



**HAL**  
open science

## Induced broadcasting algorithms in iterated line digraphs

Jean-Claude Bermond, Xavier Muñoz, Alberto Marchetti-Spaccamela

► **To cite this version:**

Jean-Claude Bermond, Xavier Muñoz, Alberto Marchetti-Spaccamela. Induced broadcasting algorithms in iterated line digraphs. Proc. Conference EUROPAR96, Aug 1996, Lyon, France. pp.313-324. ⟨hal-03838874⟩

**HAL Id: hal-03838874**

**<https://hal.science/hal-03838874v1>**

Submitted on 3 Nov 2022

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



HAL Authorization

# Induced Broadcasting Algorithms in Iterated Line Digraphs

Jean-Claude Bermond<sup>1</sup>, Xavier Muñoz<sup>2</sup> and Alberto Marchetti-Spaccamela<sup>3</sup>

<sup>1</sup> Project SLOOP I3S-CNRS/INRIA/Université de Nice-Sophia Antipolis  
930 route des Colles, F-06903 Sophia-Antipolis, France. (bermond@unice.fr)

<sup>2</sup> Departament de Matemàtica Aplicada i Telemàtica, Universitat Politècnica de Catalunya. Campus Nord C3, c/Gran Capità s/n, 08034 Barcelona, Spain.  
(xml@mat.upc.es)

<sup>3</sup> Dipartimento di Informatica e Sistemistica, Università di Roma "La Sapienza".  
via Salaria 113 - 00198 - Roma, Italy. (alberto@dis.uniroma1.it)

**Abstract.** We propose broadcasting algorithms for line digraphs in the telephone model. The new protocols use a broadcasting protocol for a graph  $G$  to obtain a broadcasting protocol for the graph  $L^k G$ , the graph obtained by applying  $k$  times, the line digraph operation to  $G$ . As a consequence improved bounds for the broadcasting time in De Bruijn, Kautz and Wrapped Butterfly digraphs are obtained.

## 1 Introduction

Considerable attention has been recently devoted to the dissemination of information in parallel computing, since the time needed to disseminate information among the processors in a network often determines the running time of the whole algorithm. Broadcasting is the process in which a node of the network (the originator) sends one piece of information to all members of the network by placing calls over the communication lines of the network. This is to be completed as quickly as possible subjected to the constraints of the considered model. This problem has been widely studied both for its theoretical and practical interest for different network configurations and under different models (see [7, 10] for recent surveys or the book [11]).

In this paper we are concerned with the telephone model, that is, each call involves only two vertices of the network and requires one unit of time (round); moreover we require that a vertex can participate in at most one call per unit of time by either sending or receiving a call and we restrict links to be one-way, that is, information will always flow in the same direction.

---

\* Work partially supported by the HCM Network of Excellence project MAP. Part of this work was developed while J-C.Bermond and X.Muñoz were visiting the University of Rome "La Sapienza". J-C. Bermond has been also supported by the French action RUMEUR of the GDR PRS, X. Muñoz by the Spanish Research Council (Comisión Interministerial de Ciencia y Tecnología, CICYT) under project TIC 92-1228-E and A. Marchetti-Spaccamela by the project MURST 40% Algoritmi e strutture di calcolo.

Following the literature we model the network topology by a digraph  $G = (X, U)$ ; a vertex  $x \in X$  corresponds to a node in the network, and an arc  $(x, y) \in U$ , corresponds to a link in which  $x$  is the sender and  $y$  is the receiver.  $b(G, x)$  represents the minimum number of time units (number of rounds) necessary to complete broadcasting from vertex  $x$ ;  $b(G)$  is defined as the maximum of  $b(G, x)$  taken over all vertices  $x$  of  $G$ . It is well known that for any digraph  $G$  with  $n$  vertices  $b(G) \geq \lceil \log_2 n \rceil$  and that the problem of determining  $b(G)$  is a NP-hard problem. Values of  $b(G)$  are known for many usual interconnection networks; however for Butterfly digraphs and related networks the order is still unknown.

In this paper we propose a general methodology for designing broadcasting algorithms in line digraphs and iterated line digraphs. In Section 3.1 we give a simple protocol for broadcasting in  $LG$  (the graph obtained by applying the line digraph operation to the graph  $G$ ) using a broadcast protocol in  $G$  where  $G$  is a regular digraph with degree  $d$ ; if broadcast in  $G$  can be performed in time  $t$ , then the algorithm performs broadcasting in  $LG$  in time  $t + \lceil \log_2 d \rceil + 1$ . In section 3.2 we show that if  $G$  satisfies an additional property, that we call  $A$ , then there exists a broadcasting protocol in  $LG$  that runs in time  $t + \lceil \log_2 d \rceil + 1$ .

By iterating the algorithms of Sections 3.1 and 3.2 we obtain protocols for iterated line digraphs ( $L(L^{n-1}G) = L^n G$ ). In Section 4.1 we show an improved algorithm for broadcasting in  $L^2G$  using a broadcasting protocol in  $G$ . For some values of  $d$ , the direct derivation of the protocol for  $L^2G$  has running time  $t + 2\lceil \log_2 d \rceil + 1$  if  $G$  satisfies property  $A$ . In Section 5 we sketch the extension to  $L^kG$ . For example we obtain for  $d = 2^\alpha$ ,  $\alpha > 1$ , a broadcasting algorithm in  $L^3G$  running in time  $t + 3\log_2 d + 1$ .

The iteration of the line digraph operation is a good method to obtain large digraphs with fixed degree and diameter. Besides, many other good properties are observed in such digraphs when used as models of communication networks ([3, 9]). In fact some of the best known families of digraphs are indeed line digraphs: De Bruijn, Kautz, Wrapped Butterfly, among others.

