} \right] \right) \\ &= \lim_{T \rightarrow +\infty} \left(\frac{1}{x} - \frac{x}{T} \right) = \frac{1}{x} \end{aligned} \quad (213)$$

In both cases, whether $x > 0$ or $x < 0$, when $x \rightarrow 0$, the conditioned drift behaves as $\mu_{A_\infty^*}^*(x \rightarrow 0, A = A_\infty^*) \simeq 1/x$. The origin $x = 0$ is thus an entrance boundary that the process cannot cross, therefore in the second region (ii), the local time cannot increase any further, as expected.

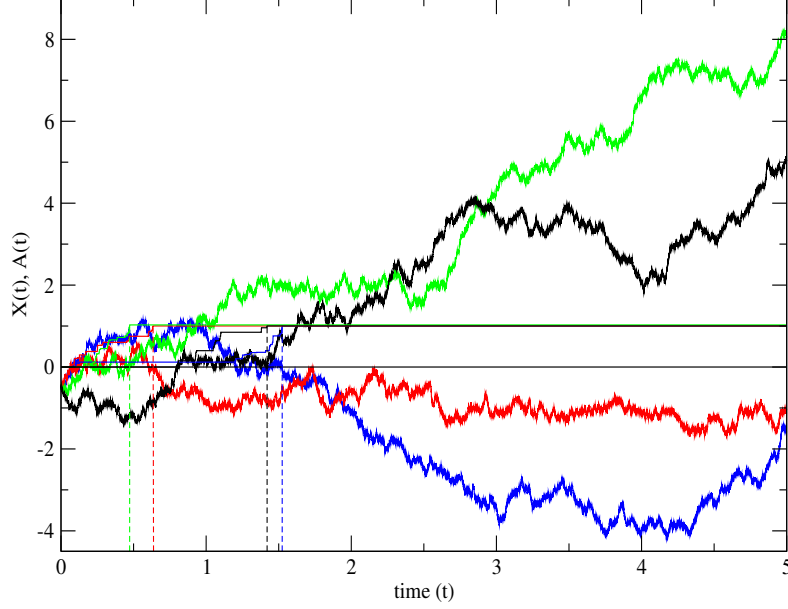


Figure 4. Examples of realization of the Brownian process with drift $\mu > 0$ where the local time is conditioned towards the finite asymptotic value $A_\infty^* = 1$ at the infinite time horizon $T = +\infty$ (see the conditioned drift of Eqs. 211 and 212). For each trajectory, the associated local time $A(t)$ is shown as a function of the time t . The dashed vertical lines indicate the time when the local time reaches the finite asymptotic value $A_\infty^* = 1$. The time step used in the discretization is $dt = 10^{-5}$.

VIII. APPLICATION TO THE DRIFT $\mu(x) = -\mu \text{sgn}(x)$ OF PARAMETER $\mu > 0$

In this section, we consider the drift directed towards the origin $x = 0$ of amplitude $\mu > 0$

$$\mu(x) = -\mu \text{sgn}(x) \quad (214)$$

This process is sometimes called Brownian motion with alternating drift (or bang-bang process [97, 98]) and was originally introduced by de Gennes to study dry friction [99].

The associated potential $U(x)$ of Eq. 9

$$U(x) = 2\mu \int_0^x dy \text{sgn}(y) = 2\mu|x| \quad (215)$$

corresponds to the normalizable equilibrium Boltzmann distribution of Eq. 28

$$G_{eq}(x) = \frac{e^{-U(x)}}{\int_{-\infty}^{+\infty} dy e^{-U(y)}} = \mu e^{-2\mu|x|} \quad (216)$$

So the unconditioned dynamics converges towards this equilibrium distribution for $t \rightarrow +\infty$, and the local time increment $(A - A_0)$ grows extensively in $(t - t_0)$ as discussed in subsection V B.

A. Properties of the unconditioned diffusion process $X(t)$ alone

1. Propagator $G(x, t|x_0, t_0)$ for the position alone

Via the similarity transformation of Eq. 10 based on the potential $U(x)$ of Eq. 215

$$G(x, t|x_0, t_0) = e^{\mu(|x_0| - |x|)} \psi(x, t|x_0, t_0) \quad (217)$$

the Fokker-Planck Eq. 6 for the propagator $G(x, t|x_0, t_0)$ becomes the Schrödinger Equation of Eq. 11 for $\psi(x, t|x_0, t_0)$, where the quantum Hamiltonian of Eq. 12 involves the potential of Eq. 13 that reads for the drift of Eq. 214

$$V(x) = \frac{\mu^2}{2} - \mu\delta(x) \quad (218)$$

As a consequence, the Hamiltonian of Eq. 12 can be decomposed

$$H = H_0 + H_1 \quad (219)$$

into the two contributions

$$\begin{aligned} H_0 &\equiv -\frac{1}{2}\partial_x^2 + \frac{\mu^2}{2} \\ H_1 &\equiv -\mu\delta(x) \end{aligned} \quad (220)$$

When the contribution H_1 is absent, the Schrödinger propagator $\psi^{[0]}(x, t|x_0, t_0)$ associated to the Hamiltonian H_0 whose potential reduces to the constant $\frac{\mu^2}{2}$ is given by

$$\psi^{[0]}(x, t|x_0, t_0) = \frac{1}{\sqrt{2\pi(t-t_0)}} e^{-\frac{(x-x_0)^2}{2(t-t_0)} - \frac{\mu^2}{2}(t-t_0)} \quad (221)$$

while its time Laplace transform reads

$$\tilde{\psi}_s^{[0]}(x|x_0) \equiv \int_{t_0}^{+\infty} dt e^{-s(t-t_0)} \psi^{[0]}(x, t|x_0, t_0) = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} d\tau \tau^{-\frac{1}{2}} e^{-\left(s + \frac{\mu^2}{2}\right)\tau} e^{-\frac{(x-x_0)^2}{2\tau}} = \frac{e^{-\sqrt{\mu^2+2s}|x-x_0|}}{\sqrt{\mu^2+2s}} \quad (222)$$

When the contribution H_1 is present, the Laplace transform $\tilde{\psi}_s(x|x_0)$ of the Schrödinger propagator $\psi(x, t|x_0, t_0)$ for the full Hamiltonian H can be computed from $\tilde{\psi}_s^{[0]}(x|x_0)$ of Eq. 222 via the Dyson formula analog to Eq. 44 to obtain

$$\begin{aligned} \tilde{\psi}_s(x|x_0) &= \tilde{\psi}_s^{[0]}(x|x_0) + \tilde{\psi}_s^{[0]}(x|0) \frac{\mu}{1 - \mu \tilde{\psi}_s^{[0]}(0|0)} \tilde{\psi}_s^{[0]}(0|x_0) \\ &= \frac{e^{-\sqrt{\mu^2+2s}|x-x_0|}}{\sqrt{\mu^2+2s}} + \frac{e^{-\sqrt{\mu^2+2s}|x|}}{\sqrt{\mu^2+2s}} \left(\frac{\mu}{1 - \frac{\mu}{\sqrt{\mu^2+2s}}} \right) \frac{e^{-\sqrt{\mu^2+2s}|x_0|}}{\sqrt{\mu^2+2s}} \\ &= \frac{1}{\sqrt{\mu^2+2s}} \left[e^{-\sqrt{\mu^2+2s}|x-x_0|} + \frac{\mu}{\sqrt{\mu^2+2s} - \mu} e^{-\sqrt{\mu^2+2s}(|x|+|x_0|)} \right] \end{aligned} \quad (223)$$

So the Laplace transform $\tilde{G}_s(x|x_0)$ of the Fokker-Planck propagator $G(x, t|x_0, t_0)$ of Eq. 217 reads

$$\begin{aligned} \tilde{G}_s(x|x_0) &\equiv \int_{t_0}^{+\infty} dt e^{-s(t-t_0)} G(x, t|x_0, t_0) = \int_{t_0}^{+\infty} dt e^{-s(t-t_0)} e^{\mu(|x_0|-|x|)} \psi(x, t|x_0, t_0) = e^{\mu(|x_0|-|x|)} \tilde{\psi}_s(x|x_0) \\ &= \frac{e^{\mu(|x_0|-|x|)}}{\sqrt{\mu^2+2s}} \left[e^{-\sqrt{\mu^2+2s}|x-x_0|} + \frac{\mu}{\sqrt{\mu^2+2s} - \mu} e^{-\sqrt{\mu^2+2s}(|x|+|x_0|)} \right] \end{aligned} \quad (224)$$

The limit $s \rightarrow 0$ of

$$\begin{aligned} \lim_{s \rightarrow 0} [s \tilde{G}_s(x|x_0)] &= \lim_{s \rightarrow 0} \left(s \frac{e^{\mu(|x_0|-|x|)}}{\sqrt{\mu^2+2s}} \left[e^{-\sqrt{\mu^2+2s}|x-x_0|} + \frac{\mu(\sqrt{\mu^2+2s} + \mu)}{2s} e^{-\sqrt{\mu^2+2s}(|x|+|x_0|)} \right] \right) \\ &= \mu e^{-2\mu|x|} = G_{eq}(x) \end{aligned} \quad (225)$$

allows to recover the equilibrium distribution $G_{eq}(x)$ of Eq. 216 as it should.

