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Abstract. We propose broadcasting algorithms for line digraphs in the telephone model. The new protocols use a broadcasting protocol for a graph G to obtain a broadcasting protocol for the graph L^kG, the graph obtained by applying k times, the line digraph operation to G. As a consequence improved bounds for the broadcasting time in De Bruijn, Kautz and Wrapped Butterfly digraphs are obtained.

1 Introduction

Considerable attention has been recently devoted to the dissemination of information in parallel computing, since the time needed to disseminate information among the processors in a network often determines the running time of the whole algorithm. Broadcasting is the process in which a node of the network (the originator) sends one piece of information to all members of the network by placing calls over the communication lines of the network. This is to be completed as quickly as possible subjected to the constraints of the considered model. This problem has been widely studied both for its theoretical and practical interest for different network configurations and under different models (see [7, 10] for recent surveys or the book [11]).

In this paper we are concerned with the telephone model, that is, each call involves only two vertices of the network and requires one unit of time (round); moreover we require that a vertex can participate in at most one call per unit of time by either sending or receiving a call and we restrict links to be one-way, that is, information will always flow in the same direction.

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Following the literature we model the network topology by a digraph $G = (X, U)$; a vertex $x \in X$ corresponds to a node in the network, and an arc $(x, y) \in U$, corresponds to a link in which x is the sender and y is the receiver. $b(G, x)$ represents the minimum number of time units (number of rounds) necessary to complete broadcasting form vertex x; $b(G)$ is defined as the maximum of $b(G, x)$ taken over all vertices x of G. It is well known that for any digraph G with n vertices $b(G) \geq \lceil \log_2 n \rceil$ and that the problem of determining $b(G)$ is a NP–hard problem. Values of $b(G)$ are known for many usual interconnection networks; however for Butterfly digraphs and related networks the order is still unknown.

In this paper we propose a general methodology for designing broadcasting algorithms in line digraphs and iterated line digraphs. In Section 3.1 we give a simple protocol for broadcasting in LG (the graph obtained by applying the line digraph operation to the graph G) using a broadcast protocol in G where G is a regular digraph with degree d; if broadcast in G can be performed in time t, then the algorithm performs broadcasting in LG in time $t + \lceil \log_2 d \rceil + 1$. In section 3.2 we show that if G satisfies an additional property, that we call A, then there exists a broadcasting protocol in LG that runs in time $t + \lfloor \log_2 d \rfloor + 1$.

By iterating the algorithms of Sections 3.1 and 3.2 we obtain protocols for iterated line digraphs ($L(L^{n-1}G) = L^nG$). In Section 4.1 we show an improved algorithm for broadcasting in L^2G using a broadcasting protocol in G. For some values of d, the direct derivation of the protocol for L^2G has running time $t + 2\lfloor \log_2 d \rfloor + 1$ if G satisfies property A. In Section 5 we sketch the extension to L^kG. For example we obtain for $d = 2^\alpha$, $\alpha > 1$, a broadcasting algorithm in L^3G running in time $t + 3\log_2 d + 1$.

The iteration of the line digraph operation is a good method to obtain large digraphs with fixed degree and diameter. Besides, many other good properties are observed in such digraphs when used as models of communication networks ([3, 9]). In fact some of the best known families of digraphs are indeed line digraphs: De Bruijn, Kautz,Wrapped Butterfly, among others.

Property A is satisfied in particular if G is itself a line digraph. So we obtain new and better protocols for De Bruijn, Kautz and Butterfly networks. In the case of De Bruijn digraphs $B(d, D)$ and Kautz digraphs $K(d, D)$ (the De Bruijn/Kautz digraph with degree d and diameter D) we obtain the following bounds

$$b(B(d, D)) \leq D(\log_2 d + f(d));$$

$$b(K(d, D)) \leq D(\log_2 d + f(d)) + 1.$$

where $f(d)$ satisfies: $\log_2(1 + \frac{1}{d}) \leq f(d) \leq 0.508..$

This improves the best current known bounds both for digraphs ([3], [9]) and graphs ([2]).

2 Notation and Preliminaries

We refer to [5, 11] for basic definitions concerning graphs and digraphs and we recall basic definitions and properties of line digraphs (see also [6, 8]).