Property  $A$  is satisfied in particular if  $G$  is itself a line digraph. So we obtain new and better protocols for De Bruijn, Kautz and Butterfly networks. In the case of De Bruijn digraphs  $B(d, D)$  and Kautz digraphs  $K(d, D)$  (the De Bruijn/Kautz digraph with degree  $d$  and diameter  $D$ ) we obtain the following bounds

$$b(B(d, D)) \leq D(\log_2 d + f(d));$$

$$b(K(d, D)) \leq D(\log_2 d + f(d)) + 1.$$

where  $f(d)$  satisfies:  $\log_2(1 + \frac{1}{d}) \leq f(d) \leq 0.508$ .

This improves the best current known bounds both for digraphs ([3], [9]) and graphs ([2]).

## 2 Notation and Preliminaries

We refer to [5, 11] for basic definitions concerning graphs and digraphs and we recall basic definitions and properties of line digraphs (see also [6, 8]).

Given a directed graph  $G = (X, U)$ , if  $(x, y)$  is an arc, then  $x$  is *adjacent to*  $y$  and  $y$  is *adjacent from*  $x$ . We also say that arc  $(x, y)$  is *incident to*  $y$  and *incident from*  $x$ , and that arc  $u = (x, y)$  is adjacent to arc  $v = (y, z)$ . We denote by  $d^+(x)$  ( $d^-(x)$ ) the number of vertices adjacent from  $x$  (adjacent to  $x$ ); a digraph is said to be *d-regular* if  $d^+(x) = d^-(x) = d$  for any vertex  $x$ .

A *dipath* of length  $h$  from a vertex  $x$  to a vertex  $y$  is a sequence of vertices  $x = x_0, x_1, \dots, x_{h-1}, x_h = y$  where  $(x_i, x_{i+1})$  is an arc of  $G$ . A digraph is *strongly connected* if, for any couple of vertices  $x, y$ , there exists a dipath from  $x$  to  $y$ . The length of a shortest dipath from  $x$  to  $y$  is the *distance* from  $x$  to  $y$ . Its maximum value over all couples of vertices is the *diameter* of the digraph denoted  $D(G)$ .

Given a digraph  $G = (X, U)$  the *line digraph* operation allows to define a new digraph  $LG$  whose vertex set is in one to one correspondence with the set of arcs of  $G$ . Vertex  $u$  of  $LG$  representing the arc  $u = (x, y)$  is adjacent to the vertex representing the arc  $v$  if and only if  $v = (y, z)$ , that is arcs of  $LG$  represent dipaths of length 2 of  $G$ . It can be easily shown that if  $G$  is  $d$ -regular with  $n$  vertices then  $LG$  is  $d$ -regular with  $dn$  vertices; furthermore if  $G$  is a strongly connected digraph different from a directed cycle and  $d > 1$ , then the diameter of  $LG$  is the diameter of  $G$  plus one. We denote by  $L^k G$  the graph  $L(L^{k-1}G)$ .

Let  $G$  be a digraph with  $n$  vertices and  $LG$  the corresponding line digraph. The arcs of  $LG$  can be partitioned into  $n$  complete bipartite digraphs isomorphic to  $\overline{K}_{d,d}$  that are in one to one correspondence with vertices of  $G$ . If  $x$  is a vertex of a digraph  $G$  then we denote by  $B_x$  the corresponding bipartite digraph in  $LG$ . Note also that each vertex of  $LG$  belongs to exactly to two of such bipartite digraphs: in one case with in-degree 0 and in the other with out-degree 0.

An *arc-labeling* for a digraph  $G$  is a labeling of its arcs such that any two arcs incident to the same vertex have different labels and any two arcs incident from the same vertex have also different labels. If  $G$  is  $d$ -regular then it is always possible to obtain such an arc-labeling with  $d$  labels. Since we always consider  $d$ -regular digraphs we simply say an arc-labeling instead of an arc-labeling with  $d$  labels. Given a digraph with an arc-labeling, we identify the labels with the elements of  $\mathbf{Z}_d$  and we refer to the arcs incoming from or outgoing to the same vertex by their labels.

The De Bruijn digraph  $B(d, D)$  is a  $d$ -regular digraph with diameter  $D$  defined as follows: vertices are the strings of length  $D$ ,  $z_1 z_2 \dots z_D$ , where  $z_i \in \mathbf{Z}_d$ ,  $i = 1, 2, \dots, D$ , and vertex  $z_1 z_2 \dots z_D$  is adjacent to vertices  $z_2 \dots z_D \alpha$ ,  $\alpha = 0, 1, 2, \dots, d-1$ . In [6] it is shown that  $B(d, D) = L^{D-1} K_d^+$ , where  $K_d^+$  denotes the complete symmetric digraph on  $d$  vertices with an additional loop in each vertex.

The Kautz digraph  $K(d, D)$  is defined analogously: vertices are all strings of length  $D$ ,  $z_1 z_2 \dots z_D$ ,  $z_i \in \mathbf{Z}_{d+1}$ ,  $i = 1, 2, \dots, D$ , such that two consecutive symbols cannot be equal. Vertex  $z_1 z_2 \dots z_D$  is adjacent to vertices  $z_2 \dots z_D \alpha$ ,  $\alpha \in \mathbf{Z}_{d+1}$ ,  $\alpha \neq z_D$ . The digraph  $K(d, D)$  is  $d$ -regular and has diameter  $D$ ; moreover  $K(d, D) = L^{D-1} K_{d+1}^*$ , where  $K_{d+1}^*$  denotes the complete symmetric digraph on  $d+1$  vertices [6].

