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# SINGULAR QUASILINEAR PROBLEMS WITH QUADRATIC GROWTH IN THE GRADIENT

BOUSSAD HAMOUR

In this paper we consider the problem

$$\begin{cases} u \in H_0^1(\Omega), \\ -\operatorname{div}(A(x)Du) = H(x, u, Du) + \frac{a_0(x)}{|u|^\theta} + \chi_{\{u \neq 0\}} f(x) \quad \text{in } \mathcal{D}'(\Omega), \end{cases}$$

where  $\Omega$  is an open bounded set of  $\mathbb{R}^N$  ( $N \geq 3$ ),  $A(x)$  is a coercive matrix with coefficients in  $L^\infty(\Omega)$ ,  $H(x, s, \xi)$  is a Carathéodory function which satisfies for a given  $\gamma > 0$  and some  $c_0 \geq 0$

$$\begin{aligned} -c_0 A(x) \xi \xi &\leq H(x, s, \xi) \operatorname{sign}(s) \leq \gamma A(x) \xi \xi \\ \text{a.e. } x \in \Omega, \quad \forall s \in \mathbb{R}, \quad \forall \xi \in \mathbb{R}^N. \end{aligned}$$

The nonnegative term  $a_0$  belongs to  $L^{N/2}(\Omega)$ ,  $\chi_{\{u \neq 0\}}$  is characteristic function,  $f$  belongs to  $L^{N/2}(\Omega)$  and  $0 < \theta < 1$ . For  $f$  and  $a_0$  sufficiently small (and more precisely when  $f$  and  $a_0$  satisfy the smallness condition (2.11)), we prove the existence of at least one solution  $u$  such that  $e^{\delta|u|} - 1$  belongs to  $H_0^1(\Omega)$  for some  $\delta \geq \gamma$ . Some a priori estimates are obtained.

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## 1. Introduction

In this paper, we consider the quasilinear problem

$$\begin{cases} u \in H_0^1(\Omega), \\ -\operatorname{div}(A(x)Du) = H(x, u, Du) + \frac{a_0(x)}{|u|^\theta} + \chi_{\{u \neq 0\}} f(x) \quad \text{in } \mathcal{D}'(\Omega), \end{cases} \quad (1.1)$$

where  $\Omega$  is a bounded open set of  $\mathbb{R}^N$ ,  $N \geq 3$ ,  $A(x)$  is a coercive matrix with bounded measurable coefficients. We assume that  $a_0 \in L^{N/2}(\Omega)$ ,  $a_0 \geq 0$ ,  $0 < \theta \leq 1$ ,  $\chi_{\{u \neq 0\}}$  is characteristic function,  $f \in L^{N/2}(\Omega)$  and  $H(x, u, Du)$  is a Carathéodory function with quadratic growth in  $Du$ , more precisely

$$|H(x, s, \xi)| \leq c|\xi|^2,$$

for some positive constant  $c$ .

Under suitable smallness conditions on  $\|a_0\|_{N/2}$  and  $\|f\|_{N/2}$  we prove the existence of solution  $u$  of (2.1) which satisfies a regularity in the following sense. If we define  $w$  by

$$w = \delta^{-1}(e^{\delta|u|} - 1)\operatorname{sign}(u),$$

then  $w$  belongs to  $H_0^1(\Omega)$  for every  $\delta$  in a certain interval  $(\gamma, \delta_0)$  which depends on  $a_0$ ,  $f$ , the bound of  $H$  and the coercivity of the matrix  $A$ .

Compared to the results obtained in the latest papers, we prove in the present paper, as said above, the existence of (only) one solution of (2.1) in the case (2.5) (i.e.  $a_0 \geq 0$ ) when  $a_0$  and  $f$  satisfy the smallness condition (2.11), but our result is obtained in the general case of a nonlinearity  $H(x, s, \xi)$  which satisfies only (2.3) with  $f \in L^{N/2}(\Omega)$  and with  $a_0$  to  $L^{N/2}(\Omega)$ .

We first review some recent results. The problem

$$\begin{cases} u \in H_0^1(\Omega), \\ -\operatorname{div}(A(x)Du) = H(x, u, Du) + a_0(x)u + f(x) \quad \text{in } \mathcal{D}'(\Omega). \end{cases} \quad (1.2)$$

has been extensively studied by many authors, among which we will quote the works in a serie of papers [8], [9], [10] and [11], investigated the case where

$$a_0(x) \leq -\alpha_0 < 0. \quad (1.3)$$

Considering general nonlinear monotone operators, they proved an existence of at least one solution when  $a_0$  satisfies (1.3), and when  $f$  belongs to  $L^q(\Omega)$  ( $q > \frac{N}{2}$ ). The solution is proven to verify some a priori estimates. The uniqueness of such a solution has been proved in [4] and [5] under some further structure assumptions.

The case where

$$a_0 = 0 \quad (1.4)$$

was considered in [2], [31], [20] and [23]. Among these papers the authors have considered nonlinear monotone operators and proved that if  $a_0$  satisfies (1.4) and  $f$  belongs to  $L^q(\Omega)$  ( $q > \frac{N}{2}$ ) with  $\|f\|_{L^q(\Omega)}$  sufficiently small then there exists at least one solution of (1.1) in  $L^\infty(\Omega)$  and satisfies a priori estimates. The case where  $a_0$  satisfies (1.4) and  $f$  only belongs to  $L^{N/2}(\Omega)$  for  $N \geq 3$  (and no longer to  $L^q(\Omega)$  ( $q > \frac{N}{2}$ )) was studied in [17] (and in [18] in the nonlinear monotone case), these authors proved that when  $\|f\|_{L^{N/2}(\Omega)}$  is sufficiently small there exists at least one solution of (1.2) such that  $e^{\delta|u|} - 1 \in H_0^1(\Omega)$  for some  $\delta > \gamma$  and satisfies an a priori estimates. Similar results were obtained in the case where  $f \in L^{N/2}(\Omega)$  in [16] for possibly unbounded domains when  $a_0$  satisfies (1.3); in this case no smallness condition is required on  $f$ , however in [19] the authors discussed (also in the case of nonlinear monotone operators) when  $a_0$  satisfies  $a_0 \leq 0$  and  $f$  belongs to the Lorentz space  $L^{N/2,\infty}(\Omega)$ ; in this case two smallness conditions should be fulfilled. Finally the case where  $a_0$  satisfies  $a_0 \leq 0$ , let us quote the paper [33] the author investigated the asymptotic behaviour of the solution  $u$  of (1.2) when  $a_0$  is a strictly positive constant sufficiently small, and proves that an ergodic constant appears at the limit  $a_0 = 0$ . Let us also mention the case where the nonlinearity  $H(x, s, \xi)$  has the “good sign property”, namely

$$-H(x, s, \xi) \operatorname{sign}(s) \geq 0. \quad (1.5)$$

The case where  $a_0 \leq 0$  and  $f$  belongs to  $H^{-1}(\Omega)$  was considered in [6] and [7], the authors proved the existence of at least one solution of (1.2) which belongs to  $H_0^1(\Omega)$ .

The case where

$$a_0 \geq 0, \quad a_0 \neq 0 \quad (1.6)$$

was considered in [3], [24] (and in [26] in the nonlinear monotone case), [25], [27] and [28]. The first attempt to study equations with a gradient term having quadratic growth, was carried out in [24] (see also [26] for extensions), the authors prove the existence of at least one solution of (1.2) when  $a_0$  satisfies (1.6), and  $f$  belongs to  $L^{N/2}(\Omega)$ , with  $\|f\|_{L^{N/2}(\Omega)}$  and  $\|a_0\|_{L^q(\Omega)}$  sufficiently small.

In [27], the authors proved a similar result when  $a_0$  satisfies (1.6) and  $f$  belongs to  $L^q(\Omega)$  ( $q > \frac{N}{2}$ ) with  $\|f\|_{L^q(\Omega)}$  and  $\|a_0\|_{L^q(\Omega)}$  sufficiently small. Moreover, they proved the existence of at least two solutions of (1.2) (which moreover belong to  $L^\infty(\Omega)$ , when  $A(x) = Id$ ,  $H(x) = \mu|\xi|^2$ ,  $\mu > 0$ ,  $f \in L^q(\Omega)$  ( $q > \frac{N}{2}$ ),  $f \geq 0$ , and  $a_0 \in L^q(\Omega)$  ( $q > \frac{N}{2}$ ) with  $\|f\|_{L^q(\Omega)}$  and  $\|a_0\|_{L^q(\Omega)}$  sufficiently small).

In [3], the authors proved the existence of a continuum  $(u, \lambda)$  of solutions (with  $u$  which belongs to  $L^\infty(\Omega)$ ) when  $A(x) = Id$ ,  $H(x, s, \xi) = \mu(x)|\xi|^2$ , with

$\mu \in L^\infty(\Omega)$ ,  $\mu(x) \geq \mu > 0$ ,  $f \in L^q(\Omega)$  ( $q > \frac{N}{2}$ ),  $f \geq 0$ ,  $f \neq 0$  and  $a_0(x) = \lambda a_0^*(x)$  with  $a_0^* \in L^q(\Omega)$ ,  $a_0^* \geq 0$  and  $a_0^* \neq 0$ . In addition, under further conditions on  $f$ , these authors proved that this continuum is defined for  $\lambda \in ]-\infty, \lambda_0]$  with  $\lambda_0 > 0$  and that there are at least two nonnegative solutions of (1.2) when  $\lambda > 0$  is sufficiently small.

