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# Zero-Sum Games with Noisy Observations

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## Abstract

In this paper,  $2 \times 2$  zero-sum games (ZSGs) are studied under the following assumptions: (1) One of the players (the leader) publicly and irrevocably commits to choose its actions by sampling a given probability measure (strategy); (2) The leader announces its action, which is observed by its opponent (the follower) through a binary channel; and (3) the follower chooses its strategy based on the knowledge of the leader's strategy and the noisy observation of the leader's action. Under these conditions, the equilibrium is shown to always exist and be often different from the Nash and Stackelberg equilibria. Even subject to noise, observing the actions of the leader is either beneficial or immaterial to the follower for all possible commitments. When the commitment is observed subject to a distortion, the equilibrium does not necessarily exist. Nonetheless, the leader might still obtain some benefit in some specific cases subject to equilibrium refinements. For instance,  $\epsilon$ -equilibria might exist in which the leader commits to suboptimal strategies that allow unequivocally predicting the best response of its opponent.

## 1 INTRODUCTION

Zero-sum games (ZSGs) are mathematical models describing the interaction of mutually adversarial decision makers. In the realm of machine learning, ZSGs have played a central role in the development of techniques such as generative adversarial networks (GANs) [Goodfellow et al., 2014] and adversarial training [Madry et al., 2018]. Within this context, two solution concepts have been adopted for predicting the outcome of ZSGs: the Nash equilibrium (NE) [Nash, 1950] and the Stackelberg equilibrium (SE) [Stackelberg, 1952]. The NE is a prediction observed under the assumption that both players simultaneously choose their strategies (probability measures over the set of pos-

sible actions or decisions). On the other hand, the SE describes the outcome in which one of the players (the leader) publicly and irrevocably commits to use a particular strategy before its opponent (the follower). In such a case, the follower chooses its strategy as a best response to the commitment of the leader.

Commitments are said to be in mixed strategies when the leader is allowed to commit to strategies whose support contains more than one action. In this case, the relevant solution concept is the SE in mixed strategies [Conitzer and Sandholm, 2006, Conitzer, 2016, Leonardos and Melolidakis, 2018]. Interestingly, in finite ZSGs, the payoffs at the NE and the SE in mixed strategies are identical, as shown in [Von Stengel and Zamir, 2010]. The commitment is said to be in pure strategies when the leader is constrained to commit to play one action with probability one. This is assimilated to the case in which the follower perfectly observes the action played by the leader. The relevant solution concept under these assumptions is the SE in pure strategies [Stackelberg, 1952, Simaan and Cruz, 1973b, Simaan and Cruz, 1973a]. The expected payoff at the SE in pure strategies is equal to the min max or max min solution, where the optimization is over the set of actions [Jin et al., 2020, Bai et al., 2021].

In adversarial training, the underlying assumption is that the follower (the attacker or adversary) perfectly observes the action played by the leader (the learner) [Huang et al., 2022, Zuo et al., 2021, Brückner and Scheffer, 2011, Gao et al., 2022, Bai et al., 2021]. Similarly, in data integrity attacks, the follower (the learner) perfectly observes the action of the leader (the attacker) [Chivukula and Liu, 2017, Liu and Chawla, 2009, Kantarcioğlu et al., 2011]. That is, adversarial training and data integrity attacks are studied using the SE in pure strategies. Alternatively, GANs are modelled by ZSGs in which the relevant solution concept is the NE (or SE in mixed strategies) [Hsieh et al., 2019, Oliehoek et al., 2018]. Essentially, ZSGs are used to predict game outcomes in terms of mixed strategies (probability measures), instead of actions (pure strategies). In a nutshell, the underlying assumption of the SE in mixed strategies is that the strategy to which the leader commits to is perfectly observed by the follower and the actions are unobservable. Alternatively, the assumption of the SE in pure

strategies is that actions are perfectly observable, which makes the notion of commitment irrelevant. These assumptions are difficult to justify in practice. Often, data is obtained via data acquisition systems that are subject to quantization errors, additive noise, and impairments due to data transmission and storage. These additional impairments are not necessarily due to malicious agents but the nature of the data acquisition and information processing [Cover and Thomas, 2012]. In real system implementations, the observations of actions and commitments, if they occur, are subject to noise. Nonetheless, the impact of noisy observations in adversarial training, GANs, and most areas of ML relying on ZSGs remains an uncharted territory, in part due to the lack of simple and adapted solution concepts. This work makes progress in this direction and proposes a game formulation that takes into account noisy observations of both actions and commitments.

### 1.1 Contributions

For pedagogical purposes, results are presented for  $2 \times 2$  ZSGs. Nonetheless, the results can be readily extended to the case of two-player ZSGs, with finite number of actions. The contributions are presented as follows.

Section 3 introduces a game formulation in which the follower obtains a noisy observation of the action played by the leader, whereas the commitment is assumed to be perfectly observed. Three results are presented: First, the set of best responses of the follower is characterized and the role of the priors formed by the follower with the available information is presented. Second, the set of optimal commitments for the leader is calculated and it is shown that, even subject to noise, observations either benefit or never make a difference to the follower. Third, the equilibrium is shown to always exist. Benefits for the follower are observed at the equilibrium exclusively when the ZSG exhibits a unique NE in mixed strategies. In all other cases, e.g., ZSG exhibiting strategic dominance, unique NE in pure strategies, or infinitely many NEs, the payoffs with and without observations are identical.

Section 4 introduces a game formulation in which the follower obtains a noisy observation of both the action played by the leader and the commitment. The commitment mismatch is modelled by a deterministic distortion (affine function) that is assumed to be known by the leader and ignored by the follower. Commitment mismatch is shown to be either beneficial or immaterial to the leader. Nonetheless, beneficial situations for the leader are shown to be not stable, in part because, an equilibrium does not necessarily exist. This phenomenon arises due to the fact that optimal commitments induce infinitely many best responses for the follower and each of them leads to a different payoff. To overcome this challenge, the leader is subject to commit to suboptimal strategies in order to induce a unique best re-

sponse that can be predicted.

The work is finalized with a discussion and final remarks presented in Section 5. The proofs of all results are presented in the supplementary material.

### Notation

Given a finite set  $\mathcal{X}$ , the notation  $2^{\mathcal{X}}$  represents the power set of  $\mathcal{X}$ . The notation  $\Delta(\mathcal{X})$  represents the set of all probability measures that can be defined on the measurable space  $(\mathcal{X}, 2^{\mathcal{X}})$ . The set of all subsets of  $\Delta(\mathcal{X})$  is denoted by  $2^{\Delta(\mathcal{X})}$ . Given two matrices  $\underline{a}$  and  $\underline{b}$ , their Hadamard product is denoted by  $\underline{a} \circ \underline{b}$ .

## 2 PRELIMINARIES

Consider a two-player two-action zero-sum game in normal form with payoff matrix

$$\underline{u} = \begin{pmatrix} u_{1,1} & u_{1,2} \\ u_{2,1} & u_{2,2} \end{pmatrix}. \quad (1)$$

Let the elements of the set  $\mathcal{K} \triangleq \{1, 2\}$  represent the indices of the players; and let the elements of the set  $\mathcal{A}_1 = \mathcal{A}_2 \triangleq \{a_1, a_2\}$  represent the actions of the players. Hence, for all  $(i, j) \in \{1, 2\}^2$ , when Player 1 plays  $a_i$  and Player 2 plays  $a_j$ , the outcome of the game is  $u_{i,j}$ . In the following, such a game is represented by the tuple

$$\mathcal{G}(\underline{u}) \triangleq (\mathcal{K}, \mathcal{A}_1, \mathcal{A}_2, \underline{u}). \quad (2)$$

The remaining part of this section relies on the following assumptions: (i) The game  $\mathcal{G}(\underline{u})$  is repeated infinitely many times; (ii) At each repetition, the players are oblivious of all previous repetitions; and (iii) actions are simultaneously chosen at each repetition. Under assumptions (i) – (iii), the average payoff achieved by the players in the *repeated game* can be expressed in terms of their *strategies*. For all  $k \in \mathcal{K}$ , the strategy of Player  $k$  is a probability measure denoted by  $P_{A_k} \in \Delta(\mathcal{A}_k)$ . At each repetition of the game, players choose their actions by sampling their probability measures (strategies). Let the average payoff be represented by the function  $u : \Delta(\mathcal{A}_1) \times \Delta(\mathcal{A}_2) \rightarrow \mathbb{R}$  such that, given the strategies  $P_{A_1}$  and  $P_{A_2}$ ,

$$u(P_{A_1}, P_{A_2}) = \sum_{(i,j) \in \{1,2\}^2} P_{A_1}(a_i) P_{A_2}(a_j) u_{i,j}. \quad (3)$$

Note that the average payoff coincides with the expected payoff under assumptions (i) – (iii). Player 1 chooses its strategy  $P_{A_1}$  aiming to maximize the expected payoff  $u(P_{A_1}, P_{A_2})$ , whereas Player 2 chooses the strategy  $P_{A_2}$  to minimize it. Interestingly, under assumptions (i) – (iii), players can calculate their optimal strategies before the beginning of the repeated game, as shown in the following sections.

When the repeated game is played without commitments, the relevant solution concept for the ZSG  $\mathcal{G}(\underline{u})$  in (2) is the NE. The following lemma characterizes the payoff at the NE and shows that  $2 \times 2$  ZSGs exhibit either a unique NE or infinitely many NEs.

**Lemma 2.1** *Let the probability measures  $P_{A_1}^* \in \Delta(\mathcal{A}_1)$  and  $P_{A_2}^* \in \Delta(\mathcal{A}_2)$  form a NE of the game  $\mathcal{G}(\underline{u})$  in (2). If the entries of the matrix  $\underline{u}$  in (1) satisfy*

$$(u_{1,1} - u_{1,2})(u_{2,2} - u_{2,1}) > 0 \text{ and} \quad (4a)$$

$$(u_{1,1} - u_{2,1})(u_{2,2} - u_{1,2}) > 0, \quad (4b)$$

then, the NE of the game  $\mathcal{G}(\underline{u})$  in (2) is unique, with

$$P_{A_1}^*(a_1) = \frac{u_{2,2} - u_{2,1}}{u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2}} \in (0, 1) \text{ and} \quad (5a)$$

$$P_{A_2}^*(a_1) = \frac{u_{2,2} - u_{1,2}}{u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2}} \in (0, 1). \quad (5b)$$

Moreover, the expected payoff at the NE is

$$u(P_{A_1}^*, P_{A_2}^*) = \frac{u_{1,1}u_{2,2} - u_{1,2}u_{2,1}}{u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2}}. \quad (6)$$

If the entries of the matrix  $\underline{u}$  in (1) satisfy

$$(u_{1,1} - u_{1,2})(u_{2,2} - u_{2,1}) \leq 0 \text{ or} \quad (7a)$$

$$(u_{1,1} - u_{2,1})(u_{2,2} - u_{1,2}) \leq 0, \quad (7b)$$

then, there exists either a unique NE or infinitely many NEs. Moreover, all NE strategies lead to the same payoff,

$$\begin{aligned} u(P_{A_1}^*, P_{A_2}^*) &= \min\{\max\{u_{1,1}, u_{2,1}\}, \max\{u_{1,2}, u_{2,2}\}\} \quad (8) \\ &= \max\{\min\{u_{1,1}, u_{1,2}\}, \min\{u_{2,1}, u_{2,2}\}\}. \quad (9) \end{aligned}$$

A payoff matrix  $\underline{u}$  that satisfies (4) represents a ZSG exhibiting a unique NE in strictly mixed strategies. Alternatively, a payoff matrix  $\underline{u}$  that satisfies (7) represents a ZSG exhibiting *strategic dominance*, a unique pure NE, or infinitely many NEs.

### 3 NOISY OBSERVATIONS OF THE ACTIONS

In this section, the repeated game is assumed to be played with commitments under the assumptions (i) and (ii) in Section 2, and a new assumption: (iv) at each repetition, the leader chooses its action and the follower obtains a noisy observation. That is, assumption (iii) is dropped, and the follower chooses a strategy at each repetition, knowing the commitment and a noisy observation of the action played by the leader.

#### 3.1 Game Formulation

Denote by  $A_1$ ,  $A_2$ , and  $\tilde{A}_2$  the random variables representing the actions of Player 1 (the follower), Player 2 (the

leader), and the noisy observation of the action played by Player 2 at each repetition of the game, respectively. Let also  $P_{A_1 \tilde{A}_2 A_2} \in \Delta(\mathcal{A}_1 \times \mathcal{A}_2 \times \mathcal{A}_2)$  be the probability measure jointly induced by  $A_1$ ,  $\tilde{A}_2$ , and  $A_2$ , which satisfies for all  $(a, \tilde{b}, b) \in \mathcal{A}_1 \times \mathcal{A}_2 \times \mathcal{A}_2$ ,

$$P_{A_1 \tilde{A}_2 A_2}(a, \tilde{b}, b) = P_{A_2}(b) P_{\tilde{A}_2 | A_2 = b}(\tilde{b}) P_{A_1 | \tilde{A}_2 = \tilde{b}}(a), \quad (10)$$

where the probability measure  $P_{A_2} \in \Delta(\mathcal{A}_2)$  is the strategy of Player 2; the pair of probability measures  $(P_{A_1 | \tilde{A}_2 = a_1}, P_{A_1 | \tilde{A}_2 = a_2}) \in \Delta(\mathcal{A}_1) \times \Delta(\mathcal{A}_1)$  form the strategy of Player 1; and the pair of measures  $(P_{\tilde{A}_2 | A_2 = a_1}, P_{\tilde{A}_2 | A_2 = a_2})$  form a binary channel through which the follower observes the action of the leader.

Using this notation, the development of the repeated game is described as follows. Before the beginning of the repetitions, Player 2 publicly and irrevocably announces its strategy  $P_{A_2}$ . At each repetition, Player 2 (the leader) plays the action  $b \in \mathcal{A}_2$  with probability  $P_{A_2}(b)$ . Player 1 observes action  $\tilde{b} \in \mathcal{A}_2$  with probability  $P_{\tilde{A}_2 | A_2 = b}(\tilde{b})$ . Finally, Player 1 plays the action  $a \in \mathcal{A}_1$ , with probability  $P_{A_1 | \tilde{A}_2 = \tilde{b}}(a)$  and both players obtain their payoffs.

The expected payoff obtained by the players is determined by the function  $v : \Delta(\mathcal{A}_1) \times \Delta(\mathcal{A}_1) \times \Delta(\mathcal{A}_2) \rightarrow \mathbb{R}$ , such that given the strategy  $(P_{A_1 | \tilde{A}_2 = a_1}, P_{A_1 | \tilde{A}_2 = a_2})$  of Player 1, often denoted by  $P_{A_1 | \tilde{A}_2}$ , and the strategy  $P_{A_2}$  of Player 2, the expected payoff  $v(P_{A_1 | \tilde{A}_2}, P_{A_2})$  is

$$v(P_{A_1 | \tilde{A}_2}, P_{A_2}) = \sum_{(i,j) \in \{1,2\}^2} u_{i,j} P_{A_1 A_2}(a_i, a_j), \quad (11)$$

where the joint probability measure  $P_{A_1 A_2}$  is the marginal probability measure on  $A_1$  and  $A_2$  of the probability measure  $P_{A_1 \tilde{A}_2 A_2}$  in (10).

The generalization of the game in normal form  $\mathcal{G}(\underline{u})$  in (2) obtained by including the binary channel formed by the measures  $P_{\tilde{A}_2 | A_2 = a_1}$  and  $P_{\tilde{A}_2 | A_2 = a_2}$  is described by the tuple

$$\mathcal{G}(\underline{u}, \underline{w}) = (\mathcal{K}, \mathcal{A}_1, \mathcal{A}_2, \underline{u}, \underline{w}), \quad (12)$$

where the  $2 \times 2$  matrix  $\underline{w}$  satisfies

$$\underline{w} = \begin{pmatrix} P_{\tilde{A}_2 | A_2 = a_1}(a_1) & P_{\tilde{A}_2 | A_2 = a_2}(a_1) \\ P_{\tilde{A}_2 | A_2 = a_1}(a_2) & P_{\tilde{A}_2 | A_2 = a_2}(a_2) \end{pmatrix}. \quad (13)$$

#### 3.2 The Set of Best Responses of Player 1

The set of best responses of Player 1 to a given strategy of Player 2 is determined by the correspondence  $\text{BR}_1 : \Delta(\mathcal{A}_2) \rightarrow 2^{\Delta(\mathcal{A}_1) \times \Delta(\mathcal{A}_1)}$ . That is, given the commitment  $P_{A_2}$ , it holds that

$$\text{BR}_1(P_{A_2}) = \arg \max_{(Q_1, Q_2) \in \Delta(\mathcal{A}_1) \times \Delta(\mathcal{A}_1)} v(Q_1, Q_2, P_{A_2}), \quad (14)$$

where the function  $v$  is defined in (11). In order to study the set  $\text{BR}_1(P_{A_2})$  in (14), consider the  $2 \times 2$  matrix

$$\underline{\mathbf{u}}^{(i)} = \underline{\mathbf{u}} \begin{pmatrix} P_{\tilde{A}_2|A_2=a_1}(a_i) & 0 \\ 0 & P_{\tilde{A}_2|A_2=a_2}(a_i) \end{pmatrix}, \quad (15)$$

with  $i \in \{1, 2\}$ , where the matrix  $\underline{\mathbf{u}}$  is defined in (1); and the probability measures  $P_{\tilde{A}_2|A_2=a_1}$  and  $P_{\tilde{A}_2|A_2=a_2}$  are defined in (13). The following lemma shows that, given a commitment  $P_{A_2}$ , the set of best responses  $\text{BR}_1(P_{A_2})$  in (14) is the Cartesian product of two sets that can be described independently.

**Lemma 3.1** *The correspondence  $\text{BR}_1$  in (14) satisfies for all  $P \in \Delta(\mathcal{A}_2)$ ,*

$$\text{BR}_1(P) = \text{BR}_{1,1}(P) \times \text{BR}_{1,2}(P), \quad (16)$$

where for all  $i \in \{1, 2\}$ , the correspondence  $\text{BR}_{1,i} : \Delta(\mathcal{A}_2) \rightarrow 2^{\Delta(\mathcal{A}_1)}$  is such that

$$\text{BR}_{1,i}(P) = \arg \max_{Q \in \Delta(\mathcal{A}_1)} \begin{pmatrix} Q(a_1) \\ Q(a_2) \end{pmatrix}^T \underline{\mathbf{u}}^{(i)} \begin{pmatrix} P(a_1) \\ P(a_2) \end{pmatrix}, \quad (17)$$

where the matrix  $\underline{\mathbf{u}}^{(i)}$  is in (15).

A first observation from Lemma 3.1 is that in the case in which the matrices  $\underline{\mathbf{u}}^{(1)}$  and  $\underline{\mathbf{u}}^{(2)}$  in (15) are identical, Player 1 chooses its actions independently of the noisy observation of the action played by Player 2. In such a case,  $\text{BR}_{1,1}(P_{A_2}) = \text{BR}_{1,2}(P_{A_2})$ , and thus, the best response of Player 1 depends exclusively on its opponent's strategy  $P_{A_2}$ .

For all  $i \in \{1, 2\}$  and for all  $P \in \Delta(\mathcal{A}_2)$ , the cardinality of set  $\text{BR}_{1,i}(P)$  is either one or infinite. In the case in which  $\text{BR}_{1,i}(P)$  is unitary, the only element is a pure strategy. Alternatively, when the cardinality is infinity, the set  $\text{BR}_{1,i}(P)$  is identical to the set of all possible probability measures on  $\mathcal{A}_1$ , i.e.,  $\text{BR}_{1,i}(P) = \Delta(\mathcal{A}_1)$ . That is, at each repetition of the game, Player 1 chooses its actions either deterministically, i.e., with probability one; or indifferently. The following lemma formalizes this observation.

**Lemma 3.2** *Given a probability measure  $P \in \Delta(\mathcal{A}_2)$ , for all  $i \in \{1, 2\}$ , the correspondence  $\text{BR}_{1,i}$  in (17) satisfies*

$$\text{BR}_{1,i}(P) = \begin{cases} \{Q \in \Delta(\mathcal{A}_1) : Q(a_1) = 1\}, & \text{if } s_i > 0, \\ \{Q \in \Delta(\mathcal{A}_1) : Q(a_1) = 0\}, & \text{if } s_i < 0, \\ \Delta(\mathcal{A}_1), & \text{if } s_i = 0, \end{cases} \quad (18)$$

where  $s_i \in \mathbb{R}$  is given by

$$s_i \triangleq (u_{1,1} - u_{2,1}) P(a_1) P_{\tilde{A}_2|A_2=a_1}(a_i) + (u_{1,2} - u_{2,2}) P(a_2) P_{\tilde{A}_2|A_2=a_2}(a_i). \quad (19)$$

For all  $(i, j) \in \{1, 2\}^2$ , at a given game repetition, the expected payoff, when Player 1 plays

$a_j$ , Player 2 has committed to  $P_{A_2}$ , and the noisy observation is  $a_i$ , is  $u_{j,1} P_{A_2}(a_1) P_{\tilde{A}_2|A_2=a_1}(a_i) + u_{j,2} P_{A_2}(a_2) P_{\tilde{A}_2|A_2=a_2}(a_i)$ . Thus, the right-hand side of the equality in (19) is the difference between the expected payoff obtained by the players when Player 1 plays  $a_1$  and when it plays  $a_2$ , subject to the observation  $a_i$  and the commitment  $P_{A_2}$ . Hence, from Lemma 3.2, the optimal action to be played at a given repetition by Player 1 is the action that maximizes the expected payoff subject to the noisy observation and the commitment. When both actions induce the same expected payoff, Player 1 chooses its actions following any strategy.

The following lemma presents a different view of the correspondences  $\text{BR}_{1,1}$  and  $\text{BR}_{1,2}$  in (17). It suggests that at each game repetition, Player 1 performs an estimation of the action played by Player 2 based on the knowledge of the commitment and the noisy observation.

**Lemma 3.3** *Given a probability measure  $P \in \Delta(\mathcal{A}_2)$ , for all  $i \in \{1, 2\}$ , the correspondence  $\text{BR}_{1,i}$  in (17) satisfies*

$$\text{BR}_{1,i}(P) = \arg \max_{Q \in \Delta(\mathcal{A}_1)} u \left( Q, P_{A_2|\tilde{A}_2=a_i} \right), \quad (20)$$

where the function  $u$  is defined in (3); the probability measure  $P_{A_2|\tilde{A}_2=a_i}$  satisfies for all  $j \in \{1, 2\}$ ,

$$P_{A_2|\tilde{A}_2=a_i}(a_j) = \frac{P_{\tilde{A}_2|A_2=a_j}(a_i) P(a_j)}{\sum_{\ell \in \{1, 2\}} P_{\tilde{A}_2|A_2=a_\ell}(a_i) P(a_\ell)}, \quad (21)$$

with the probability measures  $P_{\tilde{A}_2|A_2=a_1}$  and  $P_{\tilde{A}_2|A_2=a_2}$  defined in (13).

For all  $(i, j) \in \{1, 2\}^2$ , the likelihood with which Player 2 has chosen action  $a_j$  given the commitment  $P$  and the noisy observation  $a_i$  is  $P_{A_2|\tilde{A}_2=a_i}(a_j)$  in (21). Hence, from Lemma 3.2 and Lemma 3.3, at a given repetition of the game  $\mathcal{G}(\underline{\mathbf{u}}, \underline{\mathbf{w}})$ , the optimal action of Player 1 to the observation  $a_i$  and the commitment  $P$  is identical to the optimal action of a player in the game  $\mathcal{G}(\underline{\mathbf{u}})$  in (2) when its opponent plays the strategy  $P_{A_2|\tilde{A}_2=a_i}$  in (21).

### 3.3 The Best Strategies of Player 2

Let the function  $\hat{v} : \Delta(\mathcal{A}_2) \rightarrow \mathbb{R}$  be such that given a probability measure  $P \in \Delta(\mathcal{A}_2)$ ,

$$\hat{v}(P) = \max_{(Q_1, Q_2) \in \text{BR}_1(P)} v(Q_1, Q_2, P), \quad (22)$$

where the function  $v$  is defined in (11) and the correspondence  $\text{BR}_1$  is defined in (14). The set of best strategies for Player 2, under the assumption that Player 1 observes the commitment and obtains a noisy observation of the action played by Player 2, is the set of minimizers of the function

$\hat{v}$  in (22). Let  $P^{(1)}$  and  $P^{(2)}$  be two real numbers such that for all  $i \in \{1, 2\}$ ,

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}^\top \underline{\mathbf{u}}^{(i)} \begin{pmatrix} P^{(i)} \\ 1 - P^{(i)} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}^\top \underline{\mathbf{u}}^{(i)} \begin{pmatrix} P^{(i)} \\ 1 - P^{(i)} \end{pmatrix}. \quad (23)$$

From Lemma 3.2, it holds that if  $P^{(i)} \in [0, 1]$ , for some  $i \in \{1, 2\}$ , and Player 2 adopts a strategy  $P_{A_2} \in \Delta(\mathcal{A}_2)$  such that  $P_{A_2}(a_1) = P^{(i)}$ , then  $\text{BR}_{1,i}(P_{A_2}) = \Delta(\mathcal{A}_1)$ .

Let the function  $\hat{u} : \Delta(\mathcal{A}_2) \rightarrow \mathbb{R}$  be such that for all  $P \in \Delta(\mathcal{A}_2)$ ,

$$\hat{u}(P) = \max_{Q \in \Delta(\mathcal{A}_1)} u(Q, P), \quad (24)$$

where the function  $u$  is defined in (3).

The set of best strategies of Player 2, under the assumption that Player 1 observes the commitment but not the actual action played by the leader, is formed by the probability measures in  $\Delta(\mathcal{A}_2)$  that minimize the function  $\hat{u}$  in (24). Under this assumption, if the probability measure  $P \in \Delta(\mathcal{A}_2)$  is one of the best strategies of Player 2, then it is also a strategy of Player 2 in at least one NE. Moreover,  $\hat{u}(P)$  is the payoff at the NE.

Figure 1 depicts the functions  $\hat{v}$  in (22) and  $\hat{u}$  in (24). Note that in all cases, the function  $\hat{v}$  is lower bounded by the function  $\hat{u}$ . This implies that, when the follower is granted with an observation of the action played by the leader, even subject to noise, the payoff does not decrease, and in some cases, might significantly increase, as shown by the following lemma.

**Lemma 3.4** *Let the probability measures  $P_{A_1}^* \in \Delta(\mathcal{A}_1)$  and  $P_{A_2}^* \in \Delta(\mathcal{A}_2)$  form one of the NEs of the game  $\mathcal{G}(\underline{\mathbf{u}})$  in (2). For all  $P \in \Delta(\mathcal{A}_2)$ , it holds that*

$$u(P_{A_1}^*, P_{A_2}^*) \leq \hat{u}(P) \quad (25)$$

$$\leq \hat{v}(P) \quad (26)$$

$$\leq \sum_{k \in \{1, 2\}} P(a_k) \left( \max_{i \in \{1, 2\}} u_{i,k} \right), \quad (27)$$

where the functions  $u$ ,  $\hat{v}$ , and  $\hat{u}$  are defined in (3), (22) and (24), respectively.

The inequality in (25) follows from the definition of the function  $\hat{u}$  in (24). The inequality in (26) shows that for all the strategies that Player 2 might adopt, the payoff with noisy observations is larger than or equal to the payoff without observations. Hence, even subject to noise, granting the follower with an observation of the action played by the leader is either beneficial or immaterial for the follower. The inequality in (27) holds with equality when Player 1 is always able to best respond to the actual action played by the leader. This is for instance the case when the observation of the action played by the leader is noiseless.

### 3.4 Equilibria

The solution concept for the game  $\mathcal{G}(\underline{\mathbf{u}}, \underline{\mathbf{w}})$  in (12) is defined hereunder.

**Definition 3.1 (Equilibrium)** *The tuple  $(P_{A_1|\bar{A}_2=a_1}^\dagger, P_{A_1|\bar{A}_2=a_2}^\dagger, P_{A_2}^\dagger) \in \Delta(\mathcal{A}_1) \times \Delta(\mathcal{A}_1) \times \Delta(\mathcal{A}_2)$  is said to form an equilibrium of the game  $\mathcal{G}(\underline{\mathbf{u}}, \underline{\mathbf{w}})$  in (12) if*

$$P_{A_2}^\dagger \in \arg \min_{P \in \Delta(\mathcal{A}_2)} \hat{v}(P) \quad \text{and} \quad (28)$$

$$(P_{A_1|\bar{A}_2=a_1}^\dagger, P_{A_1|\bar{A}_2=a_2}^\dagger) \in \text{BR}_1(P_{A_2}^\dagger), \quad (29)$$

where the function  $\hat{v}$  is in (22) and the correspondence  $\text{BR}_1$  is in (16).

