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# Study of the vertical modes of a periodic plane medium immersed in water.

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## Summary

We are interested in a medium composed of  $N$  periods including two perfectly bonded plates which exhibit a high acoustic impedance contrast.  $N$  is varied from 2 to 10. Among the propagative modes in this structure, we can detect the so-called vertical modes at low frequency. Their labeling is due to the fact that their dispersion curves are vertical in the (frequency, angle) plane. In addition, their cut-off frequencies are inferior to the lowest cut-off frequency of the modes propagating in each plate. These vertical modes are linked with the coupling effect occurring between the plates which have the strongest acoustic impedance in the periodic medium. Comparisons between the dispersion curves obtained in vacuum and the zeroes of the reflection coefficient of the water loaded periodic medium are presented. Approximate analytical expressions of the cut-off frequencies of the vertical modes are obtained by developing at the fourth order the trigonometric functions appearing in the dispersion equations. It can be of use for the inverse problem, to find phase velocities or thicknesses.

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## 1. Introduction

1D periodic media may be of interest for purposes such as wave guiding, filtering or resonator modeling. The dispersion equations of a periodic media composed of  $N$  periods, each one made of  $n$  isotropic plates are obtained by using a transfer matrix method. It is well known that the dispersion curves exhibit stop bands in which no propagative modes exist and pass bands in which those ones can be detected. At low frequency, it exists the so-called vertical modes because their dispersion curves are vertical in a wide wave number domain [1-2]. They belong to the family of the structure modes which depend on the number  $N$  of periods and appear in the pass bands. The other family is named the period mode one, which only depends on the characteristics of a period and not on the number  $N$ . The period modes are strongly attenuated and are located in the stop bands or at

their limits. Another way to study the structure modes deals with the calculation of the reflection coefficient of the structure immersed in water, for instance. It exhibits minima which are linked with the propagation of guided modes. In the first section, we briefly recall how to obtain the dispersion equation for a periodic structure and its factorization in two equations, the one giving the period modes, the other the structure ones. The vertical modes are obtained from the second one, at low frequency. Approximate expressions of their cut-off frequencies are found by means of developments in Taylor series of trigonometric functions. They are good guesses for an exact calculation. In the second section, the expression of the reflection coefficient of the periodic medium immersed in water is presented. Comparisons between zeroes of the dispersion equation and zeroes of the reflection coefficient are given. In the third section, different approximate expressions are compared when the thicknesses of the layers in a period are varied.

In conclusion, some perspectives to this work are given.

## 2. Dispersion equation of a periodic plane medium in vacuum

In our study, we have considered  $N$  periods ( $2 < N < 10$ ) composed of 2 plates : one aluminum plate (Al) perfectly bonded with a polyethylene (PE) one, immersed in vacuum. No viscoelasticity has been taken into account [3]. The geometry of the problem is given in Figure 1 :

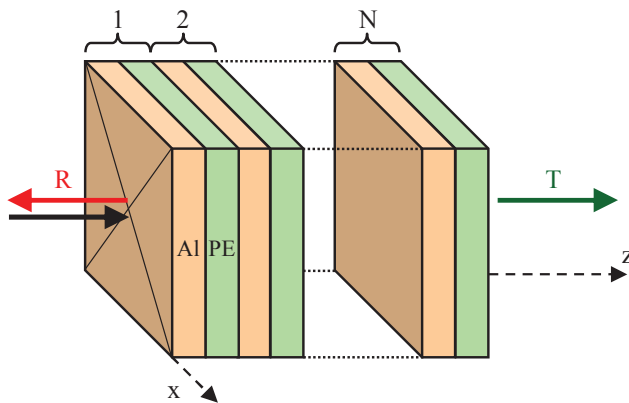


Figure 1. Geometry of the problem.

In this figure, the case of the periodic media immersed in a fluid and insonified by an incident monochromatic plane wave is also considered. The reflection and transmission at normal incidence are studied in section 3.

The values of the physical parameters are given in the following Table I :

Table I. Acoustic properties of the layers.

