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EXISTENCE OF SOLUTIONS  
FOR A NONLOCAL REACTION-DIFFUSION EQUATION  
IN BIOMEDICAL APPLICATIONS

BY

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ABSTRACT

The paper is devoted to a nonlocal semi-linear elliptic equation in  $\mathbb{R}^n$  arising in various biological and biomedical applications. The Fredholm property studied for the corresponding linear elliptic operators with discontinuous coefficients allows the application of the implicit function theorem to prove the persistence of solutions under a small perturbation of the problem. Furthermore, the existence of solutions is established by the Leray–Schauder method based on the topological degree for Fredholm and proper operators and on a priori estimates of solutions in some special weighted spaces.

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## 1. Introduction

In this work, we study the existence of solutions of the equation

$$(1.1) \quad \Delta u + au(1 - I(u)) - F(u, x) = 0$$

considered in the whole space  $\mathbb{R}^n$ ,  $n = 1, 2, 3$ , where

$$I(u) = \int_{\mathbb{R}^n} u(x) dx.$$

Such equations arise in various biological and biomedical applications where  $u(x)$  corresponds to the density of some population (animals, cells, viruses) [4, 5]. The diffusion term describes random motion of the individual entities of the population, the second term characterizes their reproduction, and the last term their mortality. The reproduction term is proportional to the population density and to the available resources  $(1 - I(u))$ . Here  $K = 1$  is dimensionless carrying capacity, and  $I(u)$  corresponds to consumed resources proportional to the total population. Thus, instead of the conventional logistic term  $u(1 - u)$  appropriate for the case of local consumption of resources, we consider the integral term for the global consumption [25]. The specific form of the mortality term depends on the applications. In particular, in a model of viral infection development,

$$F(u, x) = uf(u) + \sigma(x)u,$$

where the first term in this function describes virus elimination due to the immune response, and the second term its natural mortality.

Let us note that in the models of population dynamics the space variable  $x$  can have two different meanings. The first one corresponds to the conventional physical space where the function  $u(x, t)$  describes the distribution of animals in some habitats or cell (viruses) in the tissue. In the second interpretation, the variable  $x$  characterizes the genotype or the phenotype of the population. The population density distribution as a function of its genotype describes the existence and the evolution of biological species, cell lineages and cell clones in cancer, or virus strains. In this case, a conventional mathematical question about the existence of solutions acquires a clear and important biological significance allowing the determination of the conditions of the existence of biological species (clones, strains) (see [4, 5]).

Mathematical analysis of equation (1.1) has some specific features because it is considered in an unbounded domain, and also because of the presence of the integral term and possibly discontinuous coefficients. General elliptic problems in bounded domains with a sufficiently smooth boundary satisfy the Fredholm property if and only if the ellipticity condition, the condition of proper ellipticity and the Lopatinskii conditions are satisfied [1, 2, 19]. In the case of unbounded domains, these conditions may not be sufficient. Elliptic problems in unbounded domains are characterized by the presence of the essential spectrum determined by limiting operators [18, 20]. Solvability conditions for linear problems, their index and Fredholm property determine the properties of nonlinear operators including their properness and topological degree.

The topological degree for elliptic operators was introduced by Leray and Schauder [17] with the reduction to the operators  $I + K$ , where  $I$  is the identity operator and  $K$  is a compact operator. This construction is not applicable to elliptic problems in unbounded domains since, contrary to the case of bounded domains, the inverse to the Laplace operator is not compact. There are various degree constructions in an abstract setting [6, 8, 9, 10], [11]–[15] and in the framework of elliptic problems [7, 21]. We will use the degree construction for Fredholm and proper operators with the zero index [23]. Together with a priori estimates of solutions, the topological degree allows the application of the Leray–Schauder method to prove the existence of solutions.

We will apply these methods of linear and nonlinear analysis to study equation (1.1). The presence of the integral term implies the integrability of the solution and imposes some constraints on the function spaces. We will also be interested in equations with discontinuous coefficients because they can admit explicit analytical solutions. The latter can be used as a starting point for the continuation of solutions. However, they also require a special functional setting which will be discussed below. Altogether, this extended formulation of classical elliptic problems leads us to revisit the existing theory. Since these results can represent an independent interest for the mathematical theory, and in view of other possible applications, we will prove more general results and apply them to study equation (1.1). In the next section, we study linear operators with discontinuous coefficients. Section 3 is devoted to the application of the implicit function theorem in the case of discontinuous functions. Finally, in Section 4 we introduce the topological degree and study the existence of solutions by the Leray–Schauder method.

## 2. Linear operators with discontinuous coefficients

Fredholm property of general elliptic problems in unbounded domains is studied in [20] in the case where the coefficients of the problem belong to some Hölder spaces. In this section we will use these results in order to study a class of elliptic operators with discontinuous coefficients.

2.1. OPERATORS AND SPACES. Consider a Banach space  $E$  with the norm  $\|\cdot\|_E$ , and the space  $E_\infty$  with the norm

$$\|u\|_{E_\infty} = \sup_{y \in \mathbb{R}^n} \|u(\cdot)\omega(\cdot - y)\|_E,$$

where  $\omega(x)$  is an infinitely differentiable function,  $0 \leq \omega(x) \leq 1$  for  $x \in \mathbb{R}^n$ ,  $\omega(x) = 1$  for  $|x| \leq 1/2$  and  $\omega(x) = 0$  for  $|x| \geq 1$  (see [20] for more details). In particular, we will consider the spaces  $L_\infty^2(\mathbb{R}^n)$  and  $H_\infty^2(\mathbb{R}^n)$  for which  $E = L^2(\mathbb{R}^n)$  and  $E = H^2(\mathbb{R}^n)$ , respectively,  $n = 1, 2, 3$ .

Consider the linear second-order elliptic operator

$$Lu = \Delta u + \sum_{i=1}^n a_i(x) \frac{\partial u}{\partial x_i} + b(x)u,$$

where

$$a_i(x) = a_i^0(x) + a_i^1(x), \quad b(x) = b_0(x) + b_1(x), \quad a_i^0, b_0 \in C^\alpha(\mathbb{R}^n),$$

and  $a_i^1(x), b_1(x)$  are bounded functions (possibly discontinuous) with a bounded support. The operator acts from  $H_\infty^2(\mathbb{R}^n)$  into  $L_\infty^2(\mathbb{R}^n)$ . We will also consider the operator

$$L_0u = \Delta u + \sum_{i=1}^n a_i^0(x) \frac{\partial u}{\partial x_i} + b_0(x)u.$$

2.2. LIMITING OPERATORS AND A PRIORI ESTIMATES. Consider an arbitrary sequence  $x_k \in \mathbb{R}^n$  such that  $|x_k| \rightarrow \infty$  as  $k \rightarrow \infty$ . Set

$$a_i^k(x) = a_i^0(x + x_k), \quad b^k(x) = b_0(x + x_k), \quad k = 1, 2, \dots$$

Since these functions belong to the Hölder space, they converge locally uniformly to some functions  $a_i^*(x), b^*(x)$ . These limiting functions can depend on the choice of the sequence  $x_k$ . The operator  $\hat{L}$  with the limiting coefficients

$$\hat{L}u = \Delta u + \sum_{i=1}^m a_i^*(x) \frac{\partial u}{\partial x_i} + b^*(x)u$$

is called the limiting operator.

CONDITION NS. Any limiting equation  $\hat{L}u=0$  has only zero solution in  $H_\infty^2(\mathbb{R}^n)$ .

The following lemma is a particular case of a more general result proved in [20] (Chapter 4, Theorem 2.10, p. 155).

LEMMA 2.1: *Let Condition NS be satisfied. Then there exist numbers  $M$  and  $R$  such that the following estimate holds:*

$$(2.1) \quad \|u\|_{H_\infty^2(\mathbb{R}^n)} \leq M(\|L_0 u\|_{L_\infty^2(\mathbb{R}^n)} + \|u\|_{L^2(B_R)})$$

for any  $u \in H_\infty^2(\mathbb{R}^n)$ . Here  $B_R = \{x \in \mathbb{R}^n, |x| \leq R\}$ .

We will use this lemma to get a similar estimate for the operator  $L$ . Lemma 2.1 holds under the assumption that the coefficients of the operator  $L_0$  belong to the Hölder space. This result cannot be directly applied to the operator  $L$  with discontinuous coefficients. We prove it in the following theorem. Let us note that limiting operators for the operator  $L_0$  and  $L$  are the same.

