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# Study of period and structure modes of periodic multilayers with the S matrix formalism

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## Résumé :

*La matrice  $S$  d'une multicouche plane périodique immergée dans un fluide est construite à partir des coefficients de réflexion et de transmission. Les termes de transition obtenus à partir de ses valeurs propres sont associés soit à des modes de période soit à des modes de structures. L'influence de ces modes sur la position et la largeur des bandes interdites sont étudiées dans le cas d'une multicouche de  $N$  périodes composées de deux plaques isotropes.*

## Abstract :

*The scattering  $S$  matrix of a periodic structure is derived from its reflection and transmission coefficients. The transition terms depending on the eigenvalues of  $S$  are assigned to period or structures modes respectively. The influence of those modes on the position and width of the band gaps are studied in the case of a multilayer composed of  $N$  periods of two isotropic layers.*

**Mots clefs :** milieux périodiques, résonances, bandes interdites

## 1 Introduction

It was proved that the  $S$  matrix formalism is efficient to study the resonances of an elastic isotropic plate [1] or the ones of a porous plate [2] or those of a fluid layer embedded in semi infinite solids [3]. The aim of this paper is to show that the application of this formalism to a stack of  $N$  periods ( $2 < N < 8$ ) composed of  $n$  isotropic plates (here  $n=2$ ) permits to separate the resonances between period or structure ones. The frequency locations of those ones may give information about the width of the stop bands. In a first part, it is shown how to form the  $S$  matrix from the reflection and transmission coefficients. Those ones are obtained using the stiffness matrix method [4,5]. The transition terms are then derived from the eigenvalues of  $S$ . They permit to characterize the resonances of the structure (frequency and width) which are linked with the propagative waves. In a second part, plots of transitions term moduli for multilayers composed of different numbers of period are compared in order to enhance the period modes and the structure ones.

## 2 Theoretical basis

As illustrated (Figure 1), the geometry of the problem consists in a stack of periods composed of  $n$  isotropic plates whose density is  $\rho_n$ , thickness is  $d_n$ , longitudinal and transverse phase velocities are  $c_L^n$  and  $c_T^n$ , respectively. The multilayer is embedded in a fluid whose density is  $\rho_F$  and where the waves propagate with the phase velocity  $c_F$ .

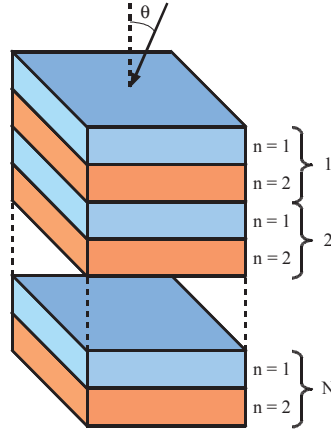


FIG. 1 – Geometry of the problem.

We may consider a plane monochromatic incident wave in the fluid at the incidence angle  $\theta$  with respect to the normal of the structure, on one side or the other. In order to construct the  $\mathbf{S}$  diffusion matrix, the reflection and transmission coefficients must be calculated. As the structure is not symmetric with regard to the median plane, the reflection coefficients are different according to the incidence side. Their moduli are identical but their numerators are complex conjugate. On the contrary, the transmission coefficient is identical whatever the incidence side. So, the  $\mathbf{S}$  matrix can be expressed as:

$$\mathbf{S} = \begin{pmatrix} -R_1 & T \\ T & -R_2 \end{pmatrix} \quad (1)$$

where  $R_1$  and  $R_2$  are the reflection coefficients on both sides and  $T$  the transmission coefficient. Those coefficients are obtained by using the stiffness matrix method. It consists in relating a stress vector composed of the stress components of two adjacent layers to a displacement vector composed of the displacement components in these layers.