The Directed Wrapped Butterfly  $WBF(d, n)$  has as vertex set the couples  $(x, l)$  where  $x$  is a string of length  $n$  ( $x_{n-1} x_{n-2} \dots x_0$ ),  $x_i \in \mathbf{Z}_d$ ,  $i = 0, 1, \dots, n-1$ , and  $l \in \mathbf{Z}_n$ . Vertex  $(x_{n-1} x_{n-2} \dots x_0, l)$  is adjacent to vertices  $(x_{n-1} \dots x_{l+1} \alpha x_{l-1} \dots x_0, l + 1)$ ,  $\alpha \in \mathbf{Z}_d$ . In [4] it is shown that  $WBF(d, n) = L^n dC_n$ , where  $dC_n$  denotes the

graph obtained from the directed cycle with  $n$  vertices by replacing each arc with  $d$  parallel arcs.

### 3 Broadcasting in line digraphs

In this section we will show that, given a broadcasting algorithm in a digraph  $G$  running in time  $t$ , it is possible to construct a broadcasting protocol in  $LG$  that runs in time  $t + \lceil \log_2 d \rceil + 1$ . The running time is  $t + \lceil \log_2 d \rceil + 1$  if  $G$  satisfies certain properties.

#### 3.1 A broadcasting protocol in a line digraph

A broadcasting algorithm in a digraph  $G = (X, U)$  with originator  $r$  can be described via the broadcast tree  $T(r)$  with root  $r$  and vertex set  $X$ .  $T(r)$  contains the arcs through which the information has been broadcasted; since each vertex in  $T(r)$  has in-degree 1, then  $T(r)$  is an arborescence.

Any broadcasting algorithm  $P$  with originator  $r$  induces a partial broadcasting in  $LG$  with originator any vertex in  $LG$  representing an arc  $(r', r)$ , which informs those vertices of  $LG$  corresponding to the arcs of the broadcast tree of  $G$ . If  $P$  is a broadcasting protocol running in time  $t$  in  $G$ , then at time  $t$ , for any  $x$ , there is at least one vertex  $u = (\cdot, x)$  in  $LG$ , where  $\cdot$  stands for an undetermined symbol, that has been informed (namely vertex corresponding to  $(r', r)$  and the vertices that correspond to the arcs of the broadcast tree  $T(r)$ ).

During round  $t + 1$ , for any  $x$  in  $G$ , the informed vertex  $u = (\cdot, x)$  in  $LG$  sends the information, denoted by  $i$ , to vertex  $(x, \cdot)$  of label 1. Since the labeling of the arcs is an arc-labeling in  $G$ , for all  $x'$ , vertices  $(\cdot, x')$  of label 1 have been informed. At round  $t + 2$ , all vertices  $(\cdot, x)$  of label 1 send  $i$  to vertices  $(x, \cdot)$  of label 2. Thus at the end of this round all vertices of  $LG$  with labels 1 or 2 will know  $i$ ; so they will be able to send  $i$  to vertices of  $LG$  of labels 3 and 4 during round  $t + 3$ . Following this pattern, we claim that at the end of round  $t + k$ , vertices of  $LG$  with label  $i$ ,  $1 \leq i \leq 2^{k-1}$ , will know  $i$ . The proof is by induction on  $k$ ; it is true for  $k = 1$  and  $k = 2$ . Suppose it is true at time  $t + k$ , then at time  $t + k + 1$  for any  $x$  the vertex  $(\cdot, x)$  of  $LG$  of label  $i$  informs the vertex  $(x, \cdot)$  of label  $i + 2^{k-1}$ . In this way all vertices with label  $i$ ,  $2^{k-1} + 1 \leq i \leq 2^k$ , get the information. Since  $G$  is  $d$ -regular at round  $t + \lceil \log_2 d \rceil + 1$ , the protocol is completed. So we can state the following theorem.

**Theorem 1.** *Given a regular digraph  $G$  with degree  $d$ , such that broadcast in  $G$  can be completed in  $t$  rounds, then there exists a broadcast protocol in  $LG$  that runs in time  $t + \lceil \log_2 d \rceil + 1$ .*

#### 3.2 An improved protocol in a line digraph

In the previous protocol we have not used the facts that  $i$  arrived at vertex  $x$  of  $G$  on some arc of a specific label and that  $i$  might arrive before time  $t$ . For example, suppose that it arrives to an arc  $(\cdot, x)$  not of label 1, then before round  $t + 2$  the

information  $i$  has reached two vertices of  $LG$ . Therefore, at round  $t + 2$ ,  $i$  could have been sent on two vertices of  $LG$  instead of only one; analogously if  $i$  reaches  $x$  at time  $t - h$  in the successive  $h$  rounds  $i$  can be sent on  $h$  new arcs.

In the sequel we obtain an improved broadcasting protocol in  $LG$  if  $G$  admits a suitable arc-labeling. To this aim let us consider the broadcasting tree  $T(r)$  given by a broadcasting algorithm  $P$  with originator vertex  $r$  and let us label its arcs in the following way: for each vertex, the last arc used to send  $i$  is labeled 0, the penultimate one is labeled  $-1$  (where we perform addition modulo  $d$ ), the preceding one is labeled with  $-2$  and so on. We will call such a labeled tree a *labeled broadcasting tree*. If it is possible to make an arc-labeling of the digraph  $G$  in which the labels of the arcs of  $T(r)$  are the ones in the labeled broadcasting tree, we say that this arc-labeling is *consistent* with the labeling of the broadcasting tree. We say that a vertex of  $G$  is *of kind  $k$*  (in the protocol  $P$ ) if it has received  $i$  though an arc of label  $-k$  (modulo  $d$ ).

