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Abstract

The broad Cauchy problems with nonlocal differential operators, including fractal, fractional, and fractal-fractional derivatives, have been taken into consideration in this work. We have presented the existence and uniqueness of each case with fractal-fractional within the framework of Carathéodory existence and uniqueness conditions because fractional derivatives can be recovered by taking the fractal order to one, and fractal derivatives can be recovered by taking the fractional order to be one. We have changed the parametrized numerical techniques for solving nonlinear ordinary differential equations in order to produce greater accuracy. For each scenario, we presented the stability and consistency analyses. For purposes of illustration, a few examples are given. It has been found that our adjustment still produces a better approximation than the original, even when the parameter $\rho = 0.1$.

Keywords: Nonlinear ODE, nonlocal differential operators, Carathéodory existence and uniqueness, modified parametrized approximation.

1 Introduction

A strong mathematical tool for simulating processes with nonlocal behavior in nature is a nonlinear ordinary differential equation with nonlocal operators, such as fractional differential operators, fractal operators and fractal-fractional differential operators [[1]-[4]]. Their applications can be found in a variety of academic disciplines, such as chaos, where researchers are concentrating on underlying patterns and deterministic laws of dynamical systems that are highly sensitive to initial conditions and were previously believed to have completely random states of disorder and irregularities. In epidemiological modeling, the idea focuses conceptually on the complex shift in health and illness patterns as well as the connections between these patterns and their demographic, economic and sociologic antecedents and effects. In the field of signaling theory, a body of theoretical research on interspecies and intraspecies communication is the core area of study. Aquifers, turbulence, and other media commonly show fractal features in porous media. The conventional diffusion or dispersion equations, which are based on random walks in empty space, do not apply to fractal media. To solve this, scales for space and time need to be modified, as well as ideas like velocity and distance need to be redefined for fractal media. Following, fractal space-time redefines fundamental physics concepts like velocity. Researchers from various backgrounds have adopted these notions as a result of their

broad applicability to create theoretical and practical investigations. These differential equations are used in modeling to forecast how particular processes will behave in the future as a function of time or space. These mathematical equations must, however, be resolved analytically or numerically in order to accomplish this. However, most often because of nonlinearities, they cannot be resolved by analytical approaches. Therefore, numerical schemes make good candidates for generating numerical results that may then be used for simulations. However, theory on fixed points has been created, such as the Carathéodory method, which was well-developed for ordinary differential equations, to ensure that such models have precise solutions. There are undoubtedly a lot more. Several techniques for solving conventional ordinary differential equations have been expanded in recent years to include nonlocal differential operators. One first-order numerical method is the Euler method, which can be used to solve ordinary differential equations with a specified initial value. It is the simplest Runge-Kutta method and the most fundamental explicit method for the numerical integration of ordinary differential equations. In order to identify an unknown function that satisfies a given differential equation, predictor-corrector techniques are used to integrate ordinary differential equations. Perhaps less well-known is the parametrized approximation, which is a modified Runge Kutta method or a more broad approximation capable of recovering the Heun, Ralston, and Midpoint approximations [[6]-[10]]. Lagrange, linear, and Newton interpolation are only a few examples of polynomial interpolation techniques that have been utilized extensively to create various numerical schemes. We should highlight that polynomial interpolation, in the context of numerical analysis, refers to the interpolation of a given data set by the polynomial of the lowest degree that traverses the data set's points. We will first examine the effectiveness of the parametrized technique for conventional differential ordinary differential equations in this study, and then we will broaden the applicability of this method to nonlocal differential operators.

2 Definitions, problems and hypothesis

In this section, we present some definitions, theorem and properties that will be used in the paper. Let f be differentiable within $[0,T]$ then

$$ {}_0^{F_c}D_t^{\alpha} f(t) = \frac{1}{\gamma(\alpha)} \int_0^t f\prime(\tau) K(t-\tau)\, d\tau \tag{1} $$

where

$$ \begin{aligned} K(t) &= t^{-\alpha}, \exp\left(-\frac{\alpha}{1-\alpha}t\right) \text{ or } E_{\alpha}\left(-\frac{\alpha}{1-\alpha}t^{\alpha}\right), \\ \gamma(\alpha) &= \frac{1}{\Gamma(\alpha)} \text{ or } \frac{1}{1-\alpha}, 0 < \alpha \leq 1. \end{aligned} \tag{2} $$

$$ {}_0^{F}D_t^{\beta} f(t) = \lim_{t_1 \to t} \frac{f(t_1) - f(t)}{t_1^{\beta} - t^{\beta}}, \beta > 0, \tag{3} $$

$$ {}_0^{FF}D_t^{\alpha,\beta} f(t) = \frac{1}{\gamma(\alpha)} \, {}_0^{F}D_t^{\beta} \int_0^t f(\tau) K(t-\tau)\, d\tau. \tag{4} $$

In the rest of the paper, we will consider a general Cauchy problem

$$ \begin{cases} {}_0^{FF}D_t^{\alpha,\beta} y(t) = F(t, y(t)) & \text{if } t > 0 \\ y(0) = y_0, & \text{if } t = 0 \end{cases}. \tag{5} $$

For this problem, we will assume that the function $F(t, y(t))$ is Carathéodory, that is to say

i) $F(t, y(t))$ is continuous in y for all fixed t.

ii) $F(t, y(t))$ is measurable in t for each fixed y.

iii) There exists a function $m(t)$ positive such that

$$|F(t,y(t))| \leq m(t), \forall (t,y) \in \mathbb{R} \tag{6}$$

where the function $m(t)$ is Lebesgue integrable in t.

In the case of singular kernel, we shall assume that $F(t,y(t))$ is $\alpha-$continuous that is to say if $F(t,y(t))$ is continuous in y for all fixed t, there exists a function $G(t,y(t))$ such that

$$G(t,y(t)) = (t-\tau)^{\alpha-1} F(t,y(t)) \tag{7}$$

is continuous.

But above condition without be necessary if we consider the nonlocal operator at $t > 0$, which indeed is the case in general. The reason is that if $F(t,y(t))$ is continuous on y for each t, then since $t^{\alpha-1}$ is absolutely continuous, it is concluded that $(t-\tau)^{\alpha-1} F(t,y(t))$ is continuous where $0 \leq \tau < t$.

3 Existence and uniqueness

In this section, we consider the general Cauchy problem with fractal-fractional

$$\begin{cases} {}_0^{FF}D_t^{\alpha,\beta} y(t) = f(t,y(t)) & \text{if } t > 0 \\ y(0) = y_0, & \text{if } t = 0 \end{cases}. \tag{8}$$

We consider this case since when $\beta = 1$, we recover all fractional derivatives, when $\alpha = 1$, we obtain a fractal derivative and if $\alpha = \beta - 1$, we recover classical.

To cover all cases, we consider the case with the following kernels

$$\delta(t), \frac{1}{1-\alpha}\exp\left(-\frac{\alpha}{1-\alpha}t\right), \frac{1}{1-\alpha}E_\alpha\left(-\frac{\alpha}{1-\alpha}t^\alpha\right) \text{ and } \frac{t^{-\alpha}}{\Gamma(\alpha)}.$$

In this section, we shall present only the last point of the Carathéodory principles, since first and two are obtained directly. Additionally, we will also present the Lipschitz condition to insure the uniqueness.

$$\begin{cases} {}_0^{F}D_t^{\beta} y(t) = f(t,y(t)) & \text{if } t > 0 \\ y(0) = y_0, & \text{if } t = 0 \end{cases}. \tag{9}$$

We assume that $f(t,y(t))$ is Carathéodory.

$$\begin{cases} y(t) = y_0 + \int_0^t \beta\tau^{\beta-1} f(\tau,y(\tau))\, d\tau, & \text{if } t > 0 \\ y(0) = y_0 \end{cases}. \tag{10}$$

We define

$$\begin{cases} \Lambda y(t) = y_0 + \int_0^t \beta\tau^{\beta-1} f(\tau,y(\tau))\, d\tau, & \text{if } t > 0 \\ y(0) = y_0 \end{cases}. \tag{11}$$

Applying the absolute value on both sides yields

$$|\Lambda y(t)| = |y_0| + \beta \int_0^t \tau^{\beta-1} |f(\tau,y(\tau))|\, d\tau. \tag{12}$$

Since $f(t,y(t))$ is Carathéodory, we can find $m(t) > 0, \forall t \in [0,T]$ such that

$$|f(\tau,y(\tau))| < m(t). \tag{13}$$

Therefore

$$|\Lambda y(t)| = |y_0| + \beta \int_0^t \tau^{\beta-1} m(\tau)\, d\tau. \tag{14}$$

But $m(t)$ is Lebesgue integrable also $t^{\beta-1}m(t)$ is Lebesgue integrable. Thus,

$$|\Lambda y(t)| = |y_0| + \beta M(t) = \overline{M}(t) \tag{15}$$

where $\overline{M}(t)$ is Lebesgue integrable. Therefore, Λ satisfies the last condition.

$$|\Lambda y_1 - \Lambda y_2| = \beta \int_0^t \tau^{\beta-1} |f(\tau, y_1) - f(\tau, y_2)|\, d\tau. \tag{16}$$

Since $f(t, y(t))$ is Carathéodory, then there exists $k(t) > 0$ such that

$$|f(t, y_1) - f(t, y_2)| \leq k(t) |y_1 - y_2|. \tag{17}$$

Therefore,

$$\begin{aligned} \|\Lambda y_1 - \Lambda y_2\|_\infty &\leq \beta \int_0^t \tau^{\beta-1} \|k(\tau)\|_\infty \|y_1 - y_2\|_\infty d\tau \\ &\leq \|k(\tau)\|_\infty \|y_1 - y_2\|_\infty T^\beta \\ &\leq \Omega \|y_1 - y_2\|_\infty \end{aligned} \tag{18}$$

which completes the proof.

We consider next the following problem

$$\begin{cases} {}_0^{FFE}D_t^{\alpha,\beta} y(t) = f(t, y(t)) & \text{if } t > 0 \\ y(0) = y_0, \quad \text{if } t = 0 \end{cases}. \tag{19}$$

We convert above into

$$\begin{cases} y(t) = (1-\alpha)\beta t^{\beta-1} f(t, y(t)) + \alpha\beta \int_0^t \tau^{\beta-1} f(\tau, y(\tau))\, d\tau, & \text{if } t > 0 \\ y(0) = y_0 \end{cases}. \tag{20}$$

We define the mapping

$$\Lambda y(t) = (1-\alpha)\beta t^{\beta-1} f(t, y(t)) + \alpha\beta \int_0^t \tau^{\beta-1} f(\tau, y(\tau))\, d\tau. \tag{21}$$

We evaluate

$$\begin{aligned} |\Lambda y(t)| &\leq (1-\alpha)\beta t^{\beta-1} |f(t, y(t))| + \alpha\beta \int_0^t \tau^{\beta-1} |f(\tau, y(\tau))|\, d\tau \\ &\leq (1-\alpha)\beta t^{\beta-1} m(t) + \alpha T^\beta \int_0^t m(\tau)\, d\tau \\ &\leq \overline{M}(t) \end{aligned} \tag{22}$$

where $\overline{M}(t)$ is Lebesgue integrable.

$$\begin{aligned} |\Lambda y_1 - \Lambda y_2| &\leq (1-\alpha)\beta t^{\beta-1} |f(t, y_1) - f(t, y_2)| \\ &\quad + \alpha\beta \int_0^t \tau^{\beta-1} |f(\tau, y_1) - f(\tau, y_2)|\, d\tau \\ &\leq (1-\alpha)\beta t^{\beta-1} k(t) |y_1 - y_2| \\ &\quad + \alpha\beta \int_0^t \tau^{\beta-1} k(\tau) |y_1 - y_2|\, d\tau \end{aligned} \tag{23}$$

$$\begin{aligned} &\leq (1-\alpha)\beta t^{\beta-1}\|k\|_\infty \|y_1-y_2\|_\infty + \alpha T^\beta \|k\|_\infty \|y_1-y_2\|_\infty \\ &\leq \left((1-\alpha)\beta t^{\beta-1}\|k\|_\infty + \alpha T^\beta \|k\|_\infty\right)\|y_1-y_2\|_\infty \\ &\leq \overline{K}\|y_1-y_2\|_\infty \end{aligned}$$

which concludes the proof.

We consider now the following Cauchy problem

$$\begin{cases} {}_0^{FFP}D_t^{\alpha,\beta} y(t) = f(t,y(t)) & \text{if } t>0 \\ y(0)=y_0, & \text{if } t=0 \end{cases}. \tag{24}$$

We convert above into

$$\begin{cases} y(t) = \frac{\beta}{\Gamma(\alpha)}\int_0^t \tau^{\beta-1} f(\tau,y(\tau))(t-\tau)^{\alpha-1}d\tau, & \text{if } t>0 \\ y(0)=y_0 \end{cases}. \tag{25}$$

We define the following mapping

$$\Lambda y(t) = \frac{\beta}{\Gamma(\alpha)}\int_0^t \tau^{\beta-1} f(\tau,y(\tau))(t-\tau)^{\alpha-1}d\tau. \tag{26}$$

We evaluate

$$\begin{aligned} |\Lambda y(t)| &\leq \frac{\beta}{\Gamma(\alpha)}\int_0^t \tau^{\beta-1}|f(\tau,y(\tau))|(t-\tau)^{\alpha-1}d\tau \\ &\leq \frac{\beta}{\Gamma(\alpha)}\int_0^t \tau^{\beta-1}(t-\tau)^{\alpha-1}d\tau \int_0^t |f(\tau,y(\tau))|d\tau \\ &\leq \frac{\beta}{\Gamma(\alpha)}B(t,\beta,\alpha)\int_0^t m(\tau)d\tau \\ &\leq \frac{\beta}{\Gamma(\alpha)}B(t,\beta,\alpha)M(t) \\ &\leq \overline{M}(t) \end{aligned} \tag{27}$$

which is also Lebesgue integrable. Therefore

$$\|\Lambda y_1 - \Lambda y_2\|_\infty \leq \frac{\beta}{\Gamma(\alpha)}B(t,\beta,\alpha)\|k\|_\infty \|y_1-y_2\|_\infty. \tag{28}$$

which completes the proof.

We now consider the following problem

$$\begin{cases} {}_0^{FFE}D_t^{\alpha,\beta} y(t) = f(t,y(t)) & \text{if } t>0 \\ y(0)=y_0, & \text{if } t=0 \end{cases}. \tag{29}$$

We convert above into

$$\begin{cases} y(t) = (1-\alpha)\beta t^{\beta-1} f(t,y(t)) + \frac{\alpha\beta}{\Gamma(\alpha)}\int_0^t \tau^{\beta-1} f(\tau,y(\tau))(t-\tau)^{\alpha-1}d\tau, & \text{if } t>0 \\ y(0)=y_0 \end{cases}. \tag{30}$$

We define the mapping

$$\Lambda y(t) = (1-\alpha)\beta t^{\beta-1} f(t,y(t)) + \frac{\alpha\beta}{\Gamma(\alpha)}\int_0^t \tau^{\beta-1} f(\tau,y(\tau))(t-\tau)^{\alpha-1}d\tau. \tag{31}$$

We evaluate

$$\begin{aligned} |\Lambda y(t)| &\leq (1-\alpha)\beta t^{\beta-1}|f(t,y(t))| + \frac{\alpha\beta}{\Gamma(\alpha)}\int_0^t \tau^{\beta-1}(t-\tau)^{\alpha-1}d\tau \\ &\quad \times \int_0^t |f(\tau,y(\tau))|\,d\tau \\ &\leq (1-\alpha)\beta t^{\beta-1}m(t) + \frac{\alpha\beta}{\Gamma(\alpha)}B(t,\beta,\alpha)\int_0^t m(\tau)\,d\tau \\ &\leq (1-\alpha)\beta t^{\beta-1}m(t) + \frac{\alpha\beta}{\Gamma(\alpha)}B(t,\beta,\alpha)M(t) \\ &\leq \overline{M}(t) \end{aligned} \tag{32}$$

where $\overline{M}(t)$ is Lebesgue integrable. Therefore

$$\begin{aligned} |\Lambda y_1 - \Lambda y_2| &\leq (1-\alpha)\beta T^{\beta-1}\|k\|_\infty \|y_1-y_2\|_\infty \\ &\quad + \frac{\alpha\beta}{\Gamma(\alpha)}B(t,\beta,\alpha)\|k\|_\infty\|y_1-y_2\|_\infty \\ &\leq \left(\begin{array}{c}(1-\alpha)\beta T^{\beta-1}\|k\|_\infty \\ +\frac{\alpha\beta}{\Gamma(\alpha)}B(t,\beta,\alpha)\|k\|_\infty\end{array}\right)\|y_1-y_2\|_\infty \\ &\leq \overline{k}(t)\|y_1-y_2\|_\infty \end{aligned} \tag{33}$$

which concludes the proof.

In the next section, we shall derive the numerical scheme for the considered Cauchy problems based on the parametrized approach. To yield accuracy, we will modify the scheme.

4 Background for parametrized method for ordinary differential equations

In this section, we present background of the parametrized method. We shall present exactly the procedure when the derivative is classical. We shall note that this method using the classical derivative was already given in [7]. To derive this method, the following general nonlinear equation is considered

$$\left\{\begin{array}{c} u\prime(t) = \gamma(t,u(t)) \\ u(0) = u_0 \end{array}\right. . \tag{34}$$

Applying the integral on both side to obtain

$$\left\{\begin{array}{c} u(t) = u(0) + \int_0^t \gamma(\tau,u(\tau))\,d\tau, \\ u(0) = u_0 \end{array}\right. . \tag{35}$$

We consider $t = t_{k+1}$ and $t = t_k$, to have after substracting

$$u(t_{k+1}) = u(t_k) + \int_{t_k}^{t_{k+1}} \gamma(\tau,u(\tau))\,d\tau. \tag{36}$$

Within $[t_k, t_{k+1}]$, it was suggested that the function $\gamma(\tau,u(\tau))$ could be approximated

$$\gamma(\tau,u(\tau)) \simeq \left(1-\frac{1}{2\rho}\right)\gamma(t_k,u_k) + \frac{1}{2\rho}\gamma(t_{k+1},u_{k+1}). \tag{37}$$

Replacing yields

$$u_{k+1} = u_k + \left(1 - \frac{1}{2\rho}\right) h\gamma\left(t_k, u_k\right) + \frac{h}{2\rho}\gamma\left(t_{k+1}, u_{k+1}\right). \tag{38}$$

The u_{k+1} and t_{k+1} that appeared on the other side of the equation were replaced, and their expressions were specified in [7] as a consequence of the observation that the obtained results led to an implicit scheme that was occasionally burdensome when doing simulation. Thus, we get

$$u_{k+1} = u_k + \left(1 - \frac{1}{2\rho}\right) h\gamma\left(t_k, u_k\right) + \frac{h}{2\rho}\gamma\left(\widetilde{t}_{k+1}, \widetilde{u}_{k+1}\right) \tag{39}$$

where $\widetilde{t}_{k+1} = t_k + \rho h$. For the function $\widetilde{u}_{k+1}$, we have

$$\widetilde{u}_{k+1} = u_k + \int_{t_k}^{t_k+\rho h} \gamma\left(\tau, u\left(\tau\right)\right) d\tau. \tag{40}$$

Here is approximated by Euler approach to obtain

$$\widetilde{u}_{k+1} = u_k + \rho h\gamma\left(t_k, u_k\right). \tag{41}$$

Therefore, the numerical scheme is given as

$$u_{k+1} = u_k + \left(1 - \frac{1}{2\rho}\right) h\gamma\left(t_k, u_k\right) + \frac{h}{2\rho}\gamma\left(t_k + \rho h, u_k + \rho h\gamma\left(t_k, u_k\right)\right). \tag{42}$$

Note that when $\alpha = 1, \alpha = 1/2$ and $\alpha = 2/3$, the parametrized method corresponds to Heun's, midpoint and the Ralston's methods for the Cauchy problem, respectively [[6]-[10]].

