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► **To cite this version:**

Romain Abraham, Jean-François Delmas, Michel Nassif. CONDITIONING (SUB)CRITICAL LÉVY TREES BY THEIR MAXIMAL DEGREE: DECOMPOSITION AND LOCAL LIMIT. 2022. hal-03838037

**HAL Id: hal-03838037**

**<https://hal.science/hal-03838037v1>**

Preprint submitted on 3 Nov 2022

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# CONDITIONING (SUB)CRITICAL LÉVY TREES BY THEIR MAXIMAL DEGREE: DECOMPOSITION AND LOCAL LIMIT

ROMAIN ABRAHAM<sup>1</sup>, JEAN-FRANÇOIS DELMAS<sup>2</sup>, AND MICHEL NASSIF<sup>3</sup>

ABSTRACT. We study the maximal degree of (sub)critical Lévy trees which arise as the scaling limits of Bienaymé-Galton-Watson trees. We determine the genealogical structure of large nodes and establish a Poissonian decomposition of the tree along those nodes. Furthermore, we make sense of the distribution of the Lévy tree conditioned to have a fixed maximal degree. In the case where the Lévy measure is diffuse, we show that the maximal degree is realized by a unique node whose height is exponentially distributed and we also prove that the conditioned Lévy tree can be obtained by grafting a Lévy forest on an independent size-biased Lévy tree with a degree constraint at a uniformly chosen leaf. Finally, we show that the Lévy tree conditioned on having large maximal degree converges locally to an immortal tree (which is the continuous analogue of the Kesten tree) in the critical case and to a condensation tree in the subcritical case. Our results are formulated in terms of the exploration process which allows to drop the Grey condition.

## 1. INTRODUCTION AND MAIN RESULTS

Lévy trees are random metric spaces that encode the genealogical structure of continuous-state branching processes (CB processes for short). As such, they arise as the scaling limits of Bienaymé-Galton-Watson trees. Lévy trees were introduced by Le Gall and Le Jan [28] and Duquesne and Le Gall [12] in order to generalize Aldous' Brownian tree [7]. They also appear as scaling limits of various models of trees and graphs, see e.g. Haas and Miermont [18], and are naturally related to fragmentation processes, see Miermont [29, 30], Haas and Miermont [17], Abraham and Delmas [1].

In the present paper, we study the maximal degree of a general Lévy tree. More precisely, we first establish a Poissonian decomposition of the Lévy tree along large nodes. Then, we make sense of the distribution of the Lévy tree conditioned to have a fixed maximal

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*Date:* November 3, 2022.

*2010 Mathematics Subject Classification.* 60J80, 60F17, 60J25.

*Key words and phrases.* Lévy trees, maximal degree, local limit, condensation, immortal tree.

degree. In the case where the Lévy measure is diffuse, we show that the maximal degree is realized by a unique node, and we describe how to reconstruct the tree by grafting a Lévy forest on an independent size-biased Lévy tree (with a restriction on the maximal degree) at a uniform leaf. Finally, we investigate the asymptotic behavior of the Lévy tree conditioned to have large maximal degree.

These questions arise naturally in the study of random trees and have been thoroughly investigated in the case of Bienaymé-Galton-Watson trees. The first results in this direction were obtained by Jonsson and Stefánsson [25] who showed that a condensation phenomenon appears for a certain class of subcritical Bienaymé-Galton-Watson trees conditioned to have a large size, in the sense that with high probability there exists a unique node with degree proportional to the size. Furthermore, the tree converges locally to a condensation tree consisting of a finite spine with random geometric length onto which independent and identically distributed Bienaymé-Galton-Watson trees are grafted. This was later generalized by Janson [24], with further results by Kortchemski [26], Abraham and Delmas [4], Stuffer [31]. On the other hand, He [19] shows that Bienaymé-Galton-Watson trees conditioned on having large maximal degree converge locally to Kesten's tree (which consists of an infinite spine onto which independent and identically distributed Bienaymé-Galton-Watson trees are grafted) in the critical case and to a condensation tree in the subcritical case.

In the continuum setting, Bertoin [9] determined the distribution of the maximal degree of a stable Lévy tree (his result is formulated in terms of Lévy processes). Using the formalism of CB processes, He and Li [22] treated the case of a general branching mechanism (in fact their result is more general as they considered CB processes with immigration). In [21], they also studied the local limit of a CB process conditioned to have large maximal degree (i.e. large maximal jump). In the critical case, they showed that it converges locally to a CB process with immigration. Later, He [20] extended the local convergence result to the whole genealogy: more precisely, he showed that a critical Lévy tree conditioned on having large maximal degree converges locally to an immortal tree (which is the continuous counterpart of Kesten's tree, consisting of an infinite spine onto which trees are grafted according to a Poisson point process). We improve these results by considering the density version of the conditioning instead of the tail version: more explicitly, we study the asymptotic behavior of critical Lévy trees conditioned to have maximal degree equal to (and not larger than) a given value. Density versions are finer than their tail counterparts and are usually more difficult to prove.

The existing literature in the subcritical case is less developed. He and Li [21] showed that a subcritical CB process conditioned to have large maximal degree converges locally to a CB process with immigration which is killed (i.e. sent to infinity) at an independent exponential time, thus underlining a condensation phenomenon. We improve this result in several directions. Again we consider the density version of the conditioning instead of the tail version. We also extend the convergence result to the whole genealogical structure

instead of the population size at a given time: this gives more information and, as an example, allows us to see that only one large node emerges. Finally, we are also able to describe precisely what happens above the condensation node.

For the sake of clarity, we shall formulate our results in terms of Lévy trees in the introduction. This requires an additional assumption on the branching mechanism, namely the Grey condition (see below), in order to have a nice topology on the set of trees. Indeed, this condition ensures that the Lévy tree is a *compact* real tree. However, it is superfluous and will be dropped in the rest of the paper where we will deal with the exploration process instead. Let us mention that a forthcoming work by Duquesne and Winkel [14] should allow us to use the formalism of real trees even for a general branching mechanism not necessarily satisfying the Grey condition.

Before stating our main results, we need to recall some definitions and to set notations.

**1.1. Real trees.** We recall the formalism of real trees, see [16]. A quadruple  $(T, d, \emptyset, \mu)$  is called a real tree if  $(T, d)$  is a metric space equipped with a distinguished vertex  $\emptyset \in T$  called the root and a nonnegative finite measure  $\mu$  on  $T$  and if the following two properties hold for every  $x, y \in T$ :

- (i) (Unique geodesics). There exists a unique isometric map  $f_{x,y}: [0, d(x, y)] \rightarrow T$  such that  $f_{x,y}(0) = x$  and  $f_{x,y}(d(x, y)) = y$ .
- (ii) (Loop-free). If  $\varphi$  is a continuous injective map from  $[0, 1]$  into  $T$  such that  $\varphi(0) = x$  and  $\varphi(1) = y$ , then we have  $\varphi([0, 1]) = f_{x,y}([0, d(x, y)])$ .

For every vertex  $x \in T$ , we define its height by  $H(x) = d(\emptyset, x)$ . The height of the tree is defined by  $\mathfrak{h}(T) = \sup_{x \in T} H(x)$ . Note that if  $(T, d)$  is compact, then  $\mathfrak{h}(T) < \infty$ .

We will denote by  $\mathbb{T}$  the set of (isometry classes of) *compact* real trees. Let us mention that it can be equipped with the Gromov-Hausdorff-Prokhorov distance which makes it a Polish space, see e.g. [6].

We will also need the set  $\mathbb{T}^*$  of (isometry classes of) compact real trees that are *marked*, i.e. equipped with a distinguished vertex in addition to the root  $\emptyset$ . Again,  $\mathbb{T}^*$  can be made into a Polish space when equipped with a marked variant of the Gromov-Hausdorff-Prokhorov distance.

**1.2. Local convergence of real trees.** We will make use of the notion of local convergence for *locally compact* real trees which we now recall. For every  $h > 0$ , define the restriction mapping on the set of (isometry classes of) real trees by:

$$r_h(T, d, \emptyset, \mu) = (T^h, d|_{T^h \times T^h}, \emptyset, \mu|_{T^h}) \quad \text{where } T^h = \{x \in T : H(x) \leq h\}.$$

In other words,  $r_h(T)$  is the real tree obtained from  $T$  by removing all nodes whose height is larger than  $h$ , equipped with the same metric and measure restricted to  $T^h$ . Recall that the Hopf-Rinow theorem implies that if  $T$  is a locally compact real tree, the closed ball  $r_h(T)$  is compact. We say that a sequence  $T_n$  of locally compact trees converges locally to a locally compact tree  $T$  if for every  $h > 0$ , the sequence  $r_h(T_n)$  converges for the Gromov-Hausdorff-Prokhorov distance to  $r_h(T)$ .

**1.3. Grafting procedure.** Given a real tree  $T \in \mathbb{T}$  and a finite or countable family  $((x_i, T_i), i \in I)$  of elements of  $T \times \mathbb{T}$ , we denote by

$$T \circledast_{i \in I} (x_i, T_i)$$

the real tree obtained by grafting  $T_i$  on  $T$  at the node  $x_i$ . For a precise definition, we refer the reader to [3, Section 2.4].

**1.4. Lévy trees.** Let  $\psi$  be a branching mechanism given by:

$$\psi(\lambda) = \alpha\lambda + \beta\lambda^2 + \int_{(0, \infty)} (e^{-\lambda r} - 1 + \lambda r) \pi(dr), \quad (1.1)$$

where  $\alpha, \beta \geq 0$  and  $\pi$  is a  $\sigma$ -finite measure on  $(0, \infty)$  such that  $\int_{(0, \infty)} (r \wedge r^2) \pi(dr) < \infty$ . The branching mechanism  $\psi$  is said to be critical (resp. subcritical) if  $\alpha = 0$  (resp.  $\alpha > 0$ ). In what follows, we assume that  $\pi \neq 0$  as otherwise all branching points of the Lévy tree will be binary. Whenever we are dealing with Lévy trees, we always assume that the Grey condition holds:

$$\int_0^\infty \frac{d\lambda}{\psi(\lambda)} < \infty, \quad (1.2)$$

which is equivalent to the compactness of the Lévy tree. In the rest of the paper, this condition will be relaxed to:

$$\beta > 0 \quad \text{or} \quad \int_{(0, 1)} r \pi(dr) = \infty. \quad (1.3)$$

We will consider a Lévy tree  $\mathcal{T}$  under its excursion measure which is denoted by  $\mathbf{N}^\psi$ . Here we briefly recall some results on Lévy trees but we refer the reader to Duquesne and Le Gall [12, 13] for a complete presentation on the subject. One can define a  $\sigma$ -finite measure  $\mathbf{N}^\psi$  on the space  $\mathbb{T}$ , called the excursion measure of the Lévy tree, with the following properties.

- (i) **Mass measure.** For  $\mathbf{N}^\psi$ -almost every  $\mathcal{T}$ , the mass measure  $\mu$  is supported on the set of leaves  $\text{Lf}(\mathcal{T}) := \{x \in \mathcal{T} : \mathcal{T} \setminus \{x\} \text{ is connected}\}$ . Furthermore, the total mass  $\sigma := \mu(\mathcal{T})$  satisfies:

$$\mathbf{N}^\psi [1 - e^{-\lambda\sigma}] = \psi^{-1}(\lambda). \quad (1.4)$$

- (ii) **Local times.** For  $\mathbf{N}^\psi$ -almost every  $\mathcal{T}$ , there exists a process  $(L^a, a \geq 0)$  with values in the space of finite measures on  $\mathcal{T}$  which is càdlàg for the weak topology and such that

$$\mu(dx) = \int_0^\infty da L^a(dx). \quad (1.5)$$

For every  $a \geq 0$ , the measure  $L^a$  is supported on  $\mathcal{T}(a) := \{x \in \mathcal{T} : H(x) = a\}$  the set of nodes at height  $a$ . Furthermore, the real-valued process  $(L_\sigma^a := \langle L^a, 1 \rangle, a \geq 0)$  is a  $\psi$ -CB process under its canonical measure.

- (iii) **Branching property.** For every  $a \geq 0$ , let  $(\mathcal{T}^i, i \in I_a)$  be the subtrees of  $\mathcal{T}$  originating from level  $a$ . Then, under  $\mathbf{N}^\psi$  and conditionally on  $r_a(\mathcal{T}) := \{x \in \mathcal{T} : H(x) \leq a\}$ , the measure  $\sum_{i \in I_a} \delta_{\mathcal{T}^i}$  is a Poisson point measure with intensity  $L_\sigma^a \mathbf{N}^\psi$ .
- (iv) **Branching points.** For  $\mathbf{N}^\psi$ -almost every  $\mathcal{T}$ , the branching points of  $\mathcal{T}$  are either binary or of infinite degree. The set of binary branching points is empty if  $\beta = 0$  and is a countable dense subset of  $\mathcal{T}$  if  $\beta > 0$ . The set

$$\text{Br}_\infty(\mathcal{T}) := \{x \in \mathcal{T} : \mathcal{T} \setminus \{x\} \text{ has infinitely many connected components}\}$$

of infinite branching points is nonempty with  $\mathbf{N}^\psi$ -positive measure if and only if  $\pi \neq 0$ . If  $\langle \pi, 1 \rangle = \infty$ , the set  $\text{Br}_\infty(\mathcal{T})$  is countable and dense in  $\mathcal{T}$  for  $\mathbf{N}^\psi$ -almost every  $\mathcal{T}$ . Furthermore, the set  $\{H(x), x \in \text{Br}_\infty(\mathcal{T})\}$  coincides with the set of discontinuity times of the mapping  $a \mapsto L^a$ . For every such discontinuity time  $a$ , there is a unique  $x_a \in \text{Br}_\infty(\mathcal{T}) \cap \mathcal{T}(a)$  and  $\Delta_a > 0$  such that

$$L^a = L^{a-} + \Delta_a \delta_{x_a}.$$

For convenience, we define  $\Delta_a$  for every  $a \geq 0$  by setting  $\Delta_a = 0$  if  $L^a = L^{a-}$ . In particular, we have  $L_\sigma^a = L_\sigma^{a-} + \Delta_a$ , that is  $\Delta_a$  is exactly the size of the jump of the associated CB process at time  $a$ . We will call  $\Delta_a$  the degree (or the mass) of the node  $x_a$ . This is an abuse of language since a node  $x_a \in \text{Br}_\infty(\mathcal{T})$  has infinite degree by definition.

1.5. **Main results.** We denote by  $\Delta$  the maximal degree of the Lévy tree  $\mathcal{T}$  under  $\mathbf{N}^\psi$ :

$$\Delta = \sup_{a \geq 0} \Delta_a. \quad (1.6)$$

The first result of this paper gives the joint distribution of the maximal degree  $\Delta$  and the total mass  $\sigma$  under  $\mathbf{N}^\psi$ . The distribution of the maximal degree was already obtained by Bertoin [9, Lemma 1] in the stable Lévy case then by He and Li [22] in the general case.

For the sake of notational simplicity, if  $\nu$  is a measure on  $\mathbb{R}$  we will write  $\nu(a, b)$  (resp.  $\nu[a, b)$ ) instead of  $\nu((a, b))$  (resp.  $\nu([a, b))$ ). We will also write  $\nu(a)$  for  $\nu(\{a\})$ . Denote by  $\bar{\pi} : \mathbb{R}_+ \rightarrow (0, \infty]$  the tail of the Lévy measure  $\pi$ :

$$\bar{\pi}(\delta) = \pi(\delta, \infty), \quad \forall \delta \geq 0, \quad (1.7)$$

and define the Laplace exponent  $\psi_\delta$  for every  $\delta > 0$  by:

$$\begin{aligned}\psi_\delta(\lambda) &= \left( \alpha + \int_{(\delta, \infty)} r \pi(dr) \right) \lambda + \beta \lambda^2 + \int_{(0, \delta]} \left( e^{-\lambda r} - 1 + \lambda r \right) \pi(dr) \\ &= \psi(\lambda) + \int_{(\delta, \infty)} \left( 1 - e^{-\lambda r} \right) \pi(dr).\end{aligned}\tag{1.8}$$

Observe that, in terms of the associated Lévy process, this corresponds to removing all jumps with size larger than  $\delta$ . If the Lévy measure  $\pi$  is finite, we also define:

$$\psi_0(\lambda) = \left( \alpha + \int_{(0, \infty)} r \pi(dr) \right) \lambda + \beta \lambda^2.\tag{1.9}$$

**Proposition 1.1.** *For every  $\delta > 0$  and  $\lambda \geq 0$ , we have:*

$$\mathbf{N}^\psi \left[ 1 - e^{-\lambda \sigma} \mathbf{1}_{\{\Delta \leq \delta\}} \right] = \psi_\delta^{-1}(\bar{\pi}(\delta) + \lambda).\tag{1.10}$$

Furthermore, if the Lévy measure  $\pi$  is finite, we have:

$$\mathbf{N}^\psi \left[ 1 - e^{-\lambda \sigma} \mathbf{1}_{\{\Delta = 0\}} \right] = \psi_0^{-1}(\langle \pi, 1 \rangle + \lambda).\tag{1.11}$$

The proof is given in Section 3.

**Remark 1.2.** Let us make a connection with He and Li [21]. Recall that under  $\mathbf{N}^\psi$  the process  $(L_\sigma^a, a \geq 0)$  is distributed as a  $\psi$ -CB process under its canonical measure and that the maximal degree  $\Delta$  of the Lévy tree corresponds to the maximal jump of the associated CB process. In particular, taking  $\lambda = 0$  in (1.10) gives the distribution of the maximal jump of a  $\psi$ -CB process, which was already obtained by He and Li, see [21, Corollary 4.2]. In fact, their result is much more general (see [21, Theorem 4.1]) since they consider a CB process with immigration and in this context, they compute the distribution of the *local* maximal jump which in terms of the Lévy tree corresponds to the maximal degree up to a fixed level  $h$ . However, they do not give the *joint* distribution of  $\Delta$  and  $\sigma$ , which in terms of the CB process corresponds to the total mass:

$$\sigma = \int_0^\infty L_\sigma^a da.$$

Next, we give a decomposition of the Lévy tree along the large nodes. More precisely, we identify the distribution of the pruned Lévy tree obtained by removing all nodes with degree larger than  $\delta$  (and the subtrees above them). This is again a Lévy tree with branching mechanism  $\psi_\delta$  under its excursion measure. Furthermore, one can recover the Lévy tree from the pruned one by grafting Lévy forests at uniformly chosen leaves in a Poissonian manner. Before stating the result, we first need to introduce some notations. For every  $r > 0$ , denote by  $\mathbb{P}_r^\psi$  the distribution of the random real tree  $\mathcal{T} = \{\emptyset\} \otimes_{i \in I} \mathcal{T}_i$  obtained by gluing together at their root the atoms  $(\mathcal{T}_i, i \in I)$  of a  $\mathbb{T}$ -valued Poisson point measure with intensity  $r \mathbf{N}^\psi[d\mathcal{T}]$ . This should be interpreted as the distribution of a Lévy forest

with initial degree  $r > 0$ . Furthermore, for every  $\delta > 0$  such that  $\bar{\pi}(\delta) > 0$ , set:

$$\mathbb{Q}_\delta^\psi(d\mathcal{T}) = \frac{1}{\bar{\pi}(\delta)} \int_{(\delta, \infty)} \pi(dr) \mathbb{P}_r^\psi(d\mathcal{T})$$

which is the distribution of a Lévy forest with random initial degree with distribution  $\pi$  conditioned on being larger than  $\delta$ .

**Theorem 1.3.** *Let  $\delta \geq 0$  such that  $\bar{\pi}(\delta) < \infty$ . Under  $\mathbf{N}^{\psi_\delta}$  and conditionally on  $(\mathcal{T}, \emptyset, d, \mu)$ , let  $((x_i, \mathcal{T}_i), i \in I)$  be the atoms of a Poisson point measure on  $\mathcal{T} \times \mathbb{T}$  with intensity  $\bar{\pi}(\delta) \mu(dx) \mathbb{Q}_\delta^\psi(d\tilde{\mathcal{T}})$ . Then, under  $\mathbf{N}^{\psi_\delta}$ , the random tree  $\mathcal{T} \otimes_{i \in I} (x_i, \mathcal{T}_i)$  has distribution  $\mathbf{N}^{\psi_\delta}$ .*

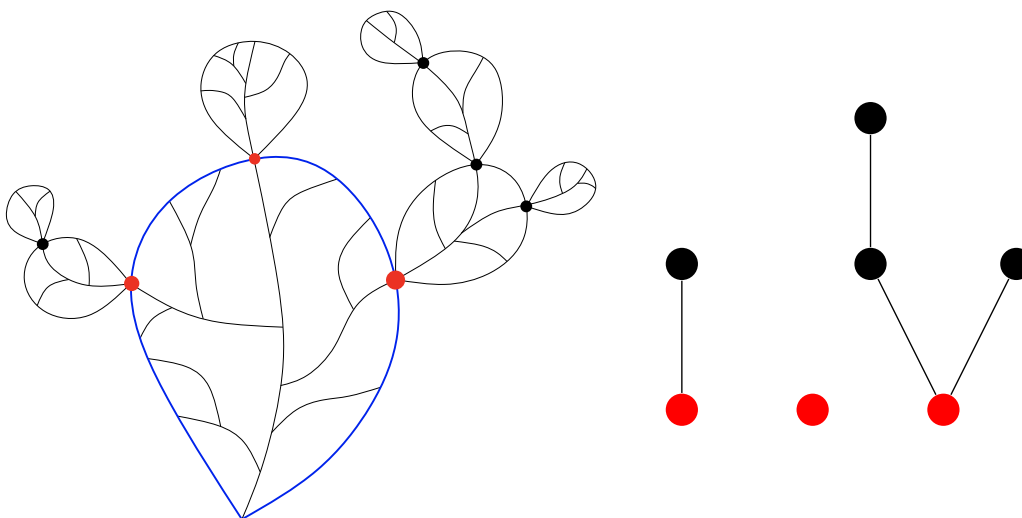


FIGURE 1. Decomposition of the Lévy tree  $\mathcal{T}$  along the nodes with degree larger than  $\delta$  (left) and the associated discrete forest (right). In blue: the pruned subtree  $\mathcal{T}^\delta$ , in red: the first-generation nodes with degree larger than  $\delta$ .

See Theorem 4.1 for a more precise statement. In particular, the pruned Lévy tree  $\mathcal{T}^\delta$  which is obtained from  $\mathcal{T}$  by removing all nodes with degree larger than  $\delta$  is again a Lévy tree with branching mechanism  $\psi_\delta$ . Thanks to this decomposition, we prove in Proposition 4.6 that the discrete forest formed by nodes with degree larger than  $\delta$  is a Bienaymé-Galton-Watson forest and we specify its initial distribution and its offspring distribution, see Figure 1.

**Remark 1.4.** Theorem 1.3 is a special case of the main result in [5]. In that paper, the authors study a pruning procedure on the Lévy tree defined as follows: they add some marks on the skeleton of the tree according to a Poisson point measure with intensity  $\alpha_1 \Lambda$  (where  $\Lambda$  is the length measure on  $\mathcal{T}$  which is the equivalent of the Lebesgue measure) and



add some other marks on the infinite branching points  $x_a$  with probability  $p(\Delta_a)$  where  $p$  is a nonnegative measurable function satisfying:

$$\int_{(0,\infty)} rp(r) \pi(dr) < \infty.$$

Then they show that the subtree  $\mathcal{T}^{\alpha_1, p}$  containing the root obtained from  $\mathcal{T}$  by removing all the marks is again a Lévy tree and identify its branching mechanism. Furthermore, they determine the distribution of the subtrees above the marks conditionally on  $\mathcal{T}^{\alpha_1, p}$ . It is obvious that the tree  $\mathcal{T}^\delta$  coincides with  $\mathcal{T}^{\alpha_1, p}$  where  $\alpha_1 = 0$  and  $p = \mathbf{1}_{(\delta, \infty)}$ . Since  $p$  satisfies the integrability assumption above (as  $\int_{(1, \infty)} r \pi(dr) < \infty$ ), their result applies and gives the joint distribution of the pruned tree  $\mathcal{T}^\delta$  and the subtrees originating from the nodes with degree larger than  $\delta$ . However, the proof is much simpler in our particular setting.

One of our main results is the next theorem giving a decomposition of the Lévy tree at its largest nodes. Under  $\mathbf{N}^\psi$ , denote by  $M_\delta$  the random variable defined by:

$$M_\delta = \frac{e^{\mathbf{g}(\delta)\sigma} - 1}{\mathbf{g}(\delta)}, \quad \text{where} \quad \mathbf{g}(\delta) = \pi(\delta)e^{-\delta \mathbf{N}^\psi[\Delta > \delta]}.$$

This should be interpreted as  $M_\delta = \sigma$  if  $\mathbf{g}(\delta) = 0$  (i.e. if  $\delta$  is not an atom of  $\pi$ ).

**Theorem 1.5.** *There exists a regular conditional probability  $\mathbf{N}^\psi[\cdot | \Delta = \delta]$  for  $\delta > 0$  such that  $\pi[\delta, \infty) > 0$ , which is given by, for every measurable and bounded  $F: \mathbb{T} \rightarrow \mathbb{R}$ :*

$$\begin{aligned} \mathbf{N}^\psi[F(\mathcal{T}) | \Delta = \delta] &= \frac{1}{\mathbf{N}^\psi[M_\delta \mathbf{1}_{\{\Delta < \delta\}}]} \sum_{k=0}^{\infty} \frac{\mathbf{g}(\delta)^k}{(k+1)!} \\ &\times \mathbf{N}^\psi \left[ \int \prod_{i=1}^{k+1} \mu(dx_i) \mathbb{P}_\delta^\psi(d\mathcal{T}_i | \Delta \leq \delta) F(\mathcal{T} \circledast_{i=1}^{k+1} (x_i, \mathcal{T}_i)) \mathbf{1}_{\{\Delta < \delta\}} \right], \end{aligned} \quad (1.12)$$

where  $\mathbf{N}^\psi[M_\delta \mathbf{1}_{\{\Delta < \delta\}}] < \infty$ . In particular, if  $\delta > 0$  is not an atom of the Lévy measure  $\pi$ , we have:

$$\mathbf{N}^\psi[F(\mathcal{T}) | \Delta = \delta] = \frac{1}{\mathbf{N}^\psi[\sigma \mathbf{1}_{\{\Delta < \delta\}}]} \mathbf{N}^\psi \left[ \int \mu(dx) \mathbb{P}_\delta^\psi(d\tilde{\mathcal{T}} | \Delta \leq \delta) F(\mathcal{T} \circledast (x, \tilde{\mathcal{T}})) \mathbf{1}_{\{\Delta < \delta\}} \right]. \quad (1.13)$$

Furthermore, if  $\langle \pi, 1 \rangle = \infty$ , then  $\mathbf{N}^\psi$ -a.e.  $\Delta > 0$ , and if  $\langle \pi, 1 \rangle < \infty$ , then we have:

$$\mathbf{N}^\psi[F(\mathcal{T}) \mathbf{1}_{\{\Delta=0\}}] = \mathbf{N}^{\psi_0}[F(\mathcal{T}) e^{-\langle \pi, 1 \rangle \sigma}]. \quad (1.14)$$

The proof is given in Section 5. Some comments are in order.

- (i) Recall that the distribution of  $\Delta$  is given in Proposition 1.1. Together with the distribution of  $\mathcal{T}$  conditionally on  $\Delta = \delta$ , we can recover the unconditional distribution of the Lévy tree via the disintegration formula:

$$\mathbf{N}^\psi[F(\mathcal{T})] = \mathbf{N}^\psi[F(\mathcal{T})\mathbf{1}_{\{\Delta=0\}}] + \int_{(0,\infty)} \mathbf{N}^\psi[\Delta \in d\delta] \mathbf{N}^\psi[F(\mathcal{T})|\Delta = \delta],$$

where the first term on the right-hand side vanishes if  $\pi$  is infinite.

- (ii) Assume that  $\delta > 0$  is not an atom of  $\pi$ . Then, conditionally on  $\Delta = \delta$ , the Lévy tree can be constructed as follows: take  $\tilde{\mathcal{T}}$  with distribution  $\mathbf{N}^\psi[\sigma\mathbf{1}_{\{\Delta \leq \delta\}}]^{-1} \mathbf{N}^\psi[\sigma\mathbf{1}_{\{\Delta \leq \delta\}} d\mathcal{T}]$ , choose a leaf uniformly at random in  $\tilde{\mathcal{T}}$  (i.e. according to its normalized mass measure  $\tilde{\sigma}^{-1}\tilde{\mu}$ ) and on this leaf graft an independent Lévy forest with initial degree  $\delta$  conditioned to have maximal degree  $\Delta \leq \delta$ . In fact, since  $\delta$  is not an atom, this random forest will have no other nodes with degree  $\delta$  besides the root. This entails that, conditionally on  $\Delta = \delta$ , there is a unique node realizing the maximum degree.
- (iii) The situation is different when  $\delta > 0$  is an atom of  $\pi$ . In that case, conditionally on  $\Delta = \delta$ , the number of first-generation nodes realizing the maximal degree has a Poisson distribution. More precisely, conditionally on  $\Delta = \delta$ , the Lévy tree can be constructed as follows: take  $\tilde{\mathcal{T}}$  with distribution  $\mathbf{N}^\psi[M_\delta\mathbf{1}_{\{\Delta < \delta\}}]^{-1} \mathbf{N}^\psi[M_\delta\mathbf{1}_{\{\Delta < \delta\}} d\mathcal{T}]$ , and, conditionally on  $\tilde{\mathcal{T}}$ , graft a Poisson point measure with intensity  $\mathbf{g}(\delta)\tilde{\mu}(dx) \mathbb{P}_\delta^\psi(d\mathcal{T}|\Delta \leq \delta)$  conditioned on containing at least one point.

