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# A CONVERSE TO PITMAN'S THEOREM FOR A SPACE-TIME BROWNIAN MOTION IN A TYPE $A_1^1$ WEYL CHAMBER

MANON DEFOSSEUX AND CHARLIE HERENT

ABSTRACT. We prove an inverse Pitman's theorem for a space-time Brownian motion conditioned in Doob's sense to remain in an affine Weyl chamber. Our theorem provides a way to recover an unconditioned space-time Brownian motion from a conditioned one by applying a sequence of path transformations.

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## 1. INTRODUCTION

Let  $\{b(t), t \geq 0\}$  be a real brownian motion then Pitman's theorem [22] asserts that

$$\mathcal{P}b(t) = b_t - 2 \inf_{0 \leq s \leq t} b_s, t \geq 0,$$

is a Bessel process of dimension 3, which has the same distribution as a brownian motion conditioned, in Doob's sense, to remain in the positive half-line. This seminal result has given rise to many generalizations or variations, see for instance [1, 2, 7, 8, 19, 21, 24]. Let us briefly describe one of the most accomplished one, due to Ph. Biane, Ph. Bougerol and N. O'Connell [4, 5]. In Pitman's theorem, the unconditioned Brownian motion lives on  $\mathbb{R}$  and the conditioned one lives on  $\mathbb{R}_+$ . Actually  $\mathbb{R}_+$  can be seen as the fundamental chamber of the group generated by the reflection through 0 acting on  $\mathbb{R}$ . This group is the simplest one among the class of Coxeter groups. In [4] the authors have shown how to obtain a brownian motion conditioned to remain in the fundamental chamber of a finite Coxeter group by applying a sequence of Pitman type transformations associated to a set of generators of the Coxeter group, according to the order of appearance of the generators in a reduced decomposition of the longest element in the group. This paper has brought to light deep connections between the Pitman transform and the Littelmann path model [18] which is a combinatorial model that describes the representations of a Kac-Moody Lie algebra.

The affine Coxeter group of type  $A_1^1$  is the Weyl group of a rank one affine Kac–Moody algebra. In [6] another Pitman type theorem has been established for a conditioned random process living in the fundamental chamber of the latter group, whose interior is the subset  $C_{\text{aff}}$  defined below. Pitman’s theorem in that case involves two Pitman type transformations corresponding to the generators of the group and is only asymptotic. Since there is no longest element in that case one has to apply an infinite number of transformations. Moreover, quite surprisingly, the conditioned process is not obtained by applying successively and infinitely the two Pitman transforms to an unconditioned process : a correction has to be applied, which involves two Lévy type transformations.

One can formulate a converse to Pitman’s theorem, indeed given for  $T \geq 0$  a nonnegative continuous real trajectory  $\{\pi(t), t \in [0, T]\}$  starting at 0, and a real number  $x \in [0, \pi(T)]$ , there is a unique real trajectory  $\eta$  starting at 0 such that

$$\mathcal{P}\eta = \pi \text{ and } x = - \inf_{0 \leq s \leq T} \eta(s).$$

It satisfies  $\eta(t) = \pi(t) - 2 \min(x, \inf_{t \leq s \leq T} \pi(s))$ ,  $t \in [0, T]$ . In other words, a path defined on  $[0, T]$  is entirely determined by its image by the Pitman transform and a real number that we will call a string coordinate, according to the terminology of Littelmann. It follows that one can construct a standard real Brownian motion starting from a Bessel 3 process and a suitable real random variable. This construction generalizes to the case of finite Coxeter groups [4].

We propose to give an analog of this reconstruction for the case of the conditioned Brownian motion of [6]. Let us nevertheless notice that our reconstruction is of a very different nature from the one previously described in the context of a finite Coxeter group. In the latter case actually the reconstruction is a direct consequence of a deterministic result, whereas our result is a purely probabilistic one. This is a reconstruction in law.

We use results obtained in [6] but our approach is quite different from the one adopted in this last paper. Indeed the proof of Pitman’s theorem in [6] relies on some approximations of Brownian motions in the fundamental Weyl chamber of the affine Coxeter group in type  $A_1^1$  by Brownian motions in fundamental chambers of dihedral groups and the version of Pitman’s theorem for these groups established in [5]. Instead we use approximations by random walks defined using the Littelmann path model for the affine Kac-Moody algebra  $A_1^1$ . Such random walks have been originally introduced by C. Lecouvey, E. Lesigne and M. Peigné in [16].

It has been proved in [9] (see also [11]) that these last processes can also be approximated by random walks defined using the Littelmann path model for the affine Kac-Moody algebra  $A_1^1$ . Such random walks have been originally introduced by C. Lecouvey, E. Lesigne and M. Peigné in [16]. These are the approximations we use here. Their laws offer the advantage of being given by explicit formulas coming from representation theory, which allows to make computations. This is a huge advantage and makes our paper fall in the large category of the so-called integrable probability.

Demazure crystals play a crucial role in our paper. These crystals have beautiful combinatorial properties. Nevertheless, as far as we know, they haven’t been used before in the framework of integrable probability, which maybe can be explained by the fact that they do not form a tensor category, so that they do not

define an hypergroup structure which could naturally relate them to a Markov process in a usual way (see for instance [25] and references therein). Since the Littelmann model and Demazure character formulas that we use are available for any affine Kac-Moody algebra, they might be useful for obtaining an inverse Pitman's theorem in a more general context.

Let us make a last remark about our result. Actually, in the context of a finite Coxeter group, one can state another reconstruction theorem. In the simplest case, it states that if  $\{r_t, t \geq 0\}$  is a Doob-conditioned positive standard Brownian motion then

$$\{r_t - 2 \inf_{s \geq t} r_s, t \geq 0\}$$

is a real standard Brownian motion (see [23], chapter VI, corollary 3.7). More generally, for any finite Coxeter group, there exists such a functional transformation, which sends a conditioned Brownian motion to an unconditioned one. Such a result seems to be unattainable for  $A_1^1$ . Actually, in the finite case, the string coordinates of a Brownian motion are infinite and a Brownian motion stands morally for the lowest weight path in the Littelmann module of a Verma module. There is no such a lowest weight path in the case of  $A_1^1$ .

The paper is organized as follows. In section 2 we give a statement of an inverse Pitman's theorem for  $A_1^1$ . In section 3 we briefly recall the necessary background on representation theory of the affine Lie algebra  $A_1^1$ . The Littelmann path model for a Kac-Moody algebra  $A_1^1$  and its connection with Pitman transforms is explained in section 4. We define in section 5 random walks with increments in a Littelmann module and the associated random processes in the affine Weyl chamber. These processes can be seen as approximations of the unconditioned and conditioned Brownian motions introduced in section 6. Finally we prove an inverse Pitman's theorem for  $A_1^1$  in section 7.

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## 2. STATEMENT OF THE THEOREM

For a real  $x \geq 0$ , we define two functional transformations  $I_0^x$  and  $I_1^x$  acting on the set of continuous maps  $\eta : \mathbb{R}_+ \rightarrow \mathbb{R}^2$  such that  $\eta(t) = (t, f(t))$ , where  $f(t) \in \mathbb{R}$ , for  $t \geq 0$ , and  $\lim_{t \rightarrow \infty} f(t)/t \in (0, 1)$  as

$$I_0^x \eta(t) = (t, f(t) + 2 \min(x, \inf_{s \geq t} (s - f(s)))) ,$$

$$I_1^x \eta(t) = (t, f(t) - 2 \min(x, \inf_{s \geq t} (f(s)))) , \quad t \geq 0.$$

Let  $\{B(t) = (t, b_t + t/2), t \geq 0\}$  be a space-time Brownian motion, where  $b$  is a standard Brownian motion, and a space-time Brownian motion  $\{A(t) = (t, a_t), t \geq 0\}$  with a drift 1/2, conditioned to remain in the domain  $C_{\text{aff}}$  defined by

$$C_{\text{aff}} = \{(t, x) \in \mathbb{R}_+ \times \mathbb{R} : 0 < x < t\}.$$

See section 6 for the definition of this process. Let  $\varepsilon_n, n \geq 0$  be a sequence of independent exponential random variables with parameter 1. Let  $p \in \mathbb{N}$ , define

$\xi_{0,p}(\infty) = \varepsilon_0$ , and, for all  $k \in \{1, \dots, p\}$ ,

$$\frac{\xi_{k,p}(\infty)}{k} = \sum_{n=k}^p \frac{2\varepsilon_n}{n(n+1)}.$$

The notational choices will be hopefully clearer later. Then one has the following reconstruction theorem.

**Theorem 2.1.** *The sequence of processes*

$$\{I_0^{\xi_{0,p}(\infty)} \dots I_p^{\xi_{p,p}(\infty)} A(t), t \geq 0\}, p \geq 0,$$

converges, in the sense of finite dimensional distributions, towards the space-time Brownian motion  $\{B(t), t \geq 0\}$ .

This theorem is a converse to Theorem 7.1 in [6]. Let us notice that there is no correction term here. Actually the correction term in the Pitman's Theorem proved in [6] comes from the fact that the sequence of string coordinates associated to a Brownian motion is a convergent sequence with limit 2. The law of the random sequence in the previous theorem is the law of the string coordinates conditioned to be ultimately equal to 0. So this is not a surprise that no correction term is needed for this reconstruction theorem.

### 3. THE AFFINE LIE ALGEBRA $A_1^1$ AND ITS REPRESENTATIONS

We recall some standard facts about the affine Lie algebra of type  $A_1^1$ . See [13] for a presentation of affine Lie algebras and their representations. For our purpose, we only need to define and consider a realization of a real Cartan subalgebra. Let  $\mathfrak{h}_{\mathbb{R}}$  and  $\mathfrak{h}_{\mathbb{R}}^*$  be two copies of  $\mathbb{R}^3$  in standard duality. One has

$$\mathfrak{h}_{\mathbb{R}} = \text{Span}_{\mathbb{R}}\{c, \alpha_1^{\vee}, d\}, \quad \mathfrak{h}_{\mathbb{R}}^* = \text{Span}_{\mathbb{R}}\{\Lambda_0, \alpha_1, \delta\},$$

where  $c = (1, 0, 0)$ ,  $\alpha_1^{\vee} = (0, 1, 0)$ ,  $d = (0, 0, 1)$ , and  $\Lambda_0 = (1, 0, 0)$ ,  $\alpha_1 = (0, 2, 0)$ ,  $\delta = (0, 0, 1)$  in  $\mathbb{R}^3$ .