We discovered that the method is less accurate when the parameter ρ is smaller than 0.5 as we can achieve a very high error when comparing the exact answer and the numerical solution by solving a few ordinary differential equations. Because of this, we reexamined the process and offered a modified version, notably for getting the predictor component. Because we stop at $t_n + \rho h$ for the predictor section, we lose $(1 - \rho) h$ step and as a result, accuracy is lost. The Euler approximation will be used to simply generate a modified version in the sections that follow.

$$u_{k+1} = u_k + \left(1 - \frac{1}{2\rho}\right) h\gamma\left(t_k, u_k\right) + \frac{h}{2\rho}\gamma\left(t_{k+1}, u_{k+1}\right). \tag{43}$$

Knowing that explicit terms make the calculation complicated, we reformulate the aforementioned equation as follows using the same concept introduced in the parametrized method:

$$u_{k+1} = u_k + \left(1 - \frac{1}{2\rho}\right) h\gamma\left(t_k, u_k\right) + \frac{h}{2\rho}\gamma\left(t_{k+1}, \widetilde{u}_{k+1}\right) \tag{44}$$

where $t_{k+1} = t_k + h$. For the function $\widetilde{u}_{k+1}$, we have

$$\widetilde{u}_{k+1} = u_k + \int_{t_k}^{t_k+h} \gamma\left(\tau, u\left(\tau\right)\right) d\tau. \tag{45}$$

Then, the predictor is obtained as

$$\widetilde{u}_{k+1} = u_k + h\gamma\left(t_k, u_k\right). \tag{46}$$

Thus, we have the following numerical scheme

$$u_{k+1} = u_k + \left(1 - \frac{1}{2\rho}\right) h\gamma\left(t_k, u_k\right) + \frac{h}{2\rho}\gamma\left(t_k + h, u_k + h\gamma\left(t_k, u_k\right)\right). \tag{47}$$

Example 1. To illustrate the major difference between the parametrized approach and our modified version, we consider the following nonlinear equation:

$$\begin{aligned} y\prime\,(t) &= t^2 \\ y\,(0) &= 0. \end{aligned} \tag{48}$$

The following is the exact solution to the equation above. To check the method's accuracy, we will compare the precise solution to the parametrized and our modified versions.

$$y\,(t) = \frac{t^3}{3}.$$

The obtained graphical representations are presented in Figure 1 and 2 below. We shall note that, in Figure 1, we present the comparison between the exact solution and the numerical solution by the parametrized method. While in Figure 2, we present the comparison between the exact solution and the numerical solution obtained by the suggested method. We achieved this when the parameter is 0.4 and the step size is 0.001.

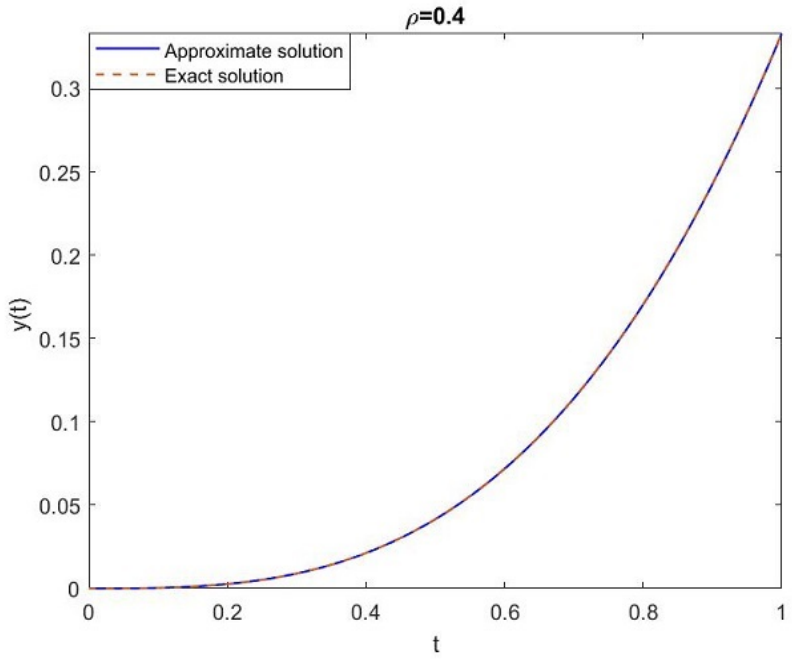

Figure 1. Numerical simulation for Cauchy problem for $\rho = 0.4$.

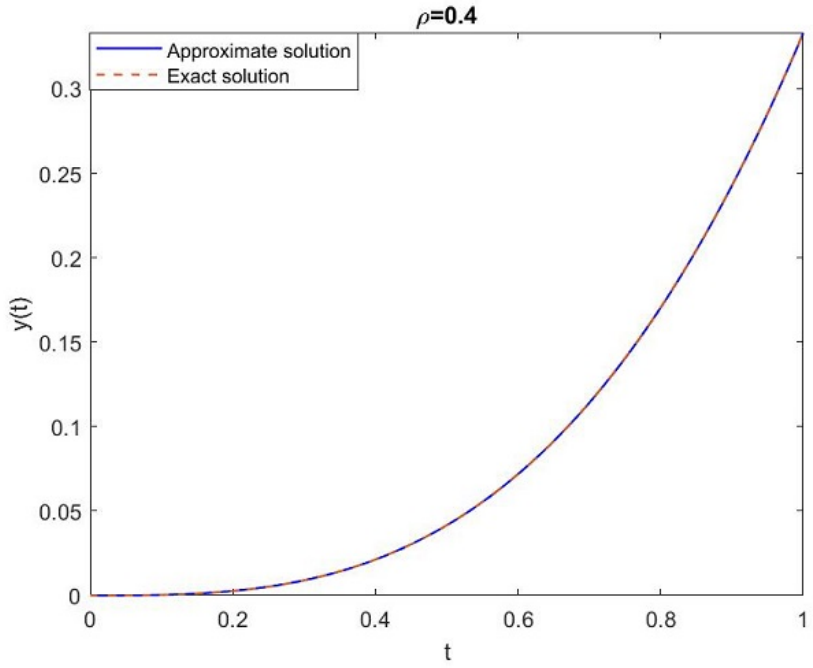

Figure 2. Numerical simulation for Cauchy problem for $\rho = 0.4$.

The error is presented in Table 1 below.

Table 1. Error for the function $y(t)$

Predictor	With ρ	Without ρ
$\rho = 1$	$1.6666\ e-07$	$1.6666\ e-07$
$\rho = 0.8$	$1.7990e-04$	$9.8333\ e-05$
$\rho = 0.6$	$3.1991\ e-04$	$1.9983\ e-04$
$\rho = 0.4$	$4.1988\ e-04$	$2.9983\ e-04$
$\rho = 0.2$	$4.7985\ e-04$	$3.9983\ e-04$

Example 2. To illustrate the major difference between the parametrized approach and our modified version, we consider the following nonlinear equation:

$$\begin{aligned} y\prime(t) &= -y \\ y(0) &= 1. \end{aligned} \tag{49}$$

The following is the exact solution to the equation above. To check the method's accuracy, we will compare the precise solution to the parametrized and our modified versions.

$$y(t) = \exp(-t)$$

The obtained graphical representations are presented in Figure 3 and 4 below. We shall note that, in Figure 3, we present the comparison between the exact solution and the numerical solution by the parametrized method. While in Figure 4, we present the comparison between the exact solution and the numerical solution

obtained by the suggested method. We achieved this when the parameter is 0.6 and the step size is 0.001.

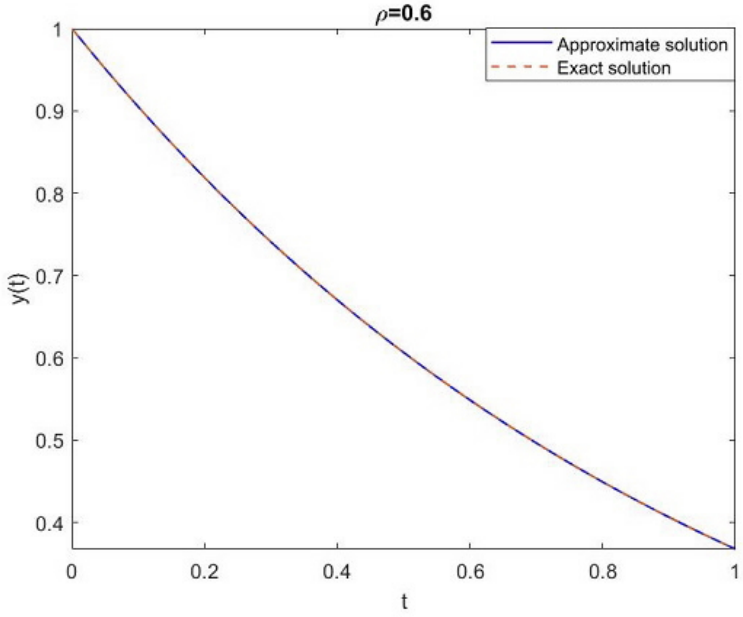

Figure 3. Numerical simulation for Cauchy problem for $\rho = 0.6$.

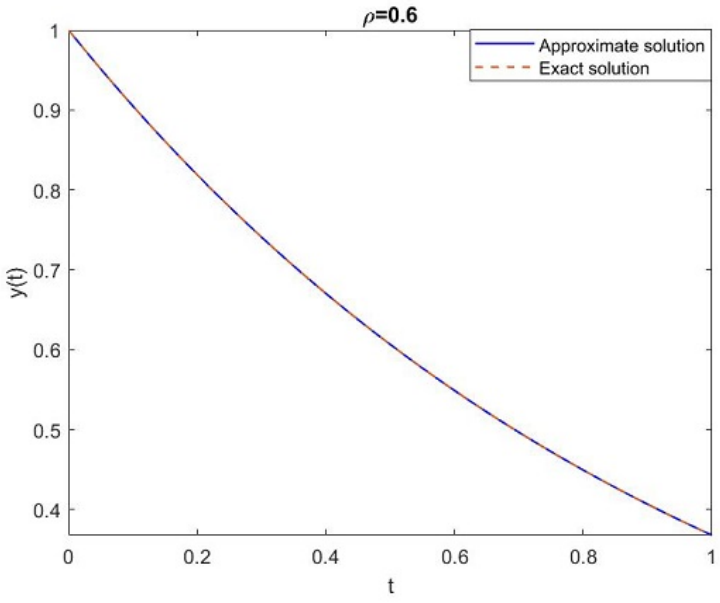

Figure 4. Numerical simulation for Cauchy problem for $\rho = 0.6$.

The error is presented in Table 2 below.

Table 2. Error for the function $y(t)$

Predictor	With ρ	Without ρ
$\rho = 1$	$6.1359\ e-08$	$6.1359\ e-08$
$\rho = 0.8$	$6.6317e-05$	$3.6761\ e-05$
$\rho = 0.6$	$1.1775\ e-04$	$7.3580\ e-05$
$\rho = 0.4$	$1.5454\ e-04$	$1.1039\ e-04$
$\rho = 0.2$	$1.7665\ e-04$	$1.4720\ e-04$

5 Parametrized method for ordinary differential equations with global derivative

We consider the following Cauchy problem

$$\begin{cases} {}_0D_g u(t) = \gamma(t, u(t)) \\ u(0) = u_0 \end{cases}. \tag{50}$$

Integrating the above equation

$$\begin{cases} u(t) = u(0) + \int_0^t \gamma(\tau, u(\tau))\, dg(\tau), \\ u(0) = u_0 \end{cases} \tag{51}$$

and

$$\begin{cases} u(t) = u(0) + \int_0^t \gamma(\tau, u(\tau))\, g'(\tau)\, d\tau \\ u(0) = u_0 \end{cases}. \tag{52}$$

At $t = t_{k+1}$ and $t = t_k$, we have

$$u(t_{k+1}) = u(t_k) + \int_{t_k}^{t_{k+1}} \gamma(\tau, u(\tau))\, d\tau \tag{53}$$

where $\gamma(\tau, u(\tau)) = \gamma(\tau, u(\tau))\, g'(\tau)$. Within $[t_k, t_{k+1}]$, $\gamma(\tau, u(\tau))$ is approximated by

$$\gamma(\tau, u(\tau)) \simeq \left(1 - \frac{1}{2\rho}\right)\gamma(t_k, u_k) + \frac{1}{2\rho}\gamma(t_{k+1}, u_{k+1}). \tag{54}$$

Adding this approximation to above integral yields

$$u_{k+1} = u_k + h\left\{\left(1 - \frac{1}{2\rho}\right)\gamma(t_k, u_k) + \frac{1}{2\rho}\gamma(t_{k+1}, u_{k+1})\right\}. \tag{55}$$

Therefore, the numerical scheme associated to this is given as

$$u_{k+1} = u_k + h\left\{\left(1 - \frac{1}{2\rho}\right)\gamma(t_k, u_k) + \frac{1}{2\rho}\gamma(t_{k+1}, \widetilde{u}_{k+1})\right\} \tag{56}$$

where $t_{k+1} = t_k + h$ and $\widetilde{u}_{k+1} = u_k + h\gamma(t_k, u_k)$. Putting the value of the function $g'(\tau)$ yields

$$u_{k+1} = u_k + \left\{\begin{array}{c} \left(1 - \frac{1}{2\rho}\right)\gamma(t_k, u_k)\left(g(t_{k+1}) - g(t_k)\right) \\ + \frac{1}{2\rho}\gamma(t_{k+1}, \widetilde{u}_{k+1})\left(g(t_{k+1}) - g(t_k)\right) \end{array}\right\}. \tag{57}$$

Thus, we obtain the following numerical scheme

$$\begin{aligned} u_{k+1} &= u_k + \left(1 - \frac{1}{2\rho}\right) \gamma\left(t_k, u_k\right)\left(g\left(t_{k+1}\right) - g\left(t_k\right)\right) \\ &\quad + \frac{1}{2\rho}\gamma\left(t_k + \gamma, u_k + h\gamma\left(t_k, u_k\right)\right) \\ &\quad \times \left(g\left(t_{k+1}\right) - g\left(t_k\right)\right). \end{aligned} \tag{58}$$

Example 3. To illustrate the major difference between the parametrized approach and our modified version, we consider the following nonlinear equation:

$$\begin{aligned} {}_0D_g y\left(t\right) &= t \\ y\left(0\right) &= 1. \end{aligned} \tag{59}$$

where $g\left(t\right) = \sin t$ and exact solution

$$y\left(t\right) = t\sin t + \cos t.$$

The obtained graphical representation is presented in Figure 5 below. We shall note that, in Figure 5, we present the comparison between the exact solution and the numerical solution obtained by the suggested method. We achieved this when the parameter is 0.55 and the step size is 0.001.

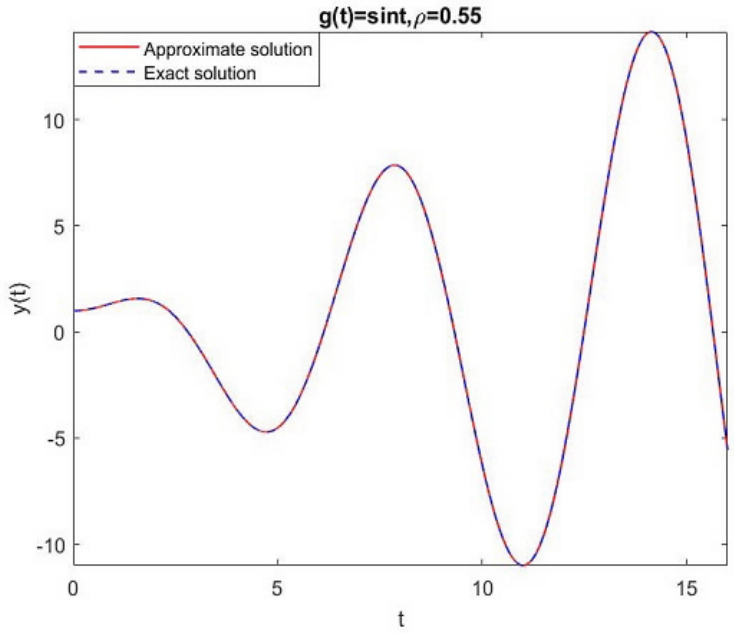

Figure 5. Numerical simulation for Cauchy problem for $g\left(t\right) = \sin t, \rho = 0.55$.

Error for the function $y\left(t\right)$ is presented in Table 3.

Table 3. Error for the function $y\left(t\right)$

ρ value	Error
$\rho = 1$	$1.7011\ e-07$
$\rho = 0.9$	$5.0082e-05$
$\rho = 0.7$	$1.5008\ e-04$
$\rho = 0.4$	$3.0008\ e-04$
$\rho = 0.3$	$3.5008\ e-04$

6 Parametrized method for ordinary differential equations with fractal derivative

We now present the method for the fractal type of Cauchy problem. We consider the following Cauchy problem

$$\left\{ \begin{array}{c} {}_0^{CH}D_t^{\beta} u(t) = \gamma(t, u(t)) \\ u(0) = u_0 \end{array} \right. \tag{60}$$

where ${}_0^{CH}D_t^{\beta}$ is the fractal derivative. Integrating the above equation, we can have

$$\left\{ \begin{array}{c} u(t) = u(0) + \beta \int_0^t \tau^{\beta-1} \gamma(\tau, u(\tau))\, d\tau \\ u(0) = u_0 \end{array} \right. . \tag{61}$$

At $t = t_{k+1}$ and $t = t_k$, we have

$$u(t_{k+1}) = u(t_k) + \beta \int_{t_k}^{t_{k+1}} \tau^{\beta-1} \gamma(\tau, u(\tau))\, d\tau. \tag{62}$$

Within $[t_k, t_{k+1}]$, $\gamma(\tau, u(\tau))$ is approximated by

$$\gamma(\tau, u(\tau)) \simeq \left(1 - \frac{1}{2\rho}\right) \gamma(t_k, u_k) + \frac{1}{2\rho} \gamma(t_{k+1}, u_{k+1}). \tag{63}$$

Adding this approximation to above integral yields

$$u_{k+1} = u_k + \beta \int_{t_k}^{t_{k+1}} \tau^{\beta-1} \left\{ \begin{array}{l} \left(1 - \frac{1}{2\rho}\right) \gamma(t_k, u_k) \\ + \frac{1}{2\rho} \gamma(t_{k+1}, u_{k+1}) \end{array} \right\} d\tau. \tag{64}$$

Then, we have

$$u_{k+1} = u_k + \beta \int_{t_k}^{t_{k+1}} \tau^{\beta-1} \left\{ \begin{array}{l} \left(1 - \frac{1}{2\rho}\right) \gamma(t_k, u_k) \\ + \frac{1}{2\rho} \gamma(t_{k+1}, u_{k+1}) \end{array} \right\} d\tau. \tag{65}$$

Now, we derive

$$\begin{aligned} u_{k+1} &= u_k + \beta \left\{ \left(1 - \frac{1}{2\rho}\right) \gamma(t_k, u_k) + \frac{1}{2\rho} \gamma(t_{k+1}, \widetilde{u}_{k+1}) \right\} \\ &\quad \times \left(\frac{t_{k+1}^{\beta}}{\beta} - \frac{t_k^{\beta}}{\beta} \right). \end{aligned} \tag{66}$$

Thus, we obtain

$$\begin{aligned} u_{k+1} &= u_k + h^{\beta} \left\{ \left(1 - \frac{1}{2\rho}\right) \gamma(t_k, u_k) + \frac{1}{2\rho} \gamma(t_{k+1}, \widetilde{u}_{k+1}) \right\} \\ &\quad \times \left((k+1)^{\beta} - k^{\beta} \right). \end{aligned} \tag{67}$$

where

$$\begin{aligned} \widetilde{u}_{k+1} &= u_k + \beta \int_{t_k}^{t_{k+1}} \tau^{\beta-1} \gamma(t_k, u_k)\, d\tau \\ &= u_k + \beta h^{\beta} \gamma(t_k, u_k) \left(\frac{(t_{k+1})^{\beta}}{\beta} - \frac{t_k^{\beta}}{\beta} \right) \\ &= u_k + h^{\beta} \gamma(t_k, u_k) \left((k+1)^{\beta} - k^{\beta} \right). \end{aligned} \tag{68}$$

One can see that when $\rho = 1$, we have

$$u_{k+1} = u_k + h\left\{\left(1 - \frac{1}{2\rho}\right)\gamma\left(t_k, u_k\right) + \frac{1}{2\rho}\gamma\left(t_{k+1}, \widetilde{u}_{k+1}\right)\right\} \tag{69}$$

where

$$\widetilde{u}_{k+1} = u_k + h\gamma\left(t_k, u_k\right). \tag{70}$$

Example 4. To illustrate the major difference between the parametrized approach and our modified version, we consider the following nonlinear equation:

$$\begin{aligned} {}_0^{CH}D_t^{\beta} y\left(t\right) &= t^3 + t \\ y\left(0\right) &= 0. \end{aligned} \tag{71}$$

where exact solution

$$y\left(t\right) = \beta\left(\frac{t^{\beta+3}}{\beta+3} + \frac{t^{\beta+1}}{\beta+1}\right).$$

The obtained graphical representation is presented in Figure 6 below. We shall note that, in Figure 6, we present the comparison between the exact solution and the numerical solution obtained by the suggested method. We achieved this when the parameter is 0.1 and the step size is 0.001.