As a consequence, we show in Proposition 5.11 that if the Lévy measure  $\pi$  is diffuse, then  $\mathbf{N}^\psi$ -a.e. there is a unique node  $X_\Delta$  with degree  $\Delta$ . Denote by  $H_\Delta = H(X_\Delta)$  its height. Then we give a decomposition of the Lévy tree conditioned on  $\Delta = \delta$  and  $H_\Delta = h$ , see Theorem 6.3.

Finally, we turn to the behavior of a Lévy tree conditioned to have a large maximal degree. Other conditionings have been considered in the past. Duquesne [11] (this is also related to Williams' decomposition, see [2]) proved that a (sub)critical Lévy tree conditioned on having a large height converges locally to the immortal tree (which consists of an infinite spine onto which trees are grafted according to a Poisson point process). Later, He [20] proved the same convergence for a critical Lévy tree conditioned on having a large maximal degree  $\Delta > \delta$  or a large width. In fact, his result is more general as it allows to condition by any measurable function of the tree satisfying a natural monotonicity property.

Here we treat both the critical and the subcritical cases and we consider the density version  $\Delta = \delta$ . Similarly to the discrete case, two drastically different types of limiting behavior appear. In the subcritical case, there is a condensation phenomenon where a node with infinite degree emerges at a finite exponentially distributed height. Denote by  $X_\Delta$  the lowest node with degree  $\Delta$  and let  $\mathcal{F}_\Delta^+$  be the forest above  $X_\Delta$ , seen as a point measure on  $\mathbb{R}_+ \times \mathbb{T}$ . To be more precise, the forest  $\mathcal{F}_\Delta^+ = \sum_{i \in I} \delta_{(\ell_i, \mathcal{T}_i)}$  is obtained by decomposing

the path of the exploration process (or the height process) into excursions away from 0, with each excursion arriving at local time  $\ell_i$  and coding a tree  $\mathcal{T}_i$ . Finally, let  $\mathcal{T}_\Delta^-$  be the pruned Lévy tree, that is the Lévy tree  $\mathcal{T}$  after removing  $X_\Delta$  and  $\mathcal{F}_\Delta^+$ . We refer the reader to Theorem 7.5 and Theorem 8.2 for a precise statement.

**Theorem 1.6.** *Assume that  $\psi$  is subcritical and that the Lévy measure  $\pi$  is unbounded. Let  $F: \mathbb{T}^* \rightarrow \mathbb{R}$  be continuous and bounded,  $\Phi: \mathbb{R}_+ \times \mathbb{T} \rightarrow \mathbb{R}_+$  be continuous with bounded support and let  $A_\delta$  be equal to any one of the following events:  $\{\Delta = \delta\}$ ,  $\{\Delta > \delta\}$ ,  $\{\mathcal{T}$  has exactly one node with degree larger than  $\delta\}$  or  $\{\mathcal{T}$  has exactly one first-generation node with degree larger than  $\delta\}$ . We have:*

$$\lim_{\delta \rightarrow \infty} \mathbf{N}^\psi \left[ F(\mathcal{T}_\Delta^-, X_\Delta) e^{-\langle \mathcal{F}_\Delta^+, \Phi \rangle} \Big| A_\delta \right] = \alpha \mathbf{N}^\psi \left[ \int_{\mathcal{T}} F(\mathcal{T}, x) \mu(dx) \right] \times \exp \left\{ - \int_0^\infty d\ell \mathbf{N}^\psi \left[ 1 - e^{-\Phi(\ell, \mathcal{T})} \right] \right\}. \quad (1.15)$$

In particular, conditionally on  $A_\delta$ , the height  $H(X_\Delta)$  of  $X_\Delta$  converges to an exponential distribution with mean  $1/\alpha$ .

The last result should be interpreted as local convergence in distribution to a “condensation tree” described as follows: start with a size-biased Lévy tree  $\tilde{\mathcal{T}}$  with distribution  $\alpha \mathbf{N}^\psi[\sigma d\mathcal{T}]$ , choose a leaf uniformly at random in  $\tilde{\mathcal{T}}$  and on this leaf graft an independent Lévy forest with infinite degree (i.e. a Poisson point measure on  $\mathbb{R}_+ \times \mathbb{T}$  with intensity  $d\ell \mathbf{N}^\psi[d\mathcal{T}]$ ). However, the limiting object is a (random) real tree which is not locally compact and the way to circumvent this difficulty is by considering the subtree above the condensation node as a point measure instead.

In the critical case, it should be no surprise that the density version  $\Delta = \delta$  gives rise to the same limiting behavior as the tail version  $\Delta > \delta$ , namely local convergence to the immortal tree. Intuitively, this means that the condensation node goes to infinity and thus becomes invisible to local convergence. Before stating the result, let us define the immortal tree. Let  $\sum_{i \in I} \delta_{(s_i, \mathcal{T}_i)}$  be a Poisson point measure on  $\mathbb{R}_+ \times \mathbb{T}$  with intensity

$$ds \left( 2\beta \mathbf{N}^\psi[d\mathcal{T}] + \int_0^\infty r \pi(dr) \mathbb{P}_r^\psi(d\mathcal{T}) \right).$$

The immortal Lévy tree  $\mathcal{T}_\infty^\psi$  with branching mechanism  $\psi$  is the real tree obtained by grafting the point measure  $\sum_{i \in I} \delta_{(s_i, \mathcal{T}_i)}$  on an infinite branch. More formally, set:

$$\mathcal{T}_\infty^\psi = \mathbb{R}_+ \otimes_{i \in I} (s_i, \mathcal{T}_i), \quad (1.16)$$

where  $\mathbb{R}_+$  is considered as a real tree rooted at 0 and equipped with the Euclidean distance and the zero measure. In particular, thanks to [13, Theorem 4.5], we have the following identity which is simply a restatement of Lemma 3.2 in [11] in terms of trees:

$$\mathbb{E} \left[ F(r_h(\mathcal{T}_\infty^\psi)) \right] = e^{-\alpha h} \mathbf{N}^\psi \left[ L_\sigma^h F(r_h(\mathcal{T})) \right], \quad \forall h > 0. \quad (1.17)$$

**Theorem 1.7.** *Assume that  $\psi$  is critical and that  $\pi$  is unbounded. Either let  $A_\delta = \{\Delta = \delta\}$  and assume that the additional assumption*

$$\lim_{\delta \rightarrow \infty} \frac{\pi(\delta)}{\mathbf{N}^\psi[\sigma \mathbf{1}_{\{\Delta < \delta\}}] \bar{\pi}(\delta) \int_{[\delta, \infty)} r \pi(dr)} = 0 \quad (1.18)$$

*holds, or let  $A_\delta$  be equal to any of the following events:  $\{\Delta > \delta\}$ ,  $\{\mathcal{T}$  has exactly one node with degree larger than  $\delta\}$  or  $\{\mathcal{T}$  has exactly one first-generation node with degree larger than  $\delta\}$ . Then, conditionally on  $A_\delta$ , the Lévy tree  $\mathcal{T}$  converges in distribution locally to the immortal Lévy tree  $\mathcal{T}_\infty^\psi$ , i.e. we have:*

$$\lim_{\delta \rightarrow \infty} \mathbf{N}^\psi [F(r_h(\mathcal{T})) | A_\delta] = \mathbb{E} [F(r_h(\mathcal{T}_\infty^\psi))]. \quad (1.19)$$

We refer to Theorem 7.8 and Theorem 8.4 for a precise statement. The assumption (1.18) is a technical condition which guarantees a fast decay for the size of the atoms of  $\pi$ . Observe that we have  $\lim_{\delta \rightarrow \infty} \mathbf{N}^\psi[\sigma \mathbf{1}_{\{\Delta < \delta\}}] = \mathbf{N}^\psi[\sigma]$  which is infinite since  $\psi$  is critical. Also notice that (1.18) is automatically satisfied if the Lévy measure  $\pi$  is diffuse.

It is worth noting that in the critical case, conditioning by the different events  $A_\delta$  yields the same limiting behavior even though in general they are not equivalent in  $\mathbf{N}^\psi$ -measure. In the stable (critical) case  $\psi(\lambda) = \lambda^\gamma$  with  $\gamma \in (1, 2)$ , these quantities can be computed explicitly, see Proposition 9.2. In that case, we also show in Proposition 9.4 that, conditionally on  $\Delta > \delta$ , the distribution of the Bienaymé-Galton-Watson forest of nodes with degree larger than  $\delta$  is independent of  $\delta$ .

The rest of the paper is organized as follows. In Section 2, we set notation and we introduce the main object we will be dealing with, namely the exploration process. We compute the distribution of the maximal degree in Section 3, then we give a Poissonian decomposition of the exploration process along the large nodes and study their structure in Section 4. In Section 5 (resp. Section 6), we make sense of the exploration process conditioned to have a fixed maximal degree (resp. a fixed maximal degree at a given height). Sections 7 and 8 deal with the local convergence of the exploration process conditioned to have large maximal degree. Finally, Section 9 is devoted to the study of the stable case  $\psi(\lambda) = \lambda^\gamma$ .

## 2. THE EXPLORATION PROCESS AND THE LÉVY TREE

In this section, we will recall the construction of the exploration process introduced in [28] and later developed in [12].

**2.1. Notation.** If  $E$  is a Polish space, let  $\mathcal{B}_+(E)$  be the set of real-valued and nonnegative measurable functions defined on  $E$  endowed with its Borel  $\sigma$ -field. For any measure  $\nu$  on  $E$

and any function  $f \in \mathcal{B}_+(E)$ , we write  $\langle \nu, f \rangle = \int_E f(x) \nu(dx)$ . We also denote by  $\text{supp}(\nu)$  the closed support of the measure  $\nu$  in  $E$ .

We denote by  $\mathcal{M}_f(E)$  the set of finite measures on  $E$  endowed with the topology of weak convergence. For every  $\nu \in \mathcal{M}_f(\mathbb{R}_+)$ , we set:

$$H(\nu) = \sup \text{supp}(\nu), \quad (2.1)$$

with the convention that  $H(0) = 0$ . Moreover, we let

$$\Delta(\nu) = \sup\{\nu(x) : x \geq 0\} \quad (2.2)$$

be the largest atomic mass of  $\nu$ . We say that  $\nu$  is diffuse if it has no atoms and set  $\Delta(\nu) = 0$  by convention.

Denote by

$$\mathcal{D} = D(\mathbb{R}_+, \mathcal{M}_f(\mathbb{R}_+)) \quad (2.3)$$

the set of  $\mathcal{M}_f(\mathbb{R}_+)$ -valued càdlàg functions equipped with the Skorokhod  $J_1$ -topology. For a function  $\mu = (\mu_t, t \geq 0) \in \mathcal{D}$ , let

$$\Delta(\mu) = \sup_{t \geq 0} \Delta(\mu_t) \quad (2.4)$$

be the largest atom of the entire path of  $\mu$ .

**2.2. The underlying Lévy process and the height process.** We consider a (sub)critical branching mechanism of the form

$$\psi(\lambda) = \alpha\lambda + \beta\lambda^2 + \int_{(0,\infty)} (e^{-\lambda r} - 1 + \lambda r) \pi(dr), \quad \forall \lambda \geq 0, \quad (2.5)$$

where  $\alpha, \beta \geq 0$  and  $\pi \neq 0$  is a  $\sigma$ -finite measure on  $(0, \infty)$  satisfying  $\int_{(0,\infty)} (r \wedge r^2) \pi(dr) < \infty$ . We consider a spectrally positive Lévy process  $X = (X_t, t \geq 0)$  with Laplace exponent  $\psi$  starting from 0. Namely, we have:

$$\mathbb{E} \left[ e^{-\lambda X_t} \right] = e^{t\psi(\lambda)}, \quad \forall t, \lambda \geq 0.$$

We assume that  $X$  is of infinite variation a.s. which is equivalent to the following condition:

$$\beta > 0 \quad \text{or} \quad \int_{(0,1)} r \pi(dr) = \infty. \quad (2.6)$$

Duquesne and Le Gall [12] proved that there exists a process  $H = (H_t, t \geq 0)$  called the  $\psi$ -height process such that for every  $t \geq 0$ , we have the following convergence in probability:

$$H_t = \liminf_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_0^t \mathbf{1}_{\{I_t^s < X_s < I_t^s + \varepsilon\}} ds, \quad (2.7)$$

where, for every  $0 \leq s \leq t$ ,  $I_t^s = \inf_{s \leq r \leq t} X_r$  is the past infimum. They also proved a Ray-Knight theorem for  $H$  which shows that the  $\psi$ -height process  $H$  describes the genealogy of the  $\psi$ -CB process, see [12, Theorem 1.4.1].

**2.3. The exploration process.** Although the height process is not Markov in general, it is a simple function of a measure-valued Markov process, the so-called exploration process that we now introduce. The exploration process  $\rho = (\rho_t, t \geq 0)$  is the  $\mathcal{M}_f(\mathbb{R}_+)$ -valued process defined as follows:

$$\rho_t(\mathrm{d}r) = \beta \mathbf{1}_{[0, H_t]}(r) \mathrm{d}r + \sum_{\substack{0 < s \leq t, \\ X_{s-} < I_t^s}} (I_t^s - X_{s-}) \delta_{H_s}(\mathrm{d}r). \quad (2.8)$$

In particular, the total mass of  $\rho_t$  is  $\langle \rho_t, 1 \rangle = X_t - I_t$ .

We will sometimes refer to  $t \geq 0$  as a *node* in reference to the corresponding real tree when it is well defined (see Section 2.9). For  $s, t \geq 0$ , we say that  $s$  is an ancestor of  $t$  and we write  $s \preceq t$  if  $s \leq t$  and  $H_s = \inf_{s \leq r \leq t} H_r$ . The set

$$\{s \geq 0: s \preceq t\} \quad (2.9)$$

is called the ancestral line of  $t$ . We say that  $t \geq 0$  is a first-generation node with property  $A \subset \mathcal{M}_f(\mathbb{R}_+)$  if  $\rho_t \in A$  and  $\rho_s \notin A$  for every (strict) ancestor  $s$  of  $t$ . For example, we will say that  $t$  is a first-generation node with mass larger than  $\delta > 0$  if  $\Delta(\rho_t) > \delta$  and  $\Delta(\rho_s) \leq \delta$  for every  $s \preceq t$  with  $s \neq t$ . Given  $0 \leq t_1 \leq \dots \leq t_n$ , there exists a unique  $s \geq 0$  such that  $r \preceq t_i$  for every  $1 \leq i \leq n$  if and only if  $r \preceq s$ . We write  $s = t_1 \wedge \dots \wedge t_n$  and call it the most recent common ancestor (MRCA for short) of  $t_1, \dots, t_n$ .

One can recover the height process from the exploration process as follows. Denote by  $\Delta X_t = X_t - X_{t-}$  the jump of the process  $X$  at time  $t$ .

**Proposition 2.1.** *Almost surely for every  $t > 0$ , we have:*

- (i)  $H(\rho_t) = H_t$ ,
- (ii)  $\rho_t = 0$  if and only if  $H_t = 0$ ,
- (iii) if  $\rho_t \neq 0$ , then  $\text{supp}(\rho_t) = [0, H_t]$ ,
- (iv)  $\rho_t = \rho_{t-} + \Delta X_t \delta_{H_t}$ .

In the definition of the exploration process, since  $X$  starts from 0, we have  $\rho_0 = 0$ . In order to state the Markov property of  $\rho$ , we have to define the process  $\rho$  starting from any initial measure  $\nu \in \mathcal{M}_f(\mathbb{R}_+)$ . To that end, for every  $a \in [0, \langle \nu, 1 \rangle]$ , we define the erased measure  $k_a \nu$  by:

$$k_a \nu[0, r] = \nu[0, r] \wedge (\langle \nu, 1 \rangle - a), \quad \forall r \geq 0.$$

If  $a > \langle \nu, 1 \rangle$ , we set  $k_a \nu = 0$ . In words, the measure  $k_a \nu$  is obtained from  $\nu$  by erasing a mass  $a$  backward starting from  $H(\nu)$ . For  $\mu \in \mathcal{M}_f(\mathbb{R}_+)$  with bounded support, we define the concatenation  $[\mu, \nu] \in \mathcal{M}_f(\mathbb{R}_+)$  of the measures  $\mu, \nu$  by:

$$\langle [\mu, \nu], f \rangle = \langle \mu, f \rangle + \langle \nu, f(H(\mu) + \cdot) \rangle, \quad \forall f \in \mathcal{B}_+(\mathbb{R}_+).$$

Finally, we set  $\rho_0^\nu = \nu$  and

$$\rho_t^\nu = [k_{-I_t} \nu, \rho_t], \quad \forall t > 0.$$

We say that  $\rho^\nu = (\rho_t^\nu, t \geq 0)$  is the exploration process started at  $\nu$  and we write  $\mathbb{P}_\nu$  for its distribution.

**Proposition 2.2.** *For any  $\nu \in \mathcal{M}_f(\mathbb{R}_+)$ , the process  $\rho^\nu = (\rho_t^\nu, t \geq 0)$  is a càdlàg strong Markov process in  $\mathcal{M}_f(\mathbb{R}_+)$ .*

**2.4. The excursion measure of the exploration process.** Let us introduce the excursion measure  $\mathbf{N}^\psi$ . Denote by  $I = (I_t, t \geq 0)$  the infimum process of  $X$ :

$$I_t = \inf_{0 \leq s \leq t} X_s. \quad (2.10)$$

Standard results (see e.g. [8]) entail that  $X - I$  is a strong Markov process with values in  $\mathbb{R}_+$  and that the point 0 is regular. Furthermore,  $-I$  is a local time at 0 for  $X - I$ . We denote by  $\mathbf{N}^\psi$  the associated excursion measure of the process  $X - I$  away from 0. It is not difficult to see from (2.7) that  $H_t$  (and thus also  $\rho_t$ ) only depends on the excursion of  $X - I$  above 0 which straddles time  $t$ . It follows that the excursion measure of  $\rho$  away from 0 is the “distribution” of  $\rho$  under  $\mathbf{N}^\psi$ . We still denote it by  $\mathbf{N}^\psi$  and we let

$$\sigma = \inf\{t > 0: \rho_t = 0\} \quad (2.11)$$

be the lifetime of  $\rho$  under  $\mathbf{N}^\psi$  (this coincides with the lifetime of  $X - I$  under  $\mathbf{N}^\psi$ ). In particular, the following holds for every  $\lambda > 0$ :

$$\mathbf{N}^\psi [1 - e^{-\lambda\sigma}] = \psi^{-1}(\lambda) \quad \text{and} \quad \mathbf{N}^\psi [\sigma e^{-\lambda\sigma}] = \frac{1}{\psi' \circ \psi^{-1}(\lambda)}, \quad (2.12)$$

where  $\psi^{-1}$  is the inverse function of  $\psi$ . By letting  $\lambda \rightarrow 0$  we obtain:

$$\mathbf{N}^\psi[\sigma] = \frac{1}{\alpha}, \quad (2.13)$$

with the convention that  $1/0 = \infty$ . Let us recall Bismut’s decomposition for the exploration process. Let  $J_a$  be the random element in  $\mathcal{M}_f(\mathbb{R}_+)$  defined by  $J_a(dr) = \mathbf{1}_{[0,a]}(r) dU_r$ , where  $U$  is a subordinator with Laplace exponent

$$\varphi(\lambda) = \frac{\psi(\lambda)}{\lambda} - \alpha = \beta\lambda + \int_0^\infty (1 - e^{-\lambda r}) \bar{\pi}(r) dr, \quad (2.14)$$

where the tail  $\bar{\pi}$  of the Lévy measure  $\pi$  is defined in (1.7).

**Proposition 2.3.** *For every  $F \in \mathcal{B}_+(\mathcal{M}_f(\mathbb{R}_+))$ , we have:*

$$\mathbf{N}^\psi \left[ \int_0^\sigma F(\rho_t) dt \right] = \int_0^\infty da e^{-\alpha a} \mathbb{E}[F(J_a)]. \quad (2.15)$$

**2.5. Local times of the height process.** Although the height process  $H$  is not Markov in general, one can show that its local time process exists under  $\mathbb{P}$  or  $\mathbf{N}^\psi$ . More precisely, for every  $a > 0$ , there exists a continuous nondecreasing process  $(L_s^a, s \geq 0)$  which can be characterized via the approximation:

$$\lim_{\varepsilon \rightarrow 0} \mathbf{N}^\psi \left[ \mathbf{1}_{\{\sup H > h\}} \sup_{0 \leq s \leq t} \left| \frac{1}{\varepsilon} \int_0^s \mathbf{1}_{\{a-\varepsilon < H_r \leq a\}} dr - L_s^a \right| \right] = 0, \quad \forall t, h \geq 0.$$

Moreover,  $\mathbf{N}^\psi$ -a.e. the support of the measure  $L^a(ds) := d_s L_s^a$  is contained in  $\{s \geq 0: H_s = a\}$  and we have the occupation time formula  $\int_0^\infty da L^a(ds) = \mathbf{1}_{[0,\sigma]}(s) ds$ . Furthermore, the process  $(L_\sigma^a, a \geq 0)$  is a  $\psi$ -CB process under its canonical measure.

Let us recall the excursion decomposition of the exploration process above level  $h > 0$ . Set  $\tau_s^h = \inf\{t > 0: \int_0^t \mathbf{1}_{\{H_r \leq h\}} dr > s\}$  and define the truncated exploration process by:

$$r_h(\rho) = (\rho_{\tau_s^h}, s \geq 0). \quad (2.16)$$

Denote by  $\mathcal{E}_h$  the  $\sigma$ -field generated by the process  $r_h(\rho)$ . Let  $(\alpha_i, \beta_i), i \in I_h$  denote the excursion intervals of  $H$  above level  $h$ . For every  $i \in I$ , we define the measure-valued process  $\rho^i$  by:

$$\langle \rho_s^i, f \rangle = \int_{(a, \infty)} f(r - a) \rho_{\alpha_i + s}(dr) \quad \text{if } 0 < s < \beta_i - \alpha_i$$

and  $\rho_s^i = 0$  if  $s = 0$  or  $s \geq \beta_i - \alpha_i$ . Finally, let  $\ell_i = L_{\alpha_i}^h$  be the local time at level  $h$  at the beginning of the excursion  $\rho^i$ .

**Proposition 2.4.** *Under  $\mathbf{N}^\psi$ , conditionally on  $\mathcal{E}_h$ , the random measure  $\sum_{i \in I} \delta_{(\ell_i, \rho^i)}$  is a Poisson point measure with intensity  $\mathbf{1}_{[0, L_h^h]}(\ell) d\ell \mathbf{N}^\psi[d\rho]$ .*

**2.6. The dual process.** We shall need the  $\mathcal{M}_f(\mathbb{R}_+)$ -valued process  $\eta = (\eta_t, t \geq 0)$  defined by:

$$\eta_t(dr) = \beta \mathbf{1}_{[0, H_t]}(r) dr + \sum_{\substack{0 < s \leq t, \\ X_s - I_t^s}} (X_s - I_t^s) \delta_{H_s}(dr). \quad (2.17)$$

The process  $\eta$  is the dual process of  $\rho$  under  $\mathbf{N}^\psi$  thanks to the following time-reversal property.

**Proposition 2.5.** *The processes  $((\rho_t, \eta_t), t \geq 0)$  and  $((\eta_{(\sigma-s)-}, \rho_{(\sigma-s)-}), t \geq 0)$  have the same distribution under  $\mathbf{N}^\psi$ .*

**2.7. Grafting procedure.** We now explain how to insert a finite collection of measured-valued processes into a measure-valued process. Let  $\mu = (\mu(t), 0 \leq t < \sigma)$  be a  $\mathcal{M}_f(\mathbb{R}_+)$ -valued function with lifetime  $\sigma \in (0, \infty]$  such that  $\mu(t)$  has bounded support for every  $t \in [0, \sigma)$  and let  $\sum_{i=1}^N \delta_{(s_i, \mu_i)}$  be a finite point measure on  $\mathbb{R}_+ \times \mathcal{D}$  where the  $s_i$  are arranged in increasing order and each  $\mu_i$  has a finite lifetime  $\sigma^i$ . Set  $s_0 = \Sigma_0 = 0$  and

$$\Sigma_i = \sum_{k=1}^i \sigma^k, \quad \forall i \geq 1.$$

Define a measure-valued process  $\tilde{\mu}$  by:

$$\tilde{\mu}(t) = \begin{cases} \mu(t - \Sigma_i) & \text{if } s_{i-1} + \Sigma_{i-1} \leq t < (s_i \wedge \sigma) + \Sigma_{i-1}, \\ [\mu(s_i), \mu_i(t - s_i - \Sigma_{i-1})] & \text{if } s_i + \Sigma_{i-1} \leq t < s_i + \Sigma_i \text{ and } s_i < \sigma. \end{cases}$$



Observe that the  $(s_i, \mu_i)$  such that  $s_i \geq \sigma$  do not play a role in this construction and that  $\tilde{\mu}$  has lifetime

$$\sigma + \sum_{i: s_i < \sigma} \sigma^k.$$

We denote this grafting procedure by:

$$\mu \circledast_{i=1}^N (s_i, \mu_i) = \tilde{\mu}. \quad (2.18)$$

In words, this is the process obtained from  $\mu$  by inserting the measure-valued process  $\mu_i$  into  $\mu$  at time  $s_i < \sigma$ .

**2.8. A Poissonian decomposition of the exploration process.** Let  $\nu \in \mathcal{M}_f(\mathbb{R}_+)$ . We write  $\mathbb{P}_\nu^{\psi,*}$  for the distribution of the exploration process  $\rho$  starting at  $\nu$  and killed when it first reaches 0. Let us introduce two probability measures on  $\mathcal{D}$  that will play a major role in the rest of the paper. For every  $r > 0$ , we will write  $\mathbb{P}_r^\psi$  for  $\mathbb{P}_{r\delta_0}^{\psi,*}$ . This should be interpreted as the distribution of the exploration processes with initial mass  $r$ . Furthermore, for every  $\delta > 0$  such that  $\bar{\pi}(\delta) > 0$ , set:

$$\mathbb{Q}_\delta^\psi(d\rho) = \frac{1}{\bar{\pi}(\delta)} \int_{(\delta, \infty)} \pi(dr) \mathbb{P}_r^\psi(d\rho), \quad (2.19)$$

which is the distribution of the exploration process starting from a random initial mass with distribution  $\pi$  conditioned on being larger than  $\delta$ .

We decompose the path of  $\rho$  under  $\mathbb{P}_\nu^{\psi,*}$  according to excursions of the total mass of  $\rho$  above its minimum. Let  $(\alpha_i, \beta_i), i \in I$  denote the excursion intervals of the process  $X - I$  away from 0 under  $\mathbb{P}_\nu^{\psi,*}$ . Define the measure-valued process  $\rho^i$  by  $\rho_{(\alpha_i+s) \wedge \beta_i} = [k_{-I\alpha_i} \nu, \rho_s^i]$ .

**Lemma 2.6.** *The random measure  $\sum_{i \in I} \delta_{(-I\alpha_i, \rho^i)}$  is under  $\mathbb{P}_\nu^{\psi,*}$  a Poisson point measure with intensity  $\mathbf{1}_{[0, (\nu, 1))}(u) du \mathbf{N}^\psi[d\rho]$ . In particular, under  $\mathbb{P}_r^\psi$ , it is a Poisson point measure with intensity  $\mathbf{1}_{[0, r]}(u) du \mathbf{N}^\psi[d\rho]$ .*

Using this decomposition, we can give another useful interpretation of the measure  $\mathbb{P}_r^\psi$ . Let  $\rho$  be the exploration process starting from 0 and let  $(L_s^0, s \geq 0)$  be its local time process at 0. Then the process  $(\tilde{\rho}_t^{(r)}, t \geq 0)$  defined by:

$$\tilde{\rho}_t^{(r)} = (r - L_t^0)_+ \delta_0 + \rho_t \mathbf{1}_{\{L_t^0 \leq r\}} \quad (2.20)$$

has distribution  $\mathbb{P}_r^\psi$ .

In the next lemma, we identify the distribution of the exploration process above level  $H(\nu)$  starting from  $\nu$ . For a measure  $\nu \in \mathcal{M}_f(\mathbb{R}_+)$  and a positive real  $a > 0$ , define  $\theta_a(\nu)$  as the measure  $\nu$  shifted by  $a$ . More formally, define a measure  $\theta_a(\nu)$  on  $\mathbb{R}_+$  by setting:

$$\langle \theta_a(\nu), f \rangle = \int_{[a, \infty)} f(x - a) \nu(dx),$$

for every  $f \in \mathcal{B}_+(\mathbb{R}_+)$  if  $a \leq H(\nu)$  and  $\theta_a(\nu) = 0$  if  $a > H(\nu)$ .

**Lemma 2.7.** *Let  $\nu \in \mathcal{M}_f(\mathbb{R}_+)$  such that  $\nu(H(\nu)) > 0$ . Under  $\mathbb{P}_\nu^{\psi,*}$ , the process  $\tilde{\rho} = (\theta_{H(\nu)}(\rho_t), t \geq 0)$  stopped at the first time it hits 0 has distribution  $\mathbb{P}_{\nu(H(\nu))}^\psi$ .*

*Proof.* We shall use the Poisson decomposition of Lemma 2.6. Using its notations, we have  $\rho_{(t+\alpha_i) \wedge \beta_i} = [k_{-I_{\alpha_i}} \nu, \rho_t^i]$  where  $\sum_{i \in I} \delta_{(-I_{\alpha_i}, \rho^i)}$  is a Poisson point measure with intensity  $\mathbf{1}_{[0, \langle \nu, 1 \rangle)}(u) du \mathbf{N}^\psi[d\rho]$ . Thus, the exploration process above level  $H(\nu)$  stopped at the first time it hits 0 satisfies:

$$\theta_{H(\nu)}(\rho_{(t+\alpha_i) \wedge \beta_i}) = (\nu(H(\nu)) + I_{\alpha_i})\delta_0 + \rho_t^i$$

if  $-I_{\alpha_i} \leq \mu(H(\nu))$  and it is zero if  $-I_{\alpha_i} > \nu(H(\nu))$ . Applying Lemma 2.6 again, it is easy to see that this is also the Poisson decomposition of  $\rho$  under  $\mathbb{P}_{\nu(H(\nu))}^\psi$ . This proves the desired result.  $\square$

**2.9. The Lévy tree.** Recall that the Grey condition

$$\int^\infty \frac{d\lambda}{\psi(\lambda)} < \infty \tag{2.21}$$

is equivalent to the almost sure extinction of the  $\psi$ -CB process in finite time. If it holds, then the height process  $H$  admits a continuous version and one can use the coding of real trees by continuous excursions (see e.g. [16]) in order to define the Lévy tree  $\mathcal{T}$  as the tree coded by the height process  $H$  under its excursion measure  $\mathbf{N}^\psi$ . Then the Grey condition implies that  $\mathcal{T}$  is a *compact* real tree. In the rest of the paper we forego this assumption, but we still interpret the results in terms of trees as they are easier to understand.