Given a directed graph $G = (X, U)$, if (x, y) is an arc, then x is *adjacent to* y and y is *adjacent from* x. We also say that arc (x, y) is *incident to* y and *incident from* x, and that arc $u = (x, y)$ is adjacent to arc $v = (y, z)$. We denote by $d^+(x)$ ($d^-(x)$) the number of vertices adjacent from x (adjacent to x); a digraph is said to be *d-regular* if $d^+(x) = d^-(x) = d$ for any vertex x.

A *dipath* of *length* h from a vertex x to a vertex y is a sequence of vertices $x = x_0, x_1, \ldots, x_{h-1}, x_h = y$ where (x_i, x_{i+1}) is an arc of G. A digraph is *strongly connected* if, for any couple of vertices x, y, there exists a dipath from x to y. The length of a shortest dipath from x to y is the *distance* from x to y. Its maximum value over all couples of vertices is the *diameter* of the digraph denoted $D(G)$.

Given a digraph $G = (X, U)$ the *line digraph* operation allows to define a new digraph LG whose vertex set is in one to one correspondence with the set of arcs of G. Vertex u of LG representing the arc $u = (x, y)$ is adjacent to the vertex representing the arc v if and only if $v = (y, z)$, that is arcs of LG represent dipaths of length 2 of G. It can be easily shown that if G is d–regular with n vertices then LG is d–regular with dn vertices; furthermore if G is a strongly connected digraph different from a directed cycle and $d > 1$, then the diameter of LG is the diameter of G plus one. We denote by L^kG the graph $L(L^{k-1}G)$.

Let G be a digraph with n vertices and LG the corresponding line digraph. The arcs of LG can be partitioned into n complete bipartite digraphs isomorphic to $\overrightarrow{K}_{d,d}$ that are in one to one correspondence with vertices of G. If x is a vertex of a digraph G then we denote by B_x the corresponding bipartite digraph in LG. Note also that each vertex of LG belongs to exactly to two of such bipartite digraphs: in one case with in-degree 0 and in the other with out-degree 0.

An *arc–labeling* for a digraph G is a labeling of its arcs such that any two arcs incident to the same vertex have different labels and any two arcs incident from the same vertex have also different labels. If G is d-regular then it is always possible to obtain such an arc–labeling with d labels. Since we always consider d-regular digraphs we simply say an arc–labeling instead of an arc–labeling with d labels. Given a digraph with an arc–labeling, we identify the labels with the elements of $\mathbf{Z}_d$ and we refer to the arcs incoming from or outgoing to the same vertex by their labels.

The De Bruijn digraph $B(d, D)$ is a d-regular digraph with diameter D defined as follows: vertices are the strings of length D, $z_1 z_2 \ldots z_D$, where $z_i \in \mathbf{Z}_d$, $i = 1, 2, \ldots, D$, and vertex $z_1 z_2 \ldots z_D$ is adjacent to vertices $z_2 \ldots z_D \alpha$, $\alpha = 0, 1, 2, \ldots, d-1$. In [6] it is shown that $B(d, D) = L^{D-1}K_d^+$, where K_d^+ denotes the complete symmetric digraph on d vertices with an additional loop in each vertex.

The Kautz digraph $K(d, D)$ is defined analogously: vertices are all strings of length D, $z_1 z_2 \ldots z_D$, $z_i \in \mathbf{Z}_{d+1}$, $i = 1, 2, \ldots, D$, such that two consecutive symbols cannot be equal. Vertex $z_1 z_2 \ldots z_D$ is adjacent to vertices $z_2 \ldots z_D \alpha$, $\alpha \in \mathbf{Z}_{d+1}$, $\alpha \neq z_D$. The digraph $K(d, D)$ is d-regular and has diameter D; moreover $K(d, D) = L^{D-1}K_{d+1}^*$, where K_{d+1}^* denotes the complete symmetric digraph on $d+1$ vertices [6].

The Directed Wrapped Butterfly $\overrightarrow{WBF}(d, n)$ has as vertex set the couples (x, l) where x is a string of length n $(x_{n-1} x_{n-2} \ldots x_0)$, $x_i \in \mathbf{Z}_d$, $i = 0, 1, \ldots, n-1$, and $l \in \mathbf{Z}_n$. Vertex $(x_{n-1} x_{n-2} \ldots x_0, l)$ is adjacent to vertices $(x_{n-1} \ldots x_{l+1} \alpha x_{l-1} \ldots x_0, l+1)$, $\alpha \in \mathbf{Z}_d$. In [4] it is shown that $\overrightarrow{WBF}(d, n) = L^n dC_n$, where dC_n denotes the

graph obtained from the directed cycle with n vertices by replacing each arc with d parallel arcs.