THEOREM 2.2: *Let Condition NS be satisfied. Then there exist numbers  $M$  and  $R$  such that the following estimate holds:*

$$(2.2) \quad \|u\|_{H_\infty^2(\mathbb{R}^n)} \leq M(\|Lu\|_{L_\infty^2(\mathbb{R}^n)} + \|u\|_{L^2(B_R)})$$

for any  $u \in H_\infty^2(\mathbb{R}^n)$ . Here  $B_R = \{x \in \mathbb{R}^n, |x| \leq R\}$ .

*Proof.* Set

$$Bu = \sum_{i=1}^m (a_i(x) - a_i^0(x)) \frac{\partial u}{\partial x_i} + (b(x) - b_0(x))u.$$

Then  $L = L_0 + B$ , and estimate (2.1) can be written as follows:

$$(2.3) \quad \begin{aligned} \|u\|_{H_\infty^2(\mathbb{R}^n)} &\leq M(\|(L - B)u\|_{L_\infty^2(\mathbb{R}^n)} + \|u\|_{L^2(B_R)}) \\ &\leq M(\|Lu\|_{L_\infty^2(\mathbb{R}^n)} + \|Bu\|_{L_\infty^2(\mathbb{R}^n)} + \|u\|_{L^2(B_R)}). \end{aligned}$$

We need to estimate the term  $\|Bu\|_{L_\infty^2(\mathbb{R}^n)}$ . We have

$$Bu = \sum_{i=1}^m a_i^1(x) \frac{\partial u}{\partial x_i} + b_1(x)u.$$

Since  $a_i^1(x)$  and  $b_1(x)$  are bounded functions with bounded supports, then

$$|a_i^1(x), b_1(x)| \leq K$$

for some positive constant  $K$ , and  $R$  can be chosen sufficiently large such that their supports belong to the ball  $B_R$ . Hence,

$$\|b_1 u\|_{L^\infty(\mathbb{R}^n)} \leq \left( \int_{\mathbb{R}^n} (b_1(x)u(x))^2 dx \right)^{1/2} \leq K \left( \int_{B_R} u^2(x) dx \right)^{1/2},$$

$$\left\| a_i^1 \frac{\partial u}{\partial x_i} \right\|_{L^\infty(\mathbb{R}^n)} \leq K \left( \int_{B_R} \left| \frac{\partial u}{\partial x_i} \right|^2 dx \right)^{1/2} \leq \epsilon \|u\|_{H^2(B_R)} + C_\epsilon \|u\|_{L^2(B_R)}.$$

The last estimate holds for any positive  $\epsilon$  and some constant  $C_\epsilon$  depending on  $\epsilon$ . For  $\epsilon$  sufficiently small, these estimates and (2.3) provide (2.2). The theorem is proved. ■

**2.3. PROPERNESS, NORMAL SOLVABILITY, THE FREDHOLM PROPERTY.** Recall that a linear operator acting in Banach spaces is normally solvable if and only if its image is closed. The operator is called proper on closed bounded sets if the inverse image of any compact set is compact in any bounded closed set. If a linear operator is proper, then it is normally solvable with a finite-dimensional kernel (see [20], Theorem 2.13, p. 163). By definition, a linear operator satisfies the Fredholm property if it is normally solvable, the dimension of its kernel is finite and the codimension of its image is also finite. The latter is equivalent to a finite number of solvability conditions of the non-homogeneous equation.

Condition NS is a necessary and sufficient condition for general elliptic problems in unbounded domains to be normally solvable with a finite-dimensional kernel [20]. This result is proved in the case of Hölder continuous coefficients of the operator. The next theorem affirms the sufficiency of this condition for the operators with discontinuous coefficients. The necessity of Condition NS is proved in [20] by construction of non-compact sequences providing a nonzero solution of some limiting operator. A similar construction may be also applicable here since the limiting operators of the operators  $L_0$  and  $L$  are the same.

Continuous deformation in the class of normally solvable operators with a finite-dimensional kernel does not change the index of the operator [16]. Therefore, we can find the index of the operator reducing it to another operator with a known index by a continuous deformation, assuming that Condition NS is satisfied in the process of this deformation. This approach is used in Theorem 2.4 and Lemma 3.3 below. In particular, if the reduction can be applied to an invertible operator, then the index equals 0.

**THEOREM 2.3:** *Let Condition NS be satisfied. Then the operator  $L$  is normally solvable with a finite-dimensional kernel.*

*Proof.* It is sufficient to prove that the operator  $L$  is proper. This property follows from Theorem 2.2. Indeed, let  $Lu_n = f_n$ ,  $f_n \rightarrow f_0$ , and the sequence  $u_n$  be bounded in  $H_\infty^2(\mathbb{R}^n)$ . We will verify that we can choose a convergent subsequence from this sequence. Since  $H^2$  is locally compact in  $L^2$ , then we can choose a subsequence  $u_{n_k}$ , which forms a fundamental sequence in  $L^2(B_R)$ . Applying Theorem 2.2 to the equation

$$L(u_{n_k} - u_{n_j}) = f_{n_k} - f_{n_j},$$

we conclude that this sequence is also fundamental in  $H_\infty^2(\mathbb{R}^n)$ . Therefore, it converges to some function  $u_0 \in H_\infty^2(\mathbb{R}^n)$ . The theorem is proved.    ■

**THEOREM 2.4:** *The operator  $L$  satisfies the Fredholm property if the operator  $L_0$  satisfies it. In this case, their indices are equal to each other.*

*Proof.* Suppose that the operator  $L_0$  satisfies the Fredholm property. Then Condition NS is satisfied for  $L_0$ . Since the limiting operators for the operator  $L_0$  and  $L$  are the same, then Condition NS is also satisfied for the operator  $L$  and for their linear combination  $L_\tau = \tau L_0 + (1 - \tau)L$ . Hence, the operator  $L_\tau$  is normally solvable with a finite-dimensional kernel for all  $\tau \in [0, 1]$ , and its index does not depend on  $\tau$ . The theorem is proved.    ■

*Example 2.5:* Suppose that  $b_0(x) \rightarrow \beta$  as  $|x| \rightarrow \infty$ , where  $\beta$  is a negative number. Then the essential spectrum of the operator  $L_0$  lies in the sector of the complex plane which does not contain the positive half-axis [20], and its index equals 0. According to the previous theorem, the same properties hold for the operator  $L$ .

**2.4. SOLVABILITY OF THE NONLOCAL EQUATION.** Let us briefly recall some definitions and spectral properties used below (see [16]). Let  $\mathcal{L}$  be a linear operator acting in Banach spaces. The set of all complex numbers  $\lambda$  such that the operator  $\mathcal{L}_\lambda = \mathcal{L} - \lambda$  does not satisfy the Fredholm property is called the essential spectrum of the operator  $\mathcal{L}$ . The essential spectrum splits the complex plane into a finite or a countable number of connected components. Inside each such connected component, the index  $\kappa = \alpha - \beta$  of the operator  $\mathcal{L}_\lambda$ , where  $\alpha$  is the dimension of the kernel and  $\beta$  the codimension of the image,



has a constant value. The numbers  $\alpha$  and  $\beta$  are also constants except, possibly, for some discrete set of  $\lambda$ . Next, suppose that the essential spectrum of the operator  $\mathcal{L}$  belongs to a half-plane  $\operatorname{Re} \lambda \leq r$  of the complex plane, where  $r$  is a real number, and the index of the operator equals 0 in the complementing half-plane  $\operatorname{Re} \lambda > r$ . This property is satisfied for general elliptic problems in unbounded domains [20]. For any  $\epsilon > 0$ , the half-plane  $\operatorname{Re} \lambda \geq r + \epsilon$  contains a finite number of points  $\lambda$  of the spectrum. The operator  $\mathcal{L}_\lambda$  satisfies the Fredholm property for such  $\lambda$  but it is not invertible, and  $\alpha = \beta$ . These points  $\lambda$  are eigenvalues of finite multiplicity. The eigenvalue with the maximal real part is called the principal eigenvalue. Due to the Krein–Rutman theorem applicable for the scalar second-order elliptic problems in bounded domains, the principal eigenvalue is real, simple, and the corresponding eigenfunction is positive. Moreover, there are no other positive eigenfunctions. This result is generalized for unbounded domains [21] (see also [3, 22]).