### 3. DISTRIBUTION OF THE MAXIMAL DEGREE

Under  $\mathbf{N}^\psi$ , denote by  $\Delta = \Delta(\rho)$  the largest atomic mass of the exploration process. Thanks to [13, Theorem 4.6], if  $\langle \pi, 1 \rangle < \infty$  then the set of discontinuity times of  $\rho$  is  $\mathbf{N}^\psi$ -a.e. finite (and possibly empty). On the other hand, if  $\langle \pi, 1 \rangle = \infty$  then it is countable and dense in  $[0, \sigma]$ . In particular, in that case we have that  $\mathbf{N}^\psi$ -a.e.  $\Delta > 0$ . The main result of this section is the following proposition giving the joint distribution of the lifetime  $\sigma$  and the maximal degree  $\Delta$  under  $\mathbf{N}^\psi$ . Recall from (1.7) and (1.8) the definitions of  $\bar{\pi}$  and  $\psi_\delta$ , and define:

$$\psi_{\delta-}(\lambda) = \lim_{\varepsilon \uparrow 0} \psi_{\delta-\varepsilon}(\lambda) = \left( \alpha + \int_{[\delta, \infty)} r \pi(dr) \right) \lambda + \beta \lambda^2 + \int_{(0, \delta)} (e^{-\lambda r} - 1 + \lambda r) \pi(dr). \tag{3.1}$$

**Proposition 3.1.** *For every  $\delta > 0$  and  $\lambda \geq 0$ , we have:*

$$\mathbf{N}^\psi \left[ 1 - e^{-\lambda \sigma} \mathbf{1}_{\{\Delta \leq \delta\}} \right] = \psi_\delta^{-1}(\bar{\pi}(\delta) + \lambda), \tag{3.2}$$

$$\mathbf{N}^\psi \left[ 1 - e^{-\lambda \sigma} \mathbf{1}_{\{\Delta < \delta\}} \right] = \psi_{\delta-}^{-1}(\bar{\pi}[\delta, \infty) + \lambda). \tag{3.3}$$

In particular, we have:

$$\mathbf{N}^\psi[\Delta > \delta] = \psi_\delta^{-1}(\bar{\pi}(\delta)), \quad (3.4)$$

$$\mathbf{N}^\psi[\Delta \geq \delta] = \psi_\delta^{-1}(\pi[\delta, \infty)). \quad (3.5)$$

Furthermore, if  $\langle \pi, 1 \rangle < \infty$ , then we have:

$$\mathbf{N}^\psi \left[ 1 - e^{-\lambda\sigma} \mathbf{1}_{\{\Delta=0\}} \right] = \psi_0^{-1}(\langle \pi, 1 \rangle + \lambda). \quad (3.6)$$

*Proof.* We only prove (3.2), the proof of (3.3) being similar. Fix  $\delta > 0$  and let  $\lambda, \mu \geq 0$ . Let

$$A = \{\nu \in \mathcal{M}_f(\mathbb{R}_+) : \nu \text{ has an atom with mass } > \delta\}.$$

We shall compute

$$v(\lambda, \mu) = \mathbf{N}^\psi \left[ 1 - e^{-\lambda\sigma - \mu \int_0^\sigma dt \mathbf{1}_{\{\rho_t \in A\}}} \right]. \quad (3.7)$$

We have:

$$\begin{aligned} v(\lambda, \mu) &= \mathbf{N}^\psi \left[ \int_0^\sigma dt (\lambda + \mu \mathbf{1}_{\{\rho_t \in A\}}) e^{-\lambda(\sigma-t) - \mu \int_t^\sigma ds \mathbf{1}_{\{\rho_s \in A\}}} \right] \\ &= \mathbf{N}^\psi \left[ \int_0^\sigma dt (\lambda + \mu \mathbf{1}_{\{\rho_t \in A\}}) \mathbb{E}_{\rho_t}^{\psi,*} \left[ e^{-\lambda\sigma - \mu \int_0^\sigma ds \mathbf{1}_{\{\rho_s \in A\}}} \right] \right], \end{aligned}$$

where we applied the Markov property for the last equality. We shall use Lemma 2.6 to compute the last expectation.

For a measure  $\nu \in \mathcal{M}_f(\mathbb{R}_+)$ , denote by  $H'(\nu)$  the first atom of  $\nu$  with mass larger than  $\delta$ :

$$H'(\nu) = \inf\{x \geq 0 : \nu(x) > \delta\},$$

with the convention that  $\inf \emptyset = +\infty$ .

Suppose that  $\nu \in A$ . Recall from Section 2.8 the excursion decomposition of the exploration process above the minimum of its total mass under  $\mathbb{P}_\nu^{\psi,*}$ . Notice that if  $-I_{\alpha_i} < \nu([H'(\nu), H(\nu)]) - \delta$ , then  $\rho_{(\alpha_i+s) \wedge \beta_i} \in A$  for every  $s \geq 0$ . On the other hand, if  $-I_{\alpha_i} > \nu([H'(\nu), H(\nu)]) - \delta$ , then  $\rho_{(\alpha_i+s) \wedge \beta_i} \in A$  if and only if  $\rho_s^i \in A$ . It follows that

$$\begin{aligned} &\mathbb{E}_\nu^{\psi,*} \left[ e^{-\lambda\sigma - \mu \int_0^\sigma ds \mathbf{1}_{\{\rho_s \in A\}}} \right] \\ &= \mathbb{E}_\nu^{\psi,*} \left[ \exp \left\{ - \sum_{i \in I} \left( \lambda \sigma^i + \mu \int_0^{\beta_i - \alpha_i} ds \mathbf{1}_{\{\rho_{\alpha_i+s} \in A\}} \right) \right\} \right] \\ &= \exp \left\{ - \int_0^{(\nu, 1)} du \mathbf{N}^\psi \left[ 1 - e^{-(\lambda + \mu \mathbf{1}_{\{u < \nu[H'(\nu), H(\nu)] - \delta\}}) \sigma - \mu \mathbf{1}_{\{u > \nu[H'(\nu), H(\nu)] - \delta\}} \int_0^\sigma ds \mathbf{1}_{\{\rho_s \in A\}}} \right] \right\} \\ &= \exp \left\{ - (\nu[H'(\nu), H(\nu)] - \delta) \psi^{-1}(\lambda + \mu) - (\nu[0, H'(\nu)] + \delta) v(\lambda, \mu) \right\}. \end{aligned}$$

Now suppose that  $\nu \notin A$ . Then  $\mathbb{P}_\nu^{\psi,*}$ -a.s. we have the equality  $\{\rho_{(\alpha_i+s)\wedge\beta_i} \in A\} = \{\rho_s^i \in A\}$ . It follows that

$$\mathbb{E}_\nu^{\psi,*} \left[ e^{-\lambda\sigma - \mu \int_0^\sigma ds \mathbf{1}_{\{\rho_s \in A\}}} \right] = \exp \{ -\langle \nu, 1 \rangle v(\lambda, \mu) \}.$$

We deduce that  $v(\lambda, \mu)$  is equal to

$$\begin{aligned} (\lambda + \mu) \mathbf{N}^\psi \left[ \int_0^\sigma dt \mathbf{1}_{\{\rho_t \in A\}} \exp \left\{ -(\rho_t[H'_t, H_t] - \delta) \psi^{-1}(\lambda + \mu) - (\rho_t[0, H'_t] + \delta) v(\lambda, \mu) \right\} \right] \\ + \lambda \mathbf{N}^\psi \left[ \int_0^\sigma dt \mathbf{1}_{\{\rho_t \notin A\}} \exp \{ -\langle \rho_t, 1 \rangle v(\lambda, \mu) \} \right], \end{aligned} \quad (3.8)$$

where  $H'_t = H'(\rho_t)$ .

Thanks to Proposition 2.3, for every  $\theta, \omega \geq 0$  we have:

$$\mathbf{N}^\psi \left[ \int_0^\sigma dt \mathbf{1}_{\{\rho_t \in A\}} \exp \{ -\theta \rho_t[0, H'_t] - \omega \rho_t[H'_t, H_t] \} \right] = \int_0^\infty da e^{-\alpha a} f(a, \theta, \omega), \quad (3.9)$$

where we set

$$f(a, \theta, \omega) := \mathbb{E} \left[ \mathbf{1}_{\{J_a \in A\}} e^{-\theta J_a[0, H'(J_a)] - \omega J_a[H'(J_a), H(J_a)]} \right].$$

Recall that  $J_a(dr) = \mathbf{1}_{[0,a]}(r) dU_r$  where  $U$  is a subordinator with Laplace exponent  $\varphi$  defined in (2.14). Denote by  $T$  be the time of the first jump of  $U$  exceeding  $\delta$ :

$$T := \inf \{ r > 0 : \Delta U_r > \delta \} \quad (3.10)$$

Then it is clear that  $H(J_a) = a$ ,  $H'(J_a) = T$  and  $\{J_a \in A\} = \{T \leq a\}$ . Thus, we get:

$$\begin{aligned} f(a, \theta, \omega) &= \mathbb{E} \left[ \mathbf{1}_{\{T \leq a\}} e^{-\theta U_T - \omega \Delta U_T - \omega(U_a - U_T)} \right] \\ &= \mathbb{E} \left[ \mathbf{1}_{\{T \leq a\}} e^{-\theta U_T - \omega \Delta U_T - \varphi(\omega)(a - T)} \right], \end{aligned}$$

where we used the strong Markov property at time  $T$  for the last equality.

Set

$$s_\delta = \int_\delta^\infty \bar{\pi}(r) dr \quad \text{and} \quad \varphi_\delta(\lambda) = \beta\lambda + \int_0^\delta (1 - e^{-\lambda r}) \bar{\pi}(r) dr. \quad (3.11)$$

Using basic results on Poisson point processes, we have that  $T$  is exponentially distributed with mean  $1/s_\delta$ ,  $\Delta U_T$  has distribution  $s_\delta^{-1} \mathbf{1}_{[\delta, \infty)}(x) \bar{\pi}(x) dx$  and is independent of  $T$ , and the process  $(U_r, 0 \leq r < T)$  is distributed as  $(V_r, 0 \leq r < T)$ , where  $V$  is a subordinator with Laplace exponent  $\varphi_\delta$ , independent of  $(T, \Delta U_T)$ . Therefore, it follows that

$$f(a, \theta, \omega) = \int_0^a dt e^{-s_\delta t - \varphi_\delta(\theta)t - \varphi(\omega)(a-t)} \int_\delta^\infty dx \bar{\pi}(x) e^{-\omega x}.$$

We deduce from (3.9) that

$$\begin{aligned} \mathbf{N}^\psi \left[ \int_0^\sigma dt \mathbf{1}_{\{\rho_t \in A\}} \exp \{ -\theta \rho_t([0, H'_t]) - \omega \rho_t([H'_t, H_t]) \} \right] \\ = \frac{1}{(\alpha + \varphi(\omega))(s_\delta + \alpha + \varphi_\delta(\theta))} \int_\delta^\infty dx \bar{\pi}(x) e^{-\omega x}. \end{aligned} \quad (3.12)$$

Similar arguments yield

$$\begin{aligned}
\mathbf{N}^\psi \left[ \int_0^\sigma dt \mathbf{1}_{\{\rho_t \notin A\}} \exp \{-\theta \langle \rho_t, 1 \rangle\} \right] &= \int_0^\infty da e^{-\alpha a} \mathbb{E} \left[ \mathbf{1}_{\{J_a \notin A\}} e^{-\theta \langle J_a, 1 \rangle} \right] \\
&= \int_0^\infty da e^{-\alpha a} \mathbb{E} \left[ \mathbf{1}_{\{T > a\}} e^{-\theta U_a} \right] \\
&= \frac{1}{s_\delta + \alpha + \varphi_\delta(\theta)}. \tag{3.13}
\end{aligned}$$

It follows from (3.8), (3.12) and (3.13) that

$$\begin{aligned}
v(\lambda, \mu) &= \frac{(\lambda + \mu) e^{\delta(\psi^{-1}(\lambda + \mu) - v(\lambda, \mu))}}{(\alpha + \varphi \circ \psi^{-1}(\lambda + \mu))(s_\delta + \alpha + \varphi_\delta \circ v(\lambda, \mu))} \int_\delta^\infty dx \bar{\pi}(x) e^{-\psi^{-1}(\lambda + \mu)x} \\
&\quad + \frac{\lambda}{s_\delta + \alpha + \varphi_\delta \circ v(\lambda, \mu)}. \tag{3.14}
\end{aligned}$$

From (3.7), it is clear by monotone convergence that  $v(\lambda, \mu) \uparrow v(\lambda)$  as  $\mu \uparrow \infty$ , where

$$v(\lambda) := \mathbf{N}^\psi \left[ 1 - e^{-\lambda \sigma} \mathbf{1}_{\{\Delta \leq \delta\}} \right].$$

Furthermore, thanks to a Tauberian theorem, we have as  $\mu \rightarrow \infty$ :

$$\int_\delta^\infty e^{-\psi^{-1}(\lambda + \mu)x} \bar{\pi}(x) dx \sim \frac{\bar{\pi}(\delta) e^{-\delta \psi^{-1}(\lambda + \mu)}}{\psi^{-1}(\lambda + \mu)}. \tag{3.15}$$

Thus, letting  $\mu \rightarrow \infty$  in (3.14) and using that  $\psi^{-1}(x)(\alpha + \varphi \circ \psi^{-1}(x)) = x$  for every  $x > 0$ , we get:

$$v(\lambda) = \frac{\bar{\pi}(\delta) e^{-\delta v(\lambda)} + \lambda}{s_\delta + \alpha + \varphi_\delta \circ v(\lambda)}. \tag{3.16}$$

Notice that for every  $x > 0$ , we have:

$$\begin{aligned}
s_\delta + \alpha + \varphi_\delta(x) &= \alpha + \varphi(x) + \int_\delta^\infty e^{-xr} \bar{\pi}(r) dr \\
&= \frac{1}{x} \left( \psi(x) + \int_{(\delta, \infty)} (1 - e^{-xr}) \pi(dr) - \bar{\pi}(\delta) (1 - e^{-x\delta}) \right) \\
&= \frac{1}{x} \left( \psi_\delta(x) - \bar{\pi}(\delta) (1 - e^{-x\delta}) \right),
\end{aligned}$$

where we used (2.14) and Fubini's theorem for the second equality and the definition of  $\psi_\delta$  for the last. Thus (3.16) becomes

$$\psi_\delta \circ v(\lambda) = \bar{\pi}(\delta) + \lambda.$$

This yields (3.2). Then (3.6) follows by letting  $\delta \rightarrow 0$ .  $\square$

As a consequence, the following corollary states that the distribution of  $\Delta$  under  $\mathbf{N}^\psi$  on  $(0, \infty)$  and the Lévy measure  $\pi$  have the same support and the same atoms.

**Corollary 3.2.** *The measures  $\mathbf{N}^\psi[\Delta \in \cdot]_{|(0,\infty)}$  and  $\pi$  have the same support. Furthermore, for every  $\delta > 0$ ,  $\mathbf{N}^\psi[\Delta = \delta] > 0$  if and only if  $\delta$  is an atom of the Lévy measure  $\pi$ .*

*Proof.* This is clear from (3.4) and (3.5). □

**Remark 3.3.** More precisely, if  $\delta > 0$  is an atom of  $\pi$ , we have:

$$\mathbf{N}^\psi[\Delta = \delta] = \psi_{\delta^-}^{-1}(\pi[\delta, \infty)) - \psi_{\delta}^{-1}(\bar{\pi}(\delta)). \quad (3.17)$$

Furthermore, if  $\langle \pi, 1 \rangle < \infty$ , then we have:

$$\mathbf{N}^\psi[\Delta > 0] = \psi_0^{-1}(\langle \pi, 1 \rangle) > 0. \quad (3.18)$$

#### 4. DEGREE DECOMPOSITION OF THE LÉVY TREE

In this section, we give a decomposition of the Lévy tree along the large nodes. More precisely, we identify the distribution of the pruned Lévy tree obtained by removing large nodes. Furthermore, we show that the initial Lévy tree can be recovered in distribution from the pruned one by grafting Lévy forests in a Poissonian manner. We apply this decomposition to describe the structure of the discrete tree formed by large nodes.

**4.1. A Poissonian decomposition of the Lévy tree.** The main result of this section is the following Poissonian decomposition along the nodes with mass larger than  $\delta$ . Recall from (2.18) the definition of the grafting procedure  $\otimes$ .

**Theorem 4.1.** *The following holds:*

- (i) *Let  $\delta \geq 0$  such that  $\bar{\pi}(\delta) < \infty$ . Under  $\mathbf{N}^{\psi_\delta}$ , let  $((s_i, \rho_i), i \in I)$  be the atoms of a Poisson point measure with intensity  $\bar{\pi}(\delta) ds \mathbb{Q}_\delta^\psi(d\tilde{\rho})$ , independent of  $\rho$ . Then, under  $\mathbf{N}^{\psi_\delta}$ , the process  $\rho \otimes_{i \in I} (s_i, \rho_i)$  has distribution  $\mathbf{N}^\psi$ .*
- (ii) *Let  $\delta > 0$ . Under  $\mathbf{N}^{\psi_{\delta^-}}$ , let  $((s_i, \rho_i), i \in I)$  be the atoms of a Poisson point measure with intensity*

$$ds \int_{[\delta, \infty)} \pi(dr) \mathbb{P}_r^\psi(d\tilde{\rho}).$$

*Then, under  $\mathbf{N}^{\psi_{\delta^-}}$ , the process  $\rho \otimes_{i \in I} (s_i, \rho_i)$  has distribution  $\mathbf{N}^\psi$ .*

**Remark 4.2.** As mentioned in the introduction, the above theorem is a special case of the main result in [5] where the number of marks is finite. This greatly simplifies the proof which is why we choose to include it. Observe however that the decomposition in [5] is proved under  $\mathbb{P}$  and that an additional argument is needed to show that it still holds under the excursion measures, see the end of the proof below.

*Proof.* We only prove the first part, the second one being similar. Notice that the statement is trivial if  $\bar{\pi}(\delta) = 0$  since in that case we have  $\psi_\delta = \psi$  and the intensity of the Poisson

point measure is 0. Thus we may assume that  $\bar{\pi}(\delta) \in (0, \infty)$ . We shall start by proving the identity under  $\mathbb{P}$  using a coupling argument. Let  $X^\delta = (X_t^\delta, t \geq 0)$  be a Lévy process with Laplace exponent  $\psi_\delta$  and let  $e = (e_t, t \geq 0)$  be an independent Poisson point process on  $\mathbb{R}_+$  with intensity  $\mathbf{1}_{\{r > \delta\}} \pi(dr)$ . Define the process  $X = (X_t, t \geq 0)$  by:

$$X_t = X_t^\delta + \sum_{s \leq t} e_s, \quad \forall t \geq 0.$$

Then the process  $X$  is also a Lévy process with Laplace transform  $\psi_\delta(\lambda) + \int_{(\delta, \infty)} (e^{-\lambda r} - 1) \pi(dr) = \psi(\lambda)$ . In words, the process  $X^\delta$  is obtained from  $X$  by removing jumps of size larger than  $\delta$ .

Denote by  $\rho$  (resp.  $\rho^\delta$ ) the exploration process associated with  $X$  (resp.  $X^\delta$ ). Let  $T_\delta := \inf\{t > 0: \Delta(\rho_t) > \delta\}$  be the first time  $\rho$  contains an atom with mass larger than  $\delta$ . It is clear from the definition that the process  $\rho$  jumps exactly when  $X$  does, so that  $T_\delta = \inf\{t > 0: \Delta X_t > \delta\}$ . Therefore, we have that  $X_t = X_t^\delta$  for  $t < T_\delta$ , which implies that  $\rho_t = \rho_t^\delta$  for  $t < T_\delta$ .

Now, from the construction of  $X$ , we get that  $T_\delta = \inf\{t > 0: e_t > \delta\}$ , that is  $T_\delta$  is the first time that the Poisson point process  $e$  enters in  $(\delta, \infty)$ . Therefore the random time  $T_\delta$  is exponentially distributed with mean  $1/\bar{\pi}(\delta)$  and the jump  $\Delta X_{T_\delta} = e_{T_\delta}$  has distribution  $\mathbf{1}_{\{r > \delta\}} \pi(dr)/\bar{\pi}(\delta)$  and is independent of  $T_\delta$ . Furthermore, the pair  $(T_\delta, \Delta X_{T_\delta})$  is independent of  $X^\delta$ .

Recall from (2.9) the definition of the ancestral line of  $t \in [0, \sigma]$ . Let  $\Delta_t = \sup_{s \leq t} \Delta X_s = \sup_{s \leq t} \Delta(\rho_s)$  be the maximal degree of the ancestral line of  $t$ . For every  $t \geq 0$ , let

$$A(t) := \int_0^t \mathbf{1}_{\{\Delta_s \leq \delta\}} ds \tag{4.1}$$

be the Lebesgue measure of the set of individuals prior to  $t$  whose lineage does not contain any node with mass larger than  $\delta$ . Let  $C_t := \inf\{s \geq 0: A_s > t\}$  be the right-continuous inverse of  $A$  and define the pruned exploration process  $\tilde{\rho} = (\tilde{\rho}_t = \rho_{C_t}, t \geq 0)$ . In other words, we remove from the tree all the individuals above a node with mass larger than  $\delta$  and the pruned exploration process  $\tilde{\rho}$  codes the remaining tree.

Next, let us consider excursions of  $\rho$  above nodes of mass larger than  $\delta$ . Let  $T_\delta^{(1)} = T_\delta$  be the first time  $\rho$  contains an atom with mass larger than  $\delta$  and  $L_\delta^{(1)} = L_\delta = \inf\{t > T_\delta: H_t < H_{T_\delta}\}$  be the first time that atom is erased. Define recursively the stopping times  $T_\delta^{(k)} = \inf\{t > L_\delta^{(k-1)}: \Delta(\rho_t) > \delta\}$  the  $k$ -th time  $\rho$  contains a (first-generation) node with mass larger than  $\delta$  and  $L_\delta^{(k)} = \inf\{t > T_\delta^{(k)}: H_t < H_{T_\delta^{(k)}}\}$  the first time that node is erased. Finally, let  $\rho^{(k)}$  be the path of the exploration process above level  $H_{T_\delta^{(k)}}$  between times  $T_\delta^{(k)}$

and  $L_\delta^{(k)}$ , defined by:

$$\rho_t^{(k)} = \theta_{H_{T_\delta^{(k)}}}(\rho_{t+T_\delta^{(k)}}), \quad \forall 0 \leq t \leq L_\delta^{(k)} - T_\delta^{(k)}.$$

Notice that by construction, we have:

$$\rho = \tilde{\rho} \otimes_{k=1}^{\infty} (A(T_\delta^{(k)}), \rho^{(k)}).$$

Using the strong Markov property under  $\mathbb{P}$  at time  $T_\delta$  and Lemma 2.7, we get that, conditionally on  $\Delta(\rho_{T_\delta})$  (which is equal to  $\Delta X_{T_\delta}$ ), the process  $\rho^{(1)}$  has distribution  $\mathbb{P}_{\Delta X_{T_\delta}}^\psi$ .

But the random time  $T_\delta$  is exponentially distributed with mean  $1/\bar{\pi}(\delta)$ , the jump  $\Delta X_{T_\delta}$  has distribution  $\mathbf{1}_{(\delta, \infty)}(r) \pi(dr)/\bar{\pi}(\delta)$  and they are independent. We deduce that  $\rho^{(1)}$  is independent of  $T_\delta$  and has distribution  $\mathbb{Q}_\delta^\psi$ . Furthermore,  $(T_\delta, \Delta X_{T_\delta})$  is generated by the Poisson point process  $e$  while  $\tilde{\rho}$  is generated by  $X^\delta$ . These being independent, we deduce that  $\tilde{\rho}$  is independent of  $(T_\delta, \Delta X_{T_\delta})$ , and thus of  $(A(T_\delta^{(1)}) = T_\delta, \rho^{(1)})$ . Iterating this argument and using the strong Markov property, we get that the random measure

$$\sum_{k=1}^{\infty} \delta_{(A(T_\delta^{(k)}), \rho^{(k)})}$$

is a Poisson point measure with intensity  $\bar{\pi}(\delta) ds \mathbb{Q}_\delta^\psi(d\rho)$  and is independent of  $\tilde{\rho}$ .

It remains to show that  $\tilde{\rho}$  is distributed as  $\rho^\delta$ . Recall that  $\tilde{\rho}_t = \rho_{C_t}$ . From this, it is clear that the two processes are equal to  $\rho$  before time  $T_\delta$ . Furthermore, at time  $T_\delta$  we have  $\tilde{\rho}_{T_\delta} = \rho_{T_\delta-} = \rho_{L_\delta} = \rho_{T_\delta}^\delta$ . Now applying the strong Markov property to  $\rho$  at  $L_\delta$  gives that, conditionally on  $\tilde{\rho}_{T_\delta}$ , the process  $(\rho_{t+L_\delta}, t \geq 0)$  has distribution  $\mathbb{P}_{\tilde{\rho}_{T_\delta}}$ . As a consequence, conditionally on  $\tilde{\rho}_{T_\delta}$ , the process  $(\tilde{\rho}_t = \rho_{t+L_\delta-T_\delta}, A(T_\delta^{(1)}) \leq t < A(T_\delta^{(2)}))$  is distributed as  $(\rho_t^\delta, S_1 \leq t < S_2)$ , where  $0 \leq S_1 \leq S_2 \leq \dots$  are the ordered atoms of a Poisson point process on  $\mathbb{R}_+$  with intensity  $\bar{\pi}(\delta) ds$ , independent of  $\rho^\delta$ . Iterating this argument, we deduce that  $\tilde{\rho}$  and  $\rho^\delta$  have the same distribution. This proves the Poisson decomposition under  $\mathbb{P}$ . Therefore, the same decomposition holds under the excursion measures up to a normalizing constant: there exists a constant  $c > 0$  such that, under  $\mathbf{N}^{\psi_\delta}$ , the process  $\rho \otimes_{i \in I} (s_i, \rho_i)$  has distribution  $c \mathbf{N}^\psi$ , where the random measure  $\sum_{i \in I} \delta_{(s_i, \rho_i)}$  is under  $\mathbf{N}^{\psi_\delta}$  a Poisson point measure with intensity  $\bar{\pi}(\delta) ds \mathbb{Q}_\delta^\psi(d\tilde{\rho})$ . Let  $\zeta = \text{Card}\{i \in I : s_i < \sigma\}$ . Then, under  $\mathbf{N}^{\psi_\delta}$  and conditionally on  $\rho$ , the random variable  $\zeta$  has Poisson distribution with parameter  $\bar{\pi}(\delta)\sigma$ . It follows that

$$\mathbf{N}^{\psi_\delta} [\zeta \geq 1] = \mathbf{N}^{\psi_\delta} [\mathbf{N}^{\psi_\delta} [\zeta \geq 1 | \rho]] = \mathbf{N}^{\psi_\delta} [1 - e^{-\bar{\pi}(\delta)\sigma}] = \psi_\delta^{-1}(\bar{\pi}(\delta)) = \mathbf{N}^\psi[\Delta > \delta],$$

where in the last equality we used Proposition 3.1. This gives  $c = 1$  and the result readily follows.  $\square$

The following corollary is an immediate consequence of the Poissonian decomposition from Theorem 4.1.



**Corollary 4.3.** *Let  $\delta > 0$  and let  $F \in \mathcal{B}_+(\mathcal{D})$ . We have:*

$$\mathbf{N}^\psi \left[ F(\rho) \mathbf{1}_{\{\Delta \leq \delta\}} \right] = \mathbf{N}^{\psi_\delta} \left[ F(\rho) e^{-\bar{\pi}(\delta)\sigma} \right], \quad (4.2)$$

$$\mathbf{N}^\psi \left[ F(\rho) \mathbf{1}_{\{\Delta < \delta\}} \right] = \mathbf{N}^{\psi_{\delta-}} \left[ F(\rho) e^{-\pi[\delta, \infty)\sigma} \right] \quad (4.3)$$

Furthermore, if  $\langle \pi, 1 \rangle < \infty$ , then we have:

$$\mathbf{N}^\psi \left[ F(\rho) \mathbf{1}_{\{\Delta=0\}} \right] = \mathbf{N}^{\psi_0} \left[ F(\rho) e^{-\langle \pi, 1 \rangle \sigma} \right]. \quad (4.4)$$

The Poissonian decomposition of Theorem 4.1 also holds for forests.

**Proposition 4.4.** *Let  $\delta > 0$  such that  $\bar{\pi}(\delta) < \infty$  and let  $r > 0$ . Under  $\mathbb{P}_r^{\psi_\delta}$  (resp.  $\mathbb{Q}_\delta^{\psi_\delta}$ ), let  $((s_i, \rho_i), i \in I)$  be the atoms of a Poisson point measure with intensity  $\bar{\pi}(\delta) ds \mathbb{Q}_\delta^{\psi_\delta}(d\tilde{\rho})$ . Then, under  $\mathbb{P}_r^{\psi_\delta}$  (resp.  $\mathbb{Q}_\delta^{\psi_\delta}$ ), the process  $\rho \otimes_{i \in I} (s_i, \rho_i)$  has distribution  $\mathbb{P}_r^\psi$  (resp.  $\mathbb{Q}_\delta^\psi$ ).*

**4.2. Structure of nodes with mass larger than  $\delta$ .** Here, we give a description of the structure of nodes with mass larger than  $\delta$  under  $\mathbf{N}^\psi$ . Let us start by determining the distribution of the height of MRCA (see Section 2.3 for the definition) of the set of nodes with mass larger than  $\delta$ .