In the singular case, the problem (1.1) has been studied in many papers in the case where  $A(x) = Id$ ,  $H(x, \cdot, \cdot) = 0$ ,  $f = 0$  and  $a_0$  is smooth. Among these papers is a serie of papers [1], [12], [13], [14], [15] and [32].

In [21] and [22], the authors proved the existence of at least one nonnegative solution and a stability result for the following problem

$$\begin{cases} -\operatorname{div}(A(x)Du) = f(x)g(u) + l(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where  $A(x) \in L^\infty(\Omega)^{N \times N}$  is coercive matrix,  $g : [0, +\infty) \rightarrow [0, +\infty)$  is continuous and  $0 \leq g(s) \leq \frac{1}{s^\theta} + 1$ ,  $\forall s > 0$ ,  $0 < \theta \leq 1$ ; and  $f, l \in L^r(\Omega)$  where  $r$  satisfies some conditions. In [13], the authors proved the existence, regularity and nonexistence results for problems whose model is

$$-\Delta u = \frac{f(x)}{u^\theta} \quad \text{in } \Omega,$$

with  $u = 0$  on  $\partial\Omega$ ,  $\Omega$  is bounded open of  $\mathbb{R}^N$ ,  $\theta > 0$  and  $f$  is nonegative function on  $\Omega$  and belongs to some Lebesgue spaces. For this, they have introduced an approximate problem by treating the singular term  $\frac{1}{u^\theta}$  and construct an increasing sequence  $(u_n)_{n \in \mathbb{N}}$  of solutions to nonsingular problem

$$\begin{cases} -\operatorname{div}(A(x)Du_n) = \frac{f_n(x)}{(u_n + \frac{1}{n})^\theta} & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases}$$

where  $f_n = \min(f(x), n)$ . This sequence satisfies, for any  $\omega \subset \subset \Omega$ , and

$$u_n \geq u_{n-1} \geq \dots \geq u_1 \geq C_\omega, \quad \forall x \in \omega.$$

The authors discussed in [1] the solution of the elliptic problem, with a gradient term and a singular nonlinearity

$$\begin{cases} -\Delta u = |\nabla u|^q + \frac{f}{g(u)} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where  $\Omega \subset \mathbb{R}^N$  is a bounded regular domain,  $g : \mathbb{R}_+ \rightarrow \mathbb{R}$  is a continuous increasing function with additional hypotheses given,  $1 < q \leq 2$  and  $f$  is a measurable nonnegative function and obtained optimal conditions on  $g, q$  which allow to get the existence positive solution for the largest possible class of datum  $f$ .

The plan of this paper is as follows: The precise statement of our result is given in Section 1 (Theorem 2.1), as well as the precise assumptions under which we are able to prove it, these conditions in particular include the smallness condition (2.11). Our method for proving Theorem 2.1 is based on an approximate result and by applying Schauder's fixed point theorem in a classical way. We will proceed by approximating the singular term  $\frac{1}{|u|^\theta}$  and we will get some a priori estimates on the solutions of equivalent problem (4.2) (see also Proposition 4.1). The equivalence result is given in [18].

More precisely, the proof of Theorem 2.1 consists in carrying out a change of unknown function  $w = \delta^{-1}(e^{\delta|u|} - 1) \text{sign}(u)$ , by transforming equation (3.4) into equation (3.8) in order to obtain the a priori estimate and the strong convergence of  $w_n$ . Another difficulty in the proof is to obtain the a priori estimate of the singular term in the set  $\{u = 0\}$ . For that, we use the method introduced in [21] and [22].

## 2. Main result

In this paper we consider the following quasilinear problem

$$\begin{cases} u \in H_0^1(\Omega), \\ -\text{div}(A(x)Du) = H(x, u, Du) + \frac{a_0(x)}{|u|^\theta} + \chi_{\{u \neq 0\}} f(x) \quad \text{in } \mathcal{D}'(\Omega), \end{cases} \quad (2.1)$$

where  $\Omega$  is an open bounded set of  $\mathbb{R}^N$ ,  $N \geq 3$ ,  $A$  is a coercive matrix with bounded measurable coefficients, i.e.

$$\begin{cases} A \in (L^\infty(\Omega))^{N \times N}, \\ \exists \alpha > 0, \quad A(x) \xi \xi \geq \alpha |\xi|^2 \quad \text{a.e. } x \in \Omega, \quad \forall \xi \in \mathbb{R}^N, \end{cases} \quad (2.2)$$

the nonlinearity  $H(x, s, \xi)$  is a Carathéodory function with quadratic growth in  $\xi$ , and more precisely satisfies

$$\begin{cases} -c_0 A(x) \xi \xi \leq H(x, s, \xi) \text{sign}(s) \leq \gamma A(x) \xi \xi, \\ \text{a.e. } x \in \Omega, \quad \forall s \in \mathbb{R}, \quad \forall \xi \in \mathbb{R}^N, \\ \text{where } \gamma > 0 \text{ and } c_0 \geq 0, \end{cases} \quad (2.3)$$

the function  $\text{sign} : \mathbb{R} \rightarrow \mathbb{R}$  is defined by

$$\text{sign}(s) = \begin{cases} +1 & \text{if } s > 0, \\ 0 & \text{if } s = 0, \\ -1 & \text{if } s < 0, \end{cases} \quad (2.4)$$

the coefficient  $a_0$  satisfies

$$a_0 \in L^{N/2}(\Omega), \quad a_0 \geq 0, \quad a_0 \neq 0, \quad (2.5)$$

the exponent  $\theta$  satisfies

$$0 < \theta \leq 1, \quad (2.6)$$

the characteristic function is defined by

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A, \end{cases} \quad (2.7)$$

and finally

$$f \in L^{N/2}(\Omega). \quad (2.8)$$

Since  $N \geq 3$ , let  $2^*$  be the Sobolev's exponent defined by

$$\frac{1}{2^*} = \frac{1}{2} - \frac{1}{N},$$

and let  $C_N$  be the Sobolev's constant defined as the best constant such that

$$\|\varphi\|_{2^*} \leq C_N \|D\varphi\|_2, \quad \forall \varphi \in H_0^1(\Omega). \quad (2.9)$$

Since  $\Omega$  is bounded, we equip the space  $H_0^1(\Omega)$  with the norm

$$\|u\|_{H_0^1(\Omega)} = \|Du\|_{L^2(\Omega)^N}. \quad (2.10)$$

We finally assume that  $f$  and  $a_0$  are sufficiently small (see Remark 2.3), and more precisely that

$$\|f\|_{N/2} \leq \frac{\alpha}{\gamma C_N^2} - \frac{\gamma^\theta}{\log^\theta(1+\gamma)} \|a_0\|_{N/2}, \quad (2.11)$$

Our main result is the following Theorem.

**Theorem 2.1.** Assume that (2.2), (2.3), (2.5), (2.6) and (2.8) hold true. Assume moreover that the smallness condition (2.11) hold true.

Then there exists at least one solution  $u$  of (2.1), which further satisfies

$$\begin{cases} (e^{\delta|u|} - 1) \in H_0^1(\Omega), \quad \forall \delta \geq \gamma \text{ such that} \\ \|f\|_{N/2} \leq \frac{\alpha}{\delta C_N^2} - \frac{\delta^\theta}{\log^\theta(1 + \delta)} \|a_0\|_{N/2}. \end{cases} \quad (2.12)$$

**Remark 2.2.** In the case where the function  $H(x, s, \xi) = H(x, \xi)$  does not depend on  $s$ , assumption (2.3) is satisfied if and only if

$$|H(x, \xi)| \leq c A(x) \xi \xi,$$

for some  $c > 0$ .

Since  $A$  is a coercive matrix with bounded entries, the last condition is satisfied if and only if

$$|H(x, \xi)| \leq c |\xi|^2,$$

for some  $c > 0$ , which means that  $H(x, \xi)$  has a quadratic growth with respect to  $\xi$ .

When  $\gamma = 0$  in (2.3), the nonlinearity function  $H(x, \xi)$  satisfies a sign condition and existence result can be proved for every  $f \in H^{-1}(\Omega)$ .

**Remark 2.3.** In this Remark, we consider that the open set  $\Omega$ , the matrix  $A$  and the function  $H$  are fixed and we consider the functions  $a_0$  and  $f$  as parameters.

Our set of assumptions on these parameters is made of the smallness condition (2.11).

Indeed, if, for example,  $a_0$  is sufficiently small such that it satisfies

$$\frac{\alpha}{\gamma C_N^2} - \frac{\gamma^\theta}{\log^\theta(1 + \gamma)} \|a_0\|_{N/2} > 0,$$

then the smallness condition (2.11) is satisfied if  $\|f\|_{N/2}$  (and therefore  $\|f\|_{H^{-1}(\Omega)}$ , since  $L^{N/2}(\Omega) \subset H^{-1}(\Omega)$  is sufficiently small).

Similarly, if, for example,  $f$  is sufficiently small such that

$$\|f\|_{N/2} \leq \frac{\alpha}{\gamma C_N^2},$$

then the smallness condition (2.11) is always satisfied if  $\|a_0\|_{N/2}$  is sufficiently small.