The following theorem ensures the existence of an equilibrium for the game  $\mathcal{G}(\underline{\mathbf{u}}, \underline{\mathbf{w}})$  in (12).

**Theorem 3.1 (Existence)** *The game  $\mathcal{G}(\underline{\mathbf{u}}, \underline{\mathbf{w}})$  in (12) always possesses an equilibrium.*

When the game  $\mathcal{G}(\underline{\mathbf{u}}, \underline{\mathbf{w}})$  in (12) possesses several equilibria, the payoff is identical at all equilibria, as shown by the following theorem.

**Theorem 3.2 (Equilibrium Payoff)** *Let the tuple  $(P_{A_1|\bar{A}_2=a_1}^\dagger, P_{A_1|\bar{A}_2=a_2}^\dagger, P_{A_2}^\dagger) \in \Delta(\mathcal{A}_1) \times \Delta(\mathcal{A}_1) \times \Delta(\mathcal{A}_2)$  form an equilibrium of the game  $\mathcal{G}(\underline{\mathbf{u}}, \underline{\mathbf{w}})$  in (12). If the matrix  $\underline{\mathbf{u}}$  in (1) satisfies (4), then*

$$\hat{v}(P_{A_2}^\dagger) = \min\{\hat{v}(P_1), \hat{v}(P_2)\}, \quad (30)$$

where, the function  $\hat{v}$  is defined in (22), and for all  $i \in \{1, 2\}$ , the probability measure  $P_i \in \Delta(\mathcal{A}_2)$  is such that  $P_i(a_1) = P^{(i)}$ , with  $P^{(i)}$  in (23). Alternatively, if the entries of the matrix  $\underline{\mathbf{u}}$  satisfy (7), then

$$\hat{v}(P_{A_2}^\dagger) = \min\{\max\{u_{1,1}, u_{2,1}\}, \max\{u_{1,2}, u_{2,2}\}\}. \quad (31)$$

When the payoff matrix  $\underline{\mathbf{u}}$  in (1) satisfies (4), at the equilibrium, Player 2 commits to a strategy that renders Player 1 indifferent to play any of its actions for at least one of the noisy observations. That is, for some  $j \in \{1, 2\}$ , when at a given game repetition, Player 1 obtains  $a_j$  as the noisy observation, it holds that  $\text{BR}_{1,j}(P_{A_2}^\dagger) = \Delta(\mathcal{A}_1)$ . The following lemma sheds more light into this particularity.

**Lemma 3.5** *Let  $\mathcal{S} \subseteq \Delta(\mathcal{A}_1) \times \Delta(\mathcal{A}_1) \times \Delta(\mathcal{A}_2)$  be the set of equilibria of the game  $\mathcal{G}(\underline{\mathbf{u}}, \underline{\mathbf{w}})$  in (12). Let the matrix  $\underline{\mathbf{u}}$  in (1) satisfy (4), and let  $P_{A_1}^* \in \Delta(\mathcal{A}_1)$  and  $P_{A_2}^* \in \Delta(\mathcal{A}_2)$  form a NE in the game  $\mathcal{G}(\underline{\mathbf{u}})$  in (2). Then, there exists a tuple  $(Q_1, Q_2, P) \in \mathcal{S}$  such that  $P(a_1) \in \{P^{(1)}, P^{(2)}\}$ . Furthermore, if  $P(a_1) = P^{(i)}$ , with  $i \in \{1, 2\}$ , then it holds that  $P_{A_2|\bar{A}_2=a_i}(a_1) = P_{A_2}^*(a_1)$ , where  $P_{A_2|\bar{A}_2=a_i}$  is in (21).*

Lemma 3.3 and Lemma 3.5 lead to a deeper conclusion. When the payoff matrix  $\underline{u}$  in (1) satisfies (4), at the equilibrium, Player 2 (the leader) commits to a strategy such that for at least one  $i \in \{1, 2\}$ , the posterior  $P_{A_2|\tilde{A}_2=a_i}$  in (21) is equal to the strategy of Player 2 at the (unique) NE.

Finally, when the payoff matrix  $\underline{u}$  in (1) satisfies (7), the payoff at the equilibrium is achieved by committing to a pure strategy. More specifically, the payoff at the equilibrium of the game  $\mathcal{G}(\underline{u}, \underline{w})$  in (12) is the same as the payoff at the SE in pure strategies of the game  $\mathcal{G}(\underline{u})$  in (2).

### 3.5 Relevance of Noisy Observations

The following lemma presents necessary and sufficient conditions under which the follower cannot benefit from the noisy observations.

**Lemma 3.6** *Let the tuple  $(P_{A_1|\tilde{A}_2=a_1}^\dagger, P_{A_1|\tilde{A}_2=a_2}^\dagger, P_{A_2}^\dagger) \in \Delta(\mathcal{A}_1) \times \Delta(\mathcal{A}_1) \times \Delta(\mathcal{A}_2)$  form an equilibrium of the game  $\mathcal{G}(\underline{u}, \underline{w})$  in (12). Let also the tuple  $(P_{A_1}^*, P_{A_2}^*) \in \Delta(\mathcal{A}_1) \times \Delta(\mathcal{A}_2)$  form one of the NEs of the game  $\mathcal{G}(\underline{u})$  in (2). Then,*

$$v(P_{A_1|\tilde{A}_2}^\dagger, P_{A_2}^\dagger) = u(P_{A_1}^*, P_{A_2}^*), \quad (32)$$

*if and only if, (a) the matrix  $\underline{u}$  in (1) satisfies (7); or (b) the matrix  $\underline{u}$  in (1) satisfies (4) and the channel  $\underline{w}$  in (13) satisfies  $\det \underline{w} = 0$ .*

Lemma 3.6 establishes that granting the follower with noisy observations of the action played by the leader does not make any difference in two particular scenarios. First, in ZSGs with strategic dominance, NEs in pure strategies and infinitely many NEs (condition (a)). Second, in ZSGs with unique NE in strictly mixed strategies when the observation given to the follower of the action of the leader is independent of the action actually played (condition (b)). Note that a channel that satisfies  $\det \underline{w} = 0$  is a channel whose mutual information between the channel input and channel output is zero.

Lemma 3.4 and Lemma 3.6 imply that granting the follower with relevant noisy observations of the action played by the leader makes a difference exclusively for ZSGs with a unique NE in mixed strategies. In this case, relevant noisy observations refer to observations obtained through channels with positive mutual information between the channel input and the channel output ( $\det \underline{w} \neq 0$ ).

The following lemma describes the special case of channels with maximum mutual information between the channel input and the channel output. That is, channels whose output is deterministic given the channel input. These channels satisfy the condition  $|\det \underline{w}| = 1$ .

**Lemma 3.7** *Let  $(P_{A_1|\tilde{A}_2=a_1}^\dagger, P_{A_1|\tilde{A}_2=a_2}^\dagger, P_{A_2}^\dagger) \in \Delta(\mathcal{A}_1) \times \Delta(\mathcal{A}_1) \times \Delta(\mathcal{A}_2)$  form an equilibrium of the game  $\mathcal{G}(\underline{u}, \underline{w})$  in (12). If  $|\det \underline{w}| = 1$ , then*

$$\hat{v}(P_{A_2}^\dagger) = \min\{\max\{u_{1,1}, u_{2,1}\}, \max\{u_{1,2}, u_{2,2}\}\}. \quad (33)$$

Lemma 3.7 strengthens the observation that under perfect observations at each repetition, the strategy to which the leader commits to becomes irrelevant.

## 4 COMMITMENT MISMATCH

In this section, the commitment observed by the follower in the game  $\mathcal{G}(\underline{u}, \underline{w})$  in (12) is assumed to be different from the actual commitment of the leader. This scenario is referred to as *commitment mismatch*.

### 4.1 Game Formulation

Let  $\underline{t}$  be a given  $2 \times 2$  nonsingular stochastic matrix. Let  $P_{A_2} \in \Delta(\mathcal{A}_2)$  be the commitment announced by Player 2 before the beginning of the game repetitions. The commitment observed by Player 1 is denoted by  $\tilde{P}_{A_2} \in \Delta(\mathcal{A}_2)$  and satisfies,

$$\begin{pmatrix} \tilde{P}_{A_2}(a_1) \\ \tilde{P}_{A_2}(a_2) \end{pmatrix} = \underline{t} \begin{pmatrix} P_{A_2}(a_1) \\ P_{A_2}(a_2) \end{pmatrix}. \quad (34)$$

That is, the commitment observed by the follower is a deterministic distortion of the commitment announced by the leader. Note that the leader is not engaged on learning the commitment observed by the follower, nor the follower is engaged in learning the commitment announced by the leader, as in [Muthukumar and Sahai, 2019]. Here, the follower assumes that  $\tilde{P}_{A_2}$  is the commitment actually announced by the leader, and the leader is aware of this. The extension of the game  $\mathcal{G}(\underline{u}, \underline{w})$  in (12) obtained by including the binary channel represented by the stochastic matrix  $\underline{t}$  is described by the tuple

$$\mathcal{G}(\underline{u}, \underline{w}, \underline{t}) = (\mathcal{K}, \mathcal{A}_1, \mathcal{A}_2, \underline{u}, \underline{w}, \underline{t}). \quad (35)$$

### 4.2 The Best Strategies of Player 2

Let the correspondence  $\tilde{v} : \Delta(\mathcal{A}_2) \rightarrow \mathbb{R}$  be such that given the probability measure  $P_{A_2} \in \Delta(\mathcal{A}_2)$ ,

$$\tilde{v}(P_{A_2}) = \max_{(Q_1, Q_2) \in \text{BR}_1(\tilde{P}_{A_2})} v(Q_1, Q_2, P_{A_2}), \quad (36)$$

where the function  $v$  is defined in (11), the correspondence  $\text{BR}_1$  is defined in (14), and the probability measure  $\tilde{P}_{A_2}$  is in (34). The correspondence  $\tilde{v}$  in (36) determines the payoff achieved by the players. Thus, the set of best strategies for Player 2, under the assumption that Player 1 observes a distorted commitment, is the set of probability

measures that minimize  $\tilde{v}$  in (36), if such a minimum exists. More specifically, the cardinality of the set  $\text{BR}_1(\tilde{P}_{A_2})$  in (36) is either one or infinite (Lemma 3.2). Hence, there might exist two tuples  $(Q_1, Q_2) \in \text{BR}_1(\tilde{P}_{A_2})$  and  $(Q_3, Q_4) \in \text{BR}_1(\tilde{P}_{A_2})$ , for which  $v(Q_1, Q_2, P_{A_2}) \neq v(Q_3, Q_4, P_{A_2})$ . In this case,  $\tilde{v}(P_{A_2})$  corresponds to a subset of  $\mathbb{R}$  in which each element is induced by an element of the set  $\text{BR}_1(\tilde{P}_{A_2})$ . From this perspective, the minimization of  $\tilde{v}$  might not be a well posed optimization problem.

For all  $i \in \{1, 2\}$ , let  $\tilde{P}^{(i)} \in \mathbb{R}$  be such that

$$\begin{pmatrix} \tilde{P}^{(i)} \\ 1 - \tilde{P}^{(i)} \end{pmatrix} = \underline{\mathbf{t}}^{-1} \begin{pmatrix} P^{(i)} \\ 1 - P^{(i)} \end{pmatrix}, \quad (37)$$

with  $P^{(i)}$  in (23). Using this notation, the following lemma shows an explicit expression for the correspondence  $\tilde{v}$  in a special case.

**Lemma 4.1** *Assume that the matrix  $\underline{\mathbf{u}}$  in (1) satisfies (4) and  $u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2} > 0$ . Assume also that  $\det \underline{\mathbf{w}} > 0$  and  $\det \underline{\mathbf{t}} > 0$ , with  $\underline{\mathbf{w}}$  in (13) and  $\underline{\mathbf{t}}$  in (35). For all  $P \in \Delta(\mathcal{A}_2)$ , it holds that: If  $P(a_1) > \tilde{P}^{(2)}$ , with  $\tilde{P}^{(2)}$  in (37), then it follows that*

$$\tilde{v}(P) = u_{1,1}P(a_1) + u_{1,2}P(a_2). \quad (38)$$

If  $P(a_1) = \tilde{P}^{(2)}$ , then it follows that

$$\tilde{v}(P) = \left\{ (u_{1,1}P(a_1) + u_{1,2}P(a_2))\beta + \left( \begin{pmatrix} 1 \\ 1 \end{pmatrix}^T (\underline{\mathbf{u}} \circ \underline{\mathbf{w}}) \begin{pmatrix} P(a_1) \\ P(a_2) \end{pmatrix} \right) (1-\beta) : \beta \in [0, 1] \right\}. \quad (39)$$

If  $\tilde{P}^{(1)} < P(a_1) < \tilde{P}^{(2)}$ , then it follows that

$$\tilde{v}(P) = \left( \begin{pmatrix} 1 \\ 1 \end{pmatrix}^T (\underline{\mathbf{u}} \circ \underline{\mathbf{w}}) \begin{pmatrix} P(a_1) \\ P(a_2) \end{pmatrix} \right). \quad (40)$$

If  $P(a_1) = \tilde{P}^{(1)}$ , with  $\tilde{P}^{(1)}$  in (37), then it follows that

$$\tilde{v}(P) = \left\{ (u_{2,1}P(a_1) + u_{2,2}P(a_2))(1-\beta) + \left( \begin{pmatrix} 1 \\ 1 \end{pmatrix}^T (\underline{\mathbf{u}} \circ \underline{\mathbf{w}}) \begin{pmatrix} P(a_1) \\ P(a_2) \end{pmatrix} \right) \beta : \beta \in [0, 1] \right\}. \quad (41)$$

If  $P(a_1) < \tilde{P}^{(1)}$ , then it follows that

$$\tilde{v}(P) = u_{2,1}P(a_1) + u_{2,2}P(a_2). \quad (42)$$

Figure 1 depicts the correspondence  $\tilde{v}$  in (36). Note that for all  $i \in \{1, 2\}$ , if the commitment of the leader  $P_{A_2}$  satisfies

$P_{A_2}(a_1) = \tilde{P}^{(i)}$ , it holds that  $\tilde{v}(P_{A_2})$  is a closed interval. Figure 1 also depicts the existence of some commitments of Player 2 for which the expected payoff is smaller than the expected payoff achieved when no distortion of the commitment is considered. The following lemma formalizes this observation and shows that even a deterministic distortion of the commitment can benefit the leader in particular cases.

**Lemma 4.2** *Consider the following assumptions: (a) The matrix  $\underline{\mathbf{u}}$  in (1) satisfies (4); (b) For all  $i \in \{1, 2\}$ , the probability measures  $Q_i \in \Delta(\mathcal{A}_2)$  such that  $Q_i(a_1) = P^{(i)}$ , with  $P^{(i)}$  in (23), satisfy  $\hat{u}(Q_1) \neq \hat{u}(Q_2)$ , where the function  $\hat{u}$  is in (24). If  $\det \underline{\mathbf{t}} \notin \{0, 1\}$ , then, there exists a strategy  $P \in \Delta(\mathcal{A}_2)$  such that  $\tilde{v}(P) < \hat{v}(P_{A_2}^\dagger)$ , where the tuple  $(P_{A_1|\tilde{A}_2=a_1}^\dagger, P_{A_1|\tilde{A}_2=a_2}^\dagger, P_{A_2}^\dagger) \in \Delta(\mathcal{A}_1) \times \Delta(\mathcal{A}_1) \times \Delta(\mathcal{A}_2)$  is an equilibrium of the game  $\mathcal{G}(\underline{\mathbf{u}}, \underline{\mathbf{w}})$  in (12).*

### 4.3 Equilibria

The solution concept for the game  $\mathcal{G}(\underline{\mathbf{u}}, \underline{\mathbf{w}}, \underline{\mathbf{t}})$  in (35) is the following.

**Definition 4.1 (Equilibrium)** *The tuple  $(P_{A_1|\tilde{A}_2=a_1}, P_{A_1|\tilde{A}_2=a_2}, P_{A_2}) \in \Delta(\mathcal{A}_1) \times \Delta(\mathcal{A}_1) \times \Delta(\mathcal{A}_2)$  is said to form an equilibrium of the game  $\mathcal{G}(\underline{\mathbf{u}}, \underline{\mathbf{w}}, \underline{\mathbf{t}})$  in (35) if*

$$P_{A_2} \in \arg \min_{P \in \Delta(\mathcal{A}_2)} \tilde{v}(P) \quad \text{and} \quad (43)$$

$$(P_{A_1|\tilde{A}_2=a_1}, P_{A_1|\tilde{A}_2=a_2}) \in \text{BR}_1(\tilde{P}_{A_2}), \quad (44)$$

where the correspondence  $\tilde{v}$  is in (36), the correspondence  $\text{BR}_1$  is in (16), and the probability measures  $P_{A_2}$  and  $\tilde{P}_{A_2}$  satisfy (34).

The game  $\mathcal{G}(\underline{\mathbf{u}}, \underline{\mathbf{w}}, \underline{\mathbf{t}})$  in (35) does not necessarily possess an equilibrium. This is due to the fact that the minimum of  $\tilde{v}$  in (36) does not always exist, as shown hereunder.

For all  $i \in \{1, 2\}$ , let the measure  $\tilde{P}_i \in \Delta(\mathcal{A}_2)$  be such that  $\tilde{P}_i(a_1) = \tilde{P}^{(i)}$ . Let the function  $\omega : \Delta(\mathcal{A}_2) \rightarrow \mathbb{R}$  be defined for all  $P \in \Delta(\mathcal{A}_2)$  as follows: If  $P(a_1) > \tilde{P}^{(2)}$ , with  $\tilde{P}^{(2)}$  in (37), then  $\omega(P) = u_{1,1}P(a_1) + u_{1,2}P(a_2)$ . If  $\tilde{P}^{(1)} \leq P_{A_2}(a_1) \leq \tilde{P}^{(2)}$ , then  $\omega(P) = (u_{1,1}P_{\tilde{A}_2|A_2=a_1}(a_1) + u_{2,1}P_{\tilde{A}_2|A_2=a_1}(a_2))P(a_1) + (u_{1,2}P_{\tilde{A}_2|A_2=a_2}(a_1) + u_{2,2}P_{\tilde{A}_2|A_2=a_2}(a_2))P(a_2)$ . If  $P(a_1) < \tilde{P}^{(1)}$ , then  $\omega(P) = u_{2,1}P(a_1) + u_{2,2}P(a_2)$ .

In the example of Figure 1, for all  $P \in \Delta(\mathcal{A}_2) \setminus \{\tilde{P}_1, \tilde{P}_2\}$ , it holds that  $\tilde{v}(P) = \omega(P)$ , and the function  $\omega$  is discontinuous at  $\tilde{P}_1$  and  $\tilde{P}_2$  with a minimum at  $\tilde{P}_2$  (magenta triangle). Note that  $\tilde{v}(\tilde{P}_1)$  and  $\tilde{v}(\tilde{P}_2)$  are intervals, and thus, the minimization of  $\tilde{v}$  does not have a solution. Essentially, if



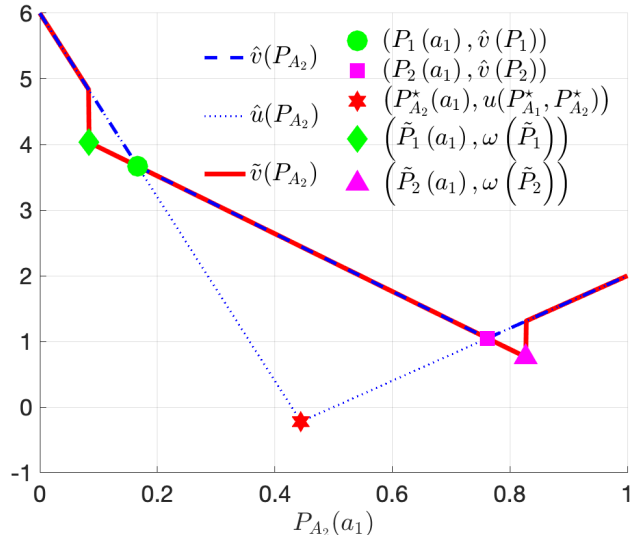


Figure 1: Plots of the functions  $\hat{v}$  in (22) and  $\hat{u}$  in (24); and the correspondence  $\tilde{v}$  in (36) as a function of the probability  $P_{A_2}(a_1)$ , with parameters  $\underline{u} = (-8, 6; 2, -2)$ ,  $\underline{w} = (0.8, 0.2; 0.2, 0.8)$  and  $\underline{t} = (0.9, 0.1; 0.1, 0.9)$ . The tuple  $(P_{A_1}^*, P_{A_2}^*)$  is the unique NE in (5) and for all  $i \in \{1, 2\}$ ,  $P_i^*(a_1) = P^{(i)}$ , with  $P^{(i)}$  in (23) and  $\tilde{P}_i(a_1) = \tilde{P}^{(i)}$ , with  $\tilde{P}^{(i)}$  in (37).

Player 2 (the leader) commits to play the probability measure  $\tilde{P}_{A_2}$  that minimizes  $\omega$ , i.e.,  $P_{A_2} = \tilde{P}_2$ , then the set of the best responses of Player 1 when the noisy observation is  $a_2$  is  $\text{BR}_{1,2}(\tilde{P}_{A_2}) = \Delta(A_1)$ , where the probability measures  $P_{A_2}$  and  $\tilde{P}_{A_2}$  satisfy (34). That is, for each game repetition in which the output of the channel is  $a_2$ , Player 1 might choose its action by sampling any probability measure in  $\Delta(A_1)$  and achieve an expected payoff in the interval  $\tilde{v}(P_2)$ . If such a payoff is larger than  $\omega(P_{A_2})$ , then Player 2 can deviate and obtain a payoff arbitrarily close to  $\omega(P_{A_2})$ . This shows the non existence of an equilibrium in the example of Figure 1.

Note that the equilibrium exists for the cases in which the strategy of Player 1 is independent of the commitment, e.g., ZSG with strategic dominance.

#### 4.4 Equilibrium Refinements

In the example in Figure 1, the function  $\omega$ , which can be minimized, is obtained from the correspondence  $\tilde{v}$  in (36) by replacing the closed intervals  $\tilde{v}(\tilde{P}_1)$  and  $\tilde{v}(\tilde{P}_2)$  by the real numbers  $\min \tilde{v}(\tilde{P}_1)$  and  $\min \tilde{v}(\tilde{P}_2)$ , respectively. Hence, the correspondence  $\tilde{v}$  and the function  $\omega$  are identical if Player 1 is forced to choose the strategy that minimizes the expected payoff every time it observes that Player 2 commits either to the strategies  $\tilde{P}_1$  or  $\tilde{P}_2$ . Under this assumption, the game  $\mathcal{G}(\underline{u}, \underline{w}, \underline{t})$  in Figure 1 possesses an equilibrium in which Player 2 commits to play

the strategy  $P_{A_2}$  that satisfies  $P_{A_2} = \tilde{P}_2$ . This observation is reminiscent of the equilibrium refinements proposed in [Leitmann, 1978] for the SE of bi-matrix games, i.e., the *strong-SE*.

Another refinement of the solution concept in Definition 4.1 can be obtained when the leader commits to a strategy  $P_{A_2}$  that satisfies  $\tilde{v}(P_{A_2}) = \omega(\tilde{P}_2) + \epsilon$ , with  $\epsilon > 0$  arbitrarily small. This refinement is reminiscent of the solution concept known as  $\epsilon$ -equilibrium for the NE [Fudenberg and Tirole, 1991]. In this case, the leader admits to committing to a suboptimal strategy in order to be able to force a unique (and predictable) best response from its opponent.

## 5 FINAL REMARKS AND DISCUSSION

The analysis of ZSG with commitments  $\tilde{P}_2$  in which the follower is granted with a noisy observation of the action and the commitment of the leader have been studied following a Bayesian approach. This approach relies on the capability of the follower to construct posterior probability measures on the actions of the leader based on the available information. The construction of posteriors is more general than the notion of incomplete information in the extensive form of ZSGs, which is limited to modelling the inability of players to distinguish between elements of the *information sets* (maximum entropy posteriors) [Von Neumann and Morgenstern, 2007]. Note also that noisy observations cannot be modelled using Bayesian games as introduced in [Harsanyi, 1967, Harsanyi, 1968]. This new game formulation is shown to always possess an equilibrium under the assumption that the commitment is observed perfectly. When this assumption is dropped, the game is shown to have an equilibrium only under strict conditions. Despite that commitment mismatch can significantly benefit the leader, such benefits are not achievable at a stable point. To benefit from commitment mismatch, the leader must admit to commit to a suboptimal strategy in order to be able to unequivocally predict the best response of its opponent.

This work relies on the assumption that actions are observed through discrete channels for which the sets of channel inputs and channel outputs are finite and identical. Nonetheless, these channels fail to model many typical data processing impairments, which calls for more elaborate channel models. For instance, the effect of erasures is modelled by the *erasure channel* [Elias, 1955]; and the effect of additive white Gaussian noise (AWGN) is modelled by the AWGN channel [Shannon, 1948a, Shannon, 1948b]. Nonetheless, the extension of this work to these and other channel models is not trivial. Finally, it is important to highlight that the conclusions of this work hold under the assumption that both players are aware of the existence of a channel through which actions are observed. Moreover, such a channel is assumed to be known by both players.

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## A PROOF OF LEMMA 3.1

For all tuples  $(Q_1, Q_2) \in \Delta(\mathcal{A}_1) \times \Delta(\mathcal{A}_1)$  and for all probability measures  $P \in \Delta(\mathcal{A}_2)$ , the function  $v$  in (11) satisfies:

$$v(Q_1, Q_2, P) = \sum_{(i,j) \in \{1,2\}^2} u_{i,j} \left( \sum_{k \in \{1,2\}} P_{\tilde{A}_2|A_2=a_j}(a_k) Q_k(a_i) \right) P(a_j), \quad (45)$$

which can be expressed as the following product of matrices,

$$v(Q_1, Q_2, P) = \begin{pmatrix} Q_1(a_1) \\ Q_1(a_2) \\ Q_2(a_1) \\ Q_2(a_2) \end{pmatrix}^T \begin{pmatrix} u_{1,1}P_{\tilde{A}_2|A_2=a_1}(a_1) & u_{1,2}P_{\tilde{A}_2|A_2=a_2}(a_1) \\ u_{2,1}P_{\tilde{A}_2|A_2=a_1}(a_1) & u_{2,2}P_{\tilde{A}_2|A_2=a_2}(a_1) \\ u_{1,1}P_{\tilde{A}_2|A_2=a_1}(a_2) & u_{1,2}P_{\tilde{A}_2|A_2=a_2}(a_2) \\ u_{2,1}P_{\tilde{A}_2|A_2=a_1}(a_2) & u_{2,2}P_{\tilde{A}_2|A_2=a_2}(a_2) \end{pmatrix} \begin{pmatrix} P(a_1) \\ P(a_2) \end{pmatrix}. \quad (46)$$

Note that that the right-hand side of the equality in (46) can also be written as follows

$$v(Q_1, Q_2, P) = \begin{pmatrix} Q_1(a_1) \\ Q_1(a_2) \end{pmatrix}^T \begin{pmatrix} u_{1,1}P_{\tilde{A}_2|A_2=a_1}(a_1) & u_{1,2}P_{\tilde{A}_2|A_2=a_2}(a_1) \\ u_{2,1}P_{\tilde{A}_2|A_2=a_1}(a_1) & u_{2,2}P_{\tilde{A}_2|A_2=a_2}(a_1) \end{pmatrix} \begin{pmatrix} P(a_1) \\ P(a_2) \end{pmatrix} \\ + \begin{pmatrix} Q_2(a_1) \\ Q_2(a_2) \end{pmatrix}^T \begin{pmatrix} u_{1,1}P_{\tilde{A}_2|A_2=a_1}(a_2) & u_{1,2}P_{\tilde{A}_2|A_2=a_2}(a_2) \\ u_{2,1}P_{\tilde{A}_2|A_2=a_1}(a_2) & u_{2,2}P_{\tilde{A}_2|A_2=a_2}(a_2) \end{pmatrix} \begin{pmatrix} P(a_1) \\ P(a_2) \end{pmatrix} \quad (47)$$

$$= \begin{pmatrix} Q_1(a_1) \\ Q_1(a_2) \end{pmatrix}^T \underline{\mathbf{u}}^{(1)} \begin{pmatrix} P(a_1) \\ P(a_2) \end{pmatrix} + \begin{pmatrix} Q_2(a_1) \\ Q_2(a_2) \end{pmatrix}^T \underline{\mathbf{u}}^{(2)} \begin{pmatrix} P(a_1) \\ P(a_2) \end{pmatrix}, \quad (48)$$

with  $\underline{\mathbf{u}}^{(i)}$  in (15).