Medium	$d$ (mm)	$\rho$ (kg/m <sup>3</sup> )	$c_L$ (m/s)	$Z$ (MRa)
Al	2	2800	6380	17.9
PE	2	940	2370	2.22
Water	–	1000	1470	1.47

$Z$  : acoustic impedance ;  $\rho$  : density ;  $c_L$  : longitudinal velocity ;  $d$  : thickness.

We consider that harmonic waves propagate in each layer and are described by the following potential :

$$\psi(x, z, t) = (Ae^{jk_z z} + Be^{-jk_z z}) e^{j(k_x x - \omega t)} \quad (1)$$

By writing the continuity of the normal displacement  $u_z = \partial\psi/\partial z$  and of the normal stress  $T_{zz} = -\rho\omega^2\psi$  at each interface between the layers, we can link between two adjacent periods state

vectors whose components are the normal displacement and stress. It is performed by using a transfer matrix [4], denoted as  $T_p$ .

This matrix has the following form :

$$T_p = \begin{pmatrix} a_p & jb_p \\ jc_p & d_p \end{pmatrix}, \quad \text{with } a_p d_p + b_p c_p = 1. \quad (2)$$

The eigenvalues  $v$  of the transfer matrix are solutions of the characteristic equation [5-6]:

$$v^2 - sv + 1 = 0, \quad \text{with } s = (a+d)/2. \quad (3)$$

The global transfer matrix of  $N$  periods, denoted as  $T_{Np}$ , which relates the output and input state vectors is the  $N^{\text{th}}$  power of the transfer matrix of a period. It is shown that  $T_{Np}$  can be obtained from the following relation :

$$T_{Np} = U_{N-1}(s)T_p - U_{N-2}(s)I, \quad (4)$$

where the functions  $U_n(s)$  are the Chebyshev polynomials of the second kind and  $I$  is the identity matrix.

The global transfer matrix  $T_{Np}$  has the same form as the period transfer matrix, but depends on elements denoted as  $a_{Np}$ ,  $b_{Np}$ ,  $c_{Np}$  and  $d_{Np}$ . The element of interest here is  $c_{Np}$  which expresses as  $c_{Np} = c_p U_{N-1}(s)$ . Indeed, when the periodic medium is in vacuum, the normal stresses at the boundary limits are equal to zero and it implies that :

$$c_{Np} = 0 \quad (5)$$

It defines the dispersion equation of the periodic medium [6]. We can note that it is factorized.

The nullity of  $c_p$  gives the period modes and the  $(N-1)$  zeroes of the Chebyshev polynomial  $U_{N-1}(s)$  correspond to the structure modes. The values of the parameter  $s_k$  corresponding to those zeroes can be written as [6] :

$$s_k = \cos\left(\frac{k}{N}\pi\right), \quad \text{with } 1 < k < (N-1) \quad (6)$$

In the following, we are mainly interested in the structure modes which are propagative whereas the period ones are strongly attenuated. Moreover, we have chosen to analyze the vertical modes at low frequency.

When a period is composed of two layers, the parameter  $s$  expresses as :

$$s = \cos\varphi_1 \cos\varphi_2 - \frac{1}{2}\left(\frac{Z_1}{Z_2} + \frac{Z_2}{Z_1}\right) \sin\varphi_1 \sin\varphi_2 \quad (7),$$

where  $\varphi_{1,2} = \frac{\omega d_{1,2}}{c_{L1,2}}$  ( $d_{1,2}$  : thicknesses of the layers,

$c_{L1,2}$  : longitudinal phase velocities) and the terms  $Z_{1,2}$  are the acoustic impedances of the layers defined as  $Z_{1,2} = \rho_{1,2}c_{L1,2}$  (Table I).