**Remark 2.4.** The smallness condition (2.11) is in some sense sharp, implies that  $\delta$  is bounded when the function  $a_0$  satisfies the assumption (2.12), we have

$$\frac{\alpha}{\delta C_N^2} - \frac{\delta^\theta}{\log^\theta(1 + \delta)} \|a_0\|_{N/2} \geq 0,$$



this implies

$$\frac{\alpha}{\delta C_N^2} \geq \frac{\delta^\theta}{\log^\theta(1+\delta)} \|a_0\|_{N/2} \geq a_0\|_{N/2}.$$

Finally

$$\gamma \leq \delta \leq \frac{\alpha}{C_N^2 \|a_0\|_{N/2}}.$$

Our proof of Theorem 2.1 is based on an approximate result which will be stated and proved in Section 3.

### 3. Proof of Theorem 2.1

The proof of Theorem 2.1 will be made in six steps.

#### Step 1: Approximation and change of unknown function

For  $n \in \mathbb{N}$ , consider two sequences  $a_n$  and  $f_n$  such that

$$a_n(x) = \begin{cases} a_0(x) & 0 \leq a_0 \leq n \\ n & \text{otherwise,} \end{cases} \quad (3.1)$$

and

$$f_n(x) = T_n f(x), \quad (3.2)$$

where  $T_n$  is defined in (3.26).

For  $n \in \mathbb{N}$ , define  $H_n(x, s, \xi)$  by

$$H_n(x, s, \xi) = \frac{H(x, s, \xi)}{1 + \frac{1}{n} |H(x, s, \xi)|}. \quad (3.3)$$

Observe that  $H_n(x, s, \xi)$  satisfies  $|H_n(x, s, \xi)| \leq H(x, s, \xi)$  as well as (2.3).

Since  $a_n(x)$ ,  $|f_n(x)|$  and  $H_n(x, s, \xi)$  are bounded, a classical result of J. Leray and J.-L. Lions [29, 30] asserts that the following approximate problem (3.4) has at least one solution.

$$\begin{cases} u_n \in H_0^1(\Omega), \\ -\operatorname{div}(A(x)Du_n) = H_n(x, u_n, Du_n) + \frac{a_n(x)}{(|u_n| + \frac{1}{n})^\theta} + \chi_{\{u_n \neq 0\}} f_n(x) \quad \text{in } \mathcal{D}'(\Omega). \end{cases} \quad (3.4)$$

Observe that  $u_n$  belongs to  $L^\infty(\Omega)$  for each  $n$  given since  $a_n(x) \in L^\infty(\Omega)$ ,  $f_n(x) \in L^\infty(\Omega)$ , and  $H_n(x, u_n, Du_n) \in L^\infty(\Omega)$ .

Let  $\delta > 0$  be fixed satisfies

$$\begin{cases} \gamma \leq \delta & \text{such that} \\ \|f\|_{N/2} \leq \frac{\alpha}{\delta C_N^2} - \frac{\delta^\theta}{\log^\theta(1+\delta)} \|a_0\|_{N/2}. \end{cases} \quad (3.5)$$

Define

$$w_n = \phi(u_n), \quad (3.6)$$

where

$$\varphi(s) = \delta^{-1}(e^{\delta|s|} - 1) \operatorname{sign}(s), \quad \forall s \in \mathbb{R}.$$

Observe that  $\varphi \in C^1(\mathbb{R})$  with  $\varphi(0) = 0$ , we have at least formally

$$\begin{cases} w_n \in H_0^1(\Omega) \cap L^\infty(\Omega), \\ e^{\delta u_n} = 1 + \delta|w_n|, \quad Dw_n = e^{\delta|u_n|} Du_n, \quad \operatorname{sign}(u_n) = \operatorname{sign}(w_n), \end{cases}$$

and

$$\begin{cases} -\operatorname{div}(A(x) Dw_n) \\ = -\delta e^{\delta|u_n|} \operatorname{sign}(u_n) A(x) Du_n Du_n \\ -e^{\delta|u_n|} \operatorname{div}(A(x) Du_n) \operatorname{sign}(u_n) \quad \text{in } \mathcal{D}'(\Omega), \end{cases} \quad (3.7)$$

where  $-e^{\delta|u_n|} \operatorname{div}(A(x) Du_n)$  is the distribution defined by

$$\mathcal{D}'(\Omega) \left\langle -e^{\delta|u_n|} \operatorname{div}(A(x) Du_n), \varphi \right\rangle_{C_0^\infty(\Omega)} = \int_\Omega A(x) Du_n D(\varphi e^{\delta|u_n|}) dx,$$

for any  $\varphi \in C_0^\infty(\Omega)$  or  $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$ .

Since  $e^{\delta|u_n|} = 1 + \delta|w_n|$ , we deduce that  $w_n$  is, at least formally, a solution (see Proposition 4.1) of the problem

$$\begin{cases} -\operatorname{div}(A(x) Dw_n) = -K_\delta(x, w_n, Dw_n) \operatorname{sign}(w_n) \\ + (1 + \delta|w_n|) \chi_{\{w_n \neq 0\}} f_n + \frac{1 + \delta|w_n|}{(\delta^{-1} \log(1 + \delta|w_n|) + \frac{1}{n})^\theta} a_n \quad \text{in } \mathcal{D}'(\Omega), \\ w_n = 0, \quad \text{on } \partial\Omega, \end{cases} \quad (3.8)$$

where the function  $K_\delta : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}$  is defined by the following formulas

$$K_\delta(x, s, \xi) = \frac{1 - \theta_n(x)}{1 + \delta|s|} \delta A(x) \xi \xi, \quad (3.9)$$

with

$$\theta_n(x) = \begin{cases} \frac{H_n(x, u_n, Du_n)}{\delta \operatorname{sign}(u_n) A(x) Dw_n Dw_n} & \text{if } \operatorname{sign}(u_n) A(x) Du_n Du_n \neq 0, \\ 0 & \text{otherwise,} \end{cases} \quad (3.10)$$

from the condition (2.3) on  $H$  and  $0 < \gamma \leq \delta$ , we have

$$-\frac{c_0}{\delta} < \theta_n \leq 1, \quad \text{a.e. } x \in \Omega. \quad (3.11)$$

When  $\gamma \leq \delta$ , this computation in particular implies that

$$K_\delta(x, s, \xi) \geq 0 \quad \text{a.e. } x \in \Omega, \quad \forall s \in \mathbb{R}, \quad \forall \xi \in \mathbb{R}^N. \quad (3.12)$$

### Step 2: *A priori estimate*

Since that the right hand side of (3.8) belongs to  $L^1(\Omega)$ , we can use  $w_n$  which belongs to  $H_0^1(\Omega) \cap L^\infty(\Omega)$ , as a test function in (3.8).

Taking into the fact that  $K_\delta(x, s, \xi) \geq 0$  (see (3.12)),  $\chi_{\{u_n \neq 0\}} \leq 1$  and using Hölder's inequality with  $\frac{1}{2^\star} + \frac{1}{2^\star} + \frac{2}{N} = 1$ , we have

$$\left\{ \begin{aligned} & \int_{\Omega} A(x) Dw_n Dw_n dx \\ & \leq \int_{\Omega} (1 + \delta|w_n|) w_n f_n(x) dx \\ & + \int_{\Omega} \frac{(1 + \delta|w_n|) w_n}{(\delta^{-1} \log(1 + \delta|w_n|) + \frac{1}{n})^\theta} a_n(x) dx \end{aligned} \right. \quad (3.13)$$

By the coercivity condition (2.2), we get

$$\left\{ \begin{aligned} & \alpha \int_{\Omega} |Dw_n|^2 dx \leq \int_{\Omega} (1 + \delta|w_n|) w_n f_n(x) dx \\ & + \int_{\Omega} \frac{(1 + \delta|w_n|) w_n}{(\delta^{-1} \log(1 + \delta|w_n|) + \frac{1}{n})^\theta} a_n(x) dx. \end{aligned} \right. \quad (3.14)$$

Using the chains of Hölder's and Sobolev's inequalities (2.9) this implies that

$$\left\{ \begin{aligned} & \int_{\Omega} (1 + \delta |w_n|) w_n f_n(x) dx \\ &= \int_{\Omega} |w_n| f_n(x) dx + \delta \int_{\Omega} |w_n|^2 f_n(x) dx \\ &\leq \|f\|_{H^{-1}(\Omega)} \|Dw_n\|_2 + \delta C_N^2 \|f\|_{N/2} \|Dw_n\|_2^2. \end{aligned} \right. \quad (3.15)$$

Splitting  $\Omega$  into  $\Omega = \{|w_n| \leq 1\} \cup \{|w_n| > 1\}$  and writing the last term of the right-hand side of (3.13) as

$$\left\{ \begin{aligned} & \int_{\Omega} \frac{(1 + \delta |w_n|) w_n}{(\delta^{-1} \log(1 + \delta |w_n|) + \frac{1}{n})^{\theta}} a_n(x) dx = \\ & \int_{\{|w_n| \leq 1\}} \frac{(1 + \delta |w_n|) w_n}{(\delta^{-1} \log(1 + \delta |w_n|) + \frac{1}{n})^{\theta}} a_n(x) dx \\ & + \int_{\{|w_n| > 1\}} \frac{(1 + \delta |w_n|) w_n}{(\delta^{-1} \log(1 + \delta |w_n|) + \frac{1}{n})^{\theta}} a_n(x) dx. \end{aligned} \right. \quad (3.16)$$