3 Broadcasting in line digraphs

In this section we will show that, given a broadcasting algorithm in a digraph G running in time t, it is possible to construct a broadcasting protocol in LG that runs in time $t + \lceil \log_2 d \rceil + 1$. The running time is $t + \lfloor \log_2 d \rfloor + 1$ if G satisfies certain properties.

3.1 A broadcasting protocol in a line digraph

A broadcasting algorithm in a digraph $G = (X, U)$ with originator r can be described via the broadcast tree $T(r)$ with root r and vertex set X. $T(r)$ contains the arcs through which the information has been broadcasted; since each vertex in $T(r)$ has in-degree 1, then $T(r)$ is an arborescence.

Any broadcasting algorithm P with originator r induces a partial broadcasting in LG with originator any vertex in LG representing an arc (r', r) , which informs those vertices of LG corresponding to the arcs of the broadcast tree of G. If P is a broadcasting protocol running in time t in G, then at time t, for any x, there is at least one vertex $u = (\cdot, x)$ in LG, where $\cdot$ stands for an undetermined symbol, that has been informed (namely vertex corresponding to (r', r) and the vertices that correspond to the arcs of the broadcast tree $T(r)$).

During round $t + 1$, for any x in G, the informed vertex $u = (\cdot, x)$ in LG sends the information, denoted by `i`, to vertex $(x, \cdot)$ of label 1. Since the labeling of the arcs is an arc–labeling in G, for all x', vertices $(\cdot, x')$ of label 1 have been informed. At round $t+2$, all vertices $(\cdot, x)$ of label 1 send `i` to vertices $(x, \cdot)$ of label 2. Thus at the end of this round all vertices of LG with labels 1 or 2 will know `i`; so they will be able to send `i` to vertices of LG of labels 3 and 4 during round $t + 3$. Following this pattern, we claim that at the end of round $t + k$, vertices of LG with label i, $1 \leq i \leq 2^{k-1}$, will know `i`. The proof is by induction on k; it is true for $k = 1$ and $k = 2$. Suppose it is true at time $t + k$, then at time $t + k + 1$ for any x the vertex $(\cdot, x)$ of LG of label i informs the vertex $(x, \cdot)$ of label $i + 2^{k-1}$. In this way all vertices with label i, $2^{k-1} + 1 \leq i \leq 2^k$, get the information. Since G is d-regular at round $t + \lceil \log_2 d \rceil + 1$, the protocol is completed. So we can state the following theorem.

Theorem 1. *Given a regular digraph G with degree d, such that broadcast in G can be completed in t rounds, then there exists a broadcast protocol in LG that runs in time $t + \lceil \log_2 d \rceil + 1$.*

3.2 An improved protocol in a line digraph

In the previous protocol we have not used the facts that `i` arrived at vertex x of G on some arc of a specific label and that `i` might arrive before time t. For example, suppose that it arrives to an arc $(\cdot, x)$ not of label 1, then before round $t + 2$ the

information i has reached two vertices of LG. Therefore, at round $t+2$, i could have been sent on two vertices of LG instead of only one; analogously if i reaches x at time $t-h$ in the successive h rounds i can be sent on h new arcs.

In the sequel we obtain an improved broadcasting protocol in LG if G admits a suitable arc–labeling. To this aim let us consider the broadcasting tree $T(r)$ given by a broadcasting algorithm P with originator vertex r and let us label its arcs in the following way: for each vertex, the last arc used to send i is labeled 0, the penultimate one is labeled -1 (where we perform addition modulo d), the preceding one is labeled with -2 and so on. We will call such a labeled tree a *labeled broadcasting tree*. If it is possible to make an arc–labeling of the digraph G in which the labels of the arcs of $T(r)$ are the ones in the labeled broadcasting tree, we say that this arc–labeling is *consistent* with the labeling of the broadcasting tree. We say that a vertex of G is *of kind* k (in the protocol P) if it has received i though an arc of label $-k$ (modulo d).

We note the following immediate lemma.

Lemma 2. *If a vertex is of kind k, then it has been informed in the protocol P at time $t-k$ or before.*

Definition 3. A d–regular digraph G satisfies property A if there exists a broadcasting protocol such that, for any vertex r, there exists an arc–labeling of G that is consistent with the labeling of the broadcasting tree $T(r)$.