For conciseness, we introduce these constants :

$$K = \frac{Z_1}{Z_2} + \frac{Z_2}{Z_1} \quad \text{and} \quad r = \frac{\phi_1}{\phi_2}$$

For a given number  $N$  of periods, equations (6) and (7) give the values of  $s_k$  and allows to obtain the values of  $\phi_2$ . In addition, with the assumption of low frequency, we can consider that the estimated value  $\phi$  of  $\phi_2$  is very small compared to unity. So, developments in Taylor series of the trigonometric functions appearing in the expression of the parameter are possible. Developments at the fourth order of equation (7) with  $s = s_k$  (equation (6)) lead to the following polynomial in order to find  $\phi$  :

$$a\phi^4 + b\phi^2 + c = 0 \quad (8)$$

with  $a = (1 + 2Kr + 6r^2 + 2Kr^3 + r^4) / 24$

$$b = -(1 + Kr + r^2) / 2$$

$$c = 1 - s_k$$

At the fourth order, the solutions have the following form :

$$\phi_k^{(4)} = \sqrt{\frac{-b - \sqrt{\Delta}}{2a}}, \quad \text{with } \Delta = b^2 - 4ac \quad (9)$$

This expression is valid since  $\Delta > 0$ , i.e. :

$$s_k < \frac{1 + 2Kr + (3K^2 - 6)r^2 + 2Kr^3 + r^4}{2(1 + 2Kr + 6r^2 + 2Kr^3 + r^4)} \quad (10)$$

For a given  $s_k$ , there are also limit values of the parameter  $r$  or  $K$ , which can lead to a negative discriminator. Especially for the  $r$  parameter, one can write :

$$\Delta = 0 \Rightarrow \prod_{i=1}^{i=4} (r - r_i) = 0 \quad (11)$$

with  $r_1 = r_+$ ,  $r_2 = 1/r_+$ ,  $r_3 = r_-$ ,  $r_4 = 1/r_-$ .

When we limit the development at the second order, one obtains :

$$\phi_k^{(2)} = \sqrt{\frac{2(1 - s_k)}{1 + Kr + r^2}} \quad (12)$$

The cut-off frequencies are given by :

$$f_k^{(2,4)} = \frac{c_{L2}}{2\pi d_2} \phi_k^{(2,4)} \quad (13)$$

An alternative method consists in using the Newton-Raphson iterative algorithm :

$$\phi_k^{(NR)}(n+1) = \phi_k^{(NR)}(n) - \frac{g(\phi_k^{(NR)}(n))}{g'(\phi_k^{(NR)}(n))} \quad (14)$$

with  $g(\phi) = \cos r\phi \cos \phi - (K/2) \sin r\phi \sin \phi - s_k$ ,  
 $\phi_k^{(NR)}(0) = \phi_k^{(2)}$ , and  $n$  the iteration number.

By this method, we quickly converge to the zeroes of the function.

### 3. Reflection coefficient of a plane periodic structure

#### 3.1. Theoretical basis

In this section, we are particularly interested in low frequency modes called vertical modes. It is justified by the plot of their dispersion curves [7] in the plane frequency  $f$  versus incidence angle  $\theta$  for a  $N = 5$  periods structure (Figure 2 (a)). They are obtained with the zeroes of  $|R|^2$ . We can observe (Figure 2 (b)) the  $N-1 = 4$  vertical dispersion curves between  $\theta = 0$  and  $10^\circ$ , what justifies the name of vertical modes.