From (14) and (48), it holds that for all probability measures  $P \in \Delta(\mathcal{A}_2)$ ,

$$\text{BR}_1(P) = \arg \max_{(Q_1, Q_2) \in \Delta(\mathcal{A}_1) \times \Delta(\mathcal{A}_1)} v(Q_1, Q_2, P) \quad (49)$$

$$= \arg \max_{Q \in \Delta(\mathcal{A}_1)} \begin{pmatrix} Q(a_1) \\ Q(a_2) \end{pmatrix}^T \underline{\mathbf{u}}^{(1)} \begin{pmatrix} P(a_1) \\ P(a_2) \end{pmatrix} \times \arg \max_{Q \in \Delta(\mathcal{A}_1)} \begin{pmatrix} Q(a_1) \\ Q(a_2) \end{pmatrix}^T \underline{\mathbf{u}}^{(2)} \begin{pmatrix} P(a_1) \\ P(a_2) \end{pmatrix} \quad (50)$$

$$= \text{BR}_{1,1}(P) \times \text{BR}_{1,2}(P), \quad (51)$$

which completes the proof.

## B PROOF OF LEMMA 3.2

From Lemma 3.1, it holds that for all  $i \in \{1, 2\}$  and for all probability measures  $P \in \Delta(\mathcal{A}_2)$ ,

$$\text{BR}_{1,i}(P) = \arg \max_{Q \in \Delta(\mathcal{A}_1)} \begin{pmatrix} Q(a_1) \\ Q(a_2) \end{pmatrix}^T \underline{\mathbf{u}}^{(i)} \begin{pmatrix} P(a_1) \\ P(a_2) \end{pmatrix}. \quad (52)$$

Moreover,

$$\begin{pmatrix} Q(a_1) \\ Q(a_2) \end{pmatrix}^T \underline{\mathbf{u}}^{(i)} \begin{pmatrix} P(a_1) \\ P(a_2) \end{pmatrix} \\ = \begin{pmatrix} Q(a_1) \\ Q(a_2) \end{pmatrix}^T \underline{\mathbf{u}} \begin{pmatrix} P_{\tilde{A}_2|A_2=a_1}(a_i) & 0 \\ 0 & P_{\tilde{A}_2|A_2=a_2}(a_i) \end{pmatrix} \begin{pmatrix} P(a_1) \\ P(a_2) \end{pmatrix} \quad (53)$$

$$= \begin{pmatrix} Q(a_1) \\ 1 - Q(a_1) \end{pmatrix}^T \begin{pmatrix} u_{1,1}P(a_1)P_{\tilde{A}_2|A_2=a_1}(a_i) + u_{1,2}P(a_2)P_{\tilde{A}_2|A_2=a_2}(a_i) \\ u_{2,1}P(a_1)P_{\tilde{A}_2|A_2=a_1}(a_i) + u_{2,2}P(a_2)P_{\tilde{A}_2|A_2=a_2}(a_i) \end{pmatrix} \quad (54)$$

$$= \begin{pmatrix} u_{1,1}P(a_1)P_{\tilde{A}_2|A_2=a_1}(a_i) + u_{1,2}P(a_2)P_{\tilde{A}_2|A_2=a_2}(a_i) - u_{2,1}P(a_1)P_{\tilde{A}_2|A_2=a_1}(a_i) - u_{2,2}P(a_2)P_{\tilde{A}_2|A_2=a_2}(a_i) \\ + u_{2,1}P(a_1)P_{\tilde{A}_2|A_2=a_1}(a_i) + u_{2,2}P(a_2)P_{\tilde{A}_2|A_2=a_2}(a_i) \end{pmatrix} Q(a_1) \quad (55)$$

$$= s_i Q(a_1) + u_{2,1}P(a_1)P_{\tilde{A}_2|A_2=a_1}(a_i) + u_{2,2}P(a_2)P_{\tilde{A}_2|A_2=a_2}(a_i), \quad (56)$$

where  $s_i$  is in (19).

Note that if  $s_i > 0$  in (56), it holds that

$$\begin{pmatrix} Q(a_1) \\ Q(a_2) \end{pmatrix}^\top \underline{\mathbf{u}}^{(i)} \begin{pmatrix} P(a_1) \\ P(a_2) \end{pmatrix} \leq s_i + u_{2,1}P(a_1)P_{\tilde{A}_2|A_2=a_1}(a_i) + u_{2,2}P(a_2)P_{\tilde{A}_2|A_2=a_2}(a_i), \quad (57)$$

where the equality holds only if  $Q(a_1) = 1$ .

Alternatively, if  $s_i = 0$  in (56), it holds that for all  $Q \in \Delta(\mathcal{A}_1)$ ,

$$\begin{pmatrix} Q(a_1) \\ Q(a_2) \end{pmatrix}^\top \underline{\mathbf{u}}^{(i)} \begin{pmatrix} P(a_1) \\ P(a_2) \end{pmatrix} = u_{2,1}P(a_1)P_{\tilde{A}_2|A_2=a_1}(a_i) + u_{2,2}P(a_2)P_{\tilde{A}_2|A_2=a_2}(a_i). \quad (58)$$

Finally, if  $s_i < 0$  in (56), it holds that

$$\begin{pmatrix} Q(a_1) \\ Q(a_2) \end{pmatrix}^\top \underline{\mathbf{u}}^{(i)} \begin{pmatrix} P(a_1) \\ P(a_2) \end{pmatrix} \leq u_{2,1}P(a_1)P_{\tilde{A}_2|A_2=a_1}(a_i) + u_{2,2}P(a_2)P_{\tilde{A}_2|A_2=a_2}(a_i), \quad (59)$$

where the equality holds only if  $Q(a_1) = 0$ .

This completes the proof.

### C PROOF OF LEMMA 3.3

From Lemma 3.1, it holds that for all  $i \in \{1, 2\}$  and for all probability measures  $P \in \Delta(\mathcal{A}_2)$ ,

$$\text{BR}_{1,i}(P) = \arg \max_{Q \in \Delta(\mathcal{A}_1)} \begin{pmatrix} Q(a_1) \\ Q(a_2) \end{pmatrix}^\top \underline{\mathbf{u}}^{(i)} \begin{pmatrix} P(a_1) \\ P(a_2) \end{pmatrix} \quad (60)$$

$$= \arg \max_{Q \in \Delta(\mathcal{A}_1)} \begin{pmatrix} Q(a_1) \\ Q(a_2) \end{pmatrix}^\top \underline{\mathbf{u}} \begin{pmatrix} P_{\tilde{A}_2|A_2=a_1}(a_i) & 0 \\ 0 & P_{\tilde{A}_2|A_2=a_2}(a_i) \end{pmatrix} \begin{pmatrix} P(a_1) \\ P(a_2) \end{pmatrix} \quad (61)$$

$$= \arg \max_{Q \in \Delta(\mathcal{A}_1)} \begin{pmatrix} Q(a_1) \\ Q(a_2) \end{pmatrix}^\top \underline{\mathbf{u}} \begin{pmatrix} P_{A_2, \tilde{A}_2}(a_1, a_i) \\ P_{A_2, \tilde{A}_2}(a_2, a_i) \end{pmatrix} \quad (62)$$

$$= \arg \max_{Q \in \Delta(\mathcal{A}_1)} \begin{pmatrix} Q(a_1) \\ Q(a_2) \end{pmatrix}^\top \underline{\mathbf{u}} \begin{pmatrix} P_{A_2|\tilde{A}_2=a_i}(a_1) \\ P_{A_2|\tilde{A}_2=a_i}(a_2) \end{pmatrix} \left( \sum_{\ell \in \{1,2\}} P_{\tilde{A}_2|A_2=a_\ell}(a_i) P_{A_2}(a_\ell) \right) \quad (63)$$

$$= \arg \max_{Q \in \Delta(\mathcal{A}_1)} \begin{pmatrix} Q(a_1) \\ Q(a_2) \end{pmatrix}^\top \underline{\mathbf{u}} \begin{pmatrix} P_{A_2|\tilde{A}_2=a_i}(a_1) \\ P_{A_2|\tilde{A}_2=a_i}(a_2) \end{pmatrix}, \quad (64)$$

where the measure  $P_{A_2, \tilde{A}_2}$  in (62) satisfies for all  $(\ell, k) \in \{1, 2\}^2$ ,

$$P_{A_2, \tilde{A}_2}(a_\ell, a_k) = P_{\tilde{A}_2|A_2=a_\ell}(a_k)P(a_\ell). \quad (65)$$

This completes the proof.

### D PROOF OF LEMMA 3.4

The proof of the inequality in (25) is as follows. For all probability measures  $P \in \Delta(\mathcal{A}_2)$ , it holds that

$$\hat{u}(P) = \max_{Q \in \Delta(\mathcal{A}_1)} u(Q, P) \quad (66)$$

$$\geq \min_{P_0 \in \Delta(\mathcal{A}_2)} \max_{Q \in \Delta(\mathcal{A}_1)} u(Q, P_0) \quad (67)$$

$$= u(P_{A_1}^*, P_{A_2}^*), \quad (68)$$

which follows from the minmax theorem [Neumann, 1928].

The proof of the inequality in (26) is as follows. Note that from (3) and (11), it holds that for all  $P \in \Delta(\mathcal{A}_2)$ ,

$$\hat{v}(P) = \max_{(Q_1, Q_2) \in \Delta(\mathcal{A}_1) \times \Delta(\mathcal{A}_1)} v(Q_1, Q_2, P) \quad (69)$$

$$\geq \max_{Q_3 \in \Delta(\mathcal{A}_1)} v(Q_3, Q_3, P) \quad (70)$$

$$= \max_{Q_3 \in \Delta(\mathcal{A}_1)} u(Q_3, P) \quad (71)$$

$$= \hat{u}(P), \quad (72)$$

which proves the inequality in (26).

The proof of the inequality in (27) is as follows. From (48), for all probability measures  $P \in \Delta(\mathcal{A}_2)$ , the function  $\hat{v}$  satisfies

$$\hat{v}(P) = \max_{Q_1 \in \Delta(\mathcal{A}_2)} \begin{pmatrix} Q_1(a_1) \\ Q_1(a_2) \end{pmatrix}^\top \underline{\mathbf{u}}^{(1)} \begin{pmatrix} P(a_1) \\ P(a_2) \end{pmatrix} + \max_{Q_2 \in \Delta(\mathcal{A}_2)} \begin{pmatrix} Q_2(a_1) \\ Q_2(a_2) \end{pmatrix}^\top \underline{\mathbf{u}}^{(2)} \begin{pmatrix} P(a_1) \\ P(a_2) \end{pmatrix} \quad (73)$$

$$= \max_{Q_1 \in \Delta(\mathcal{A}_2)} \begin{pmatrix} Q_1(a_1) \\ Q_1(a_2) \end{pmatrix}^\top \begin{pmatrix} u_{1,1} P_{\tilde{A}_2|A_2=a_1}(a_1)P(a_1) + u_{1,2} P_{\tilde{A}_2|A_2=a_2}(a_1)P(a_2) \\ u_{2,1} P_{\tilde{A}_2|A_2=a_1}(a_1)P(a_1) + u_{2,2} P_{\tilde{A}_2|A_2=a_2}(a_1)P(a_2) \end{pmatrix} \quad (74)$$

$$+ \max_{Q_2 \in \Delta(\mathcal{A}_2)} \begin{pmatrix} Q_2(a_1) \\ Q_2(a_2) \end{pmatrix}^\top \begin{pmatrix} u_{1,1} P_{\tilde{A}_2|A_2=a_1}(a_2)P(a_1) + u_{1,2} P_{\tilde{A}_2|A_2=a_2}(a_2)P(a_2) \\ u_{2,1} P_{\tilde{A}_2|A_2=a_1}(a_2)P(a_1) + u_{2,2} P_{\tilde{A}_2|A_2=a_2}(a_2)P(a_2) \end{pmatrix} \quad (75)$$

$$= \max \left\{ \begin{aligned} &u_{1,1} P_{\tilde{A}_2|A_2=a_1}(a_1)P(a_1) + u_{1,2} P_{\tilde{A}_2|A_2=a_2}(a_1)P(a_2), \\ &u_{2,1} P_{\tilde{A}_2|A_2=a_1}(a_1)P(a_1) + u_{2,2} P_{\tilde{A}_2|A_2=a_2}(a_1)P(a_2) \end{aligned} \right\} \\ + \max \left\{ \begin{aligned} &u_{1,1} P_{\tilde{A}_2|A_2=a_1}(a_2)P(a_1) + u_{1,2} P_{\tilde{A}_2|A_2=a_2}(a_2)P(a_2), \\ &u_{2,1} P_{\tilde{A}_2|A_2=a_1}(a_2)P(a_1) + u_{2,2} P_{\tilde{A}_2|A_2=a_2}(a_2)P(a_2) \end{aligned} \right\}. \quad (76)$$

Note that the equality in (76) can be written for all  $P \in \Delta(\mathcal{A}_2)$  as follows:

$$\hat{v}(P) = \max \left\{ \begin{aligned} &u_{1,1}P(a_1) + u_{1,2}P(a_2), \\ &\left( u_{1,1}P_{\tilde{A}_2|A_2=a_1}(a_1) + u_{2,1}P_{\tilde{A}_2|A_2=a_1}(a_2) \right) P(a_1) \\ &+ \left( u_{1,2}P_{\tilde{A}_2|A_2=a_2}(a_1) + u_{2,2}P_{\tilde{A}_2|A_2=a_2}(a_2) \right) P(a_2), \\ &\left( u_{1,1}P_{\tilde{A}_2|A_2=a_1}(a_2) + u_{2,1}P_{\tilde{A}_2|A_2=a_1}(a_1) \right) P(a_1) \\ &+ \left( u_{1,2}P_{\tilde{A}_2|A_2=a_2}(a_2) + u_{2,2}P_{\tilde{A}_2|A_2=a_2}(a_1) \right) P(a_2), \\ &u_{2,1}P(a_1) + u_{2,2}P(a_2) \end{aligned} \right\} \quad (77)$$

$$\leq \max \left\{ \begin{aligned} &u_{1,1}P(a_1) + u_{1,2}P(a_2), \\ &\max\{u_{1,1}, u_{2,1}\}P(a_1) + \max\{u_{1,2}, u_{2,2}\}P(a_2), \\ &u_{2,1}P(a_1) + u_{2,2}P(a_2) \end{aligned} \right\} \\ = \max\{u_{1,1}, u_{2,1}\}P(a_1) + \max\{u_{1,2}, u_{2,2}\}P(a_2), \quad (78)$$

which completes the proof.

## E PROOF OF THEOREM 3.1

Note that from Lemma 3.2, for all  $P \in \Delta(\mathcal{A}_2)$  and all  $i \in \{1, 2\}$ , it holds that the cardinality of  $\text{BR}_{1,i}(P)$  is either one or infinite. Hence, from Lemma 3.1, for all  $P \in \Delta(\mathcal{A}_2)$ , it holds that the cardinality of  $\text{BR}_1(P)$  is either one or infinite.

Let the function  $f : [0, 1] \rightarrow \mathbb{R}$  be such that

$$\begin{aligned} f(\beta) = & \max \left\{ u_{1,1} P_{\tilde{A}_2|A_2=a_1}(a_1)\beta + u_{1,2} P_{\tilde{A}_2|A_2=a_2}(a_1)(1-\beta), \right. \\ & \left. u_{2,1} P_{\tilde{A}_2|A_2=a_1}(a_1)\beta + u_{2,2} P_{\tilde{A}_2|A_2=a_2}(a_1)(1-\beta) \right\} \\ & + \max \left\{ u_{1,1} P_{\tilde{A}_2|A_2=a_1}(a_2)\beta + u_{1,2} P_{\tilde{A}_2|A_2=a_2}(a_2)(1-\beta), \right. \\ & \left. u_{2,1} P_{\tilde{A}_2|A_2=a_1}(a_2)\beta + u_{2,2} P_{\tilde{A}_2|A_2=a_2}(a_2)(1-\beta) \right\}. \end{aligned} \quad (79)$$

Then from (76) and (79), it holds that given a probability measure  $P \in \Delta(\mathcal{A}_2)$ , with  $P(a_1) = \beta \in [0, 1]$ ,

$$\hat{v}(P) = f(\beta). \quad (80)$$

Hence, it holds that

$$\arg \min_{P \in \Delta(\mathcal{A}_2)} \hat{v}(P) = \left\{ P \in \Delta(\mathcal{A}_2) : P(a_1) \in \arg \min_{\beta \in [0,1]} f(\beta) \right\}. \quad (81)$$

Given the fact that function  $f$  is continuous piecewise linear, the following optimization problem always possesses a solution [Stewart, 2015, Extreme Value Theorem]:

$$\arg \min_{\beta \in [0,1]} f(\beta). \quad (82)$$

Then, from (81), it follows that

$$\arg \min_{P \in \Delta(\mathcal{A}_2)} \hat{v}(P) \neq \emptyset. \quad (83)$$

The proof is completed by noticing that the inequality in (83) implies that the game  $\mathcal{G}(\underline{\mathbf{u}}, \underline{\mathbf{w}})$  in (12) always possesses an equilibrium.

## F PROOF OF THEOREM 3.2

The proof is divided into two parts. Subsection F.1 introduces preliminary results in the form of lemmas. Subsection F.2 presents the proof using the preliminary results.

### F.1 Preliminaries

**Lemma F.1** *The matrix  $\underline{\mathbf{w}}$  in (13) satisfies,*

$$\det \underline{\mathbf{w}} = P_{\tilde{A}_2|A_2=a_1}(a_1) - P_{\tilde{A}_2|A_2=a_2}(a_1) = P_{\tilde{A}_2|A_2=a_2}(a_2) - P_{\tilde{A}_2|A_2=a_1}(a_2). \quad (84)$$

*Proof:* From (13), it follows that

$$\det \underline{\mathbf{w}} = P_{\tilde{A}_2|A_2=a_1}(a_1)P_{\tilde{A}_2|A_2=a_2}(a_2) - P_{\tilde{A}_2|A_2=a_2}(a_1)P_{\tilde{A}_2|A_2=a_1}(a_2) \quad (85)$$

$$= \left(1 - P_{\tilde{A}_2|A_2=a_1}(a_2)\right) \left(1 - P_{\tilde{A}_2|A_2=a_2}(a_1)\right) - P_{\tilde{A}_2|A_2=a_2}(a_1)P_{\tilde{A}_2|A_2=a_1}(a_2) \quad (86)$$

$$= 1 - P_{\tilde{A}_2|A_2=a_1}(a_2) - P_{\tilde{A}_2|A_2=a_2}(a_1) \quad (87)$$

$$= P_{\tilde{A}_2|A_2=a_1}(a_1) - P_{\tilde{A}_2|A_2=a_2}(a_1) \quad (88)$$

and

$$P_{\tilde{A}_2|A_2=a_1}(a_1) - P_{\tilde{A}_2|A_2=a_2}(a_1) = P_{\tilde{A}_2|A_2=a_2}(a_2) - P_{\tilde{A}_2|A_2=a_1}(a_2), \quad (89)$$

which completes the proof.  $\blacksquare$

**Lemma F.2** *For all tuples  $(Q_1, Q_2) \in \Delta(\mathcal{A}_1) \times \Delta(\mathcal{A}_1)$  and for all probability measures  $P \in \Delta(\mathcal{A}_2)$ , the function  $v$  in (11) satisfies*

$$v(Q_1, Q_2, P) = P_{\tilde{A}_2}(a_1) \begin{pmatrix} Q_1(a_1) \\ Q_1(a_2) \end{pmatrix}^T \underline{\mathbf{u}} \begin{pmatrix} P_{A_2|\tilde{A}_2=a_1}(a_1) \\ P_{A_2|\tilde{A}_2=a_1}(a_2) \end{pmatrix} + P_{\tilde{A}_2}(a_2) \begin{pmatrix} Q_2(a_1) \\ Q_2(a_2) \end{pmatrix}^T \underline{\mathbf{u}} \begin{pmatrix} P_{A_2|\tilde{A}_2=a_2}(a_1) \\ P_{A_2|\tilde{A}_2=a_2}(a_2) \end{pmatrix}, \quad (90)$$

where the probability measure  $P_{\tilde{A}_2}$  satisfies for all  $i \in \{1, 2\}$ ,

$$P_{\tilde{A}_2}(a_i) = \sum_{\ell \in \{1, 2\}} P_{\tilde{A}_2|A_2=a_\ell}(a_i) P(a_\ell); \quad (91)$$

the probability measure  $P_{A_2|\tilde{A}_2=a_i}$  is in (21); and the probability measures  $P_{\tilde{A}_2|A_2=a_1}$  and  $P_{\tilde{A}_2|A_2=a_2}$  are in (13).

*Proof:* From (48), it holds that for all  $(Q_1, Q_2) \in \Delta(\mathcal{A}_1) \times \Delta(\mathcal{A}_1)$  and for all  $P \in \Delta(\mathcal{A}_2)$ ,

$$v(Q_1, Q_2, P) = \begin{pmatrix} Q_1(a_1) \\ Q_1(a_2) \end{pmatrix}^\top \underline{\mathbf{u}}^{(1)} \begin{pmatrix} P(a_1) \\ P(a_2) \end{pmatrix} + \begin{pmatrix} Q_2(a_1) \\ Q_2(a_2) \end{pmatrix}^\top \underline{\mathbf{u}}^{(2)} \begin{pmatrix} P(a_1) \\ P(a_2) \end{pmatrix} \quad (92)$$

$$= \begin{pmatrix} Q_1(a_1) \\ Q_1(a_2) \end{pmatrix}^\top \begin{pmatrix} u_{1,1}P_{\tilde{A}_2|A_2=a_1}(a_1) & u_{1,2}P_{\tilde{A}_2|A_2=a_2}(a_1) \\ u_{2,1}P_{\tilde{A}_2|A_2=a_1}(a_1) & u_{2,2}P_{\tilde{A}_2|A_2=a_2}(a_1) \end{pmatrix} \begin{pmatrix} P(a_1) \\ P(a_2) \end{pmatrix} \\ + \begin{pmatrix} Q_2(a_1) \\ Q_2(a_2) \end{pmatrix}^\top \begin{pmatrix} u_{1,1}P_{\tilde{A}_2|A_2=a_1}(a_2) & u_{1,2}P_{\tilde{A}_2|A_2=a_2}(a_2) \\ u_{2,1}P_{\tilde{A}_2|A_2=a_1}(a_2) & u_{2,2}P_{\tilde{A}_2|A_2=a_2}(a_2) \end{pmatrix} \begin{pmatrix} P(a_1) \\ P(a_2) \end{pmatrix} \quad (93)$$

$$= \begin{pmatrix} Q_1(a_1) \\ Q_1(a_2) \end{pmatrix}^\top \begin{pmatrix} u_{1,1} & u_{1,2} \\ u_{2,1} & u_{2,2} \end{pmatrix} \begin{pmatrix} P_{\tilde{A}_2|A_2=a_1}(a_1) & 0 \\ 0 & P_{\tilde{A}_2|A_2=a_2}(a_1) \end{pmatrix} \begin{pmatrix} P(a_1) \\ P(a_2) \end{pmatrix} \\ + \begin{pmatrix} Q_2(a_1) \\ Q_2(a_2) \end{pmatrix}^\top \begin{pmatrix} u_{1,1} & u_{1,2} \\ u_{2,1} & u_{2,2} \end{pmatrix} \begin{pmatrix} P_{\tilde{A}_2|A_2=a_1}(a_2) & 0 \\ 0 & P_{\tilde{A}_2|A_2=a_2}(a_2) \end{pmatrix} \begin{pmatrix} P(a_1) \\ P(a_2) \end{pmatrix} \quad (94)$$

$$= \begin{pmatrix} Q_1(a_1) \\ Q_2(a_2) \end{pmatrix}^\top \begin{pmatrix} u_{1,1} & u_{1,2} \\ u_{2,1} & u_{2,2} \end{pmatrix} \begin{pmatrix} P_{A_2, \tilde{A}_2}(a_1, a_1) \\ P_{A_2, \tilde{A}_2}(a_2, a_1) \end{pmatrix} + \begin{pmatrix} Q_2(a_1) \\ Q_2(a_2) \end{pmatrix}^\top \begin{pmatrix} u_{1,1} & u_{1,2} \\ u_{2,1} & u_{2,2} \end{pmatrix} \begin{pmatrix} P_{A_2, \tilde{A}_2}(a_1, a_2) \\ P_{A_2, \tilde{A}_2}(a_2, a_2) \end{pmatrix} \quad (95)$$

$$= \begin{pmatrix} Q_1(a_1) \\ Q_1(a_2) \end{pmatrix}^\top \begin{pmatrix} u_{1,1} & u_{1,2} \\ u_{2,1} & u_{2,2} \end{pmatrix} \begin{pmatrix} P_{A_2|\tilde{A}_2=a_1}(a_1) \\ P_{A_2|\tilde{A}_2=a_1}(a_2) \end{pmatrix} P_{\tilde{A}_2}(a_1) \\ + \begin{pmatrix} Q_2(a_1) \\ Q_2(a_2) \end{pmatrix}^\top \begin{pmatrix} u_{1,1} & u_{1,2} \\ u_{2,1} & u_{2,2} \end{pmatrix} \begin{pmatrix} P_{A_2|\tilde{A}_2=a_2}(a_1) \\ P_{A_2|\tilde{A}_2=a_2}(a_2) \end{pmatrix} P_{\tilde{A}_2}(a_2), \quad (96)$$

which completes the proof. ■

**Lemma F.3** Let the probability measures  $P_{A_1}^* \in \Delta(\mathcal{A}_1)$  and  $P_{A_2}^* \in \Delta(\mathcal{A}_2)$  form one of the NEs of the game  $\mathcal{G}(\underline{\mathbf{u}})$  in (2). Given a probability measure  $P \in \Delta(\mathcal{A}_2)$ , the equality

$$u(P_{A_1}^*, P_{A_2}^*) = \hat{v}(P), \quad (97)$$

with the functions  $u$  in (3) and  $v$  in (11), holds if and only if for all  $i \in \{1, 2\}$ , the probability measure  $P_{A_2|\tilde{A}_2=a_i}$  in (21) satisfy

$$P_{A_2|\tilde{A}_2=a_i} \in \arg \min_{P \in \Delta(\mathcal{A}_2)} \max_{P_0 \in \Delta(\mathcal{A}_1)} u(P_0, P). \quad (98)$$

In particular, if the game  $\mathcal{G}(\underline{\mathbf{u}})$  possesses a unique NE, the equality in (97) holds if and only if

$$P(a_1) = P_{A_2}^*(a_1) \in \{0, 1\}, \quad \text{or} \quad (99a)$$

$$P(a_1) = P_{A_2}^*(a_1) \in (0, 1) \text{ and } \det \underline{\mathbf{u}} = 0. \quad (99b)$$



*Proof:* From Lemma F.2 and (22), for all  $P \in \Delta(\mathcal{A}_2)$ , whose posterior  $P_{A_2|\tilde{A}_2}$  is in (21), it holds that

$$\hat{v}(P) = P_{\tilde{A}_2}(a_1) \max_{Q_1 \in \Delta(\mathcal{A}_1)} \left( Q_1(a_1) \right)^\top \underline{u} \left( \begin{matrix} P_{A_2|\tilde{A}_2=a_1}(a_1) \\ P_{A_2|\tilde{A}_2=a_1}(a_2) \end{matrix} \right) + P_{\tilde{A}_2}(a_2) \max_{Q_2 \in \Delta(\mathcal{A}_1)} \left( Q_2(a_1) \right)^\top \underline{u} \left( \begin{matrix} P_{A_2|\tilde{A}_2=a_2}(a_1) \\ P_{A_2|\tilde{A}_2=a_2}(a_2) \end{matrix} \right) \quad (100)$$

$$= P_{\tilde{A}_2}(a_1) \max_{Q_1 \in \Delta(\mathcal{A}_1)} u \left( Q_1, P_{A_2|\tilde{A}_2=a_1} \right) + P_{\tilde{A}_2}(a_2) \max_{Q_2 \in \Delta(\mathcal{A}_1)} u \left( Q_2, P_{A_2|\tilde{A}_2=a_2} \right) \quad (101)$$

$$\geq \min \left\{ \max_{Q_1 \in \Delta(\mathcal{A}_1)} u \left( Q_1, P_{A_2|\tilde{A}_2=a_1} \right), \max_{Q_2 \in \Delta(\mathcal{A}_1)} u \left( Q_2, P_{A_2|\tilde{A}_2=a_2} \right) \right\} \quad (102)$$

$$= \min_{i \in \{1,2\}} \max_{Q_0 \in \Delta(\mathcal{A}_1)} u \left( Q_0, P_{A_2|\tilde{A}_2=a_i} \right) \quad (103)$$

$$\geq \min_{P_0 \in \Delta(\mathcal{A}_1)} \max_{Q_0 \in \Delta(\mathcal{A}_1)} u \left( Q_0, P_0 \right) \quad (104)$$

$$= u \left( P_{A_1}^*, P_{A_2}^* \right), \quad (105)$$

where the probability measure  $P_{\tilde{A}_2}$  in (100) satisfies for all  $i \in \{1, 2\}$ ,

$$P_{\tilde{A}_2}(a_i) = \sum_{\ell \in \{1,2\}} P_{\tilde{A}_2|A_2=a_\ell}(a_i) P(a_\ell); \quad (106)$$

and, the equality in (105) follows from the minmax theorem [Neumann, 1928].