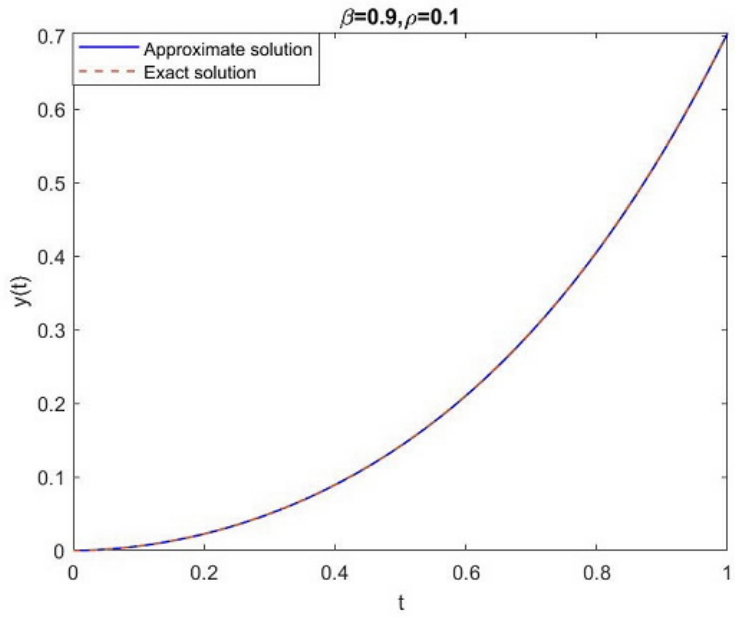

Figure 6. Numerical simulation for Cauchy problem for $\beta = 0.9, \rho = 0.3$.

Error for the function $y\left(t\right)$ is presented in Table 4.

Table 4. Error for the function $y\left(t\right)$

ρ value	Error
$\rho = 1$	$1.3013\ e-04$
$\rho = 0.9$	$2.2662e-04$
$\rho = 0.7$	$4.1960\ e-04$
$\rho = 0.4$	$7.0907e-04$
$\rho = 0.3$	$8.0556e-04$

7 Parametrized method for a general Cauchy problem with Caputo-Fabrizio derivative

Having this accurate scheme in hand, we can now extent it to the case of Caputo-Fabrizio derivative. To achieve this, we consider the following nonlinear equation

$$\begin{cases} {}_0^{CF}D_t^{\alpha} u(t) = \gamma(t, u(t)) \\ u(0) = u_0 \end{cases}. \tag{72}$$

We apply the Caputo-Fabrizio integral on both sides to obtain

$$\begin{cases} u(t) = u(0) + (1-\alpha)\gamma(t, u(t)) + \alpha \int_0^t \gamma(\tau, u(\tau))\, d\tau \end{cases}. \tag{73}$$

At $t = t_{k+1}$ and $t = t_k$, we have

$$\begin{aligned} u(t_{k+1}) &= u(t_k) + (1-\alpha)\left[\gamma(t_{k+1}, u_{k+1}) - \gamma(t_k, u_k)\right] \\ &\quad + \alpha \int_{t_k}^{t_{k+1}} \gamma(\tau, u(\tau))\, d\tau. \end{aligned} \tag{74}$$

Within $[t_k, t_{k+1}]$, $\gamma(\tau, u(\tau))$ is approximated by

$$\gamma(\tau, u(\tau)) \simeq \left(1 - \frac{1}{2\rho}\right)\gamma(t_k, u_k) + \frac{1}{2\rho}\gamma(t_{k+1}, u_{k+1}). \tag{75}$$

where $t_{k+1} = t_k + h$. Adding this approximation to above integral yields

$$\begin{aligned} u_{k+1} &= u_k + (1-\alpha)\left[\gamma(t_{k+1}, \widetilde{u}_{k+1}) - \gamma(t_k, u_k)\right] \\ &\quad + \alpha h \left\{ \left(1 - \frac{1}{2\rho}\right)\gamma(t_k, u_k) + \frac{1}{2\rho}\gamma(t_{k+1}, \widetilde{u}_{k+1}) \right\} \end{aligned} \tag{76}$$

where

$$\widetilde{u}_{k+1} = u_0 + (1-\alpha)\gamma(t_k, u_k) + \alpha h \sum_{j=0}^{k} \gamma(t_j, u_j). \tag{77}$$

Therefore, the numerical scheme associated to this is given as

$$\begin{aligned} u_{k+1} &= u_k + (1-\alpha)\left[\gamma(t_{k+1}, \widetilde{u}_{k+1}) - \gamma(t_k, u_k)\right] \\ &\quad + \alpha h \left\{ \left(1 - \frac{1}{2\rho}\right)\gamma(t_k, u_k) + \frac{1}{2\rho}\gamma(t_{k+1}, \widetilde{u}_{k+1}) \right\}. \end{aligned} \tag{78}$$

Example 5. To illustrate the major difference between the parametrized approach and our modified version, we consider the following nonlinear equation:

$$\begin{aligned} {}_0^{CF}D_t^{\alpha} y(t) &= t^2 \\ y(0) &= 0. \end{aligned} \tag{79}$$

where exact solution

$$y(t) = (1-\alpha)t^2 + \alpha\frac{t^3}{3}.$$

The obtained graphical representation is presented in Figure 7 below. We shall note that, in Figure 7, we present the comparison between the exact solution and the numerical solution obtained by the suggested method. We achieved this when the parameter is 0.3 and the step size is 0.001.

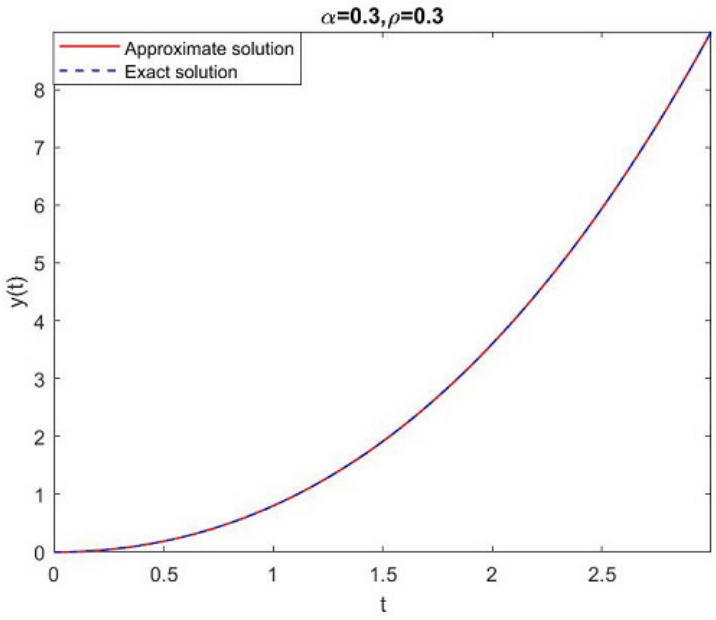

Figure 7. Numerical simulation for Cauchy problem for $\alpha = 0.3, \rho = 0.3$.

Error for the function $y(t)$ is presented in Table 5.

Table 5. Error for the function $y(t)$ for different values of α.

ρ value	Error for $\alpha = 0.3$	Error for $\alpha = 0.6$	Error for $\alpha = 0.9$
$\rho = 1$	$1.5000\ e-07$	$3.0000\ e-07$	$4.5000\ e-07$
$\rho = 0.9$	$1.3485e-05$	$2.6970\ e-04$	$4.0455\ e-04$
$\rho = 0.6$	$5.3985\ e-04$	0.0011	0.0016
$\rho = 0.4$	$8.0985\ e-04$	0.0016	0.0024
$\rho = 0.3$	$9.4485\ e-04$	0.0019	0.0028

8 Parametrized method for a general Cauchy problem with Caputo derivative

We consider next the following Cauchy problem with the Caputo derivative

$$\begin{cases} {}_0^C D_t^{\alpha} u(t) = \gamma(t, u(t)) \\ u(0) = u_0 \end{cases} . \tag{80}$$

Integrating the above equation

$$u(t) = u(0) + \frac{1}{\Gamma(\alpha)} \int_0^t \gamma(\tau, u(\tau)) (t-\tau)^{\alpha-1} d\tau \ . \tag{81}$$

At $t = t_{k+1}$, we have

$$\begin{cases} u(t_{k+1}) = u(0) + \frac{1}{\Gamma(\alpha)} \int_0^{t_{k+1}} \gamma(\tau, u(\tau)) (t_{k+1}-\tau)^{\alpha-1} d\tau \\ \quad = u(0) + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{k} \int_{t_j}^{t_{j+1}} \gamma(\tau, u(\tau)) (t_{k+1}-\tau)^{\alpha-1} d\tau \end{cases} \tag{82}$$

Applying the Riemann-Liouville integral on both sides yields

$$\left\{\begin{array}{c} u\left(t_{k+1}\right)=u\left(0\right)+\frac{1}{\Gamma(\alpha)}\sum_{j=0}^{k}\left\{\begin{array}{c}\left(1-\frac{1}{2\rho}\right)\gamma\left(t_j,u_j\right)\\ +\frac{1}{2\rho}\gamma\left(t_{j+1},u_{j+1}\right)\end{array}\right\}\\ \times\int_{t_j}^{t_{j+1}}\left(t_{k+1}-\tau\right)^{\alpha-1}d\tau\end{array}\right. \tag{83}$$

and

$$\left\{\begin{array}{c} u\left(t_{k+1}\right)=u\left(0\right)+\frac{1}{\Gamma(\alpha)}\sum_{j=0}^{k}\left\{\begin{array}{c}\left(1-\frac{1}{2\rho}\right)\gamma\left(t_j,u_j\right)\\ +\frac{1}{2\rho}\gamma\left(t_{j+1},u_{j+1}\right)\end{array}\right\}\\ \times\frac{h^{\alpha}}{\alpha}\left\{\left(k-j+1\right)^{\alpha}-\left(k-j\right)^{\alpha}\right\}\end{array}\right. . \tag{84}$$

Arranging above yields

$$\left\{\begin{array}{c} u\left(t_{k+1}\right)=u\left(0\right)+\frac{h^{\alpha}}{\Gamma(\alpha+1)}\left\{\begin{array}{c}\left(1-\frac{1}{2\rho}\right)\gamma\left(t_k,u_k\right)\\ +\frac{1}{2\rho}\gamma\left(t_{k+1},\widetilde{u}_{k+1}\right)\end{array}\right\}\\ +\frac{h^{\alpha}}{\Gamma(\alpha+1)}\sum_{j=0}^{k-1}\left\{\begin{array}{c}\left(1-\frac{1}{2\rho}\right)\gamma\left(t_j,u_j\right)\\ +\frac{1}{2\rho}\gamma\left(t_{j+1},u_{j+1}\right)\end{array}\right\}\\ \times\left\{\left(k-j+1\right)^{\alpha}-\left(k-j\right)^{\alpha}\right\}\end{array}\right. . \tag{85}$$

where $t_{k+1}=t_k+h$. Noting that

$$\begin{aligned}\widetilde{u}_{k+1} &= u\left(0\right)+\frac{1}{\Gamma\left(\alpha\right)}\sum_{j=0}^{k}\int_{t_j}^{t_{j+1}}\gamma\left(\tau,u\left(\tau\right)\right)\left(t_{k+1}-\tau\right)^{\alpha-1}d\tau \\ &= u\left(0\right)+\frac{1}{\Gamma\left(\alpha+1\right)}\sum_{j=0}^{k}\gamma\left(t_j,u_j\right)\left\{\begin{array}{c}\left(t_{k+1}-t_j\right)^{\alpha}\\ -\left(t_{k+1}-t_{j+1}\right)^{\alpha}\end{array}\right\} \\ &= u\left(0\right)+\frac{h^{\alpha}}{\Gamma\left(\alpha+1\right)}\sum_{j=0}^{k}\gamma\left(t_j,u_j\right)\left\{\begin{array}{c}\left(k-j+1\right)^{\alpha}\\ -\left(k-j\right)^{\alpha}\end{array}\right\}.\end{aligned} \tag{86}$$

Example 6. To illustrate the major difference between the parametrized approach and our modified version, we consider the following nonlinear equation:

$$\begin{aligned}{}_{0}^{C}D_{t}^{\alpha}y\left(t\right) &= t^{2} \\ y\left(0\right) &= 0.\end{aligned} \tag{87}$$

where exact solution

$$y\left(t\right)=t^{\alpha+2}\frac{\Gamma\left(3\right)}{\Gamma\left(\alpha+3\right)}.$$

The obtained graphical representation is presented in Figure 8 below. We shall note that, in Figure 8, we present the comparison between the exact solution and the numerical solution obtained by the suggested method. We achieved this when the parameter is 1 and the step size is 0.001.

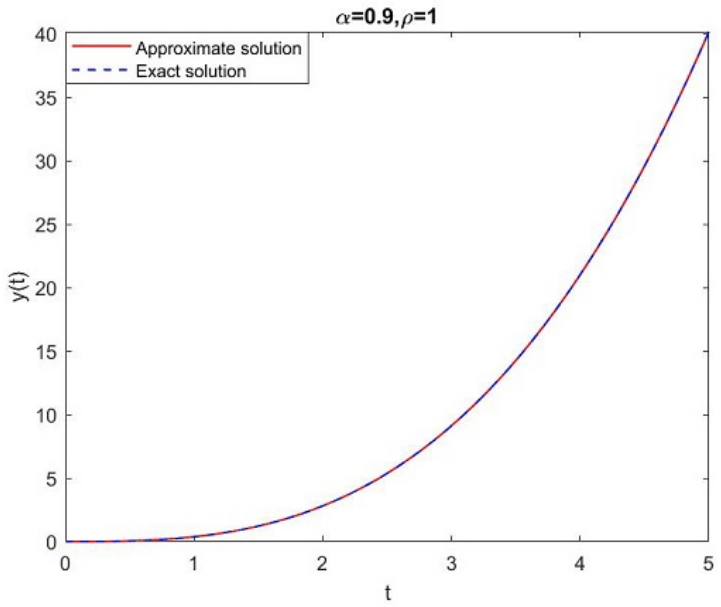

Figure 8. Numerical simulation for Cauchy problem for $\alpha = 0.9, \rho = 1$.

Error for the function $y(t)$ is presented in Table 6.

Table 6. Error for the function $y(t)$

ρ value	Error for $\alpha = 0.1$	Error for $\alpha = 0.3$	Error for $\alpha = 0.9$
$\rho = 1$	0.0022	$4.1153e-04$	$6.2403e-07$
$\rho = 0.9$	0.0028	0.0011	0.012
$\rho = 0.6$	0.0044	0.0032	0.0047
$\rho = 0.4$	0.0056	0.0046	0.0079
$\rho = 0.1$	0.0072	0.0067	0.0105

9 Parametrized method for a general Cauchy problem with Atangana-Baleanu derivative

We consider here the Cauchy problem with the Atangana-Baleanu fractional derivative

$$\begin{cases} {}_0^{ABC}D_t^{\alpha} u(t) = \gamma(t, u(t)) \\ u(0) = u_0 \end{cases}. \tag{88}$$

Integrating the above equation, we have

$$\begin{array}{c} u(t) = u(0) + (1-\alpha)\gamma(t, u(t)) \\ + \frac{\alpha}{\Gamma(\alpha)} \int_0^t \gamma(\tau, u(\tau)) (t-\tau)^{\alpha-1} d\tau \end{array}. \tag{89}$$

At $t = t_{k+1}$, we have

$$\begin{cases} u(t_{k+1}) = u(0) + (1-\alpha)\gamma(t_{k+1}, \widetilde{u}_{k+1}) \\ + \frac{\alpha}{\Gamma(\alpha)} \int_0^{t_{k+1}} \gamma(\tau, u(\tau)) (t_{k+1}-\tau)^{\alpha-1} d\tau \end{cases} \tag{90}$$

and

$$\left\{\begin{array}{c} u\left(t_{k+1}\right)=u\left(0\right)+\left(1-\alpha\right)\gamma\left(t_{k+1},\widetilde{u}_{k+1}\right) \\ +\frac{\alpha}{\Gamma(\alpha)}\sum\limits_{j=0}^{k}\int_{t_j}^{t_{j+1}}\gamma\left(\tau,u\left(\tau\right)\right)\left(t_{k+1}-\tau\right)^{\alpha-1}d\tau \end{array}\right. . \tag{91}$$

Nevertheless, the above could be implicit therefore to remove this problem, we rewrite as follows

$$\left\{\begin{array}{c} u\left(t_{k+1}\right)=u\left(0\right)+\left(1-\alpha\right)\gamma\left(t_{k+1},\widetilde{u}_{k+1}\right) \\ +\frac{\alpha h^{\alpha}}{\Gamma(a+1)}\sum\limits_{j=0}^{k}\left\{\begin{array}{c}\left(1-\frac{1}{2\rho}\right)\gamma\left(t_j,u_j\right) \\ +\frac{1}{2\rho}\gamma\left(t_{j+1},u_{j+1}\right)\end{array}\right\} \\ \times\left\{\left(k-j+1\right)^{\alpha}-\left(k-j\right)^{\alpha}\right\} \end{array}\right. \tag{92}$$

Using the approximation of the function $\gamma\left(\tau,u\left(\tau\right)\right)$ within $[t_k,t_{k+1}]$, we get

$$\left\{\begin{array}{c} u\left(t_{k+1}\right)=u\left(0\right)+\left(1-\alpha\right)\gamma\left(t_{k+1},\widetilde{u}_{k+1}\right) \\ +\frac{\alpha h^{\alpha}}{\Gamma(\alpha+1)}\left\{\begin{array}{c}\left(1-\frac{1}{2\rho}\right)\gamma\left(t_k,u_k\right) \\ +\frac{1}{2\rho}\gamma\left(t_{k+1},\widetilde{u}_{k+1}\right)\end{array}\right\} \\ +\frac{\alpha h^{\alpha}}{\Gamma(a+1)}\sum\limits_{j=0}^{k-1}\left\{\begin{array}{c}\left(1-\frac{1}{2\rho}\right)\gamma\left(t_j,u_j\right) \\ +\frac{1}{2\rho}\gamma\left(t_{j+1},u_{j+1}\right)\end{array}\right\} \\ \times\left\{\left(k-j+1\right)^{\alpha}-\left(k-j\right)^{\alpha}\right\} \end{array}\right. \tag{93}$$

where

$$\begin{aligned}\widetilde{u}_{k+1} &= u\left(0\right)+\left(1-\alpha\right)\gamma\left(t_k,u_k\right) \\ &\quad +\frac{\alpha h^{\alpha}}{\Gamma\left(\alpha+1\right)}\sum_{j=0}^{k}\gamma\left(t_j,u_j\right)\left\{\begin{array}{c}\left(k-j+1\right)^{\alpha} \\ -\left(k-j\right)^{\alpha}\end{array}\right\}.\end{aligned} \tag{94}$$

Example 7. To illustrate the major difference between the parametrized approach and our modified version, we consider the following nonlinear equation:

$$\begin{aligned}{}_{0}^{ABC}D_{t}^{\alpha}y\left(t\right) &= t^{2} \\ y\left(0\right) &= 0.\end{aligned} \tag{95}$$

where exact solution

$$y\left(t\right)=\left(1-\alpha\right)t^{2}+\alpha t^{\alpha+2}\frac{\Gamma\left(3\right)}{\Gamma\left(\alpha+3\right)}.$$

The obtained graphical representation is presented in Figure 9 below. We shall note that, in Figure 9, we present the comparison between the exact solution and the numerical solution obtained by the suggested method. We achieved this when the parameter is 0.8 and the step size is 0.001.