In the  $n^{\text{th}}$  layer, the displacement vector  $\vec{u}^n = [u_x^n \ u_z^n]^T$  (superscript T stands for transposition) can be written as the summation of four partial waves:

$$\vec{u}^n = \left( a_L^{n+} \vec{P}_L^{n+} e^{jk_{Lz}^n(z-z_n)} + a_T^{n+} \vec{P}_T^{n+} e^{jk_{Tz}^n(z-z_n)} + a_L^{n-} \vec{P}_L^{n-} e^{-jk_{Lz}^n(z-z_{n-1})} + a_T^{n-} \vec{P}_T^{n-} e^{-jk_{Tz}^n(z-z_{n-1})} \right) e^{j(k_x x - \omega t)} \quad (2)$$

The positive or negative superscripts indicate a wave propagation in the  $+z$  or  $-z$  directions.  $\vec{P}_{L,T}^{n\pm}$  are the unit displacement polarization vectors linked with longitudinal and transverse waves propagating up and down.  $k_x = \omega \cdot \sin(\theta) / c_F$  is the wave number in the  $x$  direction,  $k_{L,Tz}^n = \sqrt{(k_{L,T}^n)^2 - k_x^2}$  are the  $z$ -components of the longitudinal and transverse wave vectors  $\vec{k}_{L,T}^n$ . It is to be noted that the phase origins of the up and down partial waves are different. Without details, it is the condition which ensures the numerical stability of this method compared to transfer matrix type methods [4,5].

We can define in the  $n^{\text{th}}$  layer a displacement amplitude vector as  $\mathbf{A}^n = [a_L^{n-} \ a_T^{n-} \ a_L^{n+} \ a_T^{n+}]^T$ . The writing of the continuity of the displacement at the interfaces  $z = z_{n-1}$  and  $z = z_n$  leads to the following matrix expression:

$$\begin{bmatrix} \vec{u}^{n-1} & \vec{u}^n \end{bmatrix}^T = \mathbf{E}_u^n \mathbf{A}^n \quad (3)$$

The stress component vector  $\vec{\sigma}^n = [\sigma_{xz}^n \ \sigma_{zz}^n]^T$  can be related to the displacement amplitude vector using Hooke's law. The stress vectors  $\vec{\sigma}^n$  and  $\vec{\sigma}^{n-1}$  are related to the amplitude vector in the matrix form as:

$$\begin{bmatrix} \bar{\sigma}^{n-1} & \bar{\sigma}^n \end{bmatrix}^T = E_\sigma^n A^n \quad (4)$$

Eliminating the amplitude vector in (2) and (3), we can write:

$$\begin{bmatrix} \bar{u}^{n-1} & \bar{u}^n \end{bmatrix}^T = E_u^n (E_\sigma^n)^{-1} \begin{bmatrix} \bar{\sigma}^{n-1} & \bar{\sigma}^n \end{bmatrix}^T \quad (5)$$

The elements of the  $E_u^n$  and  $E_\sigma^n$  matrices can be recovered from Rousseau [6].

It defines the 4x4 compliance matrix:

$$S^n = E_u^n (E_\sigma^n)^{-1} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}_n, \quad \text{where } S_{ij} \text{ are } 2 \times 2 \text{ submatrices} \quad (6)$$

Rokhlin *et al.* [5] have implemented a stable recursive algorithm for the computation of the compliance matrix of multilayered media composed of N periods. The submatrices  $S_{ij}$  for N periods are deduced from the ones for (N-1) periods and those of an additional n<sup>th</sup> layer by the following recursive relations:

$$\begin{cases} S_{11}^N = S_{11}^{N-1} + S_{12}^{N-1} (S_{11}^{N-1} - S_{22}^{N-1})^{-1} S_{21}^{N-1} \\ S_{12}^N = -S_{12}^{N-1} (S_{11}^{N-1} - S_{22}^{N-1})^{-1} S_{12}^{N-1} \\ S_{21}^N = S_{21}^{N-1} (S_{11}^{N-1} - S_{22}^{N-1})^{-1} S_{21}^{N-1} \\ S_{22}^N = S_{22}^{N-1} - S_{21}^{N-1} (S_{11}^{N-1} - S_{22}^{N-1})^{-1} S_{21}^{N-1} \end{cases} \quad (7)$$

