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Stable twisted cohomology of the mapping class groups in the unit tangent bundle homology

Nariya Kawazumi and Arthur Soulié

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Abstract

We compute the stable cohomology groups of the mapping class groups of compact orientable surfaces with one boundary with twisted coefficients given by the first homology of the unit tangent bundle of the surface. The stable twisted cohomology is not free as a module over the stable cohomology algebra with constant coefficients. In fact, it is out of the scope of the traditional framework for twisted homological stability, since these twisted coefficients define a covariant functor over the classical category associated to mapping class groups to study homological stability, rather than a contravariant one. For comparison, we also compute the stable cohomology group with coefficients in the first cohomology of the unit tangent bundle of the surface, which fits into the traditional framework.

Introduction

Considering a smooth compact connected orientable surface of genus \( g \geq 0 \) and with one boundary component \( \Sigma_{g,1} \), we denote its mapping class group (that is the isotopy classes of its diffeomorphisms restricting to the identity on the boundary) by \( \Gamma_{g,1} \). The study of the (co)homology of the groups \( \{ \Gamma_{g,1}, g \in \mathbb{N} \} \) has been a very active research topic over the past decades. In particular, the full computations of their cohomology groups with twisted coefficients remain an active research topic; see [GKR19, Section 5.5] and [Hai20, Part 4]. In this paper, we compute the stable cohomology groups of the mapping class groups with twisted coefficients defined from the first (co)homology group of the unit tangent bundles of the considered surfaces; see Theorems A and B. This is the first example of stable twisted cohomology for that mapping class groups is not free over the rational stable cohomology of these groups.

Background on stable (twisted) (co)homology. A key step towards the computations of the (co)homology of the mapping class groups is their homological stability properties. Namely, we consider canonical injections \( \Gamma_{g,1} \hookrightarrow \Gamma_{g+1,1} \) induced by viewing \( \Sigma_{g,1} \) as a subsurface of \( \Sigma_{g+1,1} \) extending the diffeomorphisms of \( \Sigma_{g,1} \) by the identity on the complement \( \Sigma_{1,1} \), and a set of \( \Gamma_{g,1} \)-modules \( \{ F(g), g \in \mathbb{N} \} \) with \( \Gamma_{g,1} \)-equivariant morphisms \( F(g) \to F(g+1) \). This data defines morphisms for the homology groups \( H_i(\Gamma_{g,1}; F(g)) \to H_i(\Gamma_{g+1,1}; F(g + 1)) \). For any \( i \geq 0 \). For each \( i \geq 0 \), if this canonical morphism is an isomorphism for \( N(i, F) \leq g \) with some \( N(i, F) \in \mathbb{N} \) depending on \( i \) and \( F \), then the mapping class groups are said to satisfy homological stability (with (twisted) coefficient in \( F \)). The homological stability property with constant coefficients (i.e. for \( F(g) = \mathbb{Z} \) for all \( g \geq 0 \)) is due to Harer [Har85]. The range \( N(i, \mathbb{Z}) \) is improved by Boldsen [Bo12] and Randal-Williams [Ran16]. Furthermore, Ivanov [Iva93] proves homological stability property with twisted coefficients given \( \Gamma_{g,1} \)-modules forming a functor \( F : \mathcal{U}M_2 \to \text{Ab} \) (where \( \mathcal{U}M_2 \) is defined in §2.1 and \( \text{Ab} \) the category of abelian groups) satisfying some polynomiality conditions; we refer to §2 for further details. Finally, Randal-Williams and Wahl [RW17] and

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Galatius, Kupers and Randal-Williams [GKR19, Section 5.5] extract the general framework for proving homological stability properties (both for constant and twisted coefficients) for general families of groups, subsuming these previous studies and recovering their results.

For a fixed $i$, the colimit $\text{Colim}_{g \in [N, \infty)}(H_i(\Gamma_{g,1}; F(g)))$ is denoted by $H_i(\Gamma_{\infty,1}; F)$. When the homological stability property is satisfied, the value of the twisted homology group $H_i(\Gamma_{g,1}; F(g))$ for $g \geq N(i,d)$ corresponds to the colimit $H_i(\Gamma_{\infty,1}; F)$ and is called the stable homology of the mapping class groups. All the stable homology results obtained so far in the literature have been carried out using a field as ground ring, generally the rationals $\mathbb{Q}$. We will also mainly use this field although some of our results still hold over $\mathbb{Z}$. In particular, using the rationals allows us to equivalently consider the stable cohomology with twisted coefficients (thanks to the universal coefficient theorem with twisted coefficients, see Lemma 2.6).

By proving Mumford’s conjecture [Mum83], Madsen and Weiss [MW07] compute the rational stable homology (i.e. for $F(g) = \mathbb{Q}$ for all $g \geq 0$) of the mapping class groups. Namely, they use the standard cohomology classes defined by Mumford [Mum83], Morita [Mor84; Mor87] and Miller [Mil86] $\{e_i \in H^2(\Gamma_{\infty,1}; \mathbb{Q}); i \geq 1\}$, called the classical Mumford-Morita-Miller classes, to describe the algebra $H^*(\Gamma_{\infty,1}; \mathbb{Q})$ as follows:

$$H^*(\Gamma_{\infty,1}; \mathbb{Q}) \cong \mathbb{Q}[\{e_i, i \geq 1\}], \quad (0.1)$$

Denoting the free $\mathbb{Q}$-vector space $\bigoplus_{i \geq 1} \mathbb{Q}e_i$ by $E$ and by $\text{Sym}_\mathbb{Q}(E)$ its symmetric algebra, Madsen-Weiss theorem (0.1) may be rephrased as that there is an isomorphism $H^*(\Gamma_{\infty,1}; \mathbb{Q}) \cong \text{Sym}_\mathbb{Q}(E)$.