Lemma 4. *If G is a line digraph then it satisfies property A.*

Proof. If G is a line digraph, due to the partition of the arcs into complete bipartite digraphs, it is possible to label its arcs by arc–labeling independently the bipartite subdigraphs.

Let $T(r)$ be the broadcasting tree associated to the broadcasting protocol P in G and for a given bipartite digraph B_x of the decomposition of G, let V_1 and V_2 be the partition of the vertices of B_x. For each vertex y of G there exists exactly one arc entering in y that belongs to $T(r)$ (the arc through which i has arrived). Let $\delta_T(x)$ be the out-degree of x in T (that is the number of vertices which have been informed by x). Now we rank the vertices of V_1 and V_2 as follows.

Choose a vertex x in V_1, with $\delta_T(x) = \delta \geq 1$; label this vertex x_0 and label its out-neighbors in V_2 with $y_0, y_1, \ldots, y_{\delta-1}$ in the order they have been informed, i.e. according to their kind (the out-neighbor of kind 0 is labeled y_0, the one of kind 1, y_1 and so on. In general suppose that at some step of the algorithm we have used all labels till y_i for vertices in V_2.

If $i < d-1$, then choose a vertex x in X with out-degree $\delta_T(x) = \delta_i \geq 1$; label this vertex x_{i+1} and its out-neighbors y_{i+j}, $1 \leq j \leq \delta_i$, according to their kind. If $i = d-1$, then there might be vertices in V_1 that have not been labeled; we label them in any order using the remaining labels. Note that the labeling is not unique (in fact it depends on the order in which vertices in V_1 have been chosen).

Let us now consider the following arc–labeling of G: arc (x_i, y_j) has label $(i-j) \bmod d$. It is immediate to see that it is an arc–labeling which furthermore is consistent with the labeling of $T(r)$.

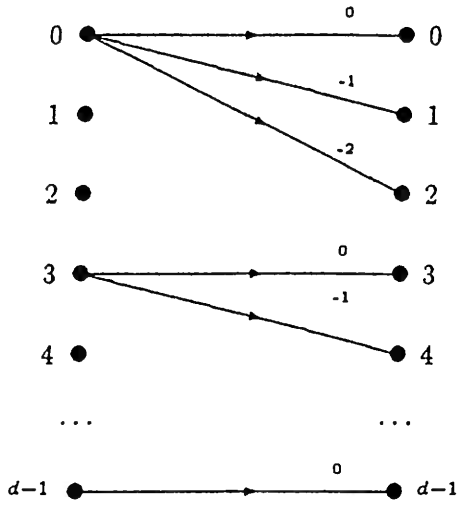

Fig. 1. Labeling of the vertices of a bipartite digraph. Drawn arcs are those in the broadcasting tree T of G

Lemma 5. *Given a d-regular digraph G that satisfies property A and a broadcasting protocol P running in G in time t, then there exists a protocol in LG such that at time $t+h$*

i) for a vertex of kind k, $-k \in \{1, 2, \ldots, 2^h - 1\}$, all its outgoing arcs have been informed;

ii) for a vertex of kind k, $-k \notin \{1, 2, \ldots, 2^h - 1\}$, all the arcs of labels j, $1 \leq j \leq 2^h - 1$, have been informed.

Proof. The proof is by induction on h and uses the following protocol Q in LG. Until time t we will use the broadcasting algorithm P in G but with the following modification. Let x be a vertex of G of kind k. By lemma 2, vertex x has received i at time less or equal to $t-k$. If we consider protocol P two possible cases arise: either x has sent i on at least k arcs or x has sent i to k' arcs, $k' < k$, and it has been idle for at least $k-k'$ rounds before t. In the modified protocol that we use until time t we will use $k-k'$ rounds to send the information to $k-k'$ arcs in such a way that, at time t, x has sent the information through arcs $0, -1, \ldots, -k+1$.

The specification of protocol Q is completed as follows. At round $t+1$ each vertex $(\cdot, x)$ of LG that corresponds to an arc of the broadcasting tree sends i to its neighbor of label 1. At the end of this round, if $x \in V(G)$ is of kind $d-1$ then $(\cdot, x)$ has informed all its neighbors in LG (in rounds before t, the vertices corresponding to arcs of label $-j$, $j = 0, 1, \ldots, d-2$, and, at round $t+1$, arc of label $1 = -(d-1)$). If x is of kind k different from $d-1$ then there exist at least two vertices of LG (y_1, x) and (y_{-k}, x) that have received i (these vertices correspond to arcs with labels 1 and $-k$); in this case these two vertices can send i to the two vertices $(x, \cdot)$ with labels 2 and 3 during round $t+2$; note that in the previous protocol at round $t+2$, i was sent to only one vertex $(x, \cdot)$.