We note the following immediate lemma.

**Lemma 2.** *If a vertex is of kind  $k$ , then it has been informed in the protocol  $P$  at time  $t - k$  or before.*

**Definition 3.** A  $d$ -regular digraph  $G$  satisfies property  $A$  if there exists a broadcasting protocol such that, for any vertex  $r$ , there exists an arc-labeling of  $G$  that is consistent with the labeling of the broadcasting tree  $T(r)$ .

**Lemma 4.** *If  $G$  is a line digraph then it satisfies property  $A$ .*

*Proof.* If  $G$  is a line digraph, due to the partition of the arcs into complete bipartite digraphs, it is possible to label its arcs by arc-labeling independently the bipartite subdigraphs.

Let  $T(r)$  be the broadcasting tree associated to the broadcasting protocol  $P$  in  $G$  and for a given bipartite digraph  $B_x$  of the decomposition of  $G$ , let  $V_1$  and  $V_2$  be the partition of the vertices of  $B_x$ . For each vertex  $y$  of  $G$  there exists exactly one arc entering in  $y$  that belongs to  $T(r)$  (the arc through which  $i$  has arrived). Let  $\delta_T(x)$  be the out-degree of  $x$  in  $T$  (that is the number of vertices which have been informed by  $x$ ). Now we rank the vertices of  $V_1$  and  $V_2$  as follows.

Choose a vertex  $x$  in  $V_1$ , with  $\delta_T(x) = \delta \geq 1$ ; label this vertex  $x_0$  and label its out-neighbors in  $V_2$  with  $y_0, y_1, \dots, y_{\delta-1}$  in the order they have been informed, i.e. according to their kind (the out-neighbor of kind 0 is labeled  $y_0$ , the one of kind 1,  $y_1$  and so on. In general suppose that at some step of the algorithm we have used all labels till  $y_i$  for vertices in  $V_2$ .

If  $i < d - 1$ , then choose a vertex  $x$  in  $X$  with out-degree  $\delta_T(x) = \delta_i \geq 1$ ; label this vertex  $x_{i+1}$  and its out-neighbors  $y_{i+j}$ ,  $1 \leq j \leq \delta_i$ , according to their kind. If  $i = d - 1$ , then there might be vertices in  $V_1$  that have not been labeled; we label them in any order using the remaining labels. Note that the labeling is not unique (in fact it depends on the order in which vertices in  $V_1$  have been chosen).

Let us now consider the following arc-labeling of  $G$ : arc  $(x_i, y_j)$  has label  $(i - j) \bmod d$ . It is immediate to see that it is an arc-labeling which furthermore is consistent with the labeling of  $T(r)$ .

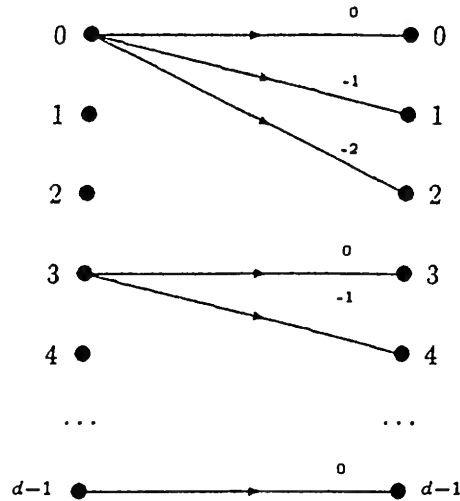


Fig. 1. Labeling of the vertices of a bipartite digraph. Drawn arcs are those in the broadcasting tree  $T$  of  $G$

**Lemma 5.** *Given a  $d$ -regular digraph  $G$  that satisfies property  $A$  and a broadcasting protocol  $P$  running in  $G$  in time  $t$ , then there exists a protocol in  $LG$  such that at time  $t + h$*

- i) for a vertex of kind  $k$ ,  $-k \in \{1, 2, \dots, 2^h - 1\}$ , all its outgoing arcs have been informed;*
- ii) for a vertex of kind  $k$ ,  $-k \notin \{1, 2, \dots, 2^h - 1\}$ , all the arcs of labels  $j$ ,  $1 \leq j \leq 2^h - 1$ , have been informed.*

*Proof.* The proof is by induction on  $h$  and uses the following protocol  $Q$  in  $LG$ . Until time  $t$  we will use the broadcasting algorithm  $P$  in  $G$  but with the following modification. Let  $x$  be a vertex of  $G$  of kind  $k$ . By lemma 2, vertex  $x$  has received  $i$  at time less or equal to  $t - k$ . If we consider protocol  $P$  two possible cases arise: either  $x$  has sent  $i$  on at least  $k$  arcs or  $x$  has sent  $i$  to  $k'$  arcs,  $k' < k$ , and it has been idle for at least  $k - k'$  rounds before  $t$ . In the modified protocol that we use until time  $t$  we will use  $k - k'$  rounds to send the information to  $k - k'$  arcs in such a way that, at time  $t$ ,  $x$  has sent the information through arcs  $0, -1, \dots, -k + 1$ .