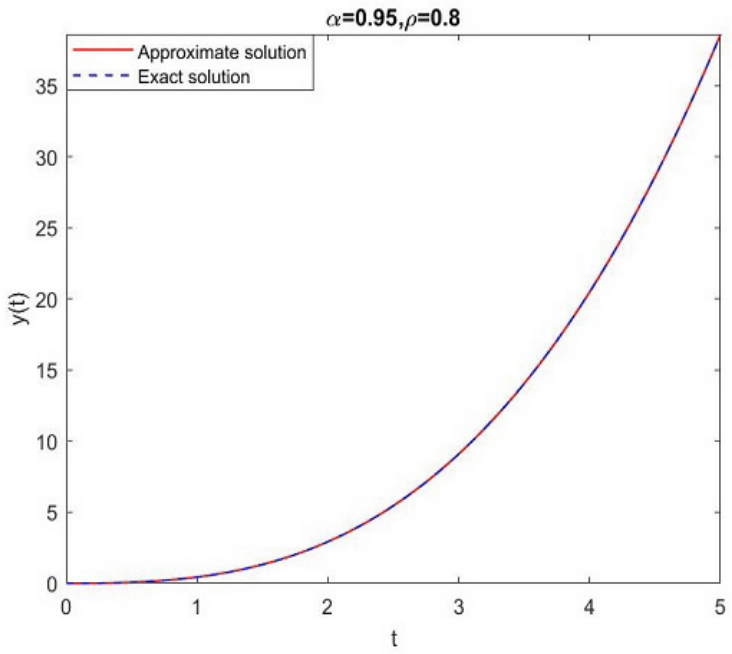

Figure 9. Numerical simulation for Cauchy problem for $\alpha = 0.95, \rho = 0.8$.

Error for the function $y(t)$ is presented in Table 7.

Table 7. Error for the function $y(t)$

ρ value	Error for $\alpha = 0.1$	Error for $\alpha = 0.5$	Error for $\alpha = 0.95$
$\rho = 1$	$2.1975\ e-04$	$3.6668\ e-05$	$2.3202\ e-07$
$\rho = 0.9$	$2.7587e-04$	$4.5719\ e-04$	0.011
$\rho = 0.6$	$4.4423e-04$	0.0017	0.0046
$\rho = 0.4$	$5.5647e-04$	0.0026	0.0069
$\rho = 0.1$	$7.2483e-04$	0.0038	0.0103

10 Parametrized method for fractal-fractional with the exponential law kernel

We can now present the following problem with fractal-fractional derivative with the exponential law kernel

$$\begin{cases} {}_0^{FFE}D_t^{\alpha,\beta} u(t) = \gamma(t, u(t)) \\ u(0) = u_0 \end{cases}. \tag{96}$$

The above problem can be converted to

$$\begin{cases} {}_0^{CF}D_t^{\alpha} u(t) = \beta t^{\beta-1}\gamma(t, u(t)) \\ u(0) = u_0 \end{cases}. \tag{97}$$

We apply the Caputo-Fabrizio integral on both sides to obtain

$$\left\{ u(t) = u(0) + (1-\alpha)\beta t^{\beta-1}\gamma(t, u(t)) + \alpha\beta \int_0^t \tau^{\beta-1}\gamma(\tau, u(\tau))\, d\tau \right. . \tag{98}$$

At $t = t_{k+1}$ and $t = t_k$, we have

$$\begin{aligned} u(t_{k+1}) &= u(t_k) + (1-\alpha)\beta\left[t_{k+1}^{\beta-1}\gamma(t_{k+1},\widetilde{u}_{k+1}) - t_k^{\beta-1}\gamma(t_k,u_k)\right] \\ &\quad +\alpha\beta\int_{t_k}^{t_{k+1}}\tau^{\beta-1}\gamma(\tau,u(\tau))\,d\tau. \end{aligned} \tag{99}$$

Within $[t_k, t_{k+1}]$, $\gamma(\tau, u(\tau))$ is approximated by

$$\gamma(\tau,u(\tau)) \simeq \left(1-\frac{1}{2\rho}\right)\gamma(t_k,u_k) + \frac{1}{2\rho}\gamma(t_{k+1},u_{k+1}). \tag{100}$$

where $t_{k+1} = t_k + h$. Adding this approximation to above integral yields

$$\begin{aligned} u_{k+1} &= u_k + (1-\alpha)\beta\left[t_{k+1}^{\beta-1}\gamma(t_{k+1},\widetilde{u}_{k+1}) - t_k^{\beta-1}\gamma(t_k,u_k)\right] \\ &\quad +\alpha\left\{\left(1-\frac{1}{2\rho}\right)\gamma(t_k,u_k) + \frac{1}{2\rho}\gamma(t_{k+1},\widetilde{u}_{k+1})\right\} \\ &\quad \times\left(t_{k+1}^{\beta} - t_k^{\beta}\right) \end{aligned} \tag{101}$$

and we get

$$\begin{aligned} u_{k+1} &= u_k + (1-\alpha)\beta\left[t_{k+1}^{\beta-1}\gamma(t_{k+1},\widetilde{u}_{k+1}) - t_k^{\beta-1}\gamma(t_k,u_k)\right] \\ &\quad +\alpha h^{\beta}\left\{\left(1-\frac{1}{2\rho}\right)\gamma(t_k,u_k) + \frac{1}{2\rho}\gamma(t_{k+1},\widetilde{u}_{k+1})\right\} \\ &\quad \times\left((k+1)^{\beta} - k^{\beta}\right) \end{aligned} \tag{102}$$

where

$$\begin{aligned} \widetilde{u}_{k+1} &= (1-\alpha)\beta t_k^{\beta-1}\gamma(t_k,u_k) \\ &\quad +\alpha h^{\beta}\sum_{j=0}^{n}\gamma(t_j,u_j)\left((j+1)^{\beta} - j^{\beta}\right). \end{aligned} \tag{103}$$

Example 8. To illustrate the major difference between the parametrized approach and our modified version, we consider the following nonlinear equation:

$$\begin{aligned} {}_0^{FFE}D_t^{\alpha}y(t) &= t^3 \\ y(0) &= 0. \end{aligned} \tag{104}$$

where exact solution

$$y(t) = (1-\alpha)\beta t^{\beta+2} + \alpha\beta t^{\beta+3}/\beta + 3. \tag{105}$$

The obtained graphical representation is presented in Figure 10 below. We shall note that, in Figure 10, we present the comparison between the exact solution and the numerical solution obtained by the suggested method. We achieved this when the parameter is 0.5 and the step size is 0.001.

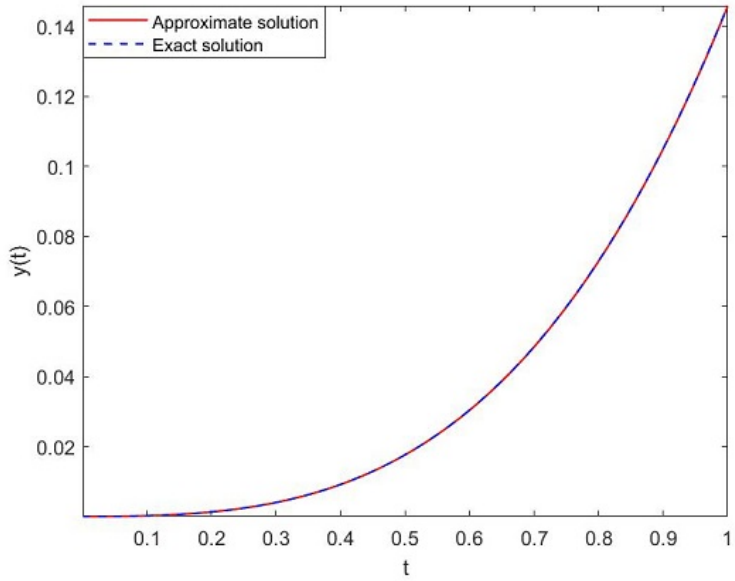

Figure 10. Numerical simulation for Cauchy problem for $\alpha = 0.9, \beta = 0.4, \rho = 0.5$.

Error for the function $y(t)$ is presented in Table 8.

Table 8. Error for the function $y(t)$

ρ value	Error for $\beta = 0.4$	Error for $\alpha = 0.6$	Error for $\alpha = 0.9$
$\rho = 1$	$1.6560 - e7$	$2.0254 - e7$	$2.2464 - e7$
$\rho = 0.9$	$2.2334 - e5$	$3.0951 - e5$	$4.1672 - e5$
$\rho = 0.6$	$8.9834 - e4$	$1.2441 - e4$	$1.6736 - e4$
$\rho = 0.4$	$1.3483 - e4$	$1.8672 - e4$	$2.5115 - e4$
$\rho = 0.1$	$2.0233 - e4$	$2.8018 - e4$	$3.7684 - e4$

11 Parametrized method for fractal-fractional with the power law kernel

We consider next the following Cauchy problem with the fractal-fractional derivative with power-law kernel

$$\begin{cases} {}_0^{FFP}D_t^{\alpha,\beta} u(t) = \gamma(t, u(t)) \\ u(0) = u_0 \end{cases}. \tag{106}$$

The above problem can be converted to

$$\begin{cases} {}_0^{C}D_t^{\alpha} u(t) = \beta t^{\beta-1} \gamma(t, u(t)) \\ u(0) = u_0 \end{cases}. \tag{107}$$

Integrating the above equation

$$u(t) = \frac{\beta}{\Gamma(\alpha)} \int_0^t \tau^{\beta-1} \gamma(\tau, u(\tau)) (t-\tau)^{\alpha-1} d\tau. \tag{108}$$

At $t = t_{k+1}$, we have

$$\left\{ \begin{array}{l} u\left(t_{k+1}\right) = \frac{\beta}{\Gamma(\alpha)} \int_0^{t_{k+1}} \tau^{\beta-1} \gamma\left(\tau, u\left(\tau\right)\right) \left(t_{k+1} - \tau\right)^{\alpha-1} d\tau \\ \quad = \frac{\beta}{\Gamma(\alpha)} \sum_{j=0}^{k} \int_{t_j}^{t_{j+1}} \tau^{\beta-1} \gamma\left(\tau, u\left(\tau\right)\right) \left(t_{k+1} - \tau\right)^{\alpha-1} d\tau \end{array} \right. \tag{109}$$

Applying the Riemann-Liouville integral on both sides yields

$$\left\{ \begin{array}{c} u\left(t_{k+1}\right) = \frac{\beta}{\Gamma(\alpha)} \sum_{j=0}^{k} \left\{ \begin{array}{c} \left(1 - \frac{1}{2\rho}\right) \gamma\left(t_j, u_j\right) \\ + \frac{1}{2\rho} \gamma\left(t_{j+1}, u_{j+1}\right) \end{array} \right\} \\ \times \int_{t_j}^{t_{j+1}} \tau^{\beta-1} \left(t_{k+1} - \tau\right)^{\alpha-1} d\tau \end{array} \right. \tag{110}$$

and

$$\left\{ \begin{array}{c} u\left(t_{k+1}\right) = \frac{\beta}{\Gamma(\alpha)} \sum_{j=0}^{k} t_{k+1}^{\alpha+\beta-1} \left\{ \begin{array}{c} \left(1 - \frac{1}{2\rho}\right) \gamma\left(t_j, u_j\right) \\ + \frac{1}{2\rho} \gamma\left(t_{j+1}, u_{j+1}\right) \end{array} \right\} \\ \times \left\{ B\left(\frac{t_{j+1}}{t_{k+1}}, \beta, \alpha\right) - B\left(\frac{t_j}{t_{k+1}}, \beta, \alpha\right) \right\} \end{array} \right. . \tag{111}$$

where

$$\begin{aligned} u_{k+1} &= \frac{\beta}{\Gamma\left(\alpha\right)} \sum_{j=0}^{n} t_{k+1}^{\alpha+\beta-1} f\left(t_j, y\left(t_j\right)\right) \\ &\quad \times \left\{ B\left(\frac{t_{j+1}}{t_{k+1}}, \beta, \alpha\right) - B\left(\frac{t_j}{t_{l+1}}, \beta, \alpha\right) \right\}. \end{aligned} \tag{112}$$

where $t_{j+1} = t_j + h$. Arranging above yields

$$\left\{ \begin{array}{c} u\left(t_{k+1}\right) = \frac{\beta}{\Gamma(\alpha)} \left\{ \begin{array}{c} \left(1 - \frac{1}{2\rho}\right) \gamma\left(t_k, u_k\right) \\ + \frac{1}{2\rho} \gamma\left(t_{k+1}, \widetilde{u}_{k+1}\right) \end{array} \right\} \\ \times t_{k+1}^{\alpha+\beta-1} \left\{ B\left(1, \beta, \alpha\right) - B\left(\frac{t_k}{t_{k+1}}, \beta, \alpha\right) \right\} \\ + \frac{\beta}{\Gamma(\alpha)} \sum_{j=0}^{k-1} \left\{ \begin{array}{c} \left(1 - \frac{1}{2\rho}\right) \gamma\left(t_j, u_j\right) \\ + \frac{1}{2\rho} \gamma\left(t_{j+1}, u_{j+1}\right) \end{array} \right\} \\ \times t_{k+1}^{\alpha+\beta-1} \left\{ B\left(\frac{t_{j+1}}{t_{k+1}}, \beta, \alpha\right) - B\left(\frac{t_j}{t_{k+1}}, \beta, \alpha\right) \right\} \end{array} \right. . \tag{113}$$

Noting that

$$\begin{aligned} \widetilde{u}_{k+1} &= \frac{\beta}{\Gamma\left(\alpha\right)} \sum_{j=0}^{k} \int_{t_j}^{t_{j+1}} \tau^{\beta-1} \gamma\left(\tau, u\left(\tau\right)\right) \left(t_{k+1} - \tau\right)^{\alpha-1} d\tau \\ &= \frac{\beta}{\Gamma\left(\alpha\right)} \sum_{j=0}^{k} t_{k+1}^{\alpha+\beta-1} \gamma\left(t_j, u_j\right) \\ &\quad \times \left\{ B\left(\frac{t_{j+1}}{t_{k+1}}, \beta, \alpha\right) - B\left(\frac{t_j}{t_{k+1}}, \beta, \alpha\right) \right\}. \end{aligned} \tag{114}$$

Example 9. To illustrate the major difference between the parametrized approach and our modified version, we consider the following nonlinear equation:

$$\begin{aligned} {}_0^{FFP}D_t^{\alpha} y\left(t\right) &= t^3 \\ y\left(0\right) &= 0. \end{aligned} \tag{115}$$

where exact solution

$$y(t) = \frac{\beta\Gamma(\beta+3)}{\Gamma(\alpha+\beta+3)} t^{\alpha+\beta+2}. \tag{116}$$

The obtained graphical representation is presented in Figure 11 below. We shall note that, in Figure 11, we present the comparison between the exact solution and the numerical solution obtained by the suggested method. We achieved this when the parameter is 0.3 and the step size is 0.001.

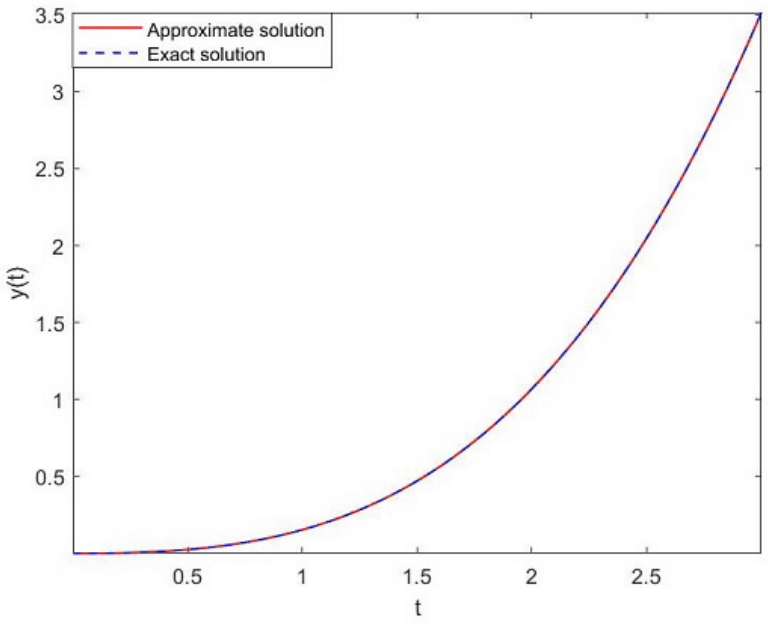

Figure 11. Numerical simulation for Cauchy problem for $\alpha = 0.7, \beta = 0.6, \rho = 0.7$.

Error for the function $y(t)$ is presented for $\alpha = 0.7$ in Table 9.