Furthermore, considering the twisted coefficient given by the first homology of the surface $\Sigma_{g,1}$ denoted by $H(g)$, the stable twisted $H^*(\Gamma_{\infty,1}; H)$ was first computed by Harer [Har91, Section 7]. Also, considering the closed oriented surfaces $\Sigma_g$ analogues (obtained from $\Sigma_{g,1}$ by capping the boundary component with a disc), Looijenga [Loo96] computed the stable cohomology of the associated mapping class groups $\{\Gamma_g, g \in \mathbb{N}\}$ with coefficients in any irreducible representation of the rational symplectic group. Looijenga did not use the stability result of [Iva93] but only that of [Har85]. Independently from these previous works, the first author [Kaw98] introduced a series of twisted cohomology classes on the mapping class group $\Gamma_{g,1}$, called the twisted Mumford-Morita-Miller classes $\{m_{i,j}; i \geq 0, j \geq 1\}$. Based on Looijenga’s idea [Loo96], one can prove that some algebraic combinations of the twisted Mumford-Morita-Miller classes define a free basis of the stable cohomology group of with coefficients in the tensor products of $H(g)$; see [Kaw08]. These computations may also be done via other methods; see Randal-Williams [Ran18, Appendix B] and [Sou20, Section 2.3.1].

The unit tangent bundle homology representations. The representation theory of the mapping class groups of surface is wild and remains an active research topic; see Margalit’s expository paper [Mar19]. In particular, there are few known representations of the mapping class groups $\Gamma_{g,1}$ apart from the first homology of the surface $H(g)$. However, other representations which appear naturally are those given by homology and cohomology of the unit tangent bundle of the surface $\Sigma_{g,1}$. We denote by $\mathcal{H}(g)$ the first integral homology group $H_1(U\Sigma_{g,1}; \mathbb{Z})$. Its dual representation is denoted by $\mathcal{H}^*(g)$, while we denote its corresponding rational homology by $\mathcal{H}_Q(g)$. These representations have been first studied by Trapp [Tra92, Theorem 2.2] and we refer the reader to §1.2 for further details.

In particular, the representation $\mathcal{H}(g)$ is a non-trivial extension of $H(g)$ by $\mathbb{Z}$; see (1.3). This extension corresponds to the twisted Mumford-Morita-Miller class $m_{1,1}$ in the stable cohomology algebra of each cohomology class $H^*(\Gamma_{\infty,1}; H)$. It is the image of the cohomology class introduced Earle [Ear78] which generates $H^1(\Gamma_{1,1}; H(g))$ for $g \geq 2$. We refer the reader to §1.2 and §2.2.1 for further details.

Results. In the present paper, we consider the cohomology of the mapping class groups with twisted coefficients given by $\mathcal{H}_Q(g)$ and $\mathcal{H}_Q^*(g)$. We consider cohomology rather than homology because of the key usefulness of the cup product structure. The pathway to make these computations is based on the short exact sequences (1.3) and (1.3) defining these modules, on the determination
of the cohomology long exact sequence connecting homomorphisms (see Lemmas 3.2 and 3.5) and the Contraction formula (2.3) between stable twisted cohomology classes. We denote by \( R \) the trivial \( \text{Sym}_R(\mathcal{E}) \)-module generated by the integral stable 0th-cohomology \( \theta \in H^0(\Gamma_{\infty,1}; \mathcal{H}) \) defined by the fiber of the unit tangent bundle.

First, the computation for the dual representations \( \tilde{H}_Q^*(g) \) is the least difficult. Indeed, these define a covariant functor \( \tilde{H}_Q^*: \mathcal{U}\mathcal{M}_2 \to \text{Ab} \) and therefore we already know from [Sou20, Theorem C] that the stable cohomology algebra \( H^*(\Gamma_{\infty,1}; \tilde{H}_Q^*) \) is free over \( \text{Sym}_R(\mathcal{E}) \). We prove:

**Theorem A** [Theorem 3.3] The stable cohomology algebra \( H^*(\Gamma_{\infty,1}; \tilde{H}_Q^*) \) is isomorphic to the free \( \text{Sym}_R(\mathcal{E}) \)-module with basis \( \{ m_{i,1}, i \geq 2 \} \). In particular, it is concentrated in odd degrees.

On the contrary, the representations \( H_Q(g) \) induce a contravariant functor \( H_Q: \mathcal{U}\mathcal{M}_2 \to \text{Ab} \). As far as the authors know, any qualitative general result and computations for such coefficients have not been realised yet.

**Theorem B** [Theorem 3.9] The stable cohomology algebra \( H^*(\Gamma_{\infty,1}; H_Q^*) \) is isomorphic to direct sum \( \mathbb{Q} \otimes \mathfrak{M} \), where \( \mathfrak{M} \) is the \( \text{Sym}_R(\mathcal{E}) \)-module generated by the classes

\[
M_{i,j} := e_im_{j,1} - e_jm_{i,1}
\]

for \( i, j \geq 1 \) and with relations

\[
e_i M_{j,k} + e_j M_{k,i} + e_k M_{i,j} \sim 0
\]

for all \( i,j,k \geq 1 \).

In particular, the stable twisted cohomology \( H^*(\Gamma_{\infty,1}; H_Q^*) \) is not free as \( \text{Sym}_R(\mathcal{E}) \)-module; see also Theorem 3.8.

**Perspectives.** The key steps of the results of Theorems A and B are proved with integral coefficients. Namely, the contraction formula and the connecting homomorphisms still hold for integral coefficients. Therefore, we might in principle be able to do the computations with \( \mathbb{Z} \) as ground ring. However, although the stable twisted cohomology module \( H^*(H) \) is free over \( H^*(\mathbb{Z}) \) (see [Kaw08, Theorem 1.3]), the stable cohomology \( H^*(\mathbb{Z}) \) is still poorly known.

On another note, a natural extension of the results of Theorems A and B consists in considering the exterior powers of the representations \( H_Q(g) \) and \( H_Q^*(g) \) respectively. This is the aim of the forthcoming work [KS]. In particular, the stable twisted cohomology algebras \( H^*(\Gamma_{\infty,1}; \Lambda^d\tilde{H}_Q^*) \) (for all \( d \geq 2 \)) and \( H^*(\Gamma_{\infty,1}; \Lambda^d H_Q^*) \) (for \( 2 \leq d \leq 5 \)) are thoroughly studied.

**Outline.** The paper is organised as follows. In §1, we make recollections on the representation theory of mapping class groups and on the classical and twisted Mumford-Morita-Miller cohomology classes. In §2, we recall the framework and properties for twisted homological stability of mapping class groups. In §3, we make the full computations of the mapping class groups stable twisted cohomology with coefficients in the first homology and cohomology of the unit tangent bundle.