Now we can start the induction. The lemma is satisfied for the cases $h = 0$ and $h = 1$. Suppose the lemma is true for h_0 and let us consider vertex x of kind

k. Then, at round $t + h_0 + 1$, if $-k \in \{1, 2, \ldots, 2^{h_0} - 1\}$ the lemma is true by induction. If $-k \notin \{1, 2, \ldots, 2^{h_0} - 1\}$, x has received at the end of round $t + h_0$ i on at least 2^{h_0} arcs: arc of label $-k$ (before time $t - k$) and arcs of label j, $j = 1, 2, \ldots, 2^{h_0} - 1$ (at rounds $t, t + 1, \ldots, t + h_0$). Therefore x can send i on 2^{h_0} arcs of label j, $j = 2^{h_0}, \ldots, 2^{h_0+1} - 1$. So if $-k \in \{1, 2, \ldots, 2^{(h_0+1)} - 1\}$ all outgoing arcs of x have been informed; otherwise the set of informed arcs is the claimed one.

The proof of the following theorem is an immediate consequence of the lemma and of the fact that $t + \lceil \log_2(d+1) \rceil = t + \lfloor \log_2(d) \rfloor + 1$.

Theorem 6. *Given a d-regular digraph G that satisfies property A and such that broadcast can be performed in time t, then there exists a protocol in LG that runs in time $t + \lfloor \log_2(d) \rfloor + 1$.*

4 Broadcasting in L^2G

Given a d-regular digraph G with $d = 2^\alpha(1+\beta)$, $0 \leq \beta < 1$, that satisfies property A, in which there exists a broadcasting algorithm running in time t, by applying twice the protocol for broadcasting in a line digraph, we obtain a broadcasting algorithm in L^2G running in time $t+2\alpha+2$. The aim of this section is to design a broadcasting protocol in L^2G directly from the protocol in G running in time $t + 2\alpha + 1$; this is possible if β is not too large.

Recall that, given a graph $G = (X, U)$, the vertex set of L^2G is the set of all possible dipaths of length 2 in G, and that broadcasting in L^2G is equivalent to broadcast the information through all paths of length 2, or equivalently, every arc of G informs every other arc adjacent in G.

Observe that, in the protocol for LG described in the previous section, most of the vertices are idle during the last unit of time. This happens because all their neighbors have been already informed and it should not make sense to send again the information. Nevertheless, if we iterate the process in order to obtain a broadcasting algorithm in L^2G, each arc must send i to any other arc and, therefore, the fact described before represents a waste of time.

4.1 The design of a protocol from small values of d

Lemma 7. *Given a d-regular graph G that satisfies property A and a broadcast protocol in G that runs in time t, then there exists a broadcast protocol in L^2G such that, at time $t + \alpha$, $\alpha = \lfloor \log_2 d \rfloor$, the information is arrived to all vertices that correspond to pairs of arcs of G with labels $(i, i + 2^j)$, where $1 \leq i \leq 2^{\alpha-1} - 1$ and $\lfloor \log_2 i \rfloor < j \leq \alpha - 1$, and also to vertices that correspond to a not completely specified pair of arcs with labels $(\cdot, 2^j)$, where $0 \leq j \leq \alpha - 1$ and $\cdot$ denotes the undetermined label of an arc (depending on the broadcast algorithm in G).*

Proof. We use the protocol Q described in Lemma 5 in the following precise way. We know that, for any vertex x, at time $t + h$, the information i is arrived on all the arcs i, $1 \leq i \leq 2^h - 1$, and on the arc $-k$ (if x is a vertex of kind k). At time

$t+h+1$, we impose that, if it has not already been done, arc i sends the information to $i+2^h$ and arc $-k$ to 2^h.

Note that if $-k \in \{1, 2, \ldots, 2^h - 1\}$ there is no conflict as arc $-k$ has already sent it to 2^h and $2^h + k'$ with $k' = -k \bmod d$, $k' \geq 0$. So the lemma follows by induction and considering the case $h = \alpha = \lfloor \log_2 d \rfloor$.