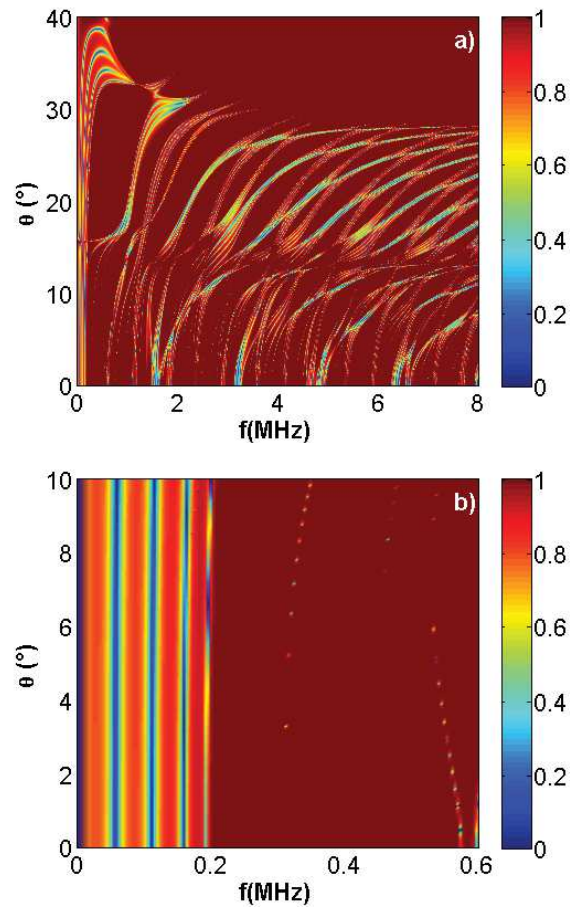


Figure 2. Theoretical dispersion curves obtained from the reflection coefficient  $|R|^2$  in the case of  $N = 5$  periods (a) for  $f = 0$  to 8 MHz and (b) for  $f = 0$  to 0,6 MHz.

The Debye series decomposition method [8] allows to write the reflection coefficient at normal incidence. It takes into account multiple reflections at the interfaces between constitutive layers. This recursive approach ensures a numerical stability.

At first, we define the Fresnel reflection and transmission coefficients at the interface between the layers indexed  $m$  and  $m+1$  :

$$R_{m,m+1} = \frac{Z_{m+1} - Z_m}{Z_{m+1} + Z_m} e^{2j\beta_m} \quad (14a)$$

and 
$$T_{m,m+1} = \frac{2Z_{m+1}}{Z_{m+1} + Z_m} e^{-j(\beta_{m+1} - \beta_m)} \quad (14b)$$

Mutually, at the interface between the layers indexed  $m+1$  and  $m$  :

$$R_{m+1,m} = \frac{Z_m - Z_{m+1}}{Z_m + Z_{m+1}} e^{-2j\beta_{m+1}} \quad (14c)$$

with 
$$T_{m+1,m} = \frac{2Z_m}{Z_{m+1} + Z_m} e^{-j(\beta_{m+1} - \beta_m)} \quad (14d)$$

with  $\beta_m = \omega z_m / c_{Lm}$  and  $\beta_{m+1} = \omega z_m / c_{L,m+1}$  are phases which depends on the interface positions  $z_m$ , between the layers  $m$  and  $m+1$ .

The global reflection coefficient of the structure is calculated from the layer  $m = M$  down to 1 :

$$R = R_{12} + \frac{T_{12}\rho_{23}T_{21}}{1 - R_{21}\rho_{23}} \quad (16)$$

with 
$$\rho_{m,m+1} = R_{m,m+1} + \frac{T_{m,m+1}\rho_{m+1,m+2}T_{m+1,m}}{1 - R_{m+1,m}\rho_{m+1,m+2}}$$

for  $2 \leq m \leq M$  and  $\rho_{M+1,M+2} = R_{M+1,M+2}$

As a result (Figure 3), the energy reflection coefficient  $|R|^2$  is compared to the second kind Chebyshev polynomial of order  $N-1$ ,  $U_{N-1}(s)$  defined in section 2. We note a remarkable coincidence between the zeroes of  $|R|^2$ , denoted as  $f_{N-1}^{(R)}$ , and those of  $U_{N-1}(s)$ . This is valid not only for the vertical modes, but also for all the structure modes at higher frequency. In this case one verifies  $N-1 = 9$  vertical modes corresponding to the roots of  $U_{N-1}(s)$ , and  $N-1$  zeroes of  $|R|^2$ .

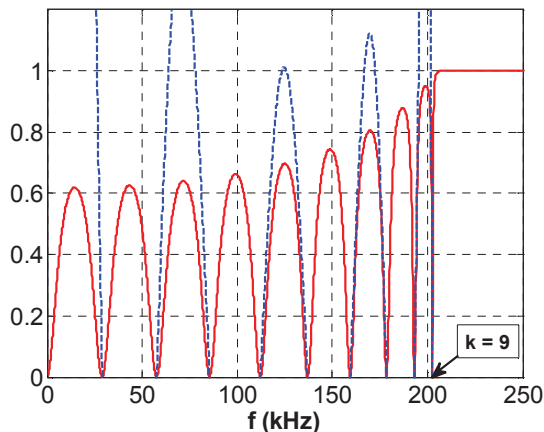


Figure 3. Theoretical reflection coefficient  $|R|^2$  (solid) and  $U_{N-1}(s)$  (dashed) in the case of  $N = 10$  periods.

