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► To cite this version:

Benoit Nennig, E. Perrey-Debain, Martin Ghienne. Localization of high order exceptional points : applications to acoustic waveguides. 16ème Congrès Français d'Acoustique, Apr 2022, Marseille, France. hal-03838635

HAL Id: hal-03838635

<https://hal.science/hal-03838635>

Submitted on 3 Nov 2022

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16^{ème} Congrès Français d'Acoustique
11-15 Avril 2022, Marseille

Localization of high order exceptional points : applications to acoustic waveguides

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In this work, a new method is presented in order to locate high order exceptional points (EP) arising in eigenvalue problems. EP correspond to a particular tuning of some complex-valued parameters which render the problem degenerate. These non-Hermitian degeneracies have raised considerable attention in the scientific community as these can have a great impact in a variety of physical problems (PT-symmetry, thermo-acoustic or fluid-structure instability, etc.) and their numerical solution. For applications dealing with dissipative acoustic waveguides, strong modal attenuation can be achieved close to EP and a maximum of attenuation occurs at EP of high order corresponding to the coalescence of more than two modes. The method is based on the automatic computation of the successive derivatives of some selected eigenpairs with respect to the parameters so that, after recombination, regular functions can be constructed. This algebraic manipulations permit to build a reduced order model allowing i) to quickly solve the eigenvalue problem for other parameters values, ii) to follow modal branches, iii) to locate higher order EPs. The method is applied to the particular case of a circular duct with a locally reacting liner at its wall which admittance varies with azimuthal position. This particular application allows to gradually test the proposed method with EP of increasing order.

1 Introduction

Acoustic treatments are often used to attenuate sound propagation in guiding structures generally encountered in HVAC, exhaust devices, and aircraft engines. In many applications, the walls of the waveguide are acoustically treated with absorbing materials, and it is usual for the analysis to distinguish between two classes of liners depending whether the latter is locally like perforated plate bounded with honeycomb or non-locally reacting like porous materials. In both cases, the propagation of sound waves is best described in terms of duct acoustic modes and the existence of exceptional point (EP) is known to be connected with very strong modal attenuation [18, 2, 21, 15, 14]. At the EP, eigenvalues and associated eigenvectors coalesce at a branch point singularity in the parametric space. Moreover, in the vicinity of the EP, perturbation analysis show that eigenvalues take the form of a Puiseux series [6] i.e. a series in fractional power of the parameter. From practical point of view, this strong attenuation can be explained by a transmission dominated by surface waves [2, 21].

Eigenvalue problems are ubiquitous in all physics branch, parametric eigenvalue problems have thus been the subject of intensive research. In acoustic waveguides, the usual parameters can be the frequency or the wall impedance. Most approaches are based on first or second perturbation order [8]. Recently, high order perturbation methods have been proposed to optimize the numerical efficiency when the parameter appears explicitly in the numerical model. In [12, 5] the bordered matrix [1] is used for obtaining the derivatives and the adjoint vector is used in [13, 9]. It is noteworthy that the bordered matrix approach is limited (at least in this form) to simple eigenvalue whereas [13] extend the adjoint method for semisimple eigenvalue, ie a repeated eigenvalue with different eigenvectors. However, high order Taylor expansion of the eigenvalue is still limited by the presence of branch point singularity. In [12], it has been proposed to recombine the Taylor expansion of the eigenvalue to build analytic functions which are not affected by the singularity at the EP. This algebraic manipulation allows to extend the radius of convergence of the eigenvalue approximation [5], the localization of EP using standard root-finding algorithms and the computation

of the associated Puiseux series up to an arbitrary order.

EP of order N (EPN) require more parameters and their localization become more challenging than EP2. In addition, the eigenvalue sensitivity increases also with N and the eigenvalue may become strongly dependent on the rounding error and split [16]. Despite this, practical realization have been done [4], although, to the best authors knowledge, no *generic* algorithm exists for finding them. The route to obtained high order EP is generally to work on simple (or reduce) system and combine them [22].