We will now use these spectral properties in order to study the solvability of the equation

$$(2.4) \quad \Delta u + uf(I(u)) - \sigma(x)u = 0,$$

where  $I(u) = \int_{\mathbb{R}^n} u(x)dx$ ,  $f(\cdot)$  is a continuous function,  $\sigma(x) \in C^\alpha(\mathbb{R}^n)$ ,  $\sigma(x) \rightarrow \sigma_0$  as  $|x| \rightarrow \infty$ ,  $\sigma_0$  is a positive constant. We will see below that the solution of equation (2.4) is an eigenfunction of a Fredholm operator. Therefore, it decays exponentially at infinity [20], and the integral  $I(u)$  is well defined. We will obtain the conditions for the existence of a positive integrable solution of this equation.

**THEOREM 2.6:** *Suppose that the principal eigenvalue  $\lambda_0$  of the operator*

$$\hat{L}_0 = \Delta u - \sigma(x)u$$

*satisfies the inequality  $\lambda_0 > -\sigma_0$ . Then equation (2.4) has a positive integrable solution if and only if the equation*

$$(2.5) \quad f(h) = -\lambda_0$$

*has a positive solution  $h$ .*

*Proof.* The essential spectrum of the operator  $\hat{L}_0$  fills the half-axis  $\lambda \leq -\sigma_0$ . Since  $\lambda_0 > -\sigma_0$ , then the principal eigenvalue  $\lambda_0$  is real and simple, the corresponding eigenfunction  $u_0(x)$  is positive, and there are no positive eigenfunctions corresponding to other eigenvalues [21].

Suppose that equation (2.5) has a positive solution  $h = h_0$ . Since  $\lambda_0$  is the principal eigenvalue of the operator  $\hat{L}_0$  with a positive eigenfunction  $u_0(x)$ , then  $u_0(x)$  is a solution of the equation

$$(2.6) \quad \Delta u - \sigma(x)u = -f(h_0)u.$$

The eigenfunction  $u_0(x)$  is exponentially decaying at infinity [20], and the integral  $I(u_0)$  is well defined. For any real constant  $c$ , the function  $u(x) = cu_0(x)$  is also a solution of equation (2.6). We can choose  $c$  in such a way that  $I(u) = h_0$ . Then  $u$  is a solution of equation (2.4).

Next, suppose that  $u(x)$  is a positive integrable solution of equation (2.4). Set  $h = I(u)$ . Then  $u(x)$  is a solution of the equation

$$\Delta u - \sigma(x)u = -f(h)u.$$

Since a positive eigenfunction corresponds only to the principal eigenvalue [25], then  $-f(h) = \lambda_0$ . The theorem is proved.    ■

*Example 2.7:* Set  $f(h) = 1 - h$ , and suppose that  $\sigma(x) \geq 0$  for all  $x \in \mathbb{R}^n$ ,  $\sigma_0 > 1$ ,  $-1 < \lambda_0 < 0$ . Then equation (2.5) has a positive solution  $h$ . According to the theorem, equation (2.4) has a positive integrable solution.

**2.5. SPECTRAL PROPERTIES OF OPERATORS WITH DISCONTINUOUS COEFFICIENTS.** In the analysis of equation (2.4) we used the spectral properties of the operator  $\hat{L}_0$ , namely, the location of its essential spectrum and the properties of the principal eigenvalue. In order to apply these properties, we assumed that the function  $\sigma(x)$  is Hölder continuous. Theorem 2.4 allows us to determine the essential spectrum for a wider class of functions including discontinuous functions if the discontinuities belong to a bounded domain on  $\mathbb{R}^n$ . Let us now discuss the properties of the principal eigenvalue in the case of discontinuous coefficients. We will verify that the corresponding eigenfunction is positive and that there are no positive eigenfunctions corresponding to other eigenvalues.

Consider the operator

$$\hat{L}_0 u = \Delta u - \sigma(x)u$$

assuming that  $\sigma(x) = \sigma_0(x) + \sigma_1(x)$ , where  $\sigma_0(x) \in C^\alpha(\mathbb{R}^n)$  and  $\sigma_1(x)$  is a bounded function with a bounded support,  $\sigma(x) \geq 0$  for all  $x \in \mathbb{R}^n$ . Suppose that  $\sigma_0(x) \rightarrow \sigma_0$  as  $|x| \rightarrow \infty$ , where  $\sigma_0$  is a positive constant, and the principal eigenvalue  $\lambda_0$  of the operator  $L_0$  satisfies the inequality  $\lambda_0 > -\sigma_0$ .

LEMMA 2.8: *The eigenfunction  $u_0(x)$  corresponding to the principle eigenvalue  $\lambda_0$  is positive.*

*Proof.* Consider a sequence of functions  $\sigma^k(x) \in C^\alpha(\mathbb{R}^n)$  which coincide with  $\sigma_0(x)$  for  $|x| \geq R$  for some  $R$  sufficiently large, and  $\sigma^k(x) \rightarrow \sigma(x)$  in  $L^2(\mathbb{R}^n)$ . We suppose that the support of the function  $\sigma_1(x)$  is inside the ball  $B_R$  of radius  $R$ . The corresponding operators

$$\hat{L}^k u = \Delta u - \sigma^k(x)u$$

have the same limiting operators and essential spectrum as the operator  $\tilde{L}_0$ , and they satisfy the Fredholm property. Denote by  $\lambda_k$  the principle eigenvalue of the operator  $\hat{L}^k$ , and by  $u_k(x)$  the corresponding eigenfunction.

Since  $\|u\|_{C(\mathbb{R}^n)} \leq \|u\|_{H_\infty^2(\mathbb{R}^n)}$  for  $u \in H_\infty^2(\mathbb{R}^n)$  ( $n = 1, 2, 3$ ), we have

$$\begin{aligned} \|\hat{L}^k u - \hat{L}_0 u\|_{L_\infty^2(\mathbb{R}^n)} &\leq \left( \int_{B_R} (\sigma^k(x) - \sigma(x))^2 u^2(x) dx \right)^{1/2} \\ &\leq \|u\|_{H_\infty^2(\mathbb{R}^n)} \left( \int_{B_R} (\sigma^k(x) - \sigma(x))^2 dx \right)^{1/2} \rightarrow 0, \quad k \rightarrow \infty. \end{aligned}$$

Therefore, the operators converge in the operator norm.

By virtue of the properties of Fredholm operators,  $\lambda_k \rightarrow \lambda_0$ , and  $u_k \rightarrow u_0$  as  $k \rightarrow \infty$ , where  $u_0(x)$  is the eigenfunction of the operator  $\hat{L}_0$  corresponding to the eigenvalue  $\lambda_0$  [16]. Since  $u_k(x) > 0$  for all  $x$ , then  $u_0(x) \geq 0$ . Let us show that this inequality is strict. Suppose that  $u_0(x_0) = 0$  for some  $x_0$ . If  $\sigma(x)$  is continuous at  $x = x_0$ , then we obtain a contradiction with the usual positiveness theorem. Therefore,  $u_0(x) > 0$  for  $|x| > R$ , where  $\sigma(x)$  is continuous. We can now apply the generalized maximum principle to affirm that  $u_0(x) > 0$  for  $|x| \leq R$ . The lemma is proved. ■

LEMMA 2.9: *The eigenfunction  $u_0(x)$  corresponding to the principle eigenvalue  $\lambda_0$  is the only positive eigenfunction.*

*Proof.* We will prove that a positive eigenfunction cannot exist for other eigenvalues and that a positive eigenfunction is unique for the eigenvalue  $\lambda_0$  up to a constant factor.

Suppose that there exists a positive eigenfunction  $u_1(x)$  for a real eigenvalue  $\lambda_1 < \lambda_0$ . As in the proof of the previous lemma, consider a sequence of smoothed problems with the corresponding sequence of eigenvalues  $\lambda_1^k \rightarrow \lambda_1$ .

Since the corresponding eigenfunctions  $u_1^k(x)$  converge to  $u_1(x)$  locally uniformly, then  $u_1^k(x) > 0$  for  $|x| = N$  for  $N$  fixed and for  $k$  sufficiently large. If  $N$  is sufficiently large, then  $u_1^k(x) > 0$  for  $|x| \geq N$  (see [24], Lemma 3.2). Therefore,  $u_1^k(x_k) \leq 0$  for some  $x_k$ ,  $|x_k| \leq N$ . Passing to the limit, we conclude that the function  $u_1(x)$  is not positive. This contradiction proves the required assertion.