**Conventions and notations.** For a ring \( R \), we denote by \( \text{R-Mod} \) the category of left \( R \)-modules. For \( R = \mathbb{Z} \), the category of \( \mathbb{Z} \)-modules is also denoted by \( \text{Ab} \). For a map \( f \), we generically (when everything is clear from the context) denote by \( f_* \) the induced map in homology and by \( f^* \) the induced map in cohomology. For \( G \) a group, \( R \) a commutative ring and \( V \) a \( R[G] \)-module, we denote by \( V^\vee \) the dual \( R[G] \)-module \( \text{Hom}_R(V, R) \). For \( \mathbb{K} \) a field and \( V \) a \( \mathbb{K} \)-vector space, we denote by \( \text{Sym}_g(V) \) the symmetric algebra on \( V \) over \( \mathbb{K} \).

Considering a \( G \)-module \( M \), if there is no risk of confusion, we generally denote the twisted cohomology groups \( H^*(G; M) \) by \( H^*(M) \) for sake of simplicity. We denote the cup product \( \cdot \), but also often abuse the notation denoting it by an empty space for simplicity when there is no risk of confusion. The first integral homology group \( H_1(\Sigma_{g,1}; \mathbb{Z}) \) is generally denoted by \( H(g) \) all along the paper, and we denote by \( H_Q(g) \) the first rational homology group \( H_1(\Sigma_{g,1}; \mathbb{Q}) \).
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1. Representations and cohomological structures

In this section, we review some representations of the mapping class groups, in particular the unit tangent bundle homology representations (see §1.2) for which we make the connection with the Earle class (see §1.1).

Let us first consider the first integral homology group of the surface \( \Sigma_{1,1} \), denoted by \( H(g) \). Since the Poincaré-Lefschetz duality is the cap product with the fundamental class, there is a \( \Gamma_{1,1} \)-module isomorphism between \( H^1(\Sigma_{1,1}; \mathbb{Z}) \) and \( H(g) \). By the universal coefficient theorem for cohomology of spaces (see [Wei94, Example 3.6.7]), we know that \( H^1(\Sigma_{1,1}; \mathbb{Z}) \) is isomorphic to \( H^\vee(g) \) as \( \Gamma_{1,1} \)-modules. A fortiori, we have a \( \Gamma_{1,1} \)-module isomorphism \( H^\vee(g) \cong H(g) \).

Let us now move on to the unit tangent bundle homology representations. We first need to recall the notion of framings of the unit tangent bundle. We denote by \( T\Sigma_{1,1} \) the tangent bundle of the surface \( \Sigma_{1,1} \) and fix a Riemannian metric \( \| \cdot \| \) on it. By definition, the unit tangent bundle \( UT\Sigma_{1,1} \) is the set of elements of \( T\Sigma_{1,1} \) whose length is 1 with respect to \( \| \cdot \| \). It can be regarded as the quotient of the complement of the zero section in \( T\Sigma_{1,1} \) by the action of the positive real numbers \( \mathbb{R}_+ \) by scalar multiplication. So, for any diffeomorphism \( \varphi \) of \( \Sigma_{1,1} \), its differential \( d\varphi \) acts on the unit tangent bundle \( UT\Sigma_{1,1} \). The canonical projection of the unit tangent bundle \( UT\Sigma_{1,1} \) onto the surface defines the locally trivial fibration \( S^1 \hookrightarrow UT\Sigma_{1,1} \cong \Sigma_{1,1} \).

Definition 1.1 A framing of \( UT\Sigma_{1,1} \) is an orientation-preserving isomorphism of vector bundles \( UT\Sigma_{1,1} \cong \Sigma_{1,1} \times S^1 \), which can be regarded as a continuous map \( \xi : UT\Sigma_{1,1} \to S^1 \) whose restriction to each fiber is an orientation-preserving homeomorphism. Since \( \Sigma_{1,1} \) has non-empty boundary, there exist framings of \( UT\Sigma_{1,1} \).

For \( \alpha : S^1 \to \Sigma_{1,1} \) an immersed loop, its rotation number \( \text{rot}\xi(\alpha) \in \mathbb{Z} \) with respect to the framing \( \xi \) is the mapping degree of the composite of \( \alpha/\|a\| : S^1 \to UT\Sigma_{1,1} \) and \( \xi : UT\Sigma_{1,1} \to S^1 \). The set \( \mathcal{F}(\Sigma_{1,1}) \) of homotopy classes (without fixing the boundary) of framings of \( UT\Sigma_{1,1} \) is an affine set modeled in the cohomology group \( H^1(\Sigma_{1,1}; \mathbb{Z}) \). More precisely, the difference of two framings \( \xi \) and \( \xi' \) is given by a cohomology class \( u \in H^1(\Sigma_{1,1}; \mathbb{Z}) \) if and only if \( \text{rot}\xi'(\alpha) - \text{rot}\xi(\alpha) = u([\alpha]) \in \mathbb{Z} \), where \([\alpha]\) is the fundamental class of the immersed loop \( \alpha \). The mapping class group \( \Gamma_{1,1} \) acts on the set \( \mathcal{F}(\Sigma_{1,1}) \) by

\[
\varphi \cdot \xi = \xi \circ d\varphi^{-1} : UT\Sigma_{1,1} \xrightarrow{d\varphi^{-1}} UT\Sigma_{1,1} \xrightarrow{\xi} S^1
\]

for \( \varphi \in \Gamma_{1,1} \) and \( \xi \in \mathcal{F}(\Sigma_{1,1}) \). For an immersed loop \( \alpha \) on \( \Sigma_{1,1} \), we have \( \text{rot}\varphi \circ \xi(\alpha) = \text{rot}\xi(\varphi^{-1} \circ \alpha) \).
1.1. Earle class

We recall here some classical facts about the first cohomology of mapping class groups with twisted coefficients in $H(g)$. For any $g \geq 1$, we recall that gluing a disc with a marked point $\Sigma_{g,1}$ on the boundary component of $\Sigma_{g,1}$ induces the following short exact sequence:

$$1 \rightarrow \mathbb{Z} \rightarrow \Gamma_{g,1} \xrightarrow{\text{cap}} \Gamma_g^1 \rightarrow 1,$$

where $\Gamma_g^1$ denotes the mapping class group of the punctured surface $\Sigma_g^1$. This is called the forgetful exact sequence or the capping short exact sequence; we refer the reader to [FM12, §4.2.5] for more details. In particular, considering the Lyndon-Hochschild-Serre spectral sequence associated to the short exact sequence (1.1), the fact that $H(g)^{\Gamma_g^1} = 0$ implies that the forgetful homomorphism $\text{Cap}$ provides an isomorphism $H^1(\Gamma_g^1; H(g)) \cong H^1(\Gamma_{g,1}; H(g))$. We denote by $e \in H^2(\Gamma_g^1; \mathbb{Z})$ the Euler class of the short exact sequence (1.1) seen as a central extension.