The aim of this work is to propose i) a new perturbation method for eigenvalue problem and ii) a generic algorithm to locate EPN. For this purpose, a new framework, based on analytic functions combining more than two eigenvalues to improve the radius of convergence and to take into account several parameters, is proposed. This can be done through the new concept of *partial characteristic polynomial* (PCP), which consists on reconstruct a part of the characteristic polynomial¹ from the Taylor expansion of some selected eigenpairs obtained at an initial value of the parameters. To compute the successive derivatives, the bordered matrix [1] is used here as in [12] (but other approaches can be used). For new parameters values, the eigenvalues can be obtained quickly as the roots of a low order polynomial. The EP of order N can then be found by imposing that the PCP and its $N - 1$ successive derivatives with respect to the eigenvalue vanish. The main ingredient of the method are given in sec. 2, an application to a 2D waveguide illustrating the accuracy and the interest of the proposed method is presented in sec. 3.

2 Partial characteristic polynomial

Let us consider a pair of eigenvalue, denoted by $\lambda_-(\nu)$ and $\lambda_+(\nu)$, dependant on the scalar parameter ν . In [5], it has been shown that the auxiliary functions

$$g(\nu) = \lambda_+ + \lambda_-, \quad (1)$$

$$h(\nu) = (\lambda_+ - \lambda_-)^2, \quad (2)$$

1. The characteristic polynomial cannot be computed for big matrices obtained after standard discretization methods like FEM.

are analytic even if the two eigenvalues present a branch point singularity and that their Taylor series (denoted by \mathcal{T}_\bullet) is not convergent. In the same way, the product of the eigenvalues is also an analytic function of ν . It follows that analytic auxiliary functions (AAF) scheme,

$$\lambda_{\pm}(\nu) = \frac{g(\nu) \pm \sqrt{h(\nu)}}{2} \approx \frac{\mathcal{T}_g(\nu) \pm \sqrt{\mathcal{T}_h(\nu)}}{2}, \quad (3)$$

behaves generally better than Puiseux or Taylor series to reconstruct eigenvalue loci as a function of ν . In the vicinity of its EP, the Puiseux series provides the good branch structure of the eigenvalue Riemann surfaces. Nevertheless, the implicit function theorem shows that Puiseux series radius of convergence is limited by the presence of the next EP even if this EP involved the same pair of eigenvalue, while in this specific case AAF scheme remains analytic and is able to accurately reconstruct the eigenvalues (but not directly the branches).

Getting the eigenvalue from the analytic auxiliary function is equivalent to find the roots of the polynomial

$$\begin{aligned} P(\lambda) &= (\lambda - \lambda_+)(\lambda - \lambda_-) \\ &= \lambda^2 - (\lambda_+ + \lambda_-)\lambda + \lambda_+\lambda_-, \end{aligned} \quad (4)$$

whom coefficients are analytic functions of the parameter ν (the explicit dependency of ν has been dropped for clarity). In particular, in (3), the link between h and the polynomial discriminant is obvious.

This approach can be generalized with the concept of partial characteristic polynomial and to vectorial parameters ν . For an analytic matrix function $\mathbf{L}(\lambda(\nu), \nu)$, its characteristic polynomial $\hat{\mathcal{P}}(\lambda, \nu)$ is also an analytic function. Assuming that $\hat{\mathcal{P}}(\lambda, \nu)$ can be split into

$$\hat{\mathcal{P}}(\lambda(\nu)) = (\lambda - \lambda_1(\nu)) \dots (\lambda - \lambda_\ell(\nu)) \dots (\lambda - \lambda_L(\nu)) \cdot f(\lambda(\nu)) \quad (5)$$

where λ_ℓ stands for all the eigenvalues (counting their multiplicities) present in a disk of the λ -complex plane and f is an analytic function vanishing outside the disk. We call this set Λ and it contains $L = |\lambda|$ elements. We denotes by

$$\mathcal{P}(\lambda, \nu) = (\lambda - \lambda_1(\nu)) \dots (\lambda - \lambda_\ell(\nu)) \dots (\lambda - \lambda_L(\nu)) \quad (6)$$

the partial characteristic polynomial of $\mathbf{L}(\lambda(\nu), \nu)$. Using the Vieta's formulas, the coefficient of this polynomial can be expressed as analytic function of the eigenvalue.