### 3.2. Validity of approximations

In this subsection, we verify the validity of approximate expressions  $f_k^{(2,4)}$  (equations (9), (12) and (13)) relatively to the exact  $f_{N-1}^{(R)}$ .

In Table II, we compare the exact value of  $f_k^{(R)}$  to the approximate values  $f_k^{(2,4)}$ . We note that the convergence of the approximated values  $f_k^{(2,4)}$  is increased between the order 2 and order 4. More precisely, the maximum relative error  $\epsilon^{(2,4)}$ , obtained for the higher frequency mode is given by :

$$\epsilon^{(2,4)} = \left| \frac{f_{N-1}^{(2,4)} - f_{N-1}^{(R)}}{f_{N-1}^{(R)}} \right| \quad (17)$$

Here, it is compared for  $N = 10$  :

$$\epsilon^{(2)} = \left| \frac{201 - 181}{201} \right| \approx 10\% \quad (18)$$

and 
$$\epsilon^{(4)} = \left| \frac{193 - 195}{193} \right| \approx 1\% \quad (19)$$

The second order development does not give sufficient accurate results, whereas the fourth order development gives correct values.

Nevertheless, the precision depends on parameters  $K$ ,  $r$  et  $s_k$ . Moreover, the case of a fourth order development, we need to verify the condition given by equation (10).

Table II. Cut-off frequencies.

N	$f_k^{(R)}$ (kHz)	$f_k^{(2)}$ (kHz)	$f_k^{(4)}$ (kHz)
2	137	131	137
5	58	57	58
	112	109	112
	159	150	160
	193	176	195
10	29	29	29
	58	57	58
	86	84	85
	112	109	112
	137	131	137
	159	150	160
	178	165	180
	193	176	195
	203	183	205

As expected (equation (6)), there are common values of parameter  $s_k$  when  $N = 2, 5$  and  $10$  periods. In a more general way, if we consider two periodic structures one having  $N_1$  periods the second having  $N_2$  periods, and if  $N_2$  is a multiple of  $N_1$ , then one obtains common roots  $s_k$  and thus identical cut-off frequencies  $f_k$ .



#### 4. Application to a variation of layer thicknesses

In this section the thicknesses of layers  $d_{1,2}$  are varied from 0 to 10 mm, whereas the other parameters are left constant. We still consider a 10 period medium. The cut-off frequency  $f_k^{(R)}$ , determined from the last zero of the reflection coefficient corresponding to the last vertical mode, i.e.  $k = N-1 = 9$ , (Figure 4) is compared to the  $f_k^{(2)}$  and  $f_k^{(4)}$  approximate values. According to equations (12) and (13), the highest cut-off frequency of vertical modes is linked with the minimum value of the parameter  $s$ , i.e.  $s_9 = \cos(9\pi/10)$ .

The frequency of the last vertical mode is plotted as a function of the thicknesses  $d_1$  and  $d_2$ , in the following Figures 4, 5 and 6.

Isovalue curves of the frequency of the last vertical mode  $f_9^{(2)}$  and  $f_9^{(4)}$  are shown in Figures 5 (a) and 6 (a), respectively. Isovalue curves of the relative errors  $\epsilon^{(2)}$  and  $\epsilon^{(4)}$ , dealing with the maximum cut-off frequency  $f_9^{(R)}$  (Figure 4) are shown in Figures 5 (b) and 6 (b), respectively.

