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Masaki Kashiwara, Pierre Schapira. A finiteness theorem for holonomic DQ-modules on Poisson manifolds. *Tunisian Journal of Mathematics*, 2021, 3 (3), pp.571-588. 10.2140/tunis.2021.3.571 . hal-03839995

HAL Id: hal-03839995

<https://hal.science/hal-03839995>

Submitted on 22 Feb 2024

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A finiteness theorem for holonomic DQ-modules on Poisson manifolds

Masaki Kashiwara and Pierre Schapira

March 6, 2020

Abstract

On a complex symplectic manifold, we prove a finiteness result for the global sections of solutions of holonomic DQ-modules in two cases: (a) by assuming that there exists a Poisson compactification (b) in the algebraic case. This extends our previous result of [KS12] in which the symplectic manifold was compact. The main tool is a finiteness theorem for \mathbb{R} -constructible sheaves on a real analytic manifold in a non proper situation.

Contents

1	Introduction and statement of the results	2
2	Finiteness results for constructible sheaves	3
3	Reminders on DQ-modules, after [KS12]	5
3.1	Cohomologically complete modules	5
3.2	Microsupport and constructible sheaves	6
3.3	DQ-modules	7
4	DQ-modules along Λ	8
4.1	A variation on a theorem of [Kas03]	8
4.2	The algebroid $\mathcal{A}_{\Lambda/X}$	9
4.3	Reminders on holonomic DQmodules	13

Key words: deformation quantization, holonomic modules, microlocal sheaf theory, constructible sheaves

MSC: 53D55, 35A27, 19L10, 32C38

The research of M.K was supported by Grant-in-Aid for Scientific Research (X) 15H03608, Japan Society for the Promotion of Science.

The research of P.S was supported by the ANR-15-CE40-0007 “MICROLOCAL”.

5	Proof of the main theorems and an example	14
5.1	Proof of Theorem 1.2	14
5.2	Proof of Theorem 1.3	14
5.3	An example	16

1 Introduction and statement of the results

Consider a complex Poisson manifold X of complex dimension d_X endowed with a DQ-algebroid \mathcal{A}_X . Recall that \mathcal{A}_X is a $\mathbb{C}[[\hbar]]$ -algebroid locally isomorphic to a star algebra $(\mathcal{O}_X[[\hbar]], \star)$ to which the Poisson structure is associated. Denote by $\mathcal{A}_X^{\text{loc}}$ the localization of \mathcal{A}_X with respect to \hbar , a $\mathbb{C}((\hbar))$ -algebroid. For short, we set

$$\mathbb{C}^{\hbar} := \mathbb{C}[[\hbar]], \quad \mathbb{C}^{\hbar, \text{loc}} := \mathbb{C}((\hbar)).$$

Hence $\mathcal{A}_X^{\text{loc}} \simeq \mathbb{C}^{\hbar, \text{loc}} \otimes_{\mathbb{C}^{\hbar}} \mathcal{A}_X$. The algebroids \mathcal{A}_X and $\mathcal{A}_X^{\text{loc}}$ are right and left Noetherian (in particular coherent) and if \mathcal{M} is a (say left) coherent $\mathcal{A}_X^{\text{loc}}$ -module, then its support is a closed complex analytic subvariety of X and it follows from Gabber's theorem that it is co-isotropic. In the extreme case where X is symplectic and the support is Lagrangian, one says that \mathcal{M} is holonomic.

Recall the following definitions (see [KS12, Def. 2.3.14, 2.3.16 and 2.7.2]).

- (a) A coherent \mathcal{A}_X -submodule \mathcal{M}_0 of a coherent $\mathcal{A}_X^{\text{loc}}$ -module \mathcal{M} is called an \mathcal{A}_X -lattice of \mathcal{M} if \mathcal{M}_0 generates \mathcal{M} .
- (b) A coherent $\mathcal{A}_X^{\text{loc}}$ -module \mathcal{M} is good if, for any relatively compact open subset U of X , there exists an $(\mathcal{A}_X|_U)$ -lattice of $\mathcal{M}|_U$.
- (c) One denotes by $\text{D}_{\text{gd}}^{\text{b}}(\mathcal{A}_X^{\text{loc}})$ the full subcategory of $\text{D}_{\text{coh}}^{\text{b}}(\mathcal{A}_X^{\text{loc}})$ consisting of objects with good cohomology.
- (d) In the algebraic case (see below) a coherent $\mathcal{A}_X^{\text{loc}}$ -module \mathcal{M} is called algebraically good if there exists an \mathcal{A}_X -lattice of \mathcal{M} . One still denotes by $\text{D}_{\text{gd}}^{\text{b}}(\mathcal{A}_X^{\text{loc}})$ the full subcategory of $\text{D}_{\text{coh}}^{\text{b}}(\mathcal{A}_X^{\text{loc}})$ consisting of objects with algebraically good cohomology.

Let $Y \subset X$. We shall consider the hypothesis

- (1.1) Y is open, relatively compact, subanalytic in X and the Poisson structure on X is symplectic on Y .

Example 1.1. Denote by X_{ns} the closed complex subvariety of X consisting of points where the Poisson bracket is not symplectic and set $Y = X \setminus X_{\text{ns}}$. Hence Y is an open subanalytic subset of X and is symplectic. If X is compact, then Y satisfies hypothesis (1.1).

In this paper we shall prove the following theorem which extends [KS12, Th. 7.2.3] in which X was symplectic, that is, $Y = X$.

Theorem 1.2. *Assume that Y satisfies hypothesis (1.1). Let \mathcal{M} and \mathcal{L} belong to $D_{\text{gd}}^{\text{b}}(\mathcal{A}_X^{\text{loc}})$ and assume that both $\mathcal{M}|_Y$ and $\mathcal{L}|_Y$ are holonomic. Then the two complexes $\text{RHom}_{\mathcal{A}_Y^{\text{loc}}}(\mathcal{M}|_Y, \mathcal{L}|_Y)$ and $\text{R}\Gamma_c(Y; \text{R}\mathcal{H}om_{\mathcal{A}_Y^{\text{loc}}}(\mathcal{L}|_Y, \mathcal{M}|_Y)) [d_X]$ have finite dimensional cohomology over $\mathbb{C}^{h, \text{loc}}$ and are dual to each other.*

We shall also obtain a similar conclusion under rather different hypotheses, namely that $X = Y$ is symplectic and all data are algebraic (see [KS12, § 2.7]). Let X be a smooth algebraic variety and let \mathcal{A}_X be a DQ-algebroid on X . We denote by X_{an} the associated complex analytic manifold and $\mathcal{A}_{X_{\text{an}}}$ the associated DQ-algebroid on X_{an} (see Lemma 5.1). For a coherent $\mathcal{A}_X^{\text{loc}}$ -module \mathcal{M} we denote by \mathcal{M}_{an} its image by the natural functor $D_{\text{coh}}^{\text{b}}(\mathcal{A}_X^{\text{loc}}) \rightarrow D_{\text{coh}}^{\text{b}}(\mathcal{A}_{X_{\text{an}}}^{\text{loc}})$.

Theorem 1.3. *Let X be a quasi-compact separated smooth symplectic algebraic variety over \mathbb{C} endowed with the Zariski topology. Let \mathcal{M} and \mathcal{L} belong to $D_{\text{gd}}^{\text{b}}(\mathcal{A}_X^{\text{loc}})$. Then the two complexes $\text{RHom}_{\mathcal{A}_{X_{\text{an}}}^{\text{loc}}}(\mathcal{M}_{\text{an}}, \mathcal{L}_{\text{an}})$ and $\text{R}\Gamma_c(X_{\text{an}}; \text{R}\mathcal{H}om_{\mathcal{A}_{X_{\text{an}}}^{\text{loc}}}(\mathcal{L}_{\text{an}}, \mathcal{M}_{\text{an}})) [d_X]$ have finite dimensional cohomology over $\mathbb{C}^{h, \text{loc}}$ and are dual to each other.*

The main tool in the proof of both theorems is Theorem 2.2 below which gives a finiteness criterion for \mathbb{R} -constructible sheaves on a real analytic manifold in a non proper situation.

This Note is motivated by the paper [GJS19] of Sam Gunningham, David Jordan and Pavel Safronov on Skein algebras, whose main theorem is based over such a finiteness result (see loc. cit. § 3). The proof of these authors uses a kind of Nakayama theorem in the case where \mathcal{M} and \mathcal{L} are simple modules over smooth Lagrangian varieties.

2 Finiteness results for constructible sheaves

In this paper, \mathbf{k} is a Noetherian commutative ring of finite global homological dimension.

We denote by $D_f^{\text{b}}(\mathbf{k})$ the full triangulated subcategory of $D^{\text{b}}(\mathbf{k})$ consisting of objects with finitely generated cohomology. We denote by D the duality functor $\text{RHom}(\cdot, \mathbf{k})$ and we say that two objects A and B of $D_f^{\text{b}}(\mathbf{k})$ are dual to each other if $DA \simeq B$, which is equivalent to $DB \simeq A$.