Table 9. Error for the function $y(t)$

ρ value	Error for $\beta = 0.25$	Error for $\beta = 0.55$	Error for $\beta = 0.85$
$\rho = 1$	$3.6762e-07$	$8.3241e-07$	$1.3197e-06$
$\rho = 0.8$	$2.7601e-04$	$7.4510e-04$	0.0014
$\rho = 0.6$	$5.5238e-04$	0.0015	0.0029
$\rho = 0.4$	$8.2876e-04$	0.0022	0.0043
$\rho = 0.2$	0.0011	0.0030	0.0057

12 Parametrized method for fractal-fractional with the Mittag-Leffler kernel

We consider here the Cauchy problem with the fractal-fractional derivative with Mittag-Leffler kernel

$$\begin{cases} {}_0^{FFM}D_t^{\alpha,\beta} u(t) = \gamma(t, u(t)) \\ u(0) = u_0 \end{cases}. \tag{117}$$

The above problem can be converted to

$$\begin{cases} {}_0^{ABR}D_t^{\alpha} u(t) = \beta t^{\beta-1}\gamma(t, u(t)) \\ u(0) = u_0, \text{ if } t = 0 \end{cases}. \tag{118}$$

Integrating the above equation, we have

$$u(t)=\beta t^{\beta-1}(1-\alpha)\gamma(t,u(t))+\frac{\alpha}{\Gamma(\alpha)}\int_0^t \beta\tau^{\beta-1}\gamma(\tau,u(\tau))(t-\tau)^{\alpha-1}d\tau. \tag{119}$$

At $t=t_{k+1}$, we have

$$\begin{cases} u(t_{k+1})=\beta(1-\alpha)t_{k+1}^{\beta-1}\gamma(t_{k+1},\widetilde{u}_{k+1}) \\ +\frac{\alpha\beta}{\Gamma(\alpha)}\int_0^{t_{k+1}}\tau^{\beta-1}\gamma(\tau,u(\tau))(t_{k+1}-\tau)^{\alpha-1}d\tau \end{cases} \tag{120}$$

and

$$\begin{cases} u(t_{k+1})=\beta(1-\alpha)t_{k+1}^{\beta-1}\gamma(t_{k+1},\widetilde{u}_{k+1}) \\ +\frac{\alpha\beta}{\Gamma(\alpha)}\sum_{j=0}^{k}\int_{t_j}^{t_{j+1}}\tau^{\beta-1}\gamma(\tau,u(\tau))(t_{k+1}-\tau)^{\alpha-1}d\tau \end{cases}. \tag{121}$$

Nevertheless, the above could be implicit therefore to remove this problem, we rewrite as follows

$$\begin{cases} u(t_{k+1})=\beta t_{k+1}^{\beta-1}(1-\alpha)\gamma(t_{k+1},\widetilde{u}_{k+1}) \\ +\frac{\alpha\beta}{\Gamma(a)}\sum_{j=0}^{k}\begin{cases}\left(1-\frac{1}{2\rho}\right)\gamma(t_j,u_j)\\ +\frac{1}{2\rho}\gamma(t_{j+1},u_{j+1})\end{cases} \\ \times t_{k+1}^{\alpha+\beta-1}\left\{B\left(\frac{t_{j+1}}{t_{k+1}},\beta,\alpha\right)-B\left(\frac{t_j}{t_{k+1}},\beta,\alpha\right)\right\} \end{cases} \tag{122}$$

Using the approximation of the function $\gamma(\tau,u(\tau))$ within $[t_k,t_{k+1}]$, we get

$$\begin{cases} u(t_{k+1})=(1-\alpha)\beta t_{k+1}^{\beta-1}\gamma(t_{k+1},\widetilde{u}_{k+1}) \\ +\frac{\alpha\beta}{\Gamma(a)}\sum_{j=0}^{k}\begin{cases}\left(1-\frac{1}{2\rho}\right)\gamma(t_j,u_j)\\ +\frac{1}{2\rho}\gamma(t_{j+1},u_{j+1})\end{cases} \\ \times t_{k+1}^{\alpha+\beta-1}\left\{B\left(\frac{t_{j+1}}{t_{k+1}},\beta,\alpha\right)-B\left(\frac{t_j}{t_{k+1}},\beta,\alpha\right)\right\} \end{cases} \tag{123}$$

where

$$\begin{aligned} \widetilde{u}_{k+1} &= (1-\alpha)\beta t_k^{\beta-1}\gamma(t_k,u_k) \\ &\quad +\frac{\alpha\beta}{\Gamma(\alpha)}\sum_{j=0}^{k}\gamma(t_j,u_j)t_{k+1}^{\alpha+\beta-1} \\ &\quad \times\left\{B\left(\frac{t_{j+1}}{t_{k+1}},\beta,\alpha\right)-B\left(\frac{t_j}{t_{k+1}},\beta,\alpha\right)\right\}. \end{aligned} \tag{124}$$

Example 10. To illustrate the major difference between the parametrized approach and our modified version, we consider the following nonlinear equation:

$$\begin{aligned} {}_0^{FFM}D_t^{\alpha}y(t) &= t^3 \\ y(0) &= 0. \end{aligned} \tag{125}$$

where exact solution

$$y(t)=(1-\alpha)\beta t^{\beta+2}+\frac{\alpha\beta\Gamma(\beta+3)}{\Gamma(\alpha+\beta+3)}t^{\alpha+\beta+2}. \tag{126}$$

The obtained graphical representation is presented in Figure 12 below. We shall note that, in Figure 12, we present the comparison between the exact solution and the numerical solution obtained by the suggested method. We achieved this when the parameter is 0.7 and the step size is 0.001.

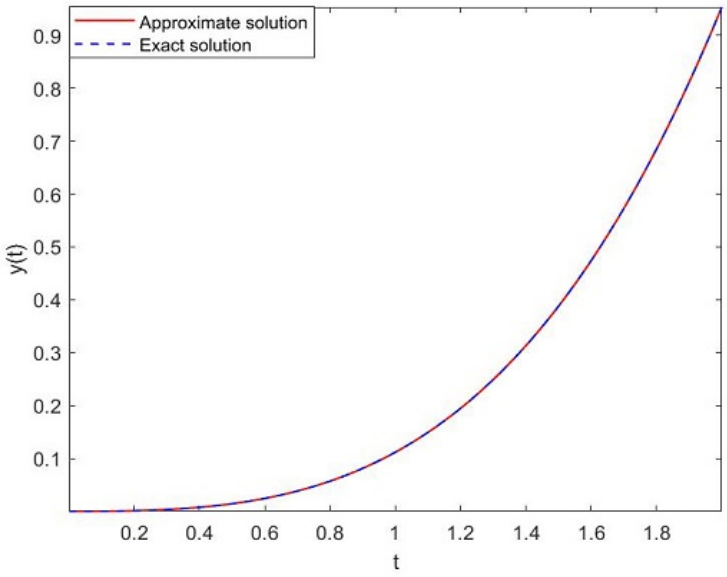

Figure 12. Numerical simulation for Cauchy problem for $\alpha = 0.9, \beta = 0.6, \rho = 0.7$.

Error for the function $y(t)$ is presented for $\alpha = 0.9$ in Table 10.

Table 10. Error for the function $y(t)$

ρ value	Error for $\beta = 0.1$	Error for $\beta = 0.6$	Error for $\beta = 0.8$
$\rho = 1$	$1.2310e-07$	$6.1391e-07$	$7.6676e-07$
$\rho = 0.9$	$2.7437e-05$	$1.8827e-04$	$2.6786e-04$
$\rho = 0.7$	$8.2557e-05$	$5.6603e-04$	$8.0511e-04$
$\rho = 0.5$	$1.3768e-04$	$9.4379e-04$	0.0013
$\rho = 0.2$	$2.2036e-04$	0.0015	0.0021

13 Consistency and stability analysis

In this section, we present the theoretical analysis of the suggested scheme. Without loss of generality, we present the analysis for fractal-fractional cases as the results of others could be obtained by setting the fractal dimension 1 and fractional order 1. We start with the fractal case. The aim in this analysis is to show that if $u(t_{n+1})$ is the exact solution and u_{n+1} is the approximate one.

$$\lim_{h \to 0} |u(t_{n+1}) - u_{n+1}| = 0. \tag{127}$$

If $\widetilde{u}_{n+1}$ is a perturbed term of u_{n+1}

$$|\widetilde{u}_{n+1}| \leq C |\widetilde{u}_0| \tag{128}$$

where C is a constant. Then, we evaluate

$$\begin{aligned} |u(t_{n+1}) - u_{n+1}| &= \left| \begin{array}{c} u(t_n) - u_n + \beta \int_{t_n}^{t_{n+1}} \tau^{\beta-1} \gamma(\tau, u(\tau))\, d\tau \\ -h^\beta \left\{ \left(1 - \frac{1}{2\rho}\right) \gamma(t_n, u_n) + \frac{1}{2\rho} \gamma(t_{n+1}, \widetilde{u}_{n+1}) \right\} \\ \times \left((n+1)^\beta - n^\beta \right) \end{array} \right| \\ &\leq |u(t_n) - u_n| + \left| \begin{array}{c} \beta \int_{t_n}^{t_{n+1}} \tau^{\beta-1} \gamma(\tau, u(\tau))\, d\tau - h^\beta \gamma(t_n, u_n) \\ \times \left((n+1)^\beta - n^\beta \right) \end{array} \right| \\ &\quad + \frac{1}{2\rho} |\gamma(t_{n+1}, u_{n+1}) - \gamma(t_n, u_n)|\, h^\beta \left((n+1)^\beta - n^\beta \right) \end{aligned} \tag{129}$$

$$\begin{aligned} &\leq |u(t_n) - u_n| + \left| \beta \int_{t_n}^{t_{n+1}} \tau^{\beta-1} \gamma(\tau, u(\tau))\, d\tau - \int_{t_n}^{t_{n+1}} \tau^{\beta-1} \gamma(t_n, u_n)\, d\tau \right| \\ &\quad + \frac{1}{2\rho} |\gamma(t_{n+1}, u_{n+1}) - \gamma(t_n, u_n)|\, h^\beta \left((n+1)^\beta - n^\beta \right) \\ &\leq |u(t_n) - u_n| + \beta \int_{t_n}^{t_{n+1}} \tau^{\beta-1} |\gamma(\tau, u(\tau)) - \gamma(t_n, u_n)|\, d\tau \\ &\quad + \frac{1}{2\rho} |\gamma(t_{n+1}, u_{n+1}) - \gamma(t_n, u_n)|\, h^\beta \left((n+1)^\beta - n^\beta \right). \end{aligned}$$

Since the function $\gamma(t, u(t))$ is differentiable, by the mean value theorem we have

$$\begin{aligned} |u(t_{n+1}) - u_{n+1}| &\leq |u(t_n) - u_n| + \beta |\gamma'(c_1, u(c_1))| \int_{t_n}^{t_{n+1}} \tau^{\beta-1} (\tau - t_n)\, d\tau \\ &\quad + \frac{1}{2\rho} |\gamma'(c_1, u(c_1))|\, h^{\beta+1} \left((n+1)^\beta - n^\beta \right) \\ &\leq |u(t_n) - u_n| + |\gamma'(c_1, u(c_1))|\, h^{\beta+1} \\ &\quad \times \left\{ \begin{array}{c} \frac{\beta}{\beta+1} \left((n+1)^{\beta+1} - n^{\beta+1} \right) \\ -n \left((n+1)^\beta - n^\beta \right) \end{array} \right\}. \end{aligned} \tag{130}$$

Then, we have

$$\lim_{\substack{h \to 0 \\ n \to \infty}} \left| \frac{u(t_{n+1}) - u_{n+1}}{u(t_n) - u_n} \right| \leq 1. \tag{131}$$

We can find $q \leq 1$ such that

$$\lim_{\substack{h \to 0 \\ n \to \infty}} |u(t_{n+1}) - u_{n+1}| \leq \lim_{\substack{h \to 0 \\ n \to \infty}} |u(t_0) - u_0|\, q^n = 0. \tag{132}$$

Thus

$$\lim_{h \to 0} |u(t_{n+1}) - u_{n+1}| = 0. \tag{133}$$

Let $\widetilde{u}_{n+1}, \widetilde{u}_n$ and $\widetilde{u}_0$ be the perturbed terms of u_{n+1}, u_n and u_0 respectively. The perturbed equation is given as

$$\begin{aligned} u_{n+1} + \widetilde{u}_{n+1} &= u_n + \widetilde{u}_n + h^\beta \left((n+1)^\beta - n^\beta \right) \\ &\quad \times \left\{ \begin{array}{c} \left(1 - \frac{1}{2\rho}\right) \gamma(t_n, u_n + \widetilde{u}_n) \\ + \frac{1}{2\rho} \gamma\left(t_{n+1}, \widetilde{u}^p_{n+1} + u^p_{n+1}\right) \end{array} \right\} \end{aligned} \tag{134}$$

Thus, we write

$$\begin{aligned}\widetilde{u}_{n+1} &= \widetilde{u}_n + h^\beta\left((n+1)^\beta - n^\beta\right) \\ &\times\left\{\begin{array}{c}\left(1-\frac{1}{2\rho}\right)\left[\gamma\left(t_n, u_n+\widetilde{u}_n\right)-\gamma\left(t_n,u_n\right)\right] \\ +\frac{1}{2\rho}\left[\gamma\left(t_{n+1},\widetilde{u}^p_{n+1}+u^p_{n+1}\right)-\gamma\left(t_{n+1},u^p_{n+1}\right)\right]\end{array}\right\}\end{aligned} \tag{135}$$

Taking the norm of both sides yields

$$\left|\widetilde{u}_{n+1}\right| \leq \left|\widetilde{u}_n\right| + h^\beta\left((n+1)^\beta - n^\beta\right)\left\{\left(1-\frac{1}{2\rho}\right)L\left|\widetilde{u}_n\right| + \frac{L}{2\rho}\left|\widetilde{u}^p_{n+1}\right|\right\} \tag{136}$$

But

$$\left|\widetilde{u}^p_{n+1}\right| \leq \left|\widetilde{u}_n\right| + h^\beta\left|\widetilde{u}_n\right| L\left((n+1)^\beta - n^\beta\right). \tag{137}$$

where L is the Lipschitz constant. Then

$$\left|\widetilde{u}_{n+1}\right| \leq \left|\widetilde{u}_n\right| + h^\beta\left((n+1)^\beta - n^\beta\right)\left|\widetilde{u}_n\right| L\left\{1-\frac{1}{2\rho}+\frac{h^\beta}{2\rho}L\left((n+1)^\beta - n^\beta\right)\right\} \tag{138}$$

Since $\beta \leq 1$, we have that

$$\begin{aligned}(n+1)^\beta &\leq n+1, \\ n^\beta &\leq n \\ (n+1)^\beta - n^\beta &\leq n+1-n=1.\end{aligned} \tag{139}$$

Therefore $\forall n \geq 0$

$$(n+1)^\beta - n^\beta \leq 1. \tag{140}$$

Then

$$\begin{aligned}\left|\widetilde{u}_{n+1}\right| &\leq \left|\widetilde{u}_n\right| + h^\beta\left|\widetilde{u}_n\right| L\left\{1-\frac{1}{2\rho}+\frac{h^\beta}{2\rho}L\right\} \\ &\leq L\left|\widetilde{u}_n\right|\left(1+h^\beta\left\{1-\frac{1}{2\rho}+\frac{h^\beta}{2\rho}L\right\}\right).\end{aligned} \tag{141}$$

On the other hand,

$$\begin{aligned}\left|\widetilde{u}_1\right| &\leq L\left|\widetilde{u}_0\right|\left(1+h^\beta\left\{1-\frac{1}{2\rho}+\frac{h^\beta}{2\rho}L\right\}\right), \\ \left|\widetilde{u}_2\right| &\leq L^2\left|\widetilde{u}_0\right|\left(1+h^\beta\left\{1-\frac{1}{2\rho}+\frac{h^\beta}{2\rho}L\right\}\right)^2, \\ &\vdots \\ \left|\widetilde{u}_{n+1}\right| &\leq L^n\left|\widetilde{u}_0\right|\left(1+h^\beta\left\{1-\frac{1}{2\rho}+\frac{h^\beta}{2\rho}L\right\}\right)^n\end{aligned} \tag{142}$$

which completes the proof.

Remark 10. When $\beta = 1$, we have

$$\left|\widetilde{u}_{n+1}\right| \leq L^n\left|\widetilde{u}_0\right|\left(1+h\left\{1-\frac{1}{2\rho}+\frac{h}{2\rho}L\right\}\right)^n. \tag{143}$$

When $\rho = 1$, we have

$$|\widetilde{u}_{n+1}| \leq L^n |\widetilde{u}_0| \left(1 + \frac{h}{2}\{1 + hL\}\right)^n. \tag{144}$$

When $\rho = 1/2$, we have

$$|\widetilde{u}_{n+1}| \leq L^n |\widetilde{u}_0| \left(1 + h^2 L\right)^n. \tag{145}$$

When $\rho = 2/3$, we have

$$|\widetilde{u}_{n+1}| \leq L^n |\widetilde{u}_0| \left(1 + h\left\{\frac{3}{4} + \frac{3h}{4}L\right\}\right)^n. \tag{146}$$

We now present consistency and stability analysis for the fractal-fractional Cauchy problem with exponential decay kernel. To achieve this, we evaluate

$$|u(t_{n+1}) - u_{n+1}| = \left| \begin{array}{c} u(t_n) + (1-\alpha)\beta \left[\begin{array}{c} t_{n+1}^{\beta-1}\gamma(t_{n+1}, u_{n+1}) \\ -t_n^{\beta-1}\gamma(t_n, u_n) \end{array} \right] \\ +\alpha\beta \int_{t_n}^{t_{n+1}} \tau^{\beta-1} f(\tau, u(\tau))\, d\tau - u_n \\ -(1-\alpha)\beta \left[t_{n+1}\gamma(t_{n+1}, u_{n+1}) - t_n\gamma(t_n, u_n)\right] \\ -\alpha h^\beta \left\{\left(1 - \frac{1}{2\rho}\right)\gamma(t_n, u_n) + \frac{1}{2\rho}\gamma(t_{n+1}, u_{n+1})\right\} \\ \times \left((n+1)^\beta - n^\beta\right) \end{array} \right| \tag{147}$$

$$\begin{aligned} \leq\ & |u(t_n) - u_n| + (1-\alpha)\beta t_{n+1}^{\beta-1} L |u(t_{n+1}) - u_{n+1}| \\ & + (1-\alpha)\beta t_n^\beta L |u(t_n) - u_n| \\ & + \left| \begin{array}{c} \alpha\beta \int_{t_n}^{t_{n+1}} \tau^{\beta-1} \left[\gamma(\tau, u(\tau)) - \gamma(t_n, u_n)\right] d\tau \\ \times \left((n+1)^\beta - n^\beta\right) \end{array} \right| \\ & + \frac{\alpha}{2\rho} h^\beta |\gamma(t_{n+1}, u_{n+1}) - \gamma(t_n, u_n)| \left((n+1)^\beta - n^\beta\right) \end{aligned}$$

Since the function $\gamma(t, u(t))$ is differentiable, by the mean value theorem we have

$$\begin{aligned} |u(t_{n+1}) - u_{n+1}| \leq\ & \left(1 + (1-\alpha)\beta n^{\beta-1} h^{\beta-1} L\right) |u(t_n) - u_n| \\ & + (1-\alpha)\beta (n+1)^{\beta-1} h^{\beta-1} L |u(t_{n+1}) - u_{n+1}| \\ & + \alpha\beta |\gamma'(c, u(c))| \int_{t_n}^{t_{n+1}} \tau^{\beta-1} (\tau - t_n)\, d\tau \\ & + \frac{\alpha}{2\rho} |\gamma'(c, u(c))| h^\beta \left((n+1)^\beta - n^\beta\right) \end{aligned} \tag{148}$$

and

$$\begin{aligned} |u(t_{n+1}) - u_{n+1}| \left(\begin{array}{c} 1 - (1-\alpha)\beta \\ \times (n+1)^{\beta-1} h^{\beta-1} L \end{array} \right) \leq\ & |u(t_n) - u_n| \left(1 + (1-\alpha)\beta (nh)^{\beta-1} L\right) \\ & + \frac{\alpha}{2\rho} |\gamma'(c, u(c))| h^\beta \left((n+1)^\beta - n^\beta\right) \\ & + \alpha\beta h^{\beta+1} |\gamma'(c, u(c))| \left\{ \begin{array}{c} \frac{\beta}{\beta+1}\left((n+1)^{\beta+1} - n^{\beta+1}\right) \\ -n^\beta \left((n+1)^\beta - n^\beta\right) \end{array} \right\}. \end{aligned} \tag{149}$$

Then, we have

$$\lim_{\substack{h \to 0 \\ n \to \infty}} \left| \frac{u(t_{n+1}) - u_{n+1}}{u(t_n) - u_n} \right| \leq 1. \tag{150}$$