For periodic multilayers, the n<sup>th</sup> layer is the period whose compliance matrix is obtained by using the same recursive relations. The global compliance matrix can be written as follows:

$$S^N = \begin{bmatrix} S_{11}^{11} & S_{11}^{12} & S_{11}^{21} & S_{11}^{22} \\ -S_{11}^{12} & S_{11}^{22} & S_{11}^{21} & S_{11}^{22} \\ -S_{12}^{11} & S_{12}^{21} & S_{12}^{11} & S_{12}^{22} \\ S_{12}^{12} & -S_{12}^{22} & -S_{12}^{21} & S_{12}^{22} \end{bmatrix} \quad (8)$$

It can be demonstrated that the  $R_{1,2}$  reflection coefficients and the T transmission coefficient express as functions of elements of the global compliance matrix  $S^N$  in the following forms:

$$\begin{aligned} R_1 &= \frac{(S_{11}^{22} - iY_F)(S_{22}^{22} - iY_F) + (S_{12}^{22})^2}{(S_{11}^{22} + iY_F)(S_{22}^{22} - iY_F) + (S_{12}^{22})^2} \\ R_2 &= \frac{(S_{11}^{22} + iY_F)(S_{22}^{22} + iY_F) + (S_{12}^{22})^2}{(S_{11}^{22} + iY_F)(S_{22}^{22} - iY_F) + (S_{12}^{22})^2}, \quad \text{where } Y_F = -\frac{\cos \theta}{\rho_F c_F \omega} \\ T &= \frac{2iY_F S_{12}^{22}}{(S_{11}^{22} + iY_F)(S_{22}^{22} - iY_F) + (S_{12}^{22})^2} \end{aligned} \quad (9)$$

The transition matrix  $\mathbf{T}$  is defined from the  $\mathbf{S}$  matrix in the following way:  $\mathbf{T} = \frac{1}{2i}(\mathbf{S} - \mathbf{I})$ , where  $\mathbf{I}$  is the identity matrix. Its eigenvalues are given by:

$$\begin{cases} T_+ = \frac{1}{2i} (\cos^2 \alpha (-R_1) + 2 \sin \alpha \cos \alpha T + \sin^2 \alpha (-R_2) - 1) \\ T_- = \frac{1}{2i} (\sin^2 \alpha (-R_1) - 2 \sin \alpha \cos \alpha T + \cos^2 \alpha (-R_2) - 1) \end{cases}, \quad \text{where } \alpha = -\frac{1}{2} \text{Arctan} \left( \frac{2S_{12}^{22}}{S_{11}^{22} + S_{22}^{22}} \right) \quad (10)$$

In the case of a plate, the reflection coefficients  $R_{1,2}$  are identical and the mixing angle  $\alpha$  is equal to 45°. The transitions terms  $T_\pm$  permit to characterize the resonances in position and width associated with the

symmetric and antisymmetric modes, respectively [1]. A periodically multilayered media being not symmetric with respect to the median plane, the waves propagating in this one can no more be separated between symmetric and antisymmetric ones. To the contrary, it exists period modes and structures ones [7]. The period modes are independent of the number of periods and they are located in the stop bands. The structures modes depend on the number of periods and are located in the pass bands.

### 3 Numerical results

In this section, the period studied is composed of an aluminum plate ( $\rho = 2800 \text{ kg/m}^3$ ,  $c_L = 6380 \text{ m/s}$ ,  $c_T = 3100 \text{ m/s}$ ) and of a polyethylene plate ( $\rho = 939 \text{ kg/m}^3$ ,  $c_L = 2371 \text{ m/s}$ ,  $c_T = 1200 \text{ m/s}$ ). The thickness of each plate is 1 mm. In the following, the period is denoted as AP for conciseness. The multilayer is embedded in a fluid (water) whose density is  $\rho_F = 1000 \text{ kg/m}^3$  and where the waves propagate with the phase velocity  $c_F = 1470 \text{ m/s}$ . In this preliminary work, we consider only normal incidence. In Figure 2, we compare the plots of the transitions terms moduli for an aluminum plate and for a period AP. The transmission coefficient plot of the period is also added. It exhibits peaks regularly spaced whose amplitude is nearly unity and smooth undulations. The plots of the transition terms of the aluminium plate show that the peaks can be assigned to symmetric and antisymmetric modes of that plate alternately. The undulation maximums are linked with modes in the polyethylene plate. The plots of the transition terms of the AP period present peaks whose amplitudes are unity, frequency locations are nearly those of the peaks and the undulation maximums of the transmission coefficient. It is also to be noted that the maximums of the two transition terms alternate.