**Proposition 4.5.** *Under  $\mathbf{N}^\psi$ , conditionally on  $\Delta > \delta$ , the height of the MRCA of the set of nodes with mass larger than  $\delta$  is exponentially distributed with mean  $1/\psi'_\delta(\mathbf{N}^\psi[\Delta > \delta])$ .*

Notice that, as  $\delta \rightarrow \infty$ ,  $\psi'_\delta(\mathbf{N}^\psi[\Delta > \delta])$  converges to  $\alpha$  which is positive in the subcritical case and 0 in the critical case (this implies that the height of the MRCA goes to infinity).

*Proof.* Under  $\mathbf{N}^{\psi_\delta}$ , denote by  $\tau_1 \leq \tau_2 \leq \dots$  the jump times of a standard Poisson process with intensity  $\bar{\pi}(\delta)$ . Denote by  $M = \sup\{i \geq 1 : \tau_i \leq \sigma\}$  the number of marks which arrive during the lifetime  $\sigma$  and set:

$$J = \begin{cases} \inf\{H_s : \tau_1 \leq s \leq \tau_M\} & \text{if } M \geq 1, \\ \infty & \text{if } M = 0. \end{cases}$$

It is clear from Theorem 4.1 that, under  $\mathbf{N}^\psi$ , the height of the MRCA of the set of nodes with mass larger than  $\delta$  is distributed as  $J$  under  $\mathbf{N}^{\psi_\delta}$ , with the convention that this height is equal to  $\infty$  if there are no such nodes. Thus, we need to determine the distribution of  $J$  under  $\mathbf{N}^{\psi_\delta}$  and conditionally on  $M \geq 1$ .

Notice that, on the event  $\{M \geq 2\}$ ,  $J$  agrees with the random variable  $K$  defined in [12, p.96]. Proposition 3.2.3 therein gives:

$$\mathbf{N}^{\psi_\delta} \left[ f(J) \mathbf{1}_{\{M \geq 2\}} \middle| M \geq 1 \right] = \left( \psi'_\delta(\mathbf{N}^\psi[\Delta > \delta]) - \frac{\bar{\pi}(\delta)}{\mathbf{N}^\psi[\Delta > \delta]} \right) \int_0^\infty f(a) e^{-a\psi'_\delta(\mathbf{N}^\psi[\Delta > \delta])} da, \quad (4.5)$$

where we used that  $\psi_\delta(\mathbf{N}^\psi[\Delta > \delta]) = \bar{\pi}(\delta)$  by (3.4).

Next, notice that under  $\mathbf{N}^{\psi_\delta}$ , conditionally on  $\rho$ ,  $M$  has Poisson distribution with parameter  $\bar{\pi}(\delta)\sigma$ . Furthermore, conditionally on  $\rho$  and on  $M = 1$ ,  $\tau_1$  is uniformly distributed on  $[0, \sigma]$ . Thus, by conditioning on  $\rho$ , we get:

$$\begin{aligned} \mathbf{N}^{\psi_\delta} \left[ f(J) \mathbf{1}_{\{M=1\}} \right] &= \bar{\pi}(\delta) \mathbf{N}^{\psi_\delta} \left[ \int_0^\sigma f(H_t) e^{-\bar{\pi}(\delta)\sigma} dt \right] \\ &= \bar{\pi}(\delta) \mathbf{N}^{\psi_\delta} \left[ \int_0^\sigma f(H_t) e^{-\bar{\pi}(\delta)t} e^{-\bar{\pi}(\delta)(\sigma-t)} dt \right] \\ &= \bar{\pi}(\delta) \mathbf{N}^{\psi_\delta} \left[ \int_0^\sigma f(H_t) e^{-\bar{\pi}(\delta)t} \mathbb{E}_{\rho_t}^{\psi_\delta, *} \left[ e^{-\bar{\pi}(\delta)\sigma} \right] dt \right], \end{aligned}$$

where we used the Markov property of the exploration process under  $\mathbf{N}^{\psi_\delta}$  for the last equality. Thanks to Lemma 2.6, for every  $\nu \in \mathcal{M}_f(\mathbb{R}_+)$  we have:

$$\mathbb{E}_\nu^{\psi_\delta, *} \left[ e^{-\bar{\pi}(\delta)\sigma} \right] = e^{-\psi_\delta^{-1}(\bar{\pi}(\delta))\langle \nu, 1 \rangle} = e^{-\mathbf{N}^\psi[\Delta > \delta]\langle \nu, 1 \rangle},$$

where we used (3.4) for the last equality.

Therefore, we get:

$$\begin{aligned} \mathbf{N}^{\psi_\delta} \left[ f(J) \mathbf{1}_{\{M=1\}} \right] &= \bar{\pi}(\delta) \mathbf{N}^{\psi_\delta} \left[ \int_0^\sigma f(H_t) e^{-\bar{\pi}(\delta)t} e^{-\langle \rho_t, 1 \rangle \mathbf{N}^\psi[\Delta > \delta]} dt \right] \\ &= \bar{\pi}(\delta) \mathbf{N}^{\psi_\delta} \left[ \int_0^\sigma f(H_t) e^{-\bar{\pi}(\delta)(\sigma-t)} e^{-\langle \eta_t, 1 \rangle \mathbf{N}^\psi[\Delta > \delta]} dt \right] \\ &= \bar{\pi}(\delta) \mathbf{N}^{\psi_\delta} \left[ \int_0^\sigma f(H_t) e^{-\langle \rho_t + \eta_t, 1 \rangle \mathbf{N}^\psi[\Delta > \delta]} dt \right], \end{aligned}$$

where we used the time-reversal property of the exploration process for the second equality and the Markov property for the last. By [12, Proposition 3.1.3], we deduce that

$$\mathbf{N}^{\psi_\delta} \left[ f(J) \mathbf{1}_{\{M=1\}} \right] = \bar{\pi}(\delta) \int_0^\infty f(a) e^{-\psi'_\delta(\mathbf{N}^\psi[\Delta > \delta])a} da.$$

Thanks to Theorem 4.1, it is clear that  $\mathbf{N}^{\psi_\delta}[M \geq 1] = \mathbf{N}^\psi[\Delta > \delta]$ . It follows that

$$\mathbf{N}^{\psi_\delta} \left[ f(J) \mathbf{1}_{\{M=1\}} \middle| M \geq 1 \right] = \frac{\bar{\pi}(\delta)}{\mathbf{N}^\psi[\Delta > \delta]} \int_0^\infty f(a) e^{-\psi'_\delta(\mathbf{N}^\psi[\Delta > \delta])a} da.$$

In conjunction with (4.5), this yields:

$$\mathbf{N}^{\psi_\delta} [f(J) | M \geq 1] = \psi'_\delta(\mathbf{N}^\psi[\Delta > \delta]) \int_0^\infty f(a) e^{-\psi'_\delta(\mathbf{N}^\psi[\Delta > \delta])a} da.$$

This shows that, under  $\mathbf{N}^{\psi_\delta}$  and conditionally on  $M \geq 1$ ,  $J$  is exponentially distributed with mean  $1/\psi'_\delta(\mathbf{N}^\psi[\Delta > \delta])$  and the proof is now complete.  $\square$

Let  $\tau_\delta$  be the (random) discrete forest spanned by nodes with mass larger than  $\delta$ . More explicitly,  $\tau_\delta$  starts with  $Z_0^\delta$  individuals, where  $Z_0^\delta$  is the number of first-generation nodes of  $\rho$  with mass larger than  $\delta$  (that is nodes of  $\rho$  with mass larger than  $\delta$  having no ancestors with mass larger than  $\delta$ ). Then, each node  $v$  of  $\tau_\delta$  gets  $\xi_v^\delta$  children, where  $\xi_v^\delta$  is the number of first-generation descendants with mass larger than  $\delta$  of the corresponding node in  $\rho$ .

Finally, denote by  $W^\delta$  the total population of  $\tau_\delta$  or equivalently the total number of nodes of  $\rho$  with mass larger than  $\delta$ . We shall identify the distribution of  $\tau_\delta$ . Given two  $\mathbb{N}$ -valued random variables  $Z_0$  and  $\xi$ , we call a  $(Z_0, \xi)$ -Bienaymé-Galton-Watson forest a collection of  $Z_0$  independent Bienaymé-Galton-Watson trees with offspring distribution (the law of)  $\xi$ .

Under  $\mathbf{N}^{\psi_\delta}$  (resp. under  $\mathbb{Q}_\delta^{\psi_\delta}$ ), let  $\sum_{i \in I} \delta_{(s_i, \rho^i)}$  be a Poisson point measure with intensity  $\bar{\pi}(\delta) ds \mathbb{Q}_\delta^\psi(d\tilde{\rho})$  independent of  $\rho$  and let

$$\zeta = \text{Card}\{i \in I: s_i < \sigma\} \quad (4.6)$$

be the number of points arriving during the lifetime  $\sigma$ . Basic properties of Poisson point measures imply that, under  $\mathbf{N}^{\psi_\delta}$  (resp. under  $\mathbb{Q}_\delta^{\psi_\delta}$ ) and conditionally on  $\rho$ , the random variable  $\zeta$  has Poisson distribution with parameter  $\bar{\pi}(\delta)\sigma$ .

**Proposition 4.6.** *Let  $\delta > 0$  such that  $\bar{\pi}(\delta) > 0$ . Under  $\mathbf{N}^\psi$ , the random forest  $\tau_\delta$  is a  $(Z_0^\delta, \xi^\delta)$ -Bienaymé-Galton-Watson forest, where  $Z_0^\delta$  is distributed as  $\zeta$  under  $\mathbf{N}^{\psi_\delta}$  and  $\xi^\delta$  is distributed as  $\zeta$  under  $\mathbb{Q}_\delta^{\psi_\delta}$ . Their Laplace transforms are given by, for every  $\lambda > 0$ :*

$$\mathbf{N}^\psi \left[ 1 - e^{-\lambda Z_0^\delta} \right] = \psi_\delta^{-1} \left( (1 - e^{-\lambda}) \bar{\pi}(\delta) \right), \quad (4.7)$$

$$\mathbf{N}^\psi \left[ e^{-\lambda \xi^\delta} \right] = \frac{1}{\bar{\pi}(\delta)} \int_{(\delta, \infty)} e^{-r \psi_\delta^{-1}((1 - e^{-\lambda}) \bar{\pi}(\delta))} \pi(dr). \quad (4.8)$$

*Proof.* That  $\tau_\delta$  is under  $\mathbf{N}^\psi$  a Bienaymé-Galton-Watson forest with the mentioned distribution is an immediate consequence of the Poissonian decompositions given in Theorem 4.1 and Proposition 4.4. Let us compute the Laplace transforms.

Recall that, under  $\mathbf{N}^{\psi_\delta}$  and conditionally on  $\rho$ ,  $\zeta$  has Poisson distribution with parameter  $\bar{\pi}(\delta)\sigma$ . Using this, we have:

$$\mathbf{N}^\psi \left[ 1 - e^{-\lambda Z_0^\delta} \right] = \mathbf{N}^{\psi_\delta} \left[ 1 - e^{-\lambda \zeta} \right] = \mathbf{N}^{\psi_\delta} \left[ 1 - e^{-(1 - e^{-\lambda}) \bar{\pi}(\delta) \sigma} \right] = \psi_\delta^{-1} \left( (1 - e^{-\lambda}) \bar{\pi}(\delta) \right). \quad (4.9)$$

This proves (4.7). Similarly, since under  $\mathbb{Q}_\delta^{\psi_\delta}$  and conditionally on  $\rho$ ,  $\zeta$  has Poisson distribution with parameter  $\bar{\pi}(\delta)\sigma$ , a similar computation yields:

$$\mathbf{N}^\psi \left[ e^{-\lambda \xi^\delta} \right] = \mathbb{Q}_\delta^{\psi_\delta} (e^{-\lambda \zeta}) = \mathbb{Q}_\delta^{\psi_\delta} \left( e^{-(1 - e^{-\lambda}) \bar{\pi}(\delta) \sigma} \right) = \frac{1}{\bar{\pi}(\delta)} \int_{(\delta, \infty)} \pi(dr) \mathbb{P}_r^{\psi_\delta} \left( e^{-(1 - e^{-\lambda}) \bar{\pi}(\delta) \sigma} \right).$$

But, using the Poisson decomposition of Lemma 2.6, we get that:

$$\mathbb{P}_r^{\psi_\delta} (e^{-x\sigma}) = \exp \left\{ -r \mathbf{N}^{\psi_\delta} \left[ 1 - e^{-x\sigma} \right] \right\} = e^{-r \psi_\delta^{-1}(x)}, \quad \forall x \geq 0, \quad (4.10)$$

and (4.8) readily follows.  $\square$

We end this section with the following result on the criticality of the random forest  $\tau_\delta$ .

**Proposition 4.7.** *Let  $\delta > 0$  such that  $\bar{\pi}(\delta) > 0$ . The mean of  $\xi^\delta$  is given by:*

$$\mathbf{N}^\psi[\xi^\delta] = \frac{\int_{(\delta, \infty)} r \pi(dr)}{\alpha + \int_{(\delta, \infty)} r \pi(dr)}. \quad (4.11)$$

*In particular, under  $\mathbf{N}^\psi$ , the Bienaymé-Galton-Watson forest  $\tau_\delta$  is critical (resp. subcritical) if  $\psi$  is critical (resp. subcritical).*

*Proof.* Thanks to Proposition 4.6, we have:

$$\mathbf{N}^\psi[\xi^\delta] = \mathbb{Q}_\delta^{\psi_\delta}(\zeta) = \bar{\pi}(\delta) \mathbb{Q}_\delta^{\psi_\delta}(\sigma) = \int_{(\delta, \infty)} \pi(dr) \mathbb{P}_r^{\psi_\delta}(\sigma).$$

But the Poissonian decomposition of  $\mathbb{P}_r^{\psi_\delta}$  gives:

$$\mathbb{P}_r^{\psi_\delta}(\sigma) = r \mathbf{N}^{\psi_\delta}[\sigma] = \frac{r}{\alpha + \int_{(\delta, \infty)} z \pi(dz)},$$

where we used (2.13) for the second equality. This yields (4.11).  $\square$

## 5. CONDITIONING ON $\Delta = \delta$

The goal of this section is to make sense of the conditional measure  $\mathbf{N}^\psi[\cdot | \Delta = \delta]$ . For every  $\delta > 0$ , we set:

$$w(\delta) = \mathbf{N}^\psi[\sigma \mathbf{1}_{\{\Delta < \delta\}}] \quad \text{and} \quad w_+(\delta) = \mathbf{N}^\psi[\sigma \mathbf{1}_{\{\Delta \leq \delta\}}]. \quad (5.1)$$

Notice that if  $\delta > 0$  is not an atom of the Lévy measure  $\pi$ , then we have  $w(\delta) = w_+(\delta)$  by Lemma 3.2. Furthermore, thanks to Corollary 4.3, (2.12) and (3.5), we have:

$$w(\delta) = \mathbf{N}^{\psi_{\delta-}} \left[ \sigma e^{-\pi[\delta, \infty)\sigma} \right] = \frac{1}{\psi'_{\delta-} \circ \psi_{\delta-}^{-1}(\pi[\delta, \infty))} = \frac{1}{\psi'_{\delta-}(\mathbf{N}^\psi[\Delta \geq \delta])}. \quad (5.2)$$

Similarly, we have:

$$w_+(\delta) = \mathbf{N}^{\psi_\delta} \left[ \sigma e^{-\bar{\pi}(\delta)\sigma} \right] = \frac{1}{\psi'_\delta \circ \psi_\delta^{-1}(\bar{\pi}(\delta))} = \frac{1}{\psi'_\delta(\mathbf{N}^\psi[\Delta > \delta])}. \quad (5.3)$$

For  $\delta > 0$ , denote by  $\mathbf{P}_\delta^\psi$  the probability measure on the space  $\mathbb{R}_+ \times \mathcal{D}$  defined by:

$$\int_{\mathbb{R}_+ \times \mathcal{D}} F d\mathbf{P}_\delta^\psi = \frac{1}{w(\delta)} \mathbf{N}^\psi \left[ \int_0^\sigma F(s, \rho) ds \mathbf{1}_{\{\Delta < \delta\}} \right], \quad (5.4)$$

for every  $F \in \mathcal{B}_+(\mathbb{R}_+ \times \mathcal{D})$ . Similarly, we set:

$$\int_{\mathbb{R}_+ \times \mathcal{D}} F d\mathbf{P}_{\delta+}^\psi = \frac{1}{w_+(\delta)} \mathbf{N}^\psi \left[ \int_0^\sigma F(s, \rho) ds \mathbf{1}_{\{\Delta \leq \delta\}} \right]. \quad (5.5)$$

Observe that  $\mathbf{P}_{\delta+}^\psi = \lim_{\varepsilon \rightarrow 0+} \mathbf{P}_{\delta+\varepsilon}^\psi$  in the sense of weak convergence of measures.

For every  $\delta, \varepsilon > 0$ , let

$$E_{\delta, \varepsilon} = \{\delta - \varepsilon < \Delta < \delta + \varepsilon, Z_0^{\delta - \varepsilon} = 1\} \quad (5.6)$$

be the event that the maximal degree is between  $\delta - \varepsilon$  and  $\delta + \varepsilon$  and there is a unique first-generation node with mass larger than  $\delta - \varepsilon$ . The next lemma states that, under the assumption that  $\delta$  is not an atom of the Lévy measure  $\pi$ , the two events  $E_{\delta, \varepsilon}$  and  $\{\delta - \varepsilon < \Delta < \delta + \varepsilon\}$  are equivalent in  $\mathbf{N}^\psi$ -measure as  $\varepsilon \rightarrow 0$ . Recall that  $\pi$  is a measure on  $(0, \infty)$  and as such, its support  $\text{supp}(\pi)$  does not contain 0.

**Lemma 5.1.** *Assume that  $\delta \in \text{supp}(\pi)$  is not an atom of the Lévy measure  $\pi$  and that  $\bar{\pi}(\delta) > 0$ . We have  $\mathbf{N}^\psi[\delta - \varepsilon < \Delta < \delta + \varepsilon] \sim \mathbf{N}^\psi[E_{\delta, \varepsilon}]$  as  $\varepsilon \rightarrow 0$ .*

*Proof.* We start by observing that, thanks to the Poissonian decomposition of  $\mathbb{P}_r^\psi$  given in Lemma 2.6, we have:

$$\mathbb{P}_r^\psi(\Delta < \delta) = \begin{cases} 0 & \text{if } r \leq \delta, \\ e^{-r} \mathbf{N}^\psi[\Delta \geq \delta] & \text{if } r > \delta. \end{cases} \quad (5.7)$$

Similarly, we have:

$$\mathbb{P}_r^\psi(\Delta \leq \delta) = \begin{cases} 0 & \text{if } r < \delta, \\ e^{-r} \mathbf{N}^\psi[\Delta > \delta] & \text{if } r \geq \delta. \end{cases} \quad (5.8)$$

We deduce that

$$\begin{aligned} \mathbb{Q}_{\delta - \varepsilon}^\psi(\Delta < \delta + \varepsilon) &= \frac{1}{\bar{\pi}(\delta - \varepsilon)} \int_{(\delta - \varepsilon, \delta + \varepsilon)} \pi(dr) \mathbb{P}_r^\psi(\Delta < \delta + \varepsilon) \\ &= \frac{1}{\bar{\pi}(\delta - \varepsilon)} \int_{(\delta - \varepsilon, \delta + \varepsilon)} e^{-r} \mathbf{N}^\psi[\Delta \geq \delta + \varepsilon] \pi(dr). \end{aligned} \quad (5.9)$$

Since  $\pi(\delta) = 0$  and  $\bar{\pi}(\delta) > 0$ , this implies that

$$\lim_{\varepsilon \rightarrow 0} \mathbb{Q}_{\delta - \varepsilon}^\psi(\Delta < \delta + \varepsilon) = 0. \quad (5.10)$$

Under  $\mathbf{N}^{\psi_{\delta - \varepsilon}}$  and conditionally on  $\rho$ , let  $\zeta$  be a Poisson random variable with parameter  $\bar{\pi}(\delta - \varepsilon)\sigma$  and let  $((s_i, \rho_i), i \geq 1)$  be independent with distribution  $\sigma^{-1} \mathbf{1}_{[0, \sigma]}(s) ds \mathbb{Q}_{\delta - \varepsilon}^\psi(d\tilde{\rho})$ , independent of  $\zeta$ . Thanks to Theorem 4.1, we have:

$$\begin{aligned} \mathbf{N}^\psi[E_{\delta, \varepsilon}] &= \mathbf{N}^{\psi_{\delta - \varepsilon}}[\zeta = 1, \Delta(\rho_1) < \delta + \varepsilon] \\ &= \mathbf{N}^{\psi_{\delta - \varepsilon}}[\bar{\pi}(\delta - \varepsilon)\sigma e^{-\bar{\pi}(\delta - \varepsilon)\sigma} \mathbb{Q}_{\delta - \varepsilon}^\psi(\Delta < \delta + \varepsilon)] \\ &= \bar{\pi}(\delta - \varepsilon) w_+(\delta - \varepsilon) \mathbb{Q}_{\delta - \varepsilon}^\psi(\Delta < \delta + \varepsilon), \end{aligned} \quad (5.11)$$

where we used (5.3) for the last equality. Similarly, we have:

$$\begin{aligned} \mathbf{N}^\psi[\delta - \varepsilon < \Delta < \delta + \varepsilon] &= \mathbf{N}^{\psi_{\delta - \varepsilon}}[\zeta \geq 1; \forall i \leq \zeta, \Delta(\rho_i) < \delta + \varepsilon] \\ &= \mathbf{N}^{\psi_{\delta - \varepsilon}}[\zeta \geq 1; \mathbb{Q}_{\delta - \varepsilon}^\psi(\Delta < \delta + \varepsilon)^\zeta] \end{aligned}$$

$$= \mathbf{N}^{\psi_{\delta-\varepsilon}} \left[ e^{-\bar{\pi}(\delta-\varepsilon)\sigma} \left( e^{\bar{\pi}(\delta-\varepsilon)\sigma} \mathbb{Q}_{\delta-\varepsilon}^{\psi}(\Delta < \delta + \varepsilon) - 1 \right) \right].$$

Therefore, using the inequality  $e^x - 1 - x \leq x^2 e^x / 2$  and the fact that the function  $x \mapsto x e^{-x}$  is bounded on  $\mathbb{R}_+$  by some constant  $C > 0$ , we deduce that

$$\begin{aligned} 0 &\leq \mathbf{N}^{\psi}[\delta - \varepsilon < \Delta < \delta + \varepsilon] - \mathbf{N}^{\psi}[E_{\delta,\varepsilon}] \\ &\leq \frac{1}{2} \bar{\pi}(\delta - \varepsilon)^2 \mathbf{N}^{\psi_{\delta-\varepsilon}} \left[ \sigma^2 e^{-\bar{\pi}(\delta-\varepsilon)\sigma} \mathbb{Q}_{\delta-\varepsilon}^{\psi}(\Delta \geq \delta + \varepsilon) \right] \mathbb{Q}_{\delta-\varepsilon}^{\psi}(\Delta < \delta + \varepsilon)^2 \\ &\leq \frac{C}{2} \bar{\pi}(\delta - \varepsilon) \mathbf{N}^{\psi_{\delta-\varepsilon}}[\sigma] \frac{\mathbb{Q}_{\delta-\varepsilon}^{\psi}(\Delta < \delta + \varepsilon)^2}{\mathbb{Q}_{\delta-\varepsilon}^{\psi}(\Delta \geq \delta + \varepsilon)} \\ &= \frac{C \bar{\pi}(\delta - \varepsilon)}{2(\alpha + \int_{(\delta-\varepsilon, \infty)} r \pi(dr))} \frac{\mathbb{Q}_{\delta-\varepsilon}^{\psi}(\Delta < \delta + \varepsilon)^2}{\mathbb{Q}_{\delta-\varepsilon}^{\psi}(\Delta \geq \delta + \varepsilon)} \\ &\leq C_{\delta} \frac{\mathbb{Q}_{\delta-\varepsilon}^{\psi}(\Delta < \delta + \varepsilon)^2}{\mathbb{Q}_{\delta-\varepsilon}^{\psi}(\Delta \geq \delta + \varepsilon)} \end{aligned}$$

for  $\varepsilon > 0$  small enough and some constant  $C_{\delta}$  which is independent of  $\varepsilon$ , where we used (2.13) for the equality.

Furthermore, it is clear from (5.3) that

$$\bar{\pi}(\delta - \varepsilon) w_+(\delta - \varepsilon) = \bar{\pi}(\delta - \varepsilon) \mathbf{N}^{\psi}[\sigma \mathbf{1}_{\{\Delta \leq \delta - \varepsilon\}}] \geq \bar{\pi}(\delta/2) \mathbf{N}^{\psi}[\sigma \mathbf{1}_{\{\Delta \leq \delta/2\}}],$$

for  $\varepsilon > 0$  small enough. In particular, it follows from (5.11) that there exists a constant  $C'_{\delta} > 0$  such that

$$0 \leq \frac{\mathbf{N}^{\psi}[\delta - \varepsilon < \Delta < \delta + \varepsilon] - \mathbf{N}^{\psi}[E_{\delta,\varepsilon}]}{\mathbf{N}^{\psi}[E_{\delta,\varepsilon}]} \leq C'_{\delta} \frac{\mathbb{Q}_{\delta-\varepsilon}^{\psi}(\Delta < \delta + \varepsilon)}{\mathbb{Q}_{\delta-\varepsilon}^{\psi}(\Delta \geq \delta + \varepsilon)},$$

where the right-hand side goes to 0 as  $\varepsilon \rightarrow 0$  thanks to (5.10). This concludes the proof.  $\square$

As a consequence, since  $E_{\delta,\varepsilon} \subset \{\delta - \varepsilon < \Delta < \delta + \varepsilon\}$ , conditioning on either event is equivalent as  $\varepsilon \rightarrow 0$ . We choose to work with the former as computations will be simpler. We shall next give a description of the exploration process conditioned on  $E_{\delta,\varepsilon}$ .

Let

$$T_{\delta} = \inf\{t > 0: \Delta(\rho_t) > \delta\} \tag{5.12}$$

be the first time that the exploration process contains an atom with mass larger than  $\delta$  and let

$$L_{\delta} = \inf\{t > T_{\delta}: H(\rho_t) < H(\rho_{T_{\delta}})\} \tag{5.13}$$

be the first time that node is erased. We split the path of the exploration process into two parts:  $\rho^{\delta,-}$  is the pruned exploration process (that is the exploration process minus the

first node with mass larger than  $\delta$ ):

$$\rho_t^{\delta,-} = \begin{cases} \rho_t & \text{if } t < T_\delta, \\ \rho_{t-T_\delta+L_\delta} & \text{if } t \geq T_\delta, \end{cases} \quad (5.14)$$

and  $\rho^{\delta,+}$  is the path of the exploration process above the unique first-generation node with mass larger than  $\delta$ :

$$\rho_t^{\delta,+} = \theta_{H_{T_\delta}}(\rho_{(t+T_\delta)\wedge L_\delta}), \quad \forall t \geq 0. \quad (5.15)$$

Notice that  $\rho_0^{\delta,+}$  is a multiple of the Dirac measure at 0.