Let  $\alpha_0^{\vee} = (1, -1, 0)$  and  $\alpha_0 = (0, -2, 1)$ , so that  $c = \alpha_0^{\vee} + \alpha_1^{\vee}$  and  $\delta = \alpha_0 + \alpha_1$ . The vectors  $\alpha_0$  and  $\alpha_1$  are the two positive simple roots of  $A_1^1$  and  $\alpha_0^{\vee}$  and  $\alpha_1^{\vee}$  their coroots. We denote by  $\langle \cdot, \cdot \rangle$  the natural pairing. The set of integral weights is

$$P = \{\lambda \in \mathfrak{h}_{\mathbb{R}}^* : \langle \lambda, \alpha_i^{\vee} \rangle \in \mathbb{Z}, i = 0, 1\},$$

and the set of dominant integral weights

$$P_+ = \{\lambda \in \mathfrak{h}_{\mathbb{R}}^* : \langle \lambda, \alpha_i^{\vee} \rangle \in \mathbb{N}, i = 0, 1\}.$$

**Highest weight representations.** For a dominant integral weight  $\lambda$ , the character of the irreducible representation  $V(\lambda)$  of  $A_1^1$  with highest weight  $\lambda$  is defined as the formal series

$$(1) \quad \text{ch}_{\lambda} = \sum_{\beta \in P} \dim(V(\lambda)_{\beta}) e^{\beta},$$

where  $V(\lambda)_{\beta}$  is the weight space corresponding to the weight  $\beta$  in  $V(\lambda)$ . Let  $e^{\beta}(h) = e^{\langle \beta, h \rangle}$  for  $h \in \mathfrak{h}_{\mathbb{R}}$ . The series converges absolutely if  $\langle \delta, h \rangle > 0$  otherwise it diverges. The character can be extended to the set of  $h \in \mathfrak{h}_{\mathbb{R}} \oplus i\mathfrak{h}_{\mathbb{R}}$  such that

$\Re\langle\delta, h\rangle > 0$ . The Weyl group  $W$  is the group generated by the reflections  $s_{\alpha_i}$ , for  $i \in \{0, 1\}$ , defined on  $\mathfrak{h}_{\mathbb{R}}^*$  by

$$s_{\alpha_i}(\beta) = \beta - \langle\beta, \alpha_i^\vee\rangle\alpha_i, \beta \in \mathfrak{h}_{\mathbb{R}}^*.$$

Weyl's character formula (chapter 10 of [13]) states that

$$(2) \quad \text{ch}_\lambda = \frac{\sum_{w \in W} \det(w) e^{w(\lambda+\rho)-\rho}}{\prod_{\alpha \in R_+} (1 - e^{-\alpha})},$$

where  $\det(w)$  is the determinant of the linear map  $w$ ,  $\rho = 2\Lambda_0 + \frac{\alpha_1}{2}$  and  $R_+$  is the set of positive roots defined by

$$R_+ = \{\alpha_0 + n\delta, \alpha_1 + n\delta, (n+1)\delta, n \in \mathbb{N}\}.$$

In particular

$$(3) \quad \prod_{\alpha \in R_+} (1 - e^{-\alpha}) = \sum_{w \in W} \det(w) e^{w(\rho)-\rho}.$$

The affine Weyl group  $W$  is the semi-direct product  $T \ltimes W_0$  where  $W_0$  is the subgroup generated by  $s_{\alpha_1}$  and  $T$  is the subgroup of transformations  $t_k$ ,  $k \in \mathbb{Z}$ , defined by

$$t_k(\lambda) = \lambda + k(\lambda, \delta)\alpha_1 - (k(\lambda, \alpha_1) + k^2(\lambda, \delta))\delta, \lambda \in \mathfrak{h}^*.$$

Thus for  $\lambda \in P_+$ , one has

$$\sum_{w \in W} \det(w) e^{w(\lambda+\rho)} = \sum_{k \in \mathbb{Z}} e^{t_k(\lambda+\rho)} - e^{t_k s_{\alpha_1}(\lambda+\rho)},$$

and for  $\lambda = n\Lambda_0 + m\frac{\alpha_1}{2}$ , with  $(m, n) \in \mathbb{N}^2$  such that  $0 \leq m \leq n$ ,  $a \in \mathbb{R}$ , and  $b > 0$ , the Weyl-Kac character formula becomes here

$$(4) \quad \text{ch}_\lambda(a\alpha_1^\vee + bd) = \frac{\sum_{k \in \mathbb{Z}} \sinh(a(m+1) + 2ak(n+2)) e^{-b(k(m+1)+k^2(n+2))}}{\sum_{k \in \mathbb{Z}} \sinh(a + 4ak) e^{-b(k+2k^2)}}.$$

**Verma modules.** The character of a Verma module with highest weight 0 is denoted by  $\text{ch}_{M(0)}$ . Let us recall some various known expressions of this character. First of all, one has

$$(5) \quad \text{ch}_{M(0)} = \prod_{\alpha \in R_+} (1 - e^{-\alpha})^{-1},$$

One has also

$$(6) \quad \text{ch}_{M(0)} = \lim_{\langle\lambda, \alpha_i^\vee\rangle \rightarrow \infty, i=0,1} e^{-\lambda} \text{ch}_\lambda$$

and

$$(7) \quad \text{ch}_{M(0)} = \left( \sum_{w \in W} \det(w) e^{w(\rho)-\rho} \right)^{-1},$$

the last identity being derived from the Weyl character formula. Note that, for  $\lambda \in P_+$  and  $h \in \mathfrak{h}_{\mathbb{R}}$  such that  $\langle\delta, h\rangle > 0$ , one has the inequality

$$(8) \quad \text{ch}_\lambda(h) e^{-\langle\lambda, h\rangle} \leq \text{ch}_{M(0)}(h).$$

## 4. PITMAN TRANSFORMS AND LITTELMANN MODULES

In this section we explain connections between the Littelmann path model and Pitman transform in the context of the affine Lie algebra  $A_1^1$ . For more details about the Littelmann path model see Peter Littelmann's papers [17, 18]. Let  $\mathcal{C}$  be the cone generated by  $P_+$ , i.e.

$$\mathcal{C} = \{\lambda \in \mathfrak{h}_{\mathbb{R}}^* : \langle \lambda, \alpha_i^\vee \rangle \geq 0, i \in \{0, 1\}\}.$$

We fix  $T > 0$ . A path  $\pi$  defined on  $[0, T]$  is a continuous piecewise linear function  $\pi : [0, T] \rightarrow \mathfrak{h}_{\mathbb{R}}^*$  such that  $\pi(0) = 0$ . It is called dominant if  $\pi(t) \in \mathcal{C}$  for all  $t \in [0, T]$ . It is called integral if  $\pi(T) \in P$  and

$$\min_{t \in [0, T]} \langle \pi(t), \alpha_i^\vee \rangle \in \mathbb{Z}, \text{ for } i \in \{0, 1\}.$$

The Pitman transforms  $\mathcal{P}_{\alpha_i}$ ,  $i \in \{0, 1\}$ , are defined on the set on continuous functions  $\eta : [0, T] \rightarrow \mathfrak{h}_{\mathbb{R}}^*$ , such that  $\eta(0) = 0$ , by the formula

$$\mathcal{P}_{\alpha_i} \eta(t) = \eta(t) - \inf_{0 \leq s \leq t} \langle \eta(s), \alpha_i^\vee \rangle \alpha_i, \quad t \in [0, T].$$

Let us notice that the fact that  $\langle \alpha_i, \alpha_i^\vee \rangle = 2$  implies that the definition above coincides with the one of the original Pitman transform. For a dominant path  $\pi$  defined on  $[0, T]$ , such that  $\pi(T) \in P_+$ , the Littelmann module  $B\pi$  generated by  $\pi$  is the set of integral paths  $\eta$  defined on  $[0, T]$  such that there exists  $k \in \mathbb{N}$  such that

$$\mathcal{P}_{\alpha_k} \dots \mathcal{P}_{\alpha_0} \eta = \pi,$$

where  $\alpha_{2k} = \alpha_0$  and  $\alpha_{2k+1} = \alpha_1$ . If  $\pi$  is a dominant integral path defined on  $[0, T]$  such that  $\pi(T) = \lambda \in P_+$ , then the Littelmann path theory ensures that

$$(9) \quad \text{ch}_\lambda = \sum_{\eta \in B\pi} e^{\eta(T)}.$$

Moreover for an integral path  $\eta$  defined on  $[0, T]$  there exists  $k_0$  such that for all  $k \geq k_0$ , one has

$$\mathcal{P}_{\alpha_k} \dots \mathcal{P}_{\alpha_0} \eta(t) = \mathcal{P}_{\alpha_{k_0}} \dots \mathcal{P}_{\alpha_0} \eta(t), \quad t \in [0, T]^1.$$

Thus for an integral path  $\eta$  defined on  $[0, T]$ , one defines a dominant path  $\mathcal{P}\eta$  on  $[0, T]$ , by

$$\mathcal{P}\eta(t) = \lim_{k \rightarrow \infty} \mathcal{P}_{\alpha_k} \dots \mathcal{P}_{\alpha_0} \eta(t), \quad t \in [0, T].$$

**String coordinates.** Let  $\ell^{(\infty)}(\mathbb{N})$  be the set of sequences of nonnegative integers, almost all zero. Let  $\pi$  be a dominant path defined on  $[0, T]$  and  $\eta \in B\pi$ . There exists a unique sequence of nonnegative integers,  $\mathbf{a}(\eta) := (a_k)_{k \geq 0}$  almost all zero, such that

$$(10) \quad \mathcal{P}_{\alpha_m} \dots \mathcal{P}_{\alpha_0} \eta(T) = \eta(T) + \sum_{k=0}^m a_k \alpha_k, \quad m \geq 0.$$

Peter Littelmann proved in [18] that the map

$$\mathbf{a} : \eta \in B\pi \rightarrow \mathbf{a}(\eta) \in \ell^{(\infty)}(\mathbb{N})$$

<sup>1</sup>It has been proved in [6] that this fact remains true if  $\eta$  is a continuous, piecewise  $C^1$  trajectory in  $\mathfrak{h}_{\mathbb{R}}^*$ .

is injective. The image of this map, which depends on  $\pi$  only through  $\pi(T)$ , is the set  $B(\pi(T))$  defined below. It is the set of vertices of a Kashiwara crystal [15]. The sets  $B(\infty)$  and  $B(\lambda)$  defined below are for instance respectively described in [20] and [18].

*Definition 4.1.* The subset  $B(\infty)$  of  $\ell^{(\infty)}(\mathbb{N})$  is defined as

$$B(\infty) = \{a = (a_k)_{k \geq 0} \in \ell^{(\infty)}(\mathbb{N}) : \frac{a_k}{k} \geq \frac{a_{k+1}}{k+1}, k \geq 1\}.$$

For  $\lambda \in P_+$ , the subset  $B(\lambda)$  of  $B(\infty)$  is defined as

$$\begin{aligned} B(\lambda) &= \{a = (a_k)_{k \geq 0} \in B(\infty) : a_p \leq \langle \lambda - \sum_{k=p+1}^{\infty} a_k \alpha_k, \alpha_p^\vee \rangle, \forall p \geq 0\} \\ &= \{a = (a_k)_{k \geq 0} \in B(\infty) : a_p \leq \langle \lambda - \omega(a) + \sum_{k=0}^p a_k \alpha_k, \alpha_p^\vee \rangle, \forall p \geq 0\}, \end{aligned}$$

where  $\omega(a) = \sum_{k=0}^{\infty} a_k \alpha_k$ , which is the opposite of the weight of  $a$  as an element of the crystal  $B(\infty)$  of the Verma module of highest weight 0.

Thus identity (9) becomes

$$(11) \quad \text{ch}_\lambda = \sum_{a \in B(\lambda)} e^{\lambda - \omega(a)},$$

and the character of a Verma module is written with the string coordinates,

$$(12) \quad \text{ch}_{M(0)} = \sum_{a \in B(\infty)} e^{-\omega(a)}.$$

The inverse function of  $\mathbf{a}$  can be written using the functionals  $I_{\alpha_i}^{x,T}$ ,  $i \in \{0, 1\}$ ,  $x \geq 0$ , introduced in [4] and defined by

$$I_{\alpha_i}^{x,T} f(t) = f(t) - \min(x, \inf_{T \geq s \geq t} \langle f(s), \alpha_i^\vee \rangle) \alpha_i, \quad t \in [0, T],$$

for  $f : [0, T] \rightarrow \mathfrak{h}_{\mathbb{R}}^*$ . It may be noted that the definition coincides with that given at the beginning of part 2.

For  $a \in B(\lambda)$  and  $\pi$  an integral dominant path on  $[0, T]$  such that  $\pi(T) = \lambda$ , the only path  $\eta \in B\pi$  such that  $\mathbf{a}(\eta) = a$  is given by

$$\eta(t) = I_{\alpha_0}^{a_0, T} \dots I_{\alpha_p}^{a_p, T} \pi(t), \quad t \in [0, T],$$

where  $p$  is chosen such that  $a_k = 0$ , for all  $k \geq p+1$ . Notice that if  $f$  is a function defined on  $\mathbb{R}_+$  with values in  $\mathfrak{h}_{\mathbb{R}}^*$  such that

$$\lim_{t \rightarrow \infty} \langle f(t), \alpha_i^\vee \rangle = +\infty, \quad i \in \{0, 1\},$$

the definition of  $I_{\alpha_i}^{x,T}$ ,  $i \in \{0, 1\}$ , makes sense for  $T = +\infty$ . In the following, we write  $I_{\alpha_i}^x$  instead of  $I_{\alpha_i}^{x,+\infty}$ . We notice that if  $f$  is a map with values in  $\mathbb{R}\alpha_0 \oplus \mathbb{R}\alpha_1$  then for  $t \geq 0$ ,  $i \in \{0, 1\}$ ,

$$I_{\alpha_i}^x f(t) = I_i^x f(t) \pmod{\delta}.$$



**Demazure character.** One can find for instance in [15] an introduction to Demazure characters in the context of crystals. For an integer  $p \geq 0$  and for  $\lambda \in P_+$ , let  $w_p = s_{\alpha_p} \dots s_{\alpha_0}$ , let  $\pi$  be an integral dominant path defined on  $[0, T]$  such that  $\pi(T) = \lambda$ , and  $B^{w_p}\pi = \{\eta \in B\pi : \mathcal{P}_{\alpha_p} \dots \mathcal{P}_{\alpha_0}\eta = \pi\}$ . One defines  $\text{ch}_\lambda^{w_p}$  by the formula

$$(13) \quad \text{ch}_\lambda^{w_p} = \sum_{\eta \in B^{w_p}\pi} e^{\eta(T)},$$

The function  $\text{ch}_\lambda^{w_p}$  is a Demazure character, i.e. the character of a  $U(\mathfrak{n}^+)$ -module. Written with the string coordinates, definition (13) becomes

$$(14) \quad \text{ch}_\lambda^{w_p} = \sum_{a \in B(\lambda), a_{p+1}=0} e^{\lambda - \omega(a)}.$$

We define a Verma–Demazure character  $\text{ch}_{M(0)}^{w_p}$  by

$$(15) \quad \text{ch}_{M(0)}^{w_p} = \sum_{a \in B(\infty), a_{p+1}=0} e^{-\omega(a)}.$$

## 5. RANDOM WALKS AND LITTELMANN PATHS

In this section  $m$  is a fixed positive integer. Let  $\pi_0$  be the path defined on  $[0, 1]$  by

$$\pi_0(t) = t\Lambda_0, \quad t \in [0, 1],$$

and the Littelmann module  $B\pi_0$  generated by  $\pi_0$ . Let  $\rho^\vee = 2d + \alpha_1^\vee/2$ . We fix an integer  $m \geq 1$ . The formula

$$(16) \quad \mu^m(\eta) = \frac{e^{\frac{1}{m}\langle \eta(1), \rho^\vee \rangle}}{\text{ch}_{\Lambda_0}(\rho^\vee/m)}, \quad \eta \in B\pi_0,$$

defines a probability measure  $\mu^m$  on  $B\pi_0$ . Let  $(\eta_i^m)_{i \geq 0}$  be a sequence of i.i.d random variables with law  $\mu^m$  and let  $\{\Pi^m(t), t \geq 0\}$  be defined by

$$\Pi^m(t) = \eta_1^m(1) + \dots + \eta_{k-1}^m(1) + \eta_k^m(t - k + 1),$$

when  $t \in [k - 1, k[$ , for  $k \in \mathbb{Z}_+$ . We write  $*$  for the usual concatenation of paths, so that for an integer  $t$ , the restriction of  $\Pi^m$  to  $[0, t]$  is in  $B\pi_0^{*t}$ . For  $t \in \mathbb{N}$ , let  $(\xi_k^m(t))_{k \geq 0}$  be string coordinates of  $\Pi^m|_{[0, t]}$ . Notice that the definition makes sense for  $t = \infty$ , since each string coordinate is an increasing function of  $t$ .

We define a random process  $\{\Pi_+^m(t), t \geq 0\}$  with values in  $\mathcal{C}$  by

$$\Pi_+^m(t) = \mathcal{P}\Pi^m(t), \quad t \geq 0.$$

The next proposition follows from the properties of the Littelmann path model. It implies in particular that

$\{\Pi_+^m(k), k \geq 0\}$  is Markovian with transition probabilities given in Theorem 4.7 of [16]. It will be very useful in the whole paper as it allows to show that the Markov process  $\{\Pi_+^m(k), k \geq 0\}$  inherits many properties from the random walk  $\{\Pi^m(k), k \geq 0\}$ .

**Proposition 5.1.** *For any integers  $k$  and  $n$ , and any function  $f$  defined on the set of continuous functions  $\mathcal{C}([n, n+k], \mathbb{R})$ , one has*

$$\begin{aligned} & \mathbb{E} \left( f(\Pi_+^m(t) : n \leq t \leq n+k) | \Pi_+^m(s), s \leq n \right) \\ &= \mathbb{E} \left( f(\Pi^m(t) + \lambda, 0 \leq t \leq k) \frac{ch_{\Pi^m(k) + \lambda}(\rho^\vee/m)}{ch_\lambda(\rho^\vee/m)} e^{-\langle \Pi^m(k), \rho^\vee/m \rangle} 1_{\lambda + \Pi^m|_{[0,k]} \in \mathcal{C}} \right), \end{aligned}$$

where  $\lambda = \Pi_+^m(n)$ .

The next proposition follows from the fact that the image of a Littelmann module  $B\pi$  under  $\mathbf{a}$  depends on  $\pi$  only through the final value of  $\pi$ .

**Proposition 5.2.** *For  $u \in \mathbb{N}$  and  $f$  a real function defined on  $B(\infty)$  one has*

$$\mathbb{E} \left( f(\xi^m(u)) | \Pi_+^m(t), t \leq u \right) = \frac{\sum_{a \in B(\Pi_+^m(u))} f(a) e^{\langle \Pi_+^m(u) - \omega(a), \rho^\vee/m \rangle}}{\sum_{a \in B(\Pi_+^m(u))} e^{\langle \Pi_+^m(u) - \omega(a), \rho^\vee/m \rangle}}$$

where  $\xi^m(u) = (\xi_k^m(u))_{k \geq 0}$ .

**Lemma 5.3.** *For  $i \in \{0, 1\}$ ,  $\langle \Pi^m(k), \alpha_i^\vee \rangle / k$  almost surely converges as  $k$  goes to infinity towards a positive real number.*

*Proof.* In a more general context, it has been proved in [16], Proposition 5.4, that  $\mathbb{E}(\eta(1))$  is the interior of  $\mathcal{C}$ . In our particular case, it is easily proved using the explicit description of the weights of  $V(\Lambda_0)$  given for instance in chapter 9 of [12]. The convergence follows from a law of large numbers.  $\square$

The following lemma is a first useful application of Proposition 5.1.

**Lemma 5.4.** *For  $i \in \{0, 1\}$ , in probability,  $\lim_{k \rightarrow \infty} \langle \Pi_+^m(k), \alpha_i^\vee \rangle = +\infty$ .*

*Proof.* Lemma 5.3 implies that almost surely  $\lim_{k \rightarrow \infty} \langle \Pi^m(k), \alpha_i^\vee \rangle = +\infty$ . For  $M > 0$ ,  $i \in \{0, 1\}$  and  $k \geq 1$ , Proposition 5.1 gives

$$\begin{aligned} & \mathbb{P} \left( \langle \Pi_+^m(k), \alpha_i^\vee \rangle < M \right) = \\ & \mathbb{E} \left( 1_{\{\langle \Pi^m(k), \alpha_i^\vee \rangle < M\}} ch_{\Pi^m(k)}(\rho^\vee/m) e^{-\langle \Pi^m(k), \rho^\vee/m \rangle} 1_{\Pi^m|_{[0,k]} \in \mathcal{C}} \right). \end{aligned}$$

Upper bound (8) and Lemma 5.3 end the proof.  $\square$

**Proposition 5.5.** *The sequence of string coordinates  $\xi^m(\infty)$  is independent of  $\{\Pi_+^m(t), t \geq 0\}$  and*

$$\mathbb{P}(\xi^m(\infty) = a) = \frac{e^{-\langle \omega(a), \rho^\vee/m \rangle}}{ch_{M(0)}(\rho^\vee/m)}, \quad a \in B(\infty).$$

*Proof.* Let  $T \geq 0$ ,  $a \in B(\infty)$  and  $f$  be a real valued function defined on  $B\pi_0^{*T}$  that we suppose bounded by 1. One has

$$\mathbb{E} \left( f(\Pi_+^m|_{[0,T]}) 1_{\{\xi^m(\infty) = a\}} \right) = \lim_{u \rightarrow \infty} \mathbb{E} \left( f(\Pi_+^m|_{[0,T]}) 1_{\{\xi^m(u) = a\}} \right).$$

Let us fix  $\varepsilon > 0$ . We choose  $M \geq 0$  such that if  $\lambda \in P_+$  and satisfies

$$\langle \alpha_i^\vee, \lambda \rangle \geq M, \quad \text{for } i \in \{0, 1\},$$

then one has

$$a \in B(\lambda) \quad \text{and} \quad \left| \frac{1}{e^{-\langle \lambda, \rho^\vee/m \rangle} ch_\lambda(\rho^\vee/m)} - \frac{1}{ch_{M(0)}(\rho^\vee/m)} \right| \leq \varepsilon.$$

Lemma 5.4 implies that there exists  $u_0 \in \mathbb{N}$  such that for all integer  $u \geq u_0$

$$\mathbb{P}(\langle \alpha_i^\vee, \Pi_+^m(u) \rangle \geq M, i \in \{0, 1\}) \geq 1 - \varepsilon.$$

By conditioning on  $\{\Pi_+^m(t), 0 \leq t \leq u\}$  in the lefthand side expectation of the following identity one obtains by proposition 5.2, for an integer  $u \geq T$ ,

$$\mathbb{E}\left(f(\Pi_+^m|_{[0,T]})1_{\{\xi^m(u)=a\}}\right) = \mathbb{E}\left(f(\Pi_+^m|_{[0,T]})\frac{e^{-\langle \omega(a), \rho^\vee/m \rangle} 1_{B(\Pi_+^m(u))(a)}}{e^{-\langle \Pi_+^m(u), \rho^\vee/m \rangle} \text{ch}_{\Pi_+^m(u)}(\rho^\vee/m)}\right).$$

It implies that for an integer  $u \geq u_0$ ,

$$\left|\mathbb{E}\left(f(\Pi_+^m|_{[0,T]})1_{\{\xi^m(u)=a\}}\right) - \mathbb{E}\left(f(\Pi_+^m|_{[0,T]})\right)\frac{e^{-\langle \omega(a), \rho^\vee/m \rangle}}{\text{ch}_{M(0)}(\rho^\vee/m)}\right| \leq 2\varepsilon,$$

which gives the lemma.  $\square$

Proposition 5.5 implies immediately the following corollary.

**Corollary 5.6.** *For  $p \geq 0$ ,*

$$\mathbb{P}(\xi_{p+1}^m(\infty) = 0) = \frac{\text{ch}_{M(0)}^{w_p}(\rho^\vee/m)}{\text{ch}_{M(0)}(\rho^\vee/m)}.$$

Since

$$\{\Pi^m(t) \in \mathcal{C}, t \geq 0\} = \{\xi^m(\infty) = 0\},$$

Proposition 5.5 has a second corollary, which has already been proved in [16] by a quite different method. This corollary is not useful for our purpose, nevertheless it is worth giving it.

**Corollary 5.7.** *One has  $\mathbb{P}(\Pi^m(t) \in \mathcal{C}, t \geq 0) = (\text{ch}_{M(0)}(\rho^\vee/m))^{-1}$ .*

## 6. THE CONTINUOUS COUNTERPART

The random processes introduced in section 5 are approximations of continuous time random processes defined in this section. For this, let us define the affine cone

$$C_{\text{aff}} = \{(t, x) \in \mathbb{R}_+ \times \mathbb{R}_+ : 0 < x < t\}.$$

Let  $\{B(t) = t\Lambda_0 + (b_t + t/2)\alpha_1/2 : t \geq 0\}$ , where  $\{b_t : t \geq 0\}$  is a standard real Brownian motion starting from 0. Let  $\varphi_{1/2}$  be a function defined on  $\mathbb{R}_+^* \times \mathbb{R}$  by

$$(17) \quad \varphi_{1/2}(t, x) = e^{-x/2} \sum_{k \in \mathbb{Z}} \sinh((2kt + x)/2) e^{-2(kx + k^2 t)}, \text{ for } t > 0, x \in \mathbb{R}.$$

This is an harmonic function for the process  $B$  killed on the boundary of  $C_{\text{aff}}$ . It is positive on  $C_{\text{aff}}$  and vanishes on the boundary of  $C_{\text{aff}}$ . Let  $\{A(t), t \geq 0\}$  be the process starting from  $(0, 0)$ , whose law is the Doob transformation of the law of the process  $B$  killed on the boundary of  $C_{\text{aff}}$  by the function  $\varphi_{1/2}$ . This process has been introduced and studied in [9, 10] and carefully defined in [6] in the context of the present paper.

The convergences in the following proposition have been proved in [11]. In this proposition, as in the convergence theorems of the following sections, all the processes are considered as processes with values in the quotient space  $\mathfrak{h}_{\mathbb{R}}^* \text{ mod } \delta$ , which is identified with  $\mathbb{R}\Lambda_0 \oplus \mathbb{R}\alpha_1 = \mathbb{R}^2$ . We notice that  $\alpha_0 = -\alpha_1$  in the quotient space. The set of continuous functions from  $\mathbb{R}_+$  to  $\mathbb{R}^2$  is equipped with

the topology of uniform convergence on compact sets and we use the standard definition of convergence in distribution for a sequence of continuous processes as in Revuz and Yor ([23], XIII.1).

**Proposition 6.1.** (1) *For any  $t \geq 0$ , the random variable  $(\Pi^m(mt) - \Pi^m \lfloor mt \rfloor) / m$  goes to 0 in probability when  $m$  goes to infinity.*

(2) *The sequence of processes*

$$\left\{ \frac{1}{m} \Pi^m(mt) : t \geq 0 \right\}, \quad m \geq 1,$$

*viewed in the quotient space  $\mathfrak{h}_{\mathbb{R}}^* \bmod \delta$ , converges in distribution towards the process  $\{B(t) : t \geq 0\}$  when  $m$  goes to infinity.*

(3) *The sequence of processes*

$$\left\{ \frac{1}{m} \Pi_+^m \lfloor mt \rfloor : t \geq 0 \right\}, \quad m \geq 1,$$

*viewed in the quotient space  $\mathfrak{h}_{\mathbb{R}}^* \bmod \delta$ , converges towards  $\{A(t) : t \geq 0\}$  when  $m$  goes to infinity, in the sense of finite dimensional distributions.*

For  $t \geq 0$ , we consider the string coordinates of  $B$  on  $[0, t]$ , denoted by  $(\xi_k(t))_{k \geq 0}$ . They are defined by

$$(18) \quad \mathcal{P}_{\alpha_m} \dots \mathcal{P}_{\alpha_0} B(t) = B(t) + \sum_{k=0}^m \xi_k(t) \alpha_k, \quad m \geq 0.$$

For every  $k \geq 0$ , the function  $t \in \mathbb{R}_+ \mapsto \xi_k(t)$  is increasing, and because of the drift,  $\lim_{t \rightarrow \infty} \xi_k(t) < +\infty$ . We set  $\xi_k(\infty) = \lim_{t \rightarrow \infty} \xi_k(t)$ . For a sequence  $x = (x_k) \in \mathbb{R}_+^{\mathbb{N}}$ , we set

$$(19) \quad \omega(x) = \lim_{n \rightarrow +\infty} \sum_{k=0}^{n-1} x_k \alpha_k + \frac{1}{2} x_n \alpha_n \quad \bmod \delta,$$

when this limit exists in  $\mathbb{R}\alpha_1$ . The following sets are the continuous analogs of the Kashiwara crystals defined in definition 4.1.

*Definition 6.2.* One defines, for  $\lambda \in \bar{C}_{\text{aff}}$ ,

$$\Gamma(\infty) = \left\{ x = (x_k) \in \mathbb{R}_+^{\mathbb{N}} : \frac{x_k}{k} \geq \frac{x_{k+1}}{k+1} \geq 0, \text{ for all } k \geq 1, \omega(x) \in \mathbb{R}^2 \right\},$$

$$\Gamma(\lambda) = \left\{ x \in \Gamma(\infty) : x_k \leq \langle \lambda - \omega(x) + \sum_{i=0}^k x_i \alpha_i, \alpha_k^\vee \rangle, \text{ for every } k \geq 0 \right\}.$$

## 7. AN INVERSE PITMAN'S THEOREM

We will now prove a reconstruction theorem which allows to get a space-time Brownian motion  $B$  from a conditioned one  $A$  and a sequence of random variables properly distributed. The idea is to prove that the commutative diagram in figure 1 is valid. The convergence represented by the third arrow of the diagram will then provide a reconstruction theorem. Black arrows on the diagram stand for convergences that have been already proved. Dashed ones stand for convergences which have still to be proved at this stage. Let us first define the random variables involved in the diagram which have not been defined yet. The law of  $\xi(\infty)$  is described by the following theorem, which has been proved in [6].

**Theorem 7.1** (Ph. Bougerol, M. Defosseux [6]). *The random variables*

$$\xi_0(\infty), \quad \frac{1}{2}((k+1)\xi_k(\infty) - k\xi_{k+1}(\infty)), \quad k \geq 1,$$

*are independent exponential random variables with parameter 1.*

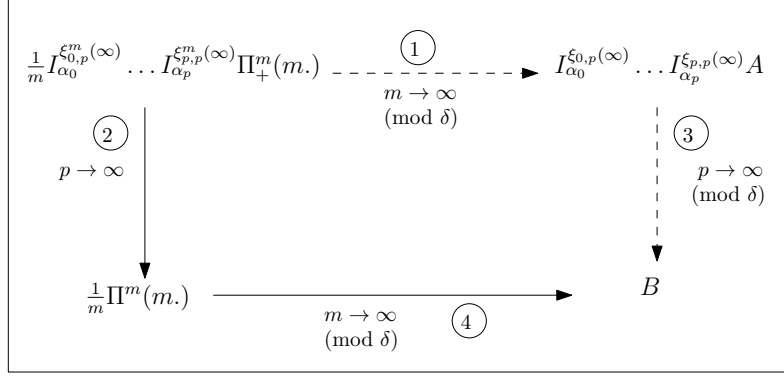


FIGURE 1. A commutative diagram of finite dimensional distributions convergences

From now on,  $\varepsilon_n, n \geq 0$ , is a sequence of independent exponential random variables with parameter 1 defined by

$$(20) \quad \varepsilon_0 = \xi_0(\infty), \quad \varepsilon_k = \frac{1}{2}((k+1)\xi_k(\infty) - k\xi_{k+1}(\infty)), \quad k \geq 1,$$

and  $\{A(t) : t \geq 0\}$  is supposed to be independent of this sequence.

*Definition 7.2.* For every  $p \geq 0$ , let  $\xi_{0,p}(\infty) = \varepsilon_0$ , and let  $\xi_{k,p}(\infty)$  be defined by

$$\frac{\xi_{k,p}(\infty)}{k} = \sum_{n=k}^p \frac{2\varepsilon_n}{n(n+1)},$$

for all  $k \in \{1, \dots, p\}$ . We write  $\xi_{\cdot,p}(\infty) = (\xi_{k,p}(\infty))_{k \in \{0, \dots, p\}}$ .

**7.1. Proof of the convergence corresponding to the first arrow of the diagram.** For every  $p \geq 0$ , let

$$(\xi_{0,p}^m(\infty), \dots, \xi_{p,p}^m(\infty))$$

be a random vector independent from  $\Pi_+^m$ , which is distributed as  $(\xi_0^m(\infty), \dots, \xi_p^m(\infty))$  conditionally on  $\xi_{p+1}^m(\infty) = 0$ . Lemma 7.3 and Propositions 7.6 and 7.7 will imply the desired convergence.

**Lemma 7.3.** *For every  $p \in \mathbb{N}$ ,  $\frac{1}{m}(\xi_{0,p}^m(\infty), \dots, \xi_{p,p}^m(\infty))$  converges in distribution towards  $(\xi_{0,p}(\infty), \dots, \xi_{k,p}(\infty))$  when  $m$  goes to  $+\infty$ .*

*Proof.* From definition 7.2, one derives that the density of  $(\xi_{0,p}(\infty), \dots, \xi_{p,p}(\infty))$  is given by

$$f_{(\xi_{0,p}, \dots, \xi_{0,p})}(x_0, \dots, x_p) = \frac{(p+1)! e^{-\sum_{k=0}^p x_k}}{2^p} 1_{x_0 \geq 0, x_1 \geq \frac{x_2}{2} \geq \dots \geq \frac{x_p}{p} \geq 0}.$$

Moreover, from Proposition 5.5 and Corollary 5.6 we deduce that for every real numbers  $t_0, \dots, t_p \geq 0$ ,

$$\mathbb{E} \left( e^{-\sum_{k=0}^p t_k \frac{\xi_{k,p}^m(\infty)}{m}} \right) = \frac{1}{\text{ch}_{M(0)}^{w_p}(\rho^\vee/m)} \sum_{(a_0, \dots, a_p) \in \mathbb{N}^{p+1}} e^{-\sum_{k=0}^p (1+t_k) \frac{a_k}{m}} \mathbf{1}_{\frac{a_1}{1} \geq \frac{a_2}{2} \geq \dots \geq \frac{a_p}{p}}.$$

Lemma follows from the fact that

$$m^{-(p+1)} \sum_{(a_0, \dots, a_p) \in \mathbb{N}^{p+1}} e^{-\sum_{k=0}^p (1+t_k) \frac{a_k}{m}} \mathbf{1}_{\frac{a_1}{1} \geq \frac{a_2}{2} \geq \dots \geq \frac{a_p}{p}}$$

converges towards the Riemann integral

$$\int_{\mathbb{R}_+^{p+1}} e^{-\sum_{k=0}^p (1+t_k) x_k} \mathbf{1}_{\frac{x_1}{1} \geq \frac{x_2}{2} \geq \dots \geq \frac{x_p}{p}} dx.$$

□

**Proposition 7.4.** *For every  $t \geq 0$ ,  $\frac{1}{m}(\Pi_+^m(mt) - \Pi_+^m[mt])$  converges in probability to 0 as  $m$  goes to infinity.*

*Proof.* Let us fix  $\varepsilon > 0$  and  $t > 0$ . We choose a compact  $K$  in  $C_{\text{aff}}$  such that

$$\mathbb{P}(A(t) \in K) > 1 - \varepsilon/2.$$

Convergences recalled in proposition 6.1 ensure that there exists  $m_0 \in \mathbb{N}^*$  such that for all  $m \geq m_0$

$$\mathbb{P} \left( \frac{1}{m}(\Pi_+^m[mt] + \rho) \in K \right) > 1 - \varepsilon.$$

We choose such an integer  $m_0$ . One has for all  $m \geq m_0$

$$(21) \quad \mathbb{E} \left( \mathbf{1}_{\left\{ \frac{1}{m} |\langle \Pi_+^m(mt) - \Pi_+^m[mt], \alpha_1^\vee \rangle| > \varepsilon \right\}} \right) \leq \mathbb{E} \left( \mathbf{1}_{\left\{ \frac{1}{m} |\langle \Pi_+^m(mt) - \Pi_+^m[mt], \alpha_1^\vee \rangle| > \varepsilon \right\} \cap K_m} \right) + \varepsilon,$$

where  $K_m = \left\{ \frac{1}{m}(\Pi_+^m[mt] + \rho) \in K \right\}$ . By proposition 5.1, one has for  $\lambda \in P_+$ ,

$$\begin{aligned} & \mathbb{E} \left( \mathbf{1}_{\left\{ \frac{1}{m} |\langle \Pi_+^m(mt) - \Pi_+^m[mt], \alpha_1^\vee \rangle| > \varepsilon \right\}} \mid \Pi_+^m[mt] = \lambda \right) \\ &= \mathbb{E} \left( \mathbf{1}_{\left\{ \frac{1}{m} |\langle \Pi^m(mt - [mt]), \alpha_1^\vee \rangle| > \varepsilon \right\}} \frac{\text{ch}_{\Pi^m(1) + \lambda}(\rho^\vee/m)}{\text{ch}_\lambda(\rho^\vee/m)} e^{-\langle \Pi^m(1), \rho^\vee/m \rangle} \mathbf{1}_{\{\lambda + \Pi_{[0,1]}^m \in C_{\text{aff}}\}} \right) \\ &\leq \mathbb{E} \left( \mathbf{1}_{\left\{ \frac{1}{m} |\langle \Pi^m(mt - [mt]), \alpha_1^\vee \rangle| > \varepsilon \right\}} \frac{\text{ch}_{M(0)}(\rho^\vee/m)}{\text{ch}_\lambda(\rho^\vee/m)} e^{-\langle \lambda, \rho^\vee/m \rangle} \right), \end{aligned}$$

the last inequality being derived from (8). Moreover the Weyl character formula gives

$$\frac{\text{ch}_\lambda(\rho^\vee/m) e^{-\langle \lambda, \rho^\vee/m \rangle}}{\text{ch}_{M(0)}(\rho^\vee/m)} = \sum_{w \in W} \det(w) e^{\langle w(\frac{\lambda + \rho^\vee}{m}) - (\frac{\lambda + \rho^\vee}{m}), \rho^\vee \rangle}.$$

The function

$$x \in C_{\text{aff}} \mapsto \sum_{w \in W} \det(w) e^{\langle w(x) - x, \rho^\vee \rangle},$$

is positive on  $K$ . We set

$$M = \max \left\{ \left( \sum_{w \in W} \det(w) e^{\langle w(x) - x, \rho^\vee \rangle} \right)^{-1} : x \in K \right\}.$$

Thus for  $\lambda \in P_+$  such that  $(\lambda + \rho)/m \in K$  one has,

$$\mathbb{E} \left( \mathbf{1}_{\{\frac{1}{m}|\langle \Pi_+^m(mt) - \Pi_+^m \lfloor mt \rfloor, \alpha_1^\vee \rangle| > \varepsilon\}} | \Pi_+^m \lfloor mt \rfloor = \lambda \right) \leq M \mathbb{E} \left( \mathbf{1}_{\{\frac{1}{m}|\langle \Pi^m(mt - \lfloor mt \rfloor), \alpha_1^\vee \rangle| > \varepsilon\}} \right).$$

As  $\frac{1}{m}\Pi^m(mt - \lfloor mt \rfloor)$  converges towards 0 in probability as it is recalled in proposition 6.1, we choose an integer  $m_1 \geq m_0$  such that for all  $m \geq m_1$ ,

$$\mathbb{E} \left( \mathbf{1}_{\{\frac{1}{m}|\langle \Pi^m(mt - \lfloor mt \rfloor), \alpha_1^\vee \rangle| > \varepsilon\}} \right) \leq \varepsilon/M.$$

Finally by conditioning by  $\Pi_+^m \lfloor mt \rfloor$  within the expectation of the righthand side of inequality (21), one obtains for  $m \geq m_1$ ,

$$(22) \quad \mathbb{E} \left( \mathbf{1}_{\{\frac{1}{m}|\langle \Pi_+^m(mt) - \Pi_+^m \lfloor mt \rfloor, \alpha_1^\vee \rangle| > \varepsilon\}} \right) \leq 2\varepsilon,$$

which proves the expected convergence.  $\square$

By proposition 5.1, we prove in the following proposition that the sequence of random processes  $\{\frac{1}{m}\Pi_+^m(mt) : t \geq 0\}$ ,  $m \geq 1$ , inherits the tightness from  $\{\frac{1}{m}\Pi^m(mt) : t \geq 0\}$ ,  $m \geq 1$ .

**Proposition 7.5.** *The sequence of processes  $\{\frac{1}{m}\Pi_+^m(mt) : t \geq 0\}$ ,  $m \geq 1$ , is tight.*

*Proof.* For  $t \geq 0$ , we set  $X^m(t) = \frac{1}{m}\Pi_+^m(mt)$ . As it has been recalled in proposition 6.1, it has been proved in [11] that  $\frac{1}{m}\Pi_+^m \lfloor mt \rfloor$  converges in law when  $m$  goes to infinity. From proposition 7.4, we deduce the convergence in law of  $X^m(t)$  for any  $t \geq 0$ . Thus it is sufficient to prove that

$$\forall T \geq 0, \forall \varepsilon > 0, \forall \eta > 0, \exists \delta > 0 \text{ s.t. } \limsup_{m \rightarrow +\infty} \mathbb{P}(w_T(X^m, \delta) \geq \eta) \leq \varepsilon,$$

where, for  $x : \mathbb{R}_+ \rightarrow \mathfrak{h}_{\mathbb{R}}^*$ ,

$$w_T(x, h) = \sup\{|\langle x(t) - x(s), \alpha_1^\vee \rangle|, s, t \in [0, T], |s - t| \leq \delta\}.$$

Let  $T, \varepsilon, \eta > 0$ . We suppose that  $T$  is greater than  $\eta$ . We set  $t_0 = \frac{\eta}{2}$  and define  $w_T^{t_0}(x, \delta)$  by

$$w_T^{t_0}(x, \delta) = \sup\{|\langle x(t) - x(s), \alpha_1^\vee \rangle|, s, t \in [t_0, T], |s - t| \leq \delta\},$$

for  $\delta \geq 0$ ,  $x : \mathbb{R}_+ \rightarrow \mathfrak{h}_{\mathbb{R}}^*$ . As for every  $t \geq 0$ ,  $X^m(t)$  is in  $C_{\text{aff}}$ , one has for  $\delta \leq t_0$ ,

$$\{w_T(X^m, \delta) \geq \eta\} \subset \{w_T^{t_0}(X^m, \delta) \geq \eta\}.$$

As in the proof of proposition 7.4 we choose a compact  $K$  in  $C_{\text{aff}}$  and  $m_0 \in \mathbb{N}$  such as for all  $m \geq m_0$

$$\mathbb{P} \left( \frac{\Pi_+^m \lfloor mt_0 \rfloor + \rho}{m} \in K \right) \geq 1 - \varepsilon.$$

Hence,

$$\mathbb{P}(w_T^{t_0}(X^m, \delta) \geq \eta) \leq \mathbb{E} \left( \mathbf{1}_{\{w_T^{t_0}(X^m, \delta) \geq \eta, \frac{\Pi_+^m \lfloor mt_0 \rfloor + \rho}{m} \in K\}} \right) + \varepsilon.$$

By conditioning by  $\Pi_+^m \lfloor mt_0 \rfloor$  in the expectation of the righthand side of the above inequality, we obtain as in the proof of the proposition 7.4 that

$$\mathbb{P} \left( w_T^{t_0}(X^m, \delta) \geq \eta, \frac{\Pi_+^m \lfloor mt_0 \rfloor + \rho}{m} \in K \right) \leq M \mathbb{P} \left( w_T \left( \frac{1}{m} \Pi^m(m \cdot), \delta \right) \geq \eta \right),$$

where

$$M = \max\left\{\left(\sum_{w \in W} \det(w) e^{(w(x)-x), \rho^\vee}\right)^{-1} : x \in K\right\}.$$

As the sequence of processes  $\{\frac{1}{m}\Pi^m(mt), t \geq 0\}$ ,  $m \geq 0$ , is tight, we choose  $m_1 \geq m_0$  and  $\delta_0 \in (0, \eta/2]$  such that for  $m \geq m_1$ ,

$$\mathbb{P}\left(w_T\left(\frac{1}{m}\Pi^m(m.), \delta_0\right) \geq \eta\right) \leq \varepsilon/M.$$

Thus for  $m \geq m_1$ , one has

$$\mathbb{P}\left(w_T^{t_0}(X^m, \delta_0) \geq \eta\right) \leq 2\varepsilon,$$

which ends the proof.  $\square$

The convergence recalled in proposition 6.1 of  $\{\frac{1}{m}\Pi_+^m\lfloor mt \rfloor : t \geq 0\}$  in the sense of finite dimensional law and the previous proposition give the following one.

**Proposition 7.6.** *The sequence of processes  $\{\frac{1}{m}\Pi_+^m(mt) : t \geq 0\}$ ,  $m \geq 1$ , converges in distribution towards  $\{A(t) : t \geq 0\}$  in the quotient space  $\mathfrak{h}_{\mathbb{R}}^* \bmod \delta$ .*

Now it remains to control the asymptotic behavior of  $\Pi_+^m$  for large time uniformly in  $m$  in order to get the convergence represented by the first arrow in the diagram. For this we show the following proposition.

**Proposition 7.7.** *For all  $\varepsilon, a > 0$  there exists  $T, m_0 \geq 0$  such as for  $i \in \{0, 1\}$  and all  $m \geq m_0$*

$$\min\left(\mathbb{P}\left(\inf_{t \geq T} \frac{1}{m} \langle \Pi^m(mt), \alpha_i^\vee \rangle \geq a\right), \mathbb{P}\left(\inf_{t \geq T} \frac{1}{m} \langle \Pi_+^m(mt), \alpha_i^\vee \rangle \geq a\right)\right) \geq 1 - \varepsilon.$$

*Proof.* Slight modifications in the proof of proposition 6.13 of [11] give the first inequality. Let  $i \in \{0, 1\}$ . As previously we choose a compact  $K$  in  $C_{\text{aff}}$  and  $m_0 \in \mathbb{N}^*$  such as for all  $m \geq m_0$

$$\mathbb{P}\left(\frac{\Pi_+^m(m) + \rho}{m} \in K\right) \geq 1 - \varepsilon.$$

Let  $T \geq 1$  that will be chosen later. For  $u > T$  and  $m \geq m_0$  one has

$$\mathbb{E}\left(1_{\{\inf\{\frac{1}{m} \langle \Pi_+^m(mt), \alpha_i^\vee \rangle, T \leq t \leq u\} \leq a\}}\right) \leq \mathbb{E}\left(1_{\{\inf\{\frac{1}{m} \langle \Pi_+^m(t), \alpha_i^\vee \rangle, \lfloor mT \rfloor \leq t \leq \lfloor mu \rfloor + 1\} \leq a\}} \cap K_m\right) + \varepsilon,$$

where  $K_m = \{\frac{1}{m}(\Pi_+^m\lfloor mt \rfloor + \rho) \in K\}$ . By conditioning by  $\Pi_+^m\lfloor m \rfloor$  in the expectation of the righthand side of the above inequality, we obtain as in the proof of Proposition 7.4 that there exists  $M \geq 0$  such that

$$\mathbb{E}\left(1_{\{\inf\{\frac{1}{m} \langle \Pi_+^m(t), \alpha_i^\vee \rangle, \lfloor mT \rfloor \leq t \leq \lfloor mu \rfloor + 1\} \leq a\}} \cap K_m\right) \leq M \mathbb{E}\left(1_{\{\inf\{\frac{1}{m} \langle \Pi^m(t), \alpha_i^\vee \rangle, \lfloor mT \rfloor - m \leq t \leq a\}}\right).$$

Thanks to the first inequality, for such an  $M \geq 0$ , we choose  $T_0 \geq 0$  and  $m_1 \geq m_0$  such as for  $m \geq m_1$

$$\mathbb{P}\left(\inf_{\lfloor mT_0 \rfloor - m \leq t} \frac{1}{m} \langle \Pi^m(t), \alpha_i^\vee \rangle \leq a\right) \leq \varepsilon/M.$$

Thus for  $u \geq T_0$ ,  $m \geq m_1$

$$\mathbb{P}\left(\inf_{T_0 \leq t \leq u} \frac{1}{m} \langle \Pi_+^m(mt), \alpha_i^\vee \rangle \leq a\right) \leq 2\varepsilon.$$



As  $m_1$  does not depend on  $u$ , we let  $u$  goes to infinity in the above inequality, which ends the proof.  $\square$

We can now state the convergence corresponding to the first arrow of the diagram of figure 1.

**Proposition 7.8.** *In the quotient space  $\mathfrak{h}_{\mathbb{R}}^* \bmod \delta$ , the sequence of random processes*

$$\left\{ \frac{1}{m} I_{\alpha_0}^{\xi_{0,p}^m(\infty)} \dots I_{\alpha_p}^{\xi_{p,p}^m(\infty)} \Pi_+^m(mt) : t \geq 0 \right\}, \quad m \geq 1,$$

*converges in distribution towards*

$$\left\{ I_{\alpha_0}^{\xi_{0,p}(\infty)} \dots I_{\alpha_p}^{\xi_{p,p}(\infty)} A(t), t \geq 0 \right\},$$

*as  $m$  goes to infinity.*

**7.2. Proof of the convergence corresponding to the third arrow of the diagram.** Let us first notice that Proposition 5.5 implies the following one.

**Proposition 7.9.** *For  $m, p \geq 1$ , the process  $\{I_{\alpha_0}^{\xi_{0,p}^m(\infty)} \dots I_{\alpha_p}^{\xi_{p,p}^m(\infty)} \Pi_+^m(t) : t \geq 0\}$  has the same law as  $\{\Pi^m(t) : t \geq 0\}$  conditionally on  $\{\xi_{p+1}^m(\infty) = 0\}$ .*

**Proposition 7.10.** *For  $u \in \mathbb{R}$ , and  $p \in \mathbb{N}$ ,*

$$(23) \quad \mathbb{E} \left( e^{iu \langle \Pi^m \lfloor mt \rfloor / m, \alpha_1^\vee \rangle} \mathbf{1}_{\xi_{p+1}^m \lfloor mt \rfloor = 0} \right) = \mathbb{E} \left( \frac{ch_{\Pi_+^m \lfloor mt \rfloor}^{w_p} \left( \frac{1}{m} (iu \alpha_1^\vee + \rho^\vee) \right)}{ch_{\Pi_+^m \lfloor mt \rfloor} \left( \frac{1}{m} \rho^\vee \right)} \right).$$

*In particular,*

$$(24) \quad \mathbb{P} \left( \xi_{p+1}^m \lfloor mt \rfloor = 0 \right) = \mathbb{E} \left( \frac{ch_{\Pi_+^m \lfloor mt \rfloor}^{w_p} \left( \frac{1}{m} \rho^\vee \right)}{ch_{\Pi_+^m \lfloor mt \rfloor} \left( \frac{1}{m} \rho^\vee \right)} \right).$$

*Proof.* First notice that Identity (24) follows by letting  $u = 0$  in (23). To prove (23), we notice that proposition 5.2 implies that

$$\mathbb{E} \left( e^{iu \langle \Pi^m \lfloor mt \rfloor / m, \alpha_1^\vee \rangle} \mathbf{1}_{\xi_{p+1}^m \lfloor mt \rfloor = 0} | \Pi_+^m \lfloor mt \rfloor = \lambda \right)$$

is equal to

$$\frac{\sum_{a \in B(\lambda)} e^{iu \langle (\lambda - \omega(a)) / m, \alpha_1^\vee \rangle} e^{\langle \lambda - \omega(a), \rho^\vee / m \rangle} \mathbf{1}_{a_{p+1} = 0}}{ch_\lambda(\rho^\vee / m)}$$

which is by (14) equal to

$$\frac{ch_\lambda^{w_p} \left( (iu \alpha_1^\vee + \rho^\vee) / m \right)}{ch_\lambda(\rho^\vee / m)}.$$

Thus (23) follows by conditioning by  $\Pi_+^m \lfloor mt \rfloor$  within the lefthand side expectation of the identity.  $\square$

The idea of the proof of the third convergence of the diagram rests on the fact that

$$\mathbb{E} \left( e^{iu \langle \Pi^m \lfloor mt \rfloor / m, \alpha_1^\vee \rangle} | \xi_{p+1}^m \lfloor mt \rfloor = 0 \right)$$

for which an explicit formula involving a Demazure character is available as we have just seen, is close to

$$\mathbb{E} \left( e^{iu \langle \Pi^m \lfloor mt \rfloor / m, \alpha_1^\vee \rangle} \mid \xi_{p+1}^m(\infty) = 0 \right)$$

whose limit we are looking for.

*Definition 7.11.* Let  $(L_p)_{p \geq 0}$  be the random sequence defined by

$$L_p = \sum_{k=0}^p \xi_{k,p}(\infty) \alpha_k, \quad p \geq 0.$$

Lemma 7.3 implies in particular that for any  $p \geq 0$ , the random variable  $\frac{1}{m} \omega(\xi_{\cdot,p}^m(\infty))$  converges in distribution towards  $L_p$  when  $m$  goes to infinity. Notice that viewed in  $\mathfrak{h}_{\mathbb{R}}^* \bmod \delta$ ,  $(L_p)_{p \geq 0}$  is a sequence of real numbers.

**Lemma 7.12.** *In  $\mathfrak{h}_{\mathbb{R}}^* \bmod \delta$ ,  $L_p$  converges almost surely and in  $L^2$  towards*

$$L = \sum_{k=0}^{\infty} \frac{\varepsilon_k}{2 \lfloor k/2 \rfloor + 1} \alpha_k \bmod \delta,$$

when  $p$  goes to infinity.

*Proof.* One has for  $p \geq 0$ ,

$$L_p = \varepsilon_0 \alpha_0 + \sum_{k=1}^p k \sum_{n=k}^p \frac{2\varepsilon_n}{n(n+1)} \alpha_k = \varepsilon_0 \alpha_0 + \sum_{n=1}^p \frac{2\varepsilon_n}{n(n+1)} \sum_{k=1}^n k \alpha_k.$$

Thus

$$\begin{aligned} \langle L_p, \alpha_1^\vee \rangle &= -2 + \sum_{n=1}^p \frac{4\varepsilon_n}{n(n+1)} \sum_{k=1}^n k (-1)^{k+1} \\ &= -2 + \sum_{n=1}^p \frac{4\varepsilon_n}{n(n+1)} (-1)^{n+1} \lfloor \frac{n+1}{2} \rfloor. \end{aligned}$$

Finally

$$L_p = \sum_{k=0}^p \frac{\varepsilon_k}{2 \lfloor k/2 \rfloor + 1} \alpha_k \bmod \delta,$$

which shows that in  $\mathfrak{h}_{\mathbb{R}}^* \bmod \delta$ ,  $(L_p)_{p \geq 0}$  is a bounded martingale in  $L^2$ , and gives the expected convergence.  $\square$

**Lemma 7.13.** *If  $(\lambda_m)$  is a sequence with values in  $P_+$  such that  $\lim_{m \rightarrow \infty} \frac{\lambda_m}{m} = \lambda \in C_{\text{aff}}$  then for  $u \in \mathbb{R}$ ,*

$$\lim_{m \rightarrow \infty} \frac{\text{ch}_{\lambda_m}^{w_p}((iu\alpha_1^\vee + \rho^\vee)/m)}{\text{ch}_{M(0)}^{w_p}(\rho^\vee/m)} = e^{\langle \lambda, \rho^\vee \rangle} \mathbb{E} \left( e^{iu \langle \lambda - L_p, \alpha_1^\vee \rangle} 1_{\xi_{\cdot,p} \in \Gamma(\lambda)} \right)$$

*Proof.* Expressions (14) and (15) for Demazure characters give

$$\frac{1}{m^{p+1}} \text{ch}_{\lambda_m}^{w_p}((iu\alpha_1^\vee + \rho^\vee)/m) = \frac{1}{m^{p+1}} \sum_{a \in B(\lambda_m), a_{p+1}=0} e^{\langle \frac{1}{m}(\lambda_m - \omega(a)), iu\alpha_1^\vee + \rho^\vee \rangle},$$

and

$$\frac{1}{m^{p+1}} \text{ch}_{M(0)}^{w_p}(\rho^\vee/m) = \frac{1}{m^{p+1}} \sum_{a \in B(\infty), a_{p+1}=0} e^{-\langle \frac{1}{m} \omega(a), \rho^\vee \rangle}.$$

Thus

$$\begin{aligned} \lim_{m \rightarrow \infty} \frac{\text{ch}_{\lambda_m}^{w_p}((iu\alpha_1^\vee + \rho^\vee)/m)}{\text{ch}_{M(0)}^{w_p}(\rho^\vee/m)} &= \frac{\int_{\mathbb{R}_+^{p+1}} e^{\langle \lambda - \omega(x), iu\alpha_1^\vee + \rho^\vee \rangle} 1_{x \in \Gamma(\lambda)} dx}{\int_{\mathbb{R}_+^{p+1}} e^{-\langle \omega(x), \rho^\vee \rangle} 1_{x \in \Gamma(\infty)} dx} \\ &= e^{\langle \lambda, \rho^\vee \rangle} \frac{\int_{\mathbb{R}_+^{p+1}} e^{iu\langle \lambda - \omega(x), \alpha_1^\vee \rangle} e^{-\langle \omega(x), \rho^\vee \rangle} 1_{x \in \Gamma(\lambda)} dx}{\int_{\mathbb{R}_+^{p+1}} e^{-\langle \omega(x), \rho^\vee \rangle} 1_{x \in \Gamma(\infty)} dx}. \end{aligned}$$

The observation of the density of  $(\xi_{0,p}(\infty), \dots, \xi_{p,p}(\infty))$  given in the proof of Lemma 7.3 allows to conclude.  $\square$

**Lemma 7.14.** *For  $\lambda \in C_{\text{aff}}$ , the random variable  $1_{\{\xi_{\cdot,p}(\infty) \in \Gamma(\lambda)\}}$  converges almost surely towards  $1_{\{\xi(\infty) \in \Gamma(\lambda)\}}$  when  $p$  goes to infinity.*

*Proof.* We know that almost surely in the quotient space  $\mathfrak{h}_{\mathbb{R}}^* \bmod \delta$

$$\lim_{k \rightarrow \infty} \sum_{i=0}^{k-1} \xi_i(\infty) \alpha_i + \frac{1}{2} \xi_k(\infty) \alpha_k = L, \text{ and } \lim_{p \rightarrow \infty} L_p = L.$$

As for every integer  $k$ , almost surely  $\lim_{p \rightarrow \infty} \xi_{k,p}(\infty) = \xi_k(\infty)$ , one obtains a first inclusion

$$\limsup_{p \rightarrow \infty} \{\xi_{\cdot,p}(\infty) \in \Gamma(\lambda)\} \subset \{\xi(\infty) \in \Gamma(\lambda)\}.$$

We set for  $k \in \{1, \dots, p\}$

$$X_{k,p} = L_p - \sum_{i=0}^{k-1} \xi_{i,p}(\infty) \alpha_i - \frac{1}{2} \xi_{k,p}(\infty) \alpha_k$$

and

$$X_k = L - \sum_{i=0}^{k-1} \xi_i(\infty) \alpha_i - \frac{1}{2} \xi_k(\infty) \alpha_k.$$