Note that if for all  $i \in \{1, 2\}$ , the probability measure  $P$  in (100) is such that

$$\max_{Q_1 \in \Delta(\mathcal{A}_1)} u \left( Q_1, P_{A_2|\tilde{A}_2=a_1} \right) = \max_{Q_2 \in \Delta(\mathcal{A}_1)} u \left( Q_2, P_{A_2|\tilde{A}_2=a_2} \right), \quad (107)$$

then the inequality in (102) holds with equality. Moreover, if there exists an  $i \in \{1, 2\}$  such that (98) holds, then the inequality in (104) holds with equality. From these observations, it holds that if for all  $i \in \{1, 2\}$ , the probability measure  $P$  in (100) satisfies (98), then the inequalities in (102) and (104) hold with equality, which implies the equality in (97).

Alternatively, if the equality in (97) holds, then, both inequalities in (102) and (104) hold with equality, which implies that for all  $i \in \{1, 2\}$ , the probability measure  $P$  in (100) satisfies (98).

In particular, under the assumption that there exists a unique NE in game  $\mathcal{G}(\underline{u})$ , the inequalities (102) and (104) hold with equality if and only if for all  $i \in \{1, 2\}$ ,

$$P_{A_2|\tilde{A}_2=a_i} = P_{A_2}^*. \quad (108)$$

The measure  $P_{A_2|\tilde{A}_2=a_1}$  in (108) satisfies

$$P_{A_2|\tilde{A}_2=a_1}(a_1) = \frac{P_{A_2,\tilde{A}_2}(a_1, a_1)}{P_{\tilde{A}_2}(a_1)} \quad (109)$$

$$= \frac{P_{A_2,\tilde{A}_2}(a_1, a_1)}{P_{A_2,\tilde{A}_2}(a_1, a_1) + P_{A_2,\tilde{A}_2}(a_2, a_1)} \quad (110)$$

$$= \frac{P(a_1)P_{\tilde{A}_2|A_2=a_1}(a_1)}{P(a_1)P_{\tilde{A}_2|A_2=a_1}(a_1) + P(a_2)P_{\tilde{A}_2|A_2=a_2}(a_1)} \quad (111)$$

$$= \frac{P(a_1)P_{\tilde{A}_2|A_2=a_1}(a_1)}{P(a_1)P_{\tilde{A}_2|A_2=a_1}(a_1)} \quad (112)$$

$$= \frac{P(a_1)P_{\tilde{A}_2|A_2=a_1}(a_1)}{P_{\tilde{A}_2|A_2=a_1}(a_1) - \left( P_{\tilde{A}_2|A_2=a_1}(a_1) - P_{\tilde{A}_2|A_2=a_2}(a_1) \right) P(a_2)} \quad (113)$$

Alternatively, the measure  $P_{A_2|\tilde{A}_2=a_2}$  in (108) satisfies

$$P_{A_2|\tilde{A}_2=a_2}(a_1) = \frac{P_{A_2,\tilde{A}_2}(a_1, a_2)}{P_{\tilde{A}_2}(a_2)} \quad (114)$$

$$= \frac{P_{A_2,\tilde{A}_2}(a_1, a_2)}{P_{A_2,\tilde{A}_2}(a_1, a_2) + P_{A_2,\tilde{A}_2}(a_2, a_2)} \quad (115)$$

$$= \frac{P(a_1)P_{\tilde{A}_2|A_2=a_1}(a_2)}{P(a_1)P_{\tilde{A}_2|A_2=a_1}(a_2) + P(a_2)P_{\tilde{A}_2|A_2=a_2}(a_2)}. \quad (116)$$

$$= \frac{P(a_1)P_{\tilde{A}_2|A_2=a_1}(a_2)}{P_{\tilde{A}_2|A_2=a_1}(a_2) + \left(P_{\tilde{A}_2|A_2=a_2}(a_2) - P_{\tilde{A}_2|A_2=a_1}(a_2)\right)P(a_2)}. \quad (117)$$

$$= \frac{P(a_1)P_{\tilde{A}_2|A_2=a_1}(a_2)}{P_{\tilde{A}_2|A_2=a_1}(a_2) + \det \underline{\mathbf{w}}P(a_2)}. \quad (118)$$

Hence, from (113) and (118), the equality  $P_{A_2|\tilde{A}_2=a_1}(a_1) = P_{A_2|\tilde{A}_2=a_2}(a_1)$  implies

$$\frac{P(a_1)P_{\tilde{A}_2|A_2=a_1}(a_1)}{P_{\tilde{A}_2|A_2=a_1}(a_1) - \det \underline{\mathbf{w}}P(a_2)} = \frac{P(a_1)P_{\tilde{A}_2|A_2=a_1}(a_2)}{P_{\tilde{A}_2|A_2=a_1}(a_2) + \det \underline{\mathbf{w}}P(a_2)}, \quad (119)$$

which can be rewritten as

$$0 = P(a_1)P_{\tilde{A}_2|A_2=a_1}(a_1) \left( P_{\tilde{A}_2|A_2=a_1}(a_2) + \det \underline{\mathbf{w}}P(a_2) \right) - P(a_1)P_{\tilde{A}_2|A_2=a_1}(a_2) \left( P_{\tilde{A}_2|A_2=a_1}(a_1) - \det \underline{\mathbf{w}}P(a_2) \right) \quad (120)$$

$$= \det \underline{\mathbf{w}}P(a_1)P(a_2) \left( P_{\tilde{A}_2|A_2=a_1}(a_1) + P_{\tilde{A}_2|A_2=a_1}(a_2) \right) \quad (121)$$

$$= \det \underline{\mathbf{w}}P(a_1)P(a_2). \quad (122)$$

Hence, the equality  $P_{A_2|\tilde{A}_2=a_1}(a_1) = P_{A_2|\tilde{A}_2=a_2}(a_1) = P_{A_2}^*(a_1)$  in (108) holds if and only if one of the conditions in (99) holds. This completes the proof.  $\blacksquare$

**Lemma F.4** *Let the probability measures  $P_{A_1}^* \in \Delta(\mathcal{A}_1)$  and  $P_{A_2}^* \in \Delta(\mathcal{A}_2)$  form one of the NEs of the game  $\mathcal{G}(\underline{\mathbf{u}})$  in (2). Under the assumption that the entries of the matrix  $\underline{\mathbf{u}}$  in (1) satisfy (4), for all  $i \in \{1, 2\}$ , the real  $P^{(i)}$  in (23) satisfies*

$$P^{(i)} = \frac{P_{A_2}^*(a_1)P_{\tilde{A}_2|A_2=a_2}(a_i)}{P_{A_2}^*(a_2)P_{\tilde{A}_2|A_2=a_1}(a_i) + P_{A_2}^*(a_1)P_{\tilde{A}_2|A_2=a_2}(a_i)} \in [0, 1]. \quad (123)$$

*Proof:* From Lemma 2.1, it follows that if the entries of the matrix  $\underline{\mathbf{u}}$  in (1) satisfy (4), the probability measures  $P_{A_1}^* \in \Delta(\mathcal{A}_1)$  and  $P_{A_2}^* \in \Delta(\mathcal{A}_2)$  form the unique NE of the game  $\mathcal{G}(\underline{\mathbf{u}})$  in (2) and moreover, for all  $(i, j) \in \{1, 2\}^2$ ,  $P_{A_i}^*(a_j) \in (0, 1)$ . Hence, for all  $i \in \{1, 2\}$ , it follows from (23) that

$$P^{(i)} = \frac{u_{2,2}P_{\tilde{A}_2|A_2=a_2}(a_i) - u_{1,2}P_{\tilde{A}_2|A_2=a_2}(a_i)}{u_{1,1}P_{\tilde{A}_2|A_2=a_1}(a_i) - u_{1,2}P_{\tilde{A}_2|A_2=a_2}(a_i) - u_{2,1}P_{\tilde{A}_2|A_2=a_1}(a_i) + u_{2,2}P_{\tilde{A}_2|A_2=a_2}(a_i)} \quad (124)$$

$$= \frac{(u_{2,2} - u_{1,2})P_{\tilde{A}_2|A_2=a_2}(a_i)}{(u_{1,1} - u_{2,1})P_{\tilde{A}_2|A_2=a_1}(a_i) + (u_{2,2} - u_{1,2})P_{\tilde{A}_2|A_2=a_2}(a_i)} \quad (125)$$

$$= \frac{\frac{u_{2,2} - u_{1,2}}{u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2}}P_{\tilde{A}_2|A_2=a_2}(a_i)}{\frac{u_{1,1} - u_{2,1}}{u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2}}P_{\tilde{A}_2|A_2=a_1}(a_i) + \frac{u_{2,2} - u_{1,2}}{u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2}}P_{\tilde{A}_2|A_2=a_2}(a_i)} \quad (126)$$

$$= \frac{P_{A_2}^*(a_1)P_{\tilde{A}_2|A_2=a_2}(a_i)}{P_{A_2}^*(a_2)P_{\tilde{A}_2|A_2=a_1}(a_i) + P_{A_2}^*(a_1)P_{\tilde{A}_2|A_2=a_2}(a_i)}, \quad (127)$$

where the equality in (126) follows from the fact that  $u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2} \neq 0$  when the entries of the matrix  $\underline{\mathbf{u}}$  satisfy (4); and the equality in (127) follows from (5). Furthermore, note that

$$0 \leq P_{A_2}^*(a_1)P_{\tilde{A}_2|A_2=a_2}(a_i) \leq P_{A_2}^*(a_1)P_{\tilde{A}_2|A_2=a_2}(a_i) + P_{A_2}^*(a_2)P_{\tilde{A}_2|A_2=a_1}(a_i), \quad (128)$$

which follows from the fact that for all  $(i, j) \in \{1, 2\}^2$ , it holds that  $P_{\tilde{A}_2|A_2=a_i}(a_j) \geq 0$  and  $P_{A_2}^*(a_i) \in (0, 1)$ . Hence, from (127) and (128), for all  $i \in \{1, 2\}$ , it holds that

$$P^{(i)} \in [0, 1]. \quad (129)$$

This completes the proof.  $\blacksquare$

**Lemma F.5** *Let the probability measures  $P_{A_1}^* \in \Delta(\mathcal{A}_1)$  and  $P_{A_2}^* \in \Delta(\mathcal{A}_2)$  form one of the NEs of the game  $\mathcal{G}(\underline{u})$  in (2). Assume that the entries of the matrix  $\underline{u}$  in (1) satisfy (4). Hence, the reals  $P^{(1)}$  and  $P^{(2)}$  in (23) satisfy*

$$0 \leq \min \{P^{(1)}, P^{(2)}\} \leq P_{A_2}^*(a_1) \leq \max \{P^{(1)}, P^{(2)}\} \leq 1. \quad (130)$$

*Proof:* From Lemma 2.1, it follows that if the entries of the matrix  $\underline{u}$  in (1) satisfy (4), the probability measures  $P_{A_1}^* \in \Delta(\mathcal{A}_1)$  and  $P_{A_2}^* \in \Delta(\mathcal{A}_2)$  form the unique NE of the game  $\mathcal{G}(\underline{u})$  in (2). Hence, from Lemma F.4, it follows that for all  $i \in \{1, 2\}$ ,

$$P^{(i)} - P_{A_2}^*(a_1) = \frac{P_{A_2}^*(a_1)P_{\tilde{A}_2|A_2=a_2}(a_i)}{P_{A_2}^*(a_2)P_{\tilde{A}_2|A_2=a_1}(a_i) + P_{A_2}^*(a_1)P_{\tilde{A}_2|A_2=a_2}(a_i)} - P_{A_2}^*(a_1) \quad (131)$$

$$= \frac{P_{A_2}^*(a_1)P_{\tilde{A}_2|A_2=a_2}(a_i) - P_{A_2}^*(a_1) \left( P_{A_2}^*(a_2)P_{\tilde{A}_2|A_2=a_1}(a_i) + P_{A_2}^*(a_1)P_{\tilde{A}_2|A_2=a_2}(a_i) \right)}{P_{A_2}^*(a_2)P_{\tilde{A}_2|A_2=a_1}(a_i) + P_{A_2}^*(a_1)P_{\tilde{A}_2|A_2=a_2}(a_i)} \quad (132)$$

$$= \frac{P_{A_2}^*(a_1) \left( P_{\tilde{A}_2|A_2=a_2}(a_i) - P_{A_2}^*(a_2)P_{\tilde{A}_2|A_2=a_1}(a_i) - P_{A_2}^*(a_1)P_{\tilde{A}_2|A_2=a_2}(a_i) \right)}{P_{A_2}^*(a_2)P_{\tilde{A}_2|A_2=a_1}(a_i) + P_{A_2}^*(a_1)P_{\tilde{A}_2|A_2=a_2}(a_i)} \quad (133)$$

$$= \frac{P_{A_2}^*(a_1) \left( P_{A_2}^*(a_2)P_{\tilde{A}_2|A_2=a_2}(a_i) - P_{A_2}^*(a_2)P_{\tilde{A}_2|A_2=a_1}(a_i) \right)}{P_{A_2}^*(a_2)P_{\tilde{A}_2|A_2=a_1}(a_i) + P_{A_2}^*(a_1)P_{\tilde{A}_2|A_2=a_2}(a_i)} \quad (134)$$

$$= \frac{P_{A_2}^*(a_1)P_{A_2}^*(a_2) \left( P_{\tilde{A}_2|A_2=a_2}(a_i) - P_{\tilde{A}_2|A_2=a_1}(a_i) \right)}{P_{A_2}^*(a_2)P_{\tilde{A}_2|A_2=a_1}(a_i) + P_{A_2}^*(a_1)P_{\tilde{A}_2|A_2=a_2}(a_i)}. \quad (135)$$

From (135), it follows that if  $P^{(1)} \geq P_{A_2}^*(a_1)$ , then, it holds that

$$P^{(2)} \leq P_{A_2}^*(a_1). \quad (136)$$

Similarly, if  $P^{(1)} \leq P_{A_2}^*(a_1)$ , then, it holds that

$$P^{(2)} \geq P_{A_2}^*(a_1). \quad (137)$$

This implies that

$$\min \{P^{(1)}, P^{(2)}\} \leq P_{A_2}^*(a_1) \leq \max \{P^{(1)}, P^{(2)}\}. \quad (138)$$

The proof is completed by noticing that if the entries of the matrix  $\underline{u}$  in (1) satisfy (4), from Lemma F.4, it holds that for all  $i \in \{1, 2\}$ ,  $P^{(i)} \in [0, 1]$ .  $\blacksquare$

**Lemma F.6** *Assume that the entries of the matrix  $\underline{u}$  in (1) satisfy (4). The reals  $P^{(1)}$  and  $P^{(2)}$  in (23) satisfy*

$$\min \{P^{(1)}, P^{(2)}\} = \begin{cases} P^{(1)}, & \text{if } \det \underline{w} > 0, \\ P^{(2)}, & \text{if } \det \underline{w} < 0, \\ P^{(1)} = P^{(2)} = P_{A_2}^*(a_1), & \text{if } \det \underline{w} = 0, \end{cases} \quad (139)$$

with  $\underline{w}$  being the matrix in (13).

*Proof:* First, consider the case in which  $\det \underline{w} < 0$ . From Lemma F.1, if  $\det \underline{w} < 0$ , then it holds that

$$P_{\tilde{A}_2|A_2=a_2}(a_1) - P_{\tilde{A}_2|A_2=a_1}(a_1) > 0. \quad (140)$$

Under the assumption that the entries of the matrix  $\underline{u}$  in (1) satisfy (4), from (135) and (140), it holds that  $P^{(1)} > P_{A_2}^*(a_1)$  and  $P_{A_2}^*(a_1) > P^{(2)}$ . Hence,

$$\min \{P^{(1)}, P^{(2)}\} = P^{(2)}. \quad (141)$$

Then consider the case in which  $\det \underline{w} > 0$ . From Lemma F.1, if  $\det \underline{w} > 0$ , then it holds that

$$P_{\tilde{A}_2|A_2=a_2}(a_2) - P_{\tilde{A}_2|A_2=a_1}(a_2) > 0. \quad (142)$$

Under the assumption that the entries of the matrix  $\underline{u}$  in (1) satisfy (4), from (135) and (142), it holds that  $P^{(2)} > P_{A_2}^*(a_1)$  and  $P_{A_2}^*(a_1) > P^{(1)}$ . Hence,

$$\min \{P^{(1)}, P^{(2)}\} = P^{(1)}. \quad (143)$$

Finally, consider the case in which  $\det \underline{w} = 0$ . From Lemma F.1, if  $\det \underline{w} = 0$ , then it holds that

$$P_{\tilde{A}_2|A_2=a_2}(a_2) - P_{\tilde{A}_2|A_2=a_1}(a_2) = 0. \quad (144)$$

Under the assumption that the entries of the matrix  $\underline{u}$  in (1) satisfy (4), from (135) and (144), it holds that  $P^{(2)} = P_{A_2}^*(a_1)$  and  $P_{A_2}^*(a_1) = P^{(1)}$ . Hence,

$$\min \{P^{(1)}, P^{(2)}\} = P^{(1)} = P^{(2)} = P_{A_2}^*(a_1), \quad (145)$$

which completes the proof.  $\blacksquare$

**Lemma F.7** Assume that the entries of the matrix  $\underline{u}$  in (1) satisfy (4). If  $u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2} > 0$ , then it holds that for all  $i \in \{1, 2\}$  and for all  $P \in \Delta(\mathcal{A}_2)$ ,

$$\text{BR}_{1,i}(P) = \begin{cases} \{Q \in \Delta(\mathcal{A}_1) : Q(a_1) = 1\}, & \text{if } P(a_1) > P^{(i)}, \\ \{Q \in \Delta(\mathcal{A}_1) : Q(a_1) = 0\}, & \text{if } P(a_1) < P^{(i)}, \\ \{Q \in \Delta(\mathcal{A}_1) : Q(a_1) = \beta, \beta \in [0, 1]\}, & \text{if } P(a_1) = P^{(i)}, \end{cases} \quad (146)$$

where  $P^{(i)}$  is in (23). Otherwise, if  $u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2} \leq 0$ , then it holds that for all  $i \in \{1, 2\}$  and for all  $P \in \Delta(\mathcal{A}_2)$ ,

$$\text{BR}_{1,i}(P) = \begin{cases} \{Q \in \Delta(\mathcal{A}_1) : Q(a_1) = 1\}, & \text{if } P(a_1) < P^{(i)}, \\ \{Q \in \Delta(\mathcal{A}_1) : Q(a_1) = 0\}, & \text{if } P(a_1) > P^{(i)}, \\ \{Q \in \Delta(\mathcal{A}_1) : Q(a_1) = \beta, \beta \in [0, 1]\}, & \text{if } P(a_1) = P^{(i)}. \end{cases} \quad (147)$$

*Proof:* Note that if the entries of the matrix  $\underline{u}$  in (1) satisfy (4), then  $u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2} \neq 0$ . Furthermore, for all  $i \in \{1, 2\}$ , the real  $s_i$  in (19) satisfies

$$s_i = (u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2}) \left( \frac{u_{1,1} - u_{2,1}}{u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2}} P(a_1) P_{\tilde{A}_2|A_2=a_1}(a_i) + \frac{u_{1,2} - u_{2,2}}{u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2}} P(a_2) P_{\tilde{A}_2|A_2=a_2}(a_i) \right) \quad (148)$$

$$= (u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2}) \left( P_{A_2}^*(a_2) P(a_1) P_{\tilde{A}_2|A_2=a_1}(a_i) - P_{A_2}^*(a_1) P(a_2) P_{\tilde{A}_2|A_2=a_2}(a_i) \right), \quad (149)$$

where (149) follows from Lemma 2.1. For all  $i \in \{1, 2\}$ , let  $B_i$  be the following constant

$$B_i = P_{A_2}^*(a_2) P(a_1) P_{\tilde{A}_2|A_2=a_1}(a_i) - P_{A_2}^*(a_1) P(a_2) P_{\tilde{A}_2|A_2=a_2}(a_i) \quad (150)$$

$$= P_{A_2}^*(a_2) P(a_1) P_{\tilde{A}_2|A_2=a_1}(a_i) - P_{A_2}^*(a_1) \left(1 - P(a_1)\right) P_{\tilde{A}_2|A_2=a_2}(a_i) \quad (151)$$

$$= P(a_1) \left( P_{A_2}^*(a_2) P_{\tilde{A}_2|A_2=a_1}(a_i) + P_{A_2}^*(a_1) P_{\tilde{A}_2|A_2=a_2}(a_i) \right) - P_{A_2}^*(a_1) P_{\tilde{A}_2|A_2=a_2}(a_i). \quad (152)$$

For all  $i \in \{1, 2\}$ , plugging  $B_i$  into (149) yields

$$s_i = (u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2}) B_i. \quad (153)$$

First, consider the case in which  $u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2} > 0$ . If  $P(a_1)$  in (152) satisfies

$$P(a_1) > \frac{P_{A_2}^*(a_1)P_{\tilde{A}_2|A_2=a_2}(a_i)}{P_{A_2}^*(a_2)P_{\tilde{A}_2|A_2=a_1}(a_i) + P_{A_2}^*(a_1)P_{\tilde{A}_2|A_2=a_2}(a_i)} \quad (154)$$

$$= P^{(i)}, \quad (155)$$

then, it holds that  $B_i > 0$ . The equality in (155) follows from Lemma F.4. Under the assumptions that  $u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2} > 0$  and (155), it holds that  $s_i > 0$ , which from Lemma 3.2, further implies that

$$\text{BR}_{1,i}(P) = \{Q \in \Delta(\mathcal{A}_1) : Q(a_1) = 1\}. \quad (156)$$

If  $P(a_1)$  in (152) satisfies

$$P(a_1) < \frac{P_{A_2}^*(a_1)P_{\tilde{A}_2|A_2=a_2}(a_i)}{P_{A_2}^*(a_2)P_{\tilde{A}_2|A_2=a_1}(a_i) + P_{A_2}^*(a_1)P_{\tilde{A}_2|A_2=a_2}(a_i)} \quad (157)$$

$$= P^{(i)}, \quad (158)$$

then, it holds that  $B_i < 0$ . The equality in (158) follows from Lemma F.4. Under the assumptions that  $u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2} > 0$  and (158), it holds that  $s_i < 0$ , which from Lemma 3.2, further implies that

$$\text{BR}_{1,i}(P) = \{Q \in \Delta(\mathcal{A}_1) : Q(a_1) = 0\}. \quad (159)$$

If  $P(a_1)$  in (152) satisfies

$$P(a_1) = \frac{P_{A_2}^*(a_1)P_{\tilde{A}_2|A_2=a_2}(a_i)}{P_{A_2}^*(a_2)P_{\tilde{A}_2|A_2=a_1}(a_i) + P_{A_2}^*(a_1)P_{\tilde{A}_2|A_2=a_2}(a_i)} = P^{(i)}, \quad (160)$$

then, it holds that  $B_i = 0$ . The equality in (160) follows from Lemma F.4. Under the assumptions that  $u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2} > 0$  and (160), it holds that  $s_i = 0$ , which from Lemma 3.2, further implies that

$$\text{BR}_{1,i}(P) = \{Q \in \Delta(\mathcal{A}_1) : Q(a_1) = \beta, \beta \in [0, 1]\}. \quad (161)$$

In a nutshell, if  $u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2} > 0$ , it holds that

$$\text{BR}_{1,i}(P) = \begin{cases} \{Q \in \Delta(\mathcal{A}_1) : Q(a_1) = 1\}, & \text{if } P(a_1) > P^{(i)}, \\ \{Q \in \Delta(\mathcal{A}_1) : Q(a_1) = 0\}, & \text{if } P(a_1) < P^{(i)}, \\ \{Q \in \Delta(\mathcal{A}_1) : Q(a_1) = \beta, \beta \in [0, 1]\}, & \text{if } P(a_1) = P^{(i)}. \end{cases} \quad (162)$$

This proves the equality in (146).

Alternatively, consider the case in which  $u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2} < 0$ . If  $P(a_1)$  in (152) satisfies

$$P(a_1) > \frac{P_{A_2}^*(a_1)P_{\tilde{A}_2|A_2=a_2}(a_i)}{P_{A_2}^*(a_2)P_{\tilde{A}_2|A_2=a_1}(a_i) + P_{A_2}^*(a_1)P_{\tilde{A}_2|A_2=a_2}(a_i)} \quad (163)$$

$$= P^{(i)}, \quad (164)$$

then, it holds that  $B_i > 0$ . The equality in (164) follows from Lemma F.4. Under the assumptions that  $u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2} < 0$  and (164), it follows that  $s_i < 0$ , which from Lemma 3.2, further implies that

$$\text{BR}_{1,i}(P) = \{Q \in \Delta(\mathcal{A}_1) : Q(a_1) = 0\}. \quad (165)$$

If  $P(a_1)$  in (152) satisfies

$$P(a_1) < \frac{P_{A_2}^*(a_1)P_{\tilde{A}_2|A_2=a_2}(a_i)}{P_{A_2}^*(a_2)P_{\tilde{A}_2|A_2=a_1}(a_i) + P_{A_2}^*(a_1)P_{\tilde{A}_2|A_2=a_2}(a_i)} \quad (166)$$

$$= P^{(i)}, \quad (167)$$

then, it holds that  $B_i < 0$ . The equality in (167) follows from Lemma F.4. Under the assumptions that  $u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2} < 0$  and (167), it holds that  $s_i > 0$ , which from Lemma 3.2, further implies that

$$\text{BR}_{1,i}(P) = \{Q \in \Delta(\mathcal{A}_1) : Q(a_1) = 1\}. \quad (168)$$

If  $P(a_1)$  in (152) satisfies

$$P(a_1) = \frac{P_{A_2}^*(a_1)P_{\bar{A}_2|A_2=a_2}(a_i)}{P_{A_2}^*(a_2)P_{\bar{A}_2|A_2=a_1}(a_i) + P_{A_2}^*(a_1)P_{\bar{A}_2|A_2=a_2}(a_i)} = P^{(i)}, \quad (169)$$

then, it holds that  $B_i = 0$ . The equality in (169) follows from Lemma F.4. Under the assumptions that  $u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2} < 0$  and (169), it holds that  $s_i = 0$ , which from Lemma 3.2, further implies that

$$\text{BR}_{1,i}(P) = \{Q \in \Delta(\mathcal{A}_1) : Q(a_1) = \beta, \beta \in [0, 1]\}. \quad (170)$$

In a nutshell, if  $u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2} < 0$ , it holds that

$$\text{BR}_{1,i}(P) = \begin{cases} \{Q \in \Delta(\mathcal{A}_1) : Q(a_1) = 1\}, & \text{if } P(a_1) < P^{(i)}, \\ \{Q \in \Delta(\mathcal{A}_1) : Q(a_1) = 0\}, & \text{if } P(a_1) > P^{(i)}, \\ \{Q \in \Delta(\mathcal{A}_1) : Q(a_1) = \beta, \beta \in [0, 1]\}, & \text{if } P(a_1) = P^{(i)}. \end{cases} \quad (171)$$

This proves the equality in (147).