2. *Properties in the presence of an absorbing boundary at the origin $x = 0$*

In the presence of an absorbing boundary at the origin $x = 0$, the present model $\mu(x) = -\mu \operatorname{sgn}(x)$ is of course very similar to the previous section concerning the model $\mu(x) = \mu$: the two models coincide for $x_0 < 0$, and the region $x_0 > 0$ could be obtained by symmetry for the present model. However, one can also use the general formula as follows.

The evaluation of Eq. 224 for the special case $x = 0$

$$\tilde{G}_s(0|x_0) = \frac{e^{\mu|x_0|}}{\sqrt{\mu^2 + 2s}} \left[e^{-\sqrt{\mu^2 + 2s}|x_0|} + \frac{\mu}{\sqrt{\mu^2 + 2s} - \mu} e^{-\sqrt{\mu^2 + 2s}|x_0|} \right] = \frac{e^{(\mu - \sqrt{\mu^2 + 2s})|x_0|}}{\sqrt{\mu^2 + 2s} - \mu} \quad (226)$$

for the special case $x_0 = 0$

$$\tilde{G}_s(x|0) = \frac{e^{-\mu|x|}}{\sqrt{\mu^2 + 2s}} \left[e^{-\sqrt{\mu^2 + 2s}|x|} + \frac{\mu}{\sqrt{\mu^2 + 2s} - \mu} e^{-\sqrt{\mu^2 + 2s}|x|} \right] = \frac{e^{-(\mu + \sqrt{\mu^2 + 2s})|x|}}{\sqrt{\mu^2 + 2s} - \mu} \quad (227)$$

and for the special case $x = 0 = x_0$

$$\tilde{G}_s(0|0) = \frac{1}{\sqrt{\mu^2 + 2s} - \mu} \quad (228)$$

allows to compute the Laplace transform $\hat{G}_s^{abs}(x|x_0)$ via Eq. 49

$$\hat{G}_s^{abs}(x|x_0) \equiv \hat{G}_s(x|x_0) - \frac{\hat{G}_s(x|0)\hat{G}_s(0|x_0)}{\hat{G}_s(0|0)} = \frac{e^{\mu(|x_0| - |x|)}}{\sqrt{\mu^2 + 2s}} \left[e^{-\sqrt{\mu^2 + 2s}|x - x_0|} - e^{-\sqrt{\mu^2 + 2s}(|x| + |x_0|)} \right] \quad (229)$$

The Laplace inversion using Eq. 151 yields the propagator $G^{abs}(x, t|x_0, t_0)$ in the presence of an absorbing boundary at the origin $x = 0$

$$\begin{aligned} G^{abs}(x, A, t|x_0, A_0, t_0) &= e^{\mu(|x_0| - |x|)} \frac{e^{-\frac{\mu^2}{2}(t - t_0)}}{\sqrt{2\pi(t - t_0)}} \left[e^{-\frac{(x - x_0)^2}{2(t - t_0)}} - e^{-\frac{(|x| + |x_0|)^2}{2(t - t_0)}} \right] \\ &= e^{\mu(|x_0| - |x|)} \frac{e^{-\frac{\mu^2}{2}(t - t_0) - \frac{x^2 + x_0^2}{2(t - t_0)}}}{\sqrt{2\pi(t - t_0)}} \left[e^{\frac{xx_0}{(t - t_0)}} - e^{-\frac{|xx_0|}{(t - t_0)}} \right] \end{aligned} \quad (230)$$

in agreement with the method of images.

The Laplace transform $\hat{\gamma}_s^{abs}(x_0)$ of Eq. 59 reads

$$\hat{\gamma}_s^{abs}(x_0) = \frac{\hat{G}_s(0|x_0)}{\hat{G}_s(0|0)} = e^{(\mu - \sqrt{\mu^2 + 2s})|x_0|} \quad (231)$$

Its Laplace inversion yields the absorption rate $\gamma^{abs}(t|x_0, t_0)$

$$\gamma^{abs}(t|x_0, t_0) = \frac{|x_0|}{\sqrt{2\pi(t - t_0)}^{\frac{3}{2}}} e^{\mu|x_0| - \frac{\mu^2}{2}(t - t_0) - \frac{x_0^2}{2(t - t_0)}} \quad (232)$$

One can check the normalization to unity for any starting point x_0

$$\int_{t_0}^{+\infty} dt \gamma^{abs}(t|x_0, t_0) = \hat{\gamma}_{s=0}^{abs}(x_0) = 1 \quad (233)$$

The survival probability $S^{abs}(t|x_0, t_0)$ of Eq. 54 can be obtained from the integral over the final position x of the propagator $G^{abs}(x, t|x_0, t_0)$ of Eq. 230

$$S^{abs}(t|x_0, t_0) \equiv \int_{-\infty}^{+\infty} dx G^{abs}(x, t|x_0, t_0) = \frac{e^{-\frac{\mu^2}{2}(t - t_0) + \mu|x_0| - \frac{x_0^2}{2(t - t_0)}}}{\sqrt{2\pi(t - t_0)}} \int_{-\infty}^{+\infty} dx e^{-\mu|x| - \frac{x^2}{2(t - t_0)}} \left[e^{\frac{xx_0}{(t - t_0)}} - e^{-\frac{|xx_0|}{(t - t_0)}} \right] \quad (234)$$

Its asymptotic decay for large time $(t - t_0)$ is given by

$$S^{abs}(t|x_0, t_0) \underset{(t - t_0) \rightarrow +\infty}{\sim} \frac{e^{-\frac{\mu^2}{2}(t - t_0) + \mu|x_0|}}{\sqrt{2\pi(t - t_0)}} \int_{-\infty}^{+\infty} dx e^{-\mu|x|} \left[\frac{xx_0 + |xx_0|}{t - t_0} \right] = \sqrt{\frac{2}{\pi}} \frac{|x_0| e^{\mu|x_0| - \frac{\mu^2}{2}(t - t_0)}}{\mu^2(t - t_0)^{\frac{3}{2}}} \quad (235)$$

B. Joint propagator $P(x, A, t|x_0, A_0, t_0)$ for the unconditioned joint process $[X(t), A(t)]$

The singular contribution of Eq. 53 involves the propagator $G^{abs}(x, t|x_0, t_0)$ of Eq. 230

$$\begin{aligned} P^{Singular}(x, A, t|x_0, A_0, t_0) &= \delta(A - A_0)G^{abs}(x, t|x_0, t_0) \\ &= \delta(A - A_0)e^{\mu(|x_0| - |x|)} \frac{e^{-\frac{\mu^2}{2}(t-t_0)}}{\sqrt{2\pi(t-t_0)}} \left[e^{-\frac{(x-x_0)^2}{2(t-t_0)}} - e^{-\frac{(|x|+|x_0|)^2}{2(t-t_0)}} \right] \end{aligned} \quad (236)$$

The Laplace transform $\hat{P}_s^{Regular}(x, A|x_0, A_0)$ of Eq. 62 reads using Eqs 226, 227 and 228

$$\begin{aligned} \hat{P}_s^{Regular}(x, A|x_0, A_0) &= \theta(A > A_0) \left[\frac{\hat{G}_s(x|0)\hat{G}_s(0|x_0)}{\hat{G}_s^2(0|0)} \right] e^{-\frac{(A-A_0)}{\hat{G}_s(0|0)}} \\ &= \theta(A > A_0)e^{\mu(|x_0| - |x| + A - A_0)} e^{-\sqrt{\mu^2 + 2s}(|x_0| + |x| + A - A_0)} \end{aligned} \quad (237)$$

Equation 152 allows to compute the Laplace inversion

$$P^{Regular}(x, A, t|x_0, A_0, t_0) = \theta(A > A_0)e^{\mu(|x_0| - |x| + A - A_0) - \frac{\mu^2}{2}(t-t_0)} \left(\frac{|x_0| + |x| + A - A_0}{\sqrt{2\pi(t-t_0)}^{\frac{3}{2}}} \right) e^{-\frac{(|x_0| + |x| + A - A_0)^2}{2(t-t_0)}} \quad (238)$$