We can write

$$\lim_{\substack{h\to 0\\ n\to\infty}} |u(t_{n+1}) - u_{n+1}| \leq \lim_{\substack{h\to 0\\ n\to\infty}} |u(t_0) - u_0| q^n \tag{151}$$

where $q \leq 1$. Thus

$$\lim_{h\to 0} |u(t_{n+1}) - u_{n+1}| = 0. \tag{152}$$

We now evaluate $|\widetilde{u}_{n+1}|$. Note that

$$\begin{aligned} |\widetilde{u}_{n+1}| &\leq |\widetilde{u}_n| + (1-\alpha)\beta \left[\begin{array}{c} t_{n+1}^{\beta-1} \left[\begin{array}{c} \gamma\left(t_{n+1}, \widetilde{u}_{n+1}^p + u_{n+1}^p\right) \\ -\gamma\left(t_{n+1}, u_{n+1}^p\right) \end{array} \right] \\ -t_n^{\beta-1} \left[\begin{array}{c} \gamma(t_n, u_n + \widetilde{u}_n) \\ -\gamma(t_n, u_n) \end{array} \right] \end{array} \right] \\ &\quad + \alpha h^\beta \left((n+1)^\beta - n^\beta\right) \left\{ \begin{array}{c} \left(1 - \frac{1}{2\rho}\right) \left[\begin{array}{c} \gamma(t_n, u_n + \widetilde{u}_n) \\ -\gamma(t_n, u_n) \end{array} \right] \\ + \frac{1}{2\rho} \left[\begin{array}{c} \gamma\left(t_{n+1}, \widetilde{u}_{n+1}^p + u_{n+1}^p\right) \\ -\gamma\left(t_{n+1}, u_{n+1}^p\right) \end{array} \right] \end{array} \right\} \end{aligned} \tag{153}$$

Using the Lipschitz condition of the function $\gamma(t, u(t))$ with respect to u, we get

$$\begin{aligned} |\widetilde{u}_{n+1}| &= |\widetilde{u}_n| + (1-\alpha)\beta(n+1)^{\beta-1} h^{\beta-1} L |\widetilde{u}_{n+1}| + (1-\alpha)\beta n^{\beta-1} h^{\beta-1} L |\widetilde{u}_n| \\ &\quad + \alpha h^\beta \left((n+1)^\beta - n^\beta\right) \left\{ \left(1 - \frac{1}{2\rho}\right) L |\widetilde{u}_n| + \frac{1}{2\rho} \widetilde{u}_{n+1}^p \right\} \end{aligned} \tag{154}$$

But

$$\left|\widetilde{u}_{n+1}^p\right| \leq |\widetilde{u}_0| + (1-\alpha)\beta t_n^\beta L |\widetilde{u}_n| + \alpha h^\beta L \sum_{j=0}^{n} |\widetilde{u}_j| \left((j+1)^\beta - j^\beta\right). \tag{155}$$

where L is the Lipschitz constant. Then

$$\begin{aligned} |\widetilde{u}_{n+1}| &\leq |\widetilde{u}_n| + (1-\alpha)\beta h^{\beta-1} (n+1)^{\beta-1} \\ &\quad \times \left\{ |\widetilde{u}_0| + (1-\alpha)\beta(nh)^{\beta-1} L + \alpha h^\beta L \sum_{j=0}^{n} |\widetilde{u}_j| \right\} \\ &\quad + (1-\alpha)\beta h^{\beta-1} n^{\beta-1} L |\widetilde{u}_n| + \alpha h^\beta |\widetilde{u}_n| L \left\{ 1 - \frac{1}{2\rho} \right\} \\ &\quad + \frac{\alpha h^\beta}{2\rho} L \left\{ |\widetilde{u}_0| + (1-\alpha)\beta(nh)^{\beta-1} L + \alpha h^\beta L \sum_{j=0}^{n} |\widetilde{u}_j| \right\}. \end{aligned} \tag{156}$$

On the other hand, we have

$$u_1 = u_0 + (1-\alpha)\beta t_0^{\beta-1} \gamma(t_0, u_0) + \alpha\beta \int_{t_0}^{t_n} \tau^{\beta-1} \gamma(t_0, u_0)\, d\tau. \tag{157}$$

For $t_0 = 0$, thus

$$u_1 = u_0 + \alpha\beta \int_0^{t_1} \tau^{\beta-1} \gamma(t_0, u_0)\, d\tau = u_0 + \alpha h^\beta \gamma(t_0, u_0). \tag{158}$$

Therefore

$$|\widetilde{u}_1| \leq |\widetilde{u}_0| \left(1 + \alpha h^\beta L\right). \tag{159}$$

We write

$$u_2 = u_1 + (1-\alpha)\,\beta h^{\beta-1}\gamma\left(t_1,u_1\right) + \alpha h^{\beta}\gamma\left(t_1,u_1\right)\left(2^{\beta}-1\right). \tag{160}$$

Then, we get

$$\begin{aligned} \left|\widetilde{u}_2\right| &\leq \left|\widetilde{u}_1\right| + (1-\alpha)\,\beta h^{\beta-1} L\left|\widetilde{u}_1\right| + \alpha h^{\beta} L\left|\widetilde{u}_1\right| \\ &\leq \left|\widetilde{u}_1\right|\left(1 + (1-\alpha)\,\beta h^{\beta-1} L + \alpha h^{\beta} L\right) \\ &\leq \left|\widetilde{u}_0\right|\left(1+\alpha h^{\beta} L\right)\left(1 + (1-\alpha)\,\beta h^{\beta-1} L + \alpha h^{\beta} L\right) \\ &\leq C\left|\widetilde{u}_0\right|. \end{aligned} \tag{161}$$

We assume by recurrence that for $n \geq 1$

$$\left|\widetilde{u}_n\right| \leq C_h^{\alpha}\left|\widetilde{u}_0\right|. \tag{162}$$

We want to show that this is also correct for $\left|\widetilde{u}_{n+1}\right|$. So, we evaluate

$$\begin{aligned} \left|\widetilde{u}_{n+1}\right| \leq\ & \left|\widetilde{u}_n\right| + (1-\alpha)\,\beta h^{\beta-1}\left(n+1\right)^{\beta-1} \\ & \times\left\{\left|\widetilde{u}_0\right| + (1-\alpha)\,\beta\left(nh\right)^{\beta-1} L + \alpha h^{\beta} L\sum_{j=0}^{n}\left|\widetilde{u}_j\right|\right\} \\ & + (1-\alpha)\,\beta h^{\beta-1} n^{\beta-1} L\left|\widetilde{u}_n\right| + \alpha h^{\beta}\left|\widetilde{u}_n\right| L\left\{1-\frac{1}{2\rho}\right\} \\ & + \frac{\alpha h^{\beta}}{2\rho} L\left\{\left|\widetilde{u}_0\right| + (1-\alpha)\,\beta\left(nh\right)^{\beta-1} L + \alpha h^{\beta} L\sum_{j=0}^{n}\left|\widetilde{u}_j\right|\right\}. \end{aligned} \tag{163}$$

Arranging above leads to

$$\left|\widetilde{u}_{n+1}\right| \leq \left|\widetilde{u}_0\right|\left\{\begin{array}{c} \widetilde{C}_n^{\alpha} + (1-\alpha)\,\beta h^{\beta-1}\left(n+1\right)^{\beta-1} \\ \times\left\{1 + (1-\alpha)\,\beta\left(nh\right)^{\beta-1} L + \alpha h^{\beta} L\widetilde{C}_n^{\alpha}\right\} \\ + (1-\alpha)\,\beta h^{\beta-1} n^{\beta-1} L\widetilde{C}_n^{\alpha} + \alpha h^{\beta}\widetilde{C}_n^{\alpha} L\left\{1-\frac{1}{2\rho}\right\} \\ + \frac{\alpha h^{\beta}}{2\rho} L\left\{1 + (1-\alpha)\,\beta\left(nh\right)^{\beta-1} L + \alpha h^{\beta} L\widetilde{C}_n^{\alpha}\right\} \end{array}\right\}. \tag{164}$$

Thus, we have

$$\left|\widetilde{u}_{n+1}\right| \leq \widetilde{C}_n^{\alpha}\left|\widetilde{u}_0\right|. \tag{165}$$

which completes the proof.

We now present the analysis of consistency and stability for the fractal-fractional Cauchy problem with the power-law kernel.

$$\left|u\left(t_{n+1}\right) - u_{n+1}\right| = \left|\begin{array}{c} \frac{\beta}{\Gamma(\alpha)}\int_{t_n}^{t_{n+1}}\tau^{\beta-1}\gamma\left(\tau,u\left(\tau\right)\right)\left(t_{n+1}-\tau\right)^{\alpha-1}d\tau \\ -\frac{\beta}{\Gamma(\alpha)}\sum_{j=0}^{n}\left\{\left(1-\frac{1}{2\rho}\right)\gamma\left(t_j,u_j\right) + \frac{1}{2\rho}\gamma\left(t_{j+1},u_{j+1}\right)\right\} \\ \times t_{n+1}^{\alpha+\beta-1}\left\{B\left(\frac{t_{j+1}}{t_{n+1}},\beta,\alpha\right) - B\left(\frac{t_j}{t_{n+1}},\beta,\alpha\right)\right\} \end{array}\right| \tag{166}$$

$$
\begin{aligned}
&= \frac{\beta}{\Gamma(\alpha)} \left| \begin{array}{c} \sum_{j=0}^{n} \int_{tj}^{t_{j+1}} \tau^{\beta-1} \gamma(\tau, u(\tau)) (t_{n+1}-\tau)^{\alpha-1} d\tau \\ -\sum_{j=0}^{n} \int_{tj}^{t_{j+1}} \tau^{\beta-1} (t_{n+1}-\tau)^{\alpha-1} \left\{ \begin{array}{c} \left(1-\frac{1}{2\rho}\right) \gamma(t_j, u_j) \\ +\frac{1}{2\rho} \gamma(t_{j+1}, u_{j+1}) \end{array} \right\} d\tau \end{array} \right| \\
&\leq \frac{\beta}{\Gamma(\alpha)} \sum_{j=0}^{n} \int_{tj}^{t_{j+1}} \tau^{\beta-1} (t_{n+1}-\tau)^{\alpha-1} |\gamma(t_j, u_j) - \gamma(\tau, u(\tau))| d\tau \\
&\quad + \frac{\beta}{2\rho\Gamma(\alpha)} \sum_{j=0}^{n} \int_{tj}^{t_{j+1}} \tau^{\beta-1} (t_{n+1}-\tau)^{\alpha-1} |\gamma(t_{j+1}, u_{j+1}) - \gamma(t_j, u_j)| d\tau \\
&\leq \sum_{j=0}^{n} \frac{\beta}{\Gamma(\alpha)} |\gamma'(c, u(c))| \int_{tj}^{t_{j+1}} \tau^{\beta-1} (t_{n+1}-\tau)^{\alpha-1} (\tau - t_j) d\tau \\
&\quad + \frac{\beta h}{2\rho\Gamma(\alpha)} |\gamma'(c_1, u(c_1))| \sum_{j=0}^{n} \int_{tj}^{t_{j+1}} \tau^{\beta-1} (t_{n+1}-\tau)^{\alpha-1} d\tau \\
&\leq \sum_{j=0}^{n} \frac{\beta}{\Gamma(\alpha)} |\gamma'(c, u(c))| \int_{tj}^{t_{j+1}} \tau^{\beta-1} (t_{n+1}-\tau)^{\alpha-1} (\tau - t_j) d\tau \\
&\quad + \frac{\beta h}{2\rho\Gamma(\alpha)} |\gamma'(c_1, u(c_1))| \sum_{j=0}^{n} t_{n+1}^{\alpha+\beta-1} \left\{ \begin{array}{c} B\left(\frac{t_{j+1}}{t_{n+1}}, \beta, \alpha\right) \\ -B\left(\frac{t_j}{t_{n+1}}, \beta, \alpha\right) \end{array} \right\}.
\end{aligned}
$$

But

$$
\int_{tj}^{t_{j+1}} \tau^{\beta-1} (t_{n+1}-\tau)^{\alpha-1} (\tau - t_j) d\tau = \left[\begin{array}{c} t_{n+1}^{\alpha+\beta} \left\{ \begin{array}{c} B\left(\frac{t_{j+1}}{t_{n+1}}, \beta+1, \alpha\right) \\ -B\left(\frac{t_j}{t_{n+1}}, \beta+1, \alpha\right) \end{array} \right\} \\ -t_j t_{n+1}^{\alpha+\beta-1} \left\{ \begin{array}{c} B\left(\frac{t_{j+1}}{t_{n+1}}, \beta+1, \alpha\right) \\ -B\left(\frac{t_j}{t_{n+1}}, \beta+1, \alpha\right) \end{array} \right\} \end{array} \right]. \tag{167}
$$

Therefore

$$
|u(t_{n+1}) - u_{n+1}| \leq \frac{\beta h^{\alpha+\beta}}{\Gamma(\alpha)} \left\{ \begin{array}{c} |\gamma'(c, u(c))| \sum_{j=0}^{n} \left\{ \begin{array}{c} B\left(\frac{t_{j+1}}{t_{n+1}}, \beta, \alpha\right) \\ -B\left(\frac{t_j}{t_{n+1}}, \beta, \alpha\right) \end{array} \right\} (n+1)^{\alpha+\beta-1} \\ +\frac{1}{2\rho} \sum_{j=0}^{n} \left[\begin{array}{c} (n+1)^{\alpha+\beta} \left\{ \begin{array}{c} B\left(\frac{t_{j+1}}{t_{n+1}}, \beta+1, \alpha\right) \\ -B\left(\frac{t_j}{t_{n+1}}, \beta+1, \alpha\right) \end{array} \right\} \\ -j(n+1)^{\alpha+\beta-1} \left\{ \begin{array}{c} B\left(\frac{t_{j+1}}{t_{n+1}}, \beta, \alpha\right) \\ -B\left(\frac{t_j}{t_{n+1}}, \beta, \alpha\right) \end{array} \right\} \end{array} \right] \end{array} \right\}. \tag{168}
$$

Thus, we have

$$
\lim_{h \to 0} |u(t_{n+1}) - u_{n+1}| = 0. \tag{169}
$$

For stability,

$$\begin{aligned}
|\widetilde{u}_{n+1}| &\leq \frac{\beta}{\Gamma(\alpha)} t_{n+1}^{\alpha+\beta-1} \left\{ \left(1-\frac{1}{2\rho}\right) L|\widetilde{u}_n| + \frac{1}{2\rho}\left|\widetilde{u}_{n+1}^p\right| \right\} \\
&\quad \times \left\{ \begin{array}{c} B(1,\beta,\alpha) \\ -B\left(\frac{t_n}{t_{n+1}},\beta,\alpha\right) \end{array} \right\} \\
&\quad + \frac{\beta}{\Gamma(\alpha)} \sum_{j=0}^{n} \left\{ \left(1-\frac{1}{2\rho}\right) L|\widetilde{u}_j| + \frac{1}{2\rho}|\widetilde{u}_{j+1}| \right\} \\
&\quad \times t_{n+1}^{\alpha+\beta-1} \left\{ \begin{array}{c} B\left(\frac{t_{j+1}}{t_{n+1}},\beta,\alpha\right) \\ -B\left(\frac{t_j}{t_{n+1}},\beta,\alpha\right) \end{array} \right\}.
\end{aligned} \tag{170}$$

Note that

$$\left|\widetilde{u}_{n+1}^p\right| \leq \frac{\beta}{\Gamma(\alpha)} (n+1)^{\alpha+\beta-1} h^{\alpha+\beta-1} L \sum_{j=0}^{n} |\widetilde{u}_j| \left\{ \begin{array}{c} B\left(\frac{t_{j+1}}{t_{n+1}},\beta,\alpha\right) \\ -B\left(\frac{t_j}{t_{n+1}},\beta,\alpha\right) \end{array} \right\}. \tag{171}$$

Then, we evaluate

$$|\widetilde{u}_1^p| \leq \frac{\beta h^{\alpha+\beta-1}}{\Gamma(\alpha)} L|\widetilde{u}_0|, \tag{172}$$

$$\begin{aligned}
|\widetilde{u}_2^p| &\leq \frac{\beta h^{\alpha+\beta-1}}{\Gamma(\alpha)} 2^{\alpha+\beta-1} L\{|\widetilde{u}_0| + |\widetilde{u}_1|\} \\
&\leq \frac{\beta h^{\alpha+\beta-1}}{\Gamma(\alpha)} 2^{\alpha+\beta-1} L|\widetilde{u}_0| \left\{ 1 + \frac{\beta h^{\alpha+\beta-1}}{\Gamma(\alpha)} L \right\} \\
&\vdots \\
|\widetilde{u}_n^p| &\leq \widetilde{C}_h^{\alpha,\beta} |\widetilde{u}_0|.
\end{aligned} \tag{173}$$

Then, we get

$$\left|\widetilde{u}_{n+1}^p\right| \leq \frac{\beta h^{\alpha+\beta-1}}{\Gamma(\alpha)} (n+1)^{\alpha+\beta-1} L C_h^{\alpha,\beta} |\widetilde{u}_0|.$$

Replacing and using the same methodology for $\widetilde{u}_{n+1}$, we have

$$\begin{aligned}
|\widetilde{u}_{n+1}| &\leq \frac{\beta}{\Gamma(\alpha)} h^{\alpha+\beta-1} (n+1)^{\alpha+\beta-1} |\widetilde{u}_0| \\
&\quad \times \left\{ \begin{array}{c} \left(1-\frac{1}{2\rho}\right) L\widetilde{C}_h^{\alpha,\beta} |\widetilde{u}_0| \\ + \frac{1}{2\rho} \frac{\beta h^{\alpha+\beta-1}}{\Gamma(\alpha)} (n+1)^{\alpha+\beta-1} L C_h^{\alpha,\beta} |\widetilde{u}_0| \\ + \frac{\beta}{\Gamma(\alpha)} \sum_{j=0}^{n-1} \left\{ \left(1-\frac{1}{2\rho}\right) L\widetilde{C}_h^{\alpha,\beta} + \frac{1}{2\rho}\widetilde{C}_h^{\alpha,\beta} \right\} \\ \times h^{\alpha+\beta-1} (n+1)^{\alpha+\beta-1} \left\{ \begin{array}{c} B\left(\frac{t_{j+1}}{t_{n+1}},\beta,\alpha\right) \\ -B\left(\frac{t_j}{t_{n+1}},\beta,\alpha\right) \end{array} \right\} \end{array} \right\}
\end{aligned} \tag{174}$$

Thus, we get

$$|\widetilde{u}_{n+1}| \leq B|\widetilde{u}_0| \tag{175}$$

which completes the proof.