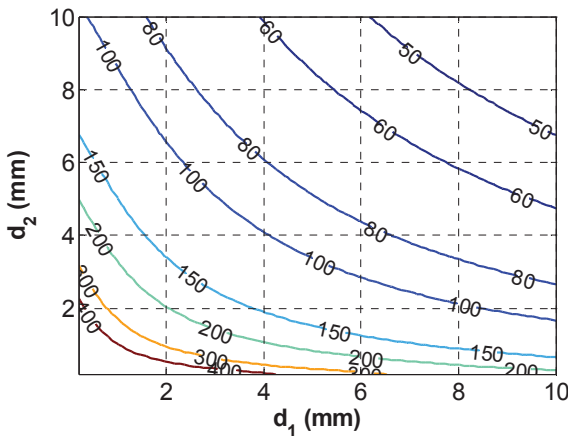


Figure 4. Exact cut-off frequency  $f_{N-1}^{(R)}$  (kHz) for  $N = 10$ .

We recall that the approximate frequency  $f_9^{(2)}$  is obtained as follows :

$$f_9^{(2)} = \frac{c_{L2}}{2\pi d_2} \sqrt{\frac{2(1-s_9)}{1+Kr+r^2}} \quad (20)$$

with  $r = \frac{c_{L2}}{c_{L1}} \frac{d_1}{d_2}$

The equation (20) explains the decreasing evolution of the isovalue curves (Figure 5 (a)).

We can notice that there is an important relative error  $\epsilon^{(2)}$  (Figure 5 (b)) for small values of  $d_1$  and  $d_2$ . Moreover,  $\epsilon^{(2)}$  is never less than 5% whatever the thickness values.

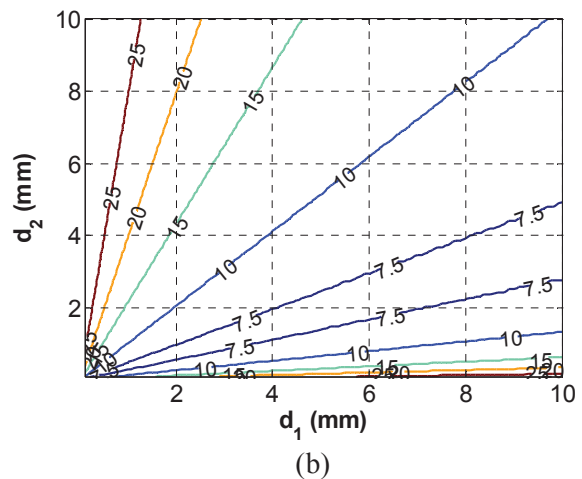
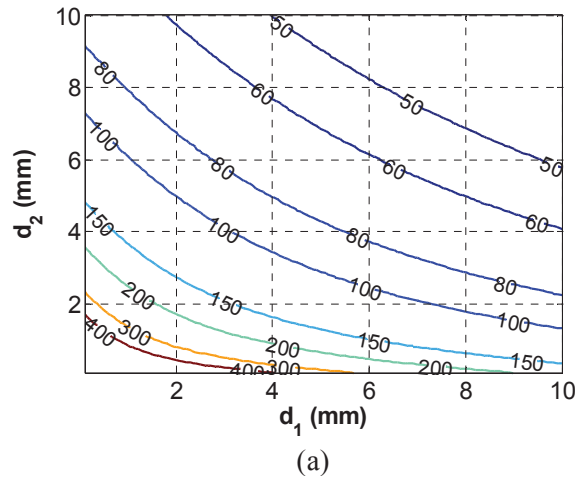


Figure 5. (a) Approximated cut-off frequency  $f_{N-1}^{(2)}$  (kHz), and (b) relative error  $\epsilon^{(2)}$  (%) dealing with the maximum cut-off frequency  $f_{N-1}^{(R)}$  for  $N = 10$ .

Dealing with the isovalue curves of  $f_9^{(4)}$  (Figure 6 (a)), the validity domain is also compared to the exact solution (Figure 4).