Furthermore, we make the following computation for the twisted first cohomology of the mapping class group of the torus with one boundary. This fact is probably known to the experts (see [CCS13] for instance), but we give a short proof for the convenience of the reader.

**Lemma 1.2** The cohomology group $H^1(\Gamma_{1,1}; H(1))$ is trivial.

**Proof.** It is a classical fact that the mapping class group $\Gamma_{1,1}$ is isomorphic to the braid group on three strands $B_3$, and that the standard generators $\sigma_1$ and $\sigma_2$ of $B_3$ act on $H_1(\Sigma_{1,1}; \mathbb{Z}) \cong \mathbb{Z}^2$ through the isomorphism by $\sigma_1 \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\sigma_2 \mapsto \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$ respectively; see for instance [Mil71, Theorem 10.5]. Let $f$ be a cocycle of $B_3$ with values in $H_1(\Sigma_{1,1}; \mathbb{Z}) \cong \mathbb{Z}^2$. It is determined by the values $f(\sigma_1) = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$ and $f(\sigma_2) = \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}$. We deduce from the braid relation $\sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2$ that $b_1 = a_2$ and $b_1 = -a_2$. Hence we have $a_2 = b_1 = 0$, and thus $f = d^1\begin{pmatrix} -b_2 \\ a_1 \end{pmatrix}$ is a coboundary. \hfill \square

Now, we assume that $g \geq 2$. Morita [Mor89a, Proposition 6.4] proves that $H^1(\Gamma_g^1; H(g)) \cong \mathbb{Z}$, and that the map induced by the push map $\pi_1(\Sigma_g; p_0) \rightarrow \Gamma_g^1$ maps its generators to $\pm (2 - 2g)1_{H(g)}$ in $H^1(\Sigma_g; H(g))$. Before this result, Earle [Ear78] constructed the generator by using theta constants, and so we call it the Earle class. The push map $\pi_1(UT\Sigma_g; p_0) \rightarrow \Gamma_g^1$ maps the generators of the latter group to $\pm (2 - 2g)$ times the projection homomorphism $H_1(UT\Sigma_g; \mathbb{Z}) \rightarrow H(g)$. Following Mikio Furuta [Mor97], one of the generators is constructed in the following explicit way. We fix a framing $\xi$ of $UT\Sigma_g$. The map $k_\xi(g) : \Gamma_{g,1} \rightarrow H^1(\Sigma_{g,1}; \mathbb{Z})$ defined by

$$k_\xi(g, \varphi) = \varphi \cdot \xi - \xi \in H^1(\Sigma_{g,1}; \mathbb{Z})$$

is a 1-cocycle of $\Gamma_{g,1}$. Kuno [Kun09] gives a combinatorial formula for the cocycle $k_\xi$. Under the Poincaré-Lefschetz duality, we have:

**Theorem 1.3** ([Mor97, §4]) For each $g \geq 2$, the cohomology class $k_g := [k_\xi(g)]$ does not depend on the choice of $\xi$ and generates the infinite cyclic group $H^1(\Gamma_{g,1}; H(g))$.

**Proof.** By the computation of [Mor89a], it suffices to compute the value of $k_\xi(g)$ at a push map. Such a computation was carried out in the original proof of [Mor97]. However, for sake of completeness, we give another explicit computation. Let $T_C \in \Gamma_{g,1}$ be the right-handed Dehn twist along an oriented simple closed curve $C$ of $\Sigma_{g,1}$. Then, it follows from some elementary considerations that

$$\text{rot}_C(T_C(\alpha)) - \text{rot}_C(\alpha) = ([\alpha] \cdot [C])\text{rot}(C),$$

where $[\alpha] : [C]$ denotes the (algebraic) intersection number of the homology classes $[\alpha]$ and $[C]$. We deduce that $\xi \circ dT_C - \xi = (\text{rot}(C))[C]$ and in particular that

$$k_\xi(g, T_C) = \xi \circ dT_C^{-1} - \xi = -(\text{rot}_C(\cdot)(C))[C] = -(\text{rot}_C(C))[C] \in H^1(\Sigma_{g,1}; \mathbb{Z}).$$
Assume that the curve $C$ passes near the boundary of $\Sigma_{g,1}$. Then, fattening the union of the boundary and the curve $C$, we obtain a pair of pants embedded in $\Sigma_{g,1}$ whose 3 boundary components are given as follows: one is parallel to the boundary of $\Sigma_{g,1}$, and the other two simple closed curves $C_1^{-1}$ and $C_2$ are parallel to $C$ except near the boundary. Then the push map along $C$ is equal to $T^{-1}C_1T^{-1}C_2 \in \Gamma_{g,1}$. By the above formula, we obtain $k_\xi(T^{-1}C_1T^{-1}C_2) = (-\rot(\partial C_2) + \rot(\partial C_1))[C]$. From the Poincaré-Hopf theorem, $-\rot(\partial C_1) + \rot(\partial C_2) + \rot(\partial \Sigma_{g,1})$ is equal to $-1$, the Euler characteristic of the pair of pants, and $\rot(\partial \Sigma_{g,1}) = \chi(\Sigma_{g,1}) = 1 - 2g$. Hence we have $k_\xi(T^{-1}C_1T^{-1}C_2) = (2 - 2g)[C]$, which ends the proof.