Let us call the above protocol a *weak protocol*. If, furthermore, we suppose that the undetermined $\cdot$ is 0 we say that the protocol is a *strong protocol*. In this case arc labeled 0 has informed arcs labeled 2^j, $0 \leq j \leq \alpha - 1$, or equivalently all the pairs of arcs$(0, 2^j)$ have been informed.

Let D_α be the set of values of d, with $\lfloor \log_2 d \rfloor = \alpha$, such that any weak protocol obtained at time $t + \alpha$ can be completed in a full protocol for L^2G in $\alpha + 1$ more steps. Since we can always find a weak protocol in time $t + \alpha$ for any d in D_α, it follows that if $d \in D_\alpha$ then there exists a protocol in L^2G running in time $t+2\alpha+1$.

Analogously let E_α be the set of values of d with $\lfloor \log_2 d \rfloor = \alpha$ such that any strong protocol obtained at time $t + \alpha$ can be completed as in the case of a weak protocol (G is a d-regular digraph) in $\alpha + 1$ more steps. Notice that $D_\alpha \subseteq E_\alpha$.

Lemma 8. *1. If $d \in D_\alpha$ then $2d \in D_{\alpha+1}$.*
2. If $d \in D_\alpha$ and $d+1 \in E_\alpha$ then $2d+1 \in D_{\alpha+1}$.
3. If $d \in E_\alpha$ then $2d \in E_{\alpha+1}$.

Proof. 1) Let $d \in D_\alpha$, thus $\alpha = \lfloor \log_2 d \rfloor$ and $\alpha + 1 = \lfloor \log_2 2d \rfloor = \lfloor \log_2(2d+1) \rfloor$. Let us recall that there exists a weak protocol for G with degree $2d$ as it is explained in the protocol for LG. Let us consider the subdigraphs G_e and G_o induced by the even and odd arcs of G respectively. The protocol for G_e (where we label arc $2i$ as i') is exactly a weak protocol for G_e (with degree d) completed in time $t + \alpha + 1$. Indeed the information is arrived on all the arcs $2i$,$2 \leq 2i \leq 2^{\alpha+1} - 1$, that is, $1 \leq i' \leq 2^\alpha - 1$, and arc i' has informed arcs $i' + 2^j$ with $\lfloor \log_2 i' \rfloor + 1 < j \leq \alpha - 1$. Similarly the protocol induced in G_o (where we identify $2i + 1$ with i'') is a strong protocol for G_o.

The protocol in G_e and in G_o can be completed in $\alpha+1$ steps more since $d \in D_\alpha$. Thus at time $t+2\alpha+2$ all the vertices in L^2G of the form $(2i, 2j)$ and $(2i+1, 2j+1)$, $0 \leq i, j \leq d - 1$ know the information. Using an extra step $(2i, 2j)$ can inform $(2j, 2i + 1)$ and $(2i + 1, 2j + 1)$ can inform $(2j + 1, 2i)$ completing the protocol in time $2(\alpha + 1) + 1$. Thus $2d \in D_{\alpha+1}$.

2) Let us consider subdigraphs G_e and G_o as in the case before, but labeling arc $2d$ as $2d + 1$, that is considering $2d$ as if it were an odd number, and thus belonging to G_o. The protocols induced in G_e and G_o are a weak protocol and a strong protocol respectively. Since $d \in D_\alpha$ and $d + 1 \in E_\alpha$ it is possible for the vertices of the form $(2i, 2j)$, $0 \leq i, j \leq d - 1$, and $(2i + 1, 2j + 1)$, $0 \leq i, j \leq d$ to be informed at time $2\alpha + 2$. Nevertheless let us change a bit the protocol in G_o in order that for each j, $0 \leq j \leq d - 1$, one of the informed vertices during last step, say $(2i + 1, 2j + 1)$ is replaced by the vertex $(2i + 1, 2j)$, that is $2i + 1$ informs $2j$ instead of informing $2j + 1$.

Notice that at time $2\alpha+2$ arcs in G_e have been informed by $d+1$ other arcs, that is $(2i, 2j)$ know the information for all $i, j \leq d - 1$ as well as $(2k + 1, 2j)$ for some k. Thus $(2j, 2i + 1)$ can get the information in the next step for all $i \leq d, j \leq d - 1$.