The specification of protocol  $Q$  is completed as follows. At round  $t + 1$  each vertex  $(\cdot, x)$  of  $LG$  that corresponds to an arc of the broadcasting tree sends  $i$  to its neighbor of label 1. At the end of this round, if  $x \in V(G)$  is of kind  $d - 1$  then  $(\cdot, x)$  has informed all its neighbors in  $LG$  (in rounds before  $t$ , the vertices corresponding to arcs of label  $-j$ ,  $j = 0, 1, \dots, d - 2$ , and, at round  $t + 1$ , arc of label  $1 = -(d - 1)$ ). If  $x$  is of kind  $k$  different from  $d - 1$  then there exist at least two vertices of  $LG$   $(y_1, x)$  and  $(y_{-k}, x)$  that have received  $i$  (these vertices correspond to arcs with labels 1 and  $-k$ ); in this case these two vertices can send  $i$  to the two vertices  $(x, \cdot)$  with labels 2 and 3 during round  $t + 2$ ; note that in the previous protocol at round  $t + 2$ ,  $i$  was sent to only one vertex  $(x, \cdot)$ .

Now we can start the induction. The lemma is satisfied for the cases  $h = 0$  and  $h = 1$ . Suppose the lemma is true for  $h_0$  and let us consider vertex  $x$  of kind

$k$ . Then, at round  $t + h_0 + 1$ , if  $-k \in \{1, 2, \dots, 2^{h_0} - 1\}$  the lemma is true by induction. If  $-k \notin \{1, 2, \dots, 2^{h_0} - 1\}$ ,  $x$  has received at the end of round  $t + h_0$   $i$  on at least  $2^{h_0}$  arcs: arc of label  $-k$  (before time  $t - k$ ) and arcs of label  $j$ ,  $j = 1, 2, \dots, 2^{h_0} - 1$  (at rounds  $t, t + 1, \dots, t + h_0$ ). Therefore  $x$  can send  $i$  on  $2^{h_0}$  arcs of label  $j$ ,  $j = 2^{h_0}, \dots, 2^{h_0+1} - 1$ . So if  $-k \in \{1, 2, \dots, 2^{(h_0+1)} - 1\}$  all outgoing arcs of  $x$  have been informed; otherwise the set of informed arcs is the claimed one.

The proof of the following theorem is an immediate consequence of the lemma and of the fact that  $t + \lceil \log_2(d + 1) \rceil = t + \lfloor \log_2(d) \rfloor + 1$ .

**Theorem 6.** *Given a  $d$ -regular digraph  $G$  that satisfies property  $A$  and such that broadcast can be performed in time  $t$ , then there exists a protocol in  $LG$  that runs in time  $t + \lfloor \log_2(d) \rfloor + 1$ .*

## 4 Broadcasting in $L^2G$

Given a  $d$ -regular digraph  $G$  with  $d = 2^\alpha(1 + \beta)$ ,  $0 \leq \beta < 1$ , that satisfies property  $A$ , in which there exists a broadcasting algorithm running in time  $t$ , by applying twice the protocol for broadcasting in a line digraph, we obtain a broadcasting algorithm in  $L^2G$  running in time  $t + 2\alpha + 2$ . The aim of this section is to design a broadcasting protocol in  $L^2G$  directly from the protocol in  $G$  running in time  $t + 2\alpha + 1$ ; this is possible if  $\beta$  is not too large.

Recall that, given a graph  $G = (X, U)$ , the vertex set of  $L^2G$  is the set of all possible dipaths of length 2 in  $G$ , and that broadcasting in  $L^2G$  is equivalent to broadcast the information through all paths of length 2, or equivalently, every arc of  $G$  informs every other arc adjacent in  $G$ .

Observe that, in the protocol for  $LG$  described in the previous section, most of the vertices are idle during the last unit of time. This happens because all their neighbors have been already informed and it should not make sense to send again the information. Nevertheless, if we iterate the process in order to obtain a broadcasting algorithm in  $L^2G$ , each arc must send  $i$  to any other arc and, therefore, the fact described before represents a waste of time.

### 4.1 The design of a protocol from small values of $d$

**Lemma 7.** *Given a  $d$ -regular graph  $G$  that satisfies property  $A$  and a broadcast protocol in  $G$  that runs in time  $t$ , then there exists a broadcast protocol in  $L^2G$  such that, at time  $t + \alpha$ ,  $\alpha = \lfloor \log_2 d \rfloor$ , the information is arrived to all vertices that correspond to pairs of arcs of  $G$  with labels  $(i, i + 2^j)$ , where  $1 \leq i \leq 2^{\alpha-1} - 1$  and  $\lfloor \log_2 i \rfloor < j \leq \alpha - 1$ , and also to vertices that correspond to a not completely specified pair of arcs with labels  $(\cdot, 2^j)$ , where  $0 \leq j \leq \alpha - 1$  and  $\cdot$  denotes the undetermined label of an arc (depending on the broadcast algorithm in  $G$ ).*

*Proof.* We use the protocol  $Q$  described in Lemma 5 in the following precise way. We know that, for any vertex  $x$ , at time  $t + h$ , the information  $i$  is arrived on all the arcs  $i$ ,  $1 \leq i \leq 2^h - 1$ , and on the arc  $-k$  (if  $x$  is a vertex of kind  $k$ ). At time

$t + h + 1$ , we impose that, if it has not already been done, arc  $i$  sends the information to  $i + 2^h$  and arc  $-k$  to  $2^h$ .

Note that if  $-k \in \{1, 2, \dots, 2^h - 1\}$  there is no conflict as arc  $-k$  has already sent  $i$  to  $2^h$  and  $2^h + k'$  with  $k' = -k \bmod d$ ,  $k' \geq 0$ . So the lemma follows by induction and considering the case  $h = \alpha = \lfloor \log_2 d \rfloor$ .

Let us call the above protocol a *weak protocol*. If, furthermore, we suppose that the undetermined  $\cdot$  is 0 we say that the protocol is a *strong protocol*. In this case arc labeled 0 has informed arcs labeled  $2^j$ ,  $0 \leq j \leq \alpha - 1$ , or equivalently all the pairs of arcs  $(0, 2^j)$  have been informed.