So the joint propagator $P(x, A, t|x_0, A_0, t_0)$ involving the two contributions of Eq. 236 and Eq. 238 reads

$$\begin{aligned} P(x, A, t|x_0, A_0, t_0) &= \delta(A - A_0)e^{\mu(|x_0| - |x|)} \frac{e^{-\frac{\mu^2}{2}(t-t_0)}}{\sqrt{2\pi(t-t_0)}} \left[e^{-\frac{(x-x_0)^2}{2(t-t_0)}} - e^{-\frac{(|x|+|x_0|)^2}{2(t-t_0)}} \right] \\ &\quad + \theta(A > A_0)e^{\mu(|x_0| - |x| + A - A_0) - \frac{\mu^2}{2}(t-t_0)} \left(\frac{|x_0| + |x| + A - A_0}{\sqrt{2\pi(t-t_0)}^{\frac{3}{2}}} \right) e^{-\frac{(|x_0| + |x| + A - A_0)^2}{2(t-t_0)}} \end{aligned} \quad (239)$$

The integration of this joint propagator $P(x, A, t|x_0, A_0, t_0)$ of Eq. 239 over the local time A

$$\begin{aligned} \int_0^{+\infty} dA P(x, A, t|x_0, A_0, t_0) &= e^{\mu(|x_0| - |x|)} \frac{e^{-\frac{\mu^2}{2}(t-t_0)}}{\sqrt{2\pi(t-t_0)}} \left[e^{-\frac{(x-x_0)^2}{2(t-t_0)}} - e^{-\frac{(|x|+|x_0|)^2}{2(t-t_0)}} \right] \\ &\quad + \int_{A_0}^{+\infty} dA e^{\mu(|x_0| - |x| + A - A_0) - \frac{\mu^2}{2}(t-t_0)} \left(\frac{|x_0| + |x| + A - A_0}{\sqrt{2\pi(t-t_0)}^{\frac{3}{2}}} \right) e^{-\frac{(|x_0| + |x| + A - A_0)^2}{2(t-t_0)}} \\ &= e^{-2\mu|x|} \left[\frac{1}{\sqrt{2\pi(t-t_0)}} e^{\mu(|x_0| + |x|) - \frac{\mu^2}{2}(t-t_0) - \frac{(x-x_0)^2}{2(t-t_0)}} + \frac{\mu}{2} \operatorname{erfc} \left(\frac{|x_0| + |x| - \mu(t-t_0)}{\sqrt{2(t-t_0)}} \right) \right] \end{aligned} \quad (240)$$

allows to recover the free propagator of the Brownian motion with alternating drift [28] as it should.

C. Probability $\Pi(A, t|x_0, A_0, t_0)$ to see the local time A at time t

The probability $\Pi(A, t|x_0, A_0, t_0)$ of Eq. 65 can be obtained via the integration of the joint propagator $P(x, A, t|x_0, A_0, t_0)$ of Eq. 239 over the final position x

$$\Pi(A, t|x_0, A_0, t_0) \equiv \int_{-\infty}^{+\infty} dx P(A, t|x_0, A_0, t_0) \quad (241)$$

Its singular contribution of Eq. 69 involves the survival probability $S^{abs}(t|x_0, t_0)$ of Eq. 234 with the asymptotic behavior of Eq. 235

$$\begin{aligned} \Pi^{Singular}(A, t|x_0, A_0, t_0) &= \delta(A - A_0)S^{abs}(t|x_0, t_0) \\ &\underset{(t-t_0) \rightarrow +\infty}{\simeq} \delta(A - A_0) \sqrt{\frac{2}{\pi}} \frac{|x_0| e^{\mu|x_0| - \frac{\mu^2}{2}(t-t_0)}}{\mu^2(t-t_0)^{\frac{3}{2}}} \end{aligned} \quad (242)$$

Its regular contribution can be obtained from the integration over x of Eq. 238

$$\begin{aligned}
\Pi^{Regular}(A, t|x_0, A_0, t_0) &\equiv \int_{-\infty}^{+\infty} dx P^{Regular}(A, t|x_0, A_0, t_0) \\
&= \theta(A > A_0) \frac{e^{-\frac{\mu^2}{2}(t-t_0) + \mu(|x_0| + A - A_0)}}{\sqrt{2\pi(t-t_0)}^{\frac{3}{2}}} \int_{-\infty}^{+\infty} dx e^{-\mu|x|} (|x| + |x_0| + A - A_0) e^{-\frac{(|x| + |x_0| + A - A_0)^2}{2(t-t_0)}} \\
&= \theta(A > A_0) \frac{e^{-\frac{\mu^2}{2}(t-t_0) + \mu(|x_0| + A - A_0)}}{\sqrt{2\pi(t-t_0)}} 2 \int_0^{+\infty} dx e^{-\mu x} \frac{(x + |x_0| + A - A_0)}{t - t_0} e^{-\frac{(x + |x_0| + A - A_0)^2}{2(t-t_0)}} \\
&= -\theta(A > A_0) \sqrt{\frac{2}{\pi(t-t_0)}} e^{-\frac{\mu^2}{2}(t-t_0) + \mu(|x_0| + A - A_0)} \left(\left[e^{-\mu x} e^{-\frac{(x + |x_0| + A - A_0)^2}{2(t-t_0)}} \right]_{x=0}^{x=+\infty} + \mu \int_0^{+\infty} dx e^{-\mu x - \frac{(x + |x_0| + A - A_0)^2}{2(t-t_0)}} \right) \\
&= \theta(A > A_0) \sqrt{\frac{2}{\pi(t-t_0)}} e^{-\frac{\mu^2}{2}(t-t_0) + \mu(|x_0| + A - A_0)} \left(e^{-\frac{(|x_0| + A - A_0)^2}{2(t-t_0)}} - \mu e^{-\frac{(|x_0| + A - A_0)^2}{2(t-t_0)}} \int_0^{+\infty} dx e^{-\frac{x^2}{2(t-t_0)} - x \left[\mu + \frac{(|x_0| + A - A_0)}{(t-t_0)} \right]} \right) \\
&= \theta(A > A_0) \sqrt{\frac{2}{\pi(t-t_0)}} e^{-\frac{(|x_0| + A - A_0 - \mu(t-t_0))^2}{2(t-t_0)}} \left(1 - \mu \int_0^{+\infty} dx e^{-\frac{x^2}{2(t-t_0)} - x \left[\mu + \frac{(|x_0| + A - A_0)}{(t-t_0)} \right]} \right) \\
&= \theta(A > A_0) e^{-\frac{(|x_0| + A - A_0 - \mu(t-t_0))^2}{2(t-t_0)}} \left[\sqrt{\frac{2}{\pi(t-t_0)}} - \mu e^{\frac{(|x_0| + A - A_0 - \mu(t-t_0))^2}{2(t-t_0)}} \operatorname{erfc} \left(\frac{|x_0| + \mu(t-t_0) + A - A_0}{\sqrt{2(t-t_0)}} \right) \right] \quad (243)
\end{aligned}$$

For large time interval $(t - t_0)$, the leading behavior is given by

$$\begin{aligned}
\Pi^{Regular}(A, t|x_0, A_0, t_0) &\underset{(t-t_0) \rightarrow +\infty}{\simeq} \theta(A > A_0) \sqrt{\frac{2}{\pi(t-t_0)}} e^{-\frac{(|x_0| + A - A_0 - \mu(t-t_0))^2}{2(t-t_0)}} \left(1 - \mu \int_0^{+\infty} dx e^{-x \left[\mu + \frac{(|x_0| + A - A_0)}{(t-t_0)} \right]} \right) \\
&\underset{(t-t_0) \rightarrow +\infty}{\simeq} \theta(A > A_0) \sqrt{\frac{2}{\pi(t-t_0)}} e^{-\frac{(|x_0| + A - A_0 - \mu(t-t_0))^2}{2(t-t_0)}} \left(1 - \frac{\mu}{\mu + \frac{(|x_0| + A - A_0)}{(t-t_0)}} \right) \\
&\underset{(t-t_0) \rightarrow +\infty}{\simeq} \theta(A > A_0) \sqrt{\frac{2}{\pi(t-t_0)}} e^{-\frac{(|x_0| + A - A_0 - \mu(t-t_0))^2}{2(t-t_0)}} \left(\frac{|x_0| + A - A_0}{\mu(t-t_0) + |x_0| + A - A_0} \right) \quad (244)
\end{aligned}$$

Note that for $\mu = 0$, we recover the expression $\Pi^{Regular}(A, t|x_0, A_0, t_0)$ of the standard Brownian motion Eq. 171, as expected.