**Lemma 5.2.** *Let  $F, G \in \mathcal{B}_+(\mathbb{R}_+ \times \mathcal{D})$ . For every  $\delta, \varepsilon > 0$  such that  $\bar{\pi}(\delta - \varepsilon) > 0$ , we have:*

$$\mathbf{N}^\psi \left[ F(T_{\delta-\varepsilon}, \rho^{\delta-\varepsilon,-}) G(\rho^{\delta-\varepsilon,+}) \mathbf{1}_{E_{\delta,\varepsilon}} \right] = \int_{\mathbb{R}_+ \times \mathcal{D}} F \, d\mathbf{P}_{(\delta-\varepsilon)+}^\psi \times \mathbb{Q}_{\delta-\varepsilon}^\psi(G(\rho) | \Delta < \delta + \varepsilon). \quad (5.16)$$

*Proof.* By Theorem 4.1, we have

$$\mathbf{N}^\psi \left[ F(T_{\delta-\varepsilon}, \rho^{\delta-\varepsilon,-}) G(\rho^{\delta-\varepsilon,+}) \mathbf{1}_{E_{\delta,\varepsilon}} \right] = \mathbf{N}^{\psi_{\delta-\varepsilon}} \left[ F(U, \rho) G(\rho^{\delta-\varepsilon}) \mathbf{1}_{\{\zeta=1, \Delta(\rho^{\delta-\varepsilon}) < \delta + \varepsilon\}} \right],$$

where, under  $\mathbf{N}^{\psi_{\delta-\varepsilon}}$ , conditionally on  $\rho$ ,  $U$  is uniformly distributed on  $[0, \sigma]$ ,  $\zeta$  is a Poisson random variable with parameter  $\bar{\pi}(\delta - \varepsilon)\sigma$ ,  $\rho^{\delta-\varepsilon}$  has distribution  $\mathbb{Q}_{\delta-\varepsilon}^\psi$  and they are independent. We deduce that

$$\begin{aligned} \mathbf{N}^\psi \left[ F(T_{\delta-\varepsilon}, \rho^{\delta-\varepsilon,-}) G(\rho^{\delta-\varepsilon,+}) \mathbf{1}_{E_{\delta,\varepsilon}} \right] \\ = \mathbf{N}^{\psi_{\delta-\varepsilon}} \left[ \bar{\pi}(\delta - \varepsilon) e^{-\bar{\pi}(\delta-\varepsilon)\sigma} \int_0^\sigma F(s, \rho) \, ds \right] \mathbb{Q}_{\delta-\varepsilon}^\psi(G(\rho) \mathbf{1}_{\{\Delta < \delta + \varepsilon\}}). \end{aligned}$$

Together with Corollary 4.3, (5.4) and (5.11), this yields the desired result.  $\square$

We now turn to the study of the asymptotic behavior of the measures appearing in the right-hand side of (5.16). Recall that the total variation distance of two probability measures  $P, Q$  on some measurable space  $(E, \mathcal{E})$  is given by:

$$d_{\text{TV}}(P, Q) = \sup\{|P(A) - Q(A)| : A \in \mathcal{E}\}.$$

**Lemma 5.3.** *Assume that  $\delta > 0$  is not an atom of the Lévy measure  $\pi$ . Then, the mapping  $r \mapsto \mathbf{P}_{r+}^\psi$  is continuous at  $\delta$  in total variation distance and  $\mathbf{P}_{\delta+}^\psi = \mathbf{P}_\delta^\psi$ .*

*Proof.* Thanks to Corollary 3.2, we have  $\mathbf{N}^\psi[\Delta = \delta] = 0$ . Then the result readily follows from the definition of the measure  $\mathbf{P}_{r+}^\psi$ .  $\square$

Recall that the space  $\mathcal{M}_f(\mathbb{R}_+)$  is equipped with the topology of weak convergence which makes it a Polish space, see [10, Section 8.3]. It can be metrized by the so-called bounded Lipschitz distance defined for every  $\mu, \nu \in \mathcal{M}_f(\mathbb{R}_+)$  by  $d_{\text{BL}}(\mu, \nu) = \sup |\langle \mu, f \rangle - \langle \nu, f \rangle|$ ,

where the supremum is taken over all Lipschitz-continuous and bounded functions  $f: \mathbb{R}_+ \rightarrow \mathbb{R}$  such that

$$\sup_{x \geq 0} |f(x)| + \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|} \leq 1.$$

Recall that  $\mathcal{D}$  is the space of càdlàg  $\mathcal{M}_f(\mathbb{R}_+)$ -valued functions defined on  $\mathbb{R}_+$ , equipped with the Skorokhod  $J_1$ -topology and let  $d_S$  be the Skorokhod distance associated with the distance  $d_{\text{BL}}$  on  $\mathcal{M}_f(\mathbb{R}_+)$ . Denote by  $\mathcal{D}_0$  the subset of  $\mathcal{D}$  consisting of excursions:

$$\mathcal{D}_0 := \left\{ \mu \in \mathcal{D} : \sigma(\mu) < \infty, \mu_t \neq 0, \forall 0 < t < \sigma(\mu) \text{ and } \mu_{\sigma(\mu)-} = 0 \text{ if } \sigma(\mu) > 0 \right\}, \quad (5.17)$$

where  $\sigma(\mu) = \inf\{t > 0 : \mu(t + \cdot) \equiv 0\}$  is the lifetime of  $\mu$ . Notice that if  $\mu \in \mathcal{D}_0$  such that  $\sigma(\mu) = 0$  then necessarily  $\mu \equiv 0$ . Observe that the mapping  $\mu \mapsto \sigma(\mu)$  is measurable with respect to the Skorokhod topology since  $\sigma(\mu) = \inf\{t \in \mathbb{Q} \cap (0, \infty) : \mu_t = 0\}$  and  $\mu \mapsto \mu_t$  is measurable. We equip  $\mathcal{D}_0$  with the following distance:

$$d_0(\mu, \nu) = d_S(\mu, \nu) + |\sigma(\mu) - \sigma(\nu)|.$$

**Lemma 5.4.** *Let  $\nu \in \mathcal{D}$  and  $s > 0$ . The mapping  $\mu \mapsto \nu \circledast (s, \mu)$  is continuous from  $(\mathcal{D}_0, d_0)$  to  $(\mathcal{D}, d_S)$ .*

*Proof.* Denote by  $\Lambda$  the set of all continuous functions  $\lambda: \mathbb{R}_+ \rightarrow \mathbb{R}_+$  that are (strictly) increasing, with  $\lambda(0) = 0$  and  $\lim_{t \rightarrow \infty} \lambda(t) = \infty$ . Let  $\mu_n$  be a sequence in  $\mathcal{D}_0$  converging to  $\mu$  with respect to the distance  $d_0$ . By definition of the Skorokhod topology (see e.g. Jacod and Shiryaev [23, Chapter VI]), this means that there exists a sequence  $\lambda_n \in \Lambda$  of time changes such that

$$\lim_{n \rightarrow \infty} |\sigma_n - \sigma| = 0, \quad \lim_{n \rightarrow \infty} \sup_{t \in \mathbb{R}_+} |\lambda_n(t) - t| = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \sup_{t \leq N} d_{\text{BL}}(\mu_n \circ \lambda_n(t), \mu(t)) = 0,$$

for every  $N \geq 1$ , where we set  $\sigma_n = \sigma(\mu_n)$  and  $\sigma = \sigma(\mu)$ .

Let  $\kappa_n = \nu \circledast (s, \mu_n)$  and  $\kappa = \nu \circledast (s, \mu)$ . Our goal is to show that  $\kappa_n$  converges to  $\kappa$  with respect to the Skorokhod topology. To this end, let  $\varepsilon_n > 0$  be a sequence converging to 0 such that  $\varepsilon_n > \lambda_n(\sigma) - \sigma_n$  and let  $\tilde{\lambda}_n \in \Lambda$  be a time change such that  $\tilde{\lambda}_n(t) = t$  if  $t \leq s$ ,  $\tilde{\lambda}_n(s + t) = s + \lambda_n(t)$  if  $t \leq \sigma$ ,  $\tilde{\lambda}_n(s + \sigma + t) = s + \sigma_n + t$  if  $t \geq \varepsilon_n$  and  $\tilde{\lambda}_n([s + \sigma, s + \sigma + \varepsilon_n]) = [s + \lambda_n(\sigma), s + \sigma_n + \varepsilon_n]$ . Notice that if  $t \in [s + \sigma, s + \sigma + \varepsilon_n]$ , we have:

$$\left| \tilde{\lambda}_n(t) - t \right| \leq |\lambda_n(\sigma) - \sigma - \varepsilon_n| + |\sigma_n + \varepsilon_n - \sigma| \leq |\lambda_n(\sigma) - \sigma| + |\sigma_n - \sigma| + 2\varepsilon_n.$$

It follows that

$$\begin{aligned} \sup_{t \in \mathbb{R}_+} \left| \tilde{\lambda}_n(t) - t \right| &\leq \sup_{s \leq t \leq s + \sigma} \left| \tilde{\lambda}_n(t) - t \right| + \sup_{s + \sigma \leq t \leq s + \sigma + \varepsilon_n} \left| \tilde{\lambda}_n(t) - t \right| + \sup_{t \geq s + \sigma + \varepsilon_n} \left| \tilde{\lambda}_n(t) - t \right| \\ &\leq \sup_{t \leq \sigma} |\lambda_n(t) - t| + |\lambda_n(\sigma) - \sigma| + 2|\sigma_n - \sigma| + 2\varepsilon_n, \end{aligned}$$

where the right-hand side goes to 0 as  $n \rightarrow \infty$ .



In order to show that  $\kappa_n$  converges to  $\kappa$  in  $\mathcal{D}$ , it is enough to check that

$$\lim_{n \rightarrow \infty} \sup_{t \leq N} d_{\text{BL}}(\kappa_n \circ \tilde{\lambda}_n(t), \kappa(t)) = 0, \quad \forall N \geq 1.$$

If  $t \leq s$ , we have  $\kappa_n \circ \tilde{\lambda}_n(t) = \kappa(t) = \nu(t)$ . If  $t \leq \sigma$  and  $\lambda_n(t) \leq \sigma_n$ , we have:

$$\kappa_n \circ \tilde{\lambda}_n(s+t) = \kappa_n(s + \lambda_n(t)) = [\nu(s), \mu_n \circ \lambda_n(t)] \quad \text{and} \quad \kappa(s+t) = [\nu(s), \mu(t)].$$

It follows that

$$d_{\text{BL}}(\kappa_n \circ \tilde{\lambda}_n(s+t), \kappa(s+t)) \leq d_{\text{BL}}(\mu_n \circ \lambda_n(t), \mu(t)).$$

On the other hand, if  $t \leq \sigma$  and  $\lambda_n(t) > \sigma_n$ , we have:

$$\kappa_n \circ \tilde{\lambda}_n(s+t) = \nu(s + \lambda_n(t) - \sigma_n) \quad \text{and} \quad \kappa(s+t) = [\nu(s), \mu(t)].$$

In that case, we get:

$$\begin{aligned} d_{\text{BL}}(\kappa_n \circ \tilde{\lambda}_n(s+t), \kappa(s+t)) &\leq d_{\text{BL}}(\nu(s + \lambda_n(t) - \sigma_n), \nu(s)) + d_{\text{BL}}(\nu(s), [\nu(s), \mu(t)]) \\ &\leq d_{\text{BL}}(\nu(s + \lambda_n(t) - \sigma_n), \nu(s)) + \langle \mu(t), 1 \rangle. \end{aligned}$$

If  $t \in [s + \sigma, s + \sigma + \varepsilon_n]$ , then  $\kappa_n \circ \tilde{\lambda}_n(t)$  is of the form  $\nu(u)$  with  $u \in [s, s + \varepsilon_n]$  or  $[\nu(s), \mu_n(u)]$  with  $u \in [\lambda_n(\sigma), \sigma_n]$ . We deduce that

$$\begin{aligned} d_{\text{BL}}(\kappa_n \circ \tilde{\lambda}_n(t), \kappa(t)) &\leq \sup_{s \leq u \leq s + \varepsilon_n} d_{\text{BL}}(\nu(u), \nu(t - \sigma)) + \sup_{\lambda_n(\sigma) \leq u \leq \sigma_n} d_{\text{BL}}([\nu(s), \mu_n(u)], \nu(t - \sigma)) \\ &\leq 3 \sup_{s \leq u \leq s + \varepsilon_n} d_{\text{BL}}(\nu(u), \nu(s)) + \sup_{\lambda_n(\sigma) \leq u \leq \sigma_n} \langle \mu_n(u), 1 \rangle. \end{aligned}$$

Finally, if  $t \geq \varepsilon_n$ , then we have  $\kappa_n \circ \tilde{\lambda}_n(s + \sigma + t) = \kappa(s + \sigma + t) = \nu(t)$ . We deduce that

$$\begin{aligned} \sup_{t \leq N} d_{\text{BL}}(\kappa_n \circ \tilde{\lambda}_n(t), \kappa(t)) &\leq \sup_{t \leq N} d_{\text{BL}}(\mu_n \circ \lambda_n(t), \mu(t)) + \sup_{s \leq u \leq s + (\lambda_n(\sigma) - \sigma_n)_+} d_{\text{BL}}(\nu(u), \nu(s)) \\ &\quad + 3 \sup_{s \leq u \leq s + \varepsilon_n} d_{\text{BL}}(\nu(u), \nu(s)) + \sup_{u \leq \sigma, \lambda_n(u) > \sigma_n} \langle \mu(u), 1 \rangle + \sup_{\sigma \leq u \leq N} \langle \mu_n \circ \lambda_n(u), 1 \rangle. \end{aligned} \tag{5.18}$$

Observe that

$$\sup_{u \leq \sigma, \lambda_n(u) > \sigma_n} \langle \mu(u), 1 \rangle = \sup_{\lambda_n^{-1}(\sigma_n) < u \leq \sigma} \langle \mu(u), 1 \rangle \rightarrow 0,$$

since  $\lambda_n^{-1}(\sigma_n) \rightarrow \sigma$  and since  $\mu$  is left-continuous at  $\sigma$  and  $\mu(\sigma) = 0$ . Furthermore, using that  $\mu(u) = 0$  for  $u \geq \sigma$ , we have:

$$\sup_{\sigma \leq u \leq N} \langle \mu_n \circ \lambda_n(u), 1 \rangle \leq \sup_{\sigma \leq u \leq N} d_{\text{BL}}(\mu_n \circ \lambda_n(u), \mu(u)) \rightarrow 0.$$

Since  $\nu$  is right-continuous at  $s$ , we deduce that the right-hand side of (5.18) converges to 0, which concludes the proof.  $\square$

**Lemma 5.5.** *For every  $\delta \in \text{supp}(\pi)$ , the measure  $\mathbb{Q}_{\delta-\varepsilon}^\psi(\cdot|\Delta < \delta + \varepsilon)$  converges weakly to  $\mathbb{P}_\delta^\psi(\cdot|\Delta \leq \delta)$  as  $\varepsilon \rightarrow 0$  on the space  $(\mathcal{D}_0, d_0)$ .*

**Remark 5.6.** Notice that if  $\delta = \inf \text{supp}(\pi)$  is positive, then the measure  $\pi$  is necessarily finite and we have:

$$\mathbb{P}_\delta^\psi(\Delta \leq \delta) \geq \mathbb{P}_\delta^\psi(\Delta = 0) = e^{-\delta \mathbf{N}^\psi[\Delta > 0]} > 0.$$

This implies that the conditional measure  $\mathbb{P}_\delta^\psi(d\rho|\Delta \leq \delta)$  is well defined.

*Proof.* It is enough to show that for every Lipschitz-continuous and bounded function  $F: \mathcal{D}_0 \rightarrow \mathbb{R}$ , the following convergence holds:

$$\lim_{\varepsilon \rightarrow 0} \mathbb{Q}_{\delta-\varepsilon}^\psi(F(\rho)|\Delta < \delta + \varepsilon) = \mathbb{P}_\delta^\psi(F(\rho)|\Delta \leq \delta).$$

Fix such a function  $F$ . From the definition of  $\mathbb{Q}_{\delta-\varepsilon}^\psi$ , we have:

$$\mathbb{Q}_{\delta-\varepsilon}^\psi(F(\rho)\mathbf{1}_{\{\Delta < \delta + \varepsilon\}}) = \frac{1}{\bar{\pi}(\delta - \varepsilon)} \int_{(\delta-\varepsilon, \delta+\varepsilon)} \pi(dr) \mathbb{P}_r^\psi(F(\rho)\mathbf{1}_{\{\Delta < \delta + \varepsilon\}}).$$

In conjunction with (5.9), this gives:

$$\begin{aligned} \mathbb{Q}_{\delta-\varepsilon}^\psi(F(\rho)|\Delta < \delta + \varepsilon) \\ = \frac{1}{\int_{(\delta-\varepsilon, \delta+\varepsilon)} e^{-r \mathbf{N}^\psi[\Delta \geq \delta + \varepsilon]} \pi(dr)} \int_{(\delta-\varepsilon, \delta+\varepsilon)} \pi(dr) \mathbb{P}_r^\psi(F(\rho)\mathbf{1}_{\{\Delta < \delta + \varepsilon\}}). \end{aligned}$$

Now it is not difficult to show that, as  $\varepsilon \rightarrow 0$ , we have:

$$\int_{(\delta-\varepsilon, \delta+\varepsilon)} e^{-r \mathbf{N}^\psi[\Delta \geq \delta + \varepsilon]} \pi(dr) \sim \pi(\delta - \varepsilon, \delta + \varepsilon) e^{-\delta \mathbf{N}^\psi[\Delta > \delta]}.$$

Thus, as  $\varepsilon \rightarrow 0$ , we have:

$$\mathbb{Q}_\delta^\psi(F(\rho)|\Delta < \delta + \varepsilon) \sim \frac{e^{\delta \mathbf{N}^\psi[\Delta > \delta]}}{\pi(\delta - \varepsilon, \delta + \varepsilon)} \int_{(\delta-\varepsilon, \delta+\varepsilon)} \pi(dr) \mathbb{P}_r^\psi(F(\rho)\mathbf{1}_{\{\Delta < \delta + \varepsilon\}}).$$

Thanks to (5.8), we have  $\mathbb{P}_\delta^\psi(\Delta \leq \delta) = e^{-\delta \mathbf{N}^\psi[\Delta > \delta]}$ . Thus, in order to prove the result, it is enough to show that

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\pi(\delta - \varepsilon, \delta + \varepsilon)} \int_{(\delta-\varepsilon, \delta+\varepsilon)} \pi(dr) \mathbb{P}_r^\psi(F(\rho)\mathbf{1}_{\{\Delta < \delta + \varepsilon\}}) = \mathbb{P}_\delta^\psi(F(\rho)\mathbf{1}_{\{\Delta \leq \delta\}}). \quad (5.19)$$

Write:

$$\begin{aligned} & \frac{1}{\pi(\delta - \varepsilon, \delta + \varepsilon)} \int_{(\delta-\varepsilon, \delta+\varepsilon)} \pi(dr) \mathbb{P}_r^\psi(F(\rho)\mathbf{1}_{\{\Delta < \delta + \varepsilon\}}) - \mathbb{P}_\delta^\psi(F(\rho)\mathbf{1}_{\{\Delta \leq \delta\}}) \\ &= \frac{1}{\pi(\delta - \varepsilon, \delta + \varepsilon)} \int_{(\delta-\varepsilon, \delta+\varepsilon)} \pi(dr) \left[ \mathbb{P}_r^\psi(F(\rho)\mathbf{1}_{\{\Delta < \delta + \varepsilon\}}) - \mathbb{P}_\delta^\psi(F(\rho)\mathbf{1}_{\{\Delta < \delta + \varepsilon\}}) \right] \\ & \quad + \mathbb{P}_\delta^\psi(F(\rho)\mathbf{1}_{\{\delta < \Delta < \delta + \varepsilon\}}). \end{aligned}$$

By dominated convergence, it is clear that the second term on the right-hand side converges to 0.

For the first term, one can couple the measures  $\mathbb{P}_r^\psi$  and  $\mathbb{P}_\delta^\psi$  in the following way. Let  $\rho$  be the exploration process with branching mechanism  $\psi$  starting from 0 and let  $(L_t^0, t \geq 0)$  be its local time process at 0. Then the process  $\tilde{\rho}^{(r)}$  defined in (2.20) has distribution  $\mathbb{P}_r^\psi$  while  $\tilde{\rho}^{(\delta)}$  has distribution  $\mathbb{P}_\delta^\psi$ . It follows that

$$\begin{aligned} & \left| \mathbb{P}_r^\psi \left( F(\rho) \mathbf{1}_{\{\Delta < \delta + \varepsilon\}} \right) - \mathbb{P}_\delta^\psi \left( F(\rho) \mathbf{1}_{\{\Delta < \delta + \varepsilon\}} \right) \right| \\ &= \left| \mathbb{E} \left[ F(\tilde{\rho}^{(r)}) \mathbf{1}_{\{\sup_{L_t^0 \leq r} \Delta(\rho_t) < \delta + \varepsilon\}} - F(\tilde{\rho}^{(\delta)}) \mathbf{1}_{\{\sup_{L_t^0 \leq \delta} \Delta(\rho_t) < \delta + \varepsilon\}} \right] \right| \\ &\leq C \mathbb{P} \left( \sup_{\delta < L_t^0 \leq r} \Delta(\rho_t) \geq \delta + \varepsilon \right) + C \mathbb{E} \left[ 1 \wedge d_0(\tilde{\rho}^{(r)}, \tilde{\rho}^{(\delta)}) \right]. \end{aligned} \quad (5.20)$$

Using the Poissonian decomposition from Lemma 2.6, we have for  $r \in (\delta, \delta + \varepsilon)$ :

$$\begin{aligned} \mathbb{P} \left( \sup_{\delta < L_t^0 \leq r} \Delta(\rho_t) \geq \delta + \varepsilon \right) &\leq \mathbb{P} \left( \sup_{\delta < L_t^0 < \delta + \varepsilon} \Delta(\rho_t) \geq \delta \right) \\ &= \mathbb{P} \left( \sup_{\delta < -I_{\alpha_t} < \delta + \varepsilon} \Delta(\rho^i) \geq \delta \right) \\ &= 1 - e^{-\varepsilon \mathbf{N}^\psi[\Delta \geq \delta]}. \end{aligned} \quad (5.21)$$

Next, by definition of  $d_0$  we have that  $d_0(\tilde{\rho}^{(r)}, \tilde{\rho}^{(\delta)}) = |\sigma(\tilde{\rho}^{(r)}) - \sigma(\tilde{\rho}^{(\delta)})| + d_S(\tilde{\rho}^{(r)}, \tilde{\rho}^{(\delta)})$ . We introduce the right-continuous inverse  $S$  of the local time process at 0 given by:

$$S_r = \inf\{t > 0: L_t^0 > r\}, \quad \forall r > 0.$$

It is well known that the process  $S$  is a subordinator. Then the process  $\tilde{\rho}^{(r)}$  has lifetime  $S_r$ . Furthermore, we have:

$$d_S(\tilde{\rho}^{(r)}, \tilde{\rho}^{(\delta)}) \leq \sup_{t \geq 0} d_{\text{BL}}(\tilde{\rho}_t^{(r)}, \tilde{\rho}_t^{(\delta)}).$$

For  $L_t^0 \leq \delta$ , the processes  $\tilde{\rho}^{(r)}$  and  $\tilde{\rho}^{(\delta)}$  differ only by their masses at 0 so that  $d_{\text{BL}}(\tilde{\rho}_t^{(r)}, \tilde{\rho}_t^{(\delta)}) \leq r - \delta \leq \varepsilon$ . On the other hand, for  $L_t^0 > \delta$ , we have  $\tilde{\rho}^{(\delta)} = 0$  so that

$$d_{\text{BL}}(\tilde{\rho}_t^{(r)}, \tilde{\rho}_t^{(\delta)}) = \langle \tilde{\rho}_t^{(r)}, 1 \rangle = (r - L_t^0) + \langle \rho_t, 1 \rangle \leq \varepsilon + \langle \rho_t, 1 \rangle,$$

where we recall from Section 2.3 that  $\langle \rho_t, 1 \rangle = X_t - I_t$ , where  $X$  is the underlying Lévy process and  $I$  is its running infimum. It follows that

$$d_0(\tilde{\rho}^{(r)}, \tilde{\rho}^{(\delta)}) \leq S_{\delta + \varepsilon} - S_\delta + \varepsilon + \sup_{\delta < L_t^0 < \delta + \varepsilon} (X_t - I_t), \quad (5.22)$$

where the right-hand side converges to 0 a.s. as  $\varepsilon \rightarrow 0$ .

Combining (5.20)–(5.22), we deduce that

$$\left| \mathbb{P}_r^\psi \left( F(\rho) \mathbf{1}_{\{\Delta < \delta + \varepsilon\}} \right) - \mathbb{P}_\delta^\psi \left( F(\rho) \mathbf{1}_{\{\Delta < \delta + \varepsilon\}} \right) \right| \leq C_1(\varepsilon),$$

for every  $r \in (\delta, \delta + \varepsilon)$ , where  $C_1(\varepsilon)$  does not depend on  $r$  and goes to 0 as  $\varepsilon \rightarrow 0$ . Similarly, for every  $r \in (\delta - \varepsilon, \delta)$ , we have:

$$\left| \mathbb{P}_r^\psi \left( F(\rho) \mathbf{1}_{\{\Delta < \delta + \varepsilon\}} \right) - \mathbb{P}_\delta^\psi \left( F(\rho) \mathbf{1}_{\{\Delta < \delta + \varepsilon\}} \right) \right| \leq C_2(\varepsilon).$$

Finally, we deduce that

$$\frac{1}{\pi(\delta - \varepsilon, \delta + \varepsilon)} \int_{(\delta - \varepsilon, \delta + \varepsilon)} \pi(dr) \left| \mathbb{P}_r^\psi \left( F(\rho) \mathbf{1}_{\{\Delta < \delta + \varepsilon\}} \right) - \mathbb{P}_\delta^\psi \left( F(\rho) \mathbf{1}_{\{\Delta < \delta + \varepsilon\}} \right) \right| \leq C_1(\varepsilon) + C_2(\varepsilon).$$

Letting  $\varepsilon \rightarrow 0$  proves (5.19) and the proof is complete.  $\square$

We are now in a position to prove the main result of this section which gives a description of the Lévy tree conditioned on having maximal degree  $\delta$ . For every atom  $\delta > 0$  of  $\pi$ , we set:

$$\mathbf{g}(\delta) = \pi(\delta) \mathbb{P}_\delta^\psi(\Delta \leq \delta) = \pi(\delta) e^{-\delta \mathbf{N}^\psi[\Delta > \delta]}, \quad (5.23)$$

where the last equality is due to (5.8). Under  $\mathbf{N}^\psi$ , denote by  $M_\delta$  the random variable defined by:

$$M_\delta = \frac{e^{\mathbf{g}(\delta)\sigma} - 1}{\mathbf{g}(\delta)}.$$

This should be interpreted as  $M_\delta = \sigma$  if  $\delta$  is not an atom of  $\pi$ .

For every atom  $\delta > 0$  of the Lévy measure  $\pi$ , we define a probability measure  $\mathbf{P}_\delta^{\psi, \mathbf{a}}$  on the space  $\mathcal{D}$  as follows. Take  $\tilde{\rho}$  with distribution  $\mathbf{N}^\psi[M_\delta \mathbf{1}_{\{\Delta < \delta\}}]^{-1} \mathbf{N}^\psi[M_\delta \mathbf{1}_{\{\Delta < \delta\}} d\rho]$ , and, conditionally on  $\tilde{\rho}$ , let  $((s_i, \rho_i), i \in I)$  be the atoms of a Poisson point measure with intensity  $\mathbf{g}(\delta) \mathbf{1}_{[0, \sigma]}(s) ds \mathbb{P}_\delta^\psi(d\hat{\rho} | \Delta \leq \delta)$  conditioned on containing at least one point. Then  $\mathbf{P}_\delta^{\psi, \mathbf{a}}$  is defined as the distribution of the process  $\tilde{\rho} \otimes_{i \in I} (s_i, \rho_i)$ .

**Theorem 5.7.** *There exists a regular conditional probability  $\mathbf{N}^\psi[\cdot | \Delta = \delta]$  for  $\delta > 0$  such that  $\pi[\delta, \infty) > 0$ , which is given by, for every  $F \in \mathcal{B}_+(\mathcal{D})$ :*

$$\begin{aligned} \mathbf{N}^\psi[F(\rho) | \Delta = \delta] &= \frac{1}{\mathbf{N}^\psi[M_\delta \mathbf{1}_{\{\Delta < \delta\}}]} \sum_{k=0}^{\infty} \frac{\mathbf{g}(\delta)^k}{(k+1)!} \\ &\times \mathbf{N}^\psi \left[ \int \prod_{i=1}^{k+1} \mathbf{1}_{[0, \sigma]}(s_i) ds_i \mathbb{P}_\delta^\psi(d\rho_i | \Delta \leq \delta) F(\rho \otimes_{i=1}^{k+1} (s_i, \rho_i)) \mathbf{1}_{\{\Delta < \delta\}} \right]. \end{aligned} \quad (5.24)$$

In particular, if  $\delta > 0$  is not an atom of the Lévy measure  $\pi$ , we have:

$$\mathbf{N}^\psi[F(\rho) | \Delta = \delta] = \int_{\mathbb{R}_+ \times \mathcal{D}} \mathbf{P}_\delta^\psi(ds, d\tilde{\rho}) \int_{\mathcal{D}} \mathbb{P}_\delta^\psi(d\hat{\rho} | \Delta \leq \delta) F(\tilde{\rho} \otimes (s, \hat{\rho})). \quad (5.25)$$

If  $\delta > 0$  is an atom of  $\pi$ , we have:

$$\mathbf{N}^\psi[F(\rho) | \Delta = \delta] = \int_{\mathcal{D}} \mathbf{P}_\delta^{\psi, \mathbf{a}}(d\tilde{\rho}) F(\tilde{\rho}). \quad (5.26)$$

**Remark 5.8.** Let  $E_\delta$  be the event that the maximal degree is  $\delta$  and there is a unique first-generation node with mass  $\delta$ . We have:

$$\mathbf{N}^\psi [F(\rho)|E_\delta] = \int_{\mathbb{R}_+ \times \mathcal{D}} \mathbf{P}_\delta^\psi(ds, d\tilde{\rho}) \int_{\mathcal{D}} \mathbb{P}_\delta^\psi(d\hat{\rho}|\Delta \leq \delta) F(\tilde{\rho} \otimes (s, \hat{\rho})). \quad (5.27)$$

When  $\delta$  is an atom of  $\pi$ , this can be proved by taking  $k = 0$  in (5.24). Indeed, we have:

$$\mathbf{N}^\psi [F(\rho)\mathbf{1}_{E_\delta}|\Delta = \delta] = \frac{1}{\mathbf{N}^\psi[M_\delta\mathbf{1}_{\{\Delta < \delta\}}]} \mathbf{N}^\psi \left[ \int \mathbf{1}_{[0, \sigma]}(s) ds \mathbb{P}_\delta^\psi(d\hat{\rho}|\Delta \leq \delta) F(\rho \otimes (s, \hat{\rho})) \right],$$

and the result follows by conditioning. When  $\delta$  is not an atom of  $\pi$ , this follows from Theorem 5.7 together with the fact that, conditionally on  $\Delta = \delta$ , there is a unique node with mass  $\delta$  (see Corollary 5.9 below). In other words, conditioning the exploration process by  $E_\delta$  when  $\delta$  is an atom of  $\pi$  yields the same distribution as conditioning by  $\Delta = \delta$  when  $\delta$  is not an atom of  $\pi$ .

*Proof.* Assume that  $\delta \in \text{supp}(\pi)$  is an atom of  $\pi$ . Then the event  $\{\Delta = \delta\}$  has positive  $\mathbf{N}^\psi$ -measure (see Corollary 3.2) and it follows from Theorem 4.1 that  $\rho$  conditioned on  $\Delta = \delta$  has distribution  $\mathbf{P}_\delta^{\psi, a}$ .