**Notation 1.4** By Smale’s theorem [Sma59, Theorem B], we have $H^1(\Gamma_{0,1}, H_1(\Sigma_{0,1}; \mathbb{Z})) = 0$ and we also assign $k(0) : \Gamma_{0,1} \to H^1(\Sigma_{0,1}; \mathbb{Z})$ and $k(1) : \Gamma_{1,1} \to H^1(\Sigma_{1,1}; \mathbb{Z})$ to be the trivial cohomology classes.

**1.2. The unit tangent bundle homology representations**

We consider the first integral homology group $\tilde{H}(g)$. The dual $\tilde{H}^\vee(g)$ is isomorphic to the first integral cohomology group $H^1(UT\Sigma_{g,1}; \mathbb{Z})$ by the universal coefficient theorem for cohomology of spaces (see [Wei94, Example 3.6.7]). Then the Serre spectral sequence of the locally trivial fibration $S^1 \hookrightarrow UT\Sigma_{g,1} \xrightarrow{\cong} \Sigma_{g,1}$ provides the following $\Gamma_{g,1}$-equivariant short exact sequences:

\[
0 \longrightarrow \mathbb{Z} \xrightarrow{i} \tilde{H}(g) \xrightarrow{\varepsilon} H(g) \xrightarrow{\partial} 0, \quad (1.3)
\]

\[
0 \longrightarrow H(g) \xrightarrow{\varepsilon} \tilde{H}^\vee(g) \xrightarrow{i} \mathbb{Z} \longrightarrow 0. \quad (1.4)
\]

We also have the analogue short exact sequences to (1.3) and (1.4) with the rational homologies $H_\mathbb{Q}(g)$, $\tilde{H}_\mathbb{Q}(g)$ and $\tilde{H}^\vee_\mathbb{Q}(g)$. We denote by $h \in \tilde{H}(g)$ the homology class of the fiber of the unit tangent bundle.

In addition, Trapp [Tra92, Theorem 2.2] describes more precisely the $\Gamma_{g,1}$-module structure. Namely, for an element $\varphi \in \Gamma_{g,1}$, the action of $\varphi$ on $\tilde{H}(g)$ is given by the matrix

\[
\begin{bmatrix}
\text{id}_\mathbb{Z} & k(g)(\varphi) \\
(0) & H(\varphi)
\end{bmatrix}
\]  

(1.5)

where $H(\varphi)$ denotes the action of $\varphi$ on $H$ and $k(g)$ is Earle’s cohomology class of Notation 1.4. In particular, the crossed homomorphism $k(g)$ is the extension class of the short exact sequences (1.3) and (1.4). We note that the $\Gamma_{g,1}$-action on $\mathbb{Z}$ is trivial representation. Also, although the sequences (1.3) and (1.4) split as abelian groups short exact sequences, they do not as $\Gamma_{g,1}$-modules. Hence, contrary to the homology and cohomology of the surface (where Poincaré-Lefschetz duality applies), the dual $\tilde{H}^\vee(g)$ is not isomorphic to $\tilde{H}(g)$ as a $\Gamma_{g,1}$-module.

**2. Homological stability framework and tools**

In this section, we review the notions of *coefficient systems*, *polynomiality* and *homological stability with twisted coefficients* with respect to the framework of the present paper. These highlight the interest in the work and mainspring of the results of §3.

**2.1. Twisted coefficient systems**

First of all, we present the suitable categories to encode *compatible representations* of the mapping class groups originally introduced in [RW17, §5.6.1]. We consider the groupoid defined by the smooth compact connected orientable surfaces $S$ with one boundary component along with a parametrized interval in the boundary, and the isotopy classes of diffeomorphisms restricting to the identity on a neighbourhood of the parametrized interval for the morphisms. We fix a 2-disc $\mathbb{D}^2$ and a torus with one boundary component that we denote by $\Sigma_{1,1}$. Let $\mathcal{M}_2$ be the skeleton of that groupoid defined as the full subgroupoid of the monoidal sums on the objects $\mathbb{D}^2$ and
The groupoid $\mathcal{M}_2$ has a (strict) braided monoidal structure induced by gluing on half of the marked interval; see [RW17, §5.6.1] for further details. Let $\mathcal{UM}_2$ be the category, called the Quillen’s bracket construction over $\mathcal{M}_2$, with the same objects as $\mathcal{M}_2$ and $\mathcal{UM}_2(\Sigma_{g,1}, \Sigma_{g',1})$ is the set Colim$_{\mathcal{M}_2}[\mathcal{M}_2(-, \Sigma_{g,1}, \Sigma_{g',1})]$. This definition is a particular output of a general construction of [Gra76]; we refer the reader to [RW17, §1.1] if additional details are required. As we will see in §2.2, this type of category is very useful to deal with homological stability questions.

We may now encode compatible representations of mapping class groups by considering functors with the category $\mathcal{UM}_2$ as source and a module category as target. We distinguish two types of such functors because of their distinct qualitative properties with respect to homological stability detailed in §2.2. A covariant system (resp. contravariant system) over $\mathcal{UM}_2$ is a functor $F: \mathcal{UM}_2 \to \mathbb{A}$ (resp. $F': \mathcal{UM}_2^{op} \to \mathbb{A}$). We now recursively define the notion of polynomiality for covariant systems as follows:

- the constant functors $\mathcal{UM}_2 \to \mathbb{A}$ are the polynomial covariant systems of degree 0;
- for a natural number $d \geq 1$, the functor $F: \mathcal{UM}_2 \to \mathbb{A}$ is a polynomial covariant system of degree less than or equal to $d$ if the morphism $F([\Sigma_{1,1}, id_{\Sigma_{1,1},2S}])$ is injective for each surface $S$ of $\mathcal{UM}_2$, and the induced functor $\delta(F): \mathcal{UM}_2 \to \mathbb{A}$ defined by $S \mapsto (\text{Coker}(F(S) \to F([\Sigma_{1,1}, id_S])))$ is a polynomial covariant system of degree less than or equal to $d - 1$.