We can conclude our analysis with Atangana-Baleanu case.

$$\begin{aligned}
|u(t_{n+1})-u_{n+1}| &\leq (1-\alpha)\,\beta t_{n+1}^{\beta-1}\left|\gamma(t_{n+1},u(t_{n+1}))-\gamma(t_{n+1},u_{n+1})\right| \\
&+\frac{\alpha\beta}{\Gamma(\alpha)}\left|\begin{array}{c}\int_0^{t_{n+1}}\tau^{\beta-1}\gamma(\tau,u(\tau))\,(t_{n+1}-\tau)^{\alpha-1}\,d\tau \\ -\sum_{j=0}^{n}t_{n+1}^{\alpha+\beta-1}\gamma(t_j,u_j) \\ \times\left\{B\left(\frac{t_{j+1}}{t_{n+1}},\beta,\alpha\right)-B\left(\frac{t_j}{t_{n+1}},\beta,\alpha\right)\right\}\end{array}\right| \\
&+\frac{\alpha\beta}{2\rho\Gamma(\alpha)}\sum_{j=0}^{n}t_{n+1}^{\alpha+\beta-1}\left|\gamma(t_{j+1},u_{j+1})-\gamma(t_j,u_j)\right| \\
&\times\left\{B\left(\frac{t_{j+1}}{t_{n+1}},\beta,\alpha\right)-B\left(\frac{t_j}{t_{n+1}},\beta,\alpha\right)\right\}.
\end{aligned}\tag{176}$$

We use the fact that $\gamma(\tau,u(\tau))$ is Lipschitz and the fact that is differentiable, then, we have

$$\begin{aligned}
|u(t_{n+1})-u_{n+1}| &\leq (1-\alpha)\,\beta h^{\alpha-1}L\left|u(t_{n+1})-u_{n+1}\right| \\
&+\frac{\alpha\beta}{\Gamma(\alpha)}\left|\sum_{j=0}^{n}\int_{t_j}^{t_{j+1}}\tau^{\beta-1}\left[\gamma(\tau,u(\tau))-\gamma(t_j,u_j)\right](t_{n+1}-\tau)^{\alpha-1}\,d\tau\right| \\
&+\frac{\alpha\beta}{2\rho\Gamma(\alpha)}\sum_{j=0}^{n}t_{n+1}^{\alpha+\beta-1}\left|\gamma(t_{j+1},u_{j+1})-\gamma(t_j,u_j)\right| \\
&\times\left\{B\left(\frac{t_{j+1}}{t_{n+1}},\beta,\alpha\right)-B\left(\frac{t_j}{t_{n+1}},\beta,\alpha\right)\right\}.
\end{aligned}\tag{177}$$

$$\begin{aligned}
&\leq (1-\alpha)\,\beta h^{\beta-1}L\left|u(t_{n+1})-u_{n+1}\right| \\
&+\frac{\alpha\beta}{\Gamma(\alpha)}\left|\gamma\prime(c,u(c))\right|\sum_{j=0}^{n}\int_{t_j}^{t_{j+1}}\tau^{\beta-1}(t_{n+1}-\tau)^{\alpha-1}(\tau-t_j)\,d\tau \\
&+\frac{\alpha\beta h^{\alpha+\beta}}{2\rho\Gamma(\alpha)}(n+1)^{\alpha+\beta-1}\sum_{j=0}^{n}L\left\{B\left(\frac{t_{j+1}}{t_{n+1}},\beta,\alpha\right)-B\left(\frac{t_j}{t_{n+1}},\beta,\alpha\right)\right\}
\end{aligned}\tag{178}$$

$$\begin{aligned}
&\leq (1-\alpha)\,\beta h^{\beta-1}L\left|u(t_{n+1})-u_{n+1}\right|(n+1)^{\beta-1} \\
&+\frac{\alpha\beta}{\Gamma(\alpha)}h^{\alpha+\beta}\left\{\begin{array}{c}\left|\gamma\prime(c,u(c))\right|\sum_{j=0}^{n}\int_{t_j}^{t_{j+1}}\left\{\begin{array}{c}B\left(\frac{t_{j+1}}{t_{n+1}},\beta,\alpha\right)\\ -B\left(\frac{t_j}{t_{n+1}},\beta,\alpha\right)\end{array}\right\}(n+1)^{\alpha+\beta-1} \\ +\frac{1}{2\rho}\sum_{j=0}^{n}\left\{\begin{array}{c}(n+1)^{\alpha+\beta}\left\{\begin{array}{c}B\left(\frac{t_{j+1}}{t_{n+1}},\beta+1,\alpha\right)\\ -B\left(\frac{t_j}{t_{n+1}},\beta+1,\alpha\right)\end{array}\right\} \\ -j\,(n+1)^{\alpha+\beta-1}\left\{\begin{array}{c}B\left(\frac{t_{j+1}}{t_{n+1}},\beta,\alpha\right)\\ -B\left(\frac{t_j}{t_{n+1}},\beta,\alpha\right)\end{array}\right\}\end{array}\right\}\end{array}\right\}
\end{aligned}\tag{179}$$

Thus, we have

$$\lim_{\substack{h\to 0\\ \text{if } \beta>1 \text{ or}\\ \text{if } \beta<1}}\left|u(t_{n+1})-u_{n+1}\right|=0\tag{180}$$

which concludes the proof. We now present the stability.

$$\begin{aligned}
|\widetilde{u}_{n+1}| &\leq (1-\alpha)\,\beta t_{n+1}^{\beta-1} L\left|\widetilde{u}_{n+1}^{p}\right| + \frac{\beta}{\Gamma(\alpha)} t_{n+1}^{\alpha+\beta-1} \left\{ \begin{array}{c} \left(1-\frac{1}{2\rho}\right) L\left|\widetilde{u}_{n}\right| \\ +\frac{1}{2\rho}\left|\widetilde{u}_{n+1}^{p}\right| \end{array} \right\} \\
&\times \left\{ \begin{array}{c} B(1,\beta,\alpha) \\ -B\left(\frac{t_n}{t_{n+1}},\beta,\alpha\right) \end{array} \right\} \\
&+\frac{\beta}{\Gamma(\alpha)} \sum_{j=0}^{n} \left\{ \left(1-\frac{1}{2\rho}\right) L\left|\widetilde{u}_{j}\right| + \frac{1}{2\rho}\left|\widetilde{u}_{j+1}\right| \right\} \\
&\times t_{n+1}^{\alpha+\beta-1} \left\{ \begin{array}{c} B\left(\frac{t_{j+1}}{t_{n+1}},\beta,\alpha\right) \\ -B\left(\frac{t_j}{t_{n+1}},\beta,\alpha\right) \end{array} \right\}.
\end{aligned} \tag{181}$$

Noting that

$$\begin{aligned}
\left|\widetilde{u}_{n+1}^{p}\right| &\leq (1-\alpha)\,\beta t_{n}^{\beta-1} L\left|\widetilde{u}_{n}\right| \\
&+\frac{\beta}{\Gamma(\alpha)} (n+1)^{\alpha+\beta-1} h^{\alpha+\beta-1} L \sum_{j=0}^{n} \left|\widetilde{u}_{j}\right| \left\{ \begin{array}{c} B\left(\frac{t_{j+1}}{t_{n+1}},\beta,\alpha\right) \\ -B\left(\frac{t_j}{t_{n+1}},\beta,\alpha\right) \end{array} \right\}.
\end{aligned} \tag{182}$$

By recursive, we show that

$$\left|\widetilde{u}_{n+1}^{p}\right| \leq |\widetilde{u}_{0}| \left\{ (1-\alpha)\,\beta t_{n}^{\beta-1} L + \frac{\alpha\beta}{\Gamma(\alpha)} (n+1)^{\alpha+\beta-1} h^{\alpha+\beta-1} L \widetilde{C}_{h}^{\alpha,\beta} \right\}. \tag{183}$$

Replacing into the original equation, we get

$$|\widetilde{u}_{n+1}| \leq |\widetilde{u}_{0}| \left\{ \begin{array}{c} (1-\alpha)\,\beta t_{n}^{\beta-1} L + \frac{\alpha\beta}{\Gamma(\alpha)} (n+1)^{\alpha+\beta-1} h^{\alpha+\beta-1} \\ +\left(1-\frac{1}{2\rho}\right) L \widetilde{C}_{h}^{\alpha,\beta} + \frac{1}{2\rho} \frac{\beta h^{\alpha+\beta-1}}{\Gamma(\alpha)} L C_{h}^{\alpha,\beta} \\ +\sum_{j=0}^{n-1} \left\{ \left(1-\frac{1}{2\rho}\right) L \widetilde{C}_{h}^{\alpha,\beta} + \frac{1}{2\rho} \widetilde{C}_{h}^{\alpha,\beta} \right\} L \\ \times \left\{ \begin{array}{c} B\left(\frac{t_{j+1}}{t_{n+1}},\beta,\alpha\right) \\ -B\left(\frac{t_j}{t_{n+1}},\beta,\alpha\right) \end{array} \right\} \end{array} \right\}. \tag{184}$$

Thus, we have

$$|\widetilde{u}_{n+1}| \leq \overline{B}\,|\widetilde{u}_{0}|. \tag{185}$$

This completes the proof.

14 Application of the modified parametrized method to nonlinear equations

In this section, we present numerical simulations for some chaotic problems with nonlocal operators to show efficacy of the modified parametrized method.

Example 1. We consider a multi-wing chaotic problem introduced in [[11]-[13]]

$$\begin{aligned}
x\prime(t) &= a(y-x) \\
y\prime(t) &= by - xz \\
z\prime(t) &= y^2 + f(y)
\end{aligned} \tag{186}$$

where

$$f(y)=\left[\begin{array}{c} D_1\left[\operatorname{sgn}(y+d_1)-\operatorname{sgn}(y-d_1)-2\right] \\ +D_2\left[\operatorname{sgn}(y+d_2)-\operatorname{sgn}(y-d_2)-2\right] \\ +D_3\left[\operatorname{sgn}(y+d_3)-\operatorname{sgn}(y-d_3)-2\right]-c \end{array}\right]. \tag{187}$$

We take the parameters

$$\begin{aligned} a &= 2, b=0.6, c=0.49, D_1=1.1, D_2=2.1, D_3=3.1, \\ d_1 &= 1.3, d_2=2.3, d_3=3.3. \end{aligned} \tag{188}$$

and initial conditions

$$x(0)=1, y(0)=0.1, z(0)=-0.7. \tag{189}$$

For simplicity, we take

$$\begin{aligned} x\prime(t) &= a(y-x) \\ y\prime(t) &= by-xz \\ z\prime(t) &= y^2+f(y) \end{aligned} \tag{190}$$

$$\begin{aligned} X(t) &= \left[\begin{array}{c} x(t) \\ y(t) \\ z(t) \end{array}\right], \\ F(t,X(t)) &= \left[\begin{array}{c} a(y-x) \\ by-xz \\ y^2+f(y) \end{array}\right]. \end{aligned} \tag{191}$$

By following the procedure of the modified parametrized method for classical case, we obtain the numerical scheme for the considered chaotic problem as follows:

$$X_{k+1}=X_k+\left(1-\frac{1}{2\rho}\right)hF(t_k,X_k)+\frac{h}{2\rho}F\left(t_{k+1},\widetilde{X}_{k+1}\right) \tag{192}$$

where

$$\widetilde{X}_{k+1}=X_k+hF(t_k,X_k). \tag{193}$$

For classical case, we depict the numerical simulations in Figure 13.

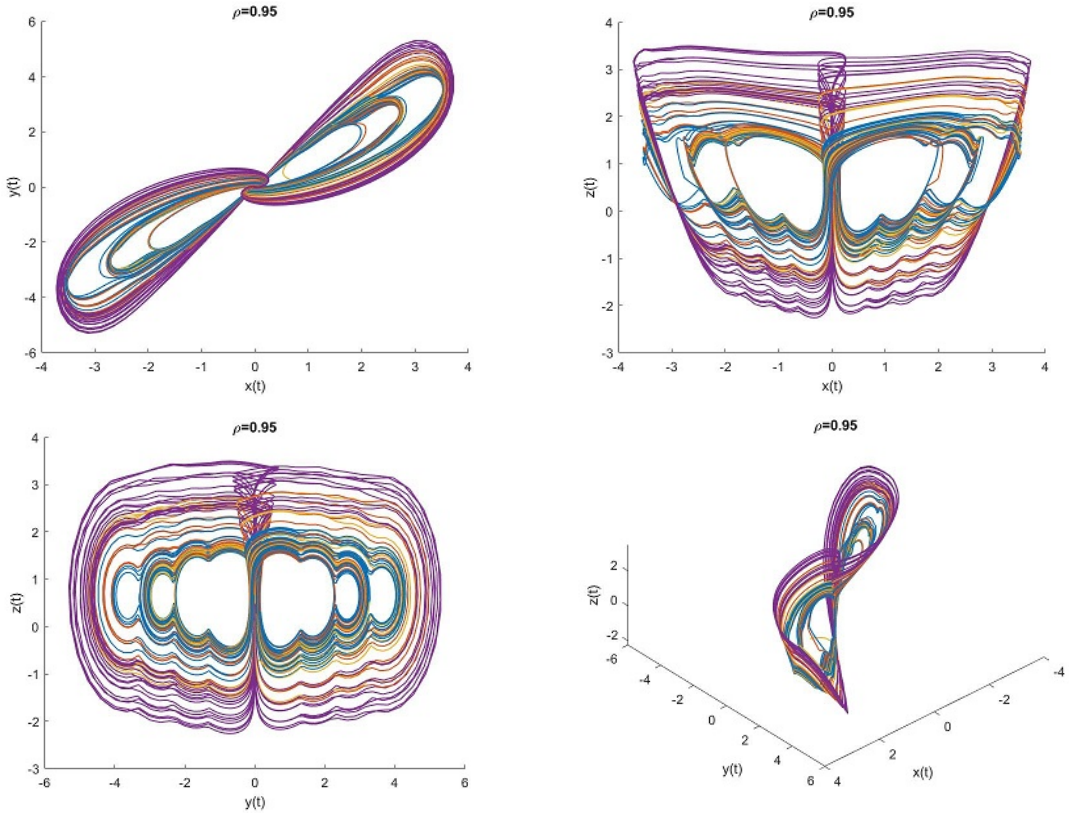

Figure 13. Numerical simulation for multi-wing chaotic system for $\rho = 0.95$.

We now consider the chaotic problem with fractal-fractional derivative with exponential decay kernel

$$\begin{cases} {}_0^{FFE}D_t^{\alpha,\beta} X(t) = F(t, X), \\ X(t) = X_0, \text{ if } t = 0 \end{cases}$$

Using the modified parametrized method for associated derivative presented earlier, we can have the following numerical scheme:

$$\begin{aligned} X_{k+1} &= X_k + (1-\alpha)\beta\left[t_{k+1}^{\beta-1} F\left(t_{k+1}, \widetilde{X}_{k+1}\right) - t_k^{\beta-1} F(t_k, X_k)\right] \\ &\quad + \alpha h^{\beta}\left\{\left(1 - \frac{1}{2\rho}\right) F(t_k, X_k) + \frac{1}{2\rho} F\left(t_{k+1}, \widetilde{X}_{k+1}\right)\right\} \\ &\quad \times \left((k+1)^{\beta} - k^{\beta}\right) \end{aligned} \tag{194}$$

where

$$\begin{aligned} \widetilde{X}_{k+1} &= (1-\alpha)\beta t_k^{\beta-1} F(t_k, X_k) \\ &\quad + \alpha h^{\beta} \sum_{j=0}^{k} F(t_j, X_j)\left((j+1)^{\beta} - j^{\beta}\right). \end{aligned} \tag{195}$$

In Figure 14, we depict the numerical simulations using same parameters and initial data.

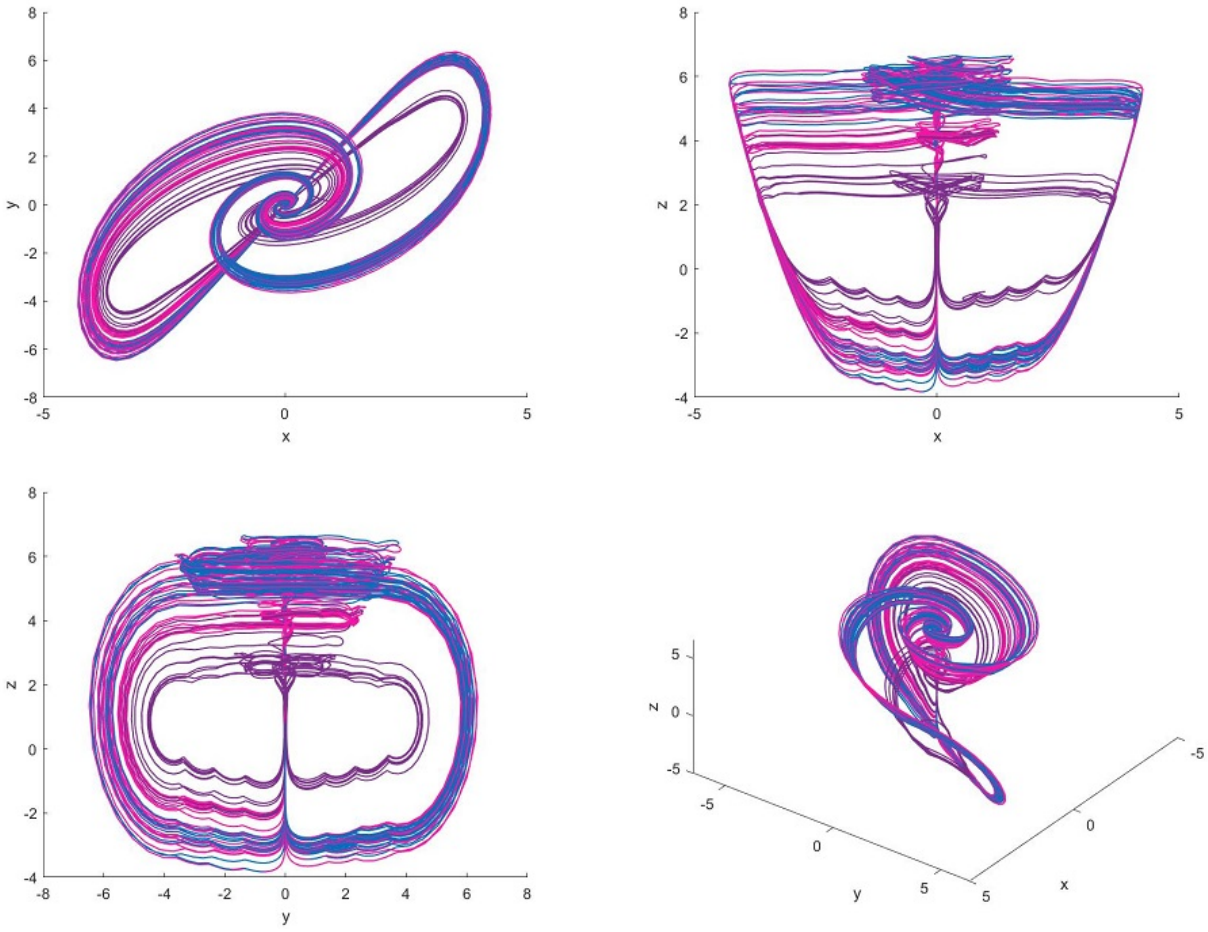

Figure 14. Numerical simulation for multi-wing chaotic system for $\rho = 0.3, \alpha = 0.99, \beta = 0.99$.

We now consider the chaotic problem with fractal-fractional derivative with Mittag-Leffler kernel

$$\begin{cases} {}_0^{FFM}D_t^{\alpha,\beta} X(t) = F(t, X), \\ X(t) = X_0, \text{ if } t = 0 \end{cases}$$

Using the modified parametrized method for associated derivative presented earlier, we can have the following numerical scheme:

$$\begin{cases} X(t_{k+1}) = (1-\alpha)\,\beta t_{k+1}^{\beta-1} F\left(t_{k+1}, \widetilde{X}_{k+1}\right) \\ \quad + \frac{\alpha\beta}{\Gamma(a)} \sum_{j=0}^{k} \begin{cases} \left(1 - \frac{1}{2\rho}\right) F(t_j, X_j) \\ + \frac{1}{2\rho} F(t_{j+1}, X_{j+1}) \end{cases} \\ \times t_{k+1}^{\alpha+\beta-1} \left\{ B\left(\frac{t_{j+1}}{t_{k+1}}, \beta, \alpha\right) - B\left(\frac{t_j}{t_{k+1}}, \beta, \alpha\right) \right\} \end{cases} \tag{196}$$

where

$$\begin{aligned}\widetilde{X}_{k+1} &= (1-\alpha)\,\beta t_k^{\beta-1} F\left(t_k, X_k\right) \\ &+ \frac{\alpha\beta}{\Gamma(\alpha)} \sum_{j=0}^{k} F\left(t_j, X_j\right) t_{k+1}^{\alpha+\beta-1} \\ &\times \left\{ B\left(\frac{t_{j+1}}{t_{k+1}}, \beta, \alpha\right) - B\left(\frac{t_j}{t_{k+1}}, \beta, \alpha\right) \right\}.\end{aligned} \tag{197}$$

With same parameters and initial data, we perform the numerical simulations in Figure 15.