D. Large deviations properties of the intensive local time $a = \frac{A-A_0}{t-t_0} \in [0, +\infty[$

The probability to see $A = A_0 + (t - t_0)a$ in Eq. 239 reads

$$\begin{aligned}
P(x, A = A_0 + (t - t_0)a, t|x_0, A_0, t_0) &= \delta((t - t_0)a) e^{\mu(|x_0| - |x|)} \frac{e^{-\frac{\mu^2}{2}(t-t_0)}}{\sqrt{2\pi(t-t_0)}} \left[e^{-\frac{(x-x_0)^2}{2(t-t_0)}} - e^{-\frac{(|x| + |x_0|)^2}{2(t-t_0)}} \right] \\
&+ \theta(a > 0) e^{\mu(|x_0| - |x| + (t-t_0)a) - \frac{\mu^2}{2}(t-t_0)} \left(\frac{|x_0| + |x| + (t-t_0)a}{\sqrt{2\pi(t-t_0)}^{\frac{3}{2}}} \right) e^{-\frac{(|x_0| + |x| + (t-t_0)a)^2}{2(t-t_0)}} \quad (245)
\end{aligned}$$

The large deviations of the intensive local time $a = \frac{A-A_0}{t-t_0} \in [0, +\infty[$

$$P(x, A = A_0 + (t - t_0)a, t|x_0, A_0, t_0) \underset{(t-t_0) \rightarrow +\infty}{\propto} e^{-(t-t_0)I(a)} \quad (246)$$

thus involve the rate function [95]

$$I(a) = -\mu a + \frac{\mu^2}{2} + \frac{a^2}{2} = \frac{(a - \mu)^2}{2} \quad \text{for } a \in [0, +\infty[\quad (247)$$

The boundary value $I(a = 0)$ at the origin $a = 0$

$$I(a = 0) = \frac{\mu^2}{2} \quad (248)$$

governs the decay of the survival probability of Eq. 97.

The equilibrium value a_{eq} of Eq. 95 where the rate function $I(a)$ of Eq. 247 and its first derivative $I'(a)$ vanish (Eq. 95) is simply $a_{eq} = \mu$. It coincides with the value $G_{eq}(0)$ of the equilibrium distribution of $G_{eq}(x)$ of Eq. 216 at the origin $x = 0$ as it should

$$a_{eq} = \mu = G_{eq}(0) \quad (249)$$

If one includes the prefactors, the leading order of the regular contribution of Eq. 245 reads

$$P^{Regular}(x, A = A_0 + (t - t_0)a, t | x_0, A_0, t_0) \underset{(t-t_0) \rightarrow +\infty}{\simeq} \frac{a}{\sqrt{2\pi(t-t_0)}} e^{-(\mu+a)|x| + (\mu-a)|x_0|} e^{-(t-t_0)I(a)} \quad (250)$$

The agreement with the general formula of Eq. 110 can be checked using Eqs 226, 227, 228 as well as Eq. 103

$$0 = a \partial_s \left[\sqrt{\mu^2 + 2s} - \mu \right] - 1 = \frac{a}{\sqrt{\mu^2 + 2s}} - 1 \quad (251)$$

that leads to the saddle-point

$$s_a = \frac{a^2 - \mu^2}{2} \quad (252)$$

E. Conditioning towards the position x_T^* and the local time A_T^* at the finite time horizon T

Let us now apply the framework described in the subsection VI A. Using the explicit joint propagator of Eq. 239

$$\begin{aligned} \ln P(x_T, A_T, T | x, A, t) &= \mu(|x| - |x_T|) - \frac{\mu^2}{2}(T - t) - \ln(\sqrt{2\pi(T-t)}) \\ &+ \ln \left(\delta(A_T - A) \left[e^{-\frac{(x_T - x)^2}{2(T-t)}} - e^{-\frac{(|x_T| + |x|)^2}{2(T-t)}} \right] + \theta(A_T > A) \left(\frac{|x| + |x_T| + A_T - A}{T - t} \right) e^{\mu(A_T - A) - \frac{(|x| + |x_T| + A_T - A)^2}{2(T-t)}} \right) \end{aligned} \quad (253)$$

one obtains the conditioned drift of Eq. 117

$$\begin{aligned} \mu_T^{[x_T^*, A_T^*]}(x, A, t) &= -\mu \operatorname{sgn}(x) + \partial_x \ln P(x_T^*, A_T^*, T | x, A, t) \\ &= \partial_x \ln \left(\delta(A_T^* - A) \left[e^{-\frac{(x_T^* - x)^2}{2(T-t)}} - e^{-\frac{(|x_T^*| + |x|)^2}{2(T-t)}} \right] + \theta(A_T^* > A) \left(\frac{|x| + |x_T^*| + A_T^* - A}{T - t} \right) e^{\mu(A_T^* - A) - \frac{(|x| + |x_T^*| + A_T^* - A)^2}{2(T-t)}} \right) \end{aligned} \quad (254)$$

It actually coincides with the conditioned drift of Eq. 199 with its two regions of Eqs 200 and 201

$$\begin{aligned} \mu_T^{[x_T^*, A_T^*]}(x, A < A_T^*, t) &= \operatorname{sgn}(x) \left[\frac{1}{|x_T^*| + |x| + A_T^* - A} - \frac{|x_T^*| + |x| + A_T^* - A}{T - t} \right] \\ \mu_T^{[x_T^*, A_T^*]}(x, A = A_T^*, t) &= \frac{\frac{x_T^* - x}{T - t} e^{-\frac{(x_T^* - x)^2}{2(T-t)}} + \frac{x + \operatorname{sgn}(x)|x_T^*|}{T - t} e^{-\frac{(|x_T^*| + |x|)^2}{2(T-t)}}}{e^{-\frac{(x_T^* - x)^2}{2(T-t)}} - e^{-\frac{(|x_T^*| + |x|)^2}{2(T-t)}}} \end{aligned} \quad (255)$$

Corresponding stochastic trajectories have already been shown on figure 2.

F. Conditioning towards the local time A_T^* at the finite time horizon T

Let us now apply the framework described in the subsection VI B.

(i) In the region $A_0 = 0 \leq A < A_T^*$ where the local time A has not yet reached its conditioned final value A_T^* , the conditioned drift of Eq. 127 involves the regular contribution $\Pi^{Regular}(A_T^*, T|x, A, t)$ of Eq. 243

$$\begin{aligned}
\mu_T^{[A_T^*]}(x, A < A_T^*, t) &= -\mu \operatorname{sgn}(x) + \partial_x \ln \Pi^{Regular}(A_T^*, T|x, A, t) \\
&= -\mu \operatorname{sgn}(x) + \partial_x \ln \left[e^{-\frac{(|x|+A_T^*-A-\mu(T-t))^2}{2(T-t)}} \left(\sqrt{\frac{2}{\pi(T-t)}} - \mu e^{\frac{(|x|+A_T^*-A+\mu(T-t))^2}{2(T-t)}} \operatorname{erfc} \left(\frac{|x| + \mu(T-t) + A_T^* - A}{\sqrt{2(T-t)}} \right) \right) \right] \\
&= -\operatorname{sgn}(x) \frac{|x| + A_T^* - A}{T-t} + \partial_x \ln \left(1 - \mu \int_0^{+\infty} dy e^{-\frac{y^2}{2(T-t)} - y \left[\mu + \frac{(|x|+A_T^*-A)}{(T-t)} \right]} \right) \\
&= \mu \operatorname{sgn}(x) - \frac{2}{(T-t)} \frac{(|x| + A_T^* - A)}{2 - \mu \sqrt{2\pi(T-t)} e^{\frac{(|x|+\mu(T-t)+A_T^*-A)^2}{2(T-t)}} \operatorname{erfc} \left(\frac{|x|+\mu(T-t)+A_T^*-A}{\sqrt{2(T-t)}} \right)} \operatorname{sgn}(x) \quad (256)
\end{aligned}$$

(ii) In the region $A = A_T^*$ where the local time A has already reached its conditioned final value A_T^* , and where the position x cannot visit the origin $x = 0$ anymore, the conditioned drift of Eq. 128 involves the survival probability $S^{abs}(T|x, t)$ of Eq. 234

$$\begin{aligned}
\mu_T^{[A_T^*]}(x, A = A_T^*, t) &= -\mu \operatorname{sgn}(x) + \partial_x \ln S^{abs}(T|x, t) \\
&= -\mu \operatorname{sgn}(x) + \partial_x \ln \left(\frac{e^{-\frac{\mu^2}{2}(T-t) + \mu|x| - \frac{x^2}{2(T-t)}}}{\sqrt{2\pi(T-t)}} \int_{-\infty}^{+\infty} dy e^{-\mu|y| - \frac{y^2}{2(T-t)}} \left[e^{\frac{yx}{(T-t)}} - e^{-\frac{|yx|}{(T-t)}} \right] \right) \\
&= \partial_x \ln \left(\int_{-\infty}^{+\infty} dy e^{-\mu|y| - \frac{y^2}{2(T-t)}} \left[e^{\frac{yx}{(T-t)}} - e^{-\frac{|yx|}{(T-t)}} \right] \right) \\
&= 2\sqrt{\frac{2}{\tau}} \frac{\operatorname{sgn}(x) \left(\frac{1}{\sqrt{\pi}} - e^{\frac{(|x|+\mu\tau)^2}{2\tau}} \mathcal{F} \left(\frac{|x|+\mu\tau}{\sqrt{2\tau}} \right) \right) + e^{\frac{(x-\mu\tau)^2}{2\tau}} \left(e^{2\mu x} \mathcal{F} \left(\frac{x+\mu\tau}{\sqrt{2\tau}} \right) - \mathcal{F} \left(\frac{\mu\tau-x}{\sqrt{2\tau}} \right) \right)}{e^{\frac{(x+\mu\tau)^2}{2\tau}} \operatorname{erfc} \left(\frac{x+\mu\tau}{\sqrt{2\tau}} \right) + e^{\frac{(x-\mu\tau)^2}{2\tau}} \operatorname{erfc} \left(\frac{\mu\tau-x}{\sqrt{2\tau}} \right) - 2e^{\frac{(|x|+\mu\tau)^2}{2\tau}} \operatorname{erfc} \left(\frac{|x|+\mu\tau}{\sqrt{2\tau}} \right)} \quad (257)
\end{aligned}$$

where $\mathcal{F}(x) = x \operatorname{erfc}(x)$ and $\tau = T - t$.