A first example of polynomial covariant system is given by the first homology group of the surfaces. Namely, assigning the first integral homology group to each surface define a functor $\mathcal{U} \to \mathbb{A}$, with the same objects as $\mathcal{UM}_2$, again $\text{poly}$, which is strong monoidal in the sense that $\mathcal{H}(\Sigma_{g'+g,1}, \Sigma_{g',1})$ and $\mathcal{H}(\Sigma_{g,1})$ and $\mathcal{H}(\varphi') \cong \mathcal{H}(\varphi') \oplus \mathcal{H}(\varphi)$ for all $g, g' \geq 1, \varphi \in \Gamma_{g,1}$ and $\varphi' \in \Gamma_{g',1}$. In particular, $\mathcal{H}: \mathcal{UM}_2 \to \mathbb{A}$ is a polynomial covariant system of degree one. Furthermore, the first homology group of the unit tangent bundle of the surfaces along with the natural action of the mapping class groups (see §1.2) define a functor $\tilde{\mathcal{H}}: \mathcal{M}_2 \to \mathbb{A}$.

Lemma 2.1 Let $g$ and $g'$ be two natural numbers. For all $\varphi \in \Gamma_{g,1}$ and $\varphi' \in \Gamma_{g',1}$, we have

$$k(g' + g, \varphi' \varphi') = k(g', \varphi') + k(g, \varphi).$$

Proof. Let $\xi$ and $\xi'$ be framings on $\Sigma_{g,1}$ and $\Sigma_{g',1}$. Then the cocycles $k_\xi(g, \varphi) = \varphi \cdot \xi - \xi$ and $k_{\xi'}(g') = \varphi' \cdot \xi' - \xi'$ represent $k(g)$ and $k(g')$ respectively; see (1.2). Let $N$ and $N'$ be neighbourhoods of the parametrized intervals $I_{\Sigma_{g,1}}$ and $I_{\Sigma_{g',1}}$, respectively. Since each of the intervals is contractible, we can choose the framings $\xi$ and $\xi'$ such that $\xi'_{|I_{\Sigma_{g',1}}^\pm} = \xi_{|I_{\Sigma_{g,1}}^\pm}$ and their union define a framing of $N\sharp N$. Hence we define an appropriate framing $\xi' \sharp \xi$ on $\Sigma_{g' + g,1}$, which induces a cocycle. We thus deduce from (1.2) that

$$k(g' + g, \varphi' \varphi') = (\varphi' \cdot \xi' - \xi')^\sharp (\varphi \cdot \xi - \xi) = k(g', \varphi') + k(g, \varphi).$$

Hence we can lift the functor $\tilde{\mathcal{H}}(g)$ to $\mathcal{UM}_2$:

**Proposition 2.2** Assigning the canonical injection $id_{\mathbb{Z}} \oplus \mathcal{H}([\Sigma_{g',1}, id_{\Sigma_{g'+g,1}}]) : \mathbb{Z} \oplus \mathcal{H}(g') \to \mathbb{Z} \oplus \mathcal{H}(g' + g)$ for the morphism $\tilde{\mathcal{H}}([\Sigma_{g',1}, id_{\Sigma_{g'+g,1}}])$ for each $g, g' \geq 1$, defines a covariant system $\tilde{\mathcal{H}}: \mathcal{UM}_2 \to \mathbb{A}$ that lifts $\tilde{\mathcal{H}}: \mathcal{M}_2 \to \mathbb{A}$ and which is polynomial of degree 1.

Proof. For two natural numbers for $g, g' \geq 1$, we consider $\varphi \in \Gamma_{g,1}$ and $\varphi' \in \Gamma_{g',1}$. It follows from the action of the mapping class group on $\tilde{\mathcal{H}}(g)$ of (1.5), from Lemma 2.1 and from the fact that $\mathcal{H}$ is a strong monoidal functor that:

$$\tilde{\mathcal{H}}([\Sigma_{g',1}, id_{\Sigma_{g'+g,1}}]) \circ \tilde{\mathcal{H}}(\varphi) = \begin{bmatrix} id_{\mathbb{Z}} & (0) & k(g, \varphi) \\ (0) & id_{\mathbb{Z}_{g',1}} & (0) \\ (0) & (0) & \mathcal{H}(\varphi) \end{bmatrix} ; \tilde{\mathcal{H}}(\varphi' \varphi') = \begin{bmatrix} id_{\mathbb{Z}} & k(g', \varphi') & k(g, \varphi) \\ (0) & \mathcal{H}(\varphi') & (0) \\ (0) & (0) & \mathcal{H}(\varphi) \end{bmatrix}.$$
Lemma 1.2]. Finally, the morphism $\tilde{H}((\Sigma_{1,1}, \text{id}_{\Sigma_{1,1}}))$ is by definition an injection for each $g \geq 0$, and the induced functor $\delta(H(g)) : \mathcal{U}M_2 \to \text{Ab}$ is the constant covariant system at $\mathbb{Z}$. 

Remark 2.3 By Proposition 2.2, the first cohomology groups of the unit tangent bundle of the surfaces define a contravariant system $\tilde{H}^\vee : \mathcal{U}M_2^{op} \to \text{Ab}$. However, as far as the authors know, there does not exist any notion of polynomiality for such object.

2.2. Twisted homological stability and stable (co)homology

In this section, we review some classical results on homological stability with twisted coefficients for mapping class groups. In particular, these prove that all the twisted coefficient systems we consider in this paper satisfy the homological stability property and thus motivate the computations of §3. Also, we recall some computations and results on the stable homology of mapping class groups with twisted coefficients, which will be used for the work of §3.

2.2.1. Classical framework

Both homological stability and stable twisted homology are already well-studied for covariant systems, and we recollect the corresponding classical results in the following paragraphs.

Stable homology general framework. The following classical result illustrates how polynomial covariant systems turn out to be very useful for (co)homological stability problems.

Theorem 2.4 ([Iva93, Theorem 4.1], [RW17, Theorem 5.26]) Let $F : \mathcal{U}M_2 \to \text{Ab}$ be polynomial covariant system of degree $d$. Then, for each $g, i \geq 0$ such that $g \geq 2i + 2d + 3$, the injection $\Gamma_{g,1} \hookrightarrow \Gamma_{g+1,1}$ induces an isomorphism $H_i(\Gamma_{g,1}; F(g)) \cong H_i(\Gamma_{g+1,1}; F(g+1))$.