Also at time $2\alpha + 2$ the arcs in G_o have been informed at least d times, that is, for every $j \leq d$ there are at least d vertices of the form $(2i + 1, 2j + 1)$ which are informed, and thus they can inform the $d - 1$ remaining vertices of the form $(2j + 1, 2i)$, $0 \leq i \leq d - 1$ which were not informed at time $2\alpha + 2$ and the vertex $(2i + 1, 2j + 1)$ which was neither informed at that time. Hence the protocol can be completed in $2(\alpha + 1) + 1$ steps, and so $2d + 1 \in D_{\alpha+1}$.
3) This case is analogous to Case 1 and it is omitted.

Lemma 9. *$4 \in D_2$ and $5 \in E_2$.*

Proof. In the following tables we show an algorithm for completing the protocol in L^2G. In the tables columns represent successive rounds and rows represent labels of arcs incident to any vertex of G. At each round it is shown what must be done by each incoming arc. For instance in table a) at time $t+3$ all arcs inform 0; thus 0 will have been informed by three different arcs and at time $t+4$ it can inform 3 different arcs (with labels 1, 2, 3).

Recall that, by Lemma 7, at time less than or equal to $t + 2$ arcs 1, 2 and 3 have been informed and moreover the incoming arc 1 has informed the outgoing arc with label 3.

	$\leq t+2$	$t+3$	$t+4$	$t+5$
0			1,2,3	0
1	3	0	1	2
2		0	2	1,3
3		0	3	1,2

a) table for $4 \in D_2$

	$\leq t+2$	$t+3$	$t+4$	$t+5$
0	1,2	4	3	0
1	3	4	2	0,1
2		2	1,4	0,3
3		3	1,4	0,2
4			2,3	0,1,4

b) table for $5 \in E_2$

It is not difficult to construct analogous tables to show that $10 \in D_3$ and $11 \in E_3$, but for a lack of space we leave them to the full paper. This result and the ones shown in Lemmas 8 and 9, imply that given d, $d = 2^\alpha(1 + \beta)$ and $\alpha \geq 3$, then $d \in D_\alpha$ if $\beta < 3/8$ and $d \in E_\alpha$ if $\beta \leq 3/8$.

We use a different analysis in order to obtain a broadcast protocol for L^2G that runs in time $t + 2\alpha + 1$, when $d = 2, 5$.

Lemma 10. *Given a d-regular line digraph with $d = 2$ or $d = 5$ there exists a protocol in L^2G running in time $t + 3$, respectively $t + 5$.*

Proof. We give only the proof for $d = 2$ (the proof for $d = 5$ uses analogous ideas). Let us consider the table below:

It seems to be a problem at time $t + 3$, because the arc of label 0 incident to vertex of kind 1 might have received the information only once, and thus cannot send twice. But if we have a little more care in our analysis we can realize that in the case it has received the information only once (i.e. the incoming arc of label 0 in a vertex of kind 1 is an outgoing arc of a vertex of kind 1) it has received the information in time less or equal than t, and thus can have sent this information previous to $t + 3$. In other case, (i.e. the incoming arc of label 0 in a vertex of kind 1 is an outgoing arc of a vertex of kind 0), it has received the information twice, and

so it can send twice the information. In any case it is possible to have a protocol for L^2G running in time $t+3$, which is $t+2\alpha+1$.

		$\leq t$	$t+1$	$t+2$	$t+3$
kind 0	0		1	0	
	1			0	1
kind 1	0				0,1
	1	0	1		

The following theorem is the main result of this section and is a straightforward consequence of previous lemmas.

Theorem 11. *Given a d-regular digraph G, $d=2^\alpha(1+\beta)$ and $\beta<3/8$, that satisfies property A and such that broadcast can be performed in time t then there exists a protocol in L^2G that runs in time $t+2\lfloor\log_2(d)\rfloor+1$.*

It is possible to improve the value of β that appears in Theorem 11 by providing tables analogous to the tables of Lemma 9 for larger values. We leave this extension to the full paper. For example in the full paper we prove that $44\in D_5$ and $45\in E_5$ and so we have a theorem analogous to Theorem 11 with $\beta=.13/32$.

We now show an upper bound for the value of β for which it possible to get this improvement by the proposed methodology. Namely it cannot be applied if $\beta>\sqrt{2}-1$. In fact the method uses for the first t rounds a broadcast protocol in G; it follows that at time t we know that only n vertices in L^2G have got the information. Since at each subsequent round the number of vertices knowing the information at most double, it follows that, at time $t+2\alpha+1$, the number of vertices being informed is at most $n2^{2\alpha+1}$. On the other hand the number of vertices in L^2G is $n2^{2\alpha}(1+\beta)^2$. The bound is obtained by comparing both expressions. Note that $13/32=0.406..$ is very close to $\sqrt{2}-1=0.414..$

4.2 Broadcasting in some families of digraphs

We consider the following families of iterated line digraphs: De Bruijn $B(d,D)$, Kautz $K(d,D)$ and Directed Wrapped Butterfly $\overrightarrow{WBF}\ (d,n)$. Since K_d^+, K_{d+1}^* and dC_n (the first cases) satisfy property A then all these families satisfy property A as well and the following corollary of theorems 6, 11 easily follows.