If a positive eigenfunction  $u_1(x)$  corresponds to the principal eigenvalue  $\lambda_0$  and it is linearly independent with the eigenfunction  $u_0(x)$ , then consider the function  $z(x) = \tau u_0(x) - u_1(x)$ . Let us take the minimal value of  $\tau$  for which  $z(x) \geq 0$  for  $|x| \leq N$  and for the same  $N$  as above. Then  $z(x) \geq 0$  for  $|x| \geq N$ . Hence  $z(x) \geq 0$  for all  $x \in \mathbb{R}^n$ . If  $z(x) > 0$  for all  $|x| \leq N$ , then  $\tau$  is not minimal, and we obtain a contradiction with the assumption above. If  $z(x) \geq 0$  and  $z(x_0) = 0$  for some  $x_0$ , then we obtain a contradiction with the maximum principle. The lemma is proved.    ■

We proved in Lemmas 2.8 and 2.9 that the properties of the principal eigenfunction remain valid for the operators with discontinuous coefficients. These properties were used in Theorem 2.6. Therefore, this theorem is now applicable for the operators with discontinuous coefficients. Similar results can be obtained for more general operators.

### 3. Persistence of solutions

3.1. OPERATORS AND SPACES. Along with the spaces  $L_\infty^2(\mathbb{R}^n)$  and  $H_\infty^2(\mathbb{R}^n)$  introduced in the previous section, we consider the corresponding weighted spaces  $L_{\infty,\mu}^2(\mathbb{R}^n)$  and  $H_{\infty,\mu}^2(\mathbb{R}^n)$  with the norms

$$\|u\|_{L_{\infty,\mu}^2(\mathbb{R}^n)} = \|u\mu\|_{L_\infty^2(\mathbb{R}^n)}, \quad \|u\|_{H_{\infty,\mu}^2(\mathbb{R}^n)} = \|u\mu\|_{H_\infty^2(\mathbb{R}^n)}$$

and the weight function  $\mu(x) = \exp(\nu\sqrt{1+|x|^2})$ . The value of constant  $\nu > 0$  will be specified below.

Consider the operator

$$A(u, \tau) = \Delta u + F(u, I(u), x, \tau)$$

depending on a parameter  $\tau \in [0, 1]$  and acting from  $E_1 = H_{\infty,\mu}^2(\mathbb{R}^n)$  into  $E_2 = L_{\infty,\mu}^2(\mathbb{R}^n)$ . Here  $I(u) = \int_{\mathbb{R}^n} u(x) dx$ ,  $F(u, v, x, \tau)$  is a real function of the variables  $u, v, x$  and of parameter  $\tau$ . Since we consider the space with an exponential weight, the integral  $I(u)$  is well defined. We will assume that the function  $F(u, v, x, \tau)$  satisfies the following conditions.

*Condition 1:* The following inequalities hold:

$$(3.1) \quad |F(u_1, v, x, \tau) - F(u_2, v, x, \tau)| \leq K|u_1 - u_2|,$$

$$(3.2) \quad |F(u, v_1, x, \tau) - F(u, v_2, x, \tau)| \leq K|v_1 - v_2||u|,$$

$$(3.3) \quad |F(u, v, x, \tau_1) - F(u, v, x, \tau_2)| \leq Kg(x, \tau_1, \tau_2)|u|,$$

where  $K$  is a positive constant, and

$$\|g(x, \tau_1, \tau_2)\|_{L^2_{\infty, \mu}(\mathbb{R}^n)} \rightarrow 0 \quad \text{as } |\tau_1 - \tau_2| \rightarrow 0.$$

These inequalities hold for any  $u_1, u_2, v_1, v_2$  such that  $|u_i|, |v_i| \leq M, i=1,2$ , where  $M$  is an arbitrary positive constant, and  $K$  depends on  $M$ , and it is independent of  $x \in \mathbb{R}^n; \tau_1, \tau_2 \in [0, 1]$ .

*Condition 2:* The following inequalities hold:

$$(3.4) \quad |F'_u(u_1, v, x, \tau) - F'_u(u_2, v, x, \tau)|, |F'_v(u_1, v, x, \tau) - F'_v(u_2, v, x, \tau)| \leq K|u_1 - u_2|,$$

$$(3.5) \quad |F'_u(u, v_1, x, \tau) - F'_u(u, v_2, x, \tau)|, |F'_v(u, v_1, x, \tau) - F'_v(u, v_2, x, \tau)| \leq K|v_1 - v_2||u|,$$

$$(3.6) \quad |F'_u(u, v, x, \tau_1) - F'_u(u, v, x, \tau_2)|, |F'_v(u, v, x, \tau_1) - F'_v(u, v, x, \tau_2)| \leq Kg(x, \tau_1, \tau_2),$$

where  $K$  is a positive constant, and

$$\|g(x, \tau_1, \tau_2)\|_{L^2_{\infty, \mu}(\mathbb{R}^n)} \rightarrow 0 \quad \text{as } |\tau_1 - \tau_2| \rightarrow 0.$$

These inequalities hold for any  $u_1, u_2, v_1, v_2$  such that  $|u_i|, |v_i| \leq M, i = 1, 2$ , where  $M$  is an arbitrary positive constant, and  $K$  depends on  $M$ , and it is independent of  $x \in \mathbb{R}^n; \tau_1, \tau_2 \in [0, 1]$ .

*Condition 3:* There exists a positive constant  $N$  such that

$$F'_u(u, v, x, \tau), F'_v(u, v, x, \tau) \in C^\alpha(\mathbb{R}_N)$$

for any  $u, v$ , and  $\tau \in [0, 1]$  fixed, where  $\mathbb{R}_N = \{x \in \mathbb{R}^n, |x| \geq N\}, \alpha \in (0, 1)$ . Furthermore,

$$F'_u(0, 0, x, \tau), F'_v(0, 0, x, \tau) < -\epsilon$$

for all  $|x| \geq N, \tau \in [0, 1]$ , and some  $\epsilon > 0$ .

LEMMA 3.1: *If Condition 1 is satisfied, then the operator*

$$A(u, \tau) : E_1 \times R \rightarrow E_2$$

*is bounded and continuous.*

*Proof.* Let  $u_1, u_2 \in B_\rho \subset E_1$ , where  $B_\rho$  is a ball with radius  $\rho$ . Then

$$\|u_i\|_{C(\mathbb{R}^n)} \leq M, \quad i = 1, 2,$$

where the constant  $M$  depends on  $\rho$ . From (3.1)–(3.3) we have the following estimates:

$$\begin{aligned} & |A(u_1, \tau_1) - A(u_2, \tau_2)| \\ & \leq |\Delta(u_1 - u_2)| + K(|u_1 - u_2| + |I(u_1) - I(u_2)||u_2| + g(x, \tau_1, \tau_2)|u_2|). \end{aligned}$$

Here and in what follows  $K$  denotes any constant which depends only on  $\rho$ . Then

$$\begin{aligned} (3.7) \quad & \|A(u_1, \tau_1) - A(u_2, \tau_2)\|_{L^2_{\infty, \mu}(\mathbb{R}^n)} \leq K\|u_1 - u_2\|_{H^2_{\infty, \mu}(\mathbb{R}^n)} \\ & + K|I(u_1) - I(u_2)|\|u_2\|_{L^2_{\infty, \mu}(\mathbb{R}^n)} \\ & + K\|u_2\|_{C(\mathbb{R}^n)}\|g(x, \tau_1, \tau_2)\|_{L^2_{\infty, \mu}(\mathbb{R}^n)}. \end{aligned}$$

Since

$$\begin{aligned} |I(u_1) - I(u_2)| & \leq \int_{\mathbb{R}^n} \frac{1}{\mu(x)} |u_1(x) - u_2(x)| \mu(x) dx \\ & \leq K\|(u_1 - u_2)\mu\|_{C(\mathbb{R}^n)} \\ & \leq \|u_1 - u_2\|_{H^2_{\infty, \mu}(\mathbb{R}^n)}, \end{aligned}$$

then it follows from (3.7) that

$$\|A(u_1, \tau_1) - A(u_2, \tau_2)\|_{L^2_{\infty, \mu}(\mathbb{R}^n)} \rightarrow 0 \quad \text{as } \|u_1 - u_2\|_{H^2_{\infty, \mu}(\mathbb{R}^n)} \rightarrow 0, \quad |\tau_1 - \tau_2| \rightarrow 0.$$