Since  $\|a_n\|_{N/2}$  converges to  $\|a_0\|_{N/2}$ ,  $\|f_n\|_{N/2}$  to  $\|f\|_{N/2}$  and using that the function  $F(x) = \frac{x}{\log(1+x)}$  is increasing in  $\mathbb{R}_+^*$ , we have

$$\left\{ \begin{aligned} & \int_{\{|w_n| \leq 1\}} \frac{(1 + \delta |w_n|) w_n}{(\delta^{-1} \log(1 + \delta |w_n|) + \frac{1}{n})^{\theta}} a_n(x) dx \\ & \leq (1 + \delta) \int_{|w_n| \leq 1} a_n(x) |w_n|^{1-\theta} \left( \frac{\delta |w_n|}{\log(1 + \delta |w_n|)} \right)^{\theta} dx \\ & \leq \frac{(1 + \delta) \delta^{\theta}}{\log^{\theta}(1 + \delta)} \int_{|w_n| \leq 1} a_n(x) |w_n|^{1-\theta} dx \\ & \leq (1 + \delta) C_{\delta}(\theta) C_N^{1-\theta} |\Omega|^{\frac{1+\theta}{2^*}} \|a_0\|_{N/2} \|Dw_n\|_2^{1-\theta}, \end{aligned} \right. \quad (3.17)$$

where  $C_{\delta}(\theta) = \frac{\delta^{\theta}}{\log^{\theta}(1 + \delta)}$

and

$$\left\{ \begin{array}{l} \int_{|w_n| \geq 1} \frac{(1 + \delta |w_n|) w_n}{(\delta^{-1} \log(1 + \delta |w_n|) + \frac{1}{n})^\theta} a_n(x) dx \\ \leq C_\delta(\theta) \int_{|w_n| \geq 1} (1 + \delta |w_n|) |w_n| a_n(x) dx \\ \leq C_\delta(\theta) \left( C_N |\Omega|^{1/2^*} \|a_0\|_{N/2} \|Dw_n\|_2 + \delta C_N^2 \|a_0\|_{N/2} \|Dw_n\|_2^2 \right). \end{array} \right. \quad (3.18)$$

From (3.14), (3.15), (3.16), (3.17) and (3.18) we have

$$\left\{ \begin{array}{l} \alpha \|Dw_n\|_2^2 \leq (1 + \delta) C_\delta(\theta) C_N^{1-\theta} |\Omega|^{\frac{1+\theta}{2^*}} \|a_0\|_{N/2} \|Dw_n\|_2^{1-\theta} \\ \quad + C_\delta(\theta) C_N |\Omega|^{1/2^*} \|a_0\|_{N/2} \|Dw_n\|_2 \\ \quad + \delta C_\delta(\theta) C_N^2 \|a_0\|_{N/2} \|Dw_n\|_2^2 \\ \quad + \|f\|_{H^{-1}(\Omega)} \|Dw_n\|_2 + \delta C_N^2 \|f\|_{N/2} \|Dw_n\|_2^2 \quad \text{if } w_n \neq 0, \end{array} \right. \quad (3.19)$$

dividing by  $\|Dw_n\|_2^{1-\theta}$ , this implies that (note that the result remains true in the case where  $w_n = 0$ )

$$\left\{ \begin{array}{l} (\alpha - \delta C_\delta(\theta) C_N^2 \|a_0\|_{N/2} - \delta C_N^2 \|f\|_{N/2}) \|Dw_n\|_2^{1+\theta} \\ \leq \left( C_\delta(\theta) C_N |\Omega|^{1/2^*} \|a_0\|_{N/2} + \|f\|_{H^{-1}(\Omega)} \right) \|Dw_n\|_2^\theta \\ \quad + (1 + \delta) C_\delta(\theta) C_N^{1-\theta} |\Omega|^{\frac{1+\theta}{2^*}} \|a_0\|_{N/2}. \end{array} \right. \quad (3.20)$$

In view of the definition of (4.6) of the function  $\Phi_\delta$  (see also Figure 1), we have proved if  $w_n$  is any solution of (3.4), one has

$$\Phi_\delta(\|Dw_n\|_2) \leq 0, \quad \text{if } \gamma \leq \delta, \quad (3.21)$$

this implies that

$$\|Dw_n\| \leq Z_\delta \quad (\text{does not depend to } n). \quad (3.22)$$

where the constant  $Z_\delta > 0$  satisfies

$$\Phi_\delta(Z_\delta) = 0. \quad (3.23)$$

Since  $u_n = \delta^{-1}(\log(1 + \delta |w_n|)) \text{sign}(w_n)$ , and from (3.22) implies that

$$u_n \quad \text{is bounded in } H_0^1(\Omega). \quad (3.24)$$

**Step 3: Proof of regularity result**

Extracting a subsequence, still denoted  $n$ , we have, for some  $u \in H_0^1(\Omega)$  and  $w \in H_0^1(\Omega)$

$$\begin{aligned} u_n &\rightharpoonup u \quad \text{weakly in } H_0^1(\Omega), \quad \text{a.e. in } \Omega, \\ w_n &\rightharpoonup w \quad \text{weakly in } H_0^1(\Omega), \quad \text{a.e. in } \Omega, \end{aligned}$$

where

$$w = \varphi(u) = \delta^{-1}(e^{\delta|u|-1}) \operatorname{sign}(u).$$

Observe that  $u$  and  $w$  do not belong to  $L^\infty(\Omega)$  in general.

If we consider another  $\delta$ , say  $\delta'$ , which also satisfies

$$\gamma \leq \delta' \quad \text{such that} \quad \|f\|_{N/2} \leq \frac{\alpha}{\delta' C_N^2} - \left( \frac{\delta'}{\log(1 + \delta')} \right)^\theta \|a_0\|_{N/2}. \quad (3.25)$$

The above a priori estimate (3.22) again shows that  $w'_n$  defined by

$$w'_n = \delta'^{-1}(e^{\delta'|u_n|-1}) \operatorname{sign}(u_n),$$

is bounded in  $H_0^1(\Omega)$ , this proves that  $u$  is such that

$$(e^{\delta'|u|-1}) \operatorname{sign}(u) \in H_0^1(\Omega), \quad \forall \delta' \text{ such that } \gamma \leq \delta' \text{ satisfies (3.25),}$$

that is (2.12).

**Step 4: An estimate for  $\int_{|w_n|>k} |Dw_n|^2$**

Let us define, for  $k \geq 0$ , the fuction  $T_k : \mathbb{R} \rightarrow \mathbb{R}$  is the usual truncation at height  $k$  defined by

$$T_k(s) = \begin{cases} -k & \text{if } s \leq -k \\ s & \text{if } -k \leq s \leq +k \\ +k & \text{if } +k \leq s, \end{cases} \quad (3.26)$$

and we define  $G_k : \mathbb{R} \rightarrow \mathbb{R}$  as the remainder of the truncation at height  $k$ , namely

$$G_k(s) = s - T_k(s), \quad \forall s \in \mathbb{R}, \quad (3.27)$$

in other terms

$$G_k(s) = \begin{cases} s + k & \text{if } s \leq -k \\ 0 & \text{if } -k \leq s \leq +k \\ s - k & \text{if } s \geq +k, \end{cases} \quad (3.28)$$

Since  $G_k(w_n) \in H_0^1(\Omega)$ , the use of  $G_k(w_n)$  as test function in (3.8) is licit. This gives

$$\left\{ \begin{aligned} & \int_{\Omega} A(x) Dw_n DG_k(w_n) dx + \int_{\Omega} K_{\delta}(x, w_n, Dw_n) \text{sign}(w_n) G_k(w_n) dx \\ &= \int_{\Omega} \frac{(1 + \delta|w_n|) G_k(w_n)}{(\delta^{-1} \log(1 + \delta|w_n|) + \frac{1}{n})^{\theta}} a_n(x) dx \\ &+ \int_{\Omega} (1 + \delta|w_n|) G_k(w_n) \chi_{\{w_n \neq 0\}} f_n(x) dx. \end{aligned} \right. \quad (3.29)$$

Using the coercivity (2.2) of the matrix  $A$ , we have for the first term of (3.29)

$$\left\{ \begin{aligned} & \int_{\Omega} A(x) Dw_n DG_k(w_n) dx = \int_{\Omega} A(x) DG_k(w_n) DG_k(w_n) dx \\ & \geq \alpha \int_{\Omega} |DG_k(w_n)|^2 dx. \end{aligned} \right. \quad (3.30)$$

On the other hand, since

$$\text{sign}(s) G_k(s) \geq 0, \quad \forall s \in \mathbb{R}, \quad (3.31)$$

and since  $K_{\delta}(x, s, \xi) \geq 0$ , in view of (3.12), this implies

$$\int_{\Omega} K_{\delta}(x, w_n, Dw_n) \text{sign}(w_n) G_k(w_n) dx \geq 0, \quad (3.32)$$

Let  $k$  be fixed, we have

$$\frac{(1 + \delta|w_n|) G_k(w_n)}{(\delta^{-1} \log(1 + \delta|w_n|) + \frac{1}{n})^{\theta}} a_n \rightarrow \frac{(1 + \delta|w|) G_k(w)}{(\delta^{-1} \log(1 + \delta|w|))^{\theta}} a_0 \quad \text{a.e. in } \Omega,$$

and

$$(1 + \delta|w_n|) G_k(w_n) \chi_{\{w_n \neq 0\}} f_n \rightarrow (1 + \delta|w|) G_k(w) \chi_{\{w \neq 0\}} f \quad \text{a.e. in } \Omega.$$