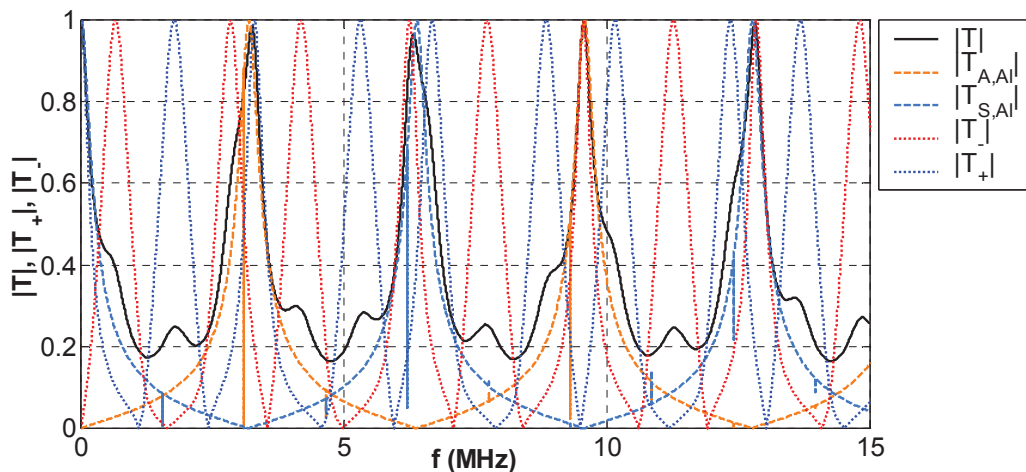


FIG. 2 – Superimposition of the transmission  $|T|$  and transition terms  $|T_{A,Al}|$ ,  $|T_{S,Al}|$ ,  $|T_{-}|$ ,  $|T_{+}|$ .

In Figure 3, the plots of the transmission coefficient for multilayers composed of different number of periods ( $1 < N < 8$ ) at normal incidence are compared. From  $N=4$  to 8, one observes stop bands in which the transmission coefficient tends to zero. It implies that no waves propagate in the multilayer, because they are strongly attenuated, so the resonances associated described by the plots of the transitions terms must have large widths.

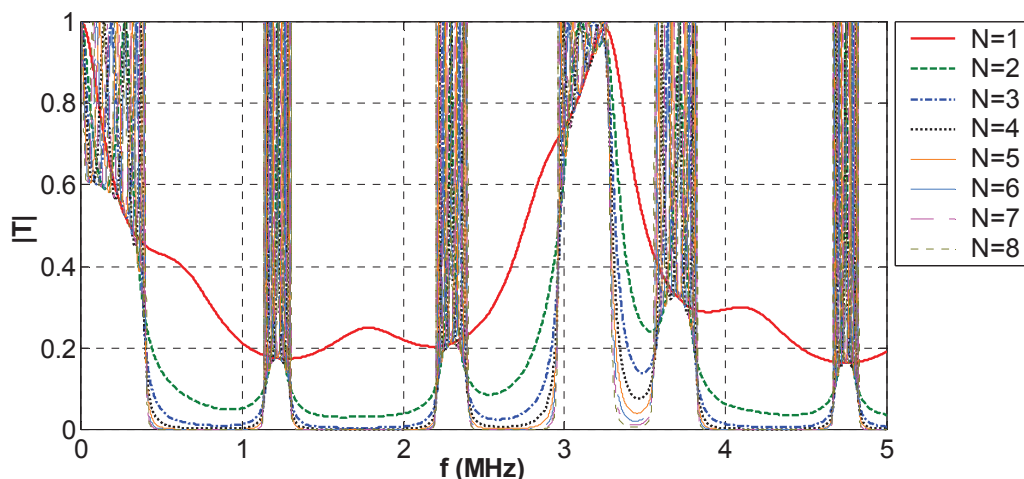


FIG. 3 – Transmission coefficients  $|T|$  obtained for  $N = 1$  to 8 periods.