For a sheaf of rings \mathcal{R} , one denotes by $D(\mathcal{R})$ the derived category of left \mathcal{R} -modules. We shall also encounter the full triangulated subcategory $D^+(\mathcal{R})$ or $D^{\text{b}}(\mathcal{R})$ of complexes whose cohomology is bounded from below or is bounded.

For a real analytic manifold M , one denotes by $D^{\text{b}}(\mathbf{k}_M)$ the bounded derived category of sheaves of \mathbf{k} -modules on M . We shall use the six Grothendieck operations. In particular, we denote by ω_M the dualizing complex. We also use the notations for $F \in D^{\text{b}}(\mathbf{k}_M)$

$$D'_M F := \text{R}\mathcal{H}om(F, \mathbf{k}_M), \quad D_M F := \text{R}\mathcal{H}om(F, \omega_M).$$

Recall that an object F of $D^{\text{b}}(\mathbf{k}_M)$ is weakly \mathbb{R} -constructible if condition (i) below is satisfied. If moreover condition (ii) is satisfied, then one says that F is \mathbb{R} -constructible.

- (i) there exists a subanalytic stratification $M = \bigsqcup_{a \in A} M_a$ such that $H^j(F)|_{M_a}$ is locally constant for all $j \in \mathbb{Z}$ and all $a \in A$

(ii) $H^j(F)_x$ is finitely generated for all $x \in M$ and all $j \in \mathbb{Z}$.

One denotes by $D_{\mathbb{R}c}^b(\mathbf{k}_M)$ the full subcategory of $D^b(\mathbf{k}_M)$ consisting of \mathbb{R} -constructible objects.

If X is a complex analytic manifold, one defines similarly the notions of (weakly) \mathbb{C} -constructible sheaf, replacing “subanalytic” with “complex analytic” and one denotes by $D_{\mathbb{C}c}^b(\mathbf{k}_X)$ the full subcategory of $D^b(\mathbf{k}_X)$ consisting of \mathbb{C} -constructible objects.

We shall use the following classical result (see [KS90, Prop. 8.4.8 and Exe. VIII.3]).

Proposition 2.1. *Let $F \in D_{\mathbb{R}c}^b(\mathbf{k}_M)$ and assume that F has compact support. Then both objects $R\Gamma(M; F)$ and $R\Gamma(M; D_M F)$ belong to $D_f^b(\mathbf{k})$ and are dual to each other.*

For $F \in D^b(\mathbf{k}_M)$, one denotes by $\text{SS}(F)$ its microsupport [KS90, Def. 5.1.2], a closed \mathbb{R}^+ -conic (*i.e.*, invariant by the \mathbb{R}^+ -action on T^*M) subset of T^*M . Recall that this set is involutive (one also says *co-isotropic*), see *loc. cit.* Def. 6.5.1.

Theorem 2.2. *Let $j: U \hookrightarrow M$ be the embedding of an open subanalytic subset U of M and let $F \in D_{\mathbb{R}c}^b(\mathbf{k}_U)$. Assume that $\text{SS}(F)$ is contained in a closed subanalytic \mathbb{R}^+ -conic Lagrangian subset Λ of T^*U which is subanalytic in T^*M . Then Rj_*F and $j_!F$ belong to $D_{\mathbb{R}c}^b(\mathbf{k}_M)$.*

Proof. (i) Let us treat first $j_!F$. The set Λ is a locally closed subanalytic subset of T^*M and is isotropic. By [KS90, Cor. 8.3.22], there exists a μ -stratification $M = \bigsqcup_{a \in A} M_a$ such that $\Lambda \subset \bigsqcup_{a \in A} T_{M_a}^*M$.

Set $U_a = U \cap M_a$. Then $U = \bigsqcup_{a \in A} U_a$ is a μ -stratification and one can apply *loc. cit.* Prop. 8.4.1. Hence, for each $a \in A$, $F|_{U_a}$ is locally constant of finite rank. Hence $(j_!F)|_{U_a}$ as well as $(j_!F)_{M \setminus U} \simeq 0$ is locally constant of finite rank. Hence $j_!F \in D_{\mathbb{R}c}^b(\mathbf{k}_M)$. (ii) Set $G = j_!F$. Then $G \in D_{\mathbb{R}c}^b(\mathbf{k}_M)$ by (i) and so does $Rj_*F \simeq R\mathcal{H}om(\mathbf{k}_U, G)$ (apply [KS90, Prop. 8.4.10]). \square

Remark 2.3. One has $\text{SS}(D_M F) = \text{SS}(F)^a$ where $(\cdot)^a$ is the antipodal map. Hence $D_M F$ satisfies the same hypotheses as F .

Corollary 2.4. *In the preceding situation, assume moreover that U is relatively compact in M . Then $R\Gamma(U; F)$ and $R\Gamma_c(U; D_U F)$ belong to $D_f^b(\mathbf{k})$ and are dual to each other.*

Proof. One has $R\Gamma(U; F) \simeq R\Gamma(M; Rj_*F)$ and $R\Gamma_c(U; D_U F) \simeq R\Gamma(M; D_M Rj_*F)$. Since Rj_*F is \mathbb{R} -constructible and has compact support, the result follows from Proposition 2.1. \square

For a complex analytic manifold X (that we identify with the real underlying manifold if necessary), one denotes by $D_{\mathbb{C}c}^b(\mathbf{k}_X)$ the full triangulated subcategory of $D^b(\mathbf{k}_X)$ consisting of \mathbb{C} -constructible sheaves.

In this paper, a smooth algebraic variety X means a quasi-compact smooth algebraic variety over \mathbb{C} endowed with the Zariski topology. We denote by X_{an} the complex analytic manifold underlying X . If X is smooth algebraic variety, we keep the notation $D_{\mathbb{C}c}^b(\mathbf{k}_X)$ to denote the category of algebraically constructible sheaves, that is, object of $D_{\mathbb{C}c}^b(\mathbf{k}_{X_{\text{an}}})$ locally constant on an algebraic stratifications. Hence, for an algebraic variety X , one shall not confuse $D_{\mathbb{C}c}^b(\mathbf{k}_X)$ and $D_{\mathbb{C}c}^b(\mathbf{k}_{X_{\text{an}}})$, although $D_{\mathbb{C}c}^b(\mathbf{k}_X)$ is a full subcategory of $D_{\mathbb{C}c}^b(\mathbf{k}_{X_{\text{an}}})$.

Corollary 2.5. *Let X be a smooth algebraic variety and let $F \in D_{\text{cc}}^b(\mathbf{k}_X)$. Then $\text{R}\Gamma(X_{\text{an}}; F)$ and $\text{R}\Gamma_c(X_{\text{an}}; D_{X_{\text{an}}}F)$ have finite dimensional cohomology over \mathbf{k} and are dual to each other.*

Proof. Let Z be a smooth algebraic compactification of X with X open in Z . By the hypothesis, Λ is a closed algebraic subvariety of T^*X . Hence, its closure in T^*Z is a closed algebraic subvariety of T^*Z . Therefore Λ is subanalytic in T^*Z_{an} .

Then the result follows from Corollary 2.4 with $M = Z_{\text{an}}$ and $U = X_{\text{an}}$. \square

3 Reminders on DQ-modules, after [KS12]

3.1 Cohomologically complete modules

In this subsection,

(3.1) X denotes a topological space and \mathcal{R} is a sheaf of $\mathbb{Z}[\hbar]$ -algebras on X with no \hbar -torsion.

Let \mathcal{M} be an \mathcal{R} -module. (Hence, a $\mathbb{Z}_X[\hbar]$ -module.) One sets

$$\begin{aligned}\mathcal{R}^{\text{loc}} &:= \mathbb{Z}_X[\hbar, \hbar^{-1}] \otimes_{\mathbb{Z}_X[\hbar]} \mathcal{R}, \\ \mathcal{M}^{\text{loc}} &:= \mathcal{R}^{\text{loc}} \otimes_{\mathcal{R}} \mathcal{M} \simeq \mathbb{Z}_X[\hbar, \hbar^{-1}] \otimes_{\mathbb{Z}_X[\hbar]} \mathcal{M}, \\ \text{gr}_{\hbar}(\mathcal{R}) &:= \mathcal{R}/\hbar\mathcal{R}, \\ \text{gr}_{\hbar}(\mathcal{M}) &:= \text{gr}_{\hbar}(\mathcal{R}) \otimes_{\mathcal{R}}^{\text{L}} \mathcal{M} \simeq \mathbb{Z}_X \otimes_{\mathbb{Z}_X[\hbar]}^{\text{L}} \mathcal{M}.\end{aligned}$$

Definition 3.1 ([KS12, Def. 1.5.5]). One says that an object \mathcal{M} of $D(\mathcal{R})$ is cohomologically complete if it belongs to $D(\mathcal{R}^{\text{loc}})^{\perp r}$, that is, $\text{Hom}_{D(\mathcal{R})}(\mathcal{N}, \mathcal{M}) \simeq 0$ for any $\mathcal{N} \in D(\mathcal{R}^{\text{loc}})$.