Hence the operator is continuous. Its boundedness obviously follows from (3.7). The lemma is proved.  $\blacksquare$

Next, consider the linearized operator

$$L(u_0, \tau_0)u = \Delta u + F'_u(u_0, I(u_0), x, \tau_0)u + F'_v(u_0, I(u_0), x, \tau_0)I(u).$$

LEMMA 3.2: *If Conditions 2 and 3 are satisfied, then the operator  $L(u_0, \tau_0)$  is continuous with respect to  $(u_0, \tau_0)$  in the operator norm.*

*Proof.* We have

$$L(u_1, \tau_1)u - L(u_0, \tau_0)u = (F'_u(u_1, I(u_1), x, \tau_1) - F'_u(u_0, I(u_0), x, \tau_0))u + (F'_v(u_1, I(u_1), x, \tau_1) - F'_v(u_0, I(u_0), x, \tau_0))I(u).$$

It follows from Condition 2 that

$$\begin{aligned} & \| (F'_u(u_1, I(u_0), x, \tau_0) - F'_u(u_0, I(u_0), x, \tau_0))u \|_{L^2_{\infty, \mu}(\mathbb{R}^n)} \\ & \leq K \| u_1 - u_2 \|_{C(\mathbb{R}^n)} \| u \|_{L^2_{\infty, \mu}(\mathbb{R}^n)}, \\ & \| (F'_u(u_1, I(u_1), x, \tau_0) - F'_u(u_1, I(u_0), x, \tau_0))u \|_{L^2_{\infty, \mu}(\mathbb{R}^n)} \\ & \leq K \| I(u_1) - I(u_0) \| \| u \|_{L^2_{\infty, \mu}(\mathbb{R}^n)}, \\ & \| (F'_u(u_1, I(u_1), x, \tau_1) - F'_u(u_1, I(u_1), x, \tau_0))u \|_{L^2_{\infty, \mu}(\mathbb{R}^n)} \\ & \leq K \| g(x, \tau_1, \tau_2)u \|_{L^2_{\infty, \mu}(\mathbb{R}^n)} \\ & \leq K \| g(x, \tau_1, \tau_2) \|_{L^2_{\infty, \mu}(\mathbb{R}^n)} \| u \|_{C(\mathbb{R}^n)} \\ & \leq K \| g(x, \tau_1, \tau_2) \|_{L^2_{\infty, \mu}(\mathbb{R}^n)} \| u \|_{H^2_{\infty, \mu}(\mathbb{R}^n)}, \end{aligned}$$

and from Condition 3 that

$$\begin{aligned} & \| (F'_v(u_1, I(u_0), x, \tau_0) - F'_v(u_0, I(u_0), x, \tau_0))I(u) \|_{L^2_{\infty, \mu}(\mathbb{R}^n)} \\ & \leq K \| I(u) \| \| u_1 - u_2 \|_{L^2_{\infty, \mu}(\mathbb{R}^n)}, \\ & \| (F'_v(u_1, I(u_1), x, \tau_0) - F'_v(u_1, I(u_0), x, \tau_0))I(u) \|_{L^2_{\infty, \mu}(\mathbb{R}^n)} \\ & \leq \| I(u_1) - I(u_0) \| \| I(u) \| \| u_1 \|_{L^2_{\infty, \mu}(\mathbb{R}^n)}, \\ & \| (F'_v(u_1, I(u_1), x, \tau_1) - F'_v(u_1, I(u_1), x, \tau_0))I(u) \|_{L^2_{\infty, \mu}(\mathbb{R}^n)} \\ & \leq K \| I(u) \| \| g(x, \tau_1, \tau_2) \|_{L^2_{\infty, \mu}(\mathbb{R}^n)}. \end{aligned}$$

Taking into account that

$$\| u_1 - u_2 \|_{C(\mathbb{R}^n)}, \| I(u_1) - I(u_0) \| \leq K \| u_1 - u_2 \|_{H^2_{\infty, \mu}(\mathbb{R}^n)},$$

we conclude from the previous estimates that

$$\begin{aligned} & \| L(u_1, \tau_1)u - L(u_0, \tau_0)u \|_{L^2_{\infty, \mu}(\mathbb{R}^n)} \\ & \leq K \| u_1 - u_2 \|_{H^2_{\infty, \mu}(\mathbb{R}^n)} \| u \|_{H^2_{\infty, \mu}(\mathbb{R}^n)} + \| g(x, \tau_1, \tau_2) \|_{L^2_{\infty, \mu}(\mathbb{R}^n)} \| u \|_{H^2_{\infty, \mu}(\mathbb{R}^n)}. \end{aligned}$$

Therefore,

$$\| L(u_1, \tau_1)u - L(u_0, \tau_0)u \| \rightarrow 0$$

in the operator norm as  $\| u_1 - u_2 \|_{H^2_{\infty, \mu}(\mathbb{R}^n)} \rightarrow 0$  and  $|\tau_1 - \tau_2| \rightarrow 0$ . The lemma is proved. ■

LEMMA 3.3: *If Conditions 2 and 3 are satisfied, then the operator  $L(u_0, \tau_0)$  satisfies the Fredholm property with the zero index.*

*Proof.* Consider, first, the operator

$$L_1 u = \Delta u + F'_u(u_0, I(u_0), x, \tau_0)u.$$

By virtue of Conditions 2 and 3, we can represent the derivative as a sum of two functions,

$$F'_u(u_0, I(u_0), x, \tau_0) = a_0(x) + a_1(x),$$

where  $a_0(x) \in C^\alpha(\mathbb{R}^n)$  and  $a_1(x)$  is a bounded function with a bounded support.

The operator  $L_1^0 u = \Delta u + a_0(x)u$  is normally solvable with a finite-dimensional kernel. Indeed, all its limiting operators  $\hat{L}u = \Delta u + \hat{a}(x)u$  are invertible since the limiting coefficients are such that  $\hat{a}(x) \leq -\epsilon < 0$  for all  $x \in \mathbb{R}^n$  [20]. Furthermore, it satisfies the Fredholm property, and it has the zero index since it can be reduced by a continuous deformation to an invertible operator in the class of normally solvable operators with a finite-dimensional kernel. From Theorem 2.4 it follows that the operator  $L_1$  is Fredholm with the zero index. It remains to note that the operator  $L(u_0, \tau_0)$  differs from the operator  $L_1$  by a finite-dimensional operator which does not change the Fredholm property and index. The lemma is proved. ■

Example 3.4: Consider the following function arising in various applications:

$$F(u, I(u), x, \tau) = u(1 - I(u)) - \sigma_\tau(x)u.$$

Suppose that  $\sigma_\tau(x) = \sigma_0(x) + \sigma_{1,\tau}(x)$ , where  $\sigma_0(x) \in C^\alpha(\mathbb{R}^n)$ , and  $\sigma_{1,\tau}(x)$  is a bounded function with a bounded support such that  $\sigma_{1,\tau_1}(x) \rightarrow \sigma_{1,\tau_0}(x)$  in  $L^2(\mathbb{R}^n)$  as  $\tau_1 \rightarrow \tau_0$ . Then the function  $F$  satisfies Condition 1, and its derivatives satisfy Condition 2. Finally, if there exist limits  $\sigma_0(\pm\infty) < 0$ , then Condition 3 is also satisfied.

3.2. IMPLICIT FUNCTION THEOREM. The results of the previous section allow us to use the implicit function theorem in order to study the persistence of solutions under small perturbations.

THEOREM 3.5: *Suppose that Conditions 1–3 are satisfied, and  $A(u_0, \tau_0) = 0$  for some  $u_0 \in H^2_{\mu,\infty}(\mathbb{R}^n)$  and  $\tau_0 \in (0, 1)$ . If the linearized operator  $A'_u(u_0, \tau_0)$  does not have a zero eigenvalue, then the equation  $A(u, \tau) = 0$  has a solution  $u \in H^2_{\mu,\infty}(\mathbb{R}^n)$  for all  $\tau$  sufficiently close to  $\tau_0$ .*



The proof of the theorem directly follows from Lemmas 3.1–3.3 and the implicit function theorem.