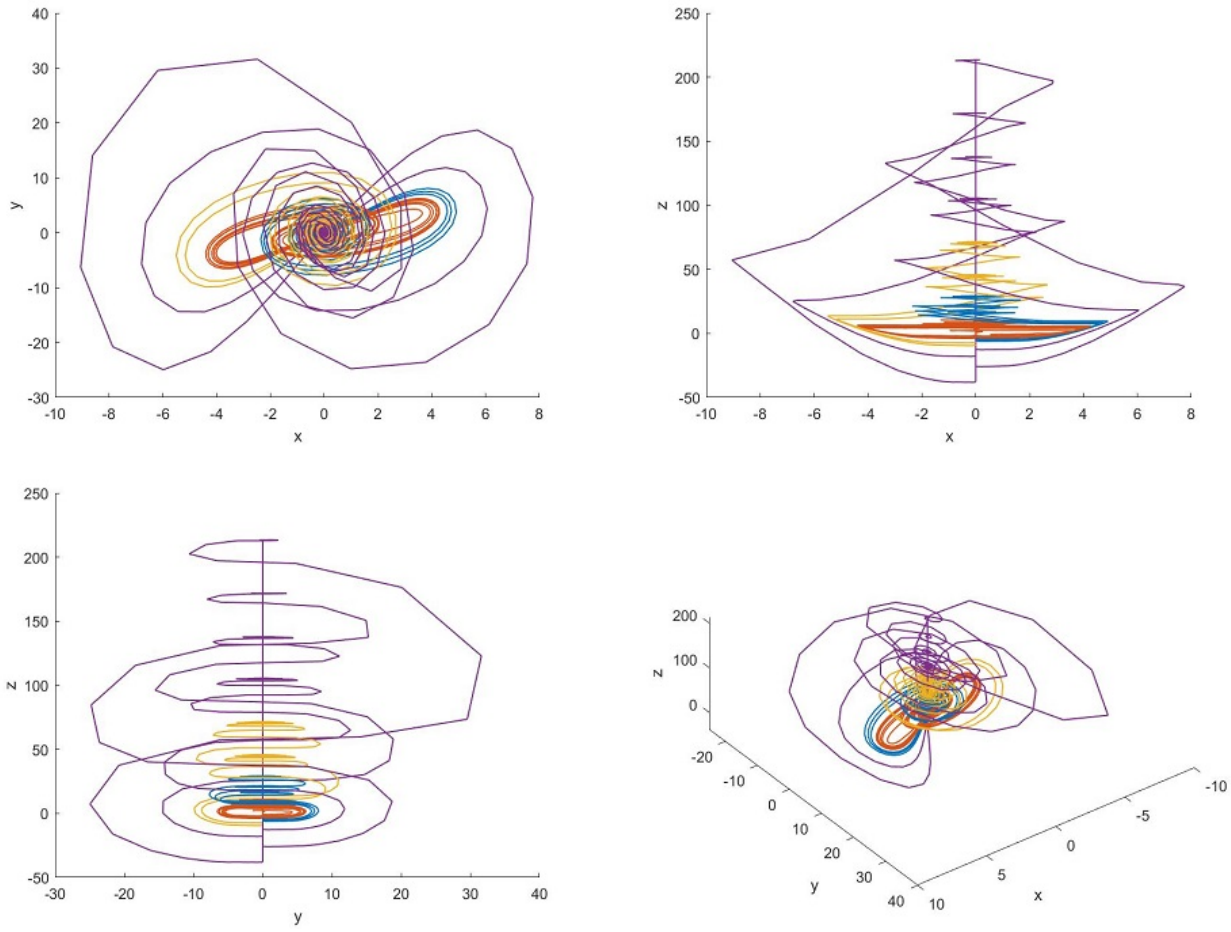

Figure 15. Numerical simulation for multi-wing chaotic system for $\rho = 0.13, \alpha = 0.99, \beta = 0.99$.

We now consider the chaotic problem with fractal-fractional derivative with power-law kernel

$$\begin{cases} {}_0^{FFP}D_t^{\alpha,\beta} X\left(t\right) = F\left(t, X\right), \\ X\left(t\right) = X_0, \text{ if } t = 0 \end{cases}$$

Using the modified parametrized method for associated derivative presented earlier, we can have the following

numerical scheme:

$$\left\{\begin{array}{l} X\left(t_{k+1}\right)=\frac{\beta}{\Gamma(\alpha)}\left\{\begin{array}{l}\left(1-\frac{1}{2\rho}\right)F\left(t_k,X_k\right)\\ +\frac{1}{2\rho}F\left(t_{k+1},\widetilde{X}_{k+1}\right)\end{array}\right\}\\ \times t_{k+1}^{\alpha+\beta-1}\left\{B\left(1,\beta,\alpha\right)-B\left(\frac{t_k}{t_{k+1}},\beta,\alpha\right)\right\}\\ +\frac{\alpha\beta}{\Gamma(a)}\sum_{j=0}^{k}\left\{\begin{array}{l}\left(1-\frac{1}{2\rho}\right)F\left(t_j,X_j\right)\\ +\frac{1}{2\rho}F\left(t_{j+1},X_{j+1}\right)\end{array}\right\}\\ \times t_{k+1}^{\alpha+\beta-1}\left\{B\left(\frac{t_{j+1}}{t_{k+1}},\beta,\alpha\right)-B\left(\frac{t_j}{t_{k+1}},\beta,\alpha\right)\right\}\end{array}\right. \tag{198}$$

where

$$\begin{aligned}\widetilde{X}_{k+1} &= \frac{\beta}{\Gamma\left(\alpha\right)}\sum_{j=0}^{k}t_{k+1}^{\alpha+\beta-1}F\left(t_j,X_j\right)\\ &\quad\times\left\{B\left(\frac{t_{j+1}}{t_{k+1}},\beta,\alpha\right)-B\left(\frac{t_j}{t_{k+1}},\beta,\alpha\right)\right\}.\end{aligned} \tag{199}$$

In Figure 16, we depict the numerical simulations using same parameters and initial data.

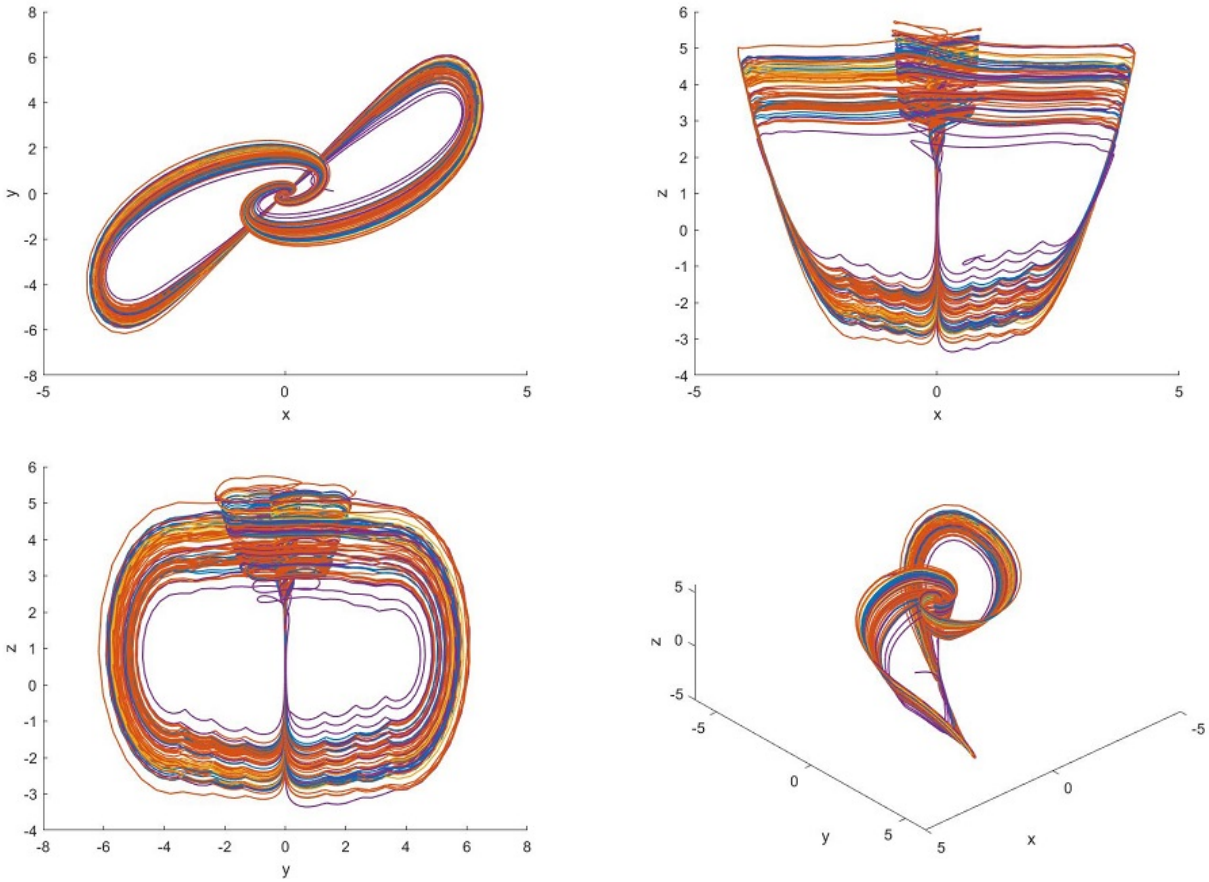

Figure 16. Numerical simulation for multi-wing chaotic system for $\rho=0.5,\alpha=0.99,\beta=0.99$.

Example 2. We consider Chen chaotic problem with time-delay term given in [14]

$$\left\{ \begin{array}{c} x\prime (t) = a\left(y\left(t-\tau \right) - x\left(t \right) \right) \\ y\prime (t) = (c-a)\, x\left(t \right) - x\left(t-\tau \right) z\left(t-\tau \right) + cy\left(t-\tau \right) \\ z\prime (t) = x\left(t-\tau \right) y\left(t-\tau \right) - bz\left(t \right) \end{array} \right. , \text{ if } t\in [0,T], \tag{200}$$

$$x\left(t \right) = \widetilde{x}\left(t \right), y\left(t \right) = \widetilde{y}\left(t \right), z\left(t \right) = \widetilde{z}\left(t \right), \text{ if } t\in [-\tau ,0]$$

We take the parameters

$$a = 34, b = 3, c = 27. \tag{201}$$

and initial conditions

$$x\left(0 \right) = 0.2, y\left(0 \right) = 0, z\left(0 \right) = 0.5. \tag{202}$$

$$\left\{ \begin{array}{l} X\prime (t) = F\left(t, X \right), \text{if } t\in [0,T] \\ X\left(t \right) = \widetilde{X}\left(t \right), \text{ if } t\in [-\tau ,0] \end{array} \right.$$

To construct the numerical scheme, we first consider uniform grid

$$\{ t_k = kh : k = -n, -n+1, ..., -1, 0, 1, ..., N\}$$

such that

$$T = hN, \tau = nh.$$

We know that

$$x\left(t_k - \tau \right) = x\left(kh - nh \right) = x\left(t_{k-n} \right), k = 0, 1, ..., N.$$

For simplicity, we take

$$\begin{array}{rcl} X\left(t \right) & = & \left[\begin{array}{c} x\left(t \right) \\ y\left(t \right) \\ z\left(t \right) \end{array} \right], \\ F\left(t, X\left(t \right), X\left(t-\tau \right) \right) & = & \left[\begin{array}{c} a\left(y\left(t-\tau \right) - x\left(t \right) \right) \\ (c-a)\, x\left(t \right) - x\left(t-\tau \right) z\left(t-\tau \right) + cy\left(t-\tau \right) \\ x\left(t-\tau \right) y\left(t-\tau \right) - bz\left(t \right) \end{array} \right]. \end{array} \tag{203}$$

By following the procedure of the modified parametrized method for classical case, we obtain the numerical scheme for the considered chaotic problem as follows:

$$X_{k+1} = X_k + \left(1 - \frac{1}{2\rho} \right) hF\left(t_k, X_k, X_{k-n} \right) + \frac{h}{2\rho} F\left(t_{k+1}, X_{k+1}, X_{k-n+1} \right)$$

where

$$\widetilde{X}_{k+1} = X_k + hF\left(t_k, X_k, X_{k-n} \right).$$

For classical case, the numerical simulations for Chen chaotic system with delay is presented in Figure 17.

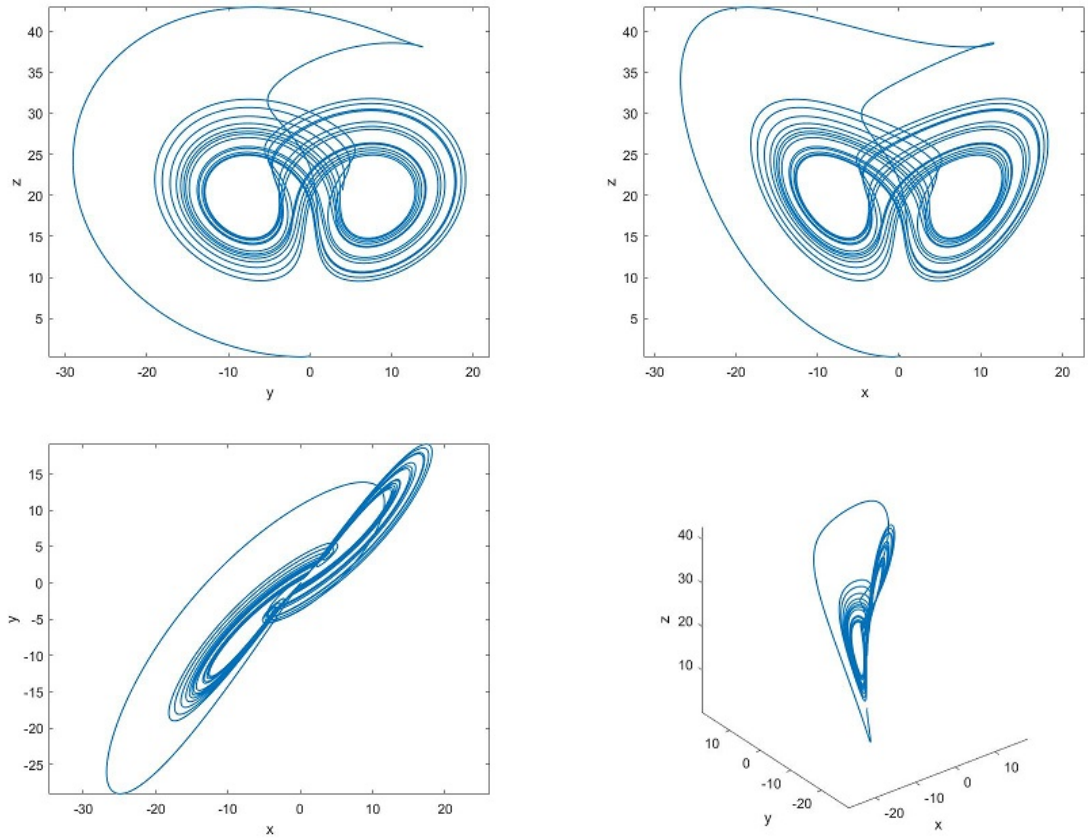

Figure 17. Numerical simulation for Chen chaotic system with delay term for $\rho = 1, \tau = 0.009$.

For Caputo case, we depict the numerical simulations for $\alpha = 0.95$ in Figure 18.

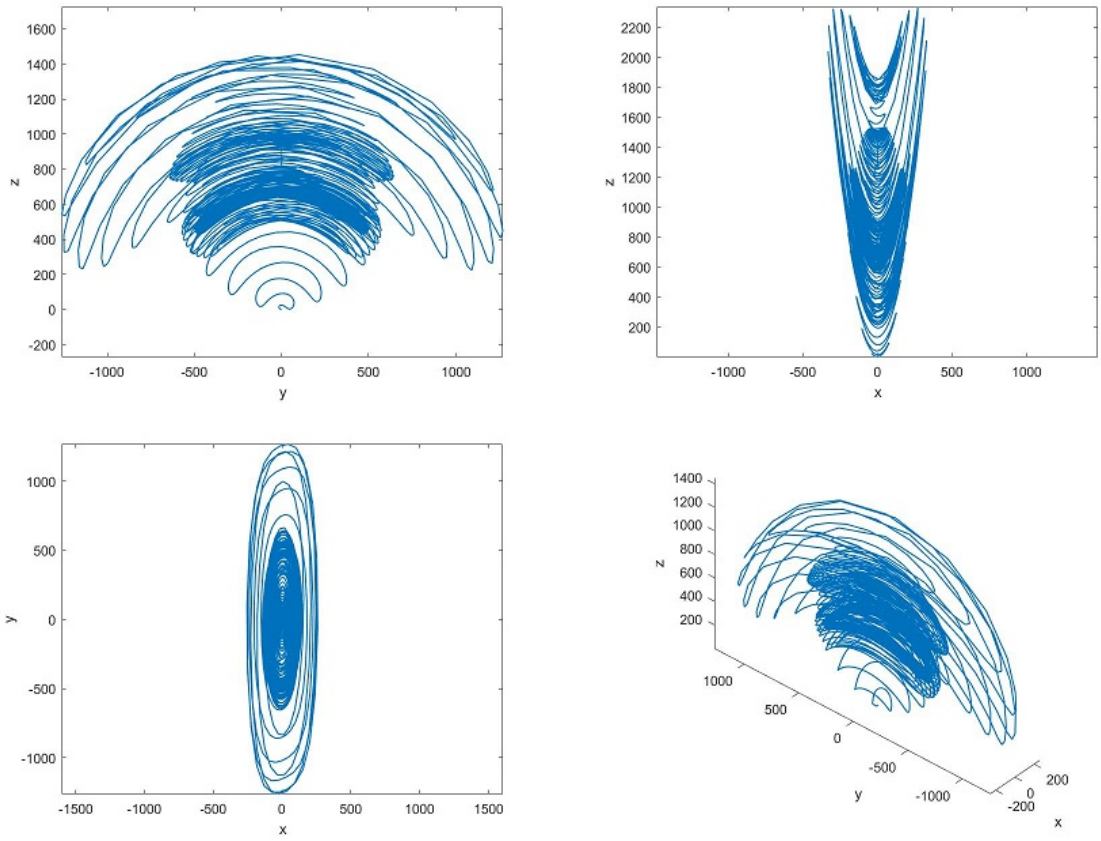

Figure 18. Numerical simulation for Chen chaotic system with delay term for $\rho = 1, \tau = 9$.

15 Conclusion

Although it is not often used, the parametrized approximation looks to be a very significant approximation that results in precise numerical techniques for solving nonlinear ordinary differential equations. For instance, the Heun's technique is recovered when the parametrized value is 1, the midway scheme is recovered when the value is 0.5, and the Ralston's method is recovered when the parametrized value is 2/3. We have found that the approach loses accuracy when the value of this parameter is less than 0.5. We changed the approach for conventional ordinary differential equations to address this issue, and the resulting system is accurate even when the parameter is smaller than 0.5. This allows us to derive a numerical technique for ordinary differential equations that are fractal, fractional, or fractal-fractional. We demonstrate the condition under which these nonlocal nonlinear ordinary differential equations admit a unique and precise solution using the Carathéodory requirements for existence and uniqueness of exact solution. We provided a theoretical examination of the obtained numerical schemes' stability and consistency. We provide some numerical simulations and instructive examples.

Conflict of interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

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