Assume then that  $\delta \in \text{supp}(\pi)$  is not an atom of  $\pi$  and let  $F: \mathcal{D} \rightarrow \mathbb{R}$  be continuous and bounded. Applying Lemma 5.1 and using the fact  $E_{\delta, \varepsilon} \subset \{\delta - \varepsilon < \Delta < \delta + \varepsilon\}$ , we have as  $\varepsilon \rightarrow 0$ :

$$\mathbf{N}^\psi [F(\rho)|\delta - \varepsilon < \Delta < \delta + \varepsilon] \sim \mathbf{N}^\psi [F(\rho)|E_{\delta, \varepsilon}].$$

But, thanks to Lemma 5.2, we have:

$$\begin{aligned} \mathbf{N}^\psi [F(\rho)|E_{\delta, \varepsilon}] &= \mathbf{N}^\psi [F(\rho^{\delta - \varepsilon, -} \otimes (T_{\delta - \varepsilon}, \rho^{\delta - \varepsilon, +})|E_{\delta, \varepsilon}] \\ &= \int_{\mathbb{R}_+ \times \mathcal{D}} \mathbf{P}_{(\delta - \varepsilon)^+}^\psi(ds, d\tilde{\rho}) \int_{\mathcal{D}} \mathbb{Q}_{\delta - \varepsilon}^\psi(d\hat{\rho}|\Delta < \delta + \varepsilon) F(\tilde{\rho} \otimes (s, \hat{\rho})). \end{aligned} \quad (5.28)$$

Recall from Lemma 5.4 that for every fixed  $(\nu, s) \in \mathcal{D} \times (0, \infty)$ , the mapping  $\mu \mapsto \nu \otimes (s, \mu)$  is continuous from  $\mathcal{D}_0$  to  $\mathcal{D}$ . Together with Lemma 5.3 and Lemma 5.5, this gives:

$$\lim_{\varepsilon \rightarrow 0} \mathbf{N}^\psi [F(\rho)|\delta - \varepsilon < \Delta < \delta + \varepsilon] = \int_{\mathbb{R}_+ \times \mathcal{D}} \mathbf{P}_\delta^\psi(ds, d\tilde{\rho}) \int_{\mathcal{D}} \mathbb{P}_\delta^\psi(d\hat{\rho}|\Delta \leq \delta) F(\tilde{\rho} \otimes (s, \hat{\rho})).$$

A standard result on measure differentiation, see e.g. [15, Theorem 1.30], yields the desired result.  $\square$

**Corollary 5.9.** *Assume that  $\delta > 0$  is not an atom of the Lévy measure  $\pi$ . Then, under  $\mathbf{N}^\psi$  and conditionally on  $\Delta = \delta$ , there is a unique node with mass  $\delta$ .*

*Proof.* Notice that  $\mathbf{P}_\delta^\psi$ -a.s.  $\Delta(\rho) < \delta$  by definition. Thus, thanks to Theorem 5.7, it is enough to show that  $\mathbb{P}_\delta^\psi(\cdot|\Delta \leq \delta)$ -a.s. there is a unique node with mass  $\delta$ . We shall use the Poissonian decomposition from Lemma 2.6. Let  $\sum_{i \in I} \delta_{(\ell_i, \rho^i)}$  be a point measure with

distribution  $\mathbb{P}_\delta^\psi$ , that is a Poisson point measure with intensity  $\mathbf{1}_{[0,\delta]}(\ell) d\ell \mathbf{N}^\psi[d\rho]$ . Then it suffices to check that, conditionally on  $\sup_{i \in I} \Delta(\rho^i) \leq \delta$ , it holds that  $\sup_{i \in I} \Delta(\rho^i) < \delta$ .

Since  $\mathbf{N}^\psi[\Delta > \delta/2] < \infty$ , only finitely many  $\rho^i$  are such that  $\Delta(\rho^i) > \delta/2$ . We deduce that

$$\mathbb{P}\left(\sup_{i \in I} \Delta(\rho^i) < \delta\right) = \mathbb{P}\left(\sup_{i \in I} \Delta(\rho^i) \leq \delta, \Delta(\rho^i) \neq \delta \text{ for all } i \in I\right).$$

Thanks to Corollary 3.2, we have  $\mathbf{N}^\psi[\Delta = \delta] = 0$ , which implies that  $\Delta(\rho^i) \neq \delta$  for all  $i \in I$  almost surely. Therefore we get:

$$\mathbb{P}\left(\sup_{i \in I} \Delta(\rho^i) < \delta\right) = \mathbb{P}\left(\sup_{i \in I} \Delta(\rho^i) \leq \delta\right).$$

This proves the result.  $\square$

As an application of Theorem 5.7, we can compute the joint distribution of the degree  $\Delta$  of the exploration process when the Lévy measure  $\pi$  is diffuse and the height  $H_\Delta$  of the (unique) node with mass  $\Delta$ . We start by determining the distribution of  $H(\rho_s)$  under  $\mathbf{P}_\delta^\psi(ds, d\rho)$ . Recall from (5.1) the definition of  $w$ .

**Lemma 5.10.** *Under  $\mathbf{P}_\delta^\psi(ds, d\rho)$  (resp.  $\mathbf{P}_{\delta+}^\psi(ds, d\rho)$ ), the random variable  $H(\rho_s)$  is exponentially distributed with mean  $w(\delta)$  (resp.  $w_+(\delta)$ ).*

*Proof.* We only prove the result under  $\mathbf{P}_\delta^\psi$ , the other being similar. By definition, we have:

$$\begin{aligned} \int_{\mathbb{R}_+ \times \mathcal{D}} \mathbf{1}_{\{H(\rho_s) > h\}} \mathbf{P}_\delta^\psi(ds, d\rho) &= \frac{1}{w(\delta)} \mathbf{N}^\psi \left[ \int_0^\sigma \mathbf{1}_{\{H_s > h\}} ds \mathbf{1}_{\{\Delta < \delta\}} \right] \\ &= \frac{1}{w(\delta)} \mathbf{N}^{\psi_{\delta-}} \left[ e^{-\pi[\delta, \infty)\sigma} \int_0^\sigma \mathbf{1}_{\{H_s > h\}} ds \right], \end{aligned}$$

where we used Corollary 4.3 for the last equality.

Thanks to Bismut's decomposition, see e.g. [3, Theorem 2.1], we have for every  $\lambda > 0$ :

$$\begin{aligned} &\mathbf{N}^{\psi_{\delta-}} \left[ e^{-\lambda\sigma} \int_0^\sigma \mathbf{1}_{\{H_s > h\}} ds \right] \\ &= \int_h^\infty dt \exp \left\{ -t \left[ \psi'_{\delta-}(0) + 2\beta \mathbf{N}^{\psi_{\delta-}}[1 - e^{-\lambda\sigma}] + \int_{(0,\delta)} r \pi(dr) \mathbb{P}_r^{\psi_{\delta-}}(1 - e^{-\lambda\sigma}) \right] \right\} \\ &= \int_h^\infty dt \exp \left\{ -t \left[ \psi'_{\delta-}(0) + 2\beta \psi_{\delta-}^{-1}(\lambda) + \int_{(0,\delta)} r(1 - e^{-r\psi_{\delta-}^{-1}(\lambda)}) \pi(dr) \right] \right\} \\ &= \int_h^\infty dt e^{-t\psi'_{\delta-} \circ \psi_{\delta-}^{-1}(\lambda)} \\ &= \frac{1}{\psi'_{\delta-} \circ \psi_{\delta-}^{-1}(\lambda)} e^{-h\psi'_{\delta-} \circ \psi_{\delta-}^{-1}(\lambda)}. \end{aligned} \tag{5.29}$$

Applying this to  $\lambda = \pi[\delta, \infty)$  and using (5.2), it follows that

$$\int_{\mathbb{R}_+ \times \mathcal{D}} \mathbf{1}_{\{H(\rho_s) > h\}} \mathbf{P}_\delta^\psi(ds, d\rho) = e^{-h/w(\delta)}.$$

This proves the result.  $\square$

Let

$$T_\Delta = \inf\{t \geq 0: \Delta(\rho_t) = \Delta\} \quad (5.30)$$

be the first time that  $\rho$  contains an atom with mass  $\Delta$  and let  $H_\Delta = H(\rho_{T_\Delta})$  be the value of the height process at that time. We shall determine the joint distribution of  $(\Delta, H_\Delta)$  assuming that the Lévy measure  $\pi$  is diffuse.

**Proposition 5.11.** *Assume that the Lévy measure  $\pi$  is diffuse. Then,  $\mathbf{N}^\psi$ -a.e. there is a unique node with mass  $\Delta$ . Furthermore, for every  $\delta, h > 0$ , we have:*

$$\mathbf{N}^\psi[\Delta > \delta, H_\Delta > h] = \int_{(\delta, \infty)} e^{-h/w(r)} \mathbf{N}^\psi[\Delta \in dr]. \quad (5.31)$$

*In other words, under  $\mathbf{N}^\psi$  and conditionally on  $\Delta = \delta$ ,  $H_\Delta$  is exponentially distributed with mean  $w(\delta)$ .*

**Question 5.12.** If  $\delta$  is an atom of  $\pi$ , what is the distribution of the height of the MRCA of the nodes with mass exactly  $\delta$  under  $\mathbf{N}^\psi$ , conditionally on  $\Delta = \delta$ ?

*Proof.* The first part follows from Corollary 5.9. Then, using Theorem 5.7, we have:

$$\mathbf{N}^\psi[\Delta > \delta, H_\Delta > h] = \int_{(\delta, \infty)} \mathbf{N}^\psi[H_\Delta > h | \Delta = r] \mathbf{N}^\psi[\Delta \in dr].$$

Now under  $\mathbf{N}^\psi[\cdot | \Delta = r]$ ,  $H_\Delta$  is distributed as  $H_s = H(\rho_s)$  under  $\mathbf{P}_r^\psi(ds, d\rho)$ . Lemma 5.10 allows to conclude.  $\square$

## 6. CONDITIONING ON $\Delta = \delta$ AND $H_\Delta = h$

In this section, we assume that the Lévy measure  $\pi$  is diffuse. Recall then from Proposition 5.11 that there is a unique node with mass  $\Delta$  and  $H_\Delta$  is its height. The goal of this section is to make sense of the conditional measure  $\mathbf{N}^\psi[\cdot | \Delta = \delta, H_\Delta = h]$ . Let

$$F_{\delta, \varepsilon} = \{\delta - \varepsilon < \Delta < \delta + \varepsilon, Z_0^{\delta - \varepsilon} = 1, h - \varepsilon < H(\rho_{T_{\delta - \varepsilon}}) < h + \varepsilon\} \quad (6.1)$$

be the event that the maximal degree is between  $\delta - \varepsilon$  and  $\delta + \varepsilon$ , there is a unique first-generation node with mass larger than  $\delta - \varepsilon$  and its height is between  $h - \varepsilon$  and  $h + \varepsilon$ . Recall from (5.1) the definition of  $w$ .

**Lemma 6.1.** *Assume that the Lévy measure  $\pi$  is diffuse. For every  $\delta \in \text{supp}(\pi)$  such that  $\bar{\pi}(\delta) > 0$  and  $h > 0$ , we have as  $\varepsilon \rightarrow 0$ :*

$$\begin{aligned} \mathbf{N}^\psi[\delta - \varepsilon < \Delta < \delta + \varepsilon, h - \varepsilon < H_\Delta < h + \varepsilon] &\sim \mathbf{N}^\psi[F_{\delta,\varepsilon}] \\ &\sim 2\varepsilon \mathbb{Q}_{\delta-\varepsilon}^\psi(\Delta < \delta + \varepsilon) \bar{\pi}(\delta) e^{-h/w(\delta)}. \end{aligned} \quad (6.2)$$

*Proof.* By Proposition 5.11, we have:

$$\mathbf{N}^\psi[\delta - \varepsilon < \Delta < \delta + \varepsilon, h - \varepsilon < H_\Delta < h + \varepsilon] = \int_{(\delta-\varepsilon, \delta+\varepsilon)} \mathbf{N}^\psi[\Delta \in dr] w(r)^{-1} \int_{h-\varepsilon}^{h+\varepsilon} e^{-t/w(r)} dt.$$

A straightforward application of the dominated convergence theorem gives:

$$\mathbf{N}^\psi[\delta - \varepsilon < \Delta < \delta + \varepsilon, h - \varepsilon < H_\Delta < h + \varepsilon] \sim 2\varepsilon \int_{(\delta-\varepsilon, \delta+\varepsilon)} \mathbf{N}^\psi[\Delta \in dr] w(r)^{-1} e^{-h/w(r)}.$$

Since  $\pi$  is diffuse, observe that  $\mathbf{N}^\psi[\sigma \mathbf{1}_{\{\Delta=r\}}] = 0$  for every  $r > 0$  thanks to Corollary 3.2. This implies that  $w$  is continuous and we deduce that

$$\mathbf{N}^\psi[\delta - \varepsilon < \Delta < \delta + \varepsilon, h - \varepsilon < H_\Delta < h + \varepsilon] \sim 2\varepsilon \mathbf{N}^\psi[\delta - \varepsilon < \Delta < \delta + \varepsilon] w(\delta)^{-1} e^{-h/w(\delta)}.$$

But Lemma 5.1 gives:

$$\mathbf{N}^\psi[\delta - \varepsilon < \Delta < \delta + \varepsilon] \sim \mathbf{N}^\psi[E_{\delta,\varepsilon}].$$

Moreover, thanks to (5.11) and the continuity of  $w$ , we have:

$$\mathbf{N}^\psi[E_{\delta,\varepsilon}] = \bar{\pi}(\delta - \varepsilon) w(\delta - \varepsilon) \mathbb{Q}_{\delta-\varepsilon}^\psi(\Delta < \delta + \varepsilon) \sim \bar{\pi}(\delta) w(\delta) \mathbb{Q}_{\delta-\varepsilon}^\psi(\Delta < \delta + \varepsilon).$$

We deduce that

$$\mathbf{N}^\psi[\delta - \varepsilon < \Delta < \delta + \varepsilon, h - \varepsilon < H_\Delta < h + \varepsilon] \sim 2\varepsilon \mathbb{Q}_{\delta-\varepsilon}^\psi(\Delta < \delta + \varepsilon) \bar{\pi}(\delta) e^{-h/w(\delta)}.$$

On the other hand, thanks to Theorem 4.1, we have:

$$\mathbf{N}^\psi[F_{\delta,\varepsilon}] = \mathbf{N}^{\psi_{\delta-\varepsilon}} \left[ \bar{\pi}(\delta - \varepsilon) e^{-\bar{\pi}(\delta-\varepsilon)\sigma} \int_0^\sigma \mathbf{1}_{\{h-\varepsilon < H_s < h+\varepsilon\}} ds \right] \mathbb{Q}_{\delta-\varepsilon}^\psi(\Delta < \delta + \varepsilon). \quad (6.3)$$

Using Bismut's decomposition as in (5.29), we get:

$$\mathbf{N}^{\psi_{\delta-\varepsilon}} \left[ e^{-\bar{\pi}(\delta-\varepsilon)\sigma} \int_0^\sigma \mathbf{1}_{\{h-\varepsilon < H_s < h+\varepsilon\}} ds \right] = \int_{h-\varepsilon}^{h+\varepsilon} e^{-t/w(\delta-\varepsilon)} dt \sim 2\varepsilon e^{-h/w(\delta)}, \quad (6.4)$$

where again we used the continuity of  $w$ . It follows that

$$\mathbf{N}^\psi[F_{\delta,\varepsilon}] \sim 2\varepsilon \mathbb{Q}_{\delta-\varepsilon}^\psi(\Delta < \delta + \varepsilon) \bar{\pi}(\delta) e^{-h/w(\delta)}.$$

□

For every  $\delta, h > 0$ , denote by  $\mathbf{P}_{\delta,h}^\psi$  the probability measure on the space  $\mathbb{R}_+ \times \mathcal{D}$  defined by:

$$\int_{\mathbb{R}_+ \times \mathcal{D}} F d\mathbf{P}_{\delta,h}^\psi = \frac{1}{\mathbf{N}^\psi[L_\sigma^h \mathbf{1}_{\{\Delta < \delta\}}]} \mathbf{N}^\psi \left[ \int_0^\sigma F(s, \rho) L^h(ds) \mathbf{1}_{\{\Delta < \delta\}} \right], \quad (6.5)$$



for every  $F \in \mathcal{B}_+(\mathbb{R}_+ \times \mathcal{D})$ . Since we are assuming that the Lévy measure  $\pi$  is diffuse, we may replace the event  $\{\Delta < \delta\}$  by  $\{\Delta \leq \delta\}$  thanks to Corollary 3.2. Thus, using Corollary 4.3, [13, Theorem 4.5] and (5.3), we have:

$$\mathbf{N}^\psi[L_\sigma^h \mathbf{1}_{\{\Delta < \delta\}}] = \mathbf{N}^{\psi_\delta} \left[ L_\sigma^h e^{-\bar{\pi}(\delta)\sigma} \right] = e^{-h\psi'_\delta(\mathbf{N}^\psi[\Delta > \delta])} = e^{-h/w(\delta)}. \quad (6.6)$$

In particular, the following identity relating the measures  $\mathbf{P}_\delta^\psi$  and  $\mathbf{P}_{\delta,h}^\psi$  holds:

$$\frac{1}{w(\delta)} \int_0^\infty dh e^{-h/w(\delta)} \mathbf{P}_{\delta,h}^\psi(ds, d\rho) = \mathbf{P}_\delta^\psi(ds, d\rho),$$

where we used that  $\mathbf{1}_{[0,\sigma]}(s) ds = \int_0^a da L^a(ds)$ , see Section 2.5. The next lemma gives an approximation of the measure  $\mathbf{P}_{\delta,h}^\psi$ .

**Lemma 6.2.** *Let  $F: \mathbb{R}_+ \times \mathcal{D} \rightarrow \mathbb{R}$  be measurable and bounded. We have for every  $\delta, h > 0$ :*

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{2\varepsilon} \mathbf{N}^\psi \left[ \int_0^\sigma F(s, \rho) \mathbf{1}_{\{h-\varepsilon < H_s < h+\varepsilon\}} ds \mathbf{1}_{\{\Delta \leq \delta-\varepsilon\}} \right] = \mathbf{N}^\psi \left[ \int_0^\sigma F(s, \rho) L^h(ds) \mathbf{1}_{\{\Delta < \delta\}} \right]. \quad (6.7)$$

*Proof.* Recall from Section 2.5 that the measure  $L^a$  is supported on the set  $\{s \in [0, \sigma]: H_s = a\}$ . Thus, we have:

$$\frac{1}{2\varepsilon} \int_0^\sigma F(s, \rho) \mathbf{1}_{\{h-\varepsilon < H_s < h+\varepsilon\}} ds = \frac{1}{2\varepsilon} \int_{h-\varepsilon}^{h+\varepsilon} da \int_0^\sigma F(s, \rho) L^a(ds). \quad (6.8)$$

Furthermore,  $h$  is a jump time for the local time process  $a \mapsto L^a$  if and only if it is a jump time for the total mass process  $a \mapsto L_\sigma^a$ . But, under  $\mathbf{N}^\psi$ , the process  $(L_\sigma^a, a \geq 0)$  is a  $\psi$ -CB process. In particular, it has no fixed jump times. As a result,  $\mathbf{N}^\psi$ -a.e. the mapping  $a \mapsto L^a$  is continuous at  $h$ . We deduce that the following convergence holds  $\mathbf{N}^\psi$ -a.e.:

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{2\varepsilon} \int_0^\sigma F(s, \rho) \mathbf{1}_{\{h-\varepsilon < H_s < h+\varepsilon\}} ds \mathbf{1}_{\{\Delta \leq \delta-\varepsilon\}} = \int_0^\sigma F(s, \rho) L^h(ds) \mathbf{1}_{\{\Delta < \delta\}}.$$

Next, using (6.8), we have:

$$\frac{1}{2\varepsilon} \left| \int_0^\sigma F(s, \rho) \mathbf{1}_{\{h-\varepsilon < H_s < h+\varepsilon\}} ds \right| \mathbf{1}_{\{\Delta \leq \delta-\varepsilon\}} \leq \frac{\|F\|_\infty}{2\varepsilon} \int_{h-\varepsilon}^{h+\varepsilon} L_\sigma^a da,$$

where the last term converges  $\mathbf{N}^\psi$ -a.e. to  $\|F\|_\infty L_\sigma^h$  thanks to the continuity of  $a \mapsto L_\sigma^a$  at  $h$ . Furthermore, by [13, Eq. (12)] we have the convergence:

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{2\varepsilon} \mathbf{N}^\psi \left[ \int_{h-\varepsilon}^{h+\varepsilon} L_\sigma^a da \right] = \mathbf{N}^\psi[L_\sigma^h].$$

Thus, the generalized dominated convergence theorem yields (6.7).  $\square$

The main result of this section is the following description of the exploration process conditioned on having maximal degree  $\delta$  at height  $h$ .

**Theorem 6.3.** *Assume that the Lévy measure  $\pi$  is diffuse. There exists a conditional probability measure  $\mathbf{N}^\psi[\cdot|\Delta = \delta, H_\Delta = h]$  for  $\delta \in \text{supp}(\pi)$ . Furthermore, for every  $F \in \mathcal{B}_+(\mathcal{D})$ , we have:*

$$\mathbf{N}^\psi[F(\rho)|\Delta = \delta, H_\Delta = h] = \int_{\mathbb{R}_+ \times \mathcal{D}} \mathbf{P}_{\delta, h}^\psi(ds, d\tilde{\rho}) \int_{\mathcal{D}} \mathbb{P}_\delta^\psi(d\hat{\rho}|\Delta \leq \delta) F(\tilde{\rho} \otimes (s, \hat{\rho})). \quad (6.9)$$

Assuming the Grey condition, this can be interpreted as follows in terms of trees. Under  $\mathbf{N}^\psi$ , conditionally on  $\Delta = \delta$ ,  $H_\Delta$  is exponentially distributed with mean  $w(\delta)$ . Moreover, conditionally on  $\Delta = \delta$  and  $H_\Delta = h$ , the Lévy tree can be constructed as follows: start with  $\tilde{\mathcal{T}}$  with distribution  $\mathbf{N}^\psi[L_\sigma^h \mathbf{1}_{\{\Delta < \delta\}}]^{-1} \mathbf{N}^{\psi_\delta}[L_\sigma^h \mathbf{1}_{\{\Delta < \delta\}} d\mathcal{T}]$ , choose a leaf uniformly at random in  $\tilde{\mathcal{T}}$  at height  $h$  (i.e. according to the probability measure  $L^h(dx)/L_\sigma^h$ ) and on this leaf graft an independent Lévy forest with initial mass  $\delta$  conditioned to have degree  $\leq \delta$ . Notice that this result generalizes Theorem 5.7 when the Lévy measure  $\pi$  is diffuse. In particular, one can recover the latter simply by integrating with respect to  $h$ .

*Proof.* Let  $\delta \in \text{supp}(\pi)$  and  $h > 0$ . Thanks to Lemma 6.1, we have as  $\varepsilon \rightarrow 0$ :

$$\mathbf{N}^\psi[F(\rho)|\delta - \varepsilon < \Delta < \delta + \varepsilon, h - \varepsilon < H_\Delta < h + \varepsilon] \sim \mathbf{N}^\psi[F(\rho)|F_{\delta, \varepsilon}]. \quad (6.10)$$

Recall from (5.14) and (5.15) the definitions of  $\rho^{\delta-\varepsilon, -}$  and  $\rho^{\delta-\varepsilon, +}$ . Using the Poissonian decomposition from Theorem 4.1 and Corollary 4.3, we have:

$$\begin{aligned} & \mathbf{N}^\psi[F(\rho)\mathbf{1}_{F_{\delta, \varepsilon}}] \\ &= \mathbf{N}^\psi[F(\rho^{\delta-\varepsilon, -} \otimes (T_{\delta-\varepsilon}, \rho^{\delta-\varepsilon, +}))\mathbf{1}_{F_{\delta, \varepsilon}}] \\ &= \bar{\pi}(\delta - \varepsilon) \mathbf{N}^\psi\left[\int \mathbf{1}_{[0, \sigma]}(s) ds \mathbb{Q}_{\delta-\varepsilon}^\psi(\mathbf{1}_{\{\Delta < \delta + \varepsilon\}} d\hat{\rho}) F(\rho \otimes (s, \hat{\rho})) \mathbf{1}_{\{h-\varepsilon < H_s < h + \varepsilon, \Delta \leq \delta - \varepsilon\}}\right]. \end{aligned}$$

By conditioning, it follows from (6.3) that

$$\begin{aligned} \mathbf{N}^\psi[F(\rho)|F_{\delta, \varepsilon}] &= \frac{1}{\mathbf{N}^\psi\left[\int_0^\sigma \mathbf{1}_{\{h-\varepsilon < H_s < h + \varepsilon\}} ds \mathbf{1}_{\{\Delta \leq \delta - \varepsilon\}}\right]} \\ &\times \mathbf{N}^\psi\left[\int \mathbf{1}_{[0, \sigma]}(s) ds \mathbb{Q}_{\delta-\varepsilon}^\psi(d\hat{\rho}|\Delta < \delta + \varepsilon) F(\rho \otimes (s, \hat{\rho})) \mathbf{1}_{\{h-\varepsilon < H_s < h + \varepsilon, \Delta \leq \delta - \varepsilon\}}\right]. \end{aligned} \quad (6.11)$$

Therefore, using Lemma 6.2, Lemma 5.4 and Lemma 5.5, we deduce that

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \mathbf{N}^\psi[F(\rho)|\delta - \varepsilon < \Delta < \delta + \varepsilon, h - \varepsilon < H_\Delta < h + \varepsilon] \\ = \int_{\mathbb{R}_+ \times \mathcal{D}} F d\mathbf{P}_{\delta, h}^\psi \times \mathbb{P}_\delta^\psi(G(\rho)|\Delta \leq \delta), \end{aligned} \quad (6.12)$$

and the result readily follows by using [15, Theorem 1.30].  $\square$

## 7. LOCAL LIMIT OF THE LÉVY TREE CONDITIONED ON LARGE MAXIMAL DEGREE

In this section, we shall investigate the behavior of the exploration process conditionally on  $\Delta = \delta$  as  $\delta \rightarrow \infty$ . We start with the subcritical case. Then recall from (2.13) that  $\mathbf{N}^\psi[\sigma] = \alpha^{-1} < \infty$ . We define a probability measure  $\mathbf{P}_\infty^\psi$  on the space  $\mathbb{R}_+ \times \mathcal{D}$  by setting:

$$\int_{\mathbb{R}_+ \times \mathcal{D}} F d\mathbf{P}_\infty^\psi = \alpha \mathbf{N}^\psi \left[ \int_0^\sigma F(s, \rho) ds \right], \quad (7.1)$$

for every  $F \in \mathcal{B}_+(\mathbb{R}_+ \times \mathcal{D})$ .

**Lemma 7.1.** *Assume that  $\psi$  is subcritical. The probability measure  $\mathbf{P}_\delta^\psi$  converges to  $\mathbf{P}_\infty^\psi$  in total variation distance on the space  $\mathbb{R}_+ \times \mathcal{D}$  as  $\delta \rightarrow \infty$ .*

*Proof.* Let  $F: \mathbb{R}_+ \times \mathcal{D}$  be measurable and bounded. We have:

$$\left| \mathbf{N}^\psi \left[ \int_0^\sigma F(s, \rho) ds \mathbf{1}_{\{\Delta < \delta\}} \right] - \mathbf{N}^\psi \left[ \int_0^\sigma F(s, \rho) ds \right] \right| \leq \|F\|_\infty \mathbf{N}^\psi[\sigma \mathbf{1}_{\{\Delta \geq \delta\}}].$$

Since  $\psi$  is subcritical, we have  $\mathbf{N}^\psi[\sigma] < \infty$  and the right-hand side converges to 0 as  $\delta \rightarrow \infty$ . This proves the result.  $\square$

For every measure-valued process  $\mu = (\mu_t, t \geq 0) \in \mathcal{D}$ , we define the measure-valued process  $R_0(\mu)$  obtained from  $\mu$  by removing any atoms at 0:

$$R_0(\mu)_t = \mu_t - \mu_t(0)\delta_0. \quad (7.2)$$

Denote by  $\mathbb{P}^\psi$  the distribution of the exploration process  $\rho$  with branching mechanism  $\psi$  starting from 0.

**Lemma 7.2.** *Assume that  $\psi$  is subcritical and that  $\pi$  is unbounded. Under  $\mathbb{P}_\delta^\psi(\cdot | \Delta \leq \delta)$ , the process  $R_0(\rho)$  converges in distribution to  $\mathbb{P}^\psi$  in the space  $(\mathcal{D}, d_S)$  as  $\delta \rightarrow \infty$ .*

*Proof.* Recall from (5.8) that  $\mathbb{P}_\delta^\psi(\Delta \leq \delta) = e^{-\delta \mathbf{N}^\psi[\Delta > \delta]}$ . Since  $\psi$  is subcritical, by [21, Proposition 3.8], we have as  $\delta \rightarrow \infty$ :

$$\mathbf{N}^\psi[\Delta > \delta] \sim \frac{\bar{\pi}(\delta)}{\alpha}. \quad (7.3)$$

But  $\delta \bar{\pi}(\delta) \leq \int_{(\delta, \infty)} r \pi(dr)$  and the last term goes to 0 as  $\delta \rightarrow \infty$ . It follows that  $\lim_{\delta \rightarrow \infty} \delta \mathbf{N}^\psi[\Delta > \delta] = 0$  and

$$\lim_{\delta \rightarrow \infty} \mathbb{P}_\delta^\psi(\Delta \leq \delta) = 1. \quad (7.4)$$

Thus, it suffices to show that for every continuous and bounded function  $F: \mathcal{D} \rightarrow \mathbb{R}$ , the following convergence holds:

$$\lim_{\delta \rightarrow \infty} \mathbb{P}_\delta^\psi(F \circ R_0(\rho)) = \mathbb{P}^\psi(F(\rho)). \quad (7.5)$$

Let  $\rho$  be the exploration process with branching mechanism  $\psi$  starting from 0, that is  $\rho$  has distribution  $\mathbb{P}^\psi$ . Then, the process  $\tilde{\rho}^{(\delta)}$  defined in (2.20) has distribution  $\mathbb{P}_\delta^\psi$ . Notice that we have  $R_0(\tilde{\rho}^{(\delta)})_t = \rho_t \mathbf{1}_{\{L_t^0 \leq r\}}$ , which implies that

$$d_S(R_0(\tilde{\rho}^{(\delta)}), \rho) \leq \sup_{t \geq 0} d_{\text{BL}}(R_0(\tilde{\rho}^{(\delta)})_t, \rho_t) = \sup_{L_t^0 > \delta} \langle \rho_t, 1 \rangle.$$

Recall that  $\langle \rho_t, 1 \rangle = X_t - I_t$ . Since the Lévy measure  $\pi$  satisfies the integrability assumption  $\int_{(0, \infty)} (r \wedge r^2) \pi(dr) < \infty$ , the process  $X$  does not drift to  $\infty$ ; see e.g. [8, Chapter VII]. This implies that the following convergence holds a.s.:

$$\lim_{\delta \rightarrow \infty} \sup_{L_t^0 > \delta} (X_t - I_t) = 0.$$

Therefore, the process  $R_0(\tilde{\rho}^{(\delta)})$  converges a.s. to  $\rho$  for the Skorokhod topology. This proves (7.5) and the proof is complete.  $\square$

**Remark 7.3.** It should be clear from (2.20) that the mass  $\tilde{\rho}_0^{(\delta)}(0)$  of the atom at 0 goes to  $\infty$  as  $\delta \rightarrow \infty$ . This corresponds to the condensation phenomenon: a node with infinite mass appears at the limit. By introducing the operator  $R_0$ , we remove this mass which allows us to study the limiting behavior above the condensation node.