In Figure 4 (a) are compared the plots of the  $T_+$  transition terms at normal incidence. They exhibit wide peaks located in the stop bands and then associated with very attenuated waves assigned to period modes and thin ones in the pass bands related to propagative waves assigned to structure modes. Some peaks on the different plots are common whatever the number of periods, as the ones located at 1.77 MHz, 3.295 MHz and 5.308 MHz. Other ones are common when the number of periods is even, such as for  $N=2$  and 4 but they are also common with peaks observed on the plots of  $T_-$  transition terms of odd  $N$  period multilayers ( $N=1, 3$ ), as shown in Figure 4 (b). Thus the period modes that are independent of the number of periods are detected by wide peaks either on the plots on  $T_+$  or  $T_-$  in the stop bands. Particularly, we can observe wide peaks common to the  $T_{\pm}$  transition terms at 650 kHz and 2.84 MHz. Moreover, it exists an alternation between wide peaks common to all period numbers and those common to even period number and to odd period number. On the contrary, we cannot observe common wide peaks on the plots of  $T_-$  transitions terms whatever the number of periods. The study of the two types of transition terms is then necessary to identify the period modes located in the stop bands. The interest of those modes is that the total width of the peaks associated, on the plots the transition terms in Figure 4 (a) and 4 (b), indicate the width of the stop bands.