Proposition 3.2 ([KS12, Prop. 1.5.6]). *Let $\mathcal{M} \in D(\mathcal{R})$. Then the conditions below are equivalent.*

- (a) \mathcal{M} is cohomologically complete,
- (b) $\text{R}\mathcal{H}om_{\mathcal{R}}(\mathcal{R}^{\text{loc}}, \mathcal{M}) \simeq 0$,
- (c) $\varinjlim_{U \ni x} \text{Ext}_{\mathbb{Z}[\hbar]}^j(\mathbb{Z}[\hbar, \hbar^{-1}], H^i(U; \mathcal{M})) \simeq 0$ for any $x \in X$, $j = 0, 1$ and any $i \in \mathbb{Z}$. Here, U ranges over an open neighborhood system of x .

Denote by $D_{\text{cc}}(\mathcal{R})$ the full subcategory of $D(\mathcal{R})$ consisting of cohomologically complete modules. Then clearly $D_{\text{cc}}(\mathcal{R})$ is triangulated.

Proposition 3.3 ([KS12, Prop. 1.5.10, Cor. 1.5.9]). *Let $\mathcal{M} \in D_{\text{cc}}(\mathcal{R})$. Then*

- (a) $\text{R}\mathcal{H}om_{\mathcal{R}}(\mathcal{N}, \mathcal{M}) \in D(\mathbb{Z}_X[\hbar])$ is cohomologically complete for any $\mathcal{N} \in D(\mathcal{R})$.
- (b) If $\text{gr}_{\hbar}(\mathcal{M}) \simeq 0$, then $\mathcal{M} \simeq 0$.

Proposition 3.4 ([KS12, Prop. 1.5.12]). *Let $f: X \rightarrow Y$ be a continuous map and let $\mathcal{M} \in D(\mathbb{Z}_X[\hbar])$. If \mathcal{M} is cohomologically complete, then so is $Rf_*\mathcal{M}$.*

Proposition 3.5. *Let $\mathcal{M} \in D(\mathcal{R})$ be a cohomologically complete object and $a \in \mathbb{Z}$. If $H^i(\mathrm{gr}_{\hbar}(\mathcal{M})) = 0$ for any $i \geq a$, then $H^i(\mathcal{M}) = 0$ for any $i > a$.*

Proof. The proof is exactly the same as that of [KS12, Prop. 1.5.8] when replacing $i > a$ with $i < a$. \square

3.2 Microsupport and constructible sheaves

Let M be a *real analytic* manifold and let \mathbf{k} be a Noetherian commutative ring of finite global homological dimension.

We shall need the next result which does not appear in [KS12].

Proposition 3.6. *Let $F \in D^b(\mathbb{Z}_M[\hbar])$. Then $\mathrm{SS}(F^{\mathrm{loc}}) \subset \mathrm{SS}(F)$.*

Proof. By using one of the equivalent definitions of the micro-support given in [KS90, Prop. 5.11], it is enough to check that for K compact, $\mathrm{R}\Gamma(K; F)^{\mathrm{loc}} \simeq \mathrm{R}\Gamma(K; F^{\mathrm{loc}})$ which follows from loc. cit. Prop. 2.6.6 and the fact that $\mathbb{Z}[\hbar, \hbar^{-1}]$ is flat over $\mathbb{Z}[\hbar]$. \square

Proposition 3.7 ([KS12, Prop. 7.1.6]). *Let $F \in D^b(\mathbb{Z}_M[\hbar])$ and assume that F is cohomologically complete. Then*

$$(3.2) \quad \mathrm{SS}(F) = \mathrm{SS}(\mathrm{gr}_{\hbar}(F)).$$

Proof. Let us recall the proof of loc. cit. The inclusion

$$\mathrm{SS}(\mathrm{gr}_{\hbar}(F)) \subset \mathrm{SS}(F)$$

follows from the distinguished triangle $F \xrightarrow{\hbar} F \rightarrow \mathrm{gr}_{\hbar}(F) \xrightarrow{+1}$. Let us prove the converse inclusion.

Using the definition of the microsupport, it is enough to prove that given two open subsets $U \subset V$ of M , $\mathrm{R}\Gamma(V; F) \rightarrow \mathrm{R}\Gamma(U; F)$ is an isomorphism as soon as $\mathrm{R}\Gamma(V; \mathrm{gr}_{\hbar}(F)) \rightarrow \mathrm{R}\Gamma(U; \mathrm{gr}_{\hbar}(F))$ is an isomorphism. Consider a distinguished triangle $\mathrm{R}\Gamma(V; F) \rightarrow \mathrm{R}\Gamma(U; F) \rightarrow G \xrightarrow{+1}$. Then we get a distinguished triangle $\mathrm{R}\Gamma(V; \mathrm{gr}_{\hbar}(F)) \rightarrow \mathrm{R}\Gamma(U; \mathrm{gr}_{\hbar}(F)) \rightarrow \mathrm{gr}_{\hbar}(G) \xrightarrow{+1}$. Therefore, $\mathrm{gr}_{\hbar}(G) \simeq 0$. On the other hand, G is cohomologically complete, thanks to Proposition 3.4 (applied to $F|_U$ and $F|_V$) and then $G \simeq 0$ by Proposition 3.3 (b). \square

Proposition 3.8 ([KS12, Prop. 7.1.7]). *Let $F \in D_{\mathbb{R}\mathbf{c}}^b(\mathbb{C}^h)$. Then F is cohomologically complete.*

Proof. Let us recall the proof of loc. cit. One has

$$\begin{aligned} \underset{U \ni x}{\text{“lim”}} \mathrm{Ext}_{\mathbb{Z}[\hbar]}^j(\mathbb{Z}[\hbar, \hbar^{-1}], H^i(U; F)) &\simeq \mathrm{Ext}_{\mathbb{Z}[\hbar]}^j(\mathbb{Z}[\hbar, \hbar^{-1}], \underset{U \ni x}{\text{“lim”}} H^i(U; F)) \\ &\simeq \mathrm{Ext}_{\mathbb{Z}[\hbar]}^j(\mathbb{Z}[\hbar, \hbar^{-1}], F_x) \simeq 0 \end{aligned}$$

where the last isomorphism follows from the fact that F_x is cohomologically complete.

Hence, hypothesis (c) of Proposition 3.2 is satisfied. \square

3.3 DQ-modules

In this subsection, X will be a complex manifold (not necessarily symplectic) of complex dimension d_X .

Set $\mathcal{O}_X^{\hbar} := \mathcal{O}_X[[\hbar]] = \varprojlim_n \mathcal{O}_X \otimes_{\mathbb{C}} (\mathbb{C}^{\hbar}/\hbar^n \mathbb{C}^{\hbar})$. An associative multiplication law \star on \mathcal{O}_X^{\hbar} is a star-product if it is \mathbb{C}^{\hbar} -bilinear and satisfies

$$(3.3) \quad f \star g = \sum_{i \geq 0} P_i(f, g) \hbar^i \quad \text{for } f, g \in \mathcal{O}_X,$$

where the P_i 's are bi-differential operators, $P_0(f, g) = fg$ and $P_i(f, 1) = P_i(1, f) = 0$ for $f \in \mathcal{O}_X$ and $i > 0$.

We call $(\mathcal{O}_X[[\hbar]], \star)$ a *star-algebra*. A \star -product defines a Poisson structure on (X, \mathcal{O}_X) by the formula

$$(3.4) \quad \{f, g\} = P_1(f, g) - P_1(g, f) \equiv \hbar^{-1}(f \star g - g \star f) \pmod{\hbar \mathcal{O}_X[[\hbar]]}.$$

Definition 3.9. A DQ-algebroid \mathcal{A} on X is a \mathbb{C}^{\hbar} -algebroid locally isomorphic to a star-algebra as a \mathbb{C}^{\hbar} -algebroid.

Remark 3.10. The data of a DQ-algebroid \mathcal{A}_X on X endows X with a structure of a complex Poisson manifold and one says that \mathcal{A}_X is a quantization of the Poisson manifold. Kontsevich's famous theorem [Kon01, Kon03] (see also [Kas96] for the case of contact manifolds) asserts that any complex Poisson manifold may be quantized.

Example 3.11. Assume that M is an open subset of \mathbb{C}^n , $X = T^*M$ and denote by $(x; u)$ the symplectic coordinates on X . In this case there is a canonical \star -algebra \mathcal{A}_X that is usually denoted by $\widehat{\mathcal{W}}_X(0)$, its localization with respect to \hbar being denoted by $\widehat{\mathcal{W}}_X$.