The proof is completed by noticing that if the entries of the matrix  $\underline{u}$  in (1) satisfy (4), then  $u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2} \neq 0$ . ■

**Lemma F.8** Assume that the entries of the matrix  $\underline{u}$  in (1) satisfy (4) and  $u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2} > 0$ . For all  $P \in \Delta(\mathcal{A}_2)$  such that  $P(a_1) \geq \max\{P^{(1)}, P^{(2)}\}$ , with  $P^{(1)}$  and  $P^{(2)}$  in (23), it holds that

$$\hat{v}(P) = u_{1,1}P(a_1) + u_{1,2}P(a_2). \quad (172)$$

If  $\det \underline{w} > 0$ , with  $\underline{w}$  in (13), then for all  $P \in \Delta(\mathcal{A}_2)$  such that  $P^{(1)} < P(a_1) < P^{(2)}$ , it holds that

$$\hat{v}(P) = \left(u_{1,1}P_{\bar{A}_2|A_2=a_1}(a_1) + u_{2,1}P_{\bar{A}_2|A_2=a_1}(a_2)\right)P(a_1) + \left(u_{1,2}P_{\bar{A}_2|A_2=a_2}(a_1) + u_{2,2}P_{\bar{A}_2|A_2=a_2}(a_2)\right)P(a_2). \quad (173)$$

If  $\det \underline{w} \leq 0$ , then for all  $P \in \Delta(\mathcal{A}_2)$  such that  $P^{(2)} < P(a_1) < P^{(1)}$ , it holds that

$$\hat{v}(P) = \left(u_{1,1}P_{\bar{A}_2|A_2=a_1}(a_2) + u_{2,1}P_{\bar{A}_2|A_2=a_1}(a_1)\right)P(a_1) + \left(u_{1,2}P_{\bar{A}_2|A_2=a_2}(a_2) + u_{2,2}P_{\bar{A}_2|A_2=a_2}(a_1)\right)P(a_2). \quad (174)$$

Finally, for all  $P \in \Delta(\mathcal{A}_2)$  such that  $P(a_1) \leq \min\{P^{(1)}, P^{(2)}\}$ , it holds that

$$\hat{v}(P) = u_{2,1}P(a_1) + u_{2,2}P(a_2). \quad (175)$$

*Proof:* For all probability measures  $P \in \Delta(\mathcal{A}_2)$  such that  $P(a_1) \geq \max\{P^{(1)}, P^{(2)}\}$ , from Lemma F.7, it holds that, for all  $i \in \{1, 2\}$ ,

$$\text{BR}_{1,i}(P) = \{Q \in \Delta(\mathcal{A}_1) : Q(a_1) = 1\}. \quad (176)$$

Furthermore, from (48), it holds that

$$\hat{v}(P) = u_{1,1}P(a_1) + u_{1,2}P(a_2), \quad (177)$$

which proves the equality in (172).

If  $\det \underline{w} > 0$ , from Lemma F.6, it holds that  $P^{(1)} < P^{(2)}$ . For all probability measures  $P \in \Delta(\mathcal{A}_2)$  such that  $P^{(1)} < P(a_1) < P^{(2)}$ , from Lemma F.7, it holds that

$$\text{BR}_{1,1}(P) = \{Q \in \Delta(\mathcal{A}_1) : Q(a_1) = 1\} \quad \text{and} \quad (178)$$

$$\text{BR}_{1,2}(P) = \{Q \in \Delta(\mathcal{A}_1) : Q(a_1) = 0\}. \quad (179)$$

From (48), it holds that

$$\begin{aligned} \hat{v}(P) &= \left( u_{1,1}P(a_1)P_{\tilde{A}_2|A_2=a_1}(a_1) + u_{1,2}P(a_2)P_{\tilde{A}_2|A_2=a_2}(a_1) \right) \\ &\quad + \left( u_{2,1}P(a_1)P_{\tilde{A}_2|A_2=a_1}(a_2) + u_{2,2}P(a_2)P_{\tilde{A}_2|A_2=a_2}(a_2) \right) \end{aligned} \quad (180)$$

$$= \left( u_{1,1}P_{\tilde{A}_2|A_2=a_1}(a_1) + u_{2,1}P_{\tilde{A}_2|A_2=a_1}(a_2) \right) P(a_1) + \left( u_{1,2}P_{\tilde{A}_2|A_2=a_2}(a_1) + u_{2,2}P_{\tilde{A}_2|A_2=a_2}(a_2) \right) P(a_2), \quad (181)$$

which proves the equality in (173).

If  $\det \mathbf{w} \leq 0$ , from Lemma F.6, it holds that  $P^{(1)} \geq P^{(2)}$ . For all probability measures  $P \in \Delta(\mathcal{A}_2)$  such that  $P^{(1)} > P(a_1) > P^{(2)}$ , from Lemma F.7, it holds that

$$\text{BR}_{1,1}(P) = \{Q \in \Delta(\mathcal{A}_1) : Q(a_1) = 0\} \quad \text{and} \quad (182)$$

$$\text{BR}_{1,2}(P) = \{Q \in \Delta(\mathcal{A}_1) : Q(a_1) = 1\}. \quad (183)$$

From (48), it holds that

$$\begin{aligned} \hat{v}(P) &= \left( u_{2,1}P(a_1)P_{\tilde{A}_2|A_2=a_1}(a_1) + u_{2,2}P(a_2)P_{\tilde{A}_2|A_2=a_2}(a_1) \right) \\ &\quad + \left( u_{1,1}P(a_1)P_{\tilde{A}_2|A_2=a_1}(a_2) + u_{1,2}P(a_2)P_{\tilde{A}_2|A_2=a_2}(a_2) \right) \end{aligned} \quad (184)$$

$$= \left( u_{1,1}P_{\tilde{A}_2|A_2=a_1}(a_2) + u_{2,1}P_{\tilde{A}_2|A_2=a_1}(a_1) \right) P(a_1) + \left( u_{1,2}P_{\tilde{A}_2|A_2=a_2}(a_2) + u_{2,2}P_{\tilde{A}_2|A_2=a_2}(a_1) \right) P(a_2), \quad (185)$$

which proves the equality in (174).

For all probability measures  $P \in \Delta(\mathcal{A}_2)$  such that  $P(a_1) \leq \min\{P^{(1)}, P^{(2)}\}$ , from Lemma F.7, it holds that, for all  $i \in \{1, 2\}$ ,

$$\text{BR}_{1,i}(P) = \{Q \in \Delta(\mathcal{A}_1) : Q(a_1) = 0\}. \quad (186)$$

From (48), it holds that

$$\hat{v}(P) = u_{2,1}P(a_1) + u_{2,2}P(a_2), \quad (187)$$

which proves the equality in (175), and completes the proof.  $\blacksquare$

**Lemma F.9** Assume that the entries of the matrix  $\underline{u}$  in (1) satisfy (4) and  $u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2} \leq 0$ . For all  $P \in \Delta(\mathcal{A}_2)$  such that  $P(a_1) \leq \min\{P^{(1)}, P^{(2)}\}$ , with  $P^{(1)}$  and  $P^{(2)}$  in (23), it holds that

$$\hat{v}(P) = u_{1,1}P(a_1) + u_{1,2}P(a_2). \quad (188)$$

If  $\det \underline{w} > 0$ , with  $\underline{w}$  in (13), then for all  $P \in \Delta(\mathcal{A}_2)$  such that  $P^{(1)} < P(a_1) < P^{(2)}$ , it holds that

$$\hat{v}(P) = \left( u_{1,1}P_{\tilde{A}_2|A_2=a_1}(a_2) + u_{2,1}P_{\tilde{A}_2|A_2=a_1}(a_1) \right) P(a_1) + \left( u_{1,2}P_{\tilde{A}_2|A_2=a_2}(a_2) + u_{2,2}P_{\tilde{A}_2|A_2=a_2}(a_1) \right) P(a_2). \quad (189)$$

If  $\det \underline{w} \leq 0$ , then for all  $P \in \Delta(\mathcal{A}_2)$  such that  $P^{(2)} < P(a_1) < P^{(1)}$ , it holds that

$$\hat{v}(P) = \left( u_{1,1}P_{\tilde{A}_2|A_2=a_1}(a_1) + u_{2,1}P_{\tilde{A}_2|A_2=a_1}(a_2) \right) P(a_1) + \left( u_{1,2}P_{\tilde{A}_2|A_2=a_2}(a_1) + u_{2,2}P_{\tilde{A}_2|A_2=a_2}(a_2) \right) P(a_2). \quad (190)$$

Finally, for all  $P \in \Delta(\mathcal{A}_2)$  such that  $P(a_1) \geq \max\{P^{(1)}, P^{(2)}\}$ , it holds that

$$\hat{v}(P) = u_{2,1}P(a_1) + u_{2,2}P(a_2). \quad (191)$$

*Proof:* For all probability measures  $P \in \Delta(\mathcal{A}_2)$  such that  $P(a_1) \leq \min\{P^{(1)}, P^{(2)}\}$ , from Lemma F.7, it holds that, for all  $i \in \{1, 2\}$ ,

$$\text{BR}_{1,i}(P) = \{Q \in \Delta(\mathcal{A}_1) : Q(a_1) = 1\}. \quad (192)$$

From (48), it holds that

$$\hat{v}(P) = u_{1,1}P(a_1) + u_{1,2}P(a_2), \quad (193)$$

which proves the equality in (188).

If  $\det \underline{w} > 0$ , from Lemma F.6, it holds that  $P^{(1)} < P^{(2)}$ . For all probability measures  $P \in \Delta(\mathcal{A}_2)$  such that  $P^{(1)} < P(a_1) < P^{(2)}$ , from Lemma F.7, it holds that

$$\text{BR}_{1,1}(P) = \{Q \in \Delta(\mathcal{A}_1) : Q(a_1) = 0\} \quad \text{and} \quad (194)$$

$$\text{BR}_{1,2}(P) = \{Q \in \Delta(\mathcal{A}_1) : Q(a_1) = 1\}. \quad (195)$$

From (48), it holds that

$$\begin{aligned} \hat{v}(P) &= \left( u_{2,1}P(a_1)P_{\tilde{A}_2|A_2=a_1}(a_1) + u_{2,2}P(a_2)P_{\tilde{A}_2|A_2=a_2}(a_1) \right) \\ &\quad + \left( u_{1,1}P(a_1)P_{\tilde{A}_2|A_2=a_1}(a_2) + u_{1,2}P(a_2)P_{\tilde{A}_2|A_2=a_2}(a_2) \right) \end{aligned} \quad (196)$$

$$= \left( u_{1,1}P_{\tilde{A}_2|A_2=a_1}(a_2) + u_{2,1}P_{\tilde{A}_2|A_2=a_1}(a_1) \right) P(a_1) + \left( u_{1,2}P_{\tilde{A}_2|A_2=a_2}(a_2) + u_{2,2}P_{\tilde{A}_2|A_2=a_2}(a_1) \right) P(a_2), \quad (197)$$

which proves the equality in (189).

If  $\det \underline{w} \leq 0$ , from Lemma F.6, it holds that  $P^{(1)} \geq P^{(2)}$ . For all probability measures  $P \in \Delta(\mathcal{A}_2)$  such that  $P^{(1)} > P(a_1) > P^{(2)}$ , from Lemma F.7, it holds that

$$\text{BR}_{1,1}(P) = \{Q \in \Delta(\mathcal{A}_1) : Q(a_1) = 1\} \quad \text{and} \quad (198)$$

$$\text{BR}_{1,2}(P) = \{Q \in \Delta(\mathcal{A}_1) : Q(a_1) = 0\}. \quad (199)$$

From (48), it holds that

$$\begin{aligned} \hat{v}(P) &= \left( u_{1,1}P(a_1)P_{\tilde{A}_2|A_2=a_1}(a_1) + u_{1,2}P(a_2)P_{\tilde{A}_2|A_2=a_2}(a_1) \right) \\ &\quad + \left( u_{2,1}P(a_1)P_{\tilde{A}_2|A_2=a_1}(a_2) + u_{2,2}P(a_2)P_{\tilde{A}_2|A_2=a_2}(a_2) \right) \end{aligned} \quad (200)$$

$$= \left( u_{1,1}P_{\tilde{A}_2|A_2=a_1}(a_1) + u_{2,1}P_{\tilde{A}_2|A_2=a_1}(a_2) \right) P(a_1) + \left( u_{1,2}P_{\tilde{A}_2|A_2=a_2}(a_1) + u_{2,2}P_{\tilde{A}_2|A_2=a_2}(a_2) \right) P(a_2), \quad (201)$$

which proves the equality in (190).

For all probability measures  $P \in \Delta(\mathcal{A}_2)$  such that  $P(a_1) \geq \max\{P^{(1)}, P^{(2)}\}$ , from Lemma F.7, it holds that, for all  $i \in \{1, 2\}$ ,

$$\text{BR}_{1,i}(P) = \{Q \in \Delta(\mathcal{A}_1) : Q(a_1) = 0\}. \quad (202)$$

From (48), it holds that

$$\hat{v}(P) = u_{2,1}P(a_1) + u_{2,2}P(a_2), \quad (203)$$

which proves the equality in (191), and completes the proof.  $\blacksquare$

**Lemma F.10** *Assume that the entries of the matrix  $\underline{u}$  in (1) satisfy (4). For all tuples  $(P, Q) \in \Delta(\mathcal{A}_2) \times \Delta(\mathcal{A}_2)$ , if  $0 \leq P(a_1) < Q(a_1) \leq \min\{P^{(1)}, P^{(2)}\}$ , then it holds that*

$$\hat{v}(P) > \hat{v}(Q). \quad (204)$$

*Alternatively, if  $\max\{P^{(1)}, P^{(2)}\} \leq P(a_1) < Q(a_1) \leq 1$ , then it holds that*

$$\hat{v}(P) < \hat{v}(Q). \quad (205)$$

*Proof:* Two cases are considered. First, the case in which  $u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2} > 0$ ; Second, the case in which  $u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2} \leq 0$ .



Consider the case in which  $u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2} > 0$ . From Lemma 2.1, if the entries of the matrix  $\underline{u}$  in (1) satisfy (4), then one of the following conditions holds:

$$u_{1,1} - u_{1,2} > 0 \quad \text{and} \quad u_{2,1} - u_{2,2} < 0, \quad \text{or} \quad (206)$$

$$u_{1,1} - u_{1,2} < 0 \quad \text{and} \quad u_{2,1} - u_{2,2} > 0. \quad (207)$$

Nonetheless, only the first condition yields  $u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2} > 0$ . Hence, from Lemma F.8, if  $0 \leq P(a_1) < Q(a_1) \leq \min\{P^{(1)}, P^{(2)}\}$ , then it holds that

$$\hat{v}(P) > \hat{v}(Q); \quad (208)$$

and if  $\max\{P^{(1)}, P^{(2)}\} \leq P(a_1) < Q(a_1) \leq 1$ , then it holds that

$$\hat{v}(P) < \hat{v}(Q). \quad (209)$$

Alternatively, consider the case in which  $u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2} \leq 0$ . From Lemma 2.1, if the entries of the matrix  $\underline{u}$  in (1) satisfy (4), then one of the following conditions holds:

$$u_{1,1} - u_{1,2} > 0 \quad \text{and} \quad u_{2,1} - u_{2,2} < 0, \quad \text{or} \quad (210)$$

$$u_{1,1} - u_{1,2} < 0 \quad \text{and} \quad u_{2,1} - u_{2,2} > 0. \quad (211)$$

Nonetheless, only the second condition yields  $u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2} \leq 0$ . Hence, from Lemma F.8, if  $0 \leq P(a_1) < Q(a_1) \leq \min\{P^{(1)}, P^{(2)}\}$ , then it holds that

$$\hat{v}(P) > \hat{v}(Q); \quad (212)$$

and if  $\max\{P^{(1)}, P^{(2)}\} \leq P(a_1) < Q(a_1) \leq 1$ , then it holds that

$$\hat{v}(P) < \hat{v}(Q). \quad (213)$$

This completes the proof. ■

## F.2 Main Proof

Two cases are considered: First, the case in which the payoff matrix  $\underline{u}$  satisfies (4); Second, the case in which the payoff matrix  $\underline{u}$  satisfies (7).

Consider the case in which the payoff matrix  $\underline{u}$  satisfies (4). From Lemma 2.1, there is a unique NE in strictly mixed strategies in the game  $\mathcal{G}(\underline{u})$ . From Lemma F.10, it holds that for all  $P \in \Delta(\mathcal{A}_2)$  such that  $0 \leq P(a_1) \leq \min\{P^{(1)}, P^{(2)}\}$ ,

$$\arg \min_{P \in \Delta(\mathcal{A}_2)} \hat{v}(P) = \left\{ Q \in \Delta(\mathcal{A}_2) : Q(a_1) = P^{(1)} \right\}; \quad (214)$$

and for all  $P \in \Delta(\mathcal{A}_2)$  such that  $\max\{P^{(1)}, P^{(2)}\} \leq P(a_1) \leq 1$ ,

$$\arg \min_{P \in \Delta(\mathcal{A}_2)} \hat{v}(P) = \left\{ Q \in \Delta(\mathcal{A}_2) : Q(a_1) = P^{(2)} \right\}. \quad (215)$$

Given the fact that the function  $f$  in (79) is continuous piecewise linear, from (81), it holds that

$$\hat{v}\left(P_{A_2}^\dagger\right) = \min_{P_{A_2} \in \Delta(\mathcal{A}_2)} \hat{v}(P_{A_2}) = \min\{\hat{v}(P_1), \hat{v}(P_2)\}, \quad (216)$$

in which for all  $i \in \{1, 2\}$ , the probability measures  $P_i \in \Delta(\mathcal{A}_2)$  are such that  $P_i(a_1) = P^{(i)}$ , with  $P^{(i)}$  in (23).

Alternatively, consider the case in which the payoff matrix  $\underline{u}$  satisfies (7). From Lemma 2.1, there is a unique NE in pure strategies or infinitely many NE in the game  $\mathcal{G}(\underline{u})$ . If there is a unique NE in pure strategies, from Lemma F.3, it holds that

$$v\left(P_{A_1|\bar{A}_2}^\dagger, P_{A_2}^\dagger\right) = u\left(P_{A_1}^*, P_{A_2}^*\right) \quad (217)$$

$$= \min\{\max\{u_{1,1}, u_{2,1}\}, \max\{u_{1,2}, u_{2,2}\}\}, \quad (218)$$

where the equality in (218) follows from (8).

If there are infinitely many NE, then there exists a probability measure  $P_{A_2}^* \in \Delta(\mathcal{A}_2)$  such that

$$P_{A_2}^* \in \{P \in \Delta(\mathcal{A}_2) : P(a_1) \in \{0, 1\}\}. \quad (219)$$

Hence, it holds that for all  $(i, j) \in \{1, 2\}^2$ ,

$$P_{A_2|\tilde{A}_2=a_i}(a_j) = \frac{P_{\tilde{A}_2|A_2=a_j}(a_i) P_{A_2}^*(a_j)}{\sum_{\ell \in \{1,2\}} P_{\tilde{A}_2|A_2=a_\ell}(a_i) P_{A_2}^*(a_\ell)} = P_{A_2}^*(a_j). \quad (220)$$

Note that the equalities in (220) imply that

$$P_{A_2|\tilde{A}_2=a_1} = P_{A_2|\tilde{A}_2=a_2} = P_{A_2}^*. \quad (221)$$

Hence, from Lemma F.3, it holds that

$$v(P_{A_1|\tilde{A}_2}^\dagger, P_{A_2}^\dagger) = u(P_{A_1}^*, P_{A_2}^*) \quad (222)$$

$$= \min\{\max\{u_{1,1}, u_{2,1}\}, \max\{u_{1,2}, u_{2,2}\}\}, \quad (223)$$

where the equality in (223) follows from (8).

This completes the proof.

## G PROOF OF LEMMA 3.5

From Lemma 2.1, if the entries of the matrix  $\underline{u}$  in (1) satisfy (4), it holds that  $P_{A_2}^*(a_1) \in (0, 1)$ . Let the tuple  $(P_{A_1|\tilde{A}_2=a_1}^\dagger, P_{A_1|\tilde{A}_2=a_2}^\dagger, P_{A_2}^\dagger) \in \Delta(\mathcal{A}_1) \times \Delta(\mathcal{A}_1) \times \Delta(\mathcal{A}_2)$  form an equilibrium of the game  $\mathcal{G}(\underline{u}, \underline{w})$  in (12). Then, from Theorem 3.2, it holds that

$$\hat{v}(P_{A_2}^\dagger) = \min\{\hat{v}(P_1), \hat{v}(P_2)\}, \quad (224)$$

where, for all  $i \in \{1, 2\}$ ,  $P_i \in \Delta(\mathcal{A}_2)$  satisfies  $P_i(a_1) = P^{(i)}$ . This proves that if the entries of the matrix  $\underline{u}$  in (1) satisfy (4), then there exists a tuple  $(Q_1, Q_2, P) \in \mathcal{S}$  such that  $P(a_1) \in \{P^{(1)}, P^{(2)}\}$ .

Note that, from (21), for all  $i \in \{1, 2\}$ , the equality  $P(a_1) = P^{(i)}$  implies

$$P_{A_2|\tilde{A}_2=a_i}(a_1) = \frac{P(a_1)P_{\tilde{A}_2|A_2=a_1}(a_i)}{P(a_1)P_{\tilde{A}_2|A_2=a_1}(a_i) + P(a_2)P_{\tilde{A}_2|A_2=a_2}(a_i)} \quad (225)$$

$$= \frac{P_{A_2}^*(a_1)P_{\tilde{A}_2|A_2=a_2}(a_i)P_{\tilde{A}_2|A_2=a_1}(a_i)}{P_{A_2}^*(a_1)P_{\tilde{A}_2|A_2=a_2}(a_i)P_{\tilde{A}_2|A_2=a_1}(a_i) + P_{A_2}^*(a_2)P_{\tilde{A}_2|A_2=a_1}(a_i)P_{\tilde{A}_2|A_2=a_2}(a_i)} \quad (226)$$

$$= \frac{P_{A_2}^*(a_1)}{P_{A_2}^*(a_2) + P_{A_2}^*(a_1)} \quad (227)$$

$$= P_{A_2}^*(a_1), \quad (228)$$

where (226) follows from (123).

This completes the proof.

## H PROOF OF LEMMA 3.6

The proof considers two cases. First, the case in which the entries in the payoff matrix  $\underline{u}$  satisfy (4). Second, the case in which the entries in the payoff matrix  $\underline{u}$  satisfy (7).

The proof of the first case is as follows. If the entries in the payoff matrix  $\underline{u}$  satisfy (4), from Lemma 2.1, it holds that

$$P_{A_2}^*(a_1) \in (0, 1). \quad (229)$$

Hence, from Lemma F.3, the equality in (32) holds if and only if  $\det \underline{w} = 0$ .

The proof of the second case is as follows. From Theorem 3.2, it follows that

$$\hat{v} \left( P_{A_2}^\dagger \right) = \min \{ \max \{ u_{1,1}, u_{2,1} \}, \max \{ u_{1,2}, u_{2,2} \} \}, \quad (230)$$

$$= u(P_{A_1}^*, P_{A_2}^*), \quad (231)$$

where the equality in (231) follows from (8). Thus, there is nothing to prove in this case.

The proof is completed by noticing that the first case and the second case form a partition of  $\mathbb{R}^{2 \times 2}$ .

## I PROOF OF LEMMA 3.7

The proof considers two cases. First, the case in which the entries in the payoff matrix  $\underline{u}$  satisfy (7). Second, the case in which the entries in the payoff matrix  $\underline{u}$  satisfy (4).

Consider the case in which the entries in the payoff matrix  $\underline{u}$  satisfy (7). From Theorem 3.2, the equality in (33) holds regardless of the value of  $\det \underline{w}$ , which ends the proof of this case.

Alternatively consider the case in which the entries in the payoff matrix  $\underline{u}$  satisfy (4). From Lemma F.1, if  $\det \underline{w} = 1$ , it holds that

$$P_{\tilde{A}_2|A_2=a_1}(a_1) = 1 \text{ and } P_{\tilde{A}_2|A_2=a_2}(a_1) = 0, \quad (232)$$

which implies that  $P^{(1)}$  and  $P^{(2)}$  in (123) satisfy

$$P^{(1)} = \frac{P_{A_2}^*(a_1)P_{\tilde{A}_2|A_2=a_2}(a_1)}{P_{A_2}^*(a_2)P_{\tilde{A}_2|A_2=a_1}(a_1) + P_{A_2}^*(a_1)P_{\tilde{A}_2|A_2=a_2}(a_1)} = 0 \quad \text{and} \quad (233)$$

$$P^{(2)} = \frac{P_{A_2}^*(a_1)P_{\tilde{A}_2|A_2=a_2}(a_2)}{P_{A_2}^*(a_2)P_{\tilde{A}_2|A_2=a_1}(a_2) + P_{A_2}^*(a_1)P_{\tilde{A}_2|A_2=a_2}(a_2)} = 1. \quad (234)$$

Hence, from Theorem 3.2, it holds that

$$\hat{v} \left( P_{A_2}^\dagger \right) = \min \{ \hat{v}(P_1), \hat{v}(P_2) \}, \quad (235)$$

$$= \min \{ \max \{ u_{1,1}, u_{2,1} \}, \max \{ u_{1,2}, u_{2,2} \} \}, \quad (236)$$

where the equality in (236) follows from (77).

On the other hand, if  $\det \underline{w} = -1$ , then it holds that

$$P_{\tilde{A}_2|A_2=a_1}(a_1) = 0 \text{ and } P_{\tilde{A}_2|A_2=a_2}(a_1) = 1, \quad (237)$$

which implies that  $P^{(1)}$  and  $P^{(2)}$  in (123) satisfy

$$P^{(1)} = \frac{P_{A_2}^*(a_1)P_{\tilde{A}_2|A_2=a_2}(a_1)}{P_{A_2}^*(a_2)P_{\tilde{A}_2|A_2=a_1}(a_1) + P_{A_2}^*(a_1)P_{\tilde{A}_2|A_2=a_2}(a_1)} = 1 \quad \text{and} \quad (238)$$

$$P^{(2)} = \frac{P_{A_2}^*(a_1)P_{\tilde{A}_2|A_2=a_2}(a_2)}{P_{A_2}^*(a_2)P_{\tilde{A}_2|A_2=a_1}(a_2) + P_{A_2}^*(a_1)P_{\tilde{A}_2|A_2=a_2}(a_2)} = 0. \quad (239)$$

Hence, from Theorem 3.2, it holds that

$$\hat{v} \left( P_{A_2}^\dagger \right) = \min \{ \hat{v}(P_1), \hat{v}(P_2) \}, \quad (240)$$

$$= \min \{ \max \{ u_{1,1}, u_{2,1} \}, \max \{ u_{1,2}, u_{2,2} \} \}, \quad (241)$$

where the equality in (241) follows from (77).

This completes the proof.