Let  $D_\alpha$  be the set of values of  $d$ , with  $\lfloor \log_2 d \rfloor = \alpha$ , such that any weak protocol obtained at time  $t + \alpha$  can be completed in a full protocol for  $L^2G$  in  $\alpha + 1$  more steps. Since we can always find a weak protocol in time  $t + \alpha$  for any  $d$  in  $D_\alpha$ , it follows that if  $d \in D_\alpha$  then there exists a protocol in  $L^2G$  running in time  $t + 2\alpha + 1$ .

Analogously let  $E_\alpha$  be the set of values of  $d$  with  $\lfloor \log_2 d \rfloor = \alpha$  such that any strong protocol obtained at time  $t + \alpha$  can be completed as in the case of a weak protocol ( $G$  is a  $d$ -regular digraph) in  $\alpha + 1$  more steps. Notice that  $D_\alpha \subseteq E_\alpha$ .

- Lemma 8.**
1. If  $d \in D_\alpha$  then  $2d \in D_{\alpha+1}$ .
  2. If  $d \in D_\alpha$  and  $d + 1 \in E_\alpha$  then  $2d + 1 \in D_{\alpha+1}$ .
  3. If  $d \in E_\alpha$  then  $2d \in E_{\alpha+1}$ .

*Proof.* 1) Let  $d \in D_\alpha$ , thus  $\alpha = \lfloor \log_2 d \rfloor$  and  $\alpha + 1 = \lfloor \log_2 2d \rfloor = \lfloor \log_2(2d + 1) \rfloor$ . Let us recall that there exists a weak protocol for  $G$  with degree  $2d$  as it is explained in the protocol for  $LG$ . Let us consider the subdigraphs  $G_e$  and  $G_o$  induced by the even and odd arcs of  $G$  respectively. The protocol for  $G_e$  (where we label arc  $2i$  as  $i'$ ) is exactly a weak protocol for  $G_e$  (with degree  $d$ ) completed in time  $t + \alpha + 1$ . Indeed the information is arrived on all the arcs  $2i, 2 \leq 2i \leq 2^{\alpha+1} - 1$ , that is,  $1 \leq i' \leq 2^\alpha - 1$ , and arc  $i'$  has informed arcs  $i' + 2^j$  with  $\lfloor \log_2 i' \rfloor + 1 < j \leq \alpha - 1$ . Similarly the protocol induced in  $G_o$  (where we identify  $2i + 1$  with  $i''$ ) is a strong protocol for  $G_o$ .

The protocol in  $G_e$  and in  $G_o$  can be completed in  $\alpha + 1$  steps more since  $d \in D_\alpha$ . Thus at time  $t + 2\alpha + 2$  all the vertices in  $L^2G$  of the form  $(2i, 2j)$  and  $(2i + 1, 2j + 1)$ ,  $0 \leq i, j \leq d - 1$  know the information. Using an extra step  $(2i, 2j)$  can inform  $(2j, 2i + 1)$  and  $(2i + 1, 2j + 1)$  can inform  $(2j + 1, 2i)$  completing the protocol in time  $2(\alpha + 1) + 1$ . Thus  $2d \in D_{\alpha+1}$ .

2) Let us consider subdigraphs  $G_e$  and  $G_o$  as in the case before, but labeling arc  $2d$  as  $2d + 1$ , that is considering  $2d$  as if it were an odd number, and thus belonging to  $G_o$ . The protocols induced in  $G_e$  and  $G_o$  are a weak protocol and a strong protocol respectively. Since  $d \in D_\alpha$  and  $d + 1 \in E_\alpha$  it is possible for the vertices of the form  $(2i, 2j)$ ,  $0 \leq i, j \leq d - 1$ , and  $(2i + 1, 2j + 1)$ ,  $0 \leq i, j \leq d$  to be informed at time  $2\alpha + 2$ . Nevertheless let us change a bit the protocol in  $G_o$  in order that for each  $j$ ,  $0 \leq j \leq d - 1$ , one of the informed vertices during last step, say  $(2i + 1, 2j + 1)$  is replaced by the vertex  $(2i + 1, 2j)$ , that is  $2i + 1$  informs  $2j$  instead of informing  $2j + 1$ .

Notice that at time  $2\alpha + 2$  arcs in  $G_e$  have been informed by  $d + 1$  other arcs, that is  $(2i, 2j)$  know the information for all  $i, j \leq d - 1$  as well as  $(2k + 1, 2j)$  for some  $k$ . Thus  $(2j, 2i + 1)$  can get the information in the next step for all  $i \leq d, j \leq d - 1$ .

Also at time  $2\alpha + 2$  the arcs in  $G_o$  have been informed at least  $d$  times, that is, for every  $j \leq d$  there are at least  $d$  vertices of the form  $(2i + 1, 2j + 1)$  which are informed, and thus they can inform the  $d - 1$  remaining vertices of the form  $(2j + 1, 2i)$ ,  $0 \leq i \leq d - 1$  which were not informed at time  $2\alpha + 2$  and the vertex  $(2i + 1, 2j + 1)$  which was neither informed at that time. Hence the protocol can be completed in  $2(\alpha + 1) + 1$  steps, and so  $2d + 1 \in D_{\alpha+1}$ .

3) This case is analogous to Case 1 and it is omitted.

**Lemma 9.**  $4 \in D_2$  and  $5 \in E_2$ .

*Proof.* In the following tables we show an algorithm for completing the protocol in  $L^2G$ . In the tables columns represent successive rounds and rows represent labels of arcs incident to any vertex of  $G$ . At each round it is shown what must be done by each incoming arc. For instance in table a) at time  $t + 3$  all arcs inform 0; thus 0 will have been informed by three different arcs and at time  $t + 4$  it can inform 3 different arcs (with labels 1, 2, 3).