Let $f, g \in \mathcal{O}_X[[\hbar]]$. Then the DQ-algebra $\widehat{\mathcal{W}}_X(0)$ is the star algebra $(\mathcal{O}_X[[\hbar]], \star)$ where:

$$(3.5) \quad f \star g = \sum_{\alpha \in \mathbb{N}^n} \frac{\hbar^{|\alpha|}}{\alpha!} (\partial_u^\alpha f)(\partial_x^\alpha g).$$

This product is similar to the product of the total symbols of differential operators on X and indeed, the morphism of \mathbb{C} -algebras $\pi_M^{-1} \mathcal{D}_M \rightarrow \widehat{\mathcal{W}}_X$ is given by

$$f(x) \mapsto f(x), \quad \partial_{x_i} \mapsto \hbar^{-1} u_i,$$

where, as usual, \mathcal{D}_M denotes the ring of finite order holomorphic differential operators and $\pi_M: T^*M \rightarrow M$ is the projection.

For a DQ-algebroid \mathcal{A}_X , there is locally an isomorphism of \mathbb{C} -algebroids $\mathcal{A}_X/\hbar \mathcal{A}_X \xrightarrow{\sim} \mathcal{O}_X$. Moreover there exists a unique isomorphism of \mathbb{C} -algebras

$$(3.6) \quad \mathcal{E}nd(\text{id}_{\text{gr}_{\hbar} \mathcal{A}_X}) \simeq \mathcal{O}_X.$$

Therefore, there is a well-defined functor

$$(3.7) \quad \bullet \otimes_{\mathcal{O}_X}^{\text{L}} \bullet : \text{D}^b(\mathcal{O}_X) \times \text{D}^b(\text{gr}_{\hbar} \mathcal{A}_X) \rightarrow \text{D}^b(\text{gr}_{\hbar} \mathcal{A}_X).$$

Theorem 3.12 ([KS12, Th. 1.2.5]). *For a DQ-algebroid \mathcal{A}_X , both \mathcal{A}_X and $\mathcal{A}_X^{\text{loc}}$ are right and left Noetherian (in particular, coherent).*

One defines the functors

$$\begin{aligned} \text{gr}_{\hbar} &: D^b(\mathcal{A}_X) \rightarrow D^b(\text{gr}_{\hbar}\mathcal{A}_X), & \mathcal{M} &\mapsto \mathbb{C}_X \otimes_{\mathbb{C}_X^{\hbar}}^L \mathcal{M}, \\ (\cdot)^{\text{loc}} &: D^b(\mathcal{A}_X) \rightarrow D^b(\mathcal{A}_X^{\text{loc}}), & \mathcal{M} &\mapsto \mathbb{C}_X^{\hbar, \text{loc}} \otimes_{\mathbb{C}_X^{\hbar}} \mathcal{M}, \\ \text{for} &: D^b(\text{gr}_{\hbar}(\mathcal{A}_X)) \rightarrow D^b(\mathcal{A}_X) \text{ associated with } \sigma_0: \mathcal{A}_X \rightarrow \text{gr}_{\hbar}(\mathcal{A}_X). \end{aligned}$$

The functor $(\cdot)^{\text{loc}}$ is exact on $\text{Mod}(\mathcal{A}_X)$. The category $\text{Mod}(\text{gr}_{\hbar}(\mathcal{A}_X))$ is equivalent to the full subcategory of $\text{Mod}(\mathcal{A}_X)$ consisting of objects M such that $\hbar: M \rightarrow M$ vanishes.

Theorem 3.13 ([KS12, Th. 1.6.1 and 1.6.4]). *Let $\mathcal{M} \in D^+(\mathcal{A}_X)$. Then the two conditions below are equivalent:*

- (a) \mathcal{M} is cohomologically complete and $\text{gr}_{\hbar}(\mathcal{M}) \in D_{\text{coh}}^+(\text{gr}_{\hbar}\mathcal{A}_X)$,
- (b) $\mathcal{M} \in D_{\text{coh}}^+(\mathcal{A}_X)$.

The next result follows from Gabber's theorem [Gab81].

Proposition 3.14 ([KS12, Prop. 2.3.18]). *Let $\mathcal{M} \in D_{\text{coh}}^b(\mathcal{A}_X^{\text{loc}})$. Then $\text{supp}(\mathcal{M})$ (the support of \mathcal{M}) is a closed complex analytic subset of X , involutive (i.e., co-isotropic) for the Poisson bracket on X .*

Remark 3.15. One shall be aware that the support of a coherent \mathcal{A}_X -module is not involutive in general. Indeed, any coherent $\text{gr}_{\hbar}\mathcal{A}_X$ -module may be regarded as an \mathcal{A}_X -module. Hence any closed analytic subset can be the support of a coherent \mathcal{A}_X -module.

4 DQ-modules along Λ

4.1 A variation on a theorem of [Kas03]

In order to prove Lemma 4.6 below, we need a slight modification of a result of [Kas03].

Let \mathcal{R} be a ring on a topological space X , and let $\{F_n(\mathcal{R})\}_{n \in \mathbb{Z}}$ be a filtration of \mathcal{R} which satisfies

- (a) $\mathcal{R} = \bigcup_{n \in \mathbb{Z}} F_n(\mathcal{R})$,
- (b) $1 \in F_0(\mathcal{R})$,
- (c) $F_m(\mathcal{R}) \cdot F_n(\mathcal{R}) \subset F_{m+n}(\mathcal{R})$.

We set

$$\text{gr}_{\geq 0}^F(\mathcal{R}) = \bigoplus_{n \geq 0} \text{gr}_n^F(\mathcal{R}).$$

Proposition 4.1. *Assume that*

- (a) $F_0(\mathcal{R})$ and $\text{gr}_{\geq 0}^F(\mathcal{R})$ are Noetherian rings,
- (b) $\text{gr}_n^F(\mathcal{R})$ is a coherent $F_0(\mathcal{R})$ -module for any $n \geq 0$.

Then \mathcal{R} is Noetherian.

Proof. Define $\tilde{F}_n(\mathcal{R})$ by

$$\tilde{F}_n(\mathcal{R}) = \begin{cases} F_n(\mathcal{R}) & \text{if } n \geq 0, \\ 0 & \text{if } n < 0. \end{cases}$$

We shall apply [Kas03, Theorem A.20] to $\tilde{F}_k(\mathcal{R})$. Hence in order to prove the theorem, it is enough to show

- (4.1) for any positive integer m and an open subset U of X , if an $\mathcal{R}|_U$ -submodule \mathcal{N} of $\mathcal{R}^{\oplus m}|_U$ has the property that $F_k(\mathcal{N}) := \mathcal{N} \cap F_k(\mathcal{R})^{\oplus m}|_U$ is a coherent $F_0(\mathcal{R})|_U$ -module for any $k \geq 0$, then \mathcal{N} is a locally finitely generated $\mathcal{R}|_U$ -module.

Since $\text{gr}_{\geq 0}^F(\mathcal{R})$ is a Noetherian ring, $\text{gr}_{\geq 0}^F(\mathcal{N}) := \bigoplus_{n \geq 0} \text{gr}_n^F(\mathcal{N})$ is a coherent $\text{gr}_{\geq 0}^F(\mathcal{R})$ -module. Hence there exists locally a finitely generated \mathcal{R} -submodule \mathcal{N}' of \mathcal{N} such that $\text{gr}_{\geq 0}^F(\mathcal{N}') = \text{gr}_{\geq 0}^F(\mathcal{N})$. Hence we have $\mathcal{N} = \mathcal{N}' + F_0(\mathcal{N})$. Since $F_0(\mathcal{N})$ is a locally finitely generated $F_0(\mathcal{R})$ -module, \mathcal{N} is locally finitely generated \mathcal{R} -module. \square

4.2 The algebroid $\mathcal{A}_{\Lambda/X}$

From now on, X is a complex manifold endowed with a DQ-algebroid \mathcal{A}_X .

Definition 4.2 ([KS12, Def. 2.3.10]). Let Λ be a smooth submanifold of X and let \mathcal{L} be a coherent \mathcal{A}_X -module supported by Λ . One says that \mathcal{L} is simple along Λ if $\text{gr}_{\hbar}(\mathcal{L})$ is concentrated in degree 0 and $H^0(\text{gr}_{\hbar}(\mathcal{L}))$ is an invertible $\mathcal{O}_{\Lambda} \otimes_{\mathcal{O}_X} \text{gr}_{\hbar}(\mathcal{A}_X)$ -module. (In particular, \mathcal{L} has no \hbar -torsion.)