*Example 3.6:* In the previous theorem, we supposed that the linearized operator does not have a zero eigenvalue. Consider an example where this condition can be explicitly verified. Suppose that the equation

$$(3.8) \quad \Delta u + au(1 - I(u)) - \sigma(x)u = 0$$

has a positive solution  $u_0(x)$  decaying at infinity. Here  $a > 0$ , and the function  $\sigma(x)$  satisfies the conditions of Example 3.4. Consider the nonlinear operator

$$A(u) = \Delta u + au(1 - I(u)) - \sigma(x)u$$

and the operator  $L$  linearized about  $u_0$ :

$$Lv = A'(u_0)v = \Delta v + av(1 - I(u_0)) - \sigma(x)v - au_0(x)I(v).$$

Suppose that it has a zero eigenvalue. Then

$$(3.9) \quad \Delta v + av(1 - I(u_0)) - \sigma(x)v = au_0(x)I(v).$$

The operator

$$\mathcal{L}v = \Delta v + av(1 - I(u_0)) - \sigma(x)v$$

has a zero eigenvalue with the eigenfunction  $u_0(x)$ . Since it is positive, then  $\lambda=0$  is the principal eigenvalue of this operator, and it is simple (Lemmas 2.8 and 2.9). Multiplying equation (3.9) by  $u_0(x)$  and integrating, we conclude that

$$I(v) \int_{-\infty}^{\infty} u_0^2(x) dx = 0.$$

Hence,  $I(v) = 0$ , and from (3.9) we obtain

$$(3.10) \quad \Delta v + av(1 - I(u_0)) - \sigma(x)v = 0.$$

Since  $v = u_0(x)$  is the only solution of this equation, and it is positive, we obtain a contradiction with the equality  $I(v) = 0$ . This contradiction shows that the linearized operator does not have a zero eigenvalue, and the implicit function theorem is applicable.

#### 4. Existence of solutions

4.1. TOPOLOGICAL DEGREE. In this section, we will prove the existence of solutions of equation (1.1) by the Leray–Schauder method using the topological degree for elliptic operators in unbounded domains [20]. The construction of the degree for Fredholm and proper operators is based on the notion of orientation  $o(\mathcal{L})$  for a linear Fredholm operator  $\mathcal{L}$ . It is given by the formula  $(-1)^\nu$ , where  $\nu$  is the sum of multiplicities of all its positive eigenvalues. The orientation of the operator is a homotopy invariant, it does not change under a continuous deformation in the class of Fredholm operators. Next, we consider the equation  $\mathcal{A}(u) = a$  for a nonlinear Fredholm operator  $\mathcal{A}$ . It is possible to choose sufficiently small  $a$  (in the norm of the space) such that this operator equation has a finite number of solutions  $u_i$ , and the sum of the orientations

$$\gamma = \sum_i o(\mathcal{A}'(u_i))$$

does not depend on  $a$ . Here  $\mathcal{A}'(u_i)$  is the operator linearized about the solution  $u_i$ . The integer value  $\gamma$  is the topological degree.

In order to define the degree, the operator  $A(u, \tau) : E_1 \times R \rightarrow E_2$  (Section 2.1) should be:

- (a) proper with respect to both variables  $u$  and  $\tau$ ,
- (b) it is assumed that the linearized operator  $Lv = A'_u(u, \tau)v$  satisfies the Fredholm property together with the operator  $L_\lambda v = Lv - \lambda v$  for all real  $\lambda \geq 0$ , and
- (c) the operator  $L_\lambda$  has a uniformly bounded inverse for all  $\lambda \geq \lambda_0$  for some  $\lambda_0$  depending on the operator  $A_\tau$ .

We will verify these conditions assuming that the function  $F(u, v, x, \tau)$  satisfies the following condition in addition to Conditions 1–3 formulated above.

*Condition 4:* Function  $F(u, v, x, \tau)$  is bounded together with its first derivative with respect to  $x \in \mathbb{R}^n$ , and up to the third derivative with respect to  $u$  in any bounded interval. It also satisfies estimates (3.2) and (3.3) in Condition 1. Furthermore,

$$|F'_v(u, v, x, \tau)| \leq K|u|$$

for any  $|u|, |v| \leq M$ ,  $\tau \in [0, 1]$ , where  $K$  depends on  $M$ .

LEMMA 4.1: *If Conditions 3 and 4 are satisfied, then the operator  $A(u, \tau)$  is proper with respect to both variables  $u$  and  $\tau$ , that is, the inverse image of any compact set in  $E_2$  is compact in any bounded closed set in  $E_1 \times R$ .*

*Proof.* First, we prove that the operator  $A(u, \tau)$  is proper for a fixed  $\tau$ . Consider a bounded sequence  $u_k \in E_1$  and suppose that it converges locally in  $C$  to some  $u_0 \in E_1$ . It is proved in [20] (Lemma 2.1, Chapter 11) that the operator is proper if the convergence

$$(4.1) \quad \|A(u_0, \tau) - A(u_k, \tau) - L(u_0)(u_0 - u_k)\|_{E_2} \rightarrow 0$$

occurs for any sequence  $u_k$  and its limiting function  $u_0$ , and the operator  $L(u_0)$  has a closed range and a finite-dimensional kernel. Here  $L(u_0) = A'_u(u_0, \tau)$ . The required properties of the operator  $L(u_0)$  follow from Condition 3.

Convergence (4.1) is proved for conventional elliptic operators without the integral term, that is, it holds for the operator

$$A_0(u, \tau) = \Delta u + F(u, I(u_0), x, \tau).$$

We have

$$A(u_0, \tau) - A(u_k, \tau) = A_0(u_0, \tau) - A_0(u_k, \tau) + A_0(u_k, \tau) - A(u_k, \tau).$$

From convergence (4.1) for the operator  $A_0$

$$\|A_0(u_0, \tau) - A_0(u_k, \tau) - L(u_0)(u_0 - u_k)\|_{E_2} \rightarrow 0,$$

and from (3.2)

$$\begin{aligned} \|A_0(u_k, \tau) - A(u_k, \tau)\|_{E_2} &= \|F(u_k, I(u_0), x, \tau) - F(u_k, I(u_k), x, \tau)\|_{E_2} \\ &\leq |I(u_0) - I(u_k)| \|u_k\|_{E_2}. \end{aligned}$$

It remains to show that  $|I(u_0) - I(u_k)| \rightarrow 0$ . Indeed,

$$I(u_0) - I(u_k) = \int_{\mathbb{R}^n} \frac{1}{\mu(x)} ((u_0(x) - u_k(x))\mu(x)) dx.$$

Since the function  $1/\mu(x)$  is integrable, and the functions

$$z_k(x) = (u_0(x) - u_k(x))\mu(x)$$

are uniformly bounded and locally converge to 0, then the desired convergence takes place. Thus, convergence (4.1) is shown.

In order to prove the properness of the operator  $A(u, \tau)$  with respect to both variables, it is sufficient to verify that the operator is continuous with respect to  $\tau$  in the operator norm (Section 2.3, Chapter 11, [20]). We have

$$\begin{aligned} \|A(u, \tau) - A(u, \tau_0)\|_{E_2} &= \|F(u, I(u), x, \tau) - F(u, I(u), x, \tau_0)\|_{E_2} \\ &\leq K \|g(x, \tau_1, \tau_2)|u|\|_{E_2}. \end{aligned}$$

It remains to note that  $|u|$  is bounded and  $\|g(x, \tau_1, \tau_2)|u|\|_{E_2} \rightarrow 0$ . The lemma is proved.    ■

LEMMA 4.2: *If Conditions 3 and 4 are satisfied, then the operator  $L_\lambda v = Lv - \lambda v$  satisfies the Fredholm property for all real  $\lambda \geq 0$ , and it has a uniformly bounded inverse for all  $\lambda \geq \lambda_0$  for some  $\lambda_0$  depending on the operator  $A_\tau$ .*

*Proof.* The operator  $L_\lambda$ ,

$$L_\lambda v = \Delta v + F'_u(u, I(u), x, \tau)v + F'_v(u, I(u), x, \tau)I(v) - \lambda v,$$

satisfies Condition 3 for any  $\lambda \geq 0$ . Therefore, this is a Fredholm operator with the zero index. The operator

$$L_{0,\lambda}v = \Delta v + F'_u(u, I(u), x, \tau)v - \lambda v$$

is sectorial ([20], Chapter 6). For all  $\lambda$  sufficiently large it satisfies the estimate

$$\|v\|_{E_1} \leq K \frac{\|L_{0,\lambda}v\|_{E_2}}{|\lambda|}.$$

Hence, by virtue of Condition 4,

$$\begin{aligned} \|v\|_{E_1} &\leq K \frac{\|L_\lambda v\|_{E_2}}{|\lambda|} + K \frac{\|F'_v(u, I(u), x, \tau)I(v)\|_{E_2}}{|\lambda|} \\ &\leq K \frac{\|L_\lambda v\|_{E_2}}{|\lambda|} + K^2 \frac{\|u\|_{E_1} \|v\|_{E_1}}{|\lambda|}. \end{aligned}$$

For all  $\lambda$  sufficiently large, this inequality provides the estimate of the inverse to the operator  $L_\lambda$ . The lemma is proved.    ■

Lemmas 4.1 and 4.2 provide the existence and uniqueness of the topological degree for the considered class of operators. Therefore, we can use the Leray–Schauder method to prove the existence of solutions. In the next section, we obtain a priori estimates of solutions.