Example 2.5 Since the first homology groups $\{H_1(\Sigma_{g,1}; \mathbb{Z}), g \in \mathbb{N}\}$ define a polynomial covariant system of degree one $H : \mathcal{U}M_2 \to \text{Ab}$, there is an isomorphism $H_i(\Gamma_{g,1}; H(g)) \cong H_i(\Gamma_{g+1,1}; H(g+1))$ for $g \geq 2i + 5$.

Furthermore, the second author proves in [Sou20] a general decomposition for the stable homology of the mapping class groups with twisted coefficients given by a covariant system. In order to rephrase this result in terms of cohomology groups, we take this opportunity to recall and prove the following version of the Universal Coefficient theorem (for which it is difficult to find a reference), in order to make the connection between the $\Gamma_{g,1}$-modules $H(g)$ and $H^\vee(g)$.

Lemma 2.6 Let $G$ be a group, $R$ be a principal ideal domain, and $M$ a left $R[G]$-module which is free as a $R$-module. We denote by $M^\vee$ the dual right $R[G]$-module $\text{Hom}_R(M, R)$. Then there is a natural short exact sequence admitting a non-canonical splitting:

$$0 \longrightarrow \text{Ext}_R^1(H_{i-1}(G, M), R) \longrightarrow H^i(G, M^\vee) \longrightarrow \text{Hom}_R(H_i(G, M), R) \longrightarrow 0.$$ 

Proof. Let $P_* \to R$ be a projective right $R[G]$-module resolution. Then $\text{Hom}_{R[G]}(P_*, M^\vee)$ is a cochain complex computing $H^*(G; M^\vee)$. The tensor-hom adjunction provides a natural isomorphism $\text{Hom}_{R[G]}(P_*, M^\vee) \cong \text{Hom}_R(P_* \otimes_{R[G]} M, R)$. Since a submodule of a free module over a principal ideal domain is free, all the terms of the resolution $P_*$ are $R$-free: since $M$ is also $R$-free, so is $P_* \otimes_{R[G]} M$. Hence, the result follows from applying the universal coefficient theorem for chain complexes over a principal ideal domain (see [Wei94, Theorem 3.6.5] for instance) on the right-hand side of the isomorphism. 

Then, using Lemma 2.6, we deduce from [Sou20, Theorem C]:

Theorem 2.7 For $\mathbb{K}$ a field and any covariant system $F : \mathcal{U}M_2 \to \mathbb{K}$-Mod, for each $g, i \geq 0$ such that $g \geq 2i + 2d + 3$, we have a natural isomorphism of $\mathbb{K}$-modules

$$H^i(\Gamma_{g,1}; F^\vee(g)) \cong \bigoplus_{k+i+1 = i} H^k(\Gamma_{g,1}; \mathbb{K}) \otimes_{\mathbb{K}} H_i(\mathcal{U}M_2; F).$$
In particular, the stable twisted cohomology algebra $H^*(\Gamma_{\infty,1}; F^\infty)$ is a free $\text{Sym}_Q(\mathcal{E})$-module.

**Remark 2.8** The result of Theorem 2.7 does not depend on any polynomiality condition, and more generally on whether there is homological stability or not: the formula holds in general for the colimit of the homology groups which always exists.

**Twisted cohomology classes with coefficients in $H_Q$.** We now review the stable twisted cohomology group with coefficients in the homology group of the surface. For the reference [KM96], we will rather quote the preprint version [KMW96] as it contains more content and details.

We recall the short exact sequence (1.1) and its Euler class $e \in H^2(\Gamma^1_g; \mathbb{Z})$. Based on the Harer stability [Har85], Harer [Har91, Theorem 7.1(b)] proves that in the stable range $H_{k-1}(\Gamma^1_g; H_Q) \cong \bigoplus_{i \geq 1} H_{k-2i}(\Gamma^1_g; \mathbb{Q})$ and $H_{k-1}(\Gamma^1_{g,1}; H_Q) \cong \bigoplus_{i \geq 1} H_{k-2i}(\Gamma_{g,1}; \mathbb{Q})$. The first isomorphism says that there is an element in $H^{2l-1}(\Gamma^1_g; H_Q)$ for each $l \geq 1$, such that these elements form a free basis of the free $H^*(\Gamma^1_g; H_Q)$-module $H^*(\Gamma^1_{g,1}; H_Q)$. The element is written by $m_{l,1} \in H^{2l-1}(\Gamma^1_{g,1}; H_Q)$, $l \geq 1$, in the terminology of the first author [Kaw98; KM96]. So we have $H^*(\Gamma^1_{g,1}; H_Q) \cong \bigoplus_{i \geq 1} H^*(\Gamma^1_g; Q)m_{l,1}$, in the stable range. Similarly the pullback of $m_{l,1}$, $l \geq 1$, on $\Gamma_{g,1}$ gives an isomorphism in the stable range:

$$H^*(\Gamma_{g,1}; H_Q) \cong \bigoplus_{l \geq 1} H^*(\Gamma_{g,1}; Q)m_{l,1} \quad (2.1)$$

For further details, see also [KMW08, Theorem 1.3].

In the original definition [Kaw98] the cohomology class $m_{l,1}$ was constructed on the group $\Gamma_{g,1}$. Later the following definition of $m_{l,1}$ on $\Gamma^1_g$ was introduced in [KM96]. Let $\pi^1: \Gamma^1_g \to \Gamma_g$ be the forgetful map of the puncture. Let $\Gamma^1_g$ be the pullback $\Gamma^1_g \times_{\Gamma_g} \Gamma^1_{g,1}$. More precisely, there is a defining fiber square

$$\begin{array}{ccc}
\Gamma^1_g & \to & \Gamma^1_g \\
\downarrow \sigma & & \downarrow \pi^1 \\
\Gamma^1_{g,1} & \to & \Gamma_{g,1}
\end{array}$$