In the second region, the asymptotic behavior near the origin $x \rightarrow 0$ is

$$\mu_T^{[A_T^*]}(x, A = A_T^*, t) \underset{x \rightarrow 0}{\simeq} \frac{1}{x} + x \left(\frac{1}{T-t} + \frac{\mu^2}{3} + \frac{1}{-3(T-t) + 3\mu \sqrt{\frac{\pi(T-t)^3}{2}} e^{\frac{(T-t)\mu^2}{2}} \operatorname{erfc} \left(\mu \sqrt{\frac{T-t}{2}} \right)} \right) \quad (258)$$

Again, the $1/x$ term prevents the process from crossing the origin and the local time cannot increase anymore.

G. Case $\mu = 0$: Conditioning towards the intensive value $a^* = \frac{A_T^*}{T}$ in the limit $T \rightarrow +\infty$

A direct consequence of Eq. 255 is that the conditioning towards the intensive value $a^* = \frac{A_T^*}{T}$ in the limit $T \rightarrow +\infty$ will give exactly the same conditioned drift of Eq. 208

$$\mu_T^{[Ta^*]}(x, A < Ta^*, t) \underset{T \rightarrow +\infty}{\simeq} -\operatorname{sgn}(x)a^* \equiv \mu_\infty^{[a^*]}(x) \quad (259)$$

The agreement with the general formula of Eq. 135 for the conditioned drift $\mu_\infty^{[a^*]}(x)$ can be checked by Eq. 226

$$\tilde{G}_s(0|x) = \frac{e^{(\mu - \sqrt{\mu^2 + 2s})|x|}}{\sqrt{\mu^2 + 2s - \mu}} \quad (260)$$

and the saddle-point value $s_{a^*} = \frac{(a^*)^2 - \mu^2}{2}$ of Eq. 252 to obtain

$$\begin{aligned}
\mu_{Bridge[a^*]}^*(x) &= -\mu \operatorname{sgn}(x) + \partial_x \ln \hat{G}_{s_{a^*}}(0|x) = -\mu \operatorname{sgn}(x) + \partial_x \left((\mu - \sqrt{\mu^2 + 2s_{a^*}})|x| - \ln(\sqrt{\mu^2 + 2s_{a^*}} - \mu) \right) \\
&= -\sqrt{\mu^2 + 2s_{a^*}} \operatorname{sgn}(x) = -\operatorname{sgn}(x)a^* \quad (261)
\end{aligned}$$

As explained in the Appendices, this result can be also recovered via the appropriate canonical conditioning leading to Eq. B20.

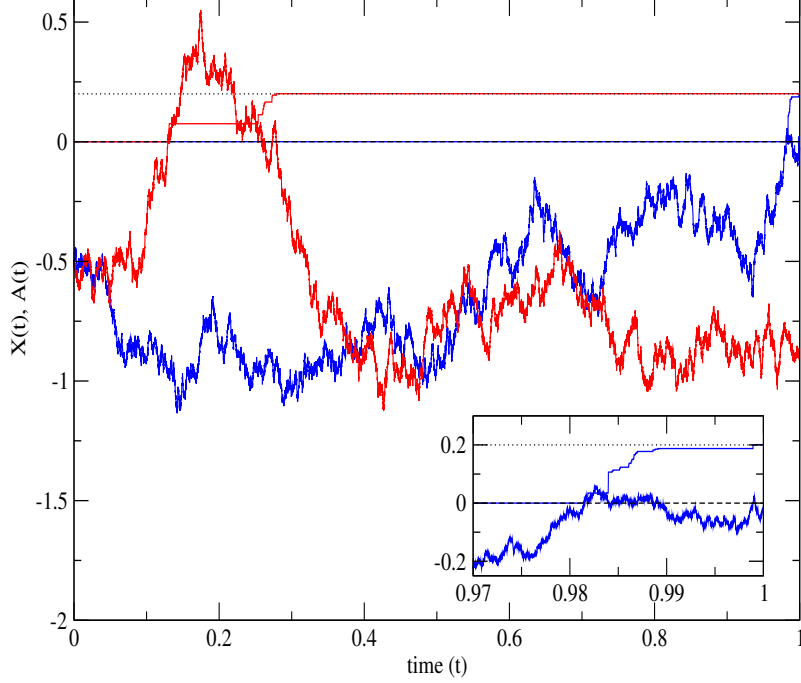


Figure 5. Examples of realization of the Brownian process with alternating drift conditioned to have the final local time $A_T^* = 0.2$ at the finite time horizon $T = 1$ (see the conditioned drift of Eqs. 256 and 257). For each trajectory, the associated local time $A(t)$ is shown as a function of the time $t \in [0, T]$. The process can end at any final position x_T , while the initial position is $x_0 = -0.5$ here. Observe that the blue realization reaches the desired local time value only at the very end of the time-window (as shown in the encapsulated plot). The time step used in the discretization is $dt = 10^{-5}$.

IX. CONCLUSION

In the present paper, we have analyzed the conditioning of a diffusion process $X(t)$ of drift $\mu(x)$ and of diffusion coefficient $D = 1/2$ with respect to its local time $A_{x=0}(t) = A(t)$ at the origin $x = 0$. Our goal was to construct various conditioned joint processes $[X^*(t), A^*(t)]$ satisfying certain conditions involving the local time $A^*(T)$ at the finite time horizon T or in the limit of the infinite time horizon $T \rightarrow +\infty$.

For the case of the finite time horizon T , we have studied :

- (a) the conditioning towards the final position $X^*(T)$ and towards the final local time $A^*(T)$. In other words, this case corresponds to conditioning a generalized Brownian bridge with respect to its local time at the final time.
- (b) the conditioning towards the final local time $A^*(T)$ alone without any condition on the final position $X^*(T)$.

In the limit of the infinite time horizon $T \rightarrow +\infty$, we have analyzed :

- (1) the conditioning towards the finite asymptotic local time $A_\infty^* < +\infty$
- (2) the conditioning towards the intensive local time a^* corresponding to the extensive behavior $A_T \simeq Ta^*$, that we have compared in the Appendices to the appropriate 'canonical conditioning' based on the generating function of the local time in the regime of large deviations.

This general construction has been applied to generate various constrained stochastic trajectories for three unconditioned diffusions with different recurrence/transience properties :

- (i) as simplest example of a transient diffusion, we have considered the uniform strictly positive drift $\mu(x) = \mu > 0$
- (ii) as simplest example of a diffusion converging towards an equilibrium, we have chosen the drift $\mu(x) = -\mu \operatorname{sgn}(x)$ of parameter $\mu > 0$
- (iii) as simplest example of a recurrent diffusion that does not converge towards an equilibrium, we have focused on the Brownian motion without drift $\mu = 0$.

The generalization of the present work to analyze the conditioning with respect to two local times is described in [100].

Appendix A: Notion of canonical conditioned process $X_p^*(t)$ of parameter p conjugated to the local time

As recalled in the Introduction, the 'canonical conditioning' based on generating functions of time-additive observables for Markov processes over a large time-window T has recently been used extensively in the field of non-equilibrium statistical physics [45–90]. Its physical meaning comes from the equivalence at the level of the large deviations for large time T between the 'canonical conditioning' and the 'microcanonical conditioning' (see the two detailed papers [68, 69] and the HDR thesis [70] with references therein). In this Appendix, it is thus interesting to analyze the 'canonical conditioning' of parameter p conjugated to the local time increment $[A(t) - A(t_0)]$ and to compare with the 'microcanonical conditioning' described in the main text.

1. Canonical conditioned process $X_p^*(t)$ of parameter p based on the Laplace transform $\tilde{P}_p(x, t|x_0, t_0)$

The canonical conditioning is based on the Laplace transform $\tilde{P}_p(x, t|x_0, t_0)$ of Eq. 35 with respect to the local time increment $(A - A_0)$, where the Laplace parameter p conjugated to the local time increment $(A - A_0)$ is fixed.