4.2. A PRIORI ESTIMATES. We consider the equation

$$(4.2) \quad \Delta u + au(1 - I(u)) - uf(u, x) = 0,$$

in  $\mathbb{R}^n$ ,  $n = 1, 2, 3$ , where  $x \in \mathbb{R}$ ,  $a > 0$ ,  $I(u) = \int_{\mathbb{R}^n} u(y)dy$ . We suppose that  $u(x)$  is integrable, and the integral  $I(u)$  is well defined.

*Condition 5:* Function  $f(u, x)$  is continuous together with its first partial derivatives,  $|f'_u(u, x)|, |f'_x(u, x)| \leq K$ , and  $0 \leq f(u, x) \leq K$  for all  $u \geq 0$ ,  $x \in \mathbb{R}^n$ .

*Condition 6:* Furthermore,  $f(u, x) - a \geq \epsilon > 0$  for all  $u \geq 0$ ,  $|x| \geq N$  and some positive  $N$ .

Under these conditions, we will obtain a priori estimates of solutions given by the following theorem.

**THEOREM 4.3:** *Suppose that the function  $f(u, x)$  satisfies Conditions 5 and 6. Then any positive solution  $u_0(x)$  of equation (4.2) decaying at infinity admits the estimate  $\|u_0\|_{H^2_{\mu, \infty}(\mathbb{R}^n)} \leq M$  with some positive constant  $M$  independent of the solution.*

The proof of the theorem will be preceded by some auxiliary results.

**LEMMA 4.4:** *Suppose that Condition 5 is satisfied and equation (4.2) has a positive integrable solution  $u_0(x)$  decaying at infinity,  $|u_0(x)| \rightarrow 0$  as  $|x| \rightarrow \infty$ . Then  $0 < I(u_0) < 1$ .*

*Proof.* The left inequality in the assertion of the lemma follows from the positiveness of the solution. If the right inequality does not hold, then we consider two cases,  $I(u_0) > 1$  and  $I(u_0) = 1$ . In the first case, we obtain a contradiction in signs in equation (4.2) at the point  $x = x_0$  where the function  $u_0(x)$  attains its maximum. In the second case, if  $f(u_0(x_0), x_0) > 0$ , we obtain the same contradiction in signs, as before. If  $f(u_0(x_0), x_0) = 0$ , then  $u_1(x) \equiv u_0(x_0)$  is a solution of equation (4.2). The function  $z(x) = u_1(x) - u_0(x)$  satisfies the corresponding linear equation

$$\Delta z + b(x)z = 0,$$

where

$$b(x) = (F(u_1(x), x) - F(u_0(x), x))/(u_1(x) - u_0(x)), \quad F(u, x) = uf(u, x).$$

Since  $z(x) \geq 0$  for all  $x \in \mathbb{R}^n$ , and  $z(x_0) = 0$ , we obtain a contradiction with the maximum principle. ■

Let us write equation (4.2) in the following form:

$$(4.3) \quad \Delta u = g(x),$$

where

$$g(x) = -au_0(x)(1 - I(u_0)) + u_0(x)f(u_0(x), x),$$

and  $u_0(x)$  is a positive solution of (4.2). Then  $u_0(x)$  is a solution of this equation, and by virtue of Theorem 3.2 (see Chapter 3, [20]), we have

$$(4.4) \quad \|u_0\|_{H^2_\infty(\mathbb{R}^n)} \leq k(\|g\|_{L^2_\infty(\mathbb{R}^n)} + \|u_0\|_{L^2_\infty(\mathbb{R}^n)}).$$

Here and below we denote by  $k$  positive constants independent of the solution.

LEMMA 4.5: *Under the conditions of Lemma 4.4, the following inequality holds:*

$$(4.5) \quad \|u_0\|_{C^{2+\alpha}(\mathbb{R}^n)} \leq k\|u_0\|_{C(\mathbb{R}^n)},$$

where  $0 < \alpha < 1$ ,  $k$  is a positive constant independent of the solution.

*Proof.* Note that

$$\|u_0\|_{L^2_\infty(\mathbb{R}^n)} \leq km,$$

where  $m = \sup_{x \in \mathbb{R}^n} |u_0(x)|$ . Due to Lemma 2.1, the function  $g(x)$  defined above admits the following estimate:

$$\|g\|_{L^2_\infty(\mathbb{R}^n)} \leq k(a + K)m.$$

Therefore, from (4.4),

$$\|u_0\|_{H^2_\infty(\mathbb{R}^n)} \leq k(a + K) \sup_{x \in \mathbb{R}^n} |u_0(x)|.$$

From the embedding theorems ( $n = 1, 2, 3$ ) it follows that

$$(4.6) \quad \|u_0\|_{C^\alpha(\mathbb{R}^n)} \leq k_1\|u_0\|_{C(\mathbb{R}^n)} \quad (0 < \alpha < 1).$$

Here and below  $k_1$  is a positive constant which depends only on  $a$  and  $K$ .

Next, we can estimate the Hölder norm of the function  $g(x)$  in (4.3):

$$\|g\|_{C^\alpha(\mathbb{R}^n)} \leq k_1\|u_0\|_{C^\alpha(\mathbb{R}^n)} \leq k_1\|u_0\|_{C(\mathbb{R}^n)}.$$

Finally, from the Schauder estimate applied for equation (4.3), we obtain estimate (4.5). ■

LEMMA 4.6: *Under the conditions of Lemma 4.4,*

$$(4.7) \quad u_0(x) \geq u_0(x_0) \left(1 - \frac{k_1}{2}(x - x_0)^2\right).$$

*Proof.* Let us recall that  $x_0$  is a global maximum of the function  $u_0(x)$ . This maximum exists since this function is positive and decaying at infinity. We consider an arbitrary straight line through the point  $x = x_0$ , and we will estimate the function  $u_0(x)$  from below along this line.

We set  $v(t) = u_0(x_0 + x^*t)$ , where  $x^*$  belongs to the unit sphere,  $|x^*| = 1$ . Then, the function  $v(t)$  is positive, decaying at infinity, it has a global maximum at  $t = 0$ ,  $v(0) = m$ , and, by virtue of Lemma 4.5,  $|v''(t)| \leq k_1 m$ . Hence,

$$|v'(t)| \leq \int_0^t |v''(y)| dy \leq k_1 m t, \quad v(t) \geq v(0) - \int_0^t |v'(y)| dy \geq v(0) - \frac{k_1 m}{2} t^2.$$

This estimate implies (4.7). ■

LEMMA 4.7: *In the conditions of Lemma 4.5,  $\|u_0\|_{C^{2+\alpha}(\mathbb{R}^n)} \leq k_1$ .*

*Proof.* From Lemmas 4.4 and 4.6, it follows that

$$1 > I(u_0) > \int_S u_0(x) dx > u_0(x_0) \int_S \left(1 - \frac{k_1}{2}(x - x_0)^2\right) dx,$$

where  $S$  is the ball  $|x - x_0| \leq \sqrt{\frac{2}{k_1}}$ . From this inequality, we get the estimate  $\|u_0\|_{C(\mathbb{R}^n)} \leq k_1$ . We recall that  $k_1$  here is an arbitrary constant which depends only on  $a$  and  $K$ . The assertion of the lemma follows now from the Schauder estimate. ■

*Remark 4.8:* A similar estimate remains valid for the more general equation

$$\Delta u + F(u, I(u), x) = 0$$

if  $F(u, v, x) < 0$  for  $v > v_0$  with some  $v_0 > 0$ , and all  $u$  and  $x$ .