Let Λ be a smooth submanifold of X and let \mathcal{L} be a coherent \mathcal{A}_X -module simple along Λ . We set for short

$$\begin{aligned} \mathcal{O}_{\Lambda}^{\hbar} &:= \mathcal{O}_{\Lambda}[[\hbar]], & \mathcal{O}_{\Lambda}^{\hbar, \text{loc}} &:= \mathcal{O}_{\Lambda}((\hbar)), \\ \mathcal{D}_{\Lambda}^{\hbar} &:= \mathcal{D}_{\Lambda}[[\hbar]], & \mathcal{D}_{\Lambda}^{\hbar, \text{loc}} &:= \mathcal{D}_{\Lambda}((\hbar)). \end{aligned}$$

One proves that there is a natural isomorphism of algebroids $\mathcal{E}nd_{\mathbb{C}^{\hbar}}(\mathcal{L}) \simeq \mathcal{E}nd_{\mathbb{C}^{\hbar}}(\mathcal{O}_{\Lambda}^{\hbar})$ ([KS12, Lem. 2.1.12]). Then the subalgebroid of $\mathcal{E}nd_{\mathbb{C}^{\hbar}}(\mathcal{L})$ corresponding to the subring $\mathcal{D}_{\Lambda}[[\hbar]]$ of $\mathcal{E}nd_{\mathbb{C}^{\hbar}}(\mathcal{O}_{\Lambda}^{\hbar})$ is well-defined. We denote it by $\mathcal{D}_{\mathcal{L}}$. Then (see [KS12, Lem. 7.1.1]):

- (a) $\mathcal{D}_{\mathcal{L}}$ is isomorphic to $\mathcal{D}_{\Lambda}^{\hbar}$ as a \mathbb{C}^{\hbar} -algebroid and $\text{gr}_{\hbar}(\mathcal{D}_{\mathcal{L}}) \simeq \mathcal{D}_{\Lambda}$.
- (b) The \mathbb{C}^{\hbar} -algebra $\mathcal{D}_{\mathcal{L}}$ is right and left Noetherian.

We denote by $I_\Lambda \subset \mathcal{O}_X$ the defining ideal of Λ . Let \mathcal{I} be the kernel of the composition

$$(4.2) \quad \hbar^{-1}\mathcal{A}_X \xrightarrow{\hbar} \mathcal{A}_X \rightarrow \mathrm{gr}_\hbar \mathcal{A}_X \rightarrow \mathcal{O}_\Lambda \overset{\mathrm{L}}{\otimes}_{\mathcal{O}_X} \mathrm{gr}_\hbar \mathcal{A}_X.$$

Then we have

$$(4.3) \quad \mathcal{I} / \mathcal{A}_X \simeq I_\Lambda \otimes_{\mathcal{O}_X} \mathrm{gr}_\hbar \mathcal{A}_X.$$

Remark 4.3. In [KS12, Ch. 7, § 1] we have used the symbol map $\sigma: \mathcal{A}_X \rightarrow \mathcal{O}_X$. This map is only defined locally, but all results of this chapter are of local nature. If nevertheless, one wants a global construction, then one has to replace the sequence two lines above Definition 7.1.2 of loc. cit. with (4.2).

Definition 4.4 ([KS12, Def. 7.1.2]). One denotes by $\mathcal{A}_{\Lambda/X}$ the \mathbb{C}^\hbar -subalgebroid of $\mathcal{A}_X^{\mathrm{loc}}$ generated by \mathcal{I} .

The ideal $\hbar\mathcal{I}$ is contained in \mathcal{A}_X , hence acts on \mathcal{L} and one sees easily that $\hbar\mathcal{I}$ sends \mathcal{L} to $\hbar\mathcal{L}$. Hence, \mathcal{I} acts on \mathcal{L} and defines a functor $\mathcal{A}_{\Lambda/X} \rightarrow \mathcal{D}_\mathcal{L}$. We thus have the morphisms of algebroids

$$\begin{array}{ccccc} \mathcal{A}_X|_\Lambda & \longrightarrow & \mathcal{A}_{\Lambda/X}|_\Lambda & \longrightarrow & \mathcal{A}_X^{\mathrm{loc}}|_\Lambda \\ & \searrow & \downarrow & & \downarrow \\ & & \mathcal{D}_\mathcal{L} & \longrightarrow & \mathcal{D}_\mathcal{L}^{\mathrm{loc}}. \end{array}$$

In particular, \mathcal{L} is naturally an $\mathcal{A}_{\Lambda/X}$ -module and $\mathrm{gr}_\hbar(\mathcal{D}_\mathcal{L}) \simeq \mathcal{D}_\Lambda$ is a $\mathrm{gr}_\hbar(\mathcal{A}_{\Lambda/X})$ -module.

Example 4.5. We follow the notations of Example 3.11. Let $\Lambda = M$. Then $\mathcal{L} := \widehat{\mathcal{W}}_X(0) / (\sum_i \widehat{\mathcal{W}}_X(0)u_i) \simeq \mathcal{O}_\Lambda^\hbar$ is simple along Λ and $\mathcal{I} \subset \hbar^{-1}\mathcal{A}_X = \mathcal{A}_X(-1)$ is generated by $\hbar^{-1}u = (\hbar^{-1}u_1, \dots, \hbar^{-1}u_n)$. Identifying $\hbar^{-1}u_i$ with $\frac{\partial}{\partial x_i}$ we get an isomorphism $\mathcal{D}_\mathcal{L} \simeq \mathcal{D}_\Lambda[[\hbar]]$.

From now on, and until the end of the proof of Proposition 4.8 we work locally on X and thus we may assume that there is an isomorphism $\mathrm{gr}_\hbar \mathcal{A}_X \xrightarrow{\sim} \mathcal{O}_X$.

We introduce a filtration $F\mathcal{A}_X^{\mathrm{loc}}$ on $\mathcal{A}_X^{\mathrm{loc}}$ by setting

$$(4.4) \quad F_k \mathcal{A}_X^{\mathrm{loc}} = \hbar^{-k} \mathcal{A}_X \text{ for } k \in \mathbb{Z}.$$

Therefore, there is a natural isomorphism

$$(4.5) \quad \mathrm{gr}_k^F \mathcal{A}_X^{\mathrm{loc}} \simeq T^{-k} \mathcal{O}_X \text{ given by } \hbar \longleftrightarrow T.$$

We endow $\mathcal{A}_{\Lambda/X}$ with the induced filtration, that is,

$$F_k \mathcal{A}_{\Lambda/X} = \mathcal{A}_{\Lambda/X} \cap F_k \mathcal{A}_X^{\mathrm{loc}}.$$

Recall (see [KS12, § 1.4]) that for a left Noetherian \mathbb{C}^h -algebra \mathcal{R} , one says that a coherent \mathcal{R} -module \mathcal{P} is locally projective if the functor

$$\mathcal{H}om_{\mathcal{R}}(\mathcal{P}, \bullet): \text{Mod}_{\text{coh}}(\mathcal{R}) \rightarrow \text{Mod}(\mathbb{C}_X^h)$$

is exact. This is equivalent to each of the following conditions: (i) for each $x \in X$, the stalk \mathcal{P}_x is projective as an \mathcal{R}_x -module, (ii) for each $x \in X$, the stalk \mathcal{P}_x is a flat \mathcal{R}_x -module, (iii) \mathcal{P} is locally a direct summand of a free \mathcal{R} -module of finite rank.

Recall that one says that a ring R has global homological dimension $\leq d$ if both $\text{Mod}(R)$ and $\text{Mod}(R^{\text{op}})$ have homological dimension $\leq d$ (see [KS90, Exe. I.28]). In such a case, we shall write for short $\text{ghd}(R) \leq d$.

Also recall that d_X denotes the complex dimension of X .

Lemma 4.6. *One has*

- (a) $(\mathcal{A}_{\Lambda/X})^{\text{loc}} \simeq \mathcal{A}_X^{\text{loc}}$.
- (b) The algebra $\text{gr}^F \mathcal{A}_{\Lambda/X}$ is a graded commutative subalgebra of $\text{gr}^F \mathcal{A}_X^{\text{loc}}$.
- (c) There are natural isomorphisms

$$\text{gr}^F \mathcal{A}_{\Lambda/X} \simeq \bigoplus_{k \in \mathbb{Z}} T^{-k} I_{\Lambda}^k \quad \text{and} \quad \text{gr}_{\geq 0}^F \mathcal{A}_{\Lambda/X} \simeq \bigoplus_{k \geq 0} T^{-k} I_{\Lambda}^k,$$

where $I_{\Lambda}^k := \mathcal{O}_X$ for $k \leq 0$.

- (d) The sheaves of algebras $\text{gr}^F \mathcal{A}_{\Lambda/X}$ and $\text{gr}_{\geq 0}^F \mathcal{A}_{\Lambda/X}$ are Noetherian.
- (e) For any $x \in X$, one has $\text{ghd}(\text{gr}^F \mathcal{A}_{\Lambda/X})_x \leq d_X + 1$.

Proof. (a) is obvious since $\mathcal{A}_X \subset \mathcal{A}_{\Lambda/X} \subset \mathcal{A}_X^{\text{loc}}$.

(b) is obvious.

(c) $\text{gr}_1^F(\mathcal{A}_{\Lambda/X}) \simeq I_{\Lambda}$. Hence, $\text{gr}_k^F \mathcal{A}_{\Lambda/X} \simeq I_{\Lambda}^k$.

(d) The commutative algebras $\text{gr}^F \mathcal{A}_{\Lambda/X}$ and $\text{gr}_{\geq 0}^F \mathcal{A}_{\Lambda/X}$ are locally finitely presented \mathcal{O}_X -algebras. Hence they are Noetherian. (Note that the associated variety with $\text{gr}^F \mathcal{A}_{\Lambda/X}$ is the deformation of normal bundle to Λ .)