Corollary 12. *Let $d=2^\alpha(1+\beta)$, $\alpha>4$, and let $c_d=1/2$ if $\beta<13/32$ and $c_d=1$ otherwise. Broadcasting in De Bruijn, Kautz and Wrapped Butterfly digraphs can be performed in time*

1. $b(B(d,D))\leq D(\lfloor\log_2 d\rfloor+c_d)$
2. $b(K(d,D))\leq D(\lfloor\log_2 d\rfloor+c_d)+1$
3. $b(\overrightarrow{WBF}\ (d,n))\leq n(\lfloor\log_2 d\rfloor+c_d)+n-1$.

Remark. For the De Bruijn digraph the broadcasting time can be written as

$$b(B(d,D))\leq D(\log_2 d+f(d))$$

where $f(d)$ is a function bounded by $\log_2(1+(1/d))\leq f(d)\leq 0.508..$

The lower bound is obtained for $d = 2^\alpha - 1$ and the upper one is obtained for $\beta = 13/32$. Nevertheless this result can be improved using protocols in L^kG as it will be sketched in the next section.

5 Protocol in L^kG

The results of the previous section can be extended by showing that, for some values of d, it is possible to construct a protocol in L^kG running in time $t + k\lfloor \log_2 d \rfloor + h$, $h < k$ directly from the broadcasting algorithm in G. In the following we omit proofs. First of all we observe that if G is a d-regular line digraph G with $d = 2^\alpha(1+\beta)$, $0 \leq \beta < 1$, with a broadcasting algorithm running in time t, it is possible to construct a broadcasting algorithm in L^kG running in time $t + k\alpha + h$, with this method, only if $1 + \beta \leq 2^{\frac{h}{k}}$.

Furthermore we can extend the notion of a weak protocol given in Section 4 as follows: let $D_\alpha^{(k,h)}$ be the set of values of d with $\lfloor \log_2 d \rfloor = \alpha$ such that any weak protocol can be completed in a full protocol for L^kG in $(k-1)\alpha + h$ steps more. The next result is an extension of Lemma 8.

Lemma 13. *If $d \in D_\alpha^{(k,h)}$ and $b(B(2^s, k-1)) = (k-1)s$, then $2^s d \in D_{s+\alpha}^{(k,h)}$.*

In [9] it is shown that $b(B(8,2)) = 3$. Moreover we have been able to complete the weak protocols for degrees 4, 8 and 16 into full protocols for L^3G in $2\alpha + 1$ time steps more. Thus we can state the following result.

Theorem 14. *Let G be a line digraph with degree $d = 2^\alpha$, $\alpha > 1$, such that there exists a broadcasting algorithm running in time t, then there exist a broadcasting algorithm in L^3G running in time $t + 3\alpha + 1$.*

6 Conclusions

We have described a constructive method to design good broadcasting protocols in line digraphs and in iterated line digraphs. The broadcasting time given by this method is in most cases better than with any other method known up to date, even considering the underlying graph of the line digraph (see [2]).

We summarize in the following table the running time for De Bruijn digraphs. The table shows for each value of d the running time in terms of $b(B(d,D)) = s_d D$. We have used the fact that we know how to construct protocols in L^3G for $d = 9$ (resp. $d = 11$) running in time $t + 10$ (resp. $t + 11$), and in L^4G for $d = 4$ (resp. $d = 8$) running in time $t + 9$ (resp. $t + 13$).

d	2	3	4	5	6	7	8	9	10	11	12
Previous bound	1.5	2	2.5	3	3.5	4	4.5	5	5.5	6	6
Undirected case	1.5	2	2.5	2.8	3	3.28	3.5	3.67	3.8	3.9	4
New result	1.5	2	2.25	2.5	3	3	3.25	3.33	3.5	3.66	4
Lower bound	1.44	1.80	2.11	2.38	2.62	2.82	3.01	3.17	3.32	3.46	3.59

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