$$\mathcal{P}(\lambda, \nu) = a_L(\nu)\lambda^L + a_{L-1}(\nu)\lambda^{L-1} + \dots + a_0(\nu), \quad (7)$$

where

$$a_k = \sum_{c \in C_k} \prod_{i \in c} \lambda_i(\nu), \quad (8)$$

and the set $C_k = \{\text{all ways to chose } (k - |\Lambda|) \text{ different eigenvalues in } \Lambda\}$. For instance, for a quintic polynomial

build using a set of 5 eigenvalues, we get

$$a_5 = 1, \quad (9a)$$

$$a_4 = -\lambda_0 - \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4 \quad (9b)$$

$$a_3 = \lambda_0\lambda_1 + \lambda_0\lambda_2 + \lambda_0\lambda_3 + \lambda_0\lambda_4 + \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4, \quad (9c)$$

$$a_2 = -\lambda_0\lambda_1\lambda_2 - \lambda_0\lambda_1\lambda_3 - \lambda_0\lambda_1\lambda_4 - \lambda_0\lambda_2\lambda_3 - \lambda_0\lambda_2\lambda_4 - \lambda_0\lambda_3\lambda_4 - \lambda_1\lambda_2\lambda_3 - \lambda_1\lambda_2\lambda_4 - \lambda_1\lambda_3\lambda_4 - \lambda_2\lambda_3\lambda_4, \quad (9d)$$

$$a_1 = \lambda_0\lambda_1\lambda_2\lambda_3 + \lambda_0\lambda_1\lambda_2\lambda_4 + \lambda_0\lambda_1\lambda_3\lambda_4 + \lambda_0\lambda_2\lambda_3\lambda_4 + \lambda_1\lambda_2\lambda_3\lambda_4, \quad (9e)$$

$$a_0 = -\lambda_0\lambda_1\lambda_2\lambda_3\lambda_4. \quad (9f)$$

Because each polynomial coefficient a_k ($k = 0, \dots, L-1$) is analytic, their Taylor series be obtained thanks to multivariate Leibniz' rule applied to (8) from the Taylor series of each eigenvalue.

The final step is to solve the partial polynomial $\mathcal{P}(\lambda, \nu)$ for a given range of ν to recover the $\lambda_i(\nu)$ ($i = 0, \dots, L-1$). Depending of the number of eigenvalue in Λ , the roots can be obtained numerically or analytically if the polynomial is lower than 4. In practice, the numerical burden due to the numerical roots is neglectable when compared to eigenvalue computation of matrices obtained after standard discretization methods like finite element method. This representation also allow to locate EP.

2.1 Higher order EP location

EP corresponds generically to multiple roots of the PCP for complex symmetric problems [17]. If the first derivative $\partial_\lambda \mathcal{P}(\lambda; \mu, \nu)$ vanishes, it corresponds to an EP2. If the second derivatives also vanishes it corresponds to an EP3 and so one. It is noteworthy that PCP derivative with respect to the eigenvalue also vanishes for semisimple eigenvalue. Thus to be sure to have an EP further checks may be required for instance by inspecting Puiseux series coefficients [19].

To locate the finite set of EP3, we need to solve the multivariate polynomial system

$$\begin{cases} \mathcal{P}(\lambda; \nu) = 0, \\ \partial_\lambda \mathcal{P}(\lambda; \nu) = 0, \\ \partial_\lambda^2 \mathcal{P}(\lambda; \nu) = 0, \end{cases} \quad (10)$$

with $\nu \in \mathbb{C}^2$. The Bezout theorem (see [20] for instance) ensures that this system has only discrete solutions.