For the bridge conditioned to end at the position x_T^* at the time horizon T , the conditioned probability for the position x at an interior time $t \in [0, T]$ reads

$$P_T^{[x_T^*:p]}(x, t) = \frac{\tilde{P}_p(x_T, T|x, t)\tilde{P}_p(x, t|x_0, 0)}{\tilde{P}_p(x_T, T|x_0, 0)} \quad (\text{A1})$$

The corresponding Ito dynamics for the conditioned process $X_p^*(t)$ of parameter p

$$dX_p^*(t) = \mu_p^*(X_p^*(t), t)dt + dB(t) \quad (\text{A2})$$

involves the conditioned drift

$$\mu_T^{[x_T^*:p]}(x, t) = \mu(x) + \partial_x \ln \tilde{P}_p(x_T, T|x, t) \quad (\text{A3})$$

2. Properties of the p -deformed propagator $\tilde{P}_p(x, t|x_0, t_0)$

The forward dynamics of the propagator $\tilde{P}_p(x, t|x_0, t_0)$ given by the Feynman-Kac formula of Eq. 36

$$\partial_t \tilde{P}_p(x, t|x_0, t_0) = -p\delta(x)\tilde{P}_p(x, t|x_0, t_0) - \partial_x \left[\mu(x)\tilde{P}_p(x, t|x_0, t_0) \right] + \frac{1}{2}\partial_x^2 \tilde{P}_p(x, t|x_0, t_0) \equiv \mathcal{F}_p \tilde{P}_p(x, t|x_0, t_0) \quad (\text{A4})$$

involves the generator

$$\mathcal{F}_p \equiv -p\delta(x) - \partial_x \mu(x) + \frac{1}{2}\partial_x^2 \quad (\text{A5})$$

Its adjoint

$$\mathcal{F}_p^\dagger = -p\delta(x) + \mu(x)\partial_x + \frac{1}{2}\partial_x^2 \quad (\text{A6})$$

governs the backward dynamics of the propagator $\tilde{P}_p(x_T, T|x, t)$

$$\partial_t \tilde{P}_p(x_T, T|x, t) = \mathcal{F}_p^\dagger \tilde{P}_p(x_T, T|x, t) = -p\delta(x)\tilde{P}_p(x_T, T|x, t) + \mu(x)\partial_x \tilde{P}_p(x_T, T|x, t) + \frac{1}{2}\partial_x^2 \tilde{P}_p(x_T, T|x, t) \quad (\text{A7})$$

a. Physical meaning of the p -deformed Fokker-Planck dynamics

With respect to the dynamics of Eq. 6 corresponding to $p = 0$, the additional term in the forward dynamics of Eq. A4 corresponds to the killing rate of amplitude $p > 0$ localized at the origin $x = 0$

$$k(x) = p\delta(x) \quad (\text{A8})$$

It is however also interesting to consider the case $p < 0$ in the Laplace transform of Eq. 35 : then the additional term in the Feynman-Kac formula of Eq. 36 corresponds instead to the reproducing rate of amplitude $(-p) > 0$ localized at the origin $x = 0$

$$r(x) = (-p)\delta(x) \quad (\text{A9})$$

b. Physical meaning of the associated p -deformed quantum Hamiltonian H_p

Via the similarity transformation analogous to Eq. 10 that involves the potential $U(x)$ of Eq. 9

$$\tilde{P}_p(x, t|x_0, t_0) = e^{-\frac{U(x)}{2}} \psi_p(x, t|x_0, t_0) e^{\frac{U(x_0)}{2}} = e^{\int_{x_0}^x dy \mu(y)} \psi_p(x, t|x_0, t_0) \quad (\text{A10})$$

the forward dynamics of Eq. A4 for $\tilde{P}_p(x, t|x_0, t_0)$ translates into the Euclidean Schrödinger Equation for $\psi_p(x, t|x_0, t_0)$

$$-\partial_t \psi_p(x, t|x_0, t_0) = H_p \psi_p(x, t|x_0, t_0) \quad (\text{A11})$$

With respect to the Hamiltonian H of Eq. 12 involving the potential $V(x)$ of Eq. 13, the quantum Hamiltonian H_p contains an additional delta potential of amplitude p localized at the origin $x = 0$

$$H_p = H + p\delta(x) = -\frac{1}{2}\partial_x^2 + V(x) + p\delta(x) = -\frac{1}{2}\partial_x^2 + \frac{\mu^2(x)}{2} + \frac{\mu'(x)}{2} + p\delta(x) \quad (\text{A12})$$

So the case $p > 0$ associated to the killing rate of Eq. A8 corresponds to an additional repulsive delta potential, while the case $p < 0$ associated to the reproducing rate of Eq. A9 corresponds to an additional attractive delta potential.

3. Canonical conditioning for large horizon T when the Hamiltonian H_p has a normalizable ground-state

a. Propagator $\tilde{P}_p(x, t|x_0, t_0)$ for large time $(t - t_0)$ when the p -deformed Hamiltonian H_p has a normalizable ground-state

When the p -deformed quantum Hamiltonian H_p has a normalizable ground-state $\phi_p^{GS}(x)$ of energy E_p

$$H_p \phi_p^{GS}(x) = E_p \phi_p^{GS}(x) \quad (\text{A13})$$

the ground state can be chosen real and positive $\phi_p^{GS}(x) \geq 0$ with the normalization

$$\langle \phi_p^{GS} | \phi_p^{GS} \rangle = \int_{-\infty}^{+\infty} dx [\phi_p^{GS}(x)]^2 = 1 \quad (\text{A14})$$

This ground-state $\phi_p^{GS}(x)$ and its energy E_p determine the leading asymptotic behavior of the quantum propagator

$$\psi_p(x, t|x_0, t_0) \underset{(t-t_0) \rightarrow +\infty}{\simeq} e^{-(t-t_0)E_p} \phi_p^{GS}(x) \phi_p^{GS}(x_0) \quad (\text{A15})$$

The corresponding asymptotic behavior of the propagator $\tilde{P}_p(x, t|x_0, t_0)$ given by the similarity transformation of Eq. A10 reads

$$\begin{aligned} \tilde{P}_p(x, t|x_0, t_0) &= e^{-\frac{U(x)}{2}} \psi_p(x, t|x_0, t_0) e^{\frac{U(x_0)}{2}} \\ &\underset{(t-t_0) \rightarrow +\infty}{\simeq} e^{-(t-t_0)E_p} \left[e^{-\frac{U(x)}{2}} \phi_p^{GS}(x) \right] \left[e^{\frac{U(x_0)}{2}} \phi_p^{GS}(x_0) \right] \equiv e^{-(t-t_0)E_p} r_p(x) l_p(x_0) \end{aligned} \quad (\text{A16})$$

where

$$\begin{aligned} r_p(x) &\equiv e^{-\frac{U(x)}{2}} \phi_p^{GS}(x) = e^{\int_0^x dy \mu(y)} \phi_p^{GS}(x) \\ l_p(x_0) &\equiv e^{\frac{U(x_0)}{2}} \phi_p^{GS}(x_0) = e^{-\int_0^{x_0} dy \mu(y)} \phi_p^{GS}(x_0) \end{aligned} \quad (\text{A17})$$

correspond to the positive right and left eigenvectors of the generator of Eq. A5 associated to the eigenvalue $(-E_p)$

$$\begin{aligned} -E_p r_p(x) &= \mathcal{F}_p r_p(x) = -p\delta(x)r_p(x) - \partial_x [\mu(x)r_p(x)] + \frac{1}{2}\partial_x^2 r_p(x) \\ -E_p l_p(x) &= \mathcal{F}_p^\dagger l_p(x) = -p\delta(x)l_p(x) + \mu(x)\partial_x l_p(x) + \frac{1}{2}\partial_x^2 l_p(x) \end{aligned} \quad (\text{A18})$$

The normalization inherited from Eq. A14 reads

$$\langle l_p | r_p \rangle = \int_{-\infty}^{+\infty} dx r_k(x) l_k(x) = \int_{-\infty}^{+\infty} dx [\phi_p^{GS}(x)]^2 = 1 \quad (\text{A19})$$

For the double Laplace transform $\hat{P}_{s,p}(x|x_0)$ of Eq. 37, the asymptotic behavior of Eq. A16 for large $(t-t_0)$ means that $\hat{P}_{s,p}(x|x_0)$ exists for $s \in]-E_p, +\infty[$ with the following pole singularity for $s \rightarrow (-E_p)^+$

$$\hat{P}_{s,p}(x|x_0) \underset{s \rightarrow (-E_p)^+}{\simeq} \int_{t_0}^{+\infty} dt e^{-(s+E_p)(t-t_0)} r_p(x) l_p(x_0) = \frac{r_p(x) l_p(x_0)}{s + E_p} \quad (\text{A20})$$

b. Simplifications for the canonical conditioning for large horizon T when H_p has a normalizable ground-state