(e) For $x \in X$, set $R_x = (\text{gr}^F \mathcal{A}_{\Lambda/X})_x$. If $x \notin \Lambda$, then $R_x \simeq \mathcal{O}_{X,x}[T, T^{-1}]$ and $\text{ghd}(R_x) \leq d_X + 1$. Assume now that $x \in \Lambda$. Then $R_x/TR_x \simeq \mathcal{O}_{\Lambda,x}[y_1, \dots, y_n]$ (with $n = \text{codim}_X \Lambda$) has global homological dimension d_X and $R_x[T^{-1}] \simeq \mathcal{O}_{X,x}[T, T^{-1}]$ has global homological dimension $d_X + 1$. Hence, $\text{ghd}(R_x) \leq d_X + 1$ by the classical Lemma 4.7 below. \square

Lemma 4.7. *Let R be a commutative Noetherian ring and let $t \in R$ be a non-zero divisor. Assume that R/tR has global homological dimension $\leq d$ and the localization $R[t^{-1}]$ has global homological dimension $\leq d + 1$. Then R has global homological dimension $\leq d + 1$.*

Proof. (i) Let $\text{Spec}(R)$ denote as usual the set of prime ideals of R . For $\mathfrak{p} \in \text{Spec}(R)$, denote by $R_{\mathfrak{p}}$ the localization of R at \mathfrak{p} . It is well-known that R has global homological dimension $\leq d$ if and only if for any $\mathfrak{p} \in \text{Spec}(R)$, $R_{\mathfrak{p}}$ has global homological dimension $\leq d$.

(ii) Let $\mathfrak{p} \in \text{Spec}(R)$ and assume that $t \notin \mathfrak{p}$. Then $R_{\mathfrak{p}} \simeq (R[t^{-1}])_{\mathfrak{p}}$ has global homological dimension $\leq d + 1$.

(iii) Let $\mathfrak{p} \in \text{Spec}(R)$ and assume that $t \in \mathfrak{p}$. In this case, $R_{\mathfrak{p}}/tR_{\mathfrak{p}} \simeq (R/tR)_{\mathfrak{p}}$ has global homological dimension $\leq d$. This implies that $R_{\mathfrak{p}}$ is a regular local ring of global homological dimension $\leq d + 1$. \square

Proposition 4.8 (see [KS12, Lem. 7.1.3] in the symplectic case). *One has*

(a) *the \mathbb{C}^h -algebroid $\mathcal{A}_{\Lambda/X}$ is right and left Noetherian,*

(b) $\text{gr}_{\hbar}(\mathcal{N}) \in \text{D}_{\text{coh}}^{\text{b}}(\text{gr}_{\hbar}\mathcal{A}_{\Lambda/X})$ *for any $\mathcal{N} \in \text{D}_{\text{coh}}^{\text{b}}(\mathcal{A}_{\Lambda/X})$.*

Proof. (a) follows from Proposition 4.1 since \mathcal{A} is Noetherian by Theorem 3.12, $\text{gr}_{\geq 0}\mathcal{A}_{\Lambda/X}$ is Noetherian by Lemma 4.6 and the I_{Λ}^k 's are coherent \mathcal{A}_X -modules since they are coherent \mathcal{O}_X -modules.

(b) Let us represent \mathcal{N} by a complex \mathcal{L}^{\bullet} bounded from above of locally free $\mathcal{A}_{\Lambda/X}$ -modules of finite rank. Then $H^i(\mathcal{L}^{\bullet}) \simeq 0$ for $i \ll 0$. Replacing \mathcal{L}^{\bullet} with $\tau^{\geq j}\mathcal{L}^{\bullet}$ for $j \ll 0$ we find a bounded complex \mathcal{L}^{\bullet} of coherent $\mathcal{A}_{\Lambda/X}$ -modules for which \hbar is injective. Now $\text{gr}_{\hbar}(\mathcal{N})$ is represented by the complex $\mathcal{L}^{\bullet}/\hbar\mathcal{L}^{\bullet}$ and the result follows.

(c) Let d denote the projective dimension \square

In the sequel, for $\mathcal{N} \in \text{D}^{\text{b}}(\mathcal{A}_{\Lambda/X})$ we set

$$(4.6) \quad \text{gr}_{\Lambda}(\mathcal{N}) := \text{gr}_{\hbar}(\mathcal{D}_{\mathcal{L}} \overset{\text{L}}{\otimes}_{\mathcal{A}_{\Lambda/X}} \mathcal{N}) \simeq \mathcal{D}_{\Lambda} \overset{\text{L}}{\otimes}_{\text{gr}_{\hbar}(\mathcal{A}_{\Lambda/X})} \text{gr}_{\hbar}(\mathcal{N}).$$

Corollary 4.9. *If $\mathcal{N} \in \text{D}_{\text{coh}}^{\text{b}}(\mathcal{A}_{\Lambda/X})$, then $\text{gr}_{\Lambda}(\mathcal{N}) \in \text{D}_{\text{coh}}^{\text{b}}(\mathcal{D}_{\Lambda})$ and $\text{char}(\text{gr}_{\Lambda}(\mathcal{N}))$ is a closed \mathbb{C}^{\times} -conic complex analytic subset of $T^*\Lambda$.*

Proof. By Proposition 4.8 (b) and Lemma 4.6 (e), $\text{gr}_{\hbar}\mathcal{N}$ is locally quasi-isomorphic to a bounded complex of projective $\text{gr}_{\hbar}\mathcal{A}_{\Lambda/X}$ -modules of finite type. To conclude, note that if \mathcal{P} is a projective $\text{gr}_{\hbar}\mathcal{A}_{\Lambda/X}$ -modules of finite type, then $\mathcal{D}_{\Lambda} \overset{\text{L}}{\otimes}_{\text{gr}_{\hbar}(\mathcal{A}_{\Lambda/X})} \text{gr}_{\hbar}(\mathcal{P})$ is concentrated in degree 0 and is \mathcal{D}_{Λ} -coherent. The result for $\text{char}(\text{gr}_{\Lambda}(\mathcal{N}))$ follows. \square

Proposition 4.10 (see [KS12, Prop. 7.1.8] in the symplectic case). *Let \mathcal{N} be a coherent $\mathcal{A}_{\Lambda/X}$ -module. Then*

$$(4.7) \quad \text{R}\mathcal{H}om_{\mathcal{A}_{\Lambda/X}}(\mathcal{N}, \mathcal{L}) \in \text{D}^{\text{b}}(\mathbb{C}_X^h),$$

$$(4.8) \quad \text{SS}(\text{R}\mathcal{H}om_{\mathcal{A}_{\Lambda/X}}(\mathcal{N}, \mathcal{L})) = \text{char}(\text{gr}_{\Lambda}\mathcal{N}).$$

Proof. (i) One has

$$\text{R}\mathcal{H}om_{\mathcal{A}_{\Lambda/X}}(\mathcal{N}, \mathcal{L}) \simeq \text{R}\mathcal{H}om_{\mathcal{D}_{\mathcal{L}}}(\mathcal{D}_{\mathcal{L}} \overset{\text{L}}{\otimes}_{\mathcal{A}_{\Lambda/X}} \mathcal{N}, \mathcal{L}).$$

Set $F = \mathbf{R}\mathcal{H}om_{\mathcal{D}_{\mathcal{L}}}(\mathcal{D}_{\mathcal{L}} \overset{\mathbf{L}}{\otimes}_{\mathcal{A}_{\Lambda/X}} \mathcal{N}, \mathcal{L})$. Then $F \in \mathbf{D}^+(\mathbb{C}_X^h)$, F is cohomologically complete by Proposition 3.3 and $\mathrm{gr}_h(F) \simeq \mathbf{R}\mathcal{H}om_{\mathcal{D}_{\Lambda}}(\mathrm{gr}_{\Lambda}\mathcal{N}, \mathcal{O}_{\Lambda})$.

- (ii) We have $\mathrm{gr}_h F \in \mathbf{D}^b(\mathbb{C}_X^h)$ by Lemma 4.6 (c). This implies (4.7) by Proposition 3.5.
- (iii) We have $\mathrm{SS}(F) = \mathrm{SS}(\mathrm{gr}_h(F))$ by Proposition 3.7. On the other hand, $\mathrm{gr}_h(F) \simeq \mathbf{R}\mathcal{H}om_{\mathcal{D}_{\Lambda}}(\mathrm{gr}_{\Lambda}\mathcal{N}, \mathcal{O}_{\Lambda})$ and the microsupport of this complex is equal to $\mathrm{char}(\mathrm{gr}_{\Lambda}\mathcal{N})$ by [KS90, Th 11.3.3]. \square

Definition 4.11. A coherent $\mathcal{A}_{\Lambda/X}$ -submodule \mathcal{N} of a coherent $\mathcal{A}_X^{\mathrm{loc}}$ -module \mathcal{M} is called an $\mathcal{A}_{\Lambda/X}$ -lattice of \mathcal{M} if \mathcal{N} generates \mathcal{M} as an $\mathcal{A}_X^{\mathrm{loc}}$ -module.