The system can be solved using

- *Newton solver*. Since such approach need an initial guess, a mesh parametric space is required and the solver need to be called for all starting points. This approach is generally fast but may miss some roots.
- *Homotopy solver* [20, 10]. This family of method are able to find *all* the solution of (10) numerically. The basic idea is to exploit analyticity to move slowly from the solution of a trivial similar problem to the solution of the real problem.

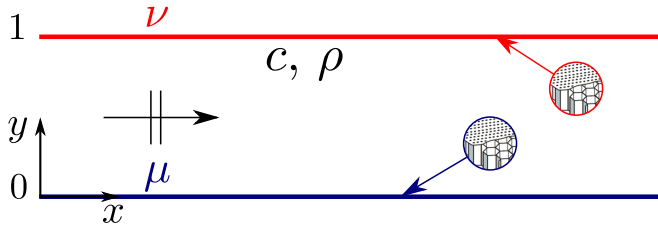


FIGURE 1 – Lined duct with two admittances.

Both approaches are efficient, but due to the truncated Taylor series, the number of spurious solutions may be huge. An estimation can be obtained from the Bezout Number [20].

To filter the spurious roots, comparing solution for two truncation orders in the a_k Taylor series is generally sufficient. It could also be combined with an estimation of radius of convergence PCP terms.

3 2D duct application

3.1 Problem statement

We consider a two-dimensional acoustic waveguide of infinite length and unit width lined with two admittances μ and ν , as described in Fig. 1. This reference problem have been already studied in [14] and can lead to a finite set of EP3 or a continuum of EP2. Their values are useful to validate the proposed approach. In the duct, the acoustic pressure satisfies the Helmholtz equation ($e^{-i\omega t}$ convention is adopted here)

$$\Delta p + k_a^2 p = 0, \quad (11)$$

where $k_a = \omega/c_a$ is the wavenumber, c_a is the sound speed and ω is the angular frequency. On both walls, the liner is assumed to be locally reacting which implies that the pressure must satisfy the Robin boundary condition

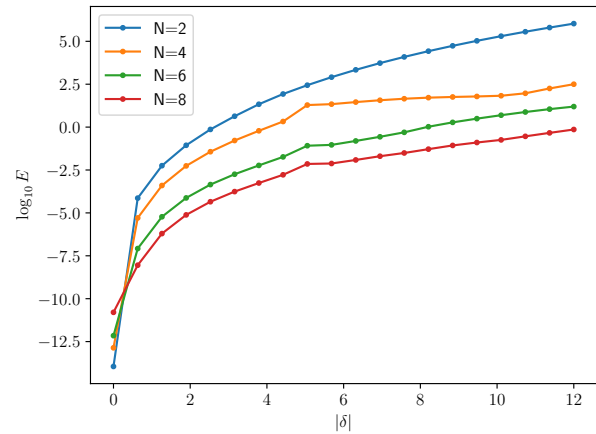
$$\partial_y p = -\mu p, \text{ at } y = 0 \quad \text{and} \quad \partial_y p = \nu p, \text{ at } y = 1 \quad (12)$$

Using invariance along the waveguide x -axis, the modal analysis is performed by assuming that the pressure field can be written in the separable form $p = \phi(y)e^{i\beta x}$. Here, function $\phi(y)$ is the mode shape and β the axial wavenumber. The weak formulation associated with the Helmholtz equation gives

$$-\int_0^1 \partial_y \psi \cdot \partial_y \phi \, dy + (k_a^2 - \beta^2) \int_0^1 \psi \phi \, dy + \mu \psi(0)\phi(0) + \nu \psi(1)\phi(1) = 0, \quad (13)$$

where ψ stands for the test function. Once the variational formulation is discretized with linear Lagrangian finite element, we obtain a generalized eigenvalue problem of the form

$$\mathbf{L}(\lambda(\nu), \nu) \mathbf{x}(\nu) = (-\mathbf{K} + (k_a^2 - \lambda) \mathbf{M} + \nu_1 \mathbf{\Gamma}_1 + \nu_2 \mathbf{\Gamma}_2) \mathbf{x}(\nu) = \mathbf{0}. \quad (14)$$

FIGURE 2 – Evolution of error on eigenvalue reconstruction according to the number of mode in the PCP and the modulus of the perturbation parameter δ .