On the other hand the following functions

$$\frac{(1 + \delta|w_n|) G_k(w_n)}{(\delta^{-1} \log(1 + \delta|w_n|) + \frac{1}{n})^{\theta}} a_n \quad \text{and} \quad (1 + \delta|w_n|) G_k(w) f_n \quad \text{are equiintegrable.}$$

Indeed, from (3.17) and (3.18), since  $a_n$  strongly converges in  $L^q(\Omega)$ , and for every

Borel set  $E \subset \Omega$ , we have

$$\left\{ \begin{aligned} & \int_E \frac{(1 + \delta|w_n|) G_k(w_n)}{(\delta^{-1} \log(1 + \delta|w_n|) + \frac{1}{n})^\theta} a_n dx \\ & \leq \int_E \frac{(1 + \delta|w_n|) |w_n|}{(\delta^{-1} \log(1 + \delta|w_n|) + \frac{1}{n})^\theta} a_n dx \\ & \leq (1 + \delta) C_\delta(\theta) C_N^{1-\theta} |\Omega|^{\left(\frac{1}{q} - \frac{1-\theta}{2^*}\right)} \|Dw_n\|_2^{1-\theta} \left( \int_E |a_n|^q dx \right)^{1/q} \\ & \quad + C_\delta(\theta) \left( C_N |\Omega|^{1/2^*} \|Dw_n\|_2 \left( \int_E |a_n|^{N/2} dx \right)^{2/N} \right. \\ & \quad \quad \left. + \delta C_N^2 \|Dw_n\|_2^2 \left( \int_E |a_n|^{N/2} dx \right)^{2/N} \right) \\ & \leq c \left( \int_E |a_n|^q dx \right)^{1/q} + c' \left( \int_E |a_n|^{N/2} dx \right)^{2/N} \end{aligned} \right. \quad (3.33)$$

Thus Vitali's Theorem implies that

$$\frac{(1 + \delta|w_n|) G_k(w_n)}{(\delta^{-1} \log(1 + \delta|w_n|) + \frac{1}{n})^\theta} a_n \rightarrow \frac{(1 + \delta|w|) G_k(w)}{(\delta^{-1} \log(1 + \delta|w_n|))^\theta} a_0 \quad \text{in } L^1(\Omega),$$

and the functions  $(1 + \delta|w_n|) G_k(w_n) f_n$  are too equiintegrable, since  $f_n$  strongly converges in  $L^{N/2}(\Omega)$ , and for Borel set  $E \subset \Omega$ , we have

$$\left\{ \begin{aligned} & \int_E (1 + \delta|w_n|) |G_k(w_n)| \chi_{\{w_n \neq 0\}} |f_n| dx \leq \int_E (1 + \delta|w_n|) |w_n| |f_n| dx \\ & \leq \|(1 + \delta|w_n|)\|_{2^*} \|w_n\|_{2^*} \left( \int_E |f_n|^{N/2} dx \right)^{2/N} \\ & \leq c \left( \int_E |f_n|^{N/2} dx \right)^{2/N}. \end{aligned} \right. \quad (3.34)$$

By Vitali's theorem implies that

$$(1 + \delta|w_n|) G_k(w_n) \chi_{\{w_n \neq 0\}} f_n \rightarrow (1 + \delta|w|) G_k(w) \chi_{\{w \neq 0\}} f \quad \text{in } L^1(\Omega).$$

Using the strong convergence of (3.1) and (3.2) of  $a_n$  and  $f_n$  in  $L^{N/2}(\Omega)$ , the almost everywhere of  $w_n$ , the bound of  $L^{2^*}(\Omega)$  of  $w_n$  and Vitali's theorem, we have for every  $k$  fixed and for  $n$  tends to infinity.

$$\frac{(1 + \delta|w_n|) G_k(w_n)}{(\delta^{-1} \log(1 + \delta|w_n|) + \frac{1}{n})^\theta} a_n \rightarrow \frac{(1 + \delta|w|) G_k(w)}{(\delta^{-1} \log(1 + \delta|w|))^\theta} a_0 \quad \text{strongly in } L^1(\Omega), \quad (3.35)$$



and

$$(1 + \delta|w_n|)G_k(w_n)\chi_{\{w_n \neq 0\}}f_n \rightarrow (1 + \delta|w|)G_k(w)\chi_{\{w \neq 0\}}f \quad \text{strongly in } L^1(\Omega). \quad (3.36)$$

Passing to the limit in (3.29), for any  $k$  fixed, we obtain

$$\left\{ \begin{array}{l} \alpha \limsup_{n \rightarrow +\infty} \int_{\Omega} |DG_k(w_n)|^2 dx \\ \leq \int_{\Omega} \frac{(1 + \delta|w|)G_k(w)}{(\delta^{-1} \log(1 + \delta|w|))^{\theta}} a_0 dx + \int_{\Omega} (1 + \delta|w|)G_k(w)\chi_{\{w \neq 0\}}f dx \end{array} \right. \quad (3.37)$$

Since  $|G_k(w)| \leq |w|$  and  $G_k(w) = 0$  in the set  $\{|w| \leq k\}$  the right-hand side of (3.37) is bounded in  $L^1(\Omega)$  and from above

$$\int_{|w| > k} \left( \frac{(1 + \delta|w|)|w|}{(\delta^{-1} \log(1 + \delta|w|))^{\theta}} a_0 + (1 + \delta|w|)|w|\chi_{\{w \neq 0\}}f \right) dx,$$

which tends to zero when  $k$  tends to infinity.

We deduce that

$$\alpha \limsup_{n \rightarrow +\infty} \int_{\Omega} |DG_k(w_n)|^2 dx \rightarrow 0 \quad \text{as } k \rightarrow +\infty. \quad (3.38)$$

**Step 5: Strong convergence of  $DT_k(w_n)$  in  $L^2(\Omega)^N$**

In this step, we will fix  $k > 0$  and prove that

$$DT_k(w_n) \rightarrow DT_k(w) \quad \text{strongly in } (L^2(\Omega))^N, \quad \text{as } n \rightarrow +\infty, \quad \text{for } k \text{ fixed.} \quad (3.39)$$

Let  $k$  be fixed, we define

$$z_n = T_k(w_n) - T_k(w), \quad (3.40)$$

and we choose an increasing,  $C^1$  function  $\psi : \mathbb{R} \rightarrow \mathbb{R}$  such that

$$\psi(0) = 0, \quad \psi'(s) - (c_0 + \delta)|\psi(s)| \geq 1/2, \quad \forall s \in \mathbb{R}, \quad (3.41)$$

where  $c_0$  is the constant which appears in the left-hand side of assumption (2.3) on  $H$  and there exist such functions  $\psi$  : for example  $\psi(s) = se^{\lambda s^2}$  with  $\lambda = (c_0 + \delta)^2/4$ .

Since  $z_n \in H_0^1(\Omega) \cap L^\infty(\Omega)$ , and since  $\psi(0) = 0$ , the function  $\psi(z_n)$  belongs to  $H_0^1(\Omega) \cap L^\infty(\Omega)$ . The use of  $\psi(z_n)$  as test function in (3.8) is licit. This gives

$$\left\{ \begin{array}{l} \int_{\Omega} A(x)Dw_n Dz_n \psi'(z_n) dx + \int_{\Omega} K_\delta(x, w_n, Dw_n) \text{sign}(w_n) \psi(z_n) dx = \\ + \int_{\Omega} \frac{(1 + \delta|w_n|)\psi(z_n)}{(\delta^{-1} \log(1 + \delta|w_n|) + \frac{1}{n})^\theta} a_n(x) dx \\ + \int_{\Omega} (1 + \delta|w_n|)\psi(z_n)\chi_{\{w_n \neq 0\}}f_n(x) dx. \end{array} \right. \quad (3.42)$$

Since

$$Dw_n = DT_k(w_n) + DG_k(w_n) = Dz_n + DT_k(w) + DG_k(w_n), \quad (3.43)$$

the first term of the left-hand side of (3.42) reads as

$$\left\{ \begin{aligned} \int_{\Omega} A(x) Dw_n Dz_n \psi'(z_n) dx &= \int_{\Omega} A(x) Dz_n Dz_n \psi'(z_n) dx \\ &+ \int_{\Omega} A(x) DT_k(w) Dz_n \psi'(z_n) dx \\ &+ \int_{\Omega} A(x) DG_k(w_n) Dz_n \psi'(z_n) dx. \end{aligned} \right. \quad (3.44)$$

On the other hand, splitting  $\Omega$  into  $\Omega = \{|w_n| > k\} \cup \{|w_n| \leq k\}$ , the second term of the left-hand side of (3.42) reads as

$$\left\{ \begin{aligned} \int_{\Omega} K_{\delta}(x, w_n, Dw_n) \operatorname{sign}(w_n) \psi(z_n) dx &= \\ \int_{\{|w_n| > k\}} K_{\delta}(x, w_n, Dw_n) \operatorname{sign}(w_n) \psi(z_n) dx & \\ + \int_{\{|w_n| \leq k\}} K_{\delta}(x, w_n, Dw_n) \operatorname{sign}(w_n) \psi(z_n) dx, & \end{aligned} \right. \quad (3.45)$$

the first term of the right-hand side of (3.45), we claim that

$$\int_{\{|w_n| > k\}} K_{\delta}(x, w_n, Dw_n) \operatorname{sign}(w_n) \psi(z_n) dx \geq 0, \quad (3.46)$$

indeed in  $\{|w_n| > k\}$ , the integrand is nonnegative since on the first hand the function  $K_{\delta}(x, w_n, Dw_n) \geq 0$  in view of (3.12) and  $\delta \geq \gamma$ , and since on the other hand one has

$$\operatorname{sign}(w_n) \psi(z_n) \geq 0 \quad \text{in } \{|w_n| > k\}, \quad (3.47)$$

indeed in  $\{|w_n| > k\}$ , one has  $z_n = T_k(w_n) - T_k(w) = k \operatorname{sign}(w_n) - T_k(w)$ , and therefore  $\operatorname{sign}(z_n) = \operatorname{sign}(w_n)$ ; this implies

$$\operatorname{sign}(w_n) \psi(z_n) = \operatorname{sign}(z_n) \psi(z_n) = |\psi(z_n)| \quad \text{in } \{|w_n| > k\}, \quad (3.48)$$

which proves (3.46).