## J PROOF OF LEMMA 4.1

Let the matrix  $\underline{t} \in \mathbb{R}^{2 \times 2}$  in (34) be such that

$$\underline{t} \triangleq \begin{pmatrix} t_{1,1} & t_{1,2} \\ t_{2,1} & t_{2,2} \end{pmatrix}. \quad (242)$$

Then it holds that for all  $(i, j) \in \{1, 2\}^2$ ,  $t_{i,j} \in [0, 1]$ , and for all  $k \in \{1, 2\}$ ,

$$\sum_{\ell=1}^2 t_{k,\ell} = 1. \quad (243)$$

From (34), for a given probability measure  $P \in \Delta(\mathcal{A}_2)$ , the commitment observed by Player 1, which is denoted by  $\tilde{P}$ , satisfies

$$\begin{pmatrix} \tilde{P}(a_1) \\ \tilde{P}(a_2) \end{pmatrix} = \begin{pmatrix} t_{1,1} & t_{1,2} \\ t_{2,1} & t_{2,2} \end{pmatrix} \begin{pmatrix} P(a_1) \\ P(a_2) \end{pmatrix}, \quad (244)$$

which yields

$$\tilde{P}(a_1) = t_{1,1}P(a_1) + t_{1,2}P(a_2) \quad (245)$$

$$= (t_{1,1} - t_{1,2})P(a_1) + t_{1,2} \quad (246)$$

$$= (1 - t_{2,1} - t_{1,2})P(a_1) + t_{1,2} \quad (247)$$

$$= P(a_1) \det \underline{t} + t_{1,2}. \quad (248)$$

Hence, for  $\tilde{P}^{(1)}$  and  $\tilde{P}^{(2)}$  in (37), it holds that

$$P^{(1)} = \tilde{P}^{(1)} \det \underline{t} + t_{1,2} \quad \text{and} \quad (249)$$

$$P^{(2)} = \tilde{P}^{(2)} \det \underline{t} + t_{1,2}, \quad (250)$$

with  $P^{(i)}$  in (23), i.e. Player 2 uses strategy  $\tilde{P}^{(i)}$  but Player 1 observes  $P^{(i)}$ . Note that under the assumption that  $\det \underline{w} > 0$ , from Lemma F.6, it holds that  $P^{(1)} < P^{(2)}$ . Then under the assumption that  $\det \underline{t} > 0$ , from (249) and (250), it holds that

$$\tilde{P}^{(1)} = \frac{1}{\det \underline{t}} (P^{(1)} - t_{1,2}) \quad (251)$$

$$< \frac{1}{\det \underline{t}} (P^{(2)} - t_{1,2}) \quad (252)$$

$$= \tilde{P}^{(2)}. \quad (253)$$

For all probability measures  $P \in \Delta(\mathcal{A}_2)$  such that  $P(a_1) > \tilde{P}^{(2)}$ , from (248), it holds that

$$\tilde{P}(a_1) = P(a_1) \det \underline{t} + t_{1,2} \quad (254)$$

$$> \tilde{P}^{(2)} \det \underline{t} + t_{1,2} \quad (255)$$

$$= P^{(2)}, \quad (256)$$

and

$$\tilde{P}(a_1) = P(a_1) \det \underline{t} + t_{1,2} \quad (257)$$

$$> \tilde{P}^{(2)} \det \underline{t} + t_{1,2} \quad (258)$$

$$> \tilde{P}^{(1)} \det \underline{t} + t_{1,2} \quad (259)$$

$$= P^{(1)}, \quad (260)$$

where the equality in (256) follows from (250); the inequality in (259) follows from (253) and the fact that  $\det \underline{t} > 0$ ; and the equality in (260) follows from (249). Hence, from Lemma F.7, it holds that, for all  $i \in \{1, 2\}$ ,

$$\text{BR}_{1,i}(\tilde{P}) = \{Q_0 \in \Delta(\mathcal{A}_1) : Q_0(a_1) = 1\}. \quad (261)$$

Furthermore, from (48), it holds that

$$\tilde{v}(P) = u_{1,1}P(a_1) + u_{1,2}P(a_2), \quad (262)$$

which proves the equality in (38).

For the probability measures  $P \in \Delta(\mathcal{A}_2)$  such that  $P(a_1) = \tilde{P}^{(2)}$ , from (248), it holds that

$$\tilde{P}(a_1) = \tilde{P}^{(2)} \det \underline{t} + t_{1,2} \quad (263)$$

$$= P^{(2)}, \quad (264)$$

where (264) follows from (250). From Lemma F.7 and (253), it holds that

$$\text{BR}_{1,1}(\tilde{P}) = \{Q_0 \in \Delta(\mathcal{A}_1) : Q_0(a_1) = 1\} \quad \text{and} \quad (265)$$

$$\text{BR}_{1,2}(\tilde{P}) = \{Q_0 \in \Delta(\mathcal{A}_1) : Q_0(a_1) = \beta, \beta \in [0, 1]\}. \quad (266)$$

Furthermore, from (48), it holds that

$$\tilde{v}(P) = \left\{ u_{1,1}P(a_1)P_{\tilde{A}_2|A_2=a_1}(a_1) + u_{1,2}P(a_2)P_{\tilde{A}_2|A_2=a_2}(a_1) \right. \quad (267)$$

$$+ \left( u_{1,1}P(a_1)P_{\tilde{A}_2|A_2=a_1}(a_2) + u_{1,2}P(a_2)P_{\tilde{A}_2|A_2=a_2}(a_2) \right) \beta$$

$$\left. + \left( u_{2,1}P(a_1)P_{\tilde{A}_2|A_2=a_1}(a_2) + u_{2,2}P(a_2)P_{\tilde{A}_2|A_2=a_2}(a_2) \right) (1 - \beta) : \beta \in [0, 1] \right\} \quad (268)$$

$$= \left\{ \beta(u_{1,1}P(a_1) + u_{1,2}P(a_2)) \right. \quad (269)$$

$$+ \left( \left( u_{1,1}P(a_1)P_{\tilde{A}_2|A_2=a_1}(a_1) + u_{1,2}P(a_2)P_{\tilde{A}_2|A_2=a_2}(a_1) \right) \right.$$

$$\left. + \left( u_{2,1}P(a_1)P_{\tilde{A}_2|A_2=a_1}(a_2) + u_{2,2}P(a_2)P_{\tilde{A}_2|A_2=a_2}(a_2) \right) \right) (1 - \beta) : \beta \in [0, 1] \left. \right\} \quad (270)$$

$$= \left\{ \beta(u_{1,1}P(a_1) + u_{1,2}P(a_2)) \right. \quad (271)$$

$$+ \left( \left( u_{1,1}P_{\tilde{A}_2|A_2=a_1}(a_1) + u_{2,1}P_{\tilde{A}_2|A_2=a_1}(a_2) \right) P(a_1) \right.$$

$$\left. + \left( u_{1,2}P_{\tilde{A}_2|A_2=a_2}(a_1) + u_{2,2}P_{\tilde{A}_2|A_2=a_2}(a_2) \right) P(a_2) \right) (1 - \beta) : \beta \in [0, 1] \left. \right\}. \quad (272)$$

Note that

$$\left( u_{1,1}P_{\tilde{A}_2|A_2=a_1}(a_1) + u_{2,1}P_{\tilde{A}_2|A_2=a_1}(a_2) \right) P(a_1) + \left( u_{1,2}P_{\tilde{A}_2|A_2=a_2}(a_1) + u_{2,2}P_{\tilde{A}_2|A_2=a_2}(a_2) \right) P(a_2)$$

$$= \left( \begin{pmatrix} 1 \\ 1 \end{pmatrix}^\top (\underline{\mathbf{u}} \circ \underline{\mathbf{w}}) \begin{pmatrix} P(a_1) \\ P(a_2) \end{pmatrix} \right).$$
(273)

As a result, plugging (273) into (272) yields

$$\tilde{v}(P) = \left\{ (u_{1,1}P(a_1) + u_{1,2}P(a_2))\beta + \left( \begin{pmatrix} 1 \\ 1 \end{pmatrix}^\top (\underline{\mathbf{u}} \circ \underline{\mathbf{w}}) \begin{pmatrix} P(a_1) \\ P(a_2) \end{pmatrix} \right) (1 - \beta) : \beta \in [0, 1] \right\}. \quad (274)$$

which proves the equality in (39).

For all probability measures  $P \in \Delta(\mathcal{A}_2)$  such that  $\tilde{P}^{(1)} < P(a_1) < \tilde{P}^{(2)}$ , from (248), it hold that

$$\tilde{P}(a_1) = P(a_1) \det \underline{t} + t_{1,2} \quad (275)$$

$$> \tilde{P}^{(1)} \det \underline{t} + t_{1,2} \quad (276)$$

$$= P^{(1)} \quad (277)$$

and

$$\tilde{P}(a_1) = P(a_1) \det \underline{t} + t_{1,2} \quad (278)$$

$$< \tilde{P}^{(2)} \det \underline{t} + t_{1,2} \quad (279)$$

$$= P^{(2)}, \quad (280)$$

where (277) and (280) follow from (249) and (250), respectively. From Lemma F.7, it holds that

$$\text{BR}_{1,1}(\tilde{P}) = \{Q_0 \in \Delta(\mathcal{A}_1) : Q_0(a_1) = 1\} \quad \text{and} \quad (281)$$

$$\text{BR}_{1,2}(\tilde{P}) = \{Q_0 \in \Delta(\mathcal{A}_1) : Q_0(a_1) = 0\}. \quad (282)$$

Furthermore, from (48), it holds that

$$\begin{aligned} \tilde{v}(P) &= \left( u_{1,1}P(a_1)P_{\tilde{A}_2|A_2=a_1}(a_1) + u_{1,2}P(a_2)P_{\tilde{A}_2|A_2=a_2}(a_1) \right) \\ &\quad + \left( u_{2,1}P(a_1)P_{\tilde{A}_2|A_2=a_1}(a_2) + u_{2,2}P(a_2)P_{\tilde{A}_2|A_2=a_2}(a_2) \right) \end{aligned} \quad (283)$$

$$\begin{aligned} &= \left( u_{1,1}P_{\tilde{A}_2|A_2=a_1}(a_1) + u_{2,1}P_{\tilde{A}_2|A_2=a_1}(a_2) \right) P(a_1) \\ &\quad + \left( u_{1,2}P_{\tilde{A}_2|A_2=a_2}(a_1) + u_{2,2}P_{\tilde{A}_2|A_2=a_2}(a_2) \right) P(a_2), \end{aligned} \quad (284)$$

which, from (273), proves the equality in (40).

For the probability measures  $P \in \Delta(\mathcal{A}_2)$  such that  $P(a_1) = \tilde{P}^{(1)}$ , from (248), it holds that

$$\tilde{P}(a_1) = \tilde{P}^{(1)} \det \underline{t} + t_{1,2} \quad (285)$$

$$= P^{(1)}, \quad (286)$$

where (286) follows from (250). From Lemma F.7 and (253), it holds that

$$\text{BR}_{1,1}(\tilde{P}) = \{Q_0 \in \Delta(\mathcal{A}_1) : Q_0(a_1) = \beta, \beta \in [0, 1]\} \quad \text{and} \quad (287)$$

$$\text{BR}_{1,2}(\tilde{P}) = \{Q_0 \in \Delta(\mathcal{A}_1) : Q_0(a_1) = 0\}. \quad (288)$$

Furthermore, from (48), it holds that

$$\tilde{v}(P) = \left\{ \left( u_{1,1}P(a_1)P_{\tilde{A}_2|A_2=a_1}(a_1) + u_{1,2}P(a_2)P_{\tilde{A}_2|A_2=a_2}(a_1) \right) \beta \right. \quad (289)$$

$$\begin{aligned} &\quad + \left( u_{2,1}P(a_1)P_{\tilde{A}_2|A_2=a_1}(a_2) + u_{2,2}P(a_2)P_{\tilde{A}_2|A_2=a_2}(a_2) \right) (1 - \beta) \\ &\quad \left. + u_{2,1}P(a_1)P_{\tilde{A}_2|A_2=a_1}(a_2) + u_{2,2}P(a_2)P_{\tilde{A}_2|A_2=a_2}(a_2) : \beta \in [0, 1] \right\} \end{aligned} \quad (290)$$

$$= \left\{ (1 - \beta) \left( u_{2,1}P(a_1) + u_{2,2}P(a_2) \right) \right. \quad (291)$$

$$\begin{aligned} &\quad + \left( \left( u_{1,1}P(a_1)P_{\tilde{A}_2|A_2=a_1}(a_1) + u_{1,2}P(a_2)P_{\tilde{A}_2|A_2=a_2}(a_1) \right) \right. \\ &\quad \left. + \left( u_{2,1}P(a_1)P_{\tilde{A}_2|A_2=a_1}(a_2) + u_{2,2}P(a_2)P_{\tilde{A}_2|A_2=a_2}(a_2) \right) \right) \beta : \beta \in [0, 1] \left. \right\} \end{aligned} \quad (292)$$

$$= \left\{ (1 - \beta) \left( u_{2,1}P(a_1) + u_{2,2}P(a_2) \right) \right. \quad (293)$$

$$\begin{aligned} &\quad + \left( \left( u_{1,1}P_{\tilde{A}_2|A_2=a_1}(a_1) + u_{2,1}P_{\tilde{A}_2|A_2=a_1}(a_2) \right) P(a_1) \right. \\ &\quad \left. + \left( u_{1,2}P_{\tilde{A}_2|A_2=a_2}(a_1) + u_{2,2}P_{\tilde{A}_2|A_2=a_2}(a_2) \right) P(a_2) \right) \beta : \beta \in [0, 1] \left. \right\}, \end{aligned} \quad (294)$$

which, from (273), proves the equality in (41).

For all probability measures  $P \in \Delta(\mathcal{A}_2)$  such that  $P(a_1) < \tilde{P}^{(1)}$ , from (248), it holds that

$$\tilde{P}(a_1) = P(a_1) \det \underline{t} + t_{1,2} \quad (295)$$

$$< \tilde{P}^{(1)} \det \underline{t} + t_{1,2} \quad (296)$$

$$= P^{(1)} \quad (297)$$

and

$$\tilde{P}(a_1) = P(a_1) \det \underline{t} + t_{1,2} \quad (298)$$

$$< \tilde{P}^{(1)} \det \underline{t} + t_{1,2} \quad (299)$$

$$< \tilde{P}^{(2)} \det \underline{t} + t_{1,2} \quad (300)$$

$$= P^{(2)}, \quad (301)$$

where the equality in (297) follows from (249); the inequality in (300) follows from (253) and the fact that  $\det \underline{t} > 0$ ; and the equality in (301) follows from (250). From Lemma F.7, it holds that for all  $i \in \{1, 2\}$ ,

$$\text{BR}_{1,i}(\tilde{P}) = \{Q_0 \in \Delta(\mathcal{A}_1) : Q_0(a_1) = 0\}. \quad (302)$$

Furthermore, from (48), it holds that

$$\tilde{v}(P) = u_{2,1}P(a_1) + u_{2,2}P(a_2), \quad (303)$$

which proves the equality in (42).

This complete the proof.

## K PROOF OF LEMMA 4.2

The proof is divided into two parts. First, a preliminary result is introduced in Subsection K.1. The proof of the theorem is presented in Subsection K.2.

### K.1 Preliminary

**Lemma K.1** *Assume that the matrix  $\underline{u}$  in (1) satisfies (4). Let  $P_{A_1}^* \in \Delta(\mathcal{A}_1)$  and  $P_{A_2}^* \in \Delta(\mathcal{A}_2)$  form a NE in the game  $(\underline{u})$  in (2). Then for all  $P \in \Delta(\mathcal{A}_2)$ , the function  $\hat{u}$  in (24) satisfies:*

$$\hat{u}(P) = \begin{cases} u_{2,1}P(a_1) + u_{2,2}P(a_2), & \text{if } P(a_1) < P_{A_2}^*(a_1) \text{ and } u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2} > 0 \text{ or} \\ & P(a_1) > P_{A_2}^*(a_1) \text{ and } u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2} \leq 0, \\ u_{1,1}P(a_1) + u_{1,2}P(a_2), & \text{if } P(a_1) > P_{A_2}^*(a_1) \text{ and } u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2} > 0 \text{ or} \\ & P(a_1) < P_{A_2}^*(a_1) \text{ and } u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2} \leq 0, \\ u(P_{A_1}^*, P_{A_2}^*), & \text{if } P(a_1) = P_{A_2}^*(a_1), \end{cases} \quad (304)$$

with  $P_{A_2}^*(a_1)$  in (5b).

*Proof:* For all  $Q \in \Delta(\mathcal{A}_1)$  and for all  $P \in \Delta(\mathcal{A}_2)$ , the function  $u$  in (3) satisfies

$$u(Q, P) = \left( Q(a_1), 1 - Q(a_1) \right) \underline{u} \left( P(a_1), 1 - P(a_1) \right)^\top \quad (305)$$

$$= \left( Q(a_1)u_{1,1} + (1 - Q(a_1))u_{2,1}, Q(a_1)u_{1,2} + (1 - Q(a_1))u_{2,2} \right) \left( P(a_1), 1 - P(a_1) \right)^\top \quad (306)$$

$$= u_{1,1}Q(a_1)P(a_1) + u_{2,1}P(a_1) - u_{2,1}Q(a_1)P(a_1) \\ + u_{1,2}Q(a_1) - u_{1,2}Q(a_1)P(a_1) + u_{2,2}(1 - Q(a_1) - P(a_1) + Q(a_1)P(a_1)) \quad (307)$$

$$= (u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2})Q(a_1)P(a_1) + (u_{1,2} - u_{2,2})Q(a_1) + (u_{2,1} - u_{2,2})P(a_1) + u_{2,2} \quad (308)$$

$$= \left( (u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2})P(a_1) + (u_{1,2} - u_{2,2}) \right) Q(a_1) + (u_{2,1} - u_{2,2})P(a_1) + u_{2,2}. \quad (309)$$

Hence, for all  $P \in \Delta(\mathcal{A}_2)$ , from (309), it holds that

$$\max_{Q \in \Delta(\mathcal{A}_1)} u(Q, P) = \max_{Q \in \Delta(\mathcal{A}_1)} \left( (u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2})P(a_1) + (u_{1,2} - u_{2,2}) \right) Q(a_1). \quad (310)$$

Under the assumption that the matrix  $\underline{u}$  in (1) satisfies (4), it follows that  $u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2} \neq 0$ . Hence, the equality in (310) can be rewritten as follows:

$$\max_{Q \in \Delta(\mathcal{A}_1)} u(Q, P) = \max_{Q \in \Delta(\mathcal{A}_1)} \left( u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2} \right) \left( P(a_1) - P_{A_2}^*(a_1) \right) Q(a_1), \quad (311)$$

where  $P_{A_2}^*(a_1)$  is given in (5b). For all  $P \in \Delta(\mathcal{A}_2)$ , it follows from (311) that:

$$\arg \max_{V \in \Delta(\mathcal{A}_1)} u(V, P) = \begin{cases} \{Q \in \Delta(\mathcal{A}_1) : Q(a_1) = 0\}, & \text{if } u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2} > 0 \text{ and } P(a_1) < P_{A_2}^*(a_1), \text{ or} \\ & u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2} < 0 \text{ and } P(a_1) > P_{A_2}^*(a_1), \\ \{Q \in \Delta(\mathcal{A}_1) : Q(a_1) = 1\}, & \text{if } u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2} > 0 \text{ and } P(a_1) > P_{A_2}^*(a_1), \text{ or} \\ & u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2} < 0 \text{ and } P(a_1) < P_{A_2}^*(a_1), \\ \{Q \in \Delta(\mathcal{A}_1) : Q(a_1) = \beta, \beta \in [0, 1]\}, & \text{if } P(a_1) = P_{A_2}^*(a_1). \end{cases} \quad (312)$$

If given a  $P \in \Delta(\mathcal{A}_2)$ , it holds that  $\arg \max_{V \in \Delta(\mathcal{A}_1)} u(V, P) = \{Q \in \Delta(\mathcal{A}_1) : Q(a_1) = 0\}$ , then, from (309), it follows that

$$\hat{u}(P) = u_{2,1}P(a_1) + u_{2,2}P(a_2). \quad (313)$$

If given a  $P \in \Delta(\mathcal{A}_2)$ , it holds that  $\arg \max_{V \in \Delta(\mathcal{A}_1)} u(V, P) = \{Q \in \Delta(\mathcal{A}_1) : Q(a_1) = 1\}$ , then, from (309), it follows that

$$\hat{u}(P) = u_{1,1}P(a_1) + u_{1,2}P(a_2). \quad (314)$$

If  $P = P_{A_2}^*$ , it holds that  $\arg \max_{V \in \Delta(\mathcal{A}_1)} u(V, P) = \{Q \in \Delta(\mathcal{A}_1) : Q(a_1) = \beta, \beta \in [0, 1]\}$ , then, from (309) and (311) follows that

$$\hat{u}(P_{A_2}^*) = \max_{Q \in \Delta(\mathcal{A}_1)} (u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2}) (P_{A_2}^*(a_1) - P_{A_2}^*(a_1)) Q(a_1) + (u_{2,1} - u_{2,2}) P_{A_2}^*(a_1) + u_{2,2} \quad (315)$$

$$= u_{2,1} P_{A_2}^*(a_1) + u_{2,2} P_{A_2}^*(a_2). \quad (316)$$

Plugging (5b) into (316) yields

$$u_{2,1} P_{A_2}^*(a_1) + u_{2,2} P_{A_2}^*(a_2) = (u_{2,1} - u_{2,2}) P_{A_2}^*(a_1) + u_{2,2} \quad (317)$$

$$= (u_{2,1} - u_{2,2}) \frac{u_{2,2} - u_{1,2}}{u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2}} + u_{2,2} \quad (318)$$

$$= \frac{(u_{2,1} - u_{2,2})(u_{2,2} - u_{1,2}) + u_{2,2}(u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2})}{u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2}} \quad (319)$$

$$= \frac{u_{1,1}u_{2,2} - u_{1,2}u_{2,1}}{u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2}}. \quad (320)$$

Note that plugging  $P(a_1) = P_{A_2}^*(a_1)$  into (314) also yields

$$u_{1,1} P_{A_2}^*(a_1) + u_{1,2} P_{A_2}^*(a_2) = (u_{1,1} - u_{1,2}) P_{A_2}^*(a_1) + u_{1,2} \quad (321)$$

$$= (u_{1,1} - u_{1,2}) \frac{u_{2,2} - u_{1,2}}{u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2}} + u_{1,2} \quad (322)$$

$$= \frac{(u_{1,1} - u_{1,2})(u_{2,2} - u_{1,2}) + u_{1,2}(u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2})}{u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2}} \quad (323)$$

$$= \frac{u_{1,1}u_{2,2} - u_{1,2}u_{2,1}}{u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2}}, \quad (324)$$

which, with (320), implies that the function  $\hat{u}$  satisfies

$$\hat{u}(P_{A_2}^*) = u_{2,1} P_{A_2}^*(a_1) + u_{2,2} P_{A_2}^*(a_2) = u_{1,1} P_{A_2}^*(a_1) + u_{1,2} P_{A_2}^*(a_2). \quad (325)$$

In a nutshell, under the assumption of the lemma, for all  $P \in \Delta(\mathcal{A}_2)$ , it holds that

$$\hat{u}(P) = \begin{cases} u_{2,1}P(a_1) + u_{2,2}P(a_2), & \text{if } P(a_1) < P_{A_2}^*(a_1) \text{ and } u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2} > 0 \text{ or} \\ & P(a_1) > P_{A_2}^*(a_1) \text{ and } u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2} < 0, \\ u_{1,1}P(a_1) + u_{1,2}P(a_2), & \text{if } P(a_1) > P_{A_2}^*(a_1) \text{ and } u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2} > 0 \text{ or} \\ & P(a_1) < P_{A_2}^*(a_1) \text{ and } u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2} < 0, \\ u(P_{A_1}^*, P_{A_2}^*), & \text{if } P(a_1) = P_{A_2}^*(a_1). \end{cases} \quad (326)$$

The proof is completed by noticing that if the matrix  $\underline{u}$  in (1) satisfies (4), it holds that  $u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2} \neq 0$ . ■



## K.2 Main Proof

The proof is divided into two parts. The first part provides the idea of the proof. The second part proves the lemma.

The first part of the proof is as follows. If Player 1 does not get access to the exact commitment and there is a unique  $P_{A_2}^\dagger$  in the game  $\mathcal{G}(\underline{u}, \underline{w})$  in (12), then there exists a probability measure  $P \in \Delta(\mathcal{A}_2)$  such that with  $i \in \{1, 2\}$ ,

$$\text{BR}_{1,i}(\tilde{P}) \neq \text{BR}_{1,i}(P), \quad (327)$$

where  $\tilde{P}$  satisfies

$$\begin{pmatrix} \tilde{P}(a_1) \\ \tilde{P}(a_2) \end{pmatrix} = \begin{pmatrix} t_{1,1} & t_{1,2} \\ t_{2,1} & t_{2,2} \end{pmatrix} \begin{pmatrix} P(a_1) \\ P(a_2) \end{pmatrix}. \quad (328)$$

Hence, for such a probability measure  $P \in \Delta(\mathcal{A}_2)$ , it holds that

$$\tilde{v}(P) < \hat{v}(P), \quad (329)$$

which, from the decreasing or increasing property of function  $\hat{v}$  in Lemma F.10, yields

$$\tilde{v}(P) < \hat{v}(P_{A_2}^\dagger). \quad (330)$$

The second part of the proof is as follows. If the entries of the matrix  $\underline{u}$  in (1) satisfy (4), i.e. condition (a) holds, from Lemma 2.1, there exists a unique NE in strictly mixed strategies. For this case, from Lemma F.5, it holds that

$$0 \leq \min \{P^{(1)}, P^{(2)}\} \leq P_{A_2}^*(a_1) \leq \max \{P^{(1)}, P^{(2)}\} \leq 1. \quad (331)$$

Furthermore, from Lemma K.1, Lemma F.8, and Lemma F.9, for all  $P \in \Delta(\mathcal{A}_2)$  such that

$$P(a_1) \in \left[0, \min\{P^{(1)}, P^{(2)}\}\right] \cup \left[\max\{P^{(1)}, P^{(2)}\}, 1\right], \quad (332)$$

it holds that

$$\hat{u}(P) = \hat{v}(P). \quad (333)$$

Hence, if  $\hat{u}(Q_1) \neq \hat{u}(Q_2)$ , i.e. condition (b) holds, then it holds that

$$\hat{v}(Q_1) \neq \hat{v}(Q_2), \quad (334)$$

which, from Lemma F.6, further implies that  $\det \underline{w} \neq 0$ . Given the fact that the function  $f$  in (79) is continuous piecewise linear, from (80), the inequality in (334) implies

$$u_{1,1}P_{\tilde{A}_2|A_2=a_1}(a_1) + u_{2,1}P_{\tilde{A}_2|A_2=a_1}(a_2) \neq u_{1,2}P_{\tilde{A}_2|A_2=a_2}(a_1) + u_{2,2}P_{\tilde{A}_2|A_2=a_2}(a_2) \quad \text{and} \quad (335)$$

$$u_{1,1}P_{\tilde{A}_2|A_2=a_1}(a_2) + u_{2,1}P_{\tilde{A}_2|A_2=a_1}(a_1) \neq u_{1,2}P_{\tilde{A}_2|A_2=a_2}(a_2) + u_{2,2}P_{\tilde{A}_2|A_2=a_2}(a_1), \quad (336)$$

where the inequality in (335) follows from (173) and (190); and the inequality in (336) follows from (174) and (189). Hence, from Lemma F.8 and Lemma F.9, for all tuples  $(P, Q) \in \Delta(\mathcal{A}_2) \times \Delta(\mathcal{A}_2)$  such that  $\min \{P^{(1)}, P^{(2)}\} \leq P(a_1) < Q(a_1) \leq \max \{P^{(1)}, P^{(2)}\}$ , one of the following inequalities holds:

$$\hat{v}(P) < \hat{v}(Q) \quad \text{or} \quad (337)$$

$$\hat{v}(P) > \hat{v}(Q). \quad (338)$$

Then from Lemma F.10, it holds that there exist a unique solution to the optimization problem given by

$$\min_{P \in \Delta(\mathcal{A}_2)} \hat{v}(P), \quad (339)$$

which, from Definition 4.1, implies that *there is a unique*  $P_{A_2}^\dagger$ .

Note that, similarly as Lemma F.1, for the determinant of matrix  $\underline{t}$ , it holds that

$$\det \underline{t} = t_{1,1} - t_{1,2} = t_{2,2} - t_{2,1}. \quad (340)$$

Then if  $\det \underline{t} \notin \{0, 1\}$ , then it holds that

$$\underline{t} \neq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, t_{1,1} \neq t_{1,2}, \text{ and } t_{2,2} \neq t_{2,1}, \quad (341)$$

which, from (249) and (250), implies that

$$\tilde{P}^{(1)} \neq P^{(1)} \quad \text{and} \quad \tilde{P}^{(2)} \neq P^{(2)}. \quad (342)$$

This implies that Player 1 cannot get the exact commitment  $P_{A_2}$ .

In the following proof, without loss of generality, assume that  $P_{A_2}^\dagger(a_1) = P^{(1)}$  and  $u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2} > 0$ . And the proof considers the following cases:

- Case I:  $\det \underline{w} > 0$ , and  $\det \underline{t} > 0$ ;
- Case II:  $\det \underline{w} > 0$ , and  $\det \underline{t} < 0$ ;
- Case III:  $\det \underline{w} < 0$ , and  $\det \underline{t} < 0$ ;
- Case IV:  $\det \underline{w} < 0$ , and  $\det \underline{t} > 0$ .

### K.2.1 Case I

The proof is divided into three parts. The first part proves the value of  $\tilde{v}(P)$  for a given probability measure  $P \in \Delta(\mathcal{A}_2)$  such that  $P(a_1) \in [0, \tilde{P}^{(1)}) \cup (\tilde{P}^{(1)}, \tilde{P}^{(2)}) \cup (\tilde{P}^{(2)}, 1]$ ; the second part proves the decreasing or increasing property of  $\tilde{v}$ ; the third part proves that there exists a probability measure  $P \in \Delta(\mathcal{A}_2)$  such that the inequality  $\tilde{v}(P) < \hat{v}(P_{A_2}^\dagger)$  holds.