To be consistent with our notation, we put $\lambda = \beta^2$ and $\nu = (\mu, \nu)$. Here, the vector \mathbf{x} contains the finite element nodal values of the acoustic pressure, the matrix $\mathbf{\Gamma}_i$ ($i = 1, 2$) stems from the admittance boundary condition and \mathbf{K} and \mathbf{M} are the standard stiffness and mass matrices respectively. We can easily calculate formally the partial derivative of (14) with respect to ν components and λ required to get the eigenvalue derivative as described in [12].

3.2 Results

This approach is implemented in EasterEig [11] open source framework dedicated to perturbation of eigenvalue problem. We use 200 linear Lagrange element in the cross section of the duct.

To illustrate the numerical efficiency of eigenvalue reconstruction based on PCP, the absolute error E between direct and PCP computation is given in Fig. 2 for increasing values of the perturbation parameter δ such $\nu = \nu_0 + e^{0.3i} \delta$. The comparison is performed by combining the Taylor series of 2, 4, 6 or 8 eigenvalues (sort by modulus) computed at $\nu_0 = (7.01265 - 4.76715i, 2.89872 - 2.47i)$. It can be observed that increasing the number of eigenvalues in the PCP strongly enhance the reconstruction quality. This can be explained by a mutual regularization of the branch point singularity in eigenvalue Taylor series. For this 1D problem, the prediction is valid up to 2 digits for perturbation parameter equal to 10, spanning most of interesting admittance values with a single direct computation.

The second aspect of the proposed approach concerns the EP location. Again, we start from $\nu_0 = (7.01265 - 4.76715i, 2.89872 - 2.47i)$ and combine the 6 first eigenvalues. The Homotopy method is used to solve (10). Solution is a set containing the triplets (λ, μ, ν) . The found ν values that yields to EP3 are presented in Fig. 3. Note that because of the symmetry of the problem, the role

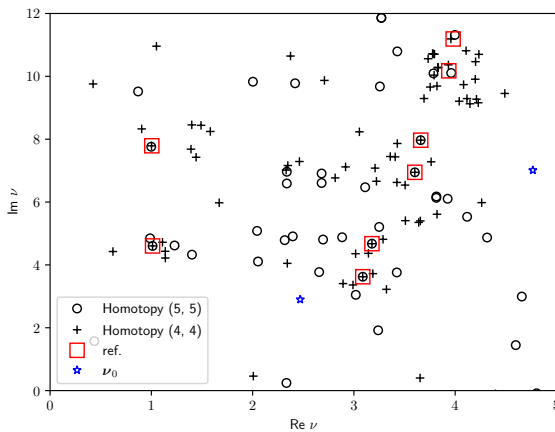


FIGURE 3 – Found EP3 solution from $\nu_0 = (7.01265 - 4.76715i, 2.89872 - 2.47i)$ and combining the 6 first eigenvalues.

of ν and μ is interchangeable thus the solution are found by pairs. It can be shown that several EP3 are accurately identified among the spurious roots when two truncation order are compared.

4 Conclusion

A new eigenvalue perturbation method has been proposed. The method is based on the *partial characteristic polynomial* concept. This approach allows to improve the radius of convergence of eigenvalue reconstruction for parametric problems depending on several parameters. This representation also permits to locate higher order EP.

This generic approach can be applied on circular ducts [7, 23]. For instance EP3 have been found for wall presenting two lined parts separated by rigid wall portions. Work is on going to better understand the merging mechanism and the link with strong attenuation as observed in 2D ducts [14].

The method can also be applied to other parametric eigenvalue problems like those arising in thermoacoustic [9, 13], in topological acoustic waveguides [3] or in structural dynamics.

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