Similarly to what was done in Section 5 (see (5.14) and (5.15)), we split the path of the exploration process into two parts around the first node with mass  $\Delta$ :  $\rho^{\Delta,-}$  is the pruned exploration process (that is the exploration process minus the first node with mass  $\Delta$ ) and  $\rho^{\Delta,+}$  is the path of the exploration process above the first node with mass  $\Delta$ . Notice that  $\rho_0^{\Delta,+}$  is equal to  $\Delta$  times the Dirac measure at 0. Let

$$E_\delta = \{\Delta = \delta, \Delta(\rho^{\Delta,-}) < \delta\} \quad (7.6)$$

be the event that the maximal degree is equal to  $\delta$  and there is a unique first-generation node with mass  $\delta$ . Recall from (5.23) the definition of  $\mathfrak{g}$ .

**Lemma 7.4.** *Assume that  $\psi$  is subcritical and that the set of atoms of the Lévy measure  $\pi$  is unbounded. The following holds as  $\delta \rightarrow \infty$  along the set of atoms of  $\pi$ :*

$$\mathbf{N}^\psi[\Delta = \delta] \sim \mathbf{N}^\psi[E_\delta] \sim \frac{\mathfrak{g}(\delta)}{\alpha}. \quad (7.7)$$

*Proof.* Under  $\mathbf{N}^{\psi_{\delta^-}}$  and conditionally on  $\rho$ , let  $\sum_{i=1}^N \delta_{(s_i, \rho_i)}$  be a Poisson point measure with intensity  $ds \int_{[\delta, \infty)} \pi(dr) \mathbb{P}_r^\psi(d\tilde{\rho})$ . Thanks to Theorem 4.1, we have:

$$\mathbf{N}^\psi[E_\delta] = \mathbf{N}^{\psi_{\delta^-}}[N = 1, \Delta(\rho_1) \leq \delta].$$

But, under  $\mathbf{N}^{\psi_{\delta^-}}$  and conditionally on  $\rho$ ,  $N$  has Poisson distribution with parameter  $\pi[\delta, \infty)\sigma$ ,  $\rho_1$  has distribution  $\pi[\delta, \infty)^{-1} \int_{[\delta, \infty)} \pi(dr) \mathbb{P}_r^\psi(d\tilde{\rho})$  and they are independent. It follows that

$$\mathbf{N}^\psi[E_\delta] = \mathbf{N}^{\psi_{\delta^-}} \left[ \sigma e^{-\pi[\delta, \infty)\sigma} \int_{[\delta, \infty)} \pi(dr) \mathbb{P}_r^\psi(\Delta \leq \delta) \right]$$

$$\begin{aligned}
&= \mathbf{N}^{\psi_{\delta^-}} \left[ \sigma e^{-\pi[\delta, \infty)\sigma} \pi(\delta) e^{-\delta \mathbf{N}^\psi[\Delta > \delta]} \right] \\
&= \mathbf{g}(\delta) w(\delta),
\end{aligned} \tag{7.8}$$

where we used (5.8) for the second equality and (5.2) for the last.

Recall from (5.1) the definition of  $w$ . Since  $\psi$  is subcritical, it follows from (2.13) that  $\lim_{\delta \rightarrow \infty} w(\delta) = \alpha^{-1}$ . This proves that

$$\mathbf{N}^\psi[E_\delta] \sim \frac{\mathbf{g}(\delta)}{\alpha}.$$

A similar computation yields:

$$\begin{aligned}
\mathbf{N}^\psi[\Delta = \delta] &= \mathbf{N}^{\psi_{\delta^-}} [N \geq 1, \Delta(\rho_i) \leq \delta, \forall 1 \leq i \leq N] \\
&= \mathbf{N}^{\psi_{\delta^-}} \left[ e^{-\pi[\delta, \infty)\sigma} \left( e^{\mathbf{g}(\delta)\sigma} - 1 \right) \right] \\
&= \mathbf{N}^\psi \left[ \left( e^{\mathbf{g}(\delta)\sigma} - 1 \right) \mathbf{1}_{\{\Delta < \delta\}} \right],
\end{aligned} \tag{7.9}$$

where we used Corollary 4.3 for the last equality.

Observe that since  $\pi(1, \infty) < \infty$ ,  $\pi(\delta)$  (and thus also  $\mathbf{g}(\delta)$ ) converges to 0 as  $\delta \rightarrow \infty$ . It is clear that

$$\lim_{\delta \rightarrow \infty} \frac{e^{\mathbf{g}(\delta)\sigma} - 1}{\mathbf{g}(\delta)} \mathbf{1}_{\{\Delta < \delta\}} = \sigma.$$

Furthermore, since  $\psi$  is subcritical, there exists  $\lambda_0 > 0$  such that  $\mathbf{N}^\psi \left[ \sigma e^{\lambda_0 \sigma} \right] < \infty$ . Thus, using Taylor's inequality, we have for  $\delta > 0$  large enough:

$$\frac{e^{\mathbf{g}(\delta)\sigma} - 1}{\mathbf{g}(\delta)} \mathbf{1}_{\{\Delta < \delta\}} \leq \sigma e^{\mathbf{g}(\delta)\sigma} \leq \sigma e^{\lambda_0 \sigma},$$

Thanks to the dominated convergence theorem, we deduce that

$$\lim_{\delta \rightarrow \infty} \frac{\mathbf{N}^\psi[\Delta = \delta]}{\mathbf{g}(\delta)} = \lim_{\delta \rightarrow \infty} \frac{1}{\mathbf{g}(\delta)} \mathbf{N}^\psi \left[ \left( e^{\mathbf{g}(\delta)\sigma} - 1 \right) \mathbf{1}_{\{\Delta < \delta\}} \right] = \mathbf{N}^\psi[\sigma] = \alpha^{-1}.$$

This finishes the proof.  $\square$

The first main result of this section concerns the limit of the subcritical Lévy tree conditioned on having a large maximal degree. Then there is a condensation phenomenon: the limit consists of a size-biased Lévy tree onto which one grafts – at a uniformly chosen leaf – an independent Lévy forest with infinite mass. In particular, the height of the condensation node is exponentially distributed. Recall from (5.30) that  $T_\Delta$  is the first time that the exploration process contains an atom with mass  $\Delta$ . Recall also that  $\rho^{\Delta,-}$  denotes the path of the exploration process after removing the first node with mass  $\Delta$  while  $\rho^{\Delta,+}$  denotes the path of the exploration process above that node. Finally, recall that  $\mathbb{P}^\psi$  is the distribution of the exploration process with branching mechanism  $\psi$  starting from 0.

**Theorem 7.5.** *Assume that  $\psi$  is subcritical and that  $\pi$  is unbounded. Let  $F: \mathbb{R}_+ \times \mathcal{D} \rightarrow \mathbb{R}$  and  $G: \mathcal{D} \rightarrow \mathbb{R}$  be continuous and bounded. We have:*

$$\lim_{\delta \rightarrow \infty} \mathbf{N}^\psi \left[ F(T_\Delta, \rho^{\Delta, -}) G \circ R_0(\rho^{\Delta, +}) \mid \Delta = \delta \right] = \alpha \mathbf{N}^\psi \left[ \int_0^\sigma F(s, \rho) ds \right] \mathbb{P}^\psi(G(\rho)). \quad (7.10)$$

*Proof.* When  $\delta \rightarrow \infty$  along the set of non-atoms  $\{\delta > 0: \pi(\delta) = 0\}$ , the convergence is a direct consequence of Theorem 5.7, Lemma 7.1 and Lemma 7.2.

Now assume that  $\delta > 0$  is an atom of  $\pi$ . Thanks to Lemma 7.4 and since the inclusion  $E_\delta \subset \{\Delta = \delta\}$  holds, it is enough to show that the result holds when conditioning by  $E_\delta$ . But, thanks to Remark 5.8, we have:

$$\mathbf{N}^\psi \left[ F(T_\Delta, \rho^{\Delta, -}) G \circ R_0(\rho^{\Delta, +}) \mid E_\delta \right] = \int_{\mathbb{R}_+ \times \mathcal{D}} F(s, \tilde{\rho}) \mathbf{P}_\delta^\psi(ds, d\tilde{\rho}) \mathbb{P}_\delta^\psi(G \circ R_0(\rho) \mid \Delta \leq \delta).$$

The result readily follows from Lemma 7.1 and Lemma 7.2.  $\square$

Next, we consider the critical case. Recall from (5.1) the definition of  $w$ . The next lemma is a key ingredient in the proof of the local convergence of the critical Lévy tree.

**Lemma 7.6.** *Assume that  $\psi$  is critical and that the Lévy measure  $\pi$  is unbounded. For every  $h > 0$ , we have*

$$\lim_{\delta \rightarrow \infty} \frac{1}{w(\delta)} \mathbf{N}^\psi \left[ \sigma F(r_h(\rho)) \mathbf{1}_{\{\Delta < \delta\}} \right] = \mathbf{N}^\psi \left[ L_\sigma^h F(r_h(\rho)) \right]. \quad (7.11)$$

*Proof.* We shall use the decomposition of the exploration process above level  $h$ , see Section 2.5. Let  $(\rho^i, i \in I_h)$  be the excursions of the exploration process above level  $h$ . For every  $i \in I_h$ , let  $\sigma^i$  (resp.  $\Delta^i$ ) be the lifetime (resp. the maximal degree) of  $\rho^i$ . Similarly, denote by  $\sigma_h$  (resp.  $\Delta_h$ ) the lifetime (resp. the maximal degree) of  $r_h(\rho)$ . Thanks to Proposition 2.4, we have:

$$\begin{aligned} \mathbf{N}^\psi \left[ \sigma F(r_h(\rho)) \mathbf{1}_{\{\Delta < \delta\}} \right] &= \mathbf{N}^\psi \left[ \left( \sigma_h + \sum_{i \in I_h} \sigma^i \right) F(r_h(\rho)); \Delta_h < \delta, \Delta^i < \delta, \forall i \in I_h \right] \\ &= \mathbf{N}^\psi \left[ F(r_h(\rho)) \mathbf{1}_{\{\Delta_h < \delta\}} \mathbf{N}^\psi \left[ \sigma_h + \sum_{i \in I_h} \sigma^i; \Delta^i < \delta, \forall i \in I_h \mid \mathcal{E}_h \right] \right]. \end{aligned}$$

Thanks to the Mecke formula for Poisson random measures, see e.g. [27, Chapter 4, Theorem 4.1], we get:

$$\mathbf{N}^\psi \left[ \sigma_h + \sum_{i \in I_h} \sigma^i; \Delta^i < \delta, \forall i \in I_h \mid \mathcal{E}_h \right] = \left( \sigma_h + L_\sigma^h \mathbf{N}^\psi[\sigma \mathbf{1}_{\{\Delta < \delta\}}] \right) e^{-L_\sigma^h \mathbf{N}^\psi[\Delta \geq \delta]}.$$

We deduce that

$$\frac{1}{w(\delta)} \mathbf{N}^\psi \left[ \sigma F(r_h(\rho)) \mathbf{1}_{\{\Delta < \delta\}} \right] = \mathbf{N}^\psi \left[ F(r_h(\rho)) \mathbf{1}_{\{\Delta_h < \delta\}} \left( L_\sigma^h + w(\delta)^{-1} \sigma_h \right) e^{-L_\sigma^h \mathbf{N}^\psi[\Delta \geq \delta]} \right]. \quad (7.12)$$

Notice that  $w(\delta) \rightarrow \infty$  as  $\delta \rightarrow \infty$  since  $\psi$  is critical. Furthermore, it is clear that  $\sigma_h = \int_0^h L_\sigma^a da$ . Now letting  $\delta \rightarrow \infty$  in (6.6) gives that  $\mathbf{N}^\psi[L_\sigma^a] = 1$ . It follows that  $\mathbf{N}^\psi[\sigma_h] = h < \infty$ . Thus, the dominated convergence theorem applies and we obtain the desired result by letting  $\delta \rightarrow \infty$  in (7.12).  $\square$

Recall from Theorem 5.7 that when the Lévy measure  $\pi$  has an atom  $\delta > 0$ , the exploration process conditioned on  $\Delta = \delta$  has a random number of first-generation nodes with mass  $\delta$ . The next lemma gives a sufficient condition for there to be exactly one with high probability as  $\delta \rightarrow \infty$ . Recall from (5.1) the definition of  $w$ . Recall also from (7.6) that  $E_\delta$  denotes the event that the maximal degree is equal to  $\delta$  and there is a unique first-generation node with mass  $\delta$ .

**Lemma 7.7.** *Assume that  $\psi$  is critical and that the Lévy measure  $\pi$  is unbounded. Furthermore, assume that*

$$\lim_{\delta \rightarrow \infty} \frac{\pi(\delta)}{w(\delta) \bar{\pi}(\delta) \int_{[\delta, \infty)} r \pi(dr)} = 0. \quad (7.13)$$

We have as  $\delta \rightarrow \infty$  along the set of atoms of  $\pi$ :

$$\mathbf{N}^\psi[\Delta = \delta] \sim \mathbf{N}^\psi[E_\delta] = \mathbf{g}(\delta)w(\delta). \quad (7.14)$$

*Proof.* Recall from (7.8) and (7.9) that

$$\mathbf{N}^\psi[E_\delta] = \mathbf{g}(\delta)w(\delta) \quad \text{and} \quad \mathbf{N}^\psi[\Delta = \delta] = \mathbf{N}^\psi \left[ \left( e^{\mathbf{g}(\delta)\sigma} - 1 \right) \mathbf{1}_{\{\Delta < \delta\}} \right].$$

Using Taylor's inequality, we deduce that

$$1 \leq \frac{\mathbf{N}^\psi[\Delta = \delta]}{\mathbf{N}^\psi[E_\delta]} \leq \frac{w_1(\delta)}{w(\delta)}, \quad (7.15)$$

where we set  $w_1(\delta) = \mathbf{N}^\psi \left[ \sigma e^{\mathbf{g}(\delta)\sigma} \mathbf{1}_{\{\Delta < \delta\}} \right]$ .

Using (5.2) and the inequality  $e^{-x} \geq 1 - x$  for every  $x \geq 0$ , we have:

$$w(\delta) = \mathbf{N}^{\psi_{\delta^-}} \left[ \sigma e^{-\pi[\delta, \infty)\sigma} \right] \geq \mathbf{N}^{\psi_{\delta^-}} \left[ \sigma (1 - \pi(\delta)\sigma) e^{-\bar{\pi}(\delta)\sigma} \right].$$

But thanks to Corollary 4.3, observe that

$$w_1(\delta) = \mathbf{N}^{\psi_{\delta^-}} \left[ \sigma e^{(\mathbf{g}(\delta) - \pi[\delta, \infty))\sigma} \right] \leq \mathbf{N}^{\psi_{\delta^-}} \left[ \sigma e^{-\bar{\pi}(\delta)\sigma} \right],$$

where we used that  $\mathbf{g}(\delta) \leq \pi(\delta)$  for the inequality. Furthermore, using that the function  $x \mapsto xe^{-x}$  is bounded on  $\mathbb{R}_+$  by some constant  $M > 0$ , we have:

$$\mathbf{N}^{\psi_{\delta^-}} \left[ \sigma^2 e^{-\bar{\pi}(\delta)\sigma} \right] \leq \frac{M}{\bar{\pi}(\delta)} \mathbf{N}^{\psi_{\delta^-}}[\sigma] = \frac{M}{\bar{\pi}(\delta) \int_{[\delta, \infty)} r \pi(dr)}.$$

We deduce that

$$w(\delta) \geq w_1(\delta) - \frac{M\pi(\delta)}{\bar{\pi}(\delta) \int_{[\delta, \infty)} r \pi(dr)}.$$

It follows from (7.15) that

$$1 \leq \frac{\mathbf{N}^\psi[\Delta = \delta]}{\mathbf{N}^\psi[E_\delta]} \leq 1 + \frac{M\pi(\delta)}{w(\delta)\bar{\pi}(\delta) \int_{[\delta, \infty)} r \pi(dr)},$$

and the result readily follows by using (7.13).  $\square$

In the critical case, the Lévy tree conditioned on having a large maximal degree converges locally to the immortal Lévy tree. Intuitively, the condensation node goes to infinity and thus becomes invisible to local convergence.

**Theorem 7.8.** *Assume that  $\psi$  is critical and that  $\pi$  is unbounded. Furthermore, assume that (7.13) holds. Let  $F: \mathcal{D} \rightarrow \mathbb{R}$  be continuous and bounded. For every  $h > 0$ , we have:*

$$\lim_{\delta \rightarrow \infty} \mathbf{N}^\psi [F(r_h(\rho)) | \Delta = \delta] = \mathbf{N}^\psi [L_\sigma^h F(r_h(\rho))]. \quad (7.16)$$

*Proof.* First assume that  $\delta > 0$  is not an atom of  $\pi$ . Thanks to Theorem 5.7, conditionally on  $\Delta = \delta$ ,  $\rho$  is distributed as  $\tilde{\rho} \circledast (s, \hat{\rho})$ , where  $(s, \tilde{\rho})$  has distribution  $\mathbf{P}_\delta^\psi$ ,  $\hat{\rho}$  has distribution  $\mathbb{P}_\delta^\psi(\cdot | \Delta \leq \delta)$  and they are independent.

Next, assume that  $\delta > 0$  is an atom of  $\pi$ . Recall that  $E_\delta$  denotes the event that  $\Delta = \delta$  and there is a unique first-generation node with mass  $\delta$ . Thanks to Lemma 7.7, since  $E_\delta \subset \{\Delta = \delta\}$ , the two conditionings are equivalent and it is enough to show that the result holds when conditioning on  $E_\delta$ . But Remark 5.8 gives that, conditionally on  $E_\delta$ ,  $\rho$  is again distributed as  $\tilde{\rho} \circledast (s, \hat{\rho})$ .

Thus, in all cases, it is enough to show that

$$\lim_{\delta \rightarrow \infty} \int_{\mathbb{R}_+ \times \mathcal{D}} \mathbf{P}_\delta^\psi(ds, d\tilde{\rho}) \int_{\mathcal{D}} \mathbb{P}_\delta^\psi(d\hat{\rho} | \Delta \leq \delta) F(r_h(\tilde{\rho} \circledast (s, \hat{\rho}))) = \mathbf{N}^\psi [L_\sigma^h F(r_h(\rho))].$$

Now, Lemma 5.10 gives that the height  $H(\tilde{\rho}_s)$  at which  $\hat{\rho}$  is grafted is exponentially distributed with mean  $w(\delta)$ . Since  $\psi$  is critical, it holds that  $\lim_{\delta \rightarrow \infty} w(\delta) = \infty$ . Thus, we deduce that  $H(\tilde{\rho}_s) > h$  with high probability as  $\delta \rightarrow \infty$  under  $\mathbf{P}_\delta^\psi(ds, d\tilde{\rho})$ , i.e. we have:

$$\lim_{\delta \rightarrow \infty} \int_{\mathbb{R}_+ \times \mathcal{D}} \mathbf{P}_\delta^\psi(ds, d\tilde{\rho}) \mathbf{1}_{\{H(\tilde{\rho}_s) \leq h\}} = 0.$$

Furthermore, on the event  $\{H(\tilde{\rho}_s) > h\}$ , it holds that  $r_h(\tilde{\rho} \circledast (s, \hat{\rho})) = r_h(\tilde{\rho})$ , and the proof reduces to showing the following convergence:

$$\lim_{\delta \rightarrow \infty} \int_{\mathbb{R}_+ \times \mathcal{D}} \mathbf{P}_\delta^\psi(ds, d\tilde{\rho}) F(r_h(\tilde{\rho})) = \mathbf{N}^\psi [L_\sigma^h F(r_h(\rho))]. \quad (7.17)$$



Recalling from (5.4) the definition of  $\mathbf{P}_\delta^\psi$ , Lemma 7.6 yields (7.17) and the proof is complete.  $\square$

We end this section with the following result dealing with the asymptotic behavior of the exploration process conditioned on having a large maximal degree at a fixed height  $h$ . Notice that this conditioning does not allow the condensation node to escape to infinity (even in the critical case as opposed to the conditioning of large maximal degree) and forces condensation to occur at a finite height. The limit consists of a Lévy tree biased by the population size at level  $h$  onto which one grafts – at a leaf chosen uniformly at random at height  $h$  – an independent Lévy forest with infinite mass.

**Theorem 7.9.** *Assume that  $\psi$  is (sub)critical and that  $\pi$  is unbounded and diffuse. Let  $F: \mathbb{R}_+ \times \mathcal{D} \rightarrow \mathbb{R}$  and  $G: \mathcal{D} \rightarrow \mathbb{R}$  be continuous and bounded. We have:*

$$\lim_{\delta \rightarrow \infty} \mathbf{N}^\psi \left[ F(T_\Delta, \rho^{\Delta, -}) G(\rho^{\Delta, +}) | \Delta = \delta, H_\Delta = h \right] = e^{\alpha h} \mathbf{N}^\psi \left[ \int_0^\sigma F(s, \rho) L^h(ds) \right] \mathbb{P}^\psi(G(\rho)). \quad (7.18)$$

*Proof.* Letting  $\delta \rightarrow \infty$  in (6.6), we have that  $\lim_{\delta \rightarrow \infty} \mathbf{N}^\psi [L_\sigma^h \mathbf{1}_{\{\Delta < \delta\}}] = e^{-\alpha h}$ . Furthermore, the dominated convergence theorem yields:

$$\lim_{\delta \rightarrow \infty} \mathbf{N}^\psi \left[ \int_0^\sigma F(s, \rho) L^h(ds) \mathbf{1}_{\{\Delta < \delta\}} \right] = \mathbf{N}^\psi \left[ \int_0^\sigma F(s, \rho) L^h(ds) \right].$$

This proves that the following convergence holds:

$$\lim_{\delta \rightarrow \infty} \int_{\mathbb{R}_+ \times \mathcal{D}} F d\mathbf{P}_{\delta, h}^\psi = e^{\alpha h} \mathbf{N}^\psi \left[ \int_0^\sigma F(s, \rho) L^h(ds) \right].$$

The result is then a direct consequence of Theorem 6.3 and Lemma 7.2.  $\square$

## 8. OTHER CONDITIONINGS OF LARGE MAXIMAL DEGREE

In this section, we look at other conditionings of large maximal degree. Recall from Section 4.2 that  $Z_0^\delta$  denotes the number of first-generation nodes with mass larger than  $\delta$  while  $W^\delta$  denotes the total number of nodes with mass larger than  $\delta$ . Specifically, we study the conditionings  $\Delta > \delta$  (which is equal to  $Z_0^\delta \geq 1$  or  $W^\delta \geq 1$ ),  $Z_0^\delta = 1$  and  $W^\delta = 1$ . We shall see that, in the subcritical and critical cases, all three give rise to the same asymptotic behavior as conditioning by  $\Delta = \delta$ .

Notice that  $\{W^\delta = 1\}$  (resp.  $\{Z_0^\delta = 1\}$ ) is the event that  $\rho$  contains exactly one node (resp. one first-generation node) with mass larger than  $\delta$ . To begin, we compute the measure of these two events. In the subcritical case, they are equivalent in  $\mathbf{N}^\psi$ -measure to  $\{\Delta > \delta\}$ . However, this is no longer the case for critical branching mechanisms, see Proposition 9.2 for the (critical) stable case.

**Proposition 8.1.** *We have:*

$$\mathbf{N}^\psi[Z_0^\delta = 1] = \frac{\bar{\pi}(\delta)}{\psi'_\delta(\mathbf{N}^\psi[\Delta > \delta])}, \quad (8.1)$$

$$\mathbf{N}^\psi[W^\delta = 1] = \frac{1}{\psi'_\delta(\mathbf{N}^\psi[\Delta > \delta])} \int_{(\delta, \infty)} e^{-r\mathbf{N}^\psi[\Delta > \delta]} \pi(dr). \quad (8.2)$$

In particular, assuming that  $\psi$  is subcritical and that  $\pi$  is unbounded, we have as  $\delta \rightarrow \infty$ :

$$\mathbf{N}^\psi[Z_0^\delta = 1] \sim \mathbf{N}^\psi[W^\delta = 1] \sim \mathbf{N}^\psi[\Delta > \delta] \sim \frac{\bar{\pi}(\delta)}{\alpha}. \quad (8.3)$$

Since we have the inclusions  $\{W^\delta = 1\} \subset \{Z_0^\delta = 1\} \subset \{\Delta > \delta\}$ , Proposition 8.1 entails that, in the subcritical case, the three conditionings are equivalent as  $\delta \rightarrow \infty$ . In particular, conditionally on  $\Delta > \delta$ , there is exactly one node with mass larger than  $\delta$  with probability tending to 1 as  $\delta \rightarrow \infty$ .

*Proof.* Notice that  $\{W^\delta = 1\}$  is the event that  $\rho$  contains only one first-generation node with mass larger than  $\delta$  and that this node has no descendants with mass larger than  $\delta$ . Thus, using the Poissonian decomposition of Theorem 4.1, we get  $\mathbf{N}^\psi[Z_0^\delta = 1] = \mathbf{N}^{\psi_\delta}[\zeta = 1]$  and

$$\mathbf{N}^\psi[W^\delta = 1] = \mathbf{N}^{\psi_\delta}[\zeta = 1] \mathbb{Q}_\delta^\psi(\Delta \leq \delta). \quad (8.4)$$

Recall that under  $\mathbf{N}^{\psi_\delta}$  and conditionally on  $\rho$ ,  $\zeta$  has Poisson distribution with parameter  $\bar{\pi}(\delta)\sigma$ . Thus we have

$$\mathbf{N}^{\psi_\delta}[\zeta = 1] = \mathbf{N}^{\psi_\delta}[\bar{\pi}(\delta)\sigma e^{-\bar{\pi}(\delta)\sigma}] = \frac{\bar{\pi}(\delta)}{\psi'_\delta \circ \psi_\delta^{-1}(\bar{\pi}(\delta))} = \frac{\bar{\pi}(\delta)}{\psi'_\delta(\mathbf{N}^\psi[\Delta > \delta])}, \quad (8.5)$$

where we used (3.4) for the last equality. This proves (8.1).

Moreover, using the Poissonian decomposition of Proposition 4.4 together with the fact that, under  $\mathbb{Q}_\delta^{\psi_\delta}$  and conditionally on  $\rho$ ,  $\xi$  has Poisson distribution with parameter  $\bar{\pi}(\delta)\sigma$ , we get:

$$\mathbb{Q}_\delta^\psi(\Delta \leq \delta) = \mathbb{Q}_\delta^{\psi_\delta}(\xi = 0) = \mathbb{Q}_\delta^{\psi_\delta}(e^{-\bar{\pi}(\delta)\sigma}) = \frac{1}{\bar{\pi}(\delta)} \int_{(\delta, \infty)} \pi(dr) \mathbb{P}_r^{\psi_\delta}(e^{-\bar{\pi}(\delta)\sigma}).$$

Thus, it follows from (4.10) and (3.4) that

$$\mathbb{Q}_\delta^\psi(\Delta \leq \delta) = \frac{1}{\bar{\pi}(\delta)} \int_{(\delta, \infty)} e^{-r\psi_\delta^{-1}(\bar{\pi}(\delta))} \pi(dr) = \frac{1}{\bar{\pi}(\delta)} \int_{(\delta, \infty)} e^{-r\mathbf{N}^\psi[\Delta > \delta]} \pi(dr). \quad (8.6)$$

Finally, combining (8.4), (8.5) and (8.6), we deduce (8.2).

Now assume that  $\psi$  is subcritical and that  $\pi$  is unbounded. Recall from (7.3) that

$$\mathbf{N}^\psi[\Delta > \delta] \sim \frac{\bar{\pi}(\delta)}{\alpha}.$$

On the other hand, differentiating (1.8), we get:

$$\psi'_\delta(\mathbf{N}^\psi[\Delta > \delta]) = \psi'(\mathbf{N}^\psi[\Delta > \delta]) + \int_{(\delta, \infty)} r e^{-r \mathbf{N}^\psi[\Delta > \delta]} \pi(dr).$$

Since  $\int_{(1, \infty)} r \pi(dr) < \infty$ , the dominated convergence theorem shows that the last integral converges to 0 as  $\delta \rightarrow \infty$ . It follows that

$$\lim_{\delta \rightarrow \infty} \psi'_\delta(\mathbf{N}^\psi[\Delta > \delta]) = \psi'(0) = \alpha. \quad (8.7)$$

In particular, we get that  $\mathbf{N}^\psi[Z_0^\delta = 1] \sim \alpha^{-1} \bar{\pi}(\delta)$ .