Recall that, by Lemma 7, at time less than or equal to  $t + 2$  arcs 1, 2 and 3 have been informed and moreover the incoming arc 1 has informed the outgoing arc with label 3.

	$\leq t + 2$	$t + 3$	$t + 4$	$t + 5$
0			1,2,3	0
1	3	0	1	2
2		0	2	1,3
3		0	3	1,2

a) table for  $4 \in D_2$

	$\leq t + 2$	$t + 3$	$t + 4$	$t + 5$
0	1,2	4	3	0
1	3	4	2	0,1
2		2	1,4	0,3
3		3	1,4	0,2
4			2,3	0,1,4

b) table for  $5 \in E_2$

It is not difficult to construct analogous tables to show that  $10 \in D_3$  and  $11 \in E_3$ , but for a lack of space we leave them to the full paper. This result and the ones shown in Lemmas 8 and 9, imply that given  $d$ ,  $d = 2^\alpha(1 + \beta)$  and  $\alpha \geq 3$ , then  $d \in D_\alpha$  if  $\beta < 3/8$  and  $d \in E_\alpha$  if  $\beta \leq 3/8$ .

We use a different analysis in order to obtain a broadcast protocol for  $L^2G$  that runs in time  $t + 2\alpha + 1$ , when  $d = 2, 5$ .

**Lemma 10.** Given a  $d$ -regular line digraph with  $d = 2$  or  $d = 5$  there exists a protocol in  $L^2G$  running in time  $t + 3$ , respectively  $t + 5$ .

*Proof.* We give only the proof for  $d = 2$  (the proof for  $d = 5$  uses analogous ideas). Let us consider the table below:

It seems to be a problem at time  $t + 3$ , because the arc of label 0 incident to vertex of kind 1 might have received the information only once, and thus cannot send twice. But if we have a little more care in our analysis we can realize that in the case it has received the information only once (i.e. the incoming arc of label 0 in a vertex of kind 1 is an outgoing arc of a vertex of kind 1) it has received the information in time less or equal than  $t$ , and thus can have sent this information previous to  $t + 3$ . In other case, (i.e. the incoming arc of label 0 in a vertex of kind 1 is an outgoing arc of a vertex of kind 0), it has received the information twice, and

so it can send twice the information. In any case it is possible to have a protocol for  $L^2G$  running in time  $t + 3$ , which is  $t + 2\alpha + 1$ .

		$\leq t$	$t + 1$	$t + 2$	$t + 3$
kind 0	0		1	0	
	1			0	1
kind 1	0				0,1
	1	0	1		

The following theorem is the main result of this section and is a straightforward consequence of previous lemmas.

**Theorem 11.** *Given a  $d$ -regular digraph  $G$ ,  $d = 2^\alpha(1+\beta)$  and  $\beta < 3/8$ , that satisfies property A and such that broadcast can be performed in time  $t$  then there exists a protocol in  $L^2G$  that runs in time  $t + 2\lfloor \log_2(d) \rfloor + 1$ .*

It is possible to improve the value of  $\beta$  that appears in Theorem 11 by providing tables analogous to the tables of Lemma 9 for larger values. We leave this extension to the full paper. For example in the full paper we prove that  $44 \in D_5$  and  $45 \in E_5$  and so we have a theorem analogous to Theorem 11 with  $\beta = .13/32$ .

We now show an upper bound for the value of  $\beta$  for which it possible to get this improvement by the proposed methodology. Namely it cannot be applied if  $\beta > \sqrt{2} - 1$ . In fact the method uses for the first  $t$  rounds a broadcast protocol in  $G$ ; it follows that at time  $t$  we know that only  $n$  vertices in  $L^2G$  have got the information. Since at each subsequent round the number of vertices knowing the information at most double, it follows that, at time  $t + 2\alpha + 1$ , the number of vertices being informed is at most  $n2^{2\alpha+1}$ . On the other hand the number of vertices in  $L^2G$  is  $n2^{2\alpha}(1+\beta)^2$ . The bound is obtained by comparing both expressions. Note that  $13/32 = 0.406..$  is very close to  $\sqrt{2} - 1 = 0.414..$

## 4.2 Broadcasting in some families of digraphs

We consider the following families of iterated line digraphs: De Bruijn  $B(d, D)$ , Kautz  $K(d, D)$  and Directed Wrapped Butterfly  $\overline{WBF}(d, n)$ . Since  $K_d^+$ ,  $K_{d+1}^*$  and  $dC_n$  (the first cases) satisfy property A then all these families satisfy property A as well and the following corollary of theorems 6, 11 easily follows.

**Corollary 12.** *Let  $d = 2^\alpha(1 + \beta)$ ,  $\alpha > 4$ , and let  $c_d = 1/2$  if  $\beta < 13/32$  and  $c_d = 1$  otherwise. Broadcasting in De Bruijn, Kautz and Wrapped Butterfly digraphs can be performed in time*

1.  $b(B(d, D)) \leq D(\lfloor \log_2 d \rfloor + c_d)$
2.  $b(K(d, D)) \leq D(\lfloor \log_2 d \rfloor + c_d) + 1$
3.  $b(\overline{WBF}(d, n)) \leq n(\lfloor \log_2 d \rfloor + c_d) + n - 1$ .