One easily proves that if \mathcal{N} is an $\mathcal{A}_{\Lambda/X}$ -lattice of \mathcal{M} , then $\mathrm{char}(\mathrm{gr}_h\mathcal{N})$ depends only on \mathcal{M} .

Notation 4.12. For a coherent $\mathcal{A}_X^{\mathrm{loc}}$ -module \mathcal{M} , one sets $\mathrm{char}_{\Lambda}(\mathcal{M}) := \mathrm{char}(\mathrm{gr}_{\Lambda}\mathcal{N})$ for \mathcal{N} a (locally defined) $\mathcal{A}_{\Lambda/X}$ -lattice of \mathcal{M} .

4.3 Reminders on holonomic DQmodules

We shall recall here the main results of [KS12, Ch. 7].

In this subsection, we assume that X is symplectic and that Λ is Lagrangian. In this case, $\mathrm{gr}_h(\mathcal{A}_{\Lambda/X}) \simeq \mathcal{D}_{\Lambda}$ as an algebroid and thus $\mathrm{gr}_{\Lambda}(\mathcal{N}) \simeq \mathrm{gr}_h(\mathcal{N})$.

Definition 4.13. Assume that X is symplectic and Λ is Lagrangian. An object \mathcal{N} of $\mathbf{D}_{\mathrm{coh}}^b(\mathcal{A}_{\Lambda/X})$ is holonomic if $\mathrm{gr}_h(\mathcal{N})$ belongs to $\mathbf{D}_{\mathrm{hol}}^b(\mathcal{D}_{\Lambda})$.

Theorem 4.14 (see [KS12, Th. 7.1.10]). *Assume that X is symplectic. Let \mathcal{N} be a holonomic $\mathcal{A}_{\Lambda/X}$ -module.*

- (a) *The objects $\mathbf{R}\mathcal{H}om_{\mathcal{A}_{\Lambda/X}}(\mathcal{N}, \mathcal{L})$ and $\mathbf{R}\mathcal{H}om_{\mathcal{A}_{\Lambda/X}}(\mathcal{L}, \mathcal{N})$ belong to $\mathbf{D}_{\mathbb{C}\mathbb{c}}^b(\mathbb{C}_{\Lambda}^h)$ and their microsupports are contained in $\mathrm{char}(\mathrm{gr}_h\mathcal{N})$.*
- (b) *There is a natural isomorphism in $\mathbf{D}_{\mathbb{C}\mathbb{c}}^b(\mathbb{C}_{\Lambda}^h)$*

$$(4.9) \quad \mathbf{R}\mathcal{H}om_{\mathcal{A}_{\Lambda/X}}(\mathcal{N}, \mathcal{L}) \xrightarrow{\simeq} \mathbf{D}'_X(\mathbf{R}\mathcal{H}om_{\mathcal{A}_{\Lambda/X}}(\mathcal{L}, \mathcal{N})) [d_X].$$

The crucial result in order to prove Theorem 4.16 below is the following.

Proposition 4.15 (see [KS12, Prop. 7.1.16]). *Assume that X is symplectic and Λ is Lagrangian. For a coherent $\mathcal{A}_X^{\mathrm{loc}}$ -module \mathcal{M} , we have*

$$\mathrm{codim} \mathrm{char}_{\Lambda}(\mathcal{M}) \geq \mathrm{codim} \mathrm{Supp}(\mathcal{M}).$$

The next result is a variation on a classical theorem of [Kas75] on holonomic D-modules.

Theorem 4.16 (see [KS12, Th. 7.2.3]). *Assume that X is symplectic. Let \mathcal{M} and \mathcal{N} be two holonomic $\mathcal{A}_X^{\mathrm{loc}}$ -modules. Then*

- (i) the object $R\mathcal{H}om_{\mathcal{A}_X^{\text{loc}}}(\mathcal{M}, \mathcal{N})$ belongs to $D_{\mathbb{C}c}^b(\mathbb{C}_X^{h,\text{loc}})$,
- (ii) there is a canonical isomorphism:

$$(4.10) \quad R\mathcal{H}om_{\mathcal{A}_X^{\text{loc}}}(\mathcal{M}, \mathcal{N}) \xrightarrow{\simeq} (D'_X R\mathcal{H}om_{\mathcal{A}_X^{\text{loc}}}(\mathcal{N}, \mathcal{M})) [d_X],$$

- (iii) the object $R\mathcal{H}om_{\mathcal{A}_X^{\text{loc}}}(\mathcal{M}, \mathcal{N})[d_X/2]$ is perverse.

5 Proof of the main theorems and an example

5.1 Proof of Theorem 1.2

In this subsection, X is again a complex Poisson manifold endowed with a DQ-algebroid \mathcal{A}_X .

By using the diagonal procedure, we may assume that $\mathcal{L} = \mathcal{L}_0^{\text{loc}}$ with \mathcal{L}_0 an \mathcal{A}_X -module simple along Λ . By the hypothesis, we may find an $\mathcal{A}_{\Lambda/X}$ -lattice \mathcal{N} of \mathcal{M} . Set

$$(5.1) \quad F_0 := R\mathcal{H}om_{\mathcal{A}_{\Lambda/X}}(\mathcal{N}, \mathcal{L}_0), \quad F := R\mathcal{H}om_{\mathcal{A}_X^{\text{loc}}}(\mathcal{M}, \mathcal{L}) \simeq F^{\text{loc}}.$$

One knows by Theorem 4.16 that $F|_Y \in D_{\mathbb{C}c}^b(\mathbb{C}_{Y \cap \Lambda}^{h,\text{loc}})$ and one knows by Proposition 4.10 and Corollary 4.9 that $\text{SS}(F_0) \times_{\Lambda} (\Lambda \cap Y)$ is Lagrangian and subanalytic in $T^*\Lambda$. Since $\text{SS}(F) \subset \text{SS}(F_0)$ by Proposition 3.6, it remains to apply Corollary 2.4.

5.2 Proof of Theorem 1.3

In this subsection, X is a quasi-compact separated smooth algebraic variety over \mathbb{C} endowed with the Zariski topology. For an algebraic variety X , one denotes by X_{an} the complex analytic manifold associated with X and by $\rho: X_{\text{an}} \rightarrow X$ the natural map. There is a natural morphism $\rho^{-1}\mathcal{O}_X \rightarrow \mathcal{O}_{X_{\text{an}}}$ and it is well-known that this morphism is faithfully flat (cf [Ser56]).

Lemma 5.1. *Let \mathcal{A}_X be a DQ-algebroid on X . Then there exists a DQ-algebroid $\mathcal{A}_{X_{\text{an}}}$ on X_{an} together with a functor $\rho^{-1}\mathcal{A}_X \rightarrow \mathcal{A}_{X_{\text{an}}}$. Moreover such an $\mathcal{A}_{X_{\text{an}}}$ is unique up to a unique isomorphism.*

Proof. First, consider a star algebra $\mathcal{A} = (\mathcal{O}_X^h, \star)$ on a smooth algebraic variety X . The star product is defined by a sequence of algebraic bidifferential operators $\{P_i\}_i$ (see [KS12, Def. 2.2.2]) and one defines a star algebra $\mathcal{A}^{\text{an}} = (\mathcal{O}_{X_{\text{an}}}^h, \star)$ on X_{an} by using the same bidifferential operators.

There exists an open (for the Zariski topology) covering $X = \bigcup_{i \in I} U_i$ such that, for each i , there exists an object s_i of the category $\mathcal{A}_X(U_i)$. Then $\mathcal{A}_i := \mathcal{E}nd(s_i)$ is a star algebra. For $i, j \in I$, since $s_i|_{U_i \cap U_j}$ and $s_j|_{U_i \cap U_j}$ are locally isomorphic, there exists an open covering $U_i \cap U_j = \bigcup_{a \in A_{ij}} U_{ij}^a$ such that setting $U_{ij} = \bigsqcup_{a \in A_{ij}} U_{ij}^a$, there exist an isomorphism $\alpha_{ij}: s_i|_{U_{ij}} \xrightarrow{\simeq} s_j|_{U_{ij}}$. Then we have

$$\alpha_{ijk} := \alpha_{ij}\alpha_{jk}\alpha_{ki} \in \text{End}(s_i|_{U_{ijk}}) = \mathcal{A}_i(U_{ijk}),$$

where $U_{ijk} = U_{ij} \times_X U_{jk} \times_X U_{ki}$.

Hence we have an isomorphism $\beta_{ij}: \mathcal{A}_i|_{U_{ij}} \xrightarrow{\simeq} \mathcal{A}_j|_{U_{ij}}$ defined by $\mathcal{A}_i \ni a \mapsto \alpha_{ij} \circ a \circ \alpha_{ij}^{-1} \in \mathcal{A}_j$. Moreover they satisfy the compatibility condition:

$$\beta_{ij}\beta_{jk}\beta_{ki} = \text{Ad}(a_{ijk}) \in \text{End}(\mathcal{A}_i|_{U_{ijk}}).$$

Then the data $(\{U_i\}, \{U_{ij}\}, \{\mathcal{A}_i\}, \{\beta_{i,j}\}, \{a_{ijk}\})$ satisfies the compatibility condition. Conversely, we can recover \mathcal{A}_X from such data (see [KS12]).