Furthermore, we have:

$$\begin{aligned} 0 \leq 1 - \frac{1}{\bar{\pi}(\delta)} \int_{(\delta, \infty)} e^{-r \mathbf{N}^\psi[\Delta > \delta]} \pi(dr) &= \frac{1}{\bar{\pi}(\delta)} \int_{(\delta, \infty)} (1 - e^{-r \mathbf{N}^\psi[\Delta > \delta]}) \pi(dr) \\ &\leq \frac{\mathbf{N}^\psi[\Delta > \delta]}{\bar{\pi}(\delta)} \int_{(\delta, \infty)} r \pi(dr). \end{aligned}$$

The dominated convergence theorem gives  $\lim_{\delta \rightarrow \infty} \int_{(\delta, \infty)} r \pi(dr) = 0$ . Since  $\lim_{\delta \rightarrow \infty} \mathbf{N}^\psi[\Delta > \delta] / \bar{\pi}(\delta) = \alpha^{-1}$ , we deduce that

$$\lim_{\delta \rightarrow \infty} \frac{1}{\bar{\pi}(\delta)} \int_{(\delta, \infty)} e^{-r \mathbf{N}^\psi[\Delta > \delta]} \pi(dr) = 1.$$

Together with (8.2) and (8.7), this yields  $\mathbf{N}^\psi[W^\delta = 1] \sim \alpha^{-1} \bar{\pi}(\delta)$ . This concludes the proof.  $\square$

In the subcritical case, the three conditionings  $\Delta > \delta$ ,  $Z_0^\delta = 1$  and  $W^\delta = 1$  are equivalent as  $\delta \rightarrow \infty$  and thus they yield the same asymptotic behavior: a condensation phenomenon occurs at the limit just like in Theorem 7.5 where we condition by  $\Delta = \delta$ . Recall from (5.30) that  $T_\Delta$  is the first time that the exploration process contains an atom with mass  $\Delta$ . Recall also from Section 7 that  $\rho^{\Delta, -}$  denotes the path of the exploration process after removing the first node with  $\Delta$  while  $\rho^{\Delta, +}$  denotes the path of the exploration process above that node.

**Theorem 8.2.** *Assume that  $\psi$  is subcritical and that  $\pi$  is unbounded. Let  $F: \mathbb{R}_+ \times \mathcal{D} \rightarrow \mathbb{R}$  and  $G: \mathcal{D} \rightarrow \mathbb{R}$  be continuous and bounded and let  $A_\delta$  be equal to  $\{\Delta > \delta\}$ ,  $\{Z_0^\delta = 1\}$  or  $\{W^\delta = 1\}$ . We have:*

$$\lim_{\delta \rightarrow \infty} \mathbf{N}^\psi \left[ F(T_\Delta, \rho^{\Delta, -}) G \circ R_0(\rho^{\Delta, +}) \middle| A_\delta \right] = \alpha \mathbf{N}^\psi \left[ \int_0^\sigma F(s, \rho) ds \right] \mathbb{P}^\psi(G(\rho)). \quad (8.8)$$

*Proof.* As the three events are equivalent it is enough to show the result for  $A_\delta = \{\Delta > \delta\}$ . Disintegrating with respect to  $\Delta$ , we have:

$$\mathbf{N}^\psi \left[ F(T_\Delta, \rho^{\Delta, -}) G \circ (\rho^{\Delta, +}) \middle| \Delta > \delta \right]$$

$$= \frac{1}{\mathbf{N}^\psi[\Delta > \delta]} \int_{(\delta, \infty)} \mathbf{N}^\psi[\Delta \in dr] \mathbf{N}^\psi \left[ F(T_\Delta, \rho^{\Delta, -}) G \circ R_0(\rho^{\Delta, +}) \mid \Delta = r \right].$$

The conclusion follows from Theorem 7.5.  $\square$

Recall from (5.12) that  $T_\delta$  is the first time  $\rho$  contains a node with mass larger than  $\delta$ . Also recall from (5.14) and (5.15) that  $\rho^{\delta, -}$  denotes the path of the exploration process after removing the first node with mass larger than  $\delta$  while  $\rho^{\delta, +}$  denotes the path of the exploration process above that node. We shall determine the joint distribution of  $(T_\delta, \rho^{\delta, -}, \rho^{\delta, +})$  conditionally on  $Z_0^\delta = 1$  and  $W^\delta = 1$ . Recall from (5.1) the definition of  $w_+$ .

**Lemma 8.3.** *Assume that  $\psi$  is (sub)critical and let  $F \in \mathcal{B}_+(\mathbb{R}_+ \times \mathcal{D})$  and  $G \in \mathcal{B}_+(\mathcal{D})$ . We have:*

$$\mathbf{N}^\psi \left[ F(T_\delta, \rho^{\delta, -}) G(\rho^{\delta, +}) \mid Z_0^\delta = 1 \right] = \frac{1}{w_+(\delta)} \mathbf{N}^\psi \left[ \int_0^\sigma F(s, \rho) ds \mathbf{1}_{\{\Delta \leq \delta\}} \right] \mathbb{Q}_\delta^\psi(G(\rho)), \quad (8.9)$$

$$\mathbf{N}^\psi \left[ F(T_\delta, \rho^{\delta, -}) G(\rho^{\delta, +}) \mid W^\delta = 1 \right] = \frac{1}{w_+(\delta)} \mathbf{N}^\psi \left[ \int_0^\sigma F(s, \rho) ds \mathbf{1}_{\{\Delta \leq \delta\}} \right] \mathbb{Q}_\delta^\psi(G(\rho) \mid \Delta \leq \delta). \quad (8.10)$$

*Proof.* We only prove the first identity, the second one being similar. Theorem 4.1 gives:

$$\mathbf{N}^\psi \left[ F(T_\delta, \rho^{\delta, -}) G(\rho^{\delta, +}) \mathbf{1}_{\{Z_0^\delta = 1\}} \right] = \mathbf{N}^{\psi_\delta} \left[ F(U, \rho) G(\mathcal{F}^\delta) \mathbf{1}_{\{\zeta = 1\}} \right],$$

where under  $\mathbf{N}^{\psi_\delta}$  and conditionally on  $\rho$ ,  $\rho^\delta$  has distribution  $\mathbb{Q}_\delta^\psi$ ,  $U$  is uniformly distributed on  $[0, \sigma]$ ,  $\zeta$  has Poisson distribution with parameter  $\bar{\pi}(\delta)\sigma$  and they are independent. Therefore, conditioning on  $\rho$  in the last term, we get:

$$\begin{aligned} \mathbf{N}^\psi \left[ F(T_\delta, \rho^{\delta, -}) G(\rho^{\delta, +}) \mathbf{1}_{\{Z_0^\delta = 1\}} \right] &= \bar{\pi}(\delta) \mathbf{N}^{\psi_\delta} \left[ e^{-\bar{\pi}(\delta)\sigma} \int_0^\sigma F(s, \rho) ds \right] \mathbb{Q}_\delta^\psi(G(\rho)) \\ &= \bar{\pi}(\delta) \mathbf{N}^\psi \left[ \int_0^\sigma F(s, \rho) ds \mathbf{1}_{\{\Delta \leq \delta\}} \right] \mathbb{Q}_\delta^\psi(G(\rho)), \end{aligned}$$

where we used Corollary 4.3 for the last equality. This in conjunction with (8.1) and (5.3) yields the desired result.  $\square$

In the critical case, the three conditionings  $\Delta > \delta$ ,  $Z_0^\delta = 1$  and  $W^\delta = 1$  are not equivalent but they still yield the same asymptotic behavior: local convergence to the immortal Lévy tree just like in Theorem 7.8 where we condition by  $\Delta = \delta$ .

**Theorem 8.4.** *Assume that  $\psi$  is critical and that  $\pi$  is unbounded. Let  $F: \mathcal{D} \rightarrow \mathbb{R}$  be continuous and bounded and let  $A_\delta$  be equal to  $\{\Delta > \delta\}$ ,  $\{Z_0^\delta = 1\}$  or  $\{W^\delta = 1\}$ . We have*

$$\lim_{\delta \rightarrow \infty} \mathbf{N}^\psi [F(r_h(\rho)) \mid A_\delta] = \mathbf{N}^\psi [L_\sigma^h F(r_h(\rho))]. \quad (8.11)$$

*Proof.* Since the conditioning by  $\Delta > \delta$  was already treated in [20], we only consider the other two. The proof uses similar arguments to that of Theorem 7.8 and we only give a

sketch. By Lemma 8.3, under  $\mathbf{N}^\psi$  and conditionally on  $Z_0^\delta = 1$ ,  $\rho$  is distributed as  $\tilde{\rho} \otimes (s, \hat{\rho})$ , where  $(s, \tilde{\rho})$  has distribution

$$\mathbb{E} [F(s, \tilde{\rho})] = \frac{1}{w_+(\delta)} \mathbf{N}^\psi \left[ \int_0^\sigma F(s, \rho) ds \mathbf{1}_{\{\Delta \leq \delta\}} \right],$$

$\hat{\rho}$  has distribution  $\mathbb{Q}_\delta^\psi$  and they are independent. But Lemma 5.10 gives that the height  $H(\tilde{\rho}_s)$  is exponentially distributed with mean  $w_+(\delta)$ . Since  $\psi$  is critical, this last quantity goes to  $\infty$  as  $\delta \rightarrow \infty$ . In particular, it holds that  $H(\tilde{\rho}_s) > h$  with high probability as  $\delta \rightarrow \infty$ . Furthermore, on the event  $\{H(\tilde{\rho}_s) > h\}$ , we have that

$$r_h(\tilde{\rho} \otimes (s, \hat{\rho})) = r_h(\tilde{\rho}). \quad (8.12)$$

As a consequence, in order to show the result, it is enough to prove that

$$\lim_{\delta \rightarrow \infty} \frac{1}{w_+(\delta)} \mathbf{N}^\psi \left[ \sigma F(r_h(\rho)) \mathbf{1}_{\{\Delta \leq \delta\}} \right] = \mathbf{N}^\psi \left[ L_\sigma^h F(r_h(\rho)) \right].$$

This last convergence holds by adapting the proof of Lemma 7.6. Finally, when conditioning on  $W^\delta = 1$ , the only change is that  $\hat{\rho}$  has distribution  $\mathbb{Q}_\delta^\psi(\cdot | \Delta \leq \delta)$  but this does not contribute to the limit because of (8.12). This completes the proof.  $\square$

## 9. STABLE CASE

We consider the stable case  $\psi(\lambda) = \lambda^\gamma$  with  $\gamma \in (1, 2)$ . Notice that the branching mechanism is critical with  $\alpha = \beta = 0$  and the Lévy measure  $\pi$  is given by:

$$\pi(dr) = a_\gamma r^{-1-\gamma} dr, \quad \text{where} \quad a_\gamma = \frac{\gamma(\gamma-1)}{\Gamma(2-\gamma)}.$$

Then we have:

$$\bar{\pi}(\delta) = \pi(\delta, \infty) = \frac{a_\gamma}{\gamma} \delta^{-\gamma}. \quad (9.1)$$

Furthermore, the Grey condition (2.21) is satisfied and we can speak of the Lévy tree  $\mathcal{T}$ , see Section 2.9.

We recall the scaling property of the stable tree. For every  $\gamma \in (1, 2)$ , define the mapping  $R_\gamma: \mathbb{T} \times (0, \infty) \rightarrow \mathbb{T}$  by:

$$R_\gamma((T, \emptyset, d, \mu), a) = (T, \emptyset, ad, a^{\gamma/(\gamma-1)} \mu), \quad \forall T \in \mathbb{T}. \quad (9.2)$$

In words, the real tree  $R_\gamma((T, \emptyset, d, \mu), a)$  is obtained from  $(T, \emptyset, d, \mu)$  by multiplying the metric by  $a$  and the measure by  $a^{\gamma/(\gamma-1)}$ . The choice of the exponent is justified by the following identity: for every  $a > 0$ ,

$$R_\gamma(\mathcal{T}, a) \quad \text{under} \quad \mathbf{N}^\psi \quad \stackrel{(d)}{=} \quad \mathcal{T} \quad \text{under} \quad a^{1/(\gamma-1)} \mathbf{N}^\psi. \quad (9.3)$$

Using this, one can define a regular conditional probability measure  $\mathbf{N}^\psi[\cdot|\sigma = a]$  such that  $\mathbf{N}^\psi[\cdot|\sigma = a]$ -a.s.  $\sigma = a$  and

$$\mathbf{N}^\psi[d\mathcal{T}] = \frac{1}{\gamma\Gamma(1-1/\gamma)} \int_0^\infty \frac{da}{a^{1+1/\gamma}} \mathbf{N}^\psi[d\mathcal{T}|\sigma = a]. \quad (9.4)$$

Furthermore, under  $\mathbf{N}^\psi[\cdot|\sigma = a]$ ,  $\mathcal{T}$  is distributed as  $R_\gamma(\mathcal{T}, a^{1-1/\gamma})$  under  $\mathbf{N}^\psi[\cdot|\sigma = 1]$ . We shall now establish the scaling property of the degree.

**Proposition 9.1.** *Let  $\psi(\lambda) = \lambda^\gamma$  with  $\gamma \in (1, 2)$ . Then, under  $\mathbf{N}^\psi[|\Delta = \delta]$ , the stable tree  $\mathcal{T}$  is distributed as  $R_\gamma(\mathcal{T}, \delta^{\gamma-1})$  under  $\mathbf{N}^\psi[|\Delta = 1]$ .*

*Proof.* Thanks to [13, Theorem 4.7], we can write the degree of the stable tree  $\mathcal{T}$  as

$$\Delta(\mathcal{T}) = \sup_{x \in \mathcal{T}} \left( \lim_{\varepsilon \rightarrow 0} ((\gamma - 1)\varepsilon)^{-1/(\gamma-1)} n_{\mathcal{T}}(x, \varepsilon) \right),$$

where  $n_{\mathcal{T}}(x, \varepsilon)$  is the number of subtrees originating from  $x$  with height greater than  $\varepsilon$ . In particular, it is straightforward to check that  $\Delta(R_\gamma(\mathcal{T}, a)) = a^{1/(\gamma-1)}\Delta(\mathcal{T})$ . Then the conclusion readily follows from (9.3).  $\square$

Denote by  $\Gamma(s, y)$  the upper incomplete gamma function:

$$\Gamma(s, y) = \int_y^\infty t^{s-1} e^{-t} dt, \quad \forall s \in \mathbb{R}, y > 0.$$

Then the Laplace exponent  $\psi_\delta$  is given by:

$$\psi_\delta(\lambda) = \lambda^\gamma + a_\gamma \int_\delta^\infty (1 - e^{-\lambda r}) \frac{dr}{r^{1+\gamma}} = \lambda^\gamma (1 - a_\gamma \Gamma(-\gamma, \lambda\delta)) + \gamma^{-1} a_\gamma \delta^{-\gamma}. \quad (9.5)$$

We will also need its derivative:

$$\psi'_\delta(\lambda) = \lambda^{\gamma-1} (\gamma + a_\gamma \Gamma(1 - \gamma, \lambda\delta)).$$

**Proposition 9.2.** *In the stable case  $\psi(\lambda) = \lambda^\gamma$ , we have:*

$$\mathbf{N}^\psi[\Delta > \delta] = c_\gamma \delta^{-1}, \quad (9.6)$$

$$\mathbf{N}^\psi[Z_0^\delta = 1] = \frac{c_\gamma}{\gamma} e^{c_\gamma} \delta^{-1}, \quad (9.7)$$

$$\mathbf{N}^\psi[W^\delta = 1] = \left( c_\gamma - \frac{\gamma c_\gamma^{\gamma+1}}{a_\gamma} e^{c_\gamma} \right) \delta^{-1}, \quad (9.8)$$

where  $c_\gamma \in (0, \infty)$  is such that  $\Gamma(-\gamma, c_\gamma) = a_\gamma^{-1}$ .

*Proof.* Thanks to (3.4), we have  $\psi_\delta(\mathbf{N}^\psi[\Delta > \delta]) = \bar{\pi}(\delta)$ . Together with (9.5), this implies that  $\delta \mathbf{N}^\psi[\Delta > \delta]$  is solution to  $\Gamma(-\gamma, x) = a_\gamma^{-1}$ . This proves (9.6).

To prove the remaining two identities, notice that

$$\psi'_\delta(\mathbf{N}^\psi[\Delta > \delta]) = \psi'_\delta(c_\gamma \delta^{-1}) = c_\gamma^{\gamma-1} \delta^{1-\gamma} (\gamma + a_\gamma \Gamma(1-\gamma, c_\gamma)) = \frac{a_\gamma}{c_\gamma} e^{-c_\gamma} \delta^{1-\gamma},$$

where we used the identity  $\Gamma(s+1, x) = s\Gamma(s, x) + x^s e^{-x}$  together with the definition of  $c_\gamma$  for the last equality. The result readily follows from Proposition 8.1 by a straightforward computation.  $\square$

**Lemma 9.3.** *For every  $\lambda \geq 0$ , there exists a constant  $c_\gamma(\lambda) \in (0, \infty)$  such that*

$$\psi_\delta^{-1}\left((1 - e^{-\lambda})\bar{\pi}(\delta)\right) = \frac{c_\gamma(\lambda)}{\delta}. \quad (9.9)$$

Moreover,  $c_\gamma(\lambda)$  is the unique positive solution to  $x^\gamma(a_\gamma \Gamma(-\gamma, x) - 1) = \gamma^{-1} a_\gamma e^{-\lambda}$ .

*Proof.* Fix  $\lambda \geq 0$  and let

$$u_\gamma^\lambda(x) = x^\gamma(1 - a_\gamma \Gamma(-\gamma, x)) + \gamma^{-1} a_\gamma e^{-\lambda}, \quad \forall x \geq 0.$$

Using the estimate  $\Gamma(-\gamma, x) \sim \gamma^{-1} x^\gamma$  as  $x \rightarrow 0$ , elementary analysis gives that  $u_\gamma^\lambda$  has a unique root which we denote by  $c_\gamma(\lambda)$ . Thanks to (9.5), we get:

$$\psi_\delta(\delta^{-1} c_\gamma(\lambda)) = (1 - e^{-\lambda}) \gamma^{-1} a_\gamma \delta^{-\gamma},$$

and the conclusion readily follows from (9.1).  $\square$

In the stable case, we can make explicit the distribution of the Bienaymé-Galton-Watson forest  $\tau_\delta$ .

**Proposition 9.4.** *Under  $\mathbf{N}^\psi$ , conditionally on  $\Delta > \delta$ , the random forest  $\tau_\delta$  consisting of nodes with mass larger than  $\delta$  is a critical  $(Z_0^\delta, \xi^\delta)$ -Bienaymé-Galton-Watson forest, where*

$$\mathbf{N}^\psi \left[1 - e^{-\lambda Z_0^\delta} \mid \Delta > \delta\right] = \frac{c_\gamma(\lambda)}{c_\gamma} \quad \text{and} \quad \mathbf{N}^\psi \left[e^{-\lambda \xi^\delta} \mid \Delta > \delta\right] = e^{-\lambda} + \frac{\gamma}{a_\gamma} c_\gamma(\lambda). \quad (9.10)$$

In particular, conditionally on  $\Delta > \delta$ , the distribution of  $\tau_\delta$  is independent of  $\delta$ .

*Proof.* Under  $\mathbf{N}^{\psi_\delta}$  and conditionally on  $\mathcal{T}$ , let  $\zeta$  be a Poisson random variable with parameter  $\bar{\pi}(\delta)\sigma$ . Notice that conditionally on  $\Delta > \delta$ ,  $Z_0^\delta$  is distributed as  $\zeta$  under  $\mathbf{N}^{\psi_\delta}$  conditionally on  $\zeta \geq 1$ . Thus we have:

$$\mathbf{N}^{\psi_\delta} \left[1 - e^{-\lambda Z_0^\delta} \mid \Delta > \delta\right] = \frac{\mathbf{N}^{\psi_\delta} \left[(1 - e^{-\lambda \zeta}) \mathbf{1}_{\{\zeta \geq 1\}}\right]}{\mathbf{N}^{\psi_\delta} [\zeta \geq 1]} = \frac{\mathbf{N}^{\psi_\delta} [1 - e^{-\lambda \zeta}]}{\mathbf{N}^{\psi_\delta} [\zeta \geq 1]}. \quad (9.11)$$

Since  $\mathbf{N}^{\psi_\delta}[\zeta \geq 1] = \mathbf{N}^\psi[\Delta > \delta]$  thanks to Theorem 4.1, it follows from (4.7) that

$$\mathbf{N}^{\psi_\delta} \left[1 - e^{-\lambda Z_0^\delta} \mid \Delta > \delta\right] = \frac{\psi_\delta^{-1}\left((1 - e^{-\lambda})\bar{\pi}(\delta)\right)}{\mathbf{N}^\psi[\Delta > \delta]}.$$

Combining (9.6) and (9.9), we deduce that

$$\mathbf{N}^\psi \left[ 1 - e^{-\lambda Z_0^\delta} \mid \Delta > \delta \right] = \frac{c_\gamma(\lambda)}{c_\gamma}.$$

Next, thanks to Theorem 4.1, it is easy to see than under  $\mathbf{N}^\psi$ , the random variables  $\xi^\delta$  and  $\mathbf{1}_{\{\Delta > \delta\}} = \mathbf{1}_{\{Z_0^\delta \geq 1\}}$  are independent. It follows from (4.8) and Lemma 9.3 that

$$\mathbf{N}^\psi \left[ e^{-\lambda \xi^\delta} \mid \Delta > \delta \right] = \mathbf{N}^\psi \left[ e^{-\lambda \xi^\delta} \right] = e^{-\lambda} + \frac{\gamma}{a_\gamma} c_\gamma(\lambda).$$

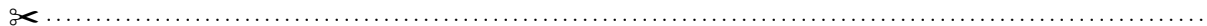
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## REFERENCES

- [1] R. Abraham and J.-F. Delmas, *Fragmentation associated with Lévy processes using snake*, Probab. Theory Related Fields **141** (2008), no. 1-2, 113–154. MR2372967
- [2] ———, *Williams' decomposition of the Lévy continuum random tree and simultaneous extinction probability for populations with neutral mutations*, Stochastic Process. Appl. **119** (2009), no. 4, 1124–1143. MR2508567
- [3] ———, *The forest associated with the record process on a Lévy tree*, Stochastic Process. Appl. **123** (2013), no. 9, 3497–3517. MR3071387
- [4] ———, *Local limits of conditioned Galton-Watson trees: the condensation case*, Electron. J. Probab. **19** (2014), no. 56, 29. MR3227065
- [5] R. Abraham, J.-F. Delmas, and G. Voisin, *Pruning a Lévy continuum random tree*, Electron. J. Probab. **15** (2010), no. 46, 1429–1473. MR2727317
- [6] L. Addario-Berry, N. Broutin, C. Goldschmidt, and G. Miermont, *The scaling limit of the minimum spanning tree of the complete graph*, Ann. Probab. **45** (2017), no. 5, 3075–3144. MR3706739
- [7] D. Aldous, *The continuum random tree. I*, Ann. Probab. (1991), 1–28. MR1085326
- [8] J. Bertoin, *Lévy processes*, Vol. 121, Cambridge University Press, Cambridge, 1996. MR1406564
- [9] ———, *On the maximal offspring in a critical branching process with infinite variance*, J. Appl. Probab. **48** (2011), no. 2, 576–582. MR2840318
- [10] V. I. Bogachev, *Measure theory. Vol. I, II*, Springer-Verlag, Berlin, 2007. MR2267655
- [11] T. Duquesne, *Continuum random trees and branching processes with immigration*, Stochastic Process. Appl. **119** (2009), no. 1, 99–129. MR2485021
- [12] T. Duquesne and J.-F. Le Gall, *Random trees, Lévy processes and spatial branching processes*, Astérisque **281** (2002), vi+147. MR1954248
- [13] ———, *Probabilistic and fractal aspects of Lévy trees*, Probab. Theory Related Fields **131** (2005), no. 4, 553–603. MR2147221
- [14] T. Duquesne and M. Winkel, *Mass erasure of Lévy trees*. In preparation.
- [15] L. C. Evans and R. F. Gariépy, *Measure theory and fine properties of functions*, Revised, Textbooks in Mathematics, CRC Press, Boca Raton, FL, 2015. MR3409135
- [16] S. N. Evans, *Probability and real trees*, Lecture Notes in Mathematics, vol. 1920, Springer, Berlin, 2008. Lectures from the 35th Summer School on Probability Theory held in Saint-Flour, July 6–23, 2005. MR2351587
- [17] B. Haas and G. Miermont, *The genealogy of self-similar fragmentations with negative index as a continuum random tree*, Electron. J. Probab. **9** (2004), no. 4, 57–97. MR2041829



- [18] ———, *Scaling limits of Markov branching trees with applications to Galton-Watson and random unordered trees*, Ann. Probab. **40** (2012), no. 6, 2589–2666. MR3050512
- [19] X. He, *Conditioning Galton-Watson trees on large maximal outdegree*, J. Theoret. Probab. **30** (2017), no. 3, 842–851. MR3687241
- [20] ———, *Local convergence of critical random trees and continuous-state branching processes*, J. Theoret. Probab. **35** (2022), no. 2, 685–713. MR4414402
- [21] X. He and Z. Li, *The maximal jump and local convergence of continuous-state branching processes*, 2014. arXiv:1412.4615.
- [22] ———, *Distributions of jumps in a continuous-state branching process with immigration*, J. Appl. Probab. **53** (2016), no. 4, 1166–1177. MR3581249
- [23] J. Jacod and A. N. Shiryaev, *Limit theorems for stochastic processes*, Second, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 288, Springer-Verlag, Berlin, 2003. MR1943877
- [24] S. Janson, *Simply generated trees, conditioned Galton-Watson trees, random allocations and condensation*, Probab. Surv. **9** (2012), 103–252. MR2908619
- [25] T. Jonsson and S. Ö. Stefánsson, *Condensation in nongeneric trees*, J. Stat. Phys. **142** (2011), no. 2, 277–313. MR2764126
- [26] I. Kortchemski, *Limit theorems for conditioned non-generic Galton-Watson trees*, Ann. Inst. Henri Poincaré Probab. Stat. **51** (2015), no. 2, 489–511. MR3335012
- [27] G. Last and M. Penrose, *Lectures on the Poisson process*, Institute of Mathematical Statistics Textbooks, vol. 7, Cambridge University Press, Cambridge, 2018. MR3791470
- [28] J.-F. Le Gall and Y. Le Jan, *Branching processes in Lévy processes: Laplace functionals of snakes and superprocesses*, Ann. Probab. **26** (1998), no. 4, 1407–1432. MR1675019
- [29] G. Miermont, *Self-similar fragmentations derived from the stable tree. I. Splitting at heights*, Probab. Theory Related Fields **127** (2003), no. 3, 423–454. MR2018924
- [30] ———, *Self-similar fragmentations derived from the stable tree. II. Splitting at nodes*, Probab. Theory Related Fields **131** (2005), no. 3, 341–375. MR2123249
- [31] B. Stuffer, *On the maximal offspring in a subcritical branching process*, Electron. J. Probab. **25** (2020), Paper No. 104, 62. MR4144886



### Index of notation

#### Spaces

$\mathcal{M}_f(E)$	space of finite measures on $E$
$\mathcal{D}$	space of càdlàg functions from $\mathbb{R}_+$ to $\mathcal{M}_f(\mathbb{R}_+)$
$\mathcal{D}_0$	space of càdlàg excursions from $\mathbb{R}_+$ to $\mathcal{M}_f(\mathbb{R}_+)$

#### Random variables

$\rho_t$	exploration process
$\eta_t$	dual process
$H_t$	height process
$\sigma$	lifetime of the exploration process
$L^h(ds)$	local time at level $h$
$\Delta$	maximal degree of the exploration process
$T_\delta$	first time the exploration process contains a node with mass larger than $\delta$
$\rho^{\delta,-}$	path of the exploration process after removing the first node with mass larger than $\delta$
$\rho^{\delta,+}$	path of the exploration process above the first node with mass larger than $\delta$
$\tau_\delta$	discrete tree consisting of nodes with mass larger than $\delta$
$W^\delta$	number of nodes with mass larger $\delta$
$Z_0^\delta$	number of first-generation nodes with mass larger than $\delta$
$T_\Delta$	first time the exploration process contains a node with mass $\Delta$
$H_\Delta$	height of the first node with mass $\Delta$
$\rho^{\Delta,-}$	path of the exploration process after removing the first node with mass $\Delta$
$\rho^{\Delta,+}$	path of the exploration process above the first node with mass $\Delta$

#### Measures

$\mathbb{P}^\psi$	distribution of the exploration process starting from 0
$\mathbf{N}^\psi$	excursion measure of the exploration process
$\mathbb{P}_\nu^{\psi,*}$	distribution of the exploration process starting at $\nu$ and killed when it first reaches 0
$\mathbb{P}_r^\psi$	distribution of the exploration process with initial degree $r$

$\mathbb{Q}_\delta^\psi$	distribution of the exploration process with random initial degree, (2.19)
$\mathbf{P}_\delta^\psi$	distribution of a marked exploration process with degree restriction, (5.4)
$\mathbf{P}_{\delta,h}^\psi$	distribution of a marked (at level $h$ ) exploration process with degree restriction, (6.5)

### Functions

$\bar{\pi}(\delta)$	tail of the Lévy measure $\pi$
$w(\delta)$	$\mathbf{N}^\psi[\sigma \mathbf{1}_{\{\Delta < \delta\}}]$
$w_+(\delta)$	$\mathbf{N}^\psi[\sigma \mathbf{1}_{\{\Delta \leq \delta\}}]$
$\mathbf{g}(\delta)$	$\pi(\delta)e^{-\delta \mathbf{N}^\psi[\Delta > \delta]}$