In Figure 6, three zones are delimited by red dashed lines. In the inner zone, there is a good agreement between the exact  $f_9^{(R)}$  and approximate  $f_9^{(4)}$  frequencies, and  $\epsilon^{(4)}$  is inferior to 10% . In the outer zones, the expression of  $f_9^{(4)}$  is not correct because the discriminator  $\Delta$  (equation (10)) is negative. In that case, the obtained frequencies are complex. Nevertheless, we have plotted their real parts.

The red dashed lines are defined by the limit values of  $r$ , which are defined by  $\Delta = 0$  (equations (10) and (11)). We obtain two physical roots (equation (11)),  $r_1 = r_+ = 0.12$  and  $r_2 = 1/r_+ = 8.16$ . The property  $r_1 r_2 = 1$  is due to the fact that the order of the layers in a period is indifferent. Although the precision of the fourth order solution (Figure 5) is improved relatively to that of the second order solution (Figure 6), it is also limited and an alternative method must be used.

The Newton-Raphson iterative algorithm offers an interesting trade-off between precision and calculation time. To implement this algorithm, the  $f_9^{(2)}$  are good guesses. The relative error dealing with the exact solution is quickly reduced. Only three iterations of the method (equation (14)) allow to reach values of  $\epsilon^{(NR)}$  inferior to 0.1%.

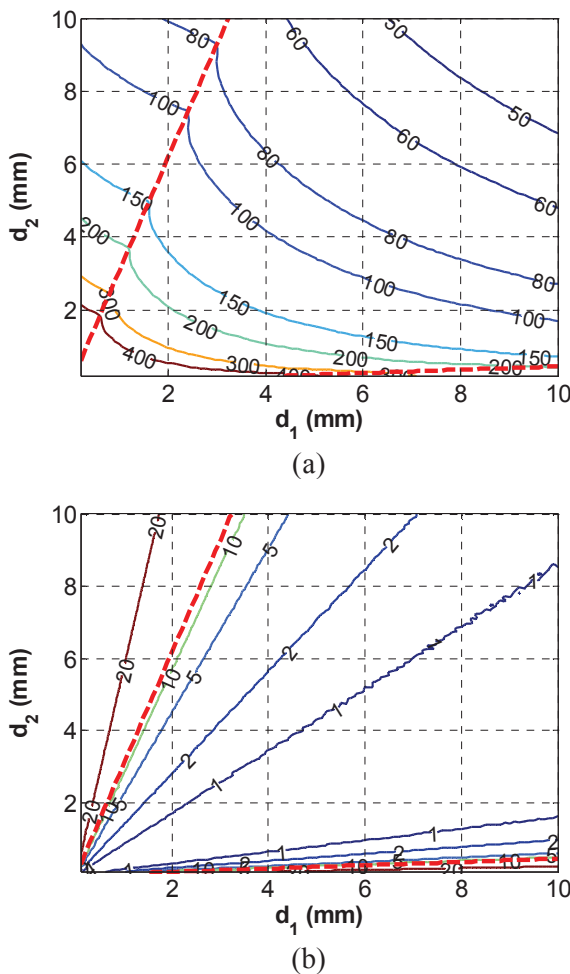


Figure 6. (a) Approximated cut-off frequency  $f_{N-1}^{(4)}$  (kHz), and (b) relative error  $\epsilon^{(4)}$  (%) dealing with the maximum cut-off frequency  $f_{N-1}^{(R)}$  for  $N = 10$ .

The association of the development at the second order resulting in  $f_k^{(2)}$  and the Newton-Raphson method permits the accurate determination of cut-off frequencies  $f_k^{(NR)}$  of the vertical modes, whatever the values of  $k$  and  $N$ .

## 5. Conclusion

This work dealt with the study of the vertical modes of periodic structures. These are propagative modes observed at low frequency. Their cut-off frequencies depend on the geometrical and physical properties of the constitutive layers. It is shown in this paper that we can obtain an approximate expression of those ones. Nevertheless, these developments at the second and fourth orders are not precise enough to accurately determine the cut-off frequencies. Numerically, using the Newton-Raphson method, the result can be as precise as wanted. This method can be used to solve an inverse problem.

As a perspective of this work, a parametric study of the impedance ratio, i.e.  $K$  parameter, would be interesting.

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