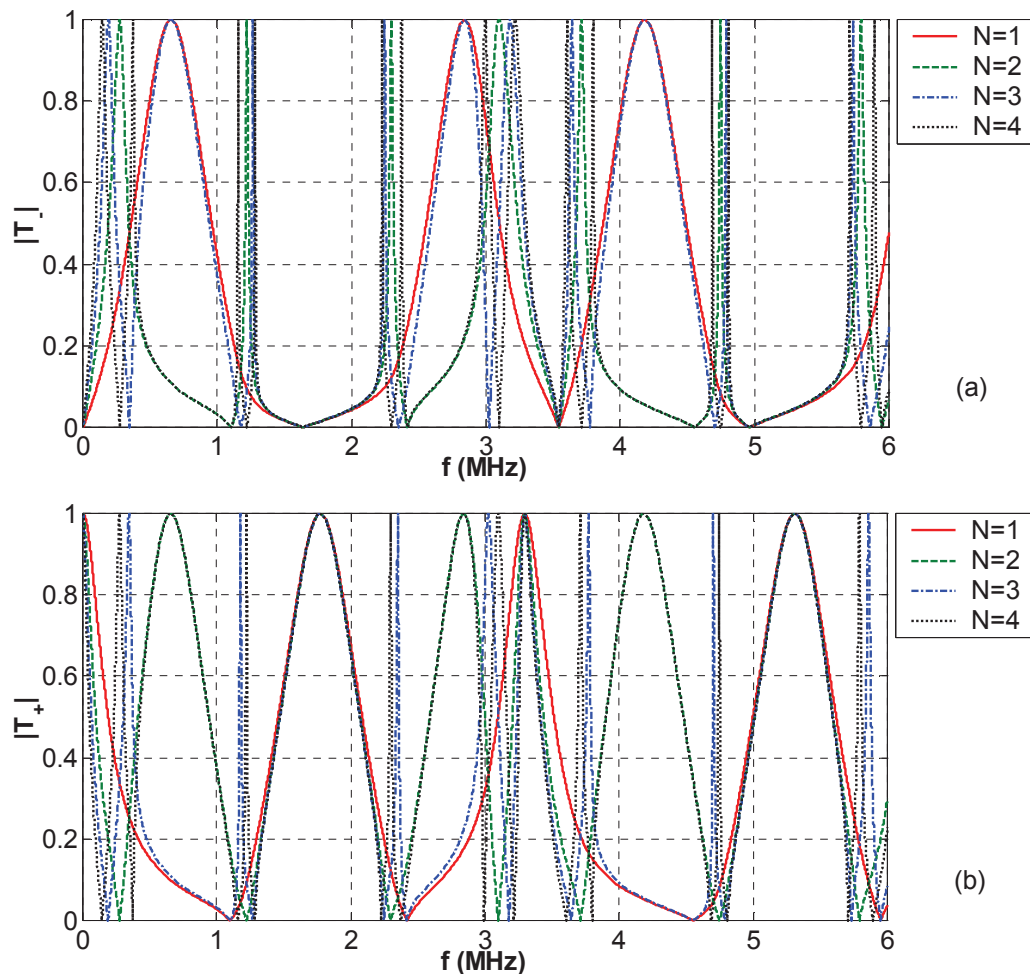


FIG. 4 – Transition terms  $|T_-|$  (a) and  $|T_+|$  (b) obtained for  $N = 1$  to 4 periods.

In Figure 5, we have plotted the two types of transition terms for  $N=8$ . In this case, the period modes are only detected by the wide peaks of the  $T_+$  transition terms. Compared to the plot of the transmission coefficient, the wide peak widths coincide with those of the stop bands. We also observe that there is an alternation of the maximums of the transition terms corresponding to maximums of the transmission coefficient in the pass bands. Moreover, for this multilayer of 8 periods, we detect 7 peaks in the pass bands. In the general case of a multilayer composed of  $N$  periods, there are  $(N-1)$  peaks related to the transition terms, so  $(N-1)$  resonances and propagating waves.

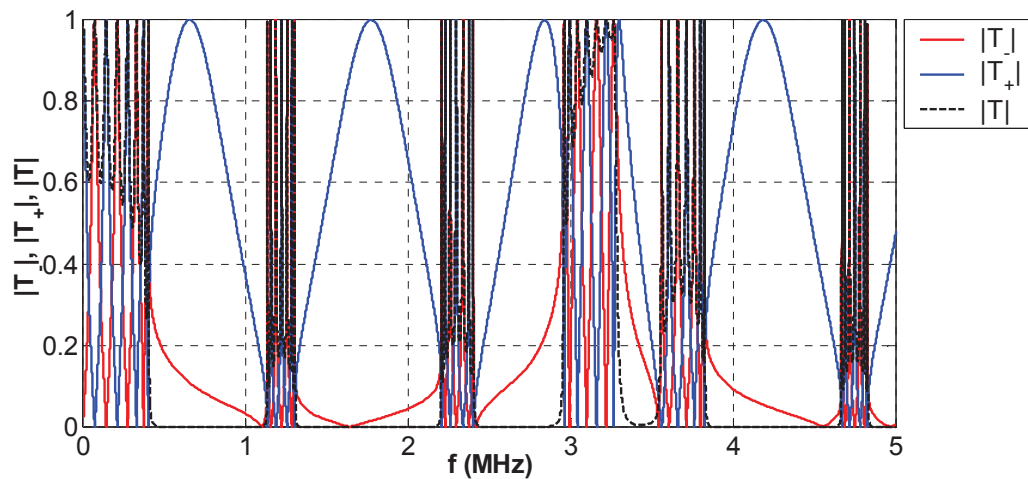


FIG. 5 – Transmission  $|T|$  and transition terms  $|T_-|$  and  $|T_+|$  obtained for  $N = 8$  periods.

## Conclusion

In this preliminary study, we have shown that the  $S$  matrix formalism efficient for plates can also be applied to periodic plane multilayers. In the latter case, the physical meaning of the transitions terms is different from the ones of single plates, but has not been established yet. However, it can be said that the wide peaks detected on the plots of the transition terms are related to period modes whose total width correspond to the widths of the stop bands of the periodic structure. In oblique incidence, similar results are obtained. Plots in frequency versus incidence angle are possible. The inclusion of default layers was also considered and the results will be presented in a future work.

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