where the section $\sigma: \Gamma^1_g \to \Gamma^1_{g,1}$ is given by $\sigma(\phi) = (\phi, \phi)$. We deduce that there is an isomorphism $\Gamma^1_g \cong \pi_1(\Sigma_g) \times \Gamma^1_{g,1}$ defined by $(\phi, \psi) \mapsto (\psi^{-1}, \phi)$. Under this isomorphism, $\sigma$ is given by $\sigma(\phi) = (1, \phi)$. Similarly to [Mor98, §7], this semi-direct product decomposition gives rise to a cocycle $\varpi \in Z^1(\Gamma^1_g; H(g))$ given by $\varpi(x, \phi) = |x|$ for all $x \in \pi_1(\Sigma_g)$ and $\phi \in \Gamma^1_g$. By an abuse of notation, we also use $\varpi$ to denote the associated element of $H^1(\Gamma^1_g; H(g))$. We denote by $\bar{e} \in H^2(\Gamma^1_g; \mathbb{Z})$ the pullback of the Euler class $e \in H^2(\Gamma^1_g; \mathbb{Z})$ by the second projection $\Gamma^1_g \to \Gamma^1_{g,1}$, $(\phi, \psi) \mapsto \psi$. The class $m_{l,1}$ is defined by

$$m_{l,1} := \pi_1(e^i \cdot \varpi) \in H^{2l-1}(\Gamma^1_g; H). \quad (2.2)$$

**Contraction formula.** Finally, we recall a classical operation on the twisted Mumford-Morita-Miller class induced by the contraction of the twisted coefficients. Let $\mu: H(g) \otimes H(g) \to \mathbb{Z}$ be the intersection pairing associated to Poincaré-Lefschetz duality. Let $M$ and $M'$ be two $\Gamma_{g,1}$-modules. The **contraction map** is the abelian group morphism $\text{id}_M \otimes \mu \otimes \text{id}_{M'}: M \otimes H(g) \otimes H(g) \otimes M' \to M \otimes M'$. The induced map for the cohomology groups is generically denoted by $(\text{id}_M \otimes \mu \otimes \text{id}_{M'})_*$.

**Proposition 2.9** ([KM96, Theorem 6.2]) For all $l, l' \geq 1$, we have

$$\mu_*(m_{l,1}m_{l',1}) = -e_{l+l'-1} \in H^*(\Gamma_{g,1}; \mathbb{Q}). \quad (2.3)$$

**Sketch of proof.** We consider the short exact sequence

$$1 \to \pi_1(\Sigma_{g,1}, x) \to \Gamma^1_{g,1} \to \Gamma_{g,1} \to 1 \quad (2.4)$$
known as the forgetful exact sequence or the Birman short exact sequence; we refer the reader to [FM12, §4.2.1] for more details. As is proved in [KM01, Theorem 5.3], the Lyndon-Hochschild-Serre spectral sequence for the group extension (2.4) induces a canonical decomposition

\[ H^*(\Gamma; M) \cong H^*(\Gamma^1_g; M) \oplus H^{*-1}(\Gamma^1_g; H(g) \otimes M) \oplus H^{*-2}(\Gamma^1_g; M) \]

for any \( \Gamma^1_g \)-module \( M \). It is multiplicative and described explicitly by using the cohomology classes \( \omega \) and \( \tau \). The following formula is then deduced from a direct computation based on the decomposition: for \( m \in H^*(\Gamma^1_g; M) \) and \( m' \in H^*(\Gamma^1_g; M') \), we have

\[ (\text{id}_M \otimes \mu \otimes \text{id}_M)_*(\pi_i(m \otimes \omega \otimes m')) = \pi_i(m \otimes m') + s^*(m) \pi_1(m') + \pi_1(m) s^*(m') - e \pi_1(m) \pi_1(m'). \]  

(2.5)

Here \( s : \Gamma^1_g \to \Gamma^1_g, \phi \mapsto (\phi, \phi) \), is the diagonal map. In particular, we have

\[ \mu_*(m \otimes m'; 1) = -e + e\nu - 1 + e' \nu - 1 - e e\nu - 1. \]

Since the Euler class \( e \) vanishes on \( \Gamma_{g,1} \), we deduce formula (2.3) \( \square \)

**Remark 2.10** Analogous formulas to (2.5) are computed in [KR19, Proposition 3.10] for mapping class groups of surfaces and higher even-dimensional manifolds. With the conventions of [KR19, §3], there is no \(-1\) sign.

### 2.2.2. Exotic situations

On the basis of current knowledge, contrary to the covariant cases, there is no general framework to deal with cohomological stability with twisted coefficients by covariant systems. However, the functor \( \tilde{H} : \mathcal{U}\mathcal{M}_2 \to \mathcal{A} \) defined by the groups \( \{H_1(UT\Sigma_{g,1}; Z), g \in \mathbb{N}\} \) define satisfies that property. Indeed, we consider the short exact sequences \( 0 \to \mathbb{Q} \to \tilde{H}(g) \to H(g) \to 0 \) and \( 0 \to \mathbb{Q} \to \tilde{H}(g+1) \to H(g+1) \to 0 \) obtained from (1.3). Then, the two induced long exact sequences in cohomology are connected by the images of the images by \( \tilde{H}, H \) and the constant group at \( \mathbb{Q} \) of the canonical arrow \( [\Sigma_{1,1}, \text{id}_{\Sigma_{g+1,1}}] \), which thus define a ladder of commutative diagrams. We deduce the homological stability with twisted coefficients in \( \tilde{H} \) from Example 2.5 with a straightforward induction on the cohomological degree and the use of the five lemma.

### 3. Twisted stable cohomology computations

In this section, we compute the stable twisted cohomology given by the first cohomology group of its unit tangent bundle \( \tilde{H}(g) \) (see §3.1) and by the first homology group of its unit tangent bundle \( \tilde{H}(g) \) (see §3.2); see Theorems 3.3 and 3.9. In both cases, the work relies on the determination of the connecting morphisms of the long exact sequences associated to the short exact sequences (1.4) and (1.3) respectively. We fix the following conventions for notations for all the remainder of the paper.

**Convention 3.1** Considering a functor \( M : \mathcal{U}\mathcal{M}_2 \to \mathcal{A} \), we denote the stable twisted cohomology groups \( H^*(\Gamma_{g,1}; M) \) by \( H^*(M) \). From now on, we implicitly assume that \( g \geq 2i + 5 \) each time we consