



**HAL**  
open science

# An inverse Pitman's theorem for a space-time Brownian motion in an affine Weyl chamber

Manon Defosseux, Charlie Herent

► **To cite this version:**

Manon Defosseux, Charlie Herent. An inverse Pitman's theorem for a space-time Brownian motion in an affine Weyl chamber. 2022. hal-03840191v1

**HAL Id: hal-03840191**

**<https://hal.science/hal-03840191v1>**

Preprint submitted on 4 Nov 2022 (v1), last revised 16 Jan 2024 (v3)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# AN INVERSE PITMAN'S THEOREM FOR A SPACE-TIME BROWNIAN MOTION IN A TYPE $A_1^1$ WEYL CHAMBER

MANON DEFOSSEUX AND CHARLIE HERENT

ABSTRACT. We prove an inverse Pitman's theorem for a space-time Brownian motion conditioned in Doob's sense to remain in an affine Weyl chamber. Our theorem provides a way to recover an unconditioned space-time Brownian motion from a conditioned one applying a sequence of path transformations.

## CONTENTS

1. Introduction	1
2. Statement of the theorem	3
3. The affine Lie algebra $A_1^1$ and its representations	4
4. Pitman transforms and Littelmann modules	6
5. Random walks and Littelmann paths	8
6. The continuous counterpart	10
7. An inverse Pitman's theorem	11
References	22

## 1. INTRODUCTION

Pitman's theorem [22] of 1975 gives a path representation of a standard real Brownian motion conditioned in Doob's sense to remain forever positive as

$$\{\mathcal{P}b(t) = b_t - 2 \inf_{0 \leq s \leq t} b_s, t \geq 0\},$$

where  $b$  is a standard Brownian motion with  $b_0 = 0$ . This seminal result has given rise to many generalizations or variations, see for instance [1, 2, 7, 8, 19, 21, 24]. Let us briefly describe one of the most accomplished one, due to Ph. Biane, Ph. Bougerol and N. O'Connell [4, 5]. In Pitman's theorem, the unconditioned Brownian motion lives on  $\mathbb{R}$  and the conditioned one lives on  $\mathbb{R}_+$ . Actually  $\mathbb{R}_+$  can be seen as the fundamental chamber of the group generated by the reflection through 0 acting on  $\mathbb{R}$ . This group is the simplest one among the class of Coxeter groups. In [4], Ph. Biane, Ph. Bougerol and N. O'Connell have established a Pitman's theorem for a Brownian motion conditioned to remain in the fundamental chamber of a finite Coxeter group. The theorem is obtained by applying to an unconditioned Brownian motion Pitman type transformations associated to a set of generators of the Coxeter group, according to the order of appearance of the generators in a reduced decomposition of the longest element in the group.

In [4] Ph. Biane, Ph. Bougerol and N. O'Connell have brought to light deep connections between the Pitman transform and the Littelmann path model [18]

which is a combinatorial model that describes the representations of a Kac–Moody Lie algebra. The affine Coxeter group of type  $A_1^1$  is the Weyl group of a rank one affine Kac–Moody algebra. In [6], a Pitman’s theorem has been established for a conditioned random process living in the fundamental chamber of the latter group. Pitman’s theorem in that case involves two Pitman type transformations corresponding to the generators of the group and is only asymptotic, due to the fact that there is no longest element in that case. Moreover, and quite surprisingly, the conditioned process is not obtained by applying successively and infinitely the two Pitman transforms to an unconditioned process : a correction has to be applied, which involves two Lévy type transformations.

In [4] a functional inverse Pitman transform is introduced. In the simplest case, given for  $T \geq 0$  a nonnegative continuous real trajectory  $\{\pi(t), t \in [0, T]\}$  starting at 0, and a real  $x \in [0, \pi(T)]$ , there is a unique real trajectory  $\eta$  starting at 0 such that

$$\mathcal{P}\eta = \pi \text{ and } x = - \inf_{0 \leq s \leq T} \eta(s).$$

It satisfies  $\eta(t) = \pi(t) - 2 \min(x, \inf_{t \leq s \leq T} \pi(s))$ ,  $t \in [0, T]$ . In other words, a path defined on  $[0, T]$  is entirely determined by its image by the Pitman transform and a real number that can be called a string coordinate, according to the terminology of Littelmann. We can reformulate this saying that there exists a functional transformation which sends a conditioned positive Brownian motion and a properly distributed real random variable on a standard real Brownian motion. And we can do the same in the framework of any finite Coxeter group.

In the framework of [6], given an unconditioned process, we can define its string coordinates and its image in the fundamental chamber of the affine Coxeter group by an analog of the Pitman transform. It is then natural to ask if it is possible to recover such an unconditioned process from a conditioned one and a sequence of string coordinates. The asymptotic nature of Pitman’s theorem in that case, and the presence of Lévy transformations make the issue nontrivial. We propose an answer to the question in this paper. It has to be noticed that our reconstruction is of a very different nature from the one previously described in the context of a finite Coxeter group. In the latter case actually, the reconstruction is a direct consequence of a deterministic result, whereas our result is a purely probabilistic one. This is a reconstruction in law.

We use results obtained in [6] but our approach is quite different as the one adopted in this last paper. Actually the proof of a Pitman’s theorem in this last paper leads on an approximation of the affine Coxeter group  $A_1^1$  by a sequence of dihedral ones. Thus, Brownian motions involved in the Pitman’s theorem for a dihedral group established in [5] provide approximations of random processes occurring in Pitman’s theorem for  $A_1^1$ . It has been proved in [9] (see also [11]) that these last processes can also be approximated by random walks defined using the Littelmann path model for the affine Kac–Moody algebra  $A_1^1$ . Such random walks have been originally introduced by C. Lecouvey, E. Lesigne and M. Peigné in [16]. These are the approximations we use here. Their laws offer the advantage of being given by explicit formulas coming from representation theory, which allows to make computations. This is a huge advantage and makes our paper fall in the large category of the so-called integrable probability. Demazure modules play a crucial role in our paper. These modules have beautiful combinatorial properties.

Nevertheless, as far as we know, they haven't been used before in the framework of integrable probability, which maybe can be explained by the fact that they do not form a tensor category, so that they do not define an hypergroup structure which could naturally relate them to a Markov process in a usual way (see for instance [25] and references therein). We notice that the Littelmann model and Demazure character formulas that we use are available for any affine Kac-Moody algebra, so that there is some hope to obtain an inverse Pitman's theorem in a more general context.

Let us make a last remark about our result. Actually, in the context of a finite Coxeter group, one can state another reconstruction theorem. In the simplest case, it states that if  $\{r_t, t \geq 0\}$  is a Doob-conditioned positive standard Brownian motion then

$$\{r_t - 2 \inf_{s \geq t} r_s, t \geq 0\}$$

is a real standard Brownian motion. More generally, for any finite Coxeter group, there exists such a functional transformation, which sends a conditioned Brownian motion to an unconditioned one. Such a result seems to be unattainable for  $A_1^1$ . Actually, in the finite case, the string coordinates of a Brownian motion are infinite and a Brownian motion stands morally for the lowest weight path in the Littelmann module of a Verma module. There is no such a lowest weight path in the case of  $A_1^1$ .

The paper is organized as follows. In section 2 we give a statement of an inverse Pitman's theorem for  $A_1^1$ . In section 3 we briefly recall the necessary background on representation theory of the affine Lie algebra  $A_1^1$ . The Littelmann path model for a Kac-Moody algebra  $A_1^1$  and its connection with Pitman transforms is explained in section 4. We define in section 5 random walks with increments in a Littelmann module and the associated random processes in the affine Weyl chamber. These processes can be seen as approximations of the unconditioned and conditioned Brownian motions introduced in section 6. Finally we prove an inverse Pitman's theorem for  $A_1^1$  in section 7.

*Acknowledgments:* This project is supported by the Agence Nationale de la Recherche funding CORTIPOM ANR-21-CE40-0019.

## 2. STATEMENT OF THE THEOREM

For a real  $x \geq 0$ , we define two functional transformations  $I_0^x$  and  $I_1^x$  acting on a continuous map  $\eta : \mathbb{R}_+ \rightarrow \mathbb{R}^2$  such that  $\eta(t) = (t, f(t))$ , where  $f(t) \in \mathbb{R}$ , for  $t \geq 0$ , and  $\lim_{t \rightarrow \infty} f(t)/t \in (0, 1)$  as

$$I_0^x \eta(t) = (t, f(t) + 2 \min(x, \inf_{s \geq t} (s - f(s)))) ,$$

$$I_1^x \eta(t) = (t, f(t) - 2 \min(x, \inf_{s \geq t} (f(s)))) , \quad t \geq 0.$$

We consider a space-time Brownian motion  $\{B(t) = (t, b_t + t/2), t \geq 0\}$ , where  $b$  is a standard Brownian motion, and a space-time Brownian motion  $\{A(t) = (t, a_t), t \geq 0\}$  with a drift  $1/2$ , conditioned to remain in the domain  $C_{\text{aff}}$  defined by

$$C_{\text{aff}} = \{(t, x) \in \mathbb{R}_+ \times \mathbb{R} : 0 < x < t\}.$$

This process will be defined more carefully later. We consider a sequence  $\varepsilon_n, n \geq 0$ , of independent exponential random variables with parameter 1 and we let  $\xi_{0,p}(\infty) = \varepsilon_0$ , and for all  $k \in \{1, \dots, p\}$ ,

$$\frac{\xi_{k,p}(\infty)}{k} = \sum_{n=k}^p \frac{2\varepsilon_n}{n(n+1)}.$$

The notational choices will be hopefully clearer later. Then one has the following reconstruction theorem.

**Theorem 1.** *The sequence of processes*

$$\{I_0^{\xi_{0,p}(\infty)} \dots I_p^{\xi_{p,p}(\infty)} A(t), t \geq 0\}, p \geq 0,$$

*converges, in the sense of finite dimensional distributions, towards the space-time Brownian motion  $\{B(t), t \geq 0\}$ .*

Let us notice that there is no correction term here. Actually the correction term in the affine Pitman's Theorem proved in [6] comes from the fact that the sequence of string coordinates associated to a Brownian motion is a convergent sequence with limit 2. The law of the random sequence in the previous theorem is the law of the string coordinates conditioned to be ultimately equal to 0. So this is not a surprise that no correction term is needed for this reconstruction theorem.

### 3. THE AFFINE LIE ALGEBRA $A_1^1$ AND ITS REPRESENTATIONS

We consider the affine Lie algebra  $A_1^1$  and recall what we need for our purpose about its representations. One can refer to the book of Victor G. Kac [13] for more details about Affine Lie algebras and their representations.

**Generalities.** For our purpose, we only need to define and consider a realization of a real Cartan subalgebra. We introduce two copies of  $\mathbb{R}^3$  in duality,

$$\mathfrak{h}_{\mathbb{R}} = \text{Span}_{\mathbb{R}}\{c, \alpha_1^{\vee}, d\}, \quad \mathfrak{h}_{\mathbb{R}}^* = \text{Span}_{\mathbb{R}}\{\Lambda_0, \alpha_1, \delta\},$$

where  $c = (1, 0, 0)$ ,  $\alpha_1^{\vee} = (0, 1, 0)$ ,  $d = (0, 0, 1)$ , and  $\Lambda_0 = (1, 0, 0)$ ,  $\alpha_1 = (0, 2, 0)$ ,  $\delta = (0, 0, 1)$  in  $\mathbb{R}^3$ . We let  $\alpha_0^{\vee} = (1, -1, 0)$  and  $\alpha_0 = (0, -2, 1)$ , so that  $c = \alpha_0^{\vee} + \alpha_1^{\vee}$  and  $\delta = \alpha_0 + \alpha_1$ . Usually  $\alpha_0$  and  $\alpha_1$  are called the two positive simple roots of  $A_1^1$  and  $\alpha_0^{\vee}$  and  $\alpha_1^{\vee}$  their coroots. One considers the set of integral weights

$$P = \{\lambda \in \mathfrak{h}_{\mathbb{R}}^* : \langle \lambda, \alpha_i^{\vee} \rangle \in \mathbb{Z}, i = 0, 1\},$$

where  $\langle \cdot, \cdot \rangle$  is the natural pairing, and the set of dominant integral weights

$$P_+ = \{\lambda \in \mathfrak{h}_{\mathbb{R}}^* : \langle \lambda, \alpha_i^{\vee} \rangle \in \mathbb{N}, i = 0, 1\}.$$

**Highest weight representations.** For a dominant integral weight  $\lambda$ , the character of the irreducible representation  $V(\lambda)$  of  $A_1^1$  with highest weight  $\lambda$  is defined as the formal series

$$(1) \quad \text{ch}_{\lambda} = \sum_{\beta \in P} \dim(V(\lambda)_{\beta}) e^{\beta},$$

where  $V(\lambda)_{\beta}$  is the weight space corresponding to the weight  $\beta$  in  $V(\lambda)$ . If we let  $e^{\beta}(h) = e^{\langle \beta, h \rangle}$  for  $h \in \mathfrak{h}_{\mathbb{R}}$ , and evaluate this formal series at  $h$ , the series converges absolutely or diverges, and it converges when  $\langle \delta, h \rangle > 0$ . For  $h \in \mathfrak{h}_{\mathbb{R}} \oplus i\mathfrak{h}_{\mathbb{R}}$  such that  $\Re \langle \delta, h \rangle > 0$ , the character is evaluated at  $h$  similarly extending linearly

weights to  $\mathfrak{h}_{\mathbb{R}} \oplus i\mathfrak{h}_{\mathbb{R}}$ . We consider the reflexions  $s_{\alpha_i}$ , for  $i \in \{0, 1\}$ , defined on  $\mathfrak{h}_{\mathbb{R}}^*$  by

$$s_{\alpha_i}(\beta) = \beta - \langle \beta, \alpha_i^\vee \rangle \alpha_i, \beta \in \mathfrak{h}_{\mathbb{R}}^*.$$

The Weyl group  $W$  is the group generated by these reflections. The Weyl character's formula (chapter 10 of [13]) states here that

$$(2) \quad \text{ch}_\lambda = \frac{\sum_{w \in W} \det(w) e^{w(\lambda + \rho) - \rho}}{\prod_{\alpha \in R_+} (1 - e^{-\alpha})},$$

where  $\rho = 2\Lambda_0 + \frac{\alpha_1}{2}$  and  $R_+$  is the set of positive roots defined by

$$R_+ = \{\alpha_0 + n\delta, \alpha_1 + n\delta, (n+1)\delta, n \in \mathbb{N}\}.$$

In particular

$$(3) \quad \prod_{\alpha \in R_+} (1 - e^{-\alpha}) = \sum_{w \in W} \det(w) e^{w(\rho) - \rho}.$$

The affine Weyl group  $W$  is the semi-direct product  $T \ltimes W_0$  where  $W_0$  is the subgroup generated by  $s_{\alpha_1}$  and  $T$  is the subgroup of transformations  $t_k$ ,  $k \in \mathbb{Z}$ , defined by

$$t_k(\lambda) = \lambda + k(\lambda, \delta)\alpha_1 - (k(\lambda, \alpha_1) + k^2(\lambda, \delta))\delta, \lambda \in \mathfrak{h}^*.$$

Thus for  $\lambda = n\Lambda_0 + m\frac{\alpha_1}{2}$ , with  $(m, n) \in \mathbb{N}^2$  such that  $0 \leq m \leq n$ ,  $a \in \mathbb{R}$ , and  $b > 0$ , the Weyl character formula becomes here

$$(4) \quad \text{ch}_\lambda(a\alpha_1^\vee + bd) = \frac{\sum_{k \in \mathbb{Z}} \sinh(a(m+1) + 2ak(n+2)) e^{-b(k(m+1) + k^2(n+2))}}{\sum_{k \in \mathbb{Z}} \sinh(a + 4ak) e^{-b(k+2k^2)}}.$$

**Verma modules.** The character of a Verma module with highest weight 0 is denoted by  $\text{ch}_{M(0)}$ . Let us recall some various known expressions of this character. First of all, one has

$$(5) \quad \text{ch}_{M(0)} = \prod_{\alpha \in R_+} (1 - e^{-\alpha})^{-1},$$

One has also

$$(6) \quad \text{ch}_{M(0)} = \lim_{\langle \lambda, \alpha_i^\vee \rangle \rightarrow \infty, i=0,1} e^{-\lambda} \text{ch}_\lambda$$

and

$$(7) \quad \text{ch}_{M(0)} = \left( \sum_{w \in W} \det(w) e^{w(\rho) - \rho} \right)^{-1},$$

the last identity being derived from the Weyl character formula. Because it will be useful in the rest of the paper, one notices that for  $\lambda \in P_+$  and  $h \in \mathfrak{h}_{\mathbb{R}}$  such that  $\langle \delta, h \rangle > 0$ , one has the inequality

$$(8) \quad \text{ch}_\lambda(h) e^{-\langle \lambda, h \rangle} \leq \text{ch}_{M(0)}(h).$$

## 4. PITMAN TRANSFORMS AND LITTELMANN MODULES

In this section we explain connections between the Littelmann path model and Pitman transform in the context of the affine Lie algebra  $A_1^1$ . For more details about the Littelmann path model one can see Peter Littelmann's papers [17, 18]. We fix  $T > 0$ . A path  $\pi$  defined on  $[0, T]$  is a continuous piecewise linear function  $\pi : [0, T] \rightarrow \mathfrak{h}_{\mathbb{R}}^*$  such that  $\pi(0) = 0$ . We consider the cone  $\mathcal{C}$  generated by  $P_+$  that is to say

$$\mathcal{C} = \{\lambda \in \mathfrak{h}_{\mathbb{R}}^* : \langle \lambda, \alpha_i^\vee \rangle \geq 0, i \in \{0, 1\}\}.$$

A path  $\pi$  defined on  $[0, T]$  is called dominant if  $\pi(t) \in \mathcal{C}$  for all  $t \in [0, T]$ . It is called integral if  $\pi(T) \in P$  and

$$\min_{t \in [0, T]} \langle \pi(t), \alpha_i^\vee \rangle \in \mathbb{Z}, \text{ for } i \in \{0, 1\}.$$

One defines Pitman transforms  $\mathcal{P}_{\alpha_i}$ ,  $i \in \{0, 1\}$ , which operate on the set on continuous functions  $\eta : [0, T] \rightarrow \mathfrak{h}_{\mathbb{R}}^*$  such that  $\eta(0) = 0$  by the formula

$$\mathcal{P}_{\alpha_i} \eta(t) = \eta(t) - \inf_{0 \leq s \leq t} \langle \eta(s), \alpha_i^\vee \rangle \alpha_i, \quad t \in [0, T].$$

For a dominant path  $\pi$  defined on  $[0, T]$ , such that  $\pi(T) \in P_+$ , the Littelmann module  $B\pi$  generated by  $\pi$  is the set of integral paths  $\eta$  defined on  $[0, T]$  such that there exists  $k \in \mathbb{N}$  such that

$$\mathcal{P}_{\alpha_k} \dots \mathcal{P}_{\alpha_0} \eta = \pi,$$

where  $\alpha_{2k} = \alpha_0$  and  $\alpha_{2k+1} = \alpha_1$ . If  $\pi$  is a dominant integral path defined on  $[0, T]$  such that  $\pi(T) = \lambda \in P_+$ , then the Littelmann path theory ensures that

$$(9) \quad \text{ch}_\lambda = \sum_{\eta \in B\pi} e^{\eta(T)}.$$

Besides for an integral path  $\eta$  defined on  $[0, T]$  there exists  $k_0$  such that for all  $k \geq k_0$ ,

$$\mathcal{P}_{\alpha_k} \dots \mathcal{P}_{\alpha_0} \eta(t) = \mathcal{P}_{\alpha_{k_0}} \dots \mathcal{P}_{\alpha_0} \eta(t), \quad t \in [0, T]^1.$$

Thus for an integral path  $\eta$  defined on  $[0, T]$ , one defines a dominant path  $\mathcal{P}\eta$  on  $[0, T]$ , letting

$$\mathcal{P}\eta(t) = \lim_{k \rightarrow \infty} \mathcal{P}_{\alpha_k} \dots \mathcal{P}_{\alpha_0} \eta(t), \quad t \in [0, T].$$

**String coordinates.** For a dominant path  $\pi$  defined on  $[0, T]$  and  $\eta \in B\pi$ , we define  $\mathbf{a}(\eta)$  as an almost-zero nonnegative integer sequences of integers  $(a_k)_{k \geq 0}$  defined by the identities

$$(10) \quad \mathcal{P}_{\alpha_m} \dots \mathcal{P}_{\alpha_0} \eta(T) = \eta(T) + \sum_{k=0}^m a_k \alpha_k, \quad m \geq 0.$$

Peter Littelmann proved in [18] that the function

$$\mathbf{a} : \eta \in B\pi \rightarrow \mathbf{a}(\eta) \in \ell^{(\infty)}(\mathbb{N})$$

is a one-to-one function. Its image set, which depends on  $\pi$  only through  $\pi(T)$ , is given by the set  $B(\pi(T))$  defined below, which is the set of vertices of a Kashiwara crystal [15]. The sets  $B(\infty)$  and  $B(\lambda)$  defined below are for instance respectively

<sup>1</sup>It has been proved in [6] that this fact remains true if  $\eta$  is a continuous, piecewise  $C^1$  trajectory in  $\mathfrak{h}_{\mathbb{R}}^*$ .

described in [20] and [18]. We let  $\ell^{(\infty)}(\mathbb{N})$  the set an almost-zero sequences of nonnegative integers.

**Definition 2.** *The subset  $B(\infty)$  of  $\ell^{(\infty)}(\mathbb{N})$  is defined as*

$$B(\infty) = \{a = (a_k)_{k \geq 0} \in \ell^{(\infty)}(\mathbb{N}) : \frac{a_k}{k} \geq \frac{a_{k+1}}{k+1}, k \geq 1\}.$$

For  $\lambda \in P_+$ , the subset  $B(\lambda)$  of  $B(\infty)$  is defined as

$$\begin{aligned} B(\lambda) &= \{a = (a_k)_{k \geq 0} \in B(\infty) : a_p \leq \langle \lambda - \sum_{k=p+1}^{\infty} a_k \alpha_k, \alpha_p^\vee \rangle, \forall p \geq 0\} \\ &= \{a = (a_k)_{k \geq 0} \in B(\infty) : a_p \leq \langle \lambda - \omega(a) + \sum_{k=0}^p a_k \alpha_k, \alpha_p^\vee \rangle, \forall p \geq 0\}, \end{aligned}$$

where  $\omega(a) = \sum_{k=0}^{\infty} a_k \alpha_k$ , which is the opposite of the weight of  $a$  as an element of the crystal  $B(\infty)$  of the Verma module of highest weight 0.

Thus identity (9) becomes

$$(11) \quad \text{ch}_\lambda = \sum_{a \in B(\lambda)} e^{\lambda - \omega(a)},$$

and the character of a Verma module is written with the string coordinates,

$$(12) \quad \text{ch}_{M(0)} = \sum_{a \in B(\infty)} e^{-\omega(a)}.$$

The inverse function of  $\mathbf{a}$  can be written using the functionals  $I_{\alpha_i}^{x,T}$ ,  $i \in \{0, 1\}$ ,  $x \geq 0$ , introduced in [4] and defined by

$$I_{\alpha_i}^{x,T} f(t) = f(t) - \min(x, \inf_{T \geq s \geq t} \langle f(s), \alpha_i^\vee \rangle) \alpha_i, \quad t \in [0, T],$$

for  $f : [0, T] \rightarrow \mathfrak{h}_{\mathbb{R}}^*$ . For  $a \in B(\lambda)$  and  $\pi$  an integral dominant path on  $[0, T]$  such that  $\pi(T) = \lambda$ , the only path  $\eta \in B\pi$  such that  $\mathbf{a}(\eta) = a$  is given by

$$\eta(t) = I_{\alpha_0}^{a_0, T} \dots I_{\alpha_p}^{a_p, T} \pi(t), \quad t \in [0, T],$$

where  $p$  is chosen such that  $a_k = 0$ , for all  $k \geq p+1$ . Notice that if  $f$  is a function defined on  $\mathbb{R}_+$  with values in  $\mathfrak{h}_{\mathbb{R}}^*$  such that

$$\lim_{t \rightarrow \infty} \langle f(t), \alpha_i^\vee \rangle = +\infty, \quad i \in \{0, 1\},$$

the definition of  $I_{\alpha_i}^{x,T}$ ,  $i \in \{0, 1\}$ , makes sense for  $T = +\infty$ . In the following, we write  $I_{\alpha_i}^x$  instead of  $I_{\alpha_i}^{x, +\infty}$ . We notice that if  $f$  is a map with values in  $\mathbb{R}\alpha_0 \oplus \mathbb{R}\alpha_1$  then for  $t \geq 0$ ,  $i \in \{0, 1\}$ ,

$$I_{\alpha_i}^x f(t) = I_i^x f(t) \quad \text{mod } \delta.$$

**Demazure modules.** A Demazure character is the character of a Demazure module. There exists a character formula for a Demazure character, which generalizes the Weyl character formula. Nevertheless this formula is not convenient to obtain asymptotic results and we do not use it here. For our purpose a formulation with the string coordinates will be more useful. One can find for instance in [15] an introduction to Demazure characters in the context of crystals. For



an integer  $p \geq 0$ , we let  $w_p = s_{\alpha_p} \dots s_{\alpha_0}$ , and for  $\lambda \in P_+$ , one considers the Demazure character  $\text{ch}_\lambda^{w_p}$  defined by

$$(13) \quad \text{ch}_\lambda^{w_p} = \sum_{\eta \in B^{w_p} \pi} e^{\eta(T)},$$

where  $\pi$  is an integral dominant path defined on  $[0, T]$  such that  $\pi(T) = \lambda$ , and  $B^{w_p} \pi = \{\eta \in B\pi : \mathcal{P}_{\alpha_p} \dots \mathcal{P}_{\alpha_0} \eta = \pi\}$ . Written with the string coordinates, definition (13) becomes

$$(14) \quad \text{ch}_\lambda^{w_p} = \sum_{a \in B(\lambda), a_{p+1}=0} e^{\lambda - \omega(a)}.$$

Actually  $B^{w_p} \pi$  can be identified with the vertices of the crystal graph of a Demazure module  $F(w_p^{-1} \lambda)$  of  $V(\lambda)$ . We define a Verma–Demazure character  $\text{ch}_{M(0)}^{w_p}$  letting

$$(15) \quad \text{ch}_{M(0)}^{w_p} = \sum_{a \in B(\infty), a_{p+1}=0} e^{-\omega(a)}.$$

Let us notice that this series is not defined as the character of a sub-module.

## 5. RANDOM WALKS AND LITTELMANN PATHS

In this whole section  $m$  is a fixed positive integer. Let us consider a path  $\pi_0$  defined on  $[0, 1]$  by

$$\pi_0(t) = t\Lambda_0, \quad t \in [0, 1],$$

and the Littelmann module  $B\pi_0$  generated by  $\pi_0$ . We consider a dual Weyl vector  $\rho^\vee = 2d + \alpha_1^\vee/2$ . We fix an integer  $m \geq 1$  and equip  $B\pi_0$  with a probability measure  $\mu^m$  letting

$$(16) \quad \mu^m(\eta) = \frac{e^{\frac{1}{m} \langle \eta(1), \rho^\vee \rangle}}{\text{ch}_{\Lambda_0}(\rho^\vee/m)}, \quad \eta \in B\pi_0.$$

One considers a sequence  $(\eta_i^m)_{i \geq 0}$  of i.i.d random variables with law  $\mu^m$  and a random path  $\{\Pi^m(t), t \geq 0\}$  defined by

$$\Pi^m(t) = \eta_1^m(1) + \dots + \eta_{k-1}^m(1) + \eta_k^m(t - k + 1),$$

when  $t \in [k - 1, k[$ , for  $k \in \mathbb{Z}_+$ . We write  $*$  for the usual concatenation of paths, so that for an integer  $t$ , the restriction of  $\Pi^m$  to  $[0, t]$  is in  $B\pi_0^{*t}$ . For  $t \in \mathbb{N}$ , we consider the string coordinates of  $\Pi^m|_{[0, t]}$ , and denote them by  $(\xi_k^m(t))_{k \geq 0}$ . Notice that the definition makes sense for  $t = \infty$ , as each string coordinate is increasing on  $\mathbb{R}_+$ . The law of  $(\xi_k^m(\infty))_{k \geq 0}$  can be described with the character of the Verma module with highest weight 0 as we will see later. We define a random process  $\{\Pi_+^m(t), t \geq 0\}$  with values in  $\mathcal{C}$  letting

$$\Pi_+^m(t) = \mathcal{P}\Pi^m(t), \quad t \geq 0.$$

The identity of the following proposition is an immediate consequence of the properties of the Littelmann path model. One can see for instance [16] for more explicit explanations. It implies in particular that  $\{\Pi_+^m(k), k \geq 0\}$  is Markovian. For us the proposition will be very useful in the whole paper as it allows to show that the Markov process  $\{\Pi_+^m(k), k \geq 0\}$  inherits many properties from the random walk  $\{\Pi^m(k), k \geq 0\}$ .

**Proposition 3.** For any integers  $k$  and  $n$ , and any fonction  $f$  defined on the set of continuous functions  $\mathcal{C}([n, n+k], \mathbb{R})$ , one has

$$\begin{aligned} & \mathbb{E} \left( f(\Pi_+^m(t) : n \leq t \leq n+k) | \Pi_+^m(s), s \leq n \right) \\ &= \mathbb{E} \left( f(\Pi^m(t) + \lambda, 0 \leq t \leq k) \frac{ch_{\Pi^m(k)+\lambda}(\rho^\vee/m)}{ch_\lambda(\rho^\vee/m)} e^{-\langle \Pi^m(k), \rho^\vee/m \rangle} 1_{\lambda + \Pi^m|_{[0,k]} \in \mathcal{C}} \right), \end{aligned}$$

where  $\lambda = \Pi_+^m(n)$ .

The identity of the following proposition is an immediate consequence of the fact that the image of a Littelmann module  $B\pi$  under  $\mathfrak{a}$  depends on  $\pi$  only through the final value of  $\pi$ . As it will also be used quite often in the rest of the paper, a proposition is devoted to it.

**Proposition 4.** For  $u \in \mathbb{N}$  and  $f$  a real function defined on  $B(\infty)$  one has

$$\mathbb{E} \left( f(\xi^m(u)) | \Pi_+^m(t), t \leq u \right) = \frac{\sum_{a \in B(\Pi_+^m(u))} f(a) e^{\langle \Pi_+^m(u) - \omega(a), \rho^\vee/m \rangle}}{\sum_{a \in B(\Pi_+^m(u))} e^{\langle \Pi_+^m(u) - \omega(a), \rho^\vee/m \rangle}}$$

**Lemma 5.** For  $i \in \{0, 1\}$ ,  $\langle \Pi^m(k), \alpha_i^\vee \rangle / k$  almost surely converges as  $k$  goes to infinity towards a positive real number.

*Proof.* In a more general context, it has been proved in [16], Proposition 5.4, that  $\mathbb{E}(\eta(1))$  is the interior of  $\mathcal{C}$ . In our particular case, it is easily proved using the explicit description of the weights of  $V(\Lambda_0)$  given for instance in chapter 9 of [12]. The convergence follows from a law of large numbers.  $\square$

The following lemma is a first useful application of Proposition 3.

**Lemma 6.** For  $i \in \{0, 1\}$ , in probability,  $\lim_{k \rightarrow \infty} \langle \Pi_+^m(k), \alpha_i^\vee \rangle = +\infty$ .

*Proof.* Lemma 5 implies in particular that almost surely  $\lim_{k \rightarrow \infty} \langle \Pi^m(k), \alpha_i^\vee \rangle = +\infty$ . For  $M > 0$ ,  $i \in \{0, 1\}$  and  $k \geq 1$ , Proposition 3 gives

$$\begin{aligned} & \mathbb{P} \left( \langle \Pi_+^m(k), \alpha_i^\vee \rangle < M \right) = \\ & \mathbb{E} \left( 1_{\{\langle \Pi^m(k), \alpha_i^\vee \rangle < M\}} ch_{\Pi^m(k)}(\rho^\vee/m) e^{-\langle \Pi^m(k), \rho^\vee/m \rangle} 1_{\Pi^m|_{[0,k]} \in \mathcal{C}} \right). \end{aligned}$$

Upper bound (8) and Lemma 5 achieve the proof.  $\square$

**Proposition 7.** The sequence of string coordinates  $\xi^m(\infty)$  is independent of  $\{\Pi_+^m(t), t \geq 0\}$  and

$$\mathbb{P}(\xi^m(\infty) = a) = \frac{e^{-\langle \omega(a), \rho^\vee/m \rangle}}{ch_{M(0)}(\rho^\vee/m)}, \quad a \in B(\infty).$$

*Proof.* Let  $T \geq 0$ ,  $a \in B(\infty)$  and  $f$  be a real valued function defined on  $B\pi_0^{*T}$  that we suppose bounded by 1. One has

$$\mathbb{E} \left( f(\Pi_+^m|_{[0,T]}) 1_{\{\xi^m(\infty)=a\}} \right) = \lim_{u \rightarrow \infty} \mathbb{E} \left( f(\Pi_+^m|_{[0,T]}) 1_{\{\xi^m(u)=a\}} \right)$$

Let us fix  $\varepsilon > 0$ . We choose  $M \geq 0$  such that if  $\lambda \in P_+$  and satisfies

$$\langle \alpha_i^\vee, \lambda \rangle \geq M, \quad \text{for } i \in \{0, 1\},$$

then one has

$$a \in B(\lambda) \quad \text{and} \quad \left| \frac{1}{e^{-\langle \lambda, \rho^\vee/m \rangle} ch_\lambda(\rho^\vee/m)} - \frac{1}{ch_{M(0)}(\rho^\vee/m)} \right| \leq \varepsilon.$$

Lemma 6 implies that there exists  $u_0 \in \mathbb{N}$  such that for all integer  $u \geq u_0$

$$\mathbb{P}(\langle \alpha_i^\vee, \Pi_+^m(u) \rangle \geq M, i \in \{0, 1\}) \geq 1 - \varepsilon.$$

By conditioning on  $\{\Pi_+^m(t), 0 \leq t \leq u\}$  in the lefthand side expectation of the following identity one obtains by proposition 4, for an integer  $u \geq T$ ,

$$\mathbb{E}\left(f(\Pi_+^m|_{[0,T]})1_{\{\xi^m(u)=a\}}\right) = \mathbb{E}\left(f(\Pi_+^m|_{[0,T]}) \frac{e^{-\langle \omega(a), \rho^\vee/m \rangle} 1_{B(\Pi_+^m(u))(a)}}{e^{-\langle \Pi_+^m(u), \rho^\vee/m \rangle} \text{ch}_{\Pi_+^m(u)}(\rho^\vee/m)}\right).$$

It implies that for an integer  $u \geq u_0$ ,

$$\left| \mathbb{E}\left(f(\Pi_+^m|_{[0,T]})1_{\{\xi^m(u)=a\}}\right) - \mathbb{E}\left(f(\Pi_+^m|_{[0,T]})\right) \frac{e^{-\langle \omega(a), \rho^\vee/m \rangle}}{\text{ch}_{M(0)}(\rho^\vee/m)} \right| \leq 2\varepsilon,$$

which gives the lemma.  $\square$

Proposition 7 implies immediately the following corollary.

**Corollary 8.** For  $p \geq 0$ ,

$$\mathbb{P}(\xi_{p+1}^m(\infty) = 0) = \frac{\text{ch}_{M(0)}^{w_p}(\rho^\vee/m)}{\text{ch}_{M(0)}(\rho^\vee/m)}$$

As  $\{\Pi^m(t) \in \mathcal{C}, t \geq 0\} = \{\xi^m(\infty) = 0\}$ , Proposition 7 has a second corollary, which has already been proved in [16] by a quite different method. This corollary is not really useful for our purpose, nevertheless it is worth giving it.

**Corollary 9.** One has  $\mathbb{P}(\Pi^m(t) \in \mathcal{C}, t \geq 0) = (\text{ch}_{M(0)}(\rho^\vee/m))^{-1}$ .

## 6. THE CONTINUOUS COUNTERPART

The random processes introduced in section 5 are approximations of continuous time random processes that we define in this section. For this, let us define the affine cone

$$C_{\text{aff}} = \{(t, x) \in \mathbb{R}_+ \times \mathbb{R}_+ : 0 < x < t\}.$$

We consider a process  $\{B(t) = t\Lambda_0 + (b_t + t/2)\alpha_1/2 : t \geq 0\}$ , where  $\{b_t : t \geq 0\}$  is a standard real Brownian motion starting from 0 and a process  $\{A(t), t \geq 0\}$  starting from  $(0, 0)$ , whose law is the Doob transformation of the law of the process  $B$  killed on the boundary of  $C_{\text{aff}}$  by the harmonic function  $\varphi_{1/2}$  defined on  $\mathbb{R}_+^* \times \mathbb{R}$  by

$$(17) \quad \varphi_{1/2}(t, x) = e^{-x/2} \sum_{k \in \mathbb{Z}} \sinh((2kt + x)/2) e^{-2(kx + k^2t)}, \text{ for } t > 0, x \in \mathbb{R},$$

which is positive on  $C_{\text{aff}}$  and vanishes on the boundary of  $C_{\text{aff}}$ . This process has been introduced and studied in [9, 10] and carefully defined in [6] in the context of the present paper.

The convergences of the following proposition have been proved in [11]. In this proposition, as in the convergence theorems of the following sections, all the processes are considered as processes with values in the quotient space  $\mathfrak{h}_{\mathbb{R}}^* \text{ mod } \delta$ , which is identified with  $\mathbb{R}\Lambda_0 \oplus \mathbb{R}\alpha_1 = \mathbb{R}^2$ . We notice that  $\alpha_0 = -\alpha_1$  in the quotient space. The set of continuous paths from  $\mathbb{R}_+$  to  $\mathbb{R}^2$  is equipped with the topology of uniform convergence on compact sets and we use the standard definition of convergence in distribution for a sequence of continuous processes as in Revuz and Yor ([23], XIII.1).

**Proposition 10.** (1) For any  $t \geq 0$ , the random variable  $\frac{1}{m}(\Pi^m(mt) - \Pi^m[mt])$  goes to 0 in probability when  $m$  goes to infinity.

(2) The sequence of processes

$$\left\{ \frac{1}{m} \Pi^m(mt) : t \geq 0 \right\}, \quad m \geq 1,$$

viewed in the quotient space  $\mathfrak{h}_{\mathbb{R}}^* \bmod \delta$ , converges in distribution towards the process  $\{B(t) : t \geq 0\}$  when  $m$  goes to infinity.

(3) The sequence of processes

$$\left\{ \frac{1}{m} \Pi_+^m[mt] : t \geq 0 \right\}, \quad m \geq 1,$$

viewed in the quotient space  $\mathfrak{h}_{\mathbb{R}}^* \bmod \delta$ , converges towards  $\{A(t) : t \geq 0\}$  when  $m$  goes to infinity, in the sense of finite dimensional distribution.

For  $t \geq 0$ , we consider the string coordinates of  $B$  on  $[0, t]$ , denoted by  $(\xi_k(t))_{k \geq 0}$ . They are defined by

$$(18) \quad \mathcal{P}_{\alpha_m} \dots \mathcal{P}_{\alpha_0} B(t) = B(t) + \sum_{k=0}^m \xi_k(t) \alpha_k, \quad m \geq 0.$$

For every  $k \geq 0$ , the function  $t \in \mathbb{R}_+ \mapsto \xi_k(t)$  is increasing, and because of the drift,  $\lim_{t \rightarrow \infty} \xi_k(t) < +\infty$ . We let  $\lim_{t \rightarrow \infty} \xi_k(t) = \xi_k(\infty)$ . For a sequence  $x = (x_k) \in \mathbb{R}_+^{\mathbb{N}}$ , we let

$$(19) \quad \omega(x) = \lim_{n \rightarrow +\infty} \sum_{k=0}^{n-1} x_k \alpha_k + \frac{1}{2} x_n \alpha_n \bmod \delta,$$

when this limit exists in  $\mathbb{R}\alpha_1$ . The following sets are the continuous analogs of the Kashiwara crystals defined in definition 2.

**Definition 11.** One defines, for  $\lambda \in \bar{C}_{\text{aff}}$ ,

$$\Gamma(\infty) = \left\{ x = (x_k) \in \mathbb{R}_+^{\mathbb{N}} : \frac{x_k}{k} \geq \frac{x_{k+1}}{k+1} \geq 0, \text{ for all } k \geq 1, \omega(x) \in \mathbb{R}^2 \right\},$$

$$\Gamma(\lambda) = \left\{ x \in \Gamma(\infty) : x_k \leq \langle \lambda - \omega(x) + \sum_{i=0}^k x_i \alpha_i, \alpha_k^\vee \rangle, \text{ for every } k \geq 0 \right\}.$$

## 7. AN INVERSE PITMAN'S THEOREM

We will now prove a reconstruction theorem which allows to get a space-time Brownian motion  $B$  from a conditioned one  $A$  and a sequence of random variables properly distributed. The idea is to prove that the commutative diagram in figure 1 is valid. The convergence represented by the third arrow of the diagram will then provides a reconstruction theorem. Black arrows on the diagram stand for convergences that have been already proved. Dashed ones stand for convergences which have still to be proved at this stage. Let us first define the random variables involved in the diagram which have not been defined yet. The law of  $\xi(\infty)$  is described by the following theorem, which has been proved in [6].

**Theorem 12** (Ph. Bougerol, M. Defosseux [6]). *The random variables*

$$\xi_0(\infty), \quad \frac{1}{2}((k+1)\xi_k(\infty) - k\xi_{k+1}(\infty)), \quad k \geq 1,$$

are independent exponential random variables with parameter 1.

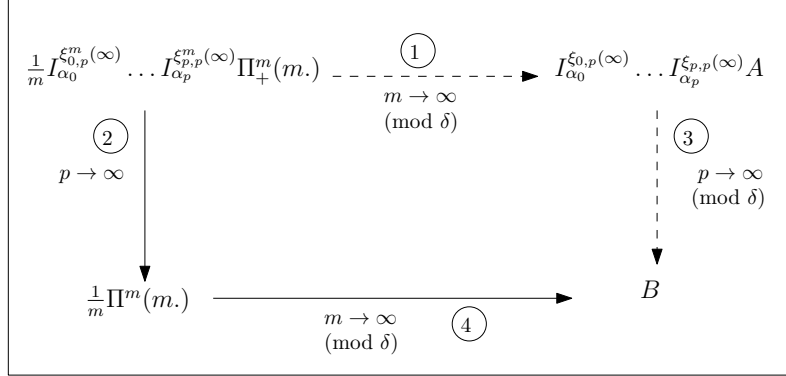


FIGURE 1. A commutative diagram of finite dimensional distribution convergences

From now on, one considers the sequence  $\varepsilon_n, n \geq 0$ , of independent exponential random variables with parameter 1 defined by

$$(20) \quad \varepsilon_0 = \xi_0(\infty), \quad \varepsilon_k = \frac{1}{2}((k+1)\xi_k(\infty) - k\xi_{k+1}(\infty)), \quad k \geq 1,$$

and  $\{A(t) : t \geq 0\}$  is supposed to be independent of this sequence.

**Definition 13.** For every  $p \geq 0$ , one lets  $\xi_{0,p}(\infty) = \varepsilon_0$ , and for all  $k \in \{1, \dots, p\}$ ,

$$\frac{\xi_{k,p}(\infty)}{k} = \sum_{n=k}^p \frac{2\varepsilon_n}{n(n+1)}.$$

We write  $\xi_{\cdot,p}(\infty) = (\xi_{k,p}(\infty))_{k \in \{0, \dots, p\}}$ .

**7.1. Proof of the convergence corresponding to the first arrow of the diagram.** For every  $p \geq 0$ , one considers a random vector

$$(\xi_{0,p}^m(\infty), \dots, \xi_{p,p}^m(\infty))$$

independent from  $\Pi_+^m$ , which is distributed as  $(\xi_0^m(\infty), \dots, \xi_p^m(\infty))$  conditionally on  $\xi_{p+1}^m(\infty) = 0$ . Lemma 14 and Propositions 17 and 18 will imply the desired convergence.

**Lemma 14.** For every  $p \in \mathbb{N}$ ,  $\frac{1}{m}(\xi_{0,p}^m(\infty), \dots, \xi_{p,p}^m(\infty))$  converges in distribution towards  $(\xi_{0,p}(\infty), \dots, \xi_{k,p}(\infty))$  when  $m$  goes to  $+\infty$ .

*Proof.* From definition 13, one derives that the density of  $(\xi_{0,p}(\infty), \dots, \xi_{p,p}(\infty))$  is given by

$$f_{(\xi_{0,p}, \dots, \xi_{0,p})}(x_0, \dots, x_p) = \frac{(p+1)! e^{-\sum_{k=0}^p x_k}}{2^p} \mathbf{1}_{x_0 \geq 0, x_1 \geq \frac{x_2}{2} \geq \dots \geq \frac{x_p}{p} \geq 0}.$$

Moreover, from Proposition 7 and Corollary 8 we deduce that for every real numbers  $t_0, \dots, t_p \geq 0$ ,

$$\mathbb{E} \left( e^{-\sum_{k=0}^p t_k \frac{\xi_{k,p}^m(\infty)}{m}} \right) = \frac{1}{\text{ch}_{M(0)}^{w_p}(\rho^V/m)} \sum_{(a_0, \dots, a_p) \in \mathbb{N}^{p+1}} e^{-\sum_{k=0}^p (1+t_k) \frac{a_k}{m}} \mathbf{1}_{\frac{a_1}{1} \geq \frac{a_2}{2} \geq \dots \geq \frac{a_p}{p}}.$$

Lemma follows from the fact that

$$m^{-(p+1)} \sum_{(a_0, \dots, a_p) \in \mathbb{N}^{p+1}} e^{-\sum_{k=0}^p (1+t_k) \frac{a_k}{m}} \mathbf{1}_{\frac{a_1}{1} \geq \frac{a_2}{2} \geq \dots \geq \frac{a_p}{p}}$$

converges towards the Riemann integral

$$\int_{\mathbb{R}_+^{p+1}} e^{-\sum_{k=0}^p (1+t_k) x_k} \mathbf{1}_{\frac{x_1}{1} \geq \frac{x_2}{2} \geq \dots \geq \frac{x_p}{p}} dx.$$

□

**Proposition 15.** *For every  $t \geq 0$ ,  $\frac{1}{m}(\Pi_+^m(mt) - \Pi_+^m[mt])$  converges in probability to 0 as  $m$  goes to infinity.*

*Proof.* Let us fix  $\varepsilon > 0$  and  $t > 0$ . We choose a compact  $K$  in  $C_{\text{aff}}$  such that

$$\mathbb{P}(A(t) \in K) > 1 - \varepsilon/2.$$

Convergences recalled in proposition 10 ensure that there exists  $m_0 \in \mathbb{N}^*$  such that for all  $m \geq m_0$

$$\mathbb{P}\left(\frac{1}{m}(\Pi_+^m[mt] + \rho) \in K\right) > 1 - \varepsilon.$$

We choose such an integer  $m_0$ . One has for all  $m \geq m_0$

$$(21) \quad \mathbb{E}\left(\mathbf{1}_{\left\{\frac{1}{m}|\langle \Pi_+^m(mt) - \Pi_+^m[mt], \alpha_1^\vee \rangle| > \varepsilon\right\}}\right) \leq \mathbb{E}\left(\mathbf{1}_{\left\{\frac{1}{m}|\langle \Pi_+^m(mt) - \Pi_+^m[mt], \alpha_1^\vee \rangle| > \varepsilon\right\} \cap K_m}\right) + \varepsilon,$$

where  $K_m = \{\frac{1}{m}(\Pi_+^m[mt] + \rho) \in K\}$ . By proposition 3, one has for  $\lambda \in P_+$ ,

$$\begin{aligned} & \mathbb{E}\left(\mathbf{1}_{\left\{\frac{1}{m}|\langle \Pi_+^m(mt) - \Pi_+^m[mt], \alpha_1^\vee \rangle| > \varepsilon\right\}} |\Pi_+^m[mt] = \lambda\right) \\ &= \mathbb{E}\left(\mathbf{1}_{\left\{\frac{1}{m}|\langle \Pi^m(mt - [mt]), \alpha_1^\vee \rangle| > \varepsilon\right\}} \frac{\text{ch}_{\Pi^m(1) + \lambda}(\rho^\vee/m)}{\text{ch}_\lambda(\rho^\vee/m)} e^{-\langle \Pi^m(1), \rho^\vee/m \rangle} \mathbf{1}_{\{\lambda + \Pi_{[0,1]}^m \in C_{\text{aff}}\}}\right) \\ &\leq \mathbb{E}\left(\mathbf{1}_{\left\{\frac{1}{m}|\langle \Pi^m(mt - [mt]), \alpha_1^\vee \rangle| > \varepsilon\right\}} \frac{\text{ch}_{M(0)}(\rho^\vee/m)}{\text{ch}_\lambda(\rho^\vee/m)} e^{-\langle \lambda, \rho^\vee/m \rangle}\right), \end{aligned}$$

the last inequality being derived from (8). Moreover the Weyl character formula gives

$$\frac{\text{ch}_\lambda(\rho^\vee/m) e^{-\langle \lambda, \rho^\vee/m \rangle}}{\text{ch}_{M(0)}(\rho^\vee/m)} = \sum_{w \in W} \det(w) e^{\langle w(\frac{\lambda + \rho^\vee}{m}) - (\frac{\lambda + \rho^\vee}{m}), \rho^\vee \rangle}.$$

The function

$$x \in C_{\text{aff}} \mapsto \sum_{w \in W} \det(w) e^{\langle w(x) - x, \rho^\vee \rangle},$$

is positive on  $K$ . We let

$$M = \max\left\{\left(\sum_{w \in W} \det(w) e^{\langle w(x) - x, \rho^\vee \rangle}\right)^{-1} : x \in K\right\}.$$

Thus for  $\lambda \in P_+$  such that  $(\lambda + \rho)/m \in K$  one has,

$$\mathbb{E}\left(\mathbf{1}_{\left\{\frac{1}{m}|\langle \Pi_+^m(mt) - \Pi_+^m[mt], \alpha_1^\vee \rangle| > \varepsilon\right\}} |\Pi_+^m[mt] = \lambda\right) \leq M \mathbb{E}\left(\mathbf{1}_{\left\{\frac{1}{m}|\langle \Pi^m(mt - [mt]), \alpha_1^\vee \rangle| > \varepsilon\right\}}\right).$$

As  $\frac{1}{m}\Pi^m(mt - \lfloor mt \rfloor)$  converges towards 0 in probability as it is recalled in proposition 10, we choose an integer  $m_1 \geq m_0$  such that for all  $m \geq m_1$ ,

$$\mathbb{E} \left( \mathbf{1}_{\left\{ \frac{1}{m} |\langle \Pi^m(mt - \lfloor mt \rfloor, \alpha_1^\vee) \rangle| > \varepsilon \right\}} \right) \leq \varepsilon/M.$$

Finally by conditioning by  $\Pi_+^m \lfloor mt \rfloor$  within the expectation of the righthand side of inequality (21), one obtains for  $m \geq m_1$ ,

$$(22) \quad \mathbb{E} \left( \mathbf{1}_{\left\{ \frac{1}{m} |\langle \Pi_+^m(mt) - \Pi_+^m \lfloor mt \rfloor, \alpha_1^\vee \rangle| > \varepsilon \right\}} \right) \leq 2\varepsilon,$$

which proves the expected convergence.  $\square$

By proposition 3, we prove in the following proposition that the sequence of random processes  $\{\frac{1}{m}\Pi_+^m(mt) : t \geq 0\}$ ,  $m \geq 1$ , inherits the tightness from  $\{\frac{1}{m}\Pi^m(mt) : t \geq 0\}$ ,  $m \geq 1$ .

**Proposition 16.** *The sequence of processes  $\{\frac{1}{m}\Pi_+^m(mt) : t \geq 0\}$ ,  $m \geq 1$ , is tight.*

*Proof.* For  $t \geq 0$ , we let  $X^m(t) = \frac{1}{m}\Pi_+^m(mt)$ . As it has been recalled in proposition 10, it has been proved in [11] that  $\frac{1}{m}\Pi_+^m \lfloor mt \rfloor$  converges in law when  $m$  goes to infinity. From proposition 15, we deduce the convergence in law of  $X^m(t)$  for any  $t \geq 0$ . Thus it is sufficient to prove that

$$\forall T \geq 0, \forall \varepsilon > 0, \forall \eta > 0, \exists \delta > 0 \text{ s.t. } \limsup_{m \rightarrow +\infty} \mathbb{P}(w_T(X^m, \delta) \geq \eta) \leq \varepsilon,$$

where, for  $x : \mathbb{R}_+ \rightarrow \mathfrak{h}_{\mathbb{R}}^*$ ,

$$w_T(x, \delta) = \sup\{|\langle x(t) - x(s), \alpha_1^\vee \rangle|, s, t \in [0, T], |s - t| \leq \delta\}.$$

Let  $T, \varepsilon, \eta > 0$ . We suppose that  $T$  is greater than  $\eta$ . We let  $t_0 = \frac{\eta}{2}$  and define  $w_T^{t_0}(x, \delta)$  by

$$w_T^{t_0}(x, \delta) = \sup\{|\langle x(t) - x(s), \alpha_1^\vee \rangle|, s, t \in [t_0, T], |s - t| \leq \delta\},$$

for  $\delta \geq 0$ ,  $x : \mathbb{R}_+ \rightarrow \mathfrak{h}_{\mathbb{R}}^*$ . As for every  $t \geq 0$ ,  $X^m(t)$  is in  $C_{\text{aff}}$ , one has for  $\delta \leq t_0$ ,

$$\{w_T(X^m, \delta) \geq \eta\} \subset \{w_T^{t_0}(X^m, \delta) \geq \eta\}.$$

As in the proof of proposition 15 we choose a compact  $K$  in  $C_{\text{aff}}$  and  $m_0 \in \mathbb{N}$  such as for all  $m \geq m_0$

$$\mathbb{P} \left( \frac{\Pi_+^m \lfloor mt_0 \rfloor + \rho}{m} \in K \right) \geq 1 - \varepsilon.$$

Hence,

$$\mathbb{P}(w_T^{t_0}(X^m, \delta) \geq \eta) \leq \mathbb{E} \left( \mathbf{1}_{\left\{ w_T^{t_0}(X^m, \delta) \geq \eta, \frac{\Pi_+^m \lfloor mt_0 \rfloor + \rho}{m} \in K \right\}} \right) + \varepsilon$$

By conditioning by  $\Pi_+^m \lfloor mt_0 \rfloor$  in the expectation of the righthand side of the above inequality, we obtain as in the proof of the proposition 15 that

$$\mathbb{P} \left( w_T^{t_0}(X^m, \delta) \geq \eta, \frac{\Pi_+^m \lfloor mt_0 \rfloor + \rho}{m} \in K \right) \leq M \mathbb{P} \left( w_T \left( \frac{1}{m} \Pi^m(m \cdot), \delta \right) \geq \eta \right),$$

where

$$M = \max_{w \in W} \left\{ \left( \sum_{w \in W} \det(w) e^{(w(x) - x) \cdot \rho^\vee} \right)^{-1} : x \in K \right\}.$$

As the sequence of processes  $\{\frac{1}{m}\Pi^m(mt), t \geq 0\}$ ,  $m \geq 0$ , is tight, we choose  $m_1 \geq m_0$  and  $\delta_0 \in (0, \eta/2]$  such that for  $m \geq m_1$ ,

$$\mathbb{P}\left(w_T\left(\frac{1}{m}\Pi^m(m.), \delta_0\right) \geq \eta\right) \leq \varepsilon/M.$$

Thus for  $m \geq m_1$ , one has

$$\mathbb{P}\left(w_T^{t_0}(X^m, \delta_0) \geq \eta\right) \leq 2\varepsilon,$$

which achieves the proof.  $\square$

The convergence recalled in proposition 10 of  $\{\frac{1}{m}\Pi_+^m\lfloor mt \rfloor : t \geq 0\}$  in the sense of finite dimensional law and the previous proposition give the following one.

**Proposition 17.** *The sequence of processes  $\{\frac{1}{m}\Pi_+^m(mt) : t \geq 0\}$ ,  $m \geq 1$ , converges in distribution towards  $\{A(t) : t \geq 0\}$  in the quotient space  $\mathfrak{h}_{\mathbb{R}}^* \bmod \delta$ .*

Now it remains to control the asymptotic behavior of  $\Pi_+^m$  for large time uniformly in  $m$  to get the convergence represented by the first arrow in the diagram. For this we show the following proposition.

**Proposition 18.** *For all  $\varepsilon, a > 0$  there exists  $T, m_0 \geq 0$  such as for  $i \in \{0, 1\}$  and all  $m \geq m_0$*

$$\mathbb{P}\left(\inf_{t \geq T} \frac{1}{m} \langle \Pi^m(mt), \alpha_i^\vee \rangle \geq a\right), \mathbb{P}\left(\inf_{t \geq T} \frac{1}{m} \langle \Pi_+^m(mt), \alpha_i^\vee \rangle \geq a\right) \geq 1 - \varepsilon.$$

*Proof.* Slight modifications in the proof of proposition 6.13 of [11] give the first inequality. Let  $i \in \{0, 1\}$ . As previously we choose a compact  $K$  in  $C_{\text{aff}}$  and  $m_0 \in \mathbb{N}^*$  such as for all  $m \geq m_0$

$$\mathbb{P}\left(\frac{\Pi_+^m(m) + \rho}{m} \in K\right) \geq 1 - \varepsilon$$

Let  $T \geq 1$  that will be chosen later. For  $u > T$  and  $m \geq m_0$  one has

$$\mathbb{E}\left(1_{\{\inf_{\frac{1}{m} \langle \Pi_+^m(mt), \alpha_i^\vee \rangle, T \leq t \leq u} \leq a\}}\right) \leq \mathbb{E}\left(1_{\{\inf_{\frac{1}{m} \langle \Pi_+^m(t), \alpha_i^\vee \rangle, \lfloor mT \rfloor \leq t \leq \lfloor mu \rfloor + 1} \leq a\}} \cap K_m\right) + \varepsilon,$$

where  $K_m = \{\frac{1}{m}(\Pi_+^m\lfloor mt \rfloor + \rho) \in K\}$ . By conditioning by  $\Pi_+^m\lfloor mT \rfloor$  in the expectation of the righthand side of the above inequality, we obtain as in the proof of Proposition 15 that there exists  $M \geq 0$  such that

$$\mathbb{E}\left(1_{\{\inf_{\frac{1}{m} \langle \Pi_+^m(t), \alpha_i^\vee \rangle, \lfloor mT \rfloor \leq t \leq \lfloor mu \rfloor + 1} \leq a\}} \cap K_m\right) \leq M \mathbb{E}\left(1_{\{\inf_{\frac{1}{m} \langle \Pi^m(t), \alpha_i^\vee \rangle, \lfloor mT \rfloor - m \leq t \leq a\}}\right)$$

Thanks to the first inequality, for such an  $M \geq 0$ , we choose  $T_0 \geq 0$  and  $m_1 \geq m_0$  such as for  $m \geq m_1$

$$\mathbb{P}\left(\inf_{\lfloor mT_0 \rfloor - m \leq t} \frac{1}{m} \langle \Pi^m(t), \alpha_i^\vee \rangle \leq a\right) \leq \varepsilon/M$$

Thus for  $u \geq T_0$ ,  $m \geq m_1$

$$\mathbb{P}\left(\inf_{T_0 \leq t \leq u} \frac{1}{m} \langle \Pi_+^m(mt), \alpha_i^\vee \rangle \leq a\right) \leq 2\varepsilon.$$

As  $m_1$  does not depend on  $u$ , we let  $u$  goes to infinity in the above inequality, which achieves the proof.  $\square$



We can now state the convergence corresponding to the first arrow of the diagram of figure 1.

**Proposition 19.** *In the quotient space  $\mathfrak{h}_{\mathbb{R}}^* \bmod (\delta)$ , the sequence of random processes*

$$\left\{ \frac{1}{m} I_{\alpha_0}^{\xi_{0,p}^m(\infty)} \dots I_{\alpha_p}^{\xi_{p,p}^m(\infty)} \Pi_+^m(mt) : t \geq 0 \right\}, \quad m \geq 1,$$

*converges in the sense of finite dimensional distribution towards*

$$\left\{ I_{\alpha_0}^{\xi_{0,p}(\infty)} \dots I_{\alpha_p}^{\xi_{p,p}(\infty)} A(t), t \geq 0 \right\},$$

*as  $m$  goes to infinity.*

**7.2. Proof of the convergence corresponding to the third arrow of the diagram.** Let us first notice that Proposition 7 implies the following one.

**Proposition 20.** *For  $m, p \geq 1$ , the process  $\{I_{\alpha_0}^{\xi_{0,p}^m(\infty)} \dots I_{\alpha_p}^{\xi_{p,p}^m(\infty)} \Pi_+^m(t) : t \geq 0\}$  has the same law as  $\{\Pi^m(t) : t \geq 0\}$  conditionally on  $\{\xi_{p+1}^m(\infty) = 0\}$ .*

**Proposition 21.** *For  $u \in \mathbb{R}$ , and  $p \in \mathbb{N}$ ,*

$$(23) \quad \mathbb{E} \left( e^{iu \langle \Pi^m \lfloor mt \rfloor / m, \alpha_1^\vee \rangle} \mathbf{1}_{\xi_{p+1}^m \lfloor mt \rfloor = 0} \right) = \mathbb{E} \left( \frac{ch_{\Pi_+^m \lfloor mt \rfloor}^{w_p} \left( \frac{1}{m} (iu \alpha_1^\vee + \rho^\vee) \right)}{ch_{\Pi_+^m \lfloor mt \rfloor} \left( \frac{1}{m} \rho^\vee \right)} \right)$$

*In particular,*

$$(24) \quad \mathbb{P} \left( \xi_{p+1}^m \lfloor mt \rfloor = 0 \right) = \mathbb{E} \left( \frac{ch_{\Pi_+^m \lfloor mt \rfloor}^{w_p} \left( \frac{1}{m} \rho^\vee \right)}{ch_{\Pi_+^m \lfloor mt \rfloor} \left( \frac{1}{m} \rho^\vee \right)} \right)$$

*Proof.* First notice that identity (24) follows letting  $u = 0$  in (23). To prove (23), we notice that proposition 4 implies that

$$\mathbb{E} \left( e^{iu \langle \Pi^m \lfloor mt \rfloor / m, \alpha_1^\vee \rangle} \mathbf{1}_{\xi_{p+1}^m \lfloor mt \rfloor = 0} \mid \Pi_+^m \lfloor mt \rfloor = \lambda \right)$$

is equal to

$$\frac{\sum_{a \in B(\lambda)} e^{iu \langle (\lambda - \omega(a)) / m, \alpha_1^\vee \rangle} e^{\langle \lambda - \omega(a), \rho^\vee \rangle / m} \mathbf{1}_{a_{p+1} = 0}}{ch_\lambda(\rho^\vee / m)}$$

which is by (14) equal to

$$\frac{ch_\lambda^{w_p} \left( (iu \alpha_1^\vee + \rho^\vee) / m \right)}{ch_\lambda(\rho^\vee / m)}.$$

Thus (23) follows by conditioning by  $\Pi_+^m \lfloor mt \rfloor$  within the lefthand side expectation of the identity.  $\square$

The idea of the proof of the third convergence of the diagram rests on the fact that

$$\mathbb{E} \left( e^{iu \langle \Pi^m \lfloor mt \rfloor / m, \alpha_1^\vee \rangle} \mid \xi_{p+1}^m \lfloor mt \rfloor = 0 \right)$$

for which an explicit formula involving a Demazure character is available as we have just seen, is close to

$$\mathbb{E} \left( e^{iu \langle \Pi^m \lfloor mt \rfloor / m, \alpha_1^\vee \rangle} \mid \xi_{p+1}^m(\infty) = 0 \right)$$

whose limit we are looking for.

**Definition 22.** We consider the random sequence  $(L_p)_{p \geq 0}$  defined by

$$L_p = \sum_{k=0}^p \xi_{k,p}(\infty) \alpha_k, \quad p \geq 0.$$

Lemma 14 implies in particular that for any  $p \geq 0$ , the random variable  $\frac{1}{m} \omega(\xi_{\cdot,p}^m(\infty))$  converges in distribution towards  $L_p$  when  $m$  goes to infinity. Moreover, we notice that viewed in  $\mathfrak{h}_{\mathbb{R}}^* \bmod \delta$ ,  $(L_p)_{p \geq 0}$  is a sequence of real numbers.

**Lemma 23.** In  $\mathfrak{h}_{\mathbb{R}}^* \bmod \delta$ ,  $L_p$  converges almost surely and in  $L^2$  towards

$$L = \sum_{k=0}^{\infty} \frac{\varepsilon_k}{2 \lfloor k/2 \rfloor + 1} \alpha_k \bmod \delta,$$

when  $p$  goes to infinity.

*Proof.* One has for  $p \geq 0$ ,

$$L_p = \varepsilon_0 \alpha_0 + \sum_{k=1}^p k \sum_{n=k}^p \frac{2\varepsilon_n}{n(n+1)} \alpha_k = \varepsilon_0 \alpha_0 + \sum_{n=1}^p \frac{2\varepsilon_n}{n(n+1)} \sum_{k=1}^n k \alpha_k$$

Thus

$$\begin{aligned} \langle L_p, \alpha_1^\vee \rangle &= -2 + \sum_{n=1}^p \frac{4\varepsilon_n}{n(n+1)} \sum_{k=1}^n k (-1)^{k+1} \\ &= -2 + \sum_{n=1}^p \frac{4\varepsilon_n}{n(n+1)} (-1)^{n+1} \lfloor \frac{n+1}{2} \rfloor. \end{aligned}$$

Finally

$$L_p = \sum_{k=0}^p \frac{\varepsilon_k}{2 \lfloor k/2 \rfloor + 1} \alpha_k \bmod \delta,$$

which shows that in  $\mathfrak{h}_{\mathbb{R}}^* \bmod \delta$ ,  $(L_p)_{p \geq 0}$  is a bounded martingale in  $L^2$ , and gives the expected convergence.  $\square$

**Lemma 24.** If  $(\lambda_m)$  is a sequence with values in  $P_+$  such that  $\lim_{m \rightarrow \infty} \frac{\lambda_m}{m} = \lambda \in C_{\text{aff}}$  then for  $u \in \mathbb{R}$ ,

$$\lim_{m \rightarrow \infty} \frac{\text{ch}_{\lambda_m}^{w_p}((iu\alpha_1^\vee + \rho^\vee)/m)}{\text{ch}_{M(0)}^{w_p}(\rho^\vee/m)} = e^{\langle \lambda, \rho^\vee \rangle} \mathbb{E} \left( e^{iu \langle \lambda - L_p, \alpha_1^\vee \rangle} 1_{\xi_{\cdot,p} \in \Gamma(\lambda)} \right)$$

*Proof.* Using (14) and (15) to write the Demazure characters one obtains

$$\frac{1}{m^{p+1}} \text{ch}_{\lambda_m}^{w_p}((iu\alpha_1^\vee + \rho^\vee)/m) = \frac{1}{m^{p+1}} \sum_{a \in B(\lambda_m), a_{p+1}=0} e^{\langle \frac{1}{m}(\lambda_m - \omega(a)), iu\alpha_1^\vee + \rho^\vee \rangle},$$

and

$$\frac{1}{m^{p+1}} \text{ch}_{M(0)}^{w_p}(\rho^\vee/m) = \frac{1}{m^{p+1}} \sum_{a \in B(\infty), a_{p+1}=0} e^{-\langle \frac{1}{m}\omega(a), \rho^\vee \rangle}.$$

Thus

$$\begin{aligned} \lim_{m \rightarrow \infty} \frac{\text{ch}_{\lambda_m}^{w_p}((iu\alpha_1^\vee + \rho^\vee)/m)}{\text{ch}_{M(0)}^{w_p}(\rho^\vee/m)} &= \frac{\int_{\mathbb{R}_+^{p+1}} e^{\langle \lambda - \omega(x), iu\alpha_1^\vee + \rho^\vee \rangle} \mathbf{1}_{x \in \Gamma(\lambda)} dx}{\int_{\mathbb{R}_+^{p+1}} e^{-\langle \omega(x), \rho^\vee \rangle} \mathbf{1}_{x \in \Gamma(\infty)} dx} \\ &= e^{\langle \lambda, \rho^\vee \rangle} \frac{\int_{\mathbb{R}_+^{p+1}} e^{iu\langle \lambda - \omega(x), \alpha_1^\vee \rangle} e^{-\langle \omega(x), \rho^\vee \rangle} \mathbf{1}_{x \in \Gamma(\lambda)} dx}{\int_{\mathbb{R}_+^{p+1}} e^{-\langle \omega(x), \rho^\vee \rangle} \mathbf{1}_{x \in \Gamma(\infty)} dx}. \end{aligned}$$

The observation of the density of  $(\xi_{0,p}(\infty), \dots, \xi_{p,p}(\infty))$  given in the proof of Lemma 14 allows to conclude.  $\square$

**Lemma 25.** *For  $\lambda \in C_{\text{aff}}$ , the random variable  $\mathbf{1}_{\{\xi_{\cdot,p}(\infty) \in \Gamma(\lambda)\}}$  converges almost surely towards  $\mathbf{1}_{\{\xi(\infty) \in \Gamma(\lambda)\}}$  when  $p$  goes to infinity.*

*Proof.* We know that almost surely in the quotient space  $\mathfrak{h}_{\mathbb{R}}^* \bmod \delta$

$$\lim_{k \rightarrow \infty} \sum_{i=0}^{k-1} \xi_i(\infty) \alpha_i + \frac{1}{2} \xi_k(\infty) \alpha_k = L, \text{ and } \lim_{p \rightarrow \infty} L_p = L.$$

As for every integer  $k$ , almost surely  $\lim_{p \rightarrow \infty} \xi_{k,p}(\infty) = \xi_k(\infty)$ , one obtains a first inclusion

$$\limsup_{p \rightarrow \infty} \{\xi_{\cdot,p}(\infty) \in \Gamma(\lambda)\} \subset \{\xi(\infty) \in \Gamma(\lambda)\}.$$

We let for  $k \in \{1, \dots, p\}$

$$X_{k,p} = L_p - \sum_{i=0}^{k-1} \xi_{i,p}(\infty) \alpha_i - \frac{1}{2} \xi_{k,p}(\infty) \alpha_k$$

and

$$X_k = L - \sum_{i=0}^{k-1} \xi_i(\infty) \alpha_i - \frac{1}{2} \xi_k(\infty) \alpha_k.$$

We notice that, in the quotient space, one has for  $k \in \{1, \dots, p\}$

$$\sum_{i=0}^{k-1} \xi_{i,p}(\infty) \alpha_i + \frac{1}{2} \xi_{k,p}(\infty) \alpha_k = \sum_{i=0}^{k-1} \xi_i \alpha_i + \frac{1}{2} \xi_k(\infty) \alpha_k - \frac{\xi_p(\infty)}{2p} \alpha_k \mathbf{1}_{k \text{ is odd}}.$$

Thus almost surely

$$\lim_{p \rightarrow \infty} \sup_{0 \leq k \leq p} |\alpha_k^\vee (X_k^p - X_k)| = 0.$$

It follows, as in the proof of proposition 5.14 of [6], that almost surely

$$\{\xi(\infty) \in \Gamma(\lambda)\} \subset \liminf_{p \rightarrow \infty} \{\xi_{\cdot,p}(\infty) \in \Gamma(\lambda)\}.$$

Finally one has,

$$\limsup_{p \rightarrow \infty} \{\xi_{\cdot,p}(\infty) \in \Gamma(\lambda)\} \subset \{\xi(\infty) \in \Gamma(\lambda)\} \subset \liminf_{p \rightarrow \infty} \{\xi_{\cdot,p}(\infty) \in \Gamma(\lambda)\},$$

from which the lemma follows.  $\square$

We define a function  $\varphi_{iu+\frac{1}{2}}$  on  $\mathbb{R}_+^* \times \mathbb{R}$  letting for  $t > 0, x \in \mathbb{R}$ .

$$(25) \quad \varphi_{iu+1/2}(t, x) = \frac{e^{-(iu+1/2)x}}{\cosh(u)} \sum_{k \in \mathbb{Z}} \sinh((iu+1/2)(2kt+x)) e^{-2(kx+k^2t)},$$

The following Lemma follows from the Poisson summation formula as it is noticed in [6]. We write here a proof following from the Euler's sine product formula.

**Lemma 26.** For  $u \in \mathbb{R}$ ,

$$\lim_{m \rightarrow \infty} \frac{\text{ch}_{M(0)}(\rho^\vee/m)}{\text{ch}_{M(0)}((iu\alpha_1^\vee + \rho^\vee)/m)} = \cosh(u)$$

*Proof.* Let  $m \geq 1$ . It follows from (5) that

$$\frac{\text{ch}_{M(0)}((iu\alpha_1^\vee + \rho^\vee)/m)}{\text{ch}_{M(0)}(\rho^\vee/m)} = \prod_{n \in \mathbb{N}} \frac{(1 - e^{-\frac{2n+1}{m}})^2}{(1 - e^{\frac{2iu-2n+1}{m}})(1 - e^{-\frac{2iu-2n+1}{m}})}$$

We write

$$\prod_{n \in \mathbb{N}} \frac{(1 - e^{-\frac{2n+1}{m}})^2}{(1 - e^{\frac{2iu-2n+1}{m}})(1 - e^{-\frac{2iu-2n+1}{m}})} = \prod_{n \in \mathbb{N}} (1 + a_{n,m})$$

where

$$a_{n,m} = \frac{2 \left( \cos\left(\frac{2u}{m}\right) - 1 \right) e^{-\frac{2n+1}{m}}}{1 + e^{-2\frac{(2n+1)}{m}} - 2 \cos\left(\frac{2u}{m}\right) e^{-\frac{2n+1}{m}}}.$$

Thus  $|a_{n,m}| = \frac{|\cos(\frac{2u}{m}) - 1|}{\cosh(\frac{2n+1}{m}) - \cos(\frac{2u}{m})}$  and by the classical inequalities

$$\forall x \in \mathbb{R} \quad \cosh(x) \geq 1 + \frac{x^2}{2} \quad \text{and} \quad |\cos(x) - 1| \leq \frac{x^2}{2}$$

we obtain the upper bound  $|a_{n,m}| \leq \frac{2(2u)^2}{(2n+1)^2}$ . It follows from the dominated convergence theorem that

$$\lim_{m \rightarrow +\infty} \prod_{n \in \mathbb{N}} (1 + a_{n,m}) = \prod_{n \in \mathbb{N}} \lim_{m \rightarrow +\infty} (1 + a_{n,m})$$

Hence

$$\lim_{m \rightarrow +\infty} \frac{\text{ch}_{M(0)}((iu\alpha_1^\vee + \rho^\vee)/m)}{\text{ch}_{M(0)}(\rho^\vee/m)} = \prod_{n \in \mathbb{N}} \frac{(2n+1)^2}{(2n+1)^2 + 4u^2}$$

By reversing the formula, we obtain

$$\frac{\text{ch}_{M(0)}(\rho^\vee/m)}{\text{ch}_{M(0)}((iu\alpha_1^\vee + \rho^\vee)/m)} \underset{m \rightarrow +\infty}{\sim} \prod_{n \in \mathbb{N}} \left( 1 + \frac{4u^2}{(2n+1)^2} \right)$$

from which follows the proposition by Euler's cosine or sine product formulas.  $\square$

**Proposition 27.** Let  $\lambda \in C_{\text{aff}}$ ,  $u \in \mathbb{R}$ . One has

$$\mathbb{E} \left( e^{-iu \langle L, \alpha_1^\vee \rangle} 1_{\xi(\infty) \in \Gamma(\lambda)} \right) = 2\varphi_{iu+1/2}(\lambda).$$

In particular,

$$\mathbb{P}(\xi(\infty) \in \Gamma(\lambda)) = 2\varphi_{\frac{1}{2}}(\lambda).$$

*Proof.* One has by Lemmas 23 and 25

$$\mathbb{E} \left( e^{-iu \langle L, \alpha_1^\vee \rangle} 1_{\xi(\infty) \in \Gamma(\lambda)} \right) = \lim_{p \rightarrow +\infty} \mathbb{E} \left( e^{-iu \langle L_p, \alpha_1^\vee \rangle} 1_{\{\xi_{\cdot, p} \in \Gamma(\lambda)\}} \right)$$

Let us consider a sequence  $(\lambda_m)_{m \geq 1}$  with values in  $P_+$  such that  $\lim_{m \rightarrow \infty} \lambda_m/m = \lambda$  and the real double-sequence  $(v_{p,m})_{p,m \geq 1}$  defined by

$$v_{p,m} = \frac{\text{ch}_{\lambda_m}^{w_p}((iu\alpha_1^\vee + \rho^\vee)/m) e^{-\langle \lambda_m, iu\alpha_1^\vee + \rho^\vee \rangle/m}}{\text{ch}_{M(0)}^{w_p}(\rho^\vee/m)}, \quad m, p \geq 1.$$

Lemma 24 says that

$$\lim_{m \rightarrow +\infty} v_{p,m} = \mathbb{E} \left( e^{-iu \langle L_p, \alpha_1^\vee \rangle} 1_{\{\xi_{\cdot, p} \in \Gamma(\lambda)\}} \right)$$

As  $(v_{p,m})_{p,m \geq 1}$  is a bounded real double-sequence then there exists a convergent double-subsequence for which we can interchange the composition of the limit in  $p$  and  $m$ . For our purpose, we can suppose that the interchange is valid for the whole sequence, i.e.

$$\lim_{p \rightarrow +\infty} \lim_{m \rightarrow +\infty} v_{p,m} = \lim_{m \rightarrow +\infty} \lim_{p \rightarrow +\infty} v_{p,m}.$$

We suppose so. Then one can write,

$$\begin{aligned} \mathbb{E} \left( e^{-iu \langle L, \alpha_1 \rangle} 1_{\xi(\infty) \in \Gamma(\lambda)} \right) &= \lim_{p \rightarrow \infty} \lim_{m \rightarrow \infty} v_{p,m} \\ &= \lim_{m \rightarrow \infty} \lim_{p \rightarrow \infty} v_{p,m} \\ &= \lim_{m \rightarrow \infty} \frac{\text{ch}_{\lambda_m}^{w_p}((iu\alpha_1^\vee + \rho^\vee)/m) e^{-\langle \lambda_m, iu\alpha_1^\vee + \rho^\vee \rangle/m}}{\text{ch}_{M(0)}^{w_p}(\rho^\vee/m)} \\ &= 2\varphi_{iu+1/2}(\lambda) \end{aligned}$$

the last limit following from the Weyl character formula and Lemma 26.  $\square$

Theorems 8.3 and 6.6 of [6] imply in particular the following proposition.

**Proposition 28.** *For  $u \in \mathbb{R}$ ,*

$$\mathbb{E} \left( e^{iu \langle B(t), \alpha_1^\vee \rangle} \right) = \mathbb{E} \left( e^{iu \langle A(t), \alpha_1^\vee \rangle} \frac{\varphi_{iu+1/2}(A(t))}{\varphi_{1/2}(A(t))} \right).$$

We have now all the ingredients needed to prove that the third convergence of the diagram is valid, i.e. to prove Theorem 1.

**Theorem 29.** *The sequence of processes*

$$\{I_{\alpha_0}^{\xi_{0,p}(\infty)} \dots I_{\alpha_p}^{\xi_{p,p}(\infty)} A(t), t \geq 0\}, p \geq 0,$$

*converges when  $p$  goes to infinity, in a sense of finite dimensional distributions, towards the space-time Brownian motion  $\{B(t), t \geq 0\}$ , in the quotient space  $\mathfrak{h}_{\mathbb{R}}^* \text{ mod } \delta$ .*

*Proof.* We first prove the convergence of  $I_{\alpha_0}^{\xi_{0,p}(\infty)} \dots I_{\alpha_p}^{\xi_{p,p}(\infty)} A(t)$  for a fixed  $t \geq 0$ . Let  $t \geq 0$ . For  $u \in \mathbb{R}$ ,  $m, p \geq 1$ , the Fourier transform

$$\mathbb{E} \left( e^{iu \langle I_{\alpha_0}^{\xi_{0,p}(\infty)} \dots I_{\alpha_p}^{\xi_{p,p}(\infty)} A(t), \alpha_1^\vee \rangle} \right)$$

is equal to

$$\lim_{m \rightarrow \infty} \mathbb{E} \left( e^{i \frac{u}{m} \langle I_{\alpha_0}^{\xi_{0,p}^m(\infty)} \dots I_{\alpha_p}^{\xi_{p,p}^m(\infty)} \Pi_+^m [mt], \alpha_1^\vee \rangle} \right),$$

which is, by Proposition 20, also equal to

$$\lim_{m \rightarrow \infty} \mathbb{E} \left( e^{iu \langle \Pi^m [mt] / m, \alpha_1^\vee \rangle} | \xi_{p+1}^m(\infty) = 0 \right).$$

We write

$$\mathbb{E} \left( e^{iu \langle \Pi^m [mt] / m, \alpha_1^\vee \rangle} | \xi_{p+1}^m(\infty) = 0 \right) = S_1(u, m, p) + S_2(m, p)$$

where

$$\begin{aligned} S_1(u, m, p) &= \mathbb{E} \left( e^{iu \langle \frac{1}{m} \Pi^m [mt], \alpha_1^\vee \rangle} 1_{\{\xi_{p+1}^m [mt] = 0\}} \right) / \mathbb{P}(\xi_{p+1}^m(\infty) = 0) \\ &= \mathbb{E} \left( \frac{\text{ch}_{\Pi_+^m [mt]}^{w_p} \left( \frac{1}{m} (iu \alpha_1^\vee + \rho^\vee) \right)}{\text{ch}_{\Pi_+^m [mt]} \left( \frac{1}{m} \rho^\vee \right)} \right) \frac{\text{ch}_{M(0)}(\rho^\vee / m)}{\text{ch}_{M(0)}^{w_p}(\rho^\vee / m)} \end{aligned}$$

and

$$S_2(m, p) = \mathbb{E} \left( e^{iu \langle \frac{1}{m} \Pi^m [mt], \alpha_1^\vee \rangle} (1_{\{\xi_{p+1}^m(\infty) = 0\}} - 1_{\{\xi_{p+1}^m [mt] = 0\}}) \right) / \mathbb{P}(\xi_{p+1}^m(\infty) = 0).$$

The convergence of  $\frac{1}{m} \Pi_+^m [mt]$  towards  $A(t)$  when  $m$  goes to infinity, and Lemma 24 imply that

$$\lim_{m \rightarrow \infty} S_1(u, m, p) = \mathbb{E}(\psi_p(u, A(t)))$$

where for  $\lambda \in C_{\text{aff}}$ ,

$$\psi_p(u, \lambda) = \frac{e^{iu \langle \lambda, \alpha_1^\vee \rangle}}{2\varphi_{1/2}(\lambda)} \mathbb{E} \left( e^{-iu \langle L_p, \alpha_1^\vee \rangle} 1_{\xi_{\cdot, p} \in \Gamma(\lambda)} \right).$$

Lemmas 25 and Propositions 27 and 28 imply that

$$\lim_{p \rightarrow \infty} \mathbb{E}(\psi_p(u, A(t))) = \mathbb{E} \left( e^{iu \langle B(t), \alpha_1^\vee \rangle} \right)$$

As  $\{\xi_{p+1}^m(\infty) = 0\} \subset \{\xi_{p+1}^m [mt] = 0\}$  one has,

$$|S_2(m, p)| \leq \frac{\mathbb{P}(\xi_{p+1}^m [mt] = 0)}{\mathbb{P}(\xi_{p+1}^m(\infty) = 0)} - 1 = S_1(0, m, p) - 1$$

which implies that  $\lim_{p \rightarrow \infty} \lim_{m \rightarrow \infty} S_2(m, p) = 0$  and achieves the proof of the convergence in law of  $I_{\alpha_0}^{\xi_{0,p}^m(\infty)} \dots I_{\alpha_p}^{\xi_{p,p}^m(\infty)} A(t)$  towards  $B(t)$  when  $p$  goes to infinity.

Let now  $t_0, \dots, t_n$  be a sequence of ordered real numbers such that  $0 = t_0 < t_1 < \dots < t_n$ , and  $u_1, \dots, u_n \in \mathbb{R}$ . For  $m, p \geq 1$ , the Fourier transform

$$(26) \quad \mathbb{E} \left( e^{i \sum_{k=1}^n u_k \langle I_{\alpha_0}^{\xi_{0,p}^m(\infty)} \dots I_{\alpha_p}^{\xi_{p,p}^m(\infty)} A(t_k), \alpha_1^\vee \rangle - \langle I_{\alpha_0}^{\xi_{0,p}^m(\infty)} \dots I_{\alpha_p}^{\xi_{p,p}^m(\infty)} A(t_{k-1}), \alpha_1^\vee \rangle} \right)$$

is equal to

$$\lim_{m \rightarrow \infty} \mathbb{E} \left( e^{i \sum_{k=1}^n \frac{u_k}{m} \langle I_{\alpha_0}^{\xi_{0,p}^m(\infty)} \dots I_{\alpha_p}^{\xi_{p,p}^m(\infty)} \Pi_+ [mt_k] - I_{\alpha_0}^{\xi_{0,p}^m(\infty)} \dots I_{\alpha_p}^{\xi_{p,p}^m(\infty)} \Pi_+ [mt_{k-1}], \alpha_1^\vee \rangle} \right).$$

We obtain as previously, introducing this time the event  $\{\xi_{p+1}^m [mt_1] = 0\}$  and using the independence of the increments, that the Fourier transform (26) converges towards

$$\mathbb{E} \left( e^{i \sum_{k=1}^n u_k \langle B(t_k) - B(t_{k-1}), \alpha_1^\vee \rangle} \right),$$

when  $p$  goes to infinity, which achieves the proof. □

#### REFERENCES

- [1] J. Bertoin. *An extension of Pitman's theorem for spectrally positive Lévy processes*. Ann. Probab., 20(3), (1992),1464–1483.
- [2] Ph. Biane, *Marches de Bernoulli quantiques*, Séminaire de probabilités (Strasbourg), tome 24 (1990), 329–344.
- [3] Ph. Biane, *Le théorème de Pitman, le groupe quantique  $SU_q(2)$ , et une question de P. A. Meyer*, Séminaire de Probabilités XXXIX, Lecture Notes in Math., 1874, Springer-Verlag (2006), 61–75.
- [4] Ph. Biane, Ph. Bougerol and N. O'Connell, *Littellmann paths and Brownian paths*, Duke Math. J. 130 (2005), 127–167.
- [5] Ph. Biane, Ph. Bougerol and N. O'Connell, *Continuous crystal and Duistermaat–Heckman measure for Coxeter groups*, Adv. Maths. 221 (2009), 1522–1583.
- [6] Ph. Bougerol and M. Defosseux, *Pitman transforms and Brownian motion in the interval viewed as an affine alcove*, arXiv:1808.09182v4
- [7] Ph. Bougerol and Th. Jeulin, *Paths in Weyl chambers and random matrices* Probability Theory and Related Fields 124(4), ( 2002), 517–543
- [8] R. Chhaibi, *Littellmann path model for geometric crystals, Whittaker functions on Lie groups and Brownian motion*, (2013, arXiv:1302.0902
- [9] M. Defosseux, *Affine Lie algebras and conditioned space-time Brownian motions in affine Weyl chambers*, Probab. Theory Relat. Fields 165 (2015), 1–17.
- [10] M. Defosseux, *Kirillov–Frenkel character formula for loop groups, radial part and Brownian sheet.*, Ann. of Probab. 47 (2019), 1036–1055 .
- [11] M. Defosseux, *Brownian paths in an alcove and the Littellmann path model*, <https://arxiv.org/pdf/2103.15656.pdf>
- [12] J. Hong and S.-J. Kang, *Introduction to Quantum groups and Crystal Bases*, American mathematical society, 2002.
- [13] V.G. Kac, *Infinite dimensional Lie algebras*, Third edition, Cambridge University Press, 1990.
- [14] M. Kashiwara, *On Crystal Bases*, Representations of groups (Banff, AB, 1994), CMS Conf. Proc., Amer. Math. Soc. 16 (1995) 155–191.
- [15] M. Kashiwara. The crystal base and Littellmann's refined Demazure character formula. Duke Math. J. 71 (1993) 839–858.
- [16] C. Lecouvey, E. Lesigne, M. Peigné, *Conditioned random walks from Kac–Moody root systems*, Trans. Am. Math. Soc. 368 (2016), 3177–3210
- [17] P. Littellmann, *Paths and root operators in representation theory*, Annals of Mathematics 142 (1995) 499–525.
- [18] P. Littellmann. Cones, crystals, and patterns. Transform. Groups 3 (1998) 145–179.
- [19] H. Matsumoto and M. Yor. *An analogue of Pitman's  $2M - X$  theorem for exponential Wiener functionals. II. The role of the generalized inverse Gaussian laws*. Nagoya Math. J. 162 65–86, 2001
- [20] T. Nakashima and A. Zelevinsky. Polyhedral realizations of crystal bases for quantized Kac–Moody algebras. Adv. Math. 131 (1997) 253–278.
- [21] N. O'Connell and M. Yor *A representation for non-colliding random walks*. Electronic Communications in Probability, (2002) Volume: 7, page 1–12
- [22] J.W. Pitman. One-dimensional Brownian motion and the three-dimensional Bessel process. Adv. Appl. Probab. 7 (1975) 511–526.
- [23] D. Revuz and M. Yor. *Continuous martingales and Brownian motion*, Second edition Springer–Verlag (1994).
- [24] B. Roynette, P. P. Vallois, M. Yor. Some extensions of Pitman's and Ray-Knight's theorems for penalized Brownian motions and their local times, IV. Studia Scientiarum Mathematicarum Hungarica, Akadémiai Kiadó, 2007, 44 (4), pp.469–516

- [25] N. J. Wildberger, *Finite commutative hypergroups and applications from group theory to conformal field theory, Applications of Hypergroups and Related Measure Algebras*, *Contemp. Math.*, 183 (1995), 413–434.

M. D. : UNIVERSITÉ PARIS CITÉ, CNRS, MAP5, F-75006 PARIS, FRANCE  
*Email address:* `manon.defosseux@parisdescartes.fr`

C. H. : UNIVERSITÉ PARIS CITÉ, CNRS, MAP5, F-75006 PARIS, FRANCE AND UNIVERSITÉ PARIS-EST, CNRS, INSTITUT GASPARD MONGE, CHAMPS-SUR-MARNE, FRANCE  
*Email address:* `charlie.herent@ens-rennes.fr`