When the p -deformed quantum Hamiltonian H_p has a normalizable ground-state $\phi_p^{GS}(x)$, the asymptotic behavior of Eq. A16 can be plugged into the three propagators of Eq. A1 to obtain that the conditioned density at any interior time $0 \ll t \ll T$

$$P_T^{[x_T^*;p]}(x, t) \underset{0 \ll t \ll T}{\simeq} \frac{e^{-E_p(T-t)} r_p(x_T) l_p(x) e^{-E_p t} r_p(x) l_p(x_0)}{e^{-E_p T} r_p(x_T) l_p(x_0)} = l_p(x) r_p(x) \equiv P_p^*(x) \quad (\text{A21})$$

does not depend on the interior time t anymore. This steady conditioned density $P_p^*(x)$ only involves the product of the left and right eigenvectors of Eq. A17 and can be thus rewritten as the square of the ground-state $\phi_p^{GS}(x)$ of the quantum Hamiltonian H_p alone

$$P_p^*(x) = l_p(x) r_p(x) = [\phi_p^{GS}(x)]^2 \quad (\text{A22})$$

The corresponding conditioned drift of Eq. A3 is also independent of the interior time t

$$\mu_T^{[x_T^*;p]}(x, t) \underset{0 \ll t \ll T}{\simeq} \mu(x) + \partial_x \ln \left[e^{-E_p(T-t)} r_p(x_T) l_p(x) \right] = \mu(x) + \partial_x \ln [l_p(x)] \equiv \mu_p^*(x) \quad (\text{A23})$$

Since $\mu_p^*(x)$ involves the initial drift $\mu(x)$ and the logarithmic derivative of the left eigenvector $l_p(x)$ of Eq. A17, it can be rewritten in terms of the logarithmic derivative of the ground-state $\phi_p^{GS}(x)$ of the quantum Hamiltonian H_p alone

$$\mu_p^*(x) = \mu(x) + \partial_x \ln \left[e^{-\int_0^x dy \mu(y)} \phi_p^{GS}(x) \right] = \partial_x \ln [\phi_p^{GS}(x)] \quad (\text{A24})$$

c. Conclusion

In summary, when the p -deformed quantum Hamiltonian H_p has a normalizable ground-state $\phi_p^{GS}(x)$, then the canonical conditioned process $X_p^*(t)$ becomes simple for large time horizon $T \rightarrow +\infty$ in the region of interior times $0 \ll t \ll T$: its steady density $P_p^*(x)$ of Eq. A22 and the conditioned drift $\mu_p^*(x)$ of Eq. A24 are time-independent and involve only the normalizable ground-state $\phi_p^{GS}(x)$ of H_p .

The physical meaning of this conditioned process $X_p^*(t)$ depends on whether $H = H_{p=0}$ has also a normalizable ground-state or not :

- (i) the case where $H = H_{p=0}$ has also a normalizable ground-state $\phi_{p=0}^{GS}(x)$, i.e. where the unconditioned process $X(t)$ converges towards an equilibrium state $G_{eq}(x)$ is discussed in Appendix B.
- (ii) the case where $H = H_{p=0}$ has no normalizable ground-state is discussed in Appendix C.

Appendix B: Canonical conditioning when the unconditioned process $X(t)$ has an equilibrium state

In this Appendix, we consider the case where the unconditioned process $X(t)$ converges towards the equilibrium state $G_{eq}(x)$ of Eq. 28, so that the quantum Hamiltonian H of Eq. 12 has a normalizable ground-state $\phi^{GS}(x)$ given by Eq. 25. Then the ground-state $\phi_p^{GS}(x)$ of the Hamiltonian H_p of Eq. A12 can be interpreted as a deformation of this unperturbed ground-state $\phi_p^{GS}(x) = \phi_0^{GS}(x)$, with the following consequences for the physical meaning of the canonical conditioning of parameter p

1. Link between E_p and the rate function $I(a)$ governing the large deviations of the intensive local time a

The ground-state energy E_p of the p -deformed Hamiltonian H_p governs the asymptotic behavior of Eq. A16 of the Laplace transform $\tilde{P}_p(x, T|x_0, 0)$ of the local time A_T introduced in Eq. 35

$$\begin{aligned} \tilde{P}_p(x, T|x_0, 0) &\equiv \int_0^{+\infty} dA e^{-pA} P(x, A, T|x_0, A_0, 0) \\ &\underset{T \rightarrow +\infty}{\simeq} e^{-TE_p} \left[e^{-\frac{U(x)}{2}} \phi_p^{GS}(x) \right] \left[e^{\frac{U(x_0)}{2}} \phi_p^{GS}(x_0) \right] \equiv e^{-TE_p} r_p(x) l_p(x_0) \end{aligned} \quad (\text{B1})$$

The large deviations properties of Eq. 94 can also be used to evaluate the generating function of $(A_T - A_0) = Ta$ via the saddle-point method for large T

$$\langle e^{-p(A_T - A_0)} \rangle = \langle e^{-pTa} \rangle = \int_0^{+\infty} da e^{-pTa} P_T(a) \underset{T \rightarrow +\infty}{\simeq} \int_0^{+\infty} da e^{-T[pa + I(a)]} \underset{T \rightarrow +\infty}{\simeq} e^{-TE_p} \quad (\text{B2})$$

So the energy E_p governing the asymptotic behavior of Eq. B1 for the propagator $\tilde{P}_p(x, t|x_0, t_0)$ is the Legendre transform of the rate function $I(a)$

$$\begin{aligned} pa + I(a) &= E_p \\ p + I'(a) &= 0 \end{aligned} \quad (\text{B3})$$

while the reciprocal Legendre transform reads

$$\begin{aligned} I(a) &= E_p - pa \\ a &= \frac{dE_p}{dp} \end{aligned} \quad (\text{B4})$$

As a consequence, the canonical conditioning of parameter p discussed in section A 3 b can be considered as asymptotically equivalent for large T to the microcanonical conditioning of subsection VIB 2 towards the intensive local time $a_p^* = \frac{dE_p}{dp}$ corresponding to the Legendre value of Eq. B4. Note that this relation $a_p^* = \frac{dE_p}{dp}$ has a very simple interpretation via the first-order perturbation theory for the energy E_p of the ground state $\phi_p^{GS}(x)$ in quantum mechanics when the parameter p is changed into $(p + \epsilon)$

$$a_p^* = \frac{dE_p}{dp} = \lim_{\epsilon \rightarrow 0} \left(\frac{E_{p+\epsilon} - E_p}{\epsilon} \right) = \langle \phi_p^{GS} | \delta(x) | \phi_p^{GS} \rangle = [\phi_p^{GS}(x=0)]^2 = P_p^*(x=0) \quad (\text{B5})$$

that corresponds to the conditioned steady state $P_p^*(x)$ of Eq. A22 at the origin $x = 0$.

Relation between the Laplace parameter p and the time-Laplace parameter s in the large deviations analysis

The comparison between

- (i) the Legendre transform of Eqs. B3, B4 between $I(a)$ and E_p
- (ii) the quasi-Legendre transform of Eqs. 100, 103 and 106 between $I(a)$ and $\frac{1}{\hat{G}_s(0|0)}$

allows to eliminate the variable a to obtain the following relations between the Laplace parameter p and the time-Laplace parameter s

$$\begin{aligned} s &= -E_p \\ p &= -\frac{1}{\hat{G}_s(0|0)} \end{aligned} \quad (\text{B6})$$

2. Example of the drift $\mu(x) = -\mu \text{sgn}(x)$ with $\mu > 0$

a. Computation of the energy E_p via the Legendre transform of the rate function $I(a)$

For the explicit rate function $I(a)$ of Eq. 247, the Legendre transform of Eq. B3 yields the following properties.