*Proof of Theorem 4.3.* From Lemma 4.7 it follows that  $\sup_{x \in \mathbb{R}^n} |u_0(x)| \leq M$ . Here and below we denote by  $M$  positive constants independent of the solutions. Consider the equation  $\Delta u - \epsilon u = 0$  in the exterior domain  $|x| > N$  with the boundary condition  $u(x) = M$  for  $|x| = N$ . Here  $N$  is determined in Condition 6. It can be easily verified that  $u_0(x) \leq u(x)$  for  $|x| \geq N$ . Therefore,  $u_0(x) \leq M e^{\nu x}$  for all  $x \in \mathbb{R}^n$  and some  $\nu > 0$ , and  $\|u_0\|_{L_{\infty, \mu}(\mathbb{R}^n)} \leq M$ .

Set  $v_0(x) = u_0(x)\mu(x)$ . Then  $\|v_0\|_{L_{\mu}(\mathbb{R}^n)} \leq M$ . Estimate (4.4) considered for the corresponding operator completes the proof of the theorem. ■

### 4.3. EXISTENCE OF SOLUTIONS.

**LERAY–SCHAUDER METHOD.** We can now prove the existence of solutions by the Leray–Schauder method. Let us briefly recall the corresponding construction. Consider an operator  $A(u, \tau) : E_1 \rightarrow E_2$  acting in Banach spaces, and a bounded domain  $\Omega \subset E_1$ . Suppose that

$$A(u, \tau) \neq 0 \quad \text{for } u \in \partial\Omega, \tau \in [0, 1].$$

Denote by  $\gamma(\tau)$  the value of the degree corresponding to the operator  $A(u, \tau)$  and to the domain  $\Omega$ . If  $\gamma(0) \neq 0$ , then, by virtue of homotopy invariance of the degree,  $\gamma(1) \neq 0$ , and there exists a solution of the equation  $A(u, 1) = 0$  inside the domain  $\Omega$ . The operator  $A(u, 1)$  is the one for which we want to prove the existence of solutions, while  $A(u, 0)$  is a model operator for which we can prove that the degree is different from 0.

Thus, in order to prove the existence of solutions we need to construct a homotopy (continuous deformation) from some model operator  $A(u, 0)$  to the given operator  $A(u, 1)$  in such a way that the operator does not vanish at the boundary of the domain  $\partial\Omega$ . Usually,  $\Omega$  is taken to be a ball  $\mathcal{B}_R \subset E_1$  of a sufficiently large radius  $R$ . Then a priori estimates of solutions, which affirm that the norm of the solution is less than  $R$ , insure that there are no solutions at the boundary of the domain. Since we want to prove the existence of only positive solutions, we will use below a more complex construction of domain  $\Omega$ .

**MODEL OPERATOR.** Consider the operator

$$A_0(u) = \Delta u + au(1 - I(u)) - f(0, x)u.$$

Suppose that the principle eigenvalue  $\lambda_0$  of the operator

$$L_0 u = \Delta u - f(0, x)u$$

satisfies the inequality  $-a < \lambda_0 < 0$ . Then equation  $A_0(u) = 0$  has a single positive solution  $u_0(x)$  (Theorem 2.6), and the linearized operator  $A'(u_0)$  does not have zero eigenvalue (Example 3.6). The index of the stationary point  $\text{ind}(u_0)$ , that is, the value of the degree with respect to a small ball containing the solution, equals  $(-1)^\nu$ , where  $\nu$  is the number of positive eigenvalues of the linearized operator. It is well defined if the linearized operator does not have a zero eigenvalue. Depending on the value of  $\nu$ , it can be  $\pm 1$ . In what follows, we will use that  $\text{ind}(u_0) \neq 0$ .



EXISTENCE THEOREM. The main existence result of this work is given by the following theorem.

THEOREM 4.9: *Suppose that a function  $f(u, x)$  satisfies Conditions 5 and 6, and the derivatives  $f''_{u^2}$  and  $f'''_{u^3}$  are bounded and continuous (cf. Condition 4). Furthermore, the principal eigenvalue  $\lambda_0$  of the operator  $L_0u = \Delta u - f(0, x)u$  satisfies the inequality  $-a < \lambda_0 < 0$ . Then the equation*

$$\Delta u + au(1 - I(u)) - uf(u, x) = 0$$

has a positive solution in the space  $H^2_{\mu, \infty}(\mathbb{R}^n)$ .

*Proof.* We consider the operator  $A(u, \tau) : E_1 \rightarrow E_2$ , where  $E_1 = H^2_{\mu, \infty}(\mathbb{R}^n)$  and  $E_2 = L^2_{\mu, \infty}(\mathbb{R}^n)$ ,

$$A(u, \tau) = \Delta u + au(1 - I(u)) - u(\tau f(u, x) + (1 - \tau)f(0, x)).$$

Due to the imposed conditions and the results of the previous sections, it is a Fredholm and proper operator with the zero index, and the topological degree can be defined.

Since we want to prove the existence of positive solutions, and not to take into account the trivial solution  $u = 0$ , we will construct a domain  $\Omega$  which contains all positive solutions and which does not contain non-positive solutions (including the trivial solution). Set

$$D = \{u \in E_1, \delta < \|u\|_{E_1} < R, u(x) > 0, x \in \mathbb{R}^n\},$$

where  $R > M$ , and  $M$  is the constant in Theorem 4.3,  $\delta$  is a positive number. The value of  $\delta$  can be chosen sufficiently small such that any positive solution of the equation  $A(u, \tau) = 0$ ,  $\tau \in [0, 1]$  belongs to  $D$ . Indeed, suppose that this assertion does not hold, and there is a sequence of positive solutions  $u_k$  for  $\tau = \tau_k$  such that  $\|u\|_{E_1} \rightarrow 0$  as  $k \rightarrow \infty$ . Without loss of generality we can assume that  $\tau_k \rightarrow \tau_0 \in [0, 1]$ . Then the linearized operator

$$A'(0, \tau_0)v = \Delta v + av - f(0, x)v$$

has a zero eigenvalue with a positive eigenfunction. Hence, this is the principal eigenvalue. This contradicts the condition of the theorem.

It can be easily verified that if  $u_0$  is a solution of the equation  $A(u, \tau) = 0$  for some  $\tau \in [0, 1]$  and  $u_0 \in D$ , then any other solution  $u_1$  (possibly for a different  $\tau$ ) sufficiently close to  $u_0$  in the norm  $E_1$  also belongs to  $D$ . Indeed,  $u_1(x)$  is positive in any bounded domain since it is sufficiently close to a positive

solution  $u_0(x)$ . Its positiveness at infinity follows from Condition 6 which ensures that the solution cannot have a negative minimum. Thus, a ball  $B_\rho(u_0)$  with center  $u_0$  and a sufficiently small radius  $\rho_0$  contains only positive solutions (no trivial solution, negative or alternating sign solutions). It should be noted that  $B_\rho(u_0)$  is not a subset of  $D$  because a function from this ball (not a solution) can become negative for large  $|x|$ .

Consider the set  $S \subset D$  of all positive solutions of the equation  $A(u, \tau) = 0$ ,  $\tau \in [0, 1]$  and its covering with balls  $B_\rho(u)$ , where  $\rho$  can depend on the solution. Since the operator  $A(u, \tau)$  is proper with respect to an ensemble of two variables, the set  $S$  is compact, and we can choose a finite sub-covering  $\Omega$  of this covering. By virtue of the properties of the set  $D$ , domain  $\Omega$  contains all positive solutions, and it does not contain non-positive solutions. We consider the topological degree with respect to the domain  $\Omega$ .

The model problem  $A(u, 0) = 0$  has a single positive solution (in  $\Omega$ ), and  $\gamma(0) \neq 0$ . Since  $A(u, \tau) \neq 0$  for  $u \in \partial\Omega$ , then  $\gamma(1) \neq 0$ , and there exists a solution of equation  $A(u, 1) = 0$  in  $\Omega$ . The theorem is proved.    ■

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