**The first part is as follows.** If  $\det \underline{w} > 0$ , from Lemma F.6, it holds that

$$P^{(1)} < P^{(2)}. \quad (343)$$

Furthermore, under the assumption that  $\det \underline{t} > 0$ , from (249) and (250), it holds that

$$\tilde{P}^{(1)} = \frac{1}{\det \underline{t}} (P^{(1)} - t_{1,2}) \quad (344)$$

$$< \frac{1}{\det \underline{t}} (P^{(2)} - t_{1,2}) \quad (345)$$

$$= \tilde{P}^{(2)}, \quad (346)$$

where the inequality in (345) follows from (343).

First, for all  $P \in \Delta(\mathcal{A}_2)$  such that  $P(a_1) > \tilde{P}^{(2)}$ , from (248), it holds that

$$\tilde{P}(a_1) = \det \underline{t} P(a_1) + t_{1,2} \quad (347)$$

$$> \det \underline{t} \tilde{P}^{(2)} + t_{1,2} \quad (348)$$

$$= P^{(2)} \quad (349)$$

and

$$\tilde{P}(a_1) = \det \underline{t} P(a_1) + t_{1,2} \quad (350)$$

$$> \det \underline{t} \tilde{P}^{(2)} + t_{1,2} \quad (351)$$

$$> \det \underline{t} \tilde{P}^{(1)} + t_{1,2} \quad (352)$$

$$= P^{(1)}, \quad (353)$$

where the inequality in (352) follows from (346). Hence, under the assumption that  $u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2} > 0$ , from Lemma F.7, it holds that for all  $i \in \{1, 2\}$ ,

$$\text{BR}_{1,i}(\tilde{P}) = \{Q \in \Delta(\mathcal{A}_1) : Q(a_1) = 1\}. \quad (354)$$

Then, from (36) and (48), the function  $\tilde{v}$  satisfies

$$\tilde{v}(P) = u_{1,1}P(a_1) + u_{1,2}P(a_2). \quad (355)$$

Second, for all  $P \in \Delta(\mathcal{A}_2)$  such that  $\tilde{P}^{(1)} < P(a_1) < \tilde{P}^{(2)}$ , from (248), it holds that

$$\tilde{P}(a_1) = \det \underline{t}P(a_1) + t_{1,2} \quad (356)$$

$$> \det \underline{t}\tilde{P}^{(1)} + t_{1,2} \quad (357)$$

$$= P^{(1)} \quad (358)$$

and

$$\tilde{P}(a_1) = \det \underline{t}P(a_1) + t_{1,2} \quad (359)$$

$$< \det \underline{t}\tilde{P}^{(2)} + t_{1,2} \quad (360)$$

$$= P^{(2)}. \quad (361)$$

Hence, under the assumption that  $u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2} > 0$ , from Lemma F.7, it holds that

$$\text{BR}_{1,1}(\tilde{P}) = \{Q \in \Delta(\mathcal{A}_1) : Q(a_1) = 1\} \quad \text{and} \quad (362)$$

$$\text{BR}_{1,2}(\tilde{P}) = \{Q \in \Delta(\mathcal{A}_1) : Q(a_1) = 0\}. \quad (363)$$

Then, from (36) and (48), the function  $\tilde{v}$  satisfies

$$\tilde{v}(P) = \left( u_{1,1}P_{\tilde{A}_2|A_2=a_1}(a_1) + u_{2,1}P_{\tilde{A}_2|A_2=a_1}(a_2) \right) P(a_1) + \left( u_{1,2}P_{\tilde{A}_2|A_2=a_2}(a_1) + u_{2,2}P_{\tilde{A}_2|A_2=a_2}(a_2) \right) P(a_2). \quad (364)$$

Last, for all  $P \in \Delta(\mathcal{A}_2)$  such that  $P(a_1) < \tilde{P}^{(1)}$ , from (248), it holds that

$$\tilde{P}(a_1) = \det \underline{t}P(a_1) + t_{1,2} \quad (365)$$

$$< \det \underline{t}\tilde{P}^{(1)} + t_{1,2} \quad (366)$$

$$= P^{(1)} \quad (367)$$

and

$$\tilde{P}(a_1) = \det \underline{t}P(a_1) + t_{1,2} \quad (368)$$

$$< \det \underline{t}\tilde{P}^{(1)} + t_{1,2} \quad (369)$$

$$< \det \underline{t}\tilde{P}^{(2)} + t_{1,2} \quad (370)$$

$$= P^{(2)}, \quad (371)$$

where the inequality in (370) follows from (346). Hence, under the assumption that  $u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2} > 0$ , from Lemma F.7, it holds that for all  $i \in \{1, 2\}$ ,

$$\text{BR}_{1,i}(\tilde{P}) = \{Q \in \Delta(\mathcal{A}_1) : Q(a_1) = 0\}. \quad (372)$$

Then, from (36) and (48), the function  $\tilde{v}$  satisfies

$$\tilde{v}(P) = u_{2,1}P(a_1) + u_{2,2}P(a_2). \quad (373)$$

In a nutshell, from (355), (364), and (373), the function  $\tilde{v}$  satisfies: for all  $P \in \Delta(\mathcal{A}_2)$  such that  $P(a_1) > \tilde{P}^{(2)}$ ,

$$\tilde{v}(P) = u_{1,1}P(a_1) + u_{1,2}P(a_2); \quad (374)$$

for all  $P \in \Delta(\mathcal{A}_2)$  such that  $\tilde{P}^{(1)} < P(a_1) < \tilde{P}^{(2)}$ ,

$$\tilde{v}(P) = \left( u_{1,1}P_{\tilde{A}_2|A_2=a_1}(a_1) + u_{2,1}P_{\tilde{A}_2|A_2=a_1}(a_2) \right) P(a_1) + \left( u_{1,2}P_{\tilde{A}_2|A_2=a_2}(a_1) + u_{2,2}P_{\tilde{A}_2|A_2=a_2}(a_2) \right) P(a_2); \quad (375)$$

for all  $P \in \Delta(\mathcal{A}_2)$  such that  $P(a_1) < \tilde{P}^{(1)}$ ,

$$\tilde{v}(P) = u_{2,1}P(a_1) + u_{2,2}P(a_2). \quad (376)$$

**The second part is as follows.** Given the fact that  $P_{A_2}^\dagger$ , with  $P_{A_2}^\dagger(a_1) = P^{(1)}$ , is the unique solution to

$$\min_{P \in \Delta(\mathcal{A}_2)} \hat{v}(P), \quad (377)$$

from Lemma F.8, it holds that for all tuples  $(P, Q) \in \Delta(\mathcal{A}_2) \times \Delta(\mathcal{A}_2)$  such that  $P^{(1)} \leq P(a_1) < Q(a_1) \leq P^{(2)}$ ,

$$\hat{v}(P) < \hat{v}(Q). \quad (378)$$

Then from Lemma F.10, it holds that for all tuples  $(P, Q) \in \Delta(\mathcal{A}_2) \times \Delta(\mathcal{A}_2)$  such that  $0 \leq P(a_1) < Q(a_1) \leq P^{(1)}$ ,

$$\hat{v}(P) > \hat{v}(Q); \quad (379)$$

and from Lemma F.10 and (378), it holds for all tuples  $(P, Q) \in \Delta(\mathcal{A}_2) \times \Delta(\mathcal{A}_2)$  such that  $P^{(1)} \leq P(a_1) < Q(a_1) \leq 1$ ,

$$\hat{v}(P) < \hat{v}(Q). \quad (380)$$

Note that, from Lemma F.8, the inequality in (379) implies that

$$u_{2,1} < u_{2,2}; \quad (381)$$

and the inequality in (380) implies that

$$u_{1,1}P_{\tilde{A}_2|A_2=a_1}(a_1) + u_{2,1}P_{\tilde{A}_2|A_2=a_1}(a_2) > u_{1,2}P_{\tilde{A}_2|A_2=a_2}(a_1) + u_{2,2}P_{\tilde{A}_2|A_2=a_2}(a_2), \quad \text{and} \quad (382)$$

$$u_{1,1} > u_{1,2}. \quad (383)$$

Hence, from (376) and (381), it holds that for all tuples  $(P, Q) \in \Delta(\mathcal{A}_2) \times \Delta(\mathcal{A}_2)$  such that  $0 \leq P(a_1) < Q(a_1) < \tilde{P}^{(1)}$ ,

$$\tilde{v}(P) > \tilde{v}(Q); \quad (384)$$

from (375) and (382), it holds that for all tuples  $(P, Q) \in \Delta(\mathcal{A}_2) \times \Delta(\mathcal{A}_2)$  such that  $\tilde{P}^{(1)} < P(a_1) < Q(a_1) < \tilde{P}^{(2)}$ ,

$$\tilde{v}(P) < \tilde{v}(Q); \quad (385)$$

from (374) and (383), it holds that for all tuples  $(P, Q) \in \Delta(\mathcal{A}_2) \times \Delta(\mathcal{A}_2)$  such that  $\tilde{P}^{(2)} < P(a_1) < Q(a_1) \leq 1$ ,

$$\tilde{v}(P) < \tilde{v}(Q). \quad (386)$$

**The third part is as follows.** Given the fact that  $P_{A_2}^\dagger(a_1) = P^{(1)}$  and the fact that  $P^{(1)} < P^{(2)}$ , from Lemma F.8, it holds that

$$\hat{v}(P_{A_2}^\dagger) = u_{2,1}P^{(1)} + u_{2,2}(1 - P^{(1)}) \quad (387)$$

$$= \left( u_{1,1}P_{\tilde{A}_2|A_2=a_1}(a_1) + u_{2,1}P_{\tilde{A}_2|A_2=a_1}(a_2) \right) P^{(1)} + \left( u_{1,2}P_{\tilde{A}_2|A_2=a_2}(a_1) + u_{2,2}P_{\tilde{A}_2|A_2=a_2}(a_2) \right) (1 - P^{(1)}), \quad (388)$$

where the equality in (388) follows from (76).

Assume that  $P^{(1)} \in (\tilde{P}^{(1)}, \tilde{P}^{(2)})$ . As a result, for all  $P \in \Delta(\mathcal{A}_2)$  such that  $P(a_1) \in (\tilde{P}^{(1)}, P^{(1)})$ , from (375), it holds that

$$\tilde{v}(P) = \left( u_{1,1}P_{\tilde{A}_2|A_2=a_1}(a_1) + u_{2,1}P_{\tilde{A}_2|A_2=a_1}(a_2) \right) P(a_1) + \left( u_{1,2}P_{\tilde{A}_2|A_2=a_2}(a_1) + u_{2,2}P_{\tilde{A}_2|A_2=a_2}(a_2) \right) P(a_2) \quad (389)$$

$$< \left( u_{1,1}P_{\tilde{A}_2|A_2=a_1}(a_1) + u_{2,1}P_{\tilde{A}_2|A_2=a_1}(a_2) \right) P^{(1)} + \left( u_{1,2}P_{\tilde{A}_2|A_2=a_2}(a_1) + u_{2,2}P_{\tilde{A}_2|A_2=a_2}(a_2) \right) (1 - P^{(1)}) \quad (390)$$

$$= \hat{v}(P_{A_2}^\dagger), \quad (391)$$

where the inequality in (390) follows from (385) and the equality in (391) follows from (388).

Assume  $P^{(1)} \notin (\tilde{P}^{(1)}, \tilde{P}^{(2)})$ . If  $P^{(1)} < \tilde{P}^{(1)}$ , for all  $P \in \Delta(\mathcal{A}_2)$  such that  $P(a_1) \in (P^{(1)}, \tilde{P}^{(1)})$ , from (376), it holds that

$$\tilde{v}(P) = u_{2,1}P(a_1) + u_{2,2}P(a_2) \quad (392)$$

$$< u_{2,1}P^{(1)} + u_{2,2}(1 - P^{(1)}) \quad (393)$$

$$= \hat{v}(P_{A_2}^\dagger), \quad (394)$$

where the inequality in (393) follows from (384) and the equality in (394) follows from (387). Alternatively, if  $P^{(1)} > \tilde{P}^{(1)}$ , for all  $P \in \Delta(\mathcal{A}_2)$  such that  $P(a_1) \in (\tilde{P}^{(1)}, \tilde{P}^{(2)})$ , from (375), it holds that

$$\tilde{v}(P) = \left( u_{1,1} P_{\tilde{A}_2|A_2=a_1}(a_1) + u_{2,1} P_{\tilde{A}_2|A_2=a_1}(a_2) \right) P(a_1) + \left( u_{1,2} P_{\tilde{A}_2|A_2=a_2}(a_1) + u_{2,2} P_{\tilde{A}_2|A_2=a_2}(a_2) \right) P(a_2) \quad (395)$$

$$< \left( u_{1,1} P_{\tilde{A}_2|A_2=a_1}(a_1) + u_{2,1} P_{\tilde{A}_2|A_2=a_1}(a_2) \right) P^{(1)} + \left( u_{1,2} P_{\tilde{A}_2|A_2=a_2}(a_1) + u_{2,2} P_{\tilde{A}_2|A_2=a_2}(a_2) \right) (1 - P^{(1)}) \quad (396)$$

$$= \hat{v}(P_{A_2}^\dagger), \quad (397)$$

where the inequality in (396) follows from (382) and the fact that if  $P^{(1)} > \tilde{P}^{(1)}$  and  $P^{(1)} \notin (\tilde{P}^{(1)}, \tilde{P}^{(2)})$ , then it holds that  $P^{(1)} > \tilde{P}^{(2)}$ ; and the equality in (397) follows from (388).

### K.2.2 Case II

The proof is divided into three parts. The first part proves the value of  $\tilde{v}(P)$  for a given probability measure  $P \in \Delta(\mathcal{A}_2)$  such that  $P(a_1) \in [0, \tilde{P}^{(2)}) \cup (\tilde{P}^{(2)}, \tilde{P}^{(1)}) \cup (\tilde{P}^{(2)}, 1]$ ; the second part proves the decreasing or increasing property of  $\tilde{v}$ ; the third part proves that there exists a probability measure  $P \in \Delta(\mathcal{A}_2)$  such that the inequality  $\tilde{v}(P) < \hat{v}(P_{A_2}^\dagger)$  holds.

**The first part is as follows.** If  $\det \underline{u} > 0$ , from Lemma F.6, it holds that

$$P^{(1)} < P^{(2)}. \quad (398)$$

Furthermore, under the assumption that  $\det \underline{t} < 0$ , from (249) and (250), it holds that

$$\tilde{P}^{(1)} = \frac{1}{\det \underline{t}} \left( P^{(1)} - t_{1,2} \right) \quad (399)$$

$$> \frac{1}{\det \underline{t}} \left( P^{(2)} - t_{1,2} \right) \quad (400)$$

$$= \tilde{P}^{(2)}, \quad (401)$$

where the inequality in (400) follows from (398).

First, for all  $P \in \Delta(\mathcal{A}_2)$  such that  $P(a_1) < \tilde{P}^{(2)}$ , from (248), it holds that

$$\tilde{P}(a_1) = \det \underline{t} P(a_1) + t_{1,2} \quad (402)$$

$$> \det \underline{t} \tilde{P}^{(2)} + t_{1,2} \quad (403)$$

$$= P^{(2)} \quad (404)$$

and

$$\tilde{P}(a_1) = \det \underline{t} P(a_1) + t_{1,2} \quad (405)$$

$$> \det \underline{t} \tilde{P}^{(2)} + t_{1,2} \quad (406)$$

$$> \det \underline{t} \tilde{P}^{(1)} + t_{1,2} \quad (407)$$

$$= P^{(1)}, \quad (408)$$

where the inequality in (407) follows from (401) and the fact that  $\det \underline{t} < 0$ .

Hence, under the assumption that  $u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2} > 0$ , from Lemma F.7, it holds that for all  $i \in \{1, 2\}$ ,

$$\text{BR}_{1,i}(\tilde{P}) = \{Q \in \Delta(\mathcal{A}_1) : Q(a_1) = 1\}. \quad (409)$$

Then, from (36) and (48), the function  $\tilde{v}$  satisfies

$$\tilde{v}(P) = u_{1,1} P(a_1) + u_{1,2} P(a_2). \quad (410)$$

Second, for all  $P \in \Delta(\mathcal{A}_2)$  such that  $\tilde{P}^{(2)} < P(a_1) < \tilde{P}^{(1)}$ , from (248), it holds that

$$\tilde{P}(a_1) = \det \underline{t} P(a_1) + t_{1,2} \quad (411)$$

$$> \det \underline{t} \tilde{P}^{(1)} + t_{1,2} \quad (412)$$

$$= P^{(1)} \quad (413)$$

and

$$\tilde{P}(a_1) = \det \underline{t}P(a_1) + t_{1,2} \quad (414)$$

$$< \det \underline{t}\tilde{P}^{(2)} + t_{1,2} \quad (415)$$

$$= P^{(2)}. \quad (416)$$

Hence, under the assumption that  $u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2} > 0$ , from Lemma F.7, it holds that

$$\text{BR}_{1,1}(\tilde{P}) = \{Q \in \Delta(\mathcal{A}_1) : Q(a_1) = 1\} \quad \text{and} \quad (417)$$

$$\text{BR}_{1,2}(\tilde{P}) = \{Q \in \Delta(\mathcal{A}_1) : Q(a_1) = 0\}. \quad (418)$$

Then, from (36) and (48), the function  $\tilde{v}$  satisfies

$$\tilde{v}(P) = \left( u_{1,1}P_{\tilde{A}_2|A_2=a_1}(a_1) + u_{2,1}P_{\tilde{A}_2|A_2=a_1}(a_2) \right) P(a_1) + \left( u_{1,2}P_{\tilde{A}_2|A_2=a_2}(a_1) + u_{2,2}P_{\tilde{A}_2|A_2=a_2}(a_2) \right) P(a_2). \quad (419)$$

Last, for all  $P \in \Delta(\mathcal{A}_2)$  such that  $P(a_1) > \tilde{P}^{(1)}$ , from (248), it holds that

$$\tilde{P}(a_1) = \det \underline{t}P(a_1) + t_{1,2} \quad (420)$$

$$< \det \underline{t}\tilde{P}^{(1)} + t_{1,2} \quad (421)$$

$$= P^{(1)} \quad (422)$$

and

$$\tilde{P}(a_1) = \det \underline{t}P(a_1) + t_{1,2} \quad (423)$$

$$< \det \underline{t}\tilde{P}^{(1)} + t_{1,2} \quad (424)$$

$$< \det \underline{t}\tilde{P}^{(2)} + t_{1,2} \quad (425)$$

$$= P^{(2)}, \quad (426)$$

where the inequality in (425) follows from (401) and the fact that  $\det \underline{t} < 0$ . Hence, under the assumption that  $u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2} > 0$ , from Lemma F.7, it holds that for all  $i \in \{1, 2\}$ ,

$$\text{BR}_{1,i}(\tilde{P}) = \{Q \in \Delta(\mathcal{A}_1) : Q(a_1) = 0\}. \quad (427)$$

Then, from (36) and (48), the function  $\tilde{v}$  satisfies

$$\tilde{v}(P) = u_{2,1}P(a_1) + u_{2,2}P(a_2). \quad (428)$$

In a nutshell, from (410), (419), and (428), the function  $\tilde{v}$  satisfies: for all  $P \in \Delta(\mathcal{A}_2)$  such that  $P(a_1) < \tilde{P}^{(2)}$ ,

$$\tilde{v}(P) = u_{1,1}P(a_1) + u_{1,2}P(a_2); \quad (429)$$

for all  $P \in \Delta(\mathcal{A}_2)$  such that  $\tilde{P}^{(2)} < P(a_1) < \tilde{P}^{(1)}$ ,

$$\tilde{v}(P) = \left( u_{1,1}P_{\tilde{A}_2|A_2=a_1}(a_1) + u_{2,1}P_{\tilde{A}_2|A_2=a_1}(a_2) \right) P(a_1) + \left( u_{1,2}P_{\tilde{A}_2|A_2=a_2}(a_1) + u_{2,2}P_{\tilde{A}_2|A_2=a_2}(a_2) \right) P(a_2); \quad (430)$$

for all  $P \in \Delta(\mathcal{A}_2)$  such that  $P(a_1) > \tilde{P}^{(1)}$ ,

$$\tilde{v}(P) = u_{2,1}P(a_1) + u_{2,2}P(a_2). \quad (431)$$

**The second part is as follows.** Given the fact that  $P_{A_2}^\dagger$ , with  $P_{A_2}^\dagger(a_1) = P^{(1)}$ , is the unique solution to

$$\arg \min_{P \in \Delta(\mathcal{A}_2)} \hat{v}(P) = P_{A_2}^\dagger, \quad (432)$$

from Lemma F.8, it holds that for all tuples  $(P, Q) \in \Delta(\mathcal{A}_2) \times \Delta(\mathcal{A}_2)$  such that  $P^{(1)} \leq P(a_1) < Q(a_1) \leq P^{(2)}$ ,

$$\hat{v}(P) < \hat{v}(Q). \quad (433)$$

Then from Lemma F.10, it holds that for all tuples  $(P, Q) \in \Delta(\mathcal{A}_2) \times \Delta(\mathcal{A}_2)$  such that  $0 \leq P(a_1) < Q(a_1) \leq P^{(1)}$ ,

$$\hat{v}(P) > \hat{v}(Q); \quad (434)$$

and from Lemma F.10 and (433), it holds for all tuples  $(P, Q) \in \Delta(\mathcal{A}_2) \times \Delta(\mathcal{A}_2)$  such that  $P^{(1)} \leq P(a_1) < Q(a_1) \leq 1$ ,

$$\hat{v}(P) < \hat{v}(Q). \quad (435)$$

Note that, from Lemma F.8, the inequality in (434) implies that

$$u_{2,1} < u_{2,2}; \quad (436)$$

and the inequality in (435) implies that

$$u_{1,1}P_{\tilde{\mathcal{A}}_2|A_2=a_1}(a_1) + u_{2,1}P_{\tilde{\mathcal{A}}_2|A_2=a_1}(a_2) > u_{1,2}P_{\tilde{\mathcal{A}}_2|A_2=a_2}(a_1) + u_{2,2}P_{\tilde{\mathcal{A}}_2|A_2=a_2}(a_2), \quad \text{and} \quad (437)$$

$$u_{1,1} > u_{1,2}. \quad (438)$$

Hence, from (429) and (438), it holds that for all tuples  $(P, Q) \in \Delta(\mathcal{A}_2) \times \Delta(\mathcal{A}_2)$  such that  $0 \leq P(a_1) < Q(a_1) < \tilde{P}^{(2)}$ ,

$$\tilde{v}(P) < \tilde{v}(Q); \quad (439)$$

from (430) and (437), it holds that for all tuples  $(P, Q) \in \Delta(\mathcal{A}_2) \times \Delta(\mathcal{A}_2)$  such that  $\tilde{P}^{(2)} < P(a_1) < Q(a_1) < \tilde{P}^{(1)}$ ,

$$\tilde{v}(P) < \tilde{v}(Q); \quad (440)$$

from (431) and (438), it holds that for all tuples  $(P, Q) \in \Delta(\mathcal{A}_2) \times \Delta(\mathcal{A}_2)$  such that  $\tilde{P}^{(1)} < P(a_1) < Q(a_1) \leq 1$ ,

$$\tilde{v}(P) > \tilde{v}(Q). \quad (441)$$

**The third part is as follows.** Given the fact that  $P_{\tilde{\mathcal{A}}_2}^\dagger(a_1) = P^{(1)}$  and the fact that  $P^{(1)} < P^{(2)}$ , from Lemma F.8, it holds that

$$\hat{v}(P_{\tilde{\mathcal{A}}_2}^\dagger) = u_{2,1}P^{(1)} + u_{2,2}(1 - P^{(1)}) \quad (442)$$

$$= \left( u_{1,1}P_{\tilde{\mathcal{A}}_2|A_2=a_1}(a_1) + u_{2,1}P_{\tilde{\mathcal{A}}_2|A_2=a_1}(a_2) \right) P^{(1)} + \left( u_{1,2}P_{\tilde{\mathcal{A}}_2|A_2=a_2}(a_1) + u_{2,2}P_{\tilde{\mathcal{A}}_2|A_2=a_2}(a_2) \right) (1 - P^{(1)}), \quad (443)$$

where the equality in (443) follows from (76).

Assume that  $P^{(1)} \in (\tilde{P}^{(2)}, \tilde{P}^{(1)})$ . As a result, for all  $P \in \Delta(\mathcal{A}_2)$  such that  $P(a_1) \in (\tilde{P}^{(2)}, P^{(1)})$ , from (430), it holds that

$$\tilde{v}(P) = \left( u_{1,1}P_{\tilde{\mathcal{A}}_2|A_2=a_1}(a_1) + u_{2,1}P_{\tilde{\mathcal{A}}_2|A_2=a_1}(a_2) \right) P(a_1) + \left( u_{1,2}P_{\tilde{\mathcal{A}}_2|A_2=a_2}(a_1) + u_{2,2}P_{\tilde{\mathcal{A}}_2|A_2=a_2}(a_2) \right) P(a_2) \quad (444)$$

$$< \left( u_{1,1}P_{\tilde{\mathcal{A}}_2|A_2=a_1}(a_1) + u_{2,1}P_{\tilde{\mathcal{A}}_2|A_2=a_1}(a_2) \right) P^{(1)} + \left( u_{1,2}P_{\tilde{\mathcal{A}}_2|A_2=a_2}(a_1) + u_{2,2}P_{\tilde{\mathcal{A}}_2|A_2=a_2}(a_2) \right) (1 - P^{(1)}) \quad (445)$$

$$= \hat{v}(P_{\tilde{\mathcal{A}}_2}^\dagger), \quad (446)$$

which the inequality in (445) follows from (440) and the equality in (446) follows from (443).

Assume  $P^{(1)} \notin (\tilde{P}^{(2)}, \tilde{P}^{(1)})$ . If  $P^{(1)} > \tilde{P}^{(2)}$ , for all probability measures  $P \in \Delta(\mathcal{A}_2)$  such that  $P(a_1) \in (\tilde{P}^{(2)}, \tilde{P}^{(1)})$ , from (430), it holds that

$$\tilde{v}(P) = \left( u_{1,1}P_{\tilde{\mathcal{A}}_2|A_2=a_1}(a_1) + u_{2,1}P_{\tilde{\mathcal{A}}_2|A_2=a_1}(a_2) \right) P(a_1) + \left( u_{1,2}P_{\tilde{\mathcal{A}}_2|A_2=a_2}(a_1) + u_{2,2}P_{\tilde{\mathcal{A}}_2|A_2=a_2}(a_2) \right) P(a_2) \quad (447)$$

$$< \left( u_{1,1}P_{\tilde{\mathcal{A}}_2|A_2=a_1}(a_1) + u_{2,1}P_{\tilde{\mathcal{A}}_2|A_2=a_1}(a_2) \right) P^{(1)} + \left( u_{1,2}P_{\tilde{\mathcal{A}}_2|A_2=a_2}(a_1) + u_{2,2}P_{\tilde{\mathcal{A}}_2|A_2=a_2}(a_2) \right) (1 - P^{(1)}) \quad (448)$$

$$= \hat{v}(P_{\tilde{\mathcal{A}}_2}^\dagger), \quad (449)$$

which the inequality in (448) follows from (437) and the fact that if  $P^{(1)} > \tilde{P}^{(2)}$  and  $P^{(1)} \notin (\tilde{P}^{(2)}, \tilde{P}^{(1)})$ , it holds that  $P^{(1)} > \tilde{P}^{(1)}$ ; and the equality in (449) follows from (443). Alternatively, if  $P^{(1)} \leq \tilde{P}^{(2)}$ , for all probability measures  $P \in \Delta(\mathcal{A}_2)$  such that  $P(a_1) \in [0, P^{(1)})$ , from (429), it holds that

$$\tilde{v}(P) = u_{1,1}P(a_1) + u_{1,2}(1 - P(a_1)) \quad (450)$$

$$< u_{1,1}P^{(1)} + u_{1,2}(1 - P^{(1)}) \quad (451)$$

$$\leq u_{1,1}P_{A_2}^*(a_1) + u_{1,2}P_{A_2}^*(a_2) \quad (452)$$

$$\leq u(P_{A_1}^*, P_{A_2}^*) \quad (453)$$

$$< \hat{v}(P_{A_2}^\dagger), \quad (454)$$

where the inequality in (451) follows from (438); the inequality in (452) follows from Lemma F.5 that  $P^{(1)} < P_{A_2}^*(a_1)$  when  $\det \underline{w} > 0$ ; the inequality in (453) follows from Lemma K.1; and the inequality in (454) follows from Lemma 3.6 and the fact that  $\det \underline{w} \neq 0$ .