(i) The microcanonical conditioning to the intensive local time a^* is asymptotically equivalent in the thermodynamic limit $T \rightarrow +\infty$ to the canonical conditioning of parameter

$$p = -I'(a^*) = \mu - a^* \quad (\text{B7})$$

so that the domain $a^* \in [0, +\infty[$ of definition for the intensive local time corresponds to the following domain for the Laplace variable p

$$p \in]-\infty, \mu] \quad (\text{B8})$$

Reciprocally, the canonical conditioning of parameter p is asymptotically equivalent in the thermodynamic limit $T \rightarrow +\infty$ to the microcanonical conditioning of the intensive local time

$$a_p^* = \mu - p \quad (\text{B9})$$

(ii) The energy E_p of Eq. B3 reads using Eq. 247 and Eq. B7

$$E_p = pa + I(a) = pa + \frac{(a - \mu)^2}{2} = p(\mu - p) + \frac{p^2}{2} = p\mu - \frac{p^2}{2} \quad (\text{B10})$$

b. Direct analysis of the ground-state of the p -deformed Hamiltonian H_p

For the drift $\mu(x) = -\mu \operatorname{sgn}(x)$ with $\mu > 0$, the Hamiltonian $H = H^{[\mu]}$ of Eq. 220 of parameter $\mu > 0$

$$H^{[\mu]} = -\frac{1}{2}\partial_x^2 + \frac{\mu^2}{2} - \mu\delta(x) \quad (\text{B11})$$

has the zero-energy normalized ground-state of Eq. 25 using Eq. 216

$$\phi^{GS[\mu]}(x) = \sqrt{G_{eq}(x)} = \sqrt{\mu}e^{-\mu|x|} \quad (\text{B12})$$

The p -deformed Hamiltonian $H_p = H_p^{[\mu]}$ of Eq. A12

$$\begin{aligned} H_p^{[\mu]} &= H^{[\mu]} + p\delta(x) = -\frac{1}{2}\partial_x^2 + \frac{\mu^2}{2} - (\mu - p)\delta(x) \\ &= -\frac{1}{2}\partial_x^2 + \frac{(\mu - p)^2}{2} - (\mu - p)\delta(x) + \frac{\mu^2 - (\mu - p)^2}{2} \\ &\equiv H^{[\mu-p]} + E_p \end{aligned} \quad (\text{B13})$$

can be thus interpreted in the domain $p \in]-\infty, \mu[$ of Eq. B8 as the sum of :

(i) the Hamiltonian $H^{[\mu-p]}$ of effective drift $(\mu - p) > 0$ in Eq. B11, with its zero-energy normalized ground-state of Eq. B12

$$\phi^{GS[\mu-p]}(x) = \sqrt{\mu - p}e^{-(\mu-p)|x|} \quad (\text{B14})$$

(ii) the remaining constant in Eq. B13

$$E_p = \frac{\mu^2 - (\mu - p)^2}{2} = p\mu - \frac{p^2}{2} \quad (\text{B15})$$

that directly represents the ground-state energy of $H_p^{[\mu]}$ and that coincides with the alternative analysis of Eq. B15 as it should.

c. Canonical conditioned process $X_p^(t)$ of parameter p*

Since Eq. B14 is the ground-state of the p -deformed Hamiltonian $H_p^{[\mu]}$

$$\phi_p^{GS[\mu]}(x) = \phi^{GS[\mu-p]}(x) = \sqrt{\mu - p}e^{-(\mu-p)|x|} \quad (\text{B16})$$

one obtains that the conditioned drift of Eq. A24 reduces to

$$\mu_p^*(x) = \partial_x \ln [\phi_p^{GS[\mu]}(x)] = -(\mu - p) \operatorname{sgn}(x) \quad (\text{B17})$$

So the canonical conditioning of parameter $p \in]-\infty, \mu[$ simply amounts to change the amplitude μ of the unconditioned drift $\mu(x) = -\mu \operatorname{sgn}(x)$ into the amplitude $(\mu - p)$. As a consequence, the corresponding conditioned equilibrium state

$$G_p^{eq[\mu]}(x) = G^{eq[\mu-p]}(x) = (\mu - p)e^{-2(\mu-p)|x|} \quad (\text{B18})$$

will produce the following equilibrium value for the intensive local time

$$a_p^{eq[\mu]} = G_p^{eq[\mu]}(x=0) = \mu - p \quad (\text{B19})$$

in agreement with the Legendre correspondence of Eq. B7.

Finally, the conditioned drift of Eq. B17 can be rewritten in terms of $a_p^* = \mu - p$ of Eq. B9 as

$$\mu_p^*(x) = -a_p^* \operatorname{sgn}(x) \quad (\text{B20})$$

in agreement with the microcanonical conditioning of Eq. 261 in the main text.

Appendix C: Canonical conditioning when the unconditioned process $X(t)$ has no equilibrium state

In this Appendix, we consider the case where the unconditioned process $X(t)$ has no equilibrium state, so that the quantum Hamiltonian H of Eq. 12 has no bound state. For the Hamiltonian H_p of Eq. A12, the presence of a bound state depends on the sign of p as follows.

(i) The case $p > 0$ corresponds to an additional repulsive delta potential at the origin $x = 0$ and will not change the range $]V_\infty, +\infty[$ of the continuous spectrum of H , since H and H_p have the same potential at $x \rightarrow \pm\infty$ in Eq. 17.

(ii) The case $p < 0$ corresponds to an additional attractive delta potential at the origin $x = 0$ that produces a normalizable ground state for H_p as we now describe.

1. Emergence of a bound state in the attractive case $p < 0$

For the double Laplace transform $\hat{P}_{s,p}(x|x_0)$ of Eq. 37, the result of Eq. 45

$$\hat{P}_{s,p}(x|x_0) = \left[\hat{G}_s(x|x_0) - \frac{\hat{G}_s(x|0)\hat{G}_s(0|x_0)}{\hat{G}_s(0|0)} \right] + \left[\frac{\hat{G}_s(x|0)\hat{G}_s(0|x_0)}{\hat{G}_s^2(0|0)} \right] \frac{1}{p + \frac{1}{\hat{G}_s(0|0)}} \quad (\text{C1})$$

shows that for $p < 0$, a new singularity will appear in $\hat{P}_{s,p}(x|x_0)$ with respect to $\hat{G}_s(x|x_0)$ when the variable s make the denominator vanish

$$0 = p + \frac{1}{\hat{G}_s(0|0)} \quad (\text{C2})$$

The comparison with the pole in Eq. A20 shows that the value of s satisfying Eq. C2 is directly related to the ground-state energy E_p of H_p

$$s = -E_p \quad (\text{C3})$$

i.e. the relations between p and s of Eqs C2 and C3 are the same as in Eq. B6.

2. Example of the uniform drift $\mu \geq 0$

a. Computation of the energy E_p via the pole analysis of Eqs C2 C3

For the case of the uniform drift $\mu \geq 0$, the Laplace transform $\hat{G}_s(0|0)$ of Eq. 147 yields that Eq. C2 reads

$$0 = p + \sqrt{\mu^2 + 2s} \quad (\text{C4})$$

So for any $p < 0$, its solution $s = -E_p$ of Eq. C3 leads to the energy

$$E_{p<0} = \frac{\mu^2 - p^2}{2} \quad (\text{C5})$$

for the ground state of H_p that emerges below the continuous spectrum $]\frac{\mu^2}{2}, +\infty[$.

b. Direct analysis of the ground-state of the p -deformed Hamiltonian H_p for $p < 0$

For the drift $\mu(x) = \mu$, the Hamiltonian H of Eq. 12

$$H = -\frac{1}{2}\partial_x^2 + \frac{\mu^2}{2} \quad (\text{C6})$$

has no bound-state, but only a continuous spectrum $]\frac{\mu^2}{2}, +\infty[$.

The p -deformed Hamiltonian H_p of Eq. A12 reads

$$H_p^{[\mu]} = H^{[\mu]} + p\delta(x) = -\frac{1}{2}\partial_x^2 + \frac{\mu^2}{2} + p\delta(x) \quad (\text{C7})$$

For the repulsive case $p > 0$, H_p keeps the continuous spectrum $]\frac{\mu^2}{2}, +\infty[$.

However, for the attractive case $p < 0$, a bound state emerges below the continuous spectrum $]\frac{\mu^2}{2}, +\infty[$. It is exponentially localized around the origin

$$\phi_{p<0}^{GS}(x) = \sqrt{(-p)}e^{-(p)|x|} \quad (\text{C8})$$

Its energy

$$E_{p<0} = \frac{\mu^2 - p^2}{2} \quad (\text{C9})$$

coincides with the other derivation of Eq. C5 as it should.

c. Canonical conditioned process $X_p^(t)$ of parameter $p < 0$*

The conditioned drift of Eq. A24 reads

$$\mu_{p<0}^*(x) = \partial_x \ln [\phi_p^{GS}(x)] = -(-p) \operatorname{sgn}(x) \quad (\text{C10})$$

The corresponding conditioned equilibrium state reads

$$G_p^{eq}(x) = [\phi_{p<0}^{GS}(x)]^2 = (-p)e^{-2(-p)|x|} \quad (\text{C11})$$

So the canonical conditioning of parameter $p < 0$ is asymptotically equivalent to the microcanonical conditioning towards the intensive local time

$$a_p^* = G_p^{eq}(x=0) = -p > 0 \quad (\text{C12})$$

So the conditioned drift of Eq. C10 can be rewritten as

$$\mu_{p<0}^*(x) = -a_p^* \operatorname{sgn}(x) \quad (\text{C13})$$

in agreement with the microcanonical conditioning of Eq. 210 in the main text.

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