The second term of the right-hand side of (3.45), in view of (3.12) and  $\delta \geq \gamma$ , we obtain

$$|K_{\delta}(x, w_n, Dw_n) \operatorname{sign}(w_n) \psi(z_n)| \leq (c_0 + \delta) |\psi(z_n)| A(x) Dw_n Dw_n. \quad (3.49)$$

Since in view of (3.43) one has

$$Dw_n = Dz_n + DT_k(w^*) \quad \text{in } \{|w_n| \leq k\},$$

and implies that

$$\left\{ \begin{array}{l} \int_{\{|w_k| \leq k\}} (K_\delta(x, w_n, Dw_n)) \operatorname{sign}(w_n) \psi(z_n) dx \\ \geq - \int_{\{|w_n| \leq k\}} (c_0 + \delta) |\psi(z_n)| A(x) Dw_n Dw_n dx \\ = - \int_{\{|w_n| \leq k\}} (c_0 + \delta) |\psi(z_n)| A(x) (Dz_n + DT_k(w)) (Dz_n + DT_k(w)) dx \\ \geq - \int_{\Omega} (c_0 + \delta) |\psi(z_n)| A(x) (Dz_n + DT_k(w)) (Dz_n + DT_k(w)) dx \\ \geq - \int_{\Omega} (c_0 + \delta) |\psi(z_n)| A(x) Dz_n Dz_n dx \\ - \int_{\Omega} (c_0 + \delta) |\psi(z_n)| \\ (A(x) DT_k(w) Dz_n + A(x) Dz_n DT_k(w) + A(x) DT_k(w) DT_k(w)) dx. \end{array} \right. \quad (3.50)$$

From (3.42), (3.44), (3.45), (3.46) and (3.50) we deduce that

$$\left\{ \begin{array}{l} \int_{\Omega} A(x) Dz_n Dz_n (\psi'(z_n) - (c_0 + \delta) |\psi(z_n)|) dx \\ \leq - \int_{\Omega} A(x) DT_k(w) Dz_n \psi'(z_n) dx \\ - \int_{\Omega} A(x) DG_k(w_n) Dz_n \psi'(z_n) dx \\ + \int_{\Omega} (c_0 + \delta) |\psi(z_n)| \\ (A(x) DT_k(w) Dz_n + A(x) Dz_n DT_k(w) + A(x) DT_k(w) DT_k(w)) dx \\ + \int_{\Omega} \left( \frac{(1 + \delta |w_n|)}{(\delta^{-1} (\log(1 + \delta |w_n|) + \frac{1}{n}))^\theta} a_n(x) + (1 + \delta |w_n|) \chi_{\{w_n \neq 0\}} f_n(x) \right) \psi(z_n) dx. \end{array} \right. \quad (3.51)$$

We claim that each term of the right-hand side of (3.51) tends to zero as  $n$  tends to infinity.

Since  $\psi'(z_n) - (c_0 + \delta) |\psi(z_n)| \geq 1/2$  by (3.41), and the matrix  $A$  is coercive (see (2.2)), this will imply that

$$z_n \rightarrow 0 \quad \text{in } H_0^1(\Omega) \quad \text{strongly,}$$

or in other terms (see the definition (3.40) of  $z_n$ ) that

$$T_k(w_n) \rightarrow T_k(w) \quad \text{in } H_0^1(\Omega) \text{ strongly as } n \rightarrow +\infty.$$

In order to prove the claim let us recall that of the definition (3.40) of  $z_n$  one has

$$z_n \rightharpoonup 0 \quad \text{in } H_0^1(\Omega) \text{ weakly, } L^\infty(\Omega) \text{ weakly star and a.e. in } \Omega \text{ as } n \rightarrow +\infty.$$

Since  $\psi(0) = 0$ , this implies that  $\psi(z_n)$  tends to zero almost everywhere in  $\Omega$  and in  $L^\infty(\Omega)$  weakly star as  $n$  tends to infinity, which in turn implies that

$$Dz_n \psi'(z_n) = D\psi(z_n) \rightharpoonup 0 \quad \text{in } L^2(\Omega)^N \text{ weakly as } n \rightarrow +\infty.$$

This implies that the first term of the right-hand side of (3.51) tends to zero as  $n$  tends to infinity.

For the second term of the right-hand side of (3.51) we observe that

$$A(x)DG_k(w_n)Dz_n = A(x)DG_k(w_n)(DT_k(w_n) - DT_k(w)) = -A(x)DG_k(w_n)DT_k(w),$$

and that by Lebesgue's dominated convergence theorem

$$DT_k(w) \psi'(z_n) \rightarrow DT_k(w) \psi'(0) \quad \text{in } L^2(\Omega)^N \text{ strongly as } n \rightarrow +\infty,$$

while  $DG_k(w_n)$  tends to  $DG_k(w)$  weakly in  $L^2(\Omega)^N$ .

Since almost everywhere one has  $A(x)DG_k(w)DT_k(w) = 0$ , the second term of the right-hand side of (3.51) tends to zero.

For the third term of the right-hand side of (3.51), we observe that

$$(c_0 + \delta)|\psi(z_n)|A(x)DT_k(w) \rightarrow 0 \quad \text{in } L^2(\Omega)^N \text{ strongly as } n \rightarrow +\infty$$

by Lebesgue's dominated convergence Theorem, since  $\psi(z_n)$  is bounded in  $L^\infty(\Omega)$  and since  $\psi(z_n)$  tends almost everywhere to zero because  $\psi(0) = 0$ . Since  $Dz_n$  is bounded in  $L^2(\Omega)^N$ , this implies that the first part of this third term tends to zero. A similar proof holds true for the two others parts of this third term.

Finally the fourth term of the right-hand side of (3.51) tends to zero, since the integrand converges almost everywhere to zero and is equiintegrable (see (3.15), (3.17) and (3.18)).

This proves that  $z_n$  tend to zero strongly in  $H_0^1(\Omega)$ , namely

$$DT_k(w_n) \rightarrow DT_k(w) \quad \text{strongly in } (L^2(\Omega))^N, \text{ as } n \rightarrow +\infty, \text{ for } k \text{ fixed}$$

Since we have

$$w_n - w = T_k(w_n) + G_k(w_n) - T_k(w) - G_k(w),$$

and using (3.38) and (3.39) we have

$$Dw_n \rightarrow Dw \quad \text{in } (L^2(\Omega))^N \text{ strongly as } n \rightarrow +\infty, \quad (3.52)$$

Thus  $w_n$  tends to  $w$  strongly in  $H_0^1(\Omega)$ . Since we have

$$u_n = \delta^{-1} \log(1 + \delta|w_n|),$$

it follows that

$$u_n \rightarrow u \quad \text{strongly in } H_0^1(\Omega).$$

**Step 6:** Control of  $\int_{|u_n| \leq \mu} \frac{a_n(x)}{(|u_n| + \frac{1}{n})^\theta} \varphi$  when  $\mu$  is small

In this step we prove that

$$\lim_n \int_{\Omega} \frac{a_n(x)}{(|u_n| + \frac{1}{n})^\theta} \varphi dx = \int_{\Omega} \frac{a_0(x)}{|u|^\theta} \varphi dx,$$

for all  $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$ .

First we observe that

$$\left\{ \begin{aligned} \int_{\Omega} \frac{a_n(x)}{(|u_n| + \frac{1}{n})^\theta} \varphi dx &= \int_{\Omega} A(x) Du_n D\varphi dx \\ &\quad - \int_{\Omega} H_n(u_n, Du_n) \varphi dx - \int_{\Omega} \chi_{\{u_n \neq 0\}} f_n \varphi dx. \end{aligned} \right. \quad (3.53)$$

Taking into account the boundness of the matrix  $A$ , using the Young's inequality and the Sobolev's inequality, we get

$$\left\{ \begin{aligned} &\int_{\Omega} \frac{a_n(x)}{(|u_n| + \frac{1}{n})^\theta} \varphi dx \\ &\leq \|A\|_\infty (\|Du_n\|_2^2 + \|D\varphi\|_2^2) \\ &\quad + (c_0 + \gamma) \|\varphi\|_\infty \|Du_n\|^2 + C_N |\Omega|^{1/2^*} \|f\|_{N/2} \|D\varphi\|_2 \\ &\leq \|A\|_\infty (\|Du_n\|_2^2 + \|D\varphi\|_2^2) \\ &\quad + (c_0 + \gamma) \|\varphi\|_\infty \|Du_n\|_2^2 + \frac{C_N^2 |\Omega|^{2/2^*}}{2} \|f\|_{N/2}^2 + \frac{1}{2} \|D\varphi\|_2^2 \\ &\leq c + (c_\varphi \|Du_n\|_2^2 + c' \|D\varphi\|_2^2), \end{aligned} \right. \quad (3.54)$$

where  $c$ ,  $c_\varphi$  and  $c'$  are the positive constants.