### K.2.3 Case III

The proof is divided into three parts. The first part proves the value of  $\tilde{v}(P)$  for a given probability measure  $P \in \Delta(\mathcal{A}_2)$  such that  $P(a_1) \in [0, \tilde{P}^{(1)}) \cup (\tilde{P}^{(1)}, \tilde{P}^{(2)}) \cup (\tilde{P}^{(2)}, 1]$ ; the second part proves the decreasing or increasing property of  $\tilde{v}$ ; the third part proves that there exists a probability measure  $P \in \Delta(\mathcal{A}_2)$  such that the inequality  $\tilde{v}(P) < \hat{v}(P_{A_2}^\dagger)$  holds.

**The first part is as follows.** If  $\det \underline{w} < 0$ , from Lemma F.6, it holds that

$$P^{(1)} > P^{(2)}. \quad (455)$$

Furthermore, under the assumption that  $\det \underline{t} < 0$ , from (249) and (250), it holds that

$$\tilde{P}^{(1)} = \frac{1}{\det \underline{t}} (P^{(1)} - t_{1,2}) \quad (456)$$

$$< \frac{1}{\det \underline{t}} (P^{(2)} - t_{1,2}) \quad (457)$$

$$= \tilde{P}^{(2)}, \quad (458)$$

where the inequality in (457) follows from (455).

First, for all  $P \in \Delta(\mathcal{A}_2)$  such that  $P(a_1) < \tilde{P}^{(1)}$ , from (248), it holds that

$$\tilde{P}(a_1) = \det \underline{t}P(a_1) + t_{1,2} \quad (459)$$

$$> \det \underline{t}\tilde{P}^{(1)} + t_{1,2} \quad (460)$$

$$= P^{(1)} \quad (461)$$

and

$$\tilde{P}(a_1) = \det \underline{t}P(a_1) + t_{1,2} \quad (462)$$

$$> \det \underline{t}\tilde{P}^{(1)} + t_{1,2} \quad (463)$$

$$> \det \underline{t}\tilde{P}^{(2)} + t_{1,2} \quad (464)$$

$$= P^{(2)}, \quad (465)$$

where the inequality in (464) follows from (458) and the fact that  $\det \underline{t} < 0$ . Hence, under the assumption that  $u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2} > 0$ , from Lemma F.7, it holds that for all  $i \in \{1, 2\}$ ,

$$\text{BR}_{1,i}(\tilde{P}) = \{Q \in \Delta(\mathcal{A}_1) : Q(a_1) = 1\}. \quad (466)$$

Then, from (36) and (48), the function  $\tilde{v}$  satisfies

$$\tilde{v}(P) = u_{1,1}P(a_1) + u_{1,2}P(a_2). \quad (467)$$



Second, for all  $P \in \Delta(\mathcal{A}_2)$  such that  $\tilde{P}^{(1)} < P(a_1) < \tilde{P}^{(2)}$ , from (248), it holds that

$$\tilde{P}(a_1) = \det \underline{t}P(a_1) + t_{1,2} \quad (468)$$

$$> \det \underline{t}\tilde{P}^{(2)} + t_{1,2} \quad (469)$$

$$= P^{(2)} \quad (470)$$

and

$$\tilde{P}(a_1) = \det \underline{t}P(a_1) + t_{1,2} \quad (471)$$

$$< \det \underline{t}\tilde{P}^{(1)} + t_{1,2} \quad (472)$$

$$= P^{(1)}. \quad (473)$$

Hence, under the assumption that  $u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2} > 0$ , from Lemma F.7, it holds that

$$\text{BR}_{1,1}(\tilde{P}) = \{Q \in \Delta(\mathcal{A}_1) : Q(a_1) = 0\} \quad \text{and} \quad (474)$$

$$\text{BR}_{1,2}(\tilde{P}) = \{Q \in \Delta(\mathcal{A}_1) : Q(a_1) = 1\}. \quad (475)$$

Then, from (36) and (48), the function  $\tilde{v}$  satisfies

$$\tilde{v}(P) = \left( u_{1,1}P_{\tilde{A}_2|A_2=a_1}(a_2) + u_{2,1}P_{\tilde{A}_2|A_2=a_1}(a_1) \right) P(a_1) + \left( u_{1,2}P_{\tilde{A}_2|A_2=a_2}(a_2) + u_{2,2}P_{\tilde{A}_2|A_2=a_2}(a_1) \right) P(a_2). \quad (476)$$

Last, for all  $P \in \Delta(\mathcal{A}_2)$  such that  $P(a_1) > \tilde{P}^{(2)}$ , from (248), it holds that

$$\tilde{P}(a_1) = \det \underline{t}P(a_1) + t_{1,2} \quad (477)$$

$$< \det \underline{t}\tilde{P}^{(2)} + t_{1,2} \quad (478)$$

$$= P^{(2)} \quad (479)$$

and

$$\tilde{P}(a_1) = \det \underline{t}P(a_1) + t_{1,2} \quad (480)$$

$$< \det \underline{t}\tilde{P}^{(2)} + t_{1,2} \quad (481)$$

$$< \det \underline{t}\tilde{P}^{(1)} + t_{1,2} \quad (482)$$

$$= P^{(1)}, \quad (483)$$

where the inequality in (482) follows from (458) and the fact that  $\det \underline{t} < 0$ . Hence, under the assumption that  $u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2} > 0$ , from Lemma F.7, it holds that for all  $i \in \{1, 2\}$ ,

$$\text{BR}_{1,i}(\tilde{P}) = \{Q \in \Delta(\mathcal{A}_1) : Q(a_1) = 0\}. \quad (484)$$

Then, from (36) and (48), the function  $\tilde{v}$  satisfies

$$\tilde{v}(P) = u_{2,1}P(a_1) + u_{2,2}P(a_2). \quad (485)$$

In a nutshell, from (467), (476), and (485), the function  $\tilde{v}$  satisfies: for all  $P \in \Delta(\mathcal{A}_2)$  such that  $P(a_1) < \tilde{P}^{(1)}$ ,

$$\tilde{v}(P) = u_{1,1}P(a_1) + u_{1,2}P(a_2); \quad (486)$$

for all  $P \in \Delta(\mathcal{A}_2)$  such that  $\tilde{P}^{(1)} < P(a_1) < \tilde{P}^{(2)}$ ,

$$\tilde{v}(P) = \left( u_{1,1}P_{\tilde{A}_2|A_2=a_1}(a_2) + u_{2,1}P_{\tilde{A}_2|A_2=a_1}(a_1) \right) P(a_1) + \left( u_{1,2}P_{\tilde{A}_2|A_2=a_2}(a_2) + u_{2,2}P_{\tilde{A}_2|A_2=a_2}(a_1) \right) P(a_2); \quad (487)$$

for all  $P \in \Delta(\mathcal{A}_2)$  such that  $P(a_1) > \tilde{P}^{(2)}$ ,

$$\tilde{v}(P) = u_{2,1}P(a_1) + u_{2,2}P(a_2). \quad (488)$$

**The second part is as follows.** Given the fact that  $P_{A_2}^\dagger$ , with  $P_{A_2}^\dagger(a_1) = P^{(1)}$ , is the unique solution to

$$\min_{P \in \Delta(\mathcal{A}_2)} \hat{v}(P), \quad (489)$$

from Lemma F.8, it holds that for all tuples  $(P, Q) \in \Delta(\mathcal{A}_2) \times \Delta(\mathcal{A}_2)$  such that  $P^{(2)} \leq P(a_1) < Q(a_1) \leq P^{(1)}$ ,

$$\hat{v}(P) > \hat{v}(Q). \quad (490)$$

Then from Lemma F.10 and (490), it holds that for all tuples  $(P, Q) \in \Delta(\mathcal{A}_2) \times \Delta(\mathcal{A}_2)$  such that  $0 \leq P(a_1) < Q(a_1) \leq P^{(1)}$ ,

$$\hat{v}(P) > \hat{v}(Q); \quad (491)$$

and from Lemma F.10, it holds for all tuples  $(P, Q) \in \Delta(\mathcal{A}_2) \times \Delta(\mathcal{A}_2)$  such that  $P^{(1)} \leq P(a_1) < Q(a_1) \leq 1$ ,

$$\hat{v}(P) < \hat{v}(Q). \quad (492)$$

Note that, from Lemma F.8, the inequality in (491) implies that

$$u_{1,1}P_{\tilde{A}_2|A_2=a_1}(a_2) + u_{2,1}P_{\tilde{A}_2|A_2=a_1}(a_1) < u_{1,2}P_{\tilde{A}_2|A_2=a_2}(a_2) + u_{2,2}P_{\tilde{A}_2|A_2=a_2}(a_1), \quad \text{and} \quad (493)$$

$$u_{2,1} < u_{2,2}; \quad (494)$$

and the inequality in (492) implies that

$$u_{1,1} > u_{1,2}. \quad (495)$$

Hence, from (486) and (495), it holds that for all tuples  $(P, Q) \in \Delta(\mathcal{A}_2) \times \Delta(\mathcal{A}_2)$  such that  $0 \leq P(a_1) < Q(a_1) < \tilde{P}^{(1)}$ ,

$$\tilde{v}(P) < \tilde{v}(Q); \quad (496)$$

from (487) and (493), it holds that for all tuples  $(P, Q) \in \Delta(\mathcal{A}_2) \times \Delta(\mathcal{A}_2)$  such that  $\tilde{P}^{(1)} < P(a_1) < Q(a_1) < \tilde{P}^{(2)}$ ,

$$\tilde{v}(P) > \tilde{v}(Q); \quad (497)$$

from (488) and (494), it holds that for all tuples  $(P, Q) \in \Delta(\mathcal{A}_2) \times \Delta(\mathcal{A}_2)$  such that  $\tilde{P}^{(2)} < P(a_1) < Q(a_1) \leq 1$ ,

$$\tilde{v}(P) > \tilde{v}(Q). \quad (498)$$

**The third part is as follows.** Given the fact that  $P_{A_2}^\dagger(a_1) = P^{(1)}$  and the fact that  $P^{(1)} > P^{(2)}$ , from Lemma F.8, it holds that

$$\hat{v}(P_{A_2}^\dagger) = u_{1,1}P^{(1)} + u_{1,2}(1 - P^{(1)}) \quad (499)$$

$$= \left( u_{1,1}P_{\tilde{A}_2|A_2=a_1}(a_2) + u_{2,1}P_{\tilde{A}_2|A_2=a_1}(a_1) \right) P^{(1)} + \left( u_{1,2}P_{\tilde{A}_2|A_2=a_2}(a_2) + u_{2,2}P_{\tilde{A}_2|A_2=a_2}(a_1) \right) (1 - P^{(1)}), \quad (500)$$

where the equality in (500) follows from (76).

Assume that  $P^{(1)} \in (\tilde{P}^{(1)}, \tilde{P}^{(2)})$ . As a result, for all  $P \in \Delta(\mathcal{A}_2)$  such that  $P(a_1) \in (P^{(1)}, \tilde{P}^{(2)})$ , from (487), it holds that

$$\tilde{v}(P_{A_2}) = \left( u_{1,1}P_{\tilde{A}_2|A_2=a_1}(a_2) + u_{2,1}P_{\tilde{A}_2|A_2=a_1}(a_1) \right) P(a_1) + \left( u_{1,2}P_{\tilde{A}_2|A_2=a_2}(a_2) + u_{2,2}P_{\tilde{A}_2|A_2=a_2}(a_1) \right) P(a_2) \quad (501)$$

$$< \left( u_{1,1}P_{\tilde{A}_2|A_2=a_1}(a_2) + u_{2,1}P_{\tilde{A}_2|A_2=a_1}(a_1) \right) P^{(1)} + \left( u_{1,2}P_{\tilde{A}_2|A_2=a_2}(a_2) + u_{2,2}P_{\tilde{A}_2|A_2=a_2}(a_1) \right) (1 - P^{(1)}) \quad (502)$$

$$= \hat{v}(P_{A_2}^\dagger), \quad (503)$$

which the inequality in (502) follows from (497); and the equality in (503) follows from (500).

Assume  $P^{(1)} \notin (\tilde{P}^{(1)}, \tilde{P}^{(2)})$ . If  $P^{(1)} > \tilde{P}^{(2)}$ , for all probability measures  $P \in \Delta(\mathcal{A}_2)$  such that  $P(a_1) \in (P^{(1)}, 1]$ , from (488), it holds that

$$\tilde{v}(P) = u_{2,1}P(a_1) + u_{2,2}P(a_2) \quad (504)$$

$$< u_{2,1}P^{(1)} + u_{2,2}(1 - P^{(1)}) \quad (505)$$

$$< u_{2,1}P_{A_2}^*(a_1) + u_{2,2}P_{A_2}^*(a_2) \quad (506)$$

$$= u(P_{A_1}^*, P_{A_2}^*), \quad (507)$$

$$< \hat{v}(P_{A_2}^\dagger), \quad (508)$$

where the inequality in (505) follows from (494); the inequality in (506) follows from Lemma F.5 that  $P^{(1)} > P_{A_2}^*(a_1)$  when  $\det \underline{w} < 0$ ; the inequality in (507) follows from Lemma K.1; and the inequality in (508) follows from Lemma 3.6 and the fact that  $\det \underline{w} \neq 0$ . Alternatively, if  $P^{(1)} \leq \tilde{P}^{(2)}$ , for all probability measures  $P \in \Delta(\mathcal{A}_2)$  such that  $P(a_1) \in [0, P^{(1)}]$ , from (486), it holds that

$$\tilde{v}(P) = u_{1,1}P(a_1) + u_{1,2}P(a_2) \quad (509)$$

$$< u_{1,1}P^{(1)} + u_{1,2}(1 - P^{(1)}) \quad (510)$$

$$= \hat{v}(P_{A_2}^\dagger), \quad (511)$$

where the equality in (509) follows from the fact that if  $P^{(1)} \leq \tilde{P}^{(2)}$  and  $P^{(1)} \notin (\tilde{P}^{(1)}, \tilde{P}^{(2)})$ , it holds that  $P^{(1)} < \tilde{P}^{(1)}$ ; the inequality in (510) follows from (495); and the inequality in (511) follows from (499).

#### K.2.4 Case IV

The proof is divided into three parts. The first part proves the value of  $\tilde{v}(P)$  for a given probability measure  $P \in \Delta(\mathcal{A}_2)$  such that  $P(a_1) \in [0, \tilde{P}^{(2)}) \cup (\tilde{P}^{(2)}, \tilde{P}^{(1)}) \cup (\tilde{P}^{(1)}, 1]$ ; the second part proves the decreasing or increasing property of  $\tilde{v}$ ; the third part proves that there exists a probability measure  $P \in \Delta(\mathcal{A}_2)$  such that the inequality  $\tilde{v}(P) < \hat{v}(P_{A_2}^\dagger)$  holds.

**The first part is as follows.** If  $\det \underline{w} < 0$ , from Lemma F.6, it holds that

$$P^{(1)} > P^{(2)}. \quad (512)$$

Furthermore, under the assumption that  $\det \underline{t} > 0$ , from (249) and (250), it holds that

$$\tilde{P}^{(1)} = \frac{1}{\det \underline{t}} (P^{(1)} - t_{1,2}) \quad (513)$$

$$> \frac{1}{\det \underline{t}} (P^{(2)} - t_{1,2}) \quad (514)$$

$$= \tilde{P}^{(2)}, \quad (515)$$

where the inequality in (514) follows from (512).

First, for all  $P \in \Delta(\mathcal{A}_2)$  such that  $P(a_1) > \tilde{P}^{(1)}$ , from (248), it holds that

$$\tilde{P}(a_1) = \det \underline{t}P(a_1) + t_{1,2} \quad (516)$$

$$> \det \underline{t}\tilde{P}^{(1)} + t_{1,2} \quad (517)$$

$$= P^{(1)} \quad (518)$$

and

$$\tilde{P}(a_1) = \det \underline{t}P(a_1) + t_{1,2} \quad (519)$$

$$> \det \underline{t}\tilde{P}^{(1)} + t_{1,2} \quad (520)$$

$$> \det \underline{t}\tilde{P}^{(2)} + t_{1,2} \quad (521)$$

$$= P^{(2)} \quad (522)$$

where the inequality in (521) follows from (515) and the fact that  $\det \underline{t} > 0$ . Hence, under the assumption that  $u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2} > 0$ , from Lemma F.7, it holds that for all  $i \in \{1, 2\}$ ,

$$\text{BR}_{1,i}(\tilde{P}) = \{Q \in \Delta(\mathcal{A}_1) : Q(a_1) = 1\}. \quad (523)$$

Then, from (36) and (48), the function  $\tilde{v}$  satisfies

$$\tilde{v}(P) = u_{1,1}P(a_1) + u_{1,2}P(a_2). \quad (524)$$

Second, for all  $P \in \Delta(\mathcal{A}_2)$  such that  $\tilde{P}^{(2)} < P(a_1) < \tilde{P}^{(1)}$ , from (248), it holds that

$$\tilde{P}(a_1) = \det \underline{t}P(a_1) + t_{1,2} \quad (525)$$

$$> \det \underline{t}\tilde{P}^{(2)} + t_{1,2} \quad (526)$$

$$= P^{(2)} \quad (527)$$

and

$$\tilde{P}(a_1) = \det \underline{t}P(a_1) + t_{1,2} \quad (528)$$

$$< \det \underline{t}\tilde{P}^{(1)} + t_{1,2} \quad (529)$$

$$= P^{(1)}. \quad (530)$$

Hence, under the assumption that  $u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2} > 0$ , from Lemma F.7, it holds that

$$\text{BR}_{1,1}(\tilde{P}) = \{Q \in \Delta(\mathcal{A}_1) : Q(a_1) = 0\} \quad \text{and} \quad (531)$$

$$\text{BR}_{1,2}(\tilde{P}) = \{Q \in \Delta(\mathcal{A}_1) : Q(a_1) = 1\}. \quad (532)$$

Then, from (36) and (48), the function  $\tilde{v}$  satisfies

$$\tilde{v}(P) = \left( u_{1,1}P_{\tilde{A}_2|A_2=a_1}(a_2) + u_{2,1}P_{\tilde{A}_2|A_2=a_1}(a_1) \right) P(a_1) + \left( u_{1,2}P_{\tilde{A}_2|A_2=a_2}(a_2) + u_{2,2}P_{\tilde{A}_2|A_2=a_2}(a_1) \right) P(a_2). \quad (533)$$

Last, for all  $P \in \Delta(\mathcal{A}_2)$  such that  $P(a_1) < \tilde{P}^{(2)}$ , from (248), it holds that

$$\tilde{P}(a_1) = \det \underline{t}P(a_1) + t_{1,2} \quad (534)$$

$$< \det \underline{t}\tilde{P}^{(2)} + t_{1,2} \quad (535)$$

$$= P^{(2)} \quad (536)$$

and

$$\tilde{P}(a_1) = \det \underline{t}P(a_1) + t_{1,2} \quad (537)$$

$$< \det \underline{t}\tilde{P}^{(2)} + t_{1,2} \quad (538)$$

$$< \det \underline{t}\tilde{P}^{(1)} + t_{1,2} \quad (539)$$

$$= P^{(1)}, \quad (540)$$

where the inequality in (539) follows from (515) and the fact that  $\det \underline{t} > 0$ . Hence, under the assumption that  $u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2} > 0$ , from Lemma F.7, it holds that for all  $i \in \{1, 2\}$ ,

$$\text{BR}_{1,i}(\tilde{P}) = \{Q \in \Delta(\mathcal{A}_1) : Q(a_1) = 0\}. \quad (541)$$

Then, from (36) and (48), the function  $\tilde{v}$  satisfies

$$\tilde{v}(P) = u_{2,1}P(a_1) + u_{2,2}P(a_2). \quad (542)$$

In a nutshell, from (524), (533), and (542), the function  $\tilde{v}$  satisfies: for all  $P \in \Delta(\mathcal{A}_2)$  such that  $P(a_1) > \tilde{P}^{(1)}$ ,

$$\tilde{v}(P) = u_{1,1}P(a_1) + u_{1,2}P(a_2); \quad (543)$$

for all  $P \in \Delta(\mathcal{A}_2)$  such that  $\tilde{P}^{(2)} < P(a_1) < \tilde{P}^{(1)}$ ,

$$\tilde{v}(P) = \left( u_{1,1}P_{\tilde{A}_2|A_2=a_1}(a_2) + u_{2,1}P_{\tilde{A}_2|A_2=a_1}(a_1) \right) P(a_1) + \left( u_{1,2}P_{\tilde{A}_2|A_2=a_2}(a_2) + u_{2,2}P_{\tilde{A}_2|A_2=a_2}(a_1) \right) P(a_2). \quad (544)$$

for all  $P \in \Delta(\mathcal{A}_2)$  such that  $P(a_1) < \tilde{P}^{(2)}$ ,

$$\tilde{v}(P) = u_{2,1}P(a_1) + u_{2,2}P(a_2). \quad (545)$$

**The second part is as follows.** Given the fact that  $P_{A_2}^\dagger$ , with  $P_{A_2}^\dagger(a_1) = P^{(1)}$ , is the unique solution to

$$\min_{P \in \Delta(\mathcal{A}_2)} \hat{v}(P), \quad (546)$$

from Lemma F.8, it holds that for all tuples  $(P, Q) \in \Delta(\mathcal{A}_2) \times \Delta(\mathcal{A}_2)$  such that  $P^{(2)} \leq P(a_1) < Q(a_1) \leq P^{(1)}$ ,

$$\hat{v}(P) > \hat{v}(Q). \quad (547)$$

Then from Lemma F.10 and (547), it holds that for all tuples  $(P, Q) \in \Delta(\mathcal{A}_2) \times \Delta(\mathcal{A}_2)$  such that  $0 \leq P(a_1) < Q(a_1) \leq P^{(1)}$ ,

$$\hat{v}(P) > \hat{v}(Q); \quad (548)$$

and from Lemma F.10, it holds for all tuples  $(P, Q) \in \Delta(\mathcal{A}_2) \times \Delta(\mathcal{A}_2)$  such that  $P^{(1)} \leq P(a_1) < Q(a_1) \leq 1$ ,

$$\hat{v}(P) < \hat{v}(Q). \quad (549)$$

Note that, from Lemma F.8, the inequality in (548) implies that

$$u_{1,1}P_{\tilde{A}_2|A_2=a_1}(a_2) + u_{2,1}P_{\tilde{A}_2|A_2=a_1}(a_1) < u_{1,2}P_{\tilde{A}_2|A_2=a_2}(a_2) + u_{2,2}P_{\tilde{A}_2|A_2=a_2}(a_1), \quad \text{and} \quad (550)$$

$$u_{2,1} < u_{2,2}; \quad (551)$$

and the inequality in (549) implies that

$$u_{1,1} > u_{1,2}. \quad (552)$$

Hence, from (545) and (551), it holds that for all tuples  $(P, Q) \in \Delta(\mathcal{A}_2) \times \Delta(\mathcal{A}_2)$  such that  $0 \leq P(a_1) < Q(a_1) < \tilde{P}^{(2)}$ ,

$$\tilde{v}(P) > \tilde{v}(Q); \quad (553)$$

from (544) and (550), it holds that for all tuples  $(P, Q) \in \Delta(\mathcal{A}_2) \times \Delta(\mathcal{A}_2)$  such that  $\tilde{P}^{(2)} < P(a_1) < Q(a_1) < \tilde{P}^{(1)}$ ,

$$\tilde{v}(P) > \tilde{v}(Q); \quad (554)$$

from (543) and (552), it holds that for all tuples  $(P, Q) \in \Delta(\mathcal{A}_2) \times \Delta(\mathcal{A}_2)$  such that  $\tilde{P}^{(1)} < P(a_1) < Q(a_1) \leq 1$ ,

$$\tilde{v}(P) < \tilde{v}(Q). \quad (555)$$

**The third part is as follows.** Given the fact that  $P_{A_2}^\dagger(a_1) = P^{(1)}$  and the fact that  $P^{(1)} > P^{(2)}$ , from Lemma F.8, it holds that

$$\hat{v}(P_{A_2}^\dagger) = u_{1,1}P^{(1)} + u_{1,2}(1 - P^{(1)}) \quad (556)$$

$$= \left( u_{1,1}P_{\tilde{A}_2|A_2=a_1}(a_2) + u_{2,1}P_{\tilde{A}_2|A_2=a_1}(a_1) \right) P^{(1)} + \left( u_{1,2}P_{\tilde{A}_2|A_2=a_2}(a_2) + u_{2,2}P_{\tilde{A}_2|A_2=a_2}(a_1) \right) (1 - P^{(1)}), \quad (557)$$

where the equality in (557) follows from (76).

Assume that  $P^{(1)} \in (\tilde{P}^{(2)}, \tilde{P}^{(1)})$ . As a result, for all  $P \in \Delta(\mathcal{A}_2)$  such that  $P(a_1) \in (P^{(1)}, \tilde{P}^{(1)})$ , from (544), it holds that

$$\tilde{v}(P) = \left( u_{1,1}P_{\tilde{A}_2|A_2=a_1}(a_2) + u_{2,1}P_{\tilde{A}_2|A_2=a_1}(a_1) \right) P(a_1) + \left( u_{1,2}P_{\tilde{A}_2|A_2=a_2}(a_2) + u_{2,2}P_{\tilde{A}_2|A_2=a_2}(a_1) \right) P(a_2) \quad (558)$$

$$< \left( u_{1,1}P_{\tilde{A}_2|A_2=a_1}(a_2) + u_{2,1}P_{\tilde{A}_2|A_2=a_1}(a_1) \right) P^{(1)} + \left( u_{1,2}P_{\tilde{A}_2|A_2=a_2}(a_2) + u_{2,2}P_{\tilde{A}_2|A_2=a_2}(a_1) \right) (1 - P^{(1)}) \quad (559)$$

$$= \hat{v}(P_{A_2}^\dagger), \quad (560)$$

which the inequality in (559) follows from (550) and the equality in (560) follows from (557).

Assume  $P^{(1)} \notin (\tilde{P}^{(2)}, \tilde{P}^{(1)})$ . If  $P^{(1)} > \tilde{P}^{(1)}$ , for all probability measures  $P \in \Delta(\mathcal{A}_2)$  such that  $P(a_1) \in (\tilde{P}^{(1)}, P^{(1)})$ , from (543), it holds that

$$\tilde{v}(P_{A_2}) = u_{1,1}P(a_1) + u_{1,2}P(a_2) \quad (561)$$

$$< u_{1,1}P^{(1)} + u_{1,2}(1 - P^{(1)}) \quad (562)$$

$$= \hat{v}(P_{A_2}^\dagger), \quad (563)$$

which the inequality in (562) follows from (552); and the equality in (563) follows from (556). Alternatively, if  $P^{(1)} < \tilde{P}^{(1)}$ , for all probability measures  $P \in \Delta(\mathcal{A}_2)$  such that  $P(a_1) \in (\tilde{P}^{(2)}, \tilde{P}^{(1)})$ , from (544), it holds that

$$\tilde{v}(P) = \left( u_{1,1} P_{\tilde{A}_2|A_2=a_1}(a_2) + u_{2,1} P_{\tilde{A}_2|A_2=a_1}(a_1) \right) P(a_1) + \left( u_{1,2} P_{\tilde{A}_2|A_2=a_2}(a_2) + u_{2,2} P_{\tilde{A}_2|A_2=a_2}(a_1) \right) P(a_2). \quad (564)$$

$$\begin{aligned} &< \left( u_{1,1} P_{\tilde{A}_2|A_2=a_1}(a_2) + u_{2,1} P_{\tilde{A}_2|A_2=a_1}(a_1) \right) P^{(1)} + \left( u_{1,2} P_{\tilde{A}_2|A_2=a_2}(a_2) + u_{2,2} P_{\tilde{A}_2|A_2=a_2}(a_1) \right) (1 - P^{(1)}). \quad (565) \\ &= \tilde{v}(P_{A_2}^\dagger), \quad (566) \end{aligned}$$

which the inequality in (565) follows from (550); and the equality in (566) follows from (557).

This completes the proof.

## L Proof of Lemma 2.1

The proof of Lemma 2.1 follows immediately from the definition of the Nash equilibrium [Nash, 1950] and simple algebraic manipulations.