We notice that, in the quotient space, one has for  $k \in \{1, \dots, p\}$

$$\sum_{i=0}^{k-1} \xi_{i,p}(\infty) \alpha_i + \frac{1}{2} \xi_{k,p}(\infty) \alpha_k = \sum_{i=0}^{k-1} \xi_i \alpha_i + \frac{1}{2} \xi_k(\infty) \alpha_k - \frac{\xi_p(\infty)}{2p} \alpha_k 1_{k \text{ is odd}}.$$

Thus almost surely

$$\lim_{p \rightarrow \infty} \sup_{0 \leq k \leq p} |\alpha_k^\vee (X_k^p - X_k)| = 0.$$

It follows, as in the proof of proposition 5.14 of [6], that almost surely

$$\{\xi(\infty) \in \Gamma(\lambda)\} \subset \liminf_{p \rightarrow \infty} \{\xi_{\cdot,p}(\infty) \in \Gamma(\lambda)\}.$$

Finally one has,

$$\limsup_{p \rightarrow \infty} \{\xi_{\cdot,p}(\infty) \in \Gamma(\lambda)\} \subset \{\xi(\infty) \in \Gamma(\lambda)\} \subset \liminf_{p \rightarrow \infty} \{\xi_{\cdot,p}(\infty) \in \Gamma(\lambda)\},$$

from which the lemma follows.  $\square$

The function  $\varphi_{iu+\frac{1}{2}}$  is defined on  $\mathbb{R}_+^* \times \mathbb{R}$  by

$$(25) \quad \varphi_{iu+1/2}(t, x) = \frac{e^{-(iu+1/2)x}}{\cosh(u)} \sum_{k \in \mathbb{Z}} \sinh((iu+1/2)(2kt+x)) e^{-2(kx+k^2t)},$$

$t > 0, x \in \mathbb{R}$ .

**Proposition 7.15.** *Let  $\lambda \in C_{\text{aff}}$ ,  $u \in \mathbb{R}$ . One has*

$$\mathbb{E}(e^{-iu\langle L, \alpha_1^\vee \rangle} | \xi(\infty) \in \Gamma(\lambda)) = \frac{\varphi_{iu+1/2}(\lambda)}{\varphi_{1/2}(\lambda)}, \quad \text{and} \quad \mathbb{P}(\xi(\infty) \in \Gamma(\lambda)) = 2\varphi_{\frac{1}{2}}(\lambda).$$

*Proof.* Notice that  $L$  is the random variable denoted by  $L^{(\mu)}(\infty)$  with  $\mu = 1/2$  in [6]. The first identity follows from Theorem 8.3 and proposition 6.7 of [6]. For the second one, we deduce from Theorems 5.2 and 5.5 of [5] a similar identity for the dihedral string coordinates  $(\xi^m(\infty)_k)_{k \geq 0}$  defined in [6]. Then we apply proposition 5.14 of [6].  $\square$

Theorems 8.3 and 6.6 of [6] imply in particular the following proposition.

**Proposition 7.16.** *For  $u \in \mathbb{R}$ ,*

$$\mathbb{E}\left(e^{iu\langle B(t), \alpha_1^\vee \rangle}\right) = \mathbb{E}\left(e^{iu\langle A(t), \alpha_1^\vee \rangle} \frac{\varphi_{iu+1/2}(A(t))}{\varphi_{1/2}(A(t))}\right).$$

We have now all the ingredients needed to prove that the third convergence of the diagram is valid, which implies Theorem 2.1.

**Theorem 7.17.** *The sequence of processes*

$$\{I_{\alpha_0}^{\xi_{0,p}(\infty)} \dots I_{\alpha_p}^{\xi_{p,p}(\infty)} A(t), t \geq 0\}, p \geq 0,$$

*converges when  $p$  goes to infinity, in a sense of finite dimensional distributions, towards the space-time Brownian motion  $\{B(t), t \geq 0\}$ , in the quotient space  $\mathfrak{h}_{\mathbb{R}}^* \bmod \delta$ .*

*Proof.* We first prove the convergence of  $I_{\alpha_0}^{\xi_{0,p}(\infty)} \dots I_{\alpha_p}^{\xi_{p,p}(\infty)} A(t)$  for a fixed  $t \geq 0$ . Let  $t \geq 0$ . For  $u \in \mathbb{R}, m, p \geq 1$ , the Fourier transform

$$\mathbb{E}\left(e^{iu\langle I_{\alpha_0}^{\xi_{0,p}(\infty)} \dots I_{\alpha_p}^{\xi_{p,p}(\infty)} A(t), \alpha_1^\vee \rangle}\right)$$

is equal to

$$\lim_{m \rightarrow \infty} \mathbb{E}\left(e^{i\frac{u}{m}\langle I_{\alpha_0}^{\xi_{0,p}^m(\infty)} \dots I_{\alpha_p}^{\xi_{p,p}^m(\infty)} \Pi_+^m[mt], \alpha_1^\vee \rangle}\right),$$

which is, by Proposition 7.9, also equal to

$$\lim_{m \rightarrow \infty} \mathbb{E}\left(e^{iu\langle \Pi^m[mt]/m, \alpha_1^\vee \rangle} | \xi_{p+1}^m(\infty) = 0\right).$$

We write

$$\mathbb{E}\left(e^{iu\langle \Pi^m[mt]/m, \alpha_1^\vee \rangle} | \xi_{p+1}^m(\infty) = 0\right) = S_1(u, m, p) + S_2(m, p)$$

where

$$\begin{aligned} S_1(u, m, p) &= \mathbb{E} \left( e^{iu \langle \frac{1}{m} \Pi^m \lfloor mt \rfloor, \alpha_1^\vee \rangle} 1_{\{\xi_{p+1}^m \lfloor mt \rfloor = 0\}} \right) / \mathbb{P} (\xi_{p+1}^m(\infty) = 0) \\ &= \mathbb{E} \left( \frac{\text{ch}_{\Pi_+^m \lfloor mt \rfloor}^{w_p} \left( \frac{1}{m} (iu \alpha_1^\vee + \rho^\vee) \right)}{\text{ch}_{\Pi_+^m \lfloor mt \rfloor} \left( \frac{1}{m} \rho^\vee \right)} \right) \frac{\text{ch}_{M(0)}(\rho^\vee / m)}{\text{ch}_{M(0)}^{w_p}(\rho^\vee / m)} \end{aligned}$$

and

$$S_2(m, p) = \mathbb{E} \left( e^{iu \langle \frac{1}{m} \Pi^m \lfloor mt \rfloor, \alpha_1^\vee \rangle} (1_{\{\xi_{p+1}^m(\infty) = 0\}} - 1_{\{\xi_{p+1}^m \lfloor mt \rfloor = 0\}}) \right) / \mathbb{P} (\xi_{p+1}^m(\infty) = 0).$$

The convergence of  $\frac{1}{m} \Pi_+^m \lfloor mt \rfloor$  towards  $A(t)$  when  $m$  goes to infinity, and Lemma 7.13 imply that

$$\lim_{m \rightarrow \infty} S_1(u, m, p) = \mathbb{E} (\psi_p(u, A(t)))$$

where for  $\lambda \in C_{\text{aff}}$ ,

$$\psi_p(u, \lambda) = \frac{e^{iu \langle \lambda, \alpha_1^\vee \rangle}}{2\varphi_{1/2}(\lambda)} \mathbb{E} \left( e^{-iu \langle L_p, \alpha_1^\vee \rangle} 1_{\xi_{\cdot, p} \in \Gamma(\lambda)} \right).$$

Lemmas 7.14 and Propositions 7.15 and 7.16 imply that

$$\lim_{p \rightarrow \infty} \mathbb{E} (\psi_p(u, A(t))) = \mathbb{E} \left( e^{iu \langle B(t), \alpha_1^\vee \rangle} \right)$$

As  $\{\xi_{p+1}^m(\infty) = 0\} \subset \{\xi_{p+1}^m \lfloor mt \rfloor = 0\}$  one has,

$$|S_2(m, p)| \leq \frac{\mathbb{P}(\xi_{p+1}^m \lfloor mt \rfloor = 0)}{\mathbb{P}(\xi_{p+1}^m(\infty) = 0)} - 1 = S_1(0, m, p) - 1$$

which implies that  $\lim_{p \rightarrow \infty} \lim_{m \rightarrow \infty} S_2(m, p) = 0$  and ends the proof of the convergence in law of  $I_{\alpha_0}^{\xi_{0,p}(\infty)} \dots I_{\alpha_p}^{\xi_{p,p}(\infty)} A(t)$  towards  $B(t)$  when  $p$  goes to infinity.

Let now  $t_0, \dots, t_n$  be a sequence of ordered real numbers such that  $0 = t_0 < t_1 < \dots < t_n$ , and  $u_1, \dots, u_n \in \mathbb{R}$ . For  $m, p \geq 1$ , the Fourier transform

$$(26) \quad \mathbb{E} \left( e^{i \sum_{k=1}^n u_k \langle I_{\alpha_0}^{\xi_{0,p}(\infty)} \dots I_{\alpha_p}^{\xi_{p,p}(\infty)} A(t_k), \alpha_1^\vee \rangle - \langle I_{\alpha_0}^{\xi_{0,p}(\infty)} \dots I_{\alpha_p}^{\xi_{p,p}(\infty)} A(t_{k-1}), \alpha_1^\vee \rangle} \right)$$

is equal to

$$\lim_{m \rightarrow \infty} \mathbb{E} \left( e^{i \sum_{k=1}^n \frac{u_k}{m} \langle I_{\alpha_0}^{\xi_{0,p}^m(\infty)} \dots I_{\alpha_p}^{\xi_{p,p}^m(\infty)} \Pi_+ \lfloor mt_k \rfloor - I_{\alpha_0}^{\xi_{0,p}^m(\infty)} \dots I_{\alpha_p}^{\xi_{p,p}^m(\infty)} \Pi_+ \lfloor mt_{k-1} \rfloor, \alpha_1^\vee \rangle} \right).$$

We obtain as previously, introducing this time the event  $\{\xi_{p+1}^m \lfloor mt_1 \rfloor = 0\}$  and using the independence of the increments, that the Fourier transform (26) converges towards

$$\mathbb{E} \left( e^{i \sum_{k=1}^n u_k \langle B(t_k) - B(t_{k-1}), \alpha_1^\vee \rangle} \right),$$

when  $p$  goes to infinity, which ends the proof.  $\square$

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