From now on, we consider a nonnegative  $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$ , applying Fatou's Lemma to the left-hand side of (3.54), we have.

$$\int_{\Omega} \frac{a_0(x)}{|u|^\theta} \varphi dx \leq C_\varphi,$$

where  $C_\varphi$  does not depend to  $n$ . Hence  $0 \leq \frac{a_0(x)}{|u|^\theta} \varphi \in L^1(\Omega)$ , for any nonnegative  $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$ .

As consequence,  $\frac{1}{|s|^\theta}$  is unbounded as  $s$  tends to 0, we deduce that

$$\{u = 0\} \subset \{a_0 = 0\},$$

up to set of zero Lebesgue measure.

From now on, we consider a nonnegative function  $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$ , and choising it was test function in the weak formulation, we have

$$\left\{ \begin{array}{l} \int_{\Omega} A(x) Du_n D\varphi dx \\ = \int_{\Omega} H_n(u_n, Du_n) \varphi dx + \int_{\Omega} \frac{a_n(x)}{(|u_n| + \frac{1}{n})^\theta} \varphi dx + \int_{\Omega} \chi_{\{u_n \neq 0\}} f_n \varphi dx, \end{array} \right. \quad (3.55)$$

we want to pass to the limit in the second right-hand side of (3.55) as  $n$  tends to infinity. For  $\mu > 0$  fixed, we consider the second right-hand side of (3.55)

$$\int_{\Omega} \frac{a_n(x)}{(|u_n| + \frac{1}{n})^\theta} \varphi dx = \int_{|u_n| \leq \mu} \frac{a_n(x)}{(|u_n| + \frac{1}{n})^\theta} \varphi dx + \int_{|u_n| > \mu} \frac{a_n(x)}{(|u_n| + \frac{1}{n})^\theta} \varphi dx \quad (3.56)$$

Applying Lemma 1.1 of [34], we have that for  $\mu > 0$ ,  $V_\mu(u_n)$  belongs to  $H_0^1(\Omega)$ , where  $V_\mu : ]-\infty, +\infty[ \rightarrow ]0, +\infty[$  is defined by

$$V_\mu(s) = \begin{cases} 0 & s < -2\mu \\ 2 + \frac{s}{\mu} & -2\mu \leq s < -\mu \\ 1 & -\mu \leq s \leq \mu \\ 2 - \frac{s}{\mu} & \mu < s < 2\mu \\ 0 & s \geq 2\mu. \end{cases}$$

Since  $V_\mu(u_n) \in H_0^1(\Omega)$ , the use of  $(V_\mu(u_n) \varphi)$  as test function in (3.4) is licit. This gives

$$\left\{ \begin{array}{l} \int_{|u_n| \leq \mu} \frac{a_n(x)}{(|u_n| + \frac{1}{n})^\theta} \varphi dx \leq \int_{\Omega} A(x) Du_n D(V_\mu(u_n) \varphi) dx \\ - \int_{\Omega} H_n(u_n, Du_n) V_\mu(u_n) \varphi dx - \int_{\Omega} \chi_{\{u_n \neq 0\}} f_n V_\mu(u_n) \varphi dx \end{array} \right. \quad (3.57)$$

The first term of the right-hand side of (3.57) can be written

$$\int_{\Omega} A(x) Du_n D(V_\mu(u_n) \varphi) dx = \int_{\Omega} A(x) Du_n D\varphi V_\mu(u_n) dx. \quad (3.58)$$

Indeed, splitting  $\Omega$  into  $\Omega = \{|u_n| \leq \mu\} \cup \{|u_n| > \mu\}$

$$\left\{ \begin{aligned} & \int_{\Omega} A(x) Du_n Du_n V_{\mu}'(u_n) \varphi dx = \\ & -\frac{1}{\mu} \int_{\{u_n \geq 0\}} A(x) Du_n Du_n \varphi dx \\ & +\frac{1}{\mu} \int_{\{u_n < 0\}} A(x) Du_n Du_n \varphi dx \end{aligned} \right. \quad (3.59)$$

$$\left\{ \begin{aligned} & \frac{1}{\mu} \int_{\{u_n < 0\}} A(x) Du_n Du_n \varphi dx \\ & = \frac{1}{\mu} \int_{\{-u_n > 0\}} A(x) D(-u_n) D(-u_n) \varphi dx \\ & = \frac{1}{\mu} \int_{\{u_n > 0\}} A(x) Du_n Du_n \varphi dx. \end{aligned} \right. \quad (3.60)$$

Finally, we have

$$\int_{\Omega} A(x) Du_n Du_n V_{\mu}'(u_n) \varphi dx = 0 \quad (3.61)$$

Since  $D\varphi V_{\mu}(u_n)$  converges to  $D\varphi V_{\mu}(u)$  strongly in  $L^2(\Omega)^N$  while  $A(x)Du_n$  converges to  $A(x)Du$  weakly in  $L^2(\Omega)^N$ , we obtain

$$\lim_n \int_{\Omega} A(x) Du_n D\varphi V_{\mu}(u_n) dx = \int_{\Omega} A(x) Du D\varphi V_{\mu}(u) dx. \quad (3.62)$$

In the second term of the right-hand side of (3.57), we observe that  $\varphi V_{\mu}(u_n)$  is bounded in  $L^{\infty}(\Omega)$  and

$$H_n(u_n, Du_n) \varphi V_{\mu}(u_n) \leq \|\varphi\|_{\infty} (c_0 + \gamma) |Du_n|^2,$$

which implies that the functions  $H_n(u_n, Du_n) \varphi V_{\mu}(u_n)$  are equiintegrable since  $Du_n$  strongly converges to  $Du$  in  $L^2(\Omega)^N$ , we have

$$\lim_n \int_{\Omega} H_n(u_n, Du_n) \varphi V_{\mu}(u_n) dx = \int_{\Omega} H(u, Du) \varphi V_{\mu}(u) dx. \quad (3.63)$$

In the third term of the right-hand side of (3.57), the functions  $f_n \varphi V_{\mu}(u_n)$  are equiintegrable, since  $f_n$  strongly converges in  $L^{N/2}(\Omega)$  and  $V_{\mu}(u_n)$  converges to  $V_{\mu}(u)$  strongly in  $L^{2^*}(\Omega)$ . Thus Vitali's theorem implies that

$$\lim_n \int_{\Omega} \chi_{\{u_n \neq 0\}} f_n \varphi V_{\mu}(u_n) dx = \int_{\Omega} \chi_{\{u \neq 0\}} f \varphi V_{\mu}(u) dx. \quad (3.64)$$

Together with (3.57), the three limits (3.62), (3.63) and (3.64) imply that

$$\left\{ \begin{aligned} & \lim_n \int_{|u_n| \leq \mu} \frac{a_n(x)}{(|u_n| + \frac{1}{n})^{\theta}} \varphi dx \leq \int_{\Omega} A(x) Du D\varphi V_{\mu}(u) dx \\ & + \int_{\Omega} H(u, Du) \varphi V_{\mu}(u) dx + \int_{\Omega} \chi_{\{u \neq 0\}} f \varphi V_{\mu}(u) dx \end{aligned} \right. \quad (3.65)$$

Since  $V_\mu(u)$  converges to  $\chi_{\{u=0\}}$  a.e. in  $\Omega$ , as  $\mu \rightarrow 0$  and since  $u \in H_0^1(\Omega)$ , then

$$(A(x)DuD\varphi + H(u, Du)\varphi + \chi_{\{u \neq 0\}}f\varphi) V_\mu(u) \rightarrow 0 \quad \text{a.e. in } \Omega, \quad \text{as } \mu \rightarrow 0. \quad (3.66)$$

Applying the Lebesgue's dominated convergence Theorem on the right-hand side of (3.65), we obtain that

$$\lim_{\mu \rightarrow 0} \lim_n \int_{\{|u_n| \leq \mu\}} \frac{a_n(x)}{(|u_n| + \frac{1}{n})^\theta} \varphi dx = 0. \quad (3.67)$$

Finally, let us pass to limit in  $n$  for  $\mu > 0$  fixed in the second term of the right-hand side of (3.56)

$$\int_{\{|u_n| > \mu\}} \frac{a_n(x)}{(|u_n| + \frac{1}{n})^\theta} \varphi dx = \int_{\Omega} \frac{a_n(x)}{(|u_n| + \frac{1}{n})^\theta} \chi_{\{|u_n| > \mu\}} \varphi dx.$$

Using that  $u_n$  converges to  $u$  a.e. on  $\Omega$ , we have

$$\frac{a_n(x)}{(|u_n| + \frac{1}{n})^\theta} \varphi \rightarrow \frac{a_0(x)}{|u|^\theta} \varphi \quad \text{a.e. on } \Omega,$$

and

$$\chi_{\{|u_n| > \mu\}} \xrightarrow{n} \chi_{\{|u| > \mu\}} \quad \text{on } \{x \in \Omega : u(x) \neq \mu\},$$

defining the set  $\mathcal{C}$  by

$$\mathcal{C} = \{\mu > 0, \quad \text{meas}\{x \in \Omega : u(x) = \mu\} > 0\},$$

and choising  $\mu \notin \mathcal{C}$ , Lebesgue's dominated convergence theorem implies that

$$\lim_n \int_{\{|u_n| > \mu\}} \frac{a_n(x)}{(|u_n| + \frac{1}{n})^\theta} \varphi dx = \int_{\{|u| > \mu\}} \frac{a_0(x)}{|u|^\theta} \varphi dx, \quad \forall \mu \notin \mathcal{C}. \quad (3.68)$$

As the set  $\mathcal{C}$  is at most countable, choising  $\mu$  such that  $\mu \notin \mathcal{C}$  and using the fact that

$$\chi_{\{|u| > \mu\}} \rightarrow \chi_{\{|u| > 0\}} \quad \text{as } \mu \rightarrow 0,$$

the fact  $\frac{a_0(x)}{|u|^\theta} \varphi$  belongs to  $L^1(\Omega)$ .

Finally, we have proved that

$$\int_{\{|u| > \mu\}} \frac{a_0(x)}{|u|^\theta} \varphi dx \rightarrow \int_{\{|u| > 0\}} \frac{a_0(x)}{|u|^\theta} \varphi dx = \int_{\Omega} \frac{a_0(x)}{|u|^\theta} \varphi dx \quad \text{as } \mu \rightarrow 0.$$

Using (3.67) and (3.68), we deduce

$$\lim_n \int_{\Omega} \frac{a_n(x)}{(|u_n| + \frac{1}{n})^\theta} \varphi dx = \int_{\Omega} \frac{a_0(x)}{|u|^\theta} \varphi dx, \quad \forall \varphi \in H_0^1(\Omega) \cap L^\infty(\Omega), \quad \varphi \geq 0. \quad (3.69)$$

Moreover, decomposing any  $\varphi = \varphi^+ - \varphi^-$  and observing that (3.69) is linear in  $\varphi$ , we deduce that (3.69) holds for every  $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$ .



**Remark 3.1.** Since  $u_n \in H_0^1(\Omega)$ , one has for every  $\mu > 0$  fixed

$$\{u_n = 0\} \subset \{|u_n| \leq \mu\},$$

this implies that

$$\int_{\{u_n=0\}} \frac{a_n(x)}{(|u_n| + \frac{1}{n})^\theta} \varphi dx = 0 \quad \text{for every } \varphi \in H_0^1(\Omega) \cap L^\infty(\Omega).$$

As  $u_n \rightarrow u$  strongly in  $H_0^1(\Omega)$ , it is then easy to pass to the limit in the approximate equation (3.4). This proves that  $u$  is a solution of (2.1). The proof of Theorem 2.1 is then complete.

## 4. Appendix

In this Appendix, we give an equivalent result of the approximate problem and the definition of the constant  $Z_\delta$  which appears in Theorem 2.1 (see Lemma 4.2).

### 4.1. An equivalence result

**Proposition 4.1.** *Assume that (2.2), (2.3), (2.5), (2.6), (2.8), (3.3), (3.1), (3.2) hold true, and let  $\delta > 0$  be fixed. Let the function  $K_\delta$  be defined in (3.9).*

*If  $u_n$  is any solution of (2.1) which satisfies*

$$(e^{\delta|u_n|} - 1) \in H_0^1(\Omega), \quad (4.1)$$

*then the function  $w_n$  defined by (3.6), namely*

$$w_n = \delta^{-1}(e^{\delta|u_n|} - 1) \text{sign}(w_n),$$

*satisfies*

$$\begin{cases} w_n \in H_0^1(\Omega), \\ -\text{div}(A(x)Dw_n) + K_\delta(x, w_n, Dw_n) \text{sign}(w_n) = \\ \quad (1 + \delta|w_n|)\chi_{\{w_n \neq 0\}} f_n + \frac{1 + \delta|w_n|}{(\delta^{-1} \log(1 + \delta|w_n| + \frac{1}{n}))^\theta} a_n(x) \quad \text{in } \mathcal{D}'(\Omega). \end{cases} \quad (4.2)$$

*Conversely, if  $w_n$  is any solution of (4.2), then the function  $u_n$  defined by*

$$u_n = \delta^{-1} \log(1 + \delta|w_n|) \text{sign}(w_n), \quad (4.3)$$

*is a solution of (3.4) which satisfies (4.3).*

*Proof.* Define the function  $\hat{f}_n$  by

$$\hat{f}_n(x) = \chi_{\{u_n \neq 0\}} f_n(x) + \frac{a_n(x)}{(|u_n| + \frac{1}{n})^\theta}.$$

In view of (3.6), one has:

$$\left\{ \begin{aligned} & (1 + \delta|w_n|) \chi_{\{w_n \neq 0\}} f_n(x) + \frac{(1 + \delta|w_n|)a_n(x)}{(\delta^{-1} \log(1 + \delta|w_n|) + \frac{1}{n})^\theta} \\ &= (1 + \delta|w_n|) \left( \chi_{\{w_n \neq 0\}} f_n(x) + \frac{a_n(x)}{(\delta^{-1} \log(1 + \delta|w_n|) + \frac{1}{n})^\theta} \right) \\ &= (1 + \delta|w_n|) \left( \chi_{\{u_n \neq 0\}} f_n(x) + \frac{a_n(x)}{(|u_n| + \frac{1}{n})^\theta} \right). \end{aligned} \right. \quad (4.4)$$

Then Proposition 4.1 becomes an immediate application of Proposition 1.8 of [17], once observes that

$$\hat{f}_n \in L^\infty(\Omega). \quad (4.5)$$

□

## 4.2. Definition of $Z_\delta$

The goal of this Subsection is to define the constant  $Z_\delta$  which appear in Theorem 2.1. We will prove the following result.

**Lemma 4.2.** *For  $\delta \geq 0$ , let  $\Phi_\delta : \mathbb{R}^+ \rightarrow \mathbb{R}$  (see Figure 1) be the function defined by*

$$\left\{ \begin{aligned} \Phi_\delta(X) &= \left( \alpha - \frac{\delta^{1+\theta}}{\log^\theta(1+\delta)} C_N^2 \|a_0\|_{N/2} - \delta C_N^2 \|f\|_{N/2} \right) X^{1+\theta} \\ &\quad - \left( \frac{\delta^\theta}{\log^\theta(1+\delta)} C_N |\Omega|^{1/2^*} \|a_0\|_{N/2} + \|f\|_{H^{-1}(\Omega)} \right) X^\theta \\ &\quad - \frac{(1+\delta)\delta^\theta}{\log^\theta(1+\delta)} C_N^{1-\theta} |\Omega|^{\frac{1+\theta}{2^*}} \|a_0\|_{N/2}, \end{aligned} \right. \quad (4.6)$$

where  $\theta$  satisfies (2.6), namely  $0 < \theta < 1$ . and where  $C_N$  is the best constant in the Sobolev's inequality (2.9).

Then, for  $\delta \geq \gamma$ , there exists a unique number  $Z_\delta$  such that

$$\Phi_\delta(Z_\delta) = 0, \quad \text{and} \quad \forall X \leq Z_\delta : \Phi_\delta(X) \leq 0. \quad (4.7)$$

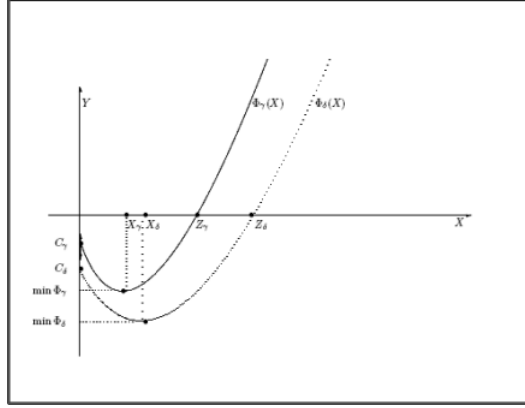


Figure 1: The graphs of the functions  $\Phi_\delta(X)$  and  $\Phi_\gamma(X)$ .

*Proof.* Let us now study the family of functions  $\Phi_\delta(X) : \mathbb{R}^+ \rightarrow \mathbb{R}$  defined by (4.6), from the smallness condition relative to  $\delta$  (see 2.11), implies that

$$\alpha - \frac{\delta^{1+\theta}}{\log^\theta(1+\delta)} C_N^2 \|a_0\|_{N/2} - \delta C_N^2 \|f\|_{N/2} \geq 0$$

Each function  $\Phi_\delta$  look like the restriction to  $\mathbb{R}^+$  of a “convex parabola”, when  $0 < \gamma \leq \delta$  this “convex parabola” has a unique minimizer in  $X_\delta$  of the function  $\Phi_\delta$ , and the minimum of  $\Phi_\delta$ , namely  $\Phi_\delta(X_\delta)$  is negative and using the intermediate value theorem, then there exists  $Z_\delta$  such that  $\Phi_\delta(Z_\delta) = 0$ .  $\square$

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