On $(U_i)_{\text{an}}$ we can define $\mathcal{A}_i^{\text{an}}$. Similarly we can extend β_{ij} to $\beta_{ij}^{\text{an}}: \mathcal{A}_i^{\text{an}}|_{(U_{ij})_{\text{an}}} \xrightarrow{\simeq} \mathcal{A}_j^{\text{an}}|_{(U_{ij})_{\text{an}}}$. Finally we have $a_{ijk} \in \mathcal{A}_i(U_{ijk}) \subset \mathcal{A}_i^{\text{an}}((U_{ijk})_{\text{an}})$, Then the data

$$(\{(U_i)_{\text{an}}\}, \{(U_{ij})_{\text{an}}\}, \{\mathcal{A}_i^{\text{an}}\}, \{\beta_{i,j}^{\text{an}}\}, \{a_{ijk}\})$$

satisfies the compatibility condition, and it defines a DQ-algebroid $\mathcal{A}_{X_{\text{an}}}$ on X_{an} . \square

Proposition 5.2. *The algebroid $\mathcal{A}_{X_{\text{an}}}$ is faithfully flat over $\rho^{-1}\mathcal{A}_X$.*

Proof. It is enough to prove that for each $x \in X$, $\mathcal{A}_{X_{\text{an}},x}$ is faithfully flat over $\mathcal{A}_{X,x}$. This follows from [KS12, Cor. 1.6.7] since $\mathcal{A}_{X,x}/\hbar\mathcal{A}_{X,x} \simeq \mathcal{O}_{X,x}$ is Noetherian, $\mathcal{A}_{X_{\text{an}},x}$ is cohomologically complete and finally $\mathcal{A}_{X_{\text{an}},x}/\hbar\mathcal{A}_{X_{\text{an}},x} \simeq \mathcal{O}_{X_{\text{an}},x}$ is faithfully flat over $\mathcal{O}_{X,x}$. \square

For an \mathcal{A}_X -module \mathcal{M} we set

$$\mathcal{M}_{\text{an}} := \mathcal{A}_{X_{\text{an}}} \otimes_{\rho^{-1}\mathcal{A}_X} \rho^{-1}\mathcal{M}.$$

Proof of Theorem 1.3. As in the proof of Theorem 1.2, we may assume that $\mathcal{L} \simeq \mathcal{L}_0^{\text{loc}}$ where \mathcal{L}_0 is a simple \mathcal{A}_X -module along a smooth algebraic Lagrangian manifold Λ , the module \mathcal{M} remaining algebraically good. Choose an $\mathcal{A}_{\Lambda/X}$ -lattice \mathcal{N} of \mathcal{M} . Let

$$(5.2) \quad F_{\text{an}} := \text{R}\mathcal{H}om_{\mathcal{A}_{X_{\text{an}}}^{\text{loc}}}(\mathcal{M}_{\text{an}}, \mathcal{L}_{\text{an}}) \simeq \text{R}\mathcal{H}om_{\mathcal{A}_{\Lambda/X_{\text{an}}}}(\mathcal{N}_{\text{an}}, (\mathcal{L}_0)_{\text{an}})^{\text{loc}}.$$

By Proposition 4.10 we know that $\text{SS}(F_{\text{an}}) \subset \text{char}(\text{gr}_{\Lambda}\mathcal{N}_{\text{an}})$ and this set is contained in $\text{char}(\text{gr}_{\Lambda}\mathcal{N})$ which is an algebraic Lagrangian subvariety of $T^*\Lambda$. To conclude, apply Corollary 2.5. \square

Remark 5.3. (i) If one assumes that \mathcal{M} and \mathcal{L} are simple modules along two smooth algebraic varieties Λ_1 and Λ_2 of X , which is the situation appearing in [GJS19], there is a much simpler proof. Indeed, it follows from [KS12, Th. 7.4.3] that in this case

$$(5.3) \quad \text{SS}(F) \subset C(\Lambda_1, \Lambda_2),$$

the Whitney normal cone of Λ_1 along Λ_2 and this set is algebraic. Hence, it remains to apply Corollary 2.5. Note that Th. 7.4.3 of loc. cit. is a variation on [KS08].

(ii) Also remark that (5.3) is no more true in the general case of irregular holonomic modules and until now, there is no estimate of $\text{SS}(F)$, except of course, the fact that it is a Lagrangian set.

5.3 An example

Consider the Poisson manifold $X = \mathbb{C}^4$ with coordinates (x_1, x_2, y_1, y_2) , the Poisson bracket being defined by:

$$(5.4) \quad \begin{aligned} \{x_1, x_2\} &= 0, \{y_1, x_1\} = \{y_2, x_2\} = x_1, \\ \{y_1, y_2\} &= y_2, \{y_1, x_2\} = y_2, \{y_1, x_2\} = \{y_2, x_1\} = 0. \end{aligned}$$

Denote by \mathcal{A}_X the DQ-algebra defined by the relations $y_1 = \hbar x_1 \partial_{x_1}$, $y_2 = \hbar x_1 \partial_{x_2}$, that is,

$$(5.5) \quad \begin{aligned} [x_1, x_2] &= 0, [y_1, x_1] = [y_2, x_2] = \hbar x_1, [y_1, y_2] = \hbar y_2, \\ [y_1, x_2] &= \hbar y_2, [y_1, x_2] = [y_2, x_1] = 0. \end{aligned}$$

Hence, $Y = \{x_1 \neq 0\}$ is the symplectic locus $X \setminus X_{\text{ns}}$ of the Poisson manifold X . Set $\Lambda = \{y_1 = y_2 = 0\}$. Then $\Lambda \cap Y$ is Lagrangian in Y .

Define the \mathcal{A}_X -module \mathcal{L} by $\mathcal{L} = \mathcal{A}_X \cdot u$ with the relations $y_1 u = y_2 u = 0$. Then $\mathcal{L} \simeq \mathcal{O}_\Lambda^h$ and for $a(x) \in \mathcal{O}_\Lambda^h$, one has

$$\begin{cases} y_1 a(x) u = \hbar x_1 \frac{\partial a}{\partial x_1} u \\ y_2 a(x) u = \hbar x_1 \frac{\partial a}{\partial x_2} u. \end{cases}$$

Now define the left \mathcal{A}_X module \mathcal{M} by $\mathcal{M} = \mathcal{A}_X \cdot v$ with the relations $(y_1 + \hbar)v = y_2 v = 0$. Then the complex below, in which the operators act on the right

$$(5.6) \quad 0 \longleftarrow \mathcal{M} \longleftarrow \mathcal{A}_X \longleftarrow \mathcal{A}_X^{\oplus 2} \xleftarrow{\bullet(y_2, -y_1)} \mathcal{A}_X \longleftarrow 0$$

$$\bullet \begin{pmatrix} y_1 + \hbar \\ y_2 \end{pmatrix}$$

is a free resolution of \mathcal{M} .

Hence, the object $\text{R}\mathcal{H}om_{\mathcal{A}_X}(\mathcal{M}, \mathcal{L}^{\text{loc}})$ is represented by the complex (the operators act on the left)

$$(5.7) \quad 0 \longrightarrow \mathcal{O}_\Lambda^{h,\text{loc}} \xrightarrow{\begin{pmatrix} x_1 \partial_{x_1} + 1 \\ x_1 \partial_{x_2} \end{pmatrix} \bullet} (\mathcal{O}_\Lambda^{h,\text{loc}})^{\oplus 2} \xrightarrow{(x_1 \partial_{x_2}, -x_1 \partial_{x_1}) \bullet} \mathcal{O}_\Lambda^{h,\text{loc}} \longrightarrow 0.$$

Since $x_1 \partial_{x_1} \mathcal{O}_\Lambda^{h,\text{loc}} + x_1 \partial_{x_2} \mathcal{O}_\Lambda^{h,\text{loc}} = x_1 \mathcal{O}_\Lambda^{h,\text{loc}}$ and $\mathcal{O}_\Lambda^{h,\text{loc}} / x_1 \mathcal{O}_\Lambda^{h,\text{loc}} \simeq \mathcal{O}_{\Lambda \cap \{x_1=0\}}^{h,\text{loc}}$, we have

$$\text{Ext}_{\mathcal{A}_X}^2(\mathcal{M}, \mathcal{L}^{\text{loc}}) \simeq \mathcal{O}_{\Lambda \cap \{x_1=0\}}^{h,\text{loc}}.$$

This example shows that $\text{R}\mathcal{H}om_{\mathcal{A}_X}(\mathcal{M}, \mathcal{L}^{\text{loc}})$ does not belong to $\text{D}_{\mathbb{C}\mathbb{C}}^b(\mathbb{C}^{h,\text{loc}})$.

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