*Remark.* For the De Bruijn digraph the broadcasting time can be written as

$$b(B(d, D)) \leq D(\log_2 d + f(d))$$

where  $f(d)$  is a function bounded by  $\log_2(1 + (1/d)) \leq f(d) \leq 0.508..$

The lower bound is obtained for  $d = 2^\alpha - 1$  and the upper one is obtained for  $\beta = 13/32$ . Nevertheless this result can be improved using protocols in  $L^k G$  as it will be sketched in the next section.

## 5 Protocol in $L^k G$

The results of the previous section can be extended by showing that, for some values of  $d$ , it is possible to construct a protocol in  $L^k G$  running in time  $t + k[\log_2 d] + h$ ,  $h < k$  directly from the broadcasting algorithm in  $G$ . In the following we omit proofs. First of all we observe that if  $G$  is a  $d$ -regular line digraph  $G$  with  $d = 2^\alpha(1 + \beta)$ ,  $0 \leq \beta < 1$ , with a broadcasting algorithm running in time  $t$ , it is possible to construct a broadcasting algorithm in  $L^k G$  running in time  $t + k\alpha + h$ , with this method, only if  $1 + \beta \leq 2^{\frac{h}{k}}$ .

Furthermore we can extend the notion of a weak protocol given in Section 4 as follows: let  $D_\alpha^{(k,h)}$  be the set of values of  $d$  with  $[\log_2 d] = \alpha$  such that any weak protocol can be completed in a full protocol for  $L^k G$  in  $(k-1)\alpha + h$  steps more. The next result is an extension of Lemma 8.

**Lemma 13.** *If  $d \in D_\alpha^{(k,h)}$  and  $b(B(2^s, k-1)) = (k-1)s$ , then  $2^s d \in D_{s+\alpha}^{(k,h)}$ .*

In [9] it is shown that  $b(B(8, 2)) = 3$ . Moreover we have been able to complete the weak protocols for degrees 4, 8 and 16 into full protocols for  $L^3 G$  in  $2\alpha + 1$  time steps more. Thus we can state the following result.

**Theorem 14.** *Let  $G$  be a line digraph with degree  $d = 2^\alpha$ ,  $\alpha > 1$ , such that there exists a broadcasting algorithm running in time  $t$ , then there exist a broadcasting algorithm in  $L^3 G$  running in time  $t + 3\alpha + 1$ .*

## 6 Conclusions

We have described a constructive method to design good broadcasting protocols in line digraphs and in iterated line digraphs. The broadcasting time given by this method is in most cases better than with any other method known up to date, even considering the underlying graph of the line digraph (see [2]).

We summarize in the following table the running time for De Bruijn digraphs. The table shows for each value of  $d$  the running time in terms of  $b(B(d, D)) = s_d D$ . We have used the fact that we know how to construct protocols in  $L^3 G$  for  $d = 9$  (resp.  $d = 11$ ) running in time  $t + 10$  (resp.  $t + 11$ ), and in  $L^4 G$  for  $d = 4$  (resp.  $d = 8$ ) running in time  $t + 9$  (resp.  $t + 13$ ).

$d$	2	3	4	5	6	7	8	9	10	11	12
Previous bound	1.5	2	2.5	3	3.5	4	4.5	5	5.5	6	6
Undirected case	1.5	2	2.5	2.8	3	3.28	3.5	3.67	3.8	3.9	4
New result	1.5	2	2.25	2.5	3	3	3.25	3.33	3.5	3.66	4
Lower bound	1.44	1.80	2.11	2.38	2.62	2.82	3.01	3.17	3.32	3.46	3.59

**Acknowledgments** The authors want to thank S. Perennes for his help.

## References

1. J-C. Bermond, P. Hell, A. L. Liestman, J. G. Peters, "Broadcasting in bounded degree graphs", *SIAM J. Discrete Math.*, 5(1), (1992), pp. 10-24.
2. J-C. Bermond, S. Perennes, "Efficient broadcast protocols on the de Bruijn and similar networks". In *Proceedings of SIROCCO'95*, Olympia, (1995), International informatics series 2, Carleton U. Press, (1995), pp. 199-204.
3. J-C. Bermond, C. Peyrat, "Broadcasting in De Bruijn networks", *Proc. of the 19th S-E Conf. on Combin., Graph Theory and Comp.*, Florida, (1988), pp. 283-292.
4. J-C. Bermond, E. Darrot, O. Delmas, S. Perennes, "Hamilton circuits in directed butterfly networks", Tech. Rep. 95-47, I3S, Univ. Nice S.A., (1995), submitted to *Discrete Applied Mathematics*.
5. G. Chartrand and L. Lesniak, *Graphs and Digraphs* Wadsworth, Monterrey, (1986).
6. M.A. Fiol, I. Alegre, J.L.A. Yebra, "Line digraph iterations and the  $(d, k)$  problem", *IEEE Trans. Comp.*, C-33, (1984), pp. 400-403.
7. P. Fraignaud, E. Lazard, "Methods and problems of communication in usual networks", *Discrete Appl. Math.*, 53, (1994), pp. 79-133.
8. R.L. Hemminger et al. *Selected topics in Graph Theory*, vol.1, Academic Press (1983).
9. M.C., Heydemann, J. Opatrny, D. Sotteau, "Broadcasting and spanning tree in De Bruijn and Kautz networks", *Discrete Appl. Math.*, 37-38, (1992), pp. 297-317.
10. J. Hromkovic, R. Klasing, B. Monien, R. Peine, "Dissemination of information in interconnection networks (broadcasting and gossiping)", in *Combinatorial Network Theory*, D-Z. Du, D.F. Hsu (Eds.), Kluwer Academic Publ., (1996).
11. Jean de RUMEUR, *Communications dans les réseaux de processeurs*, Masson, Paris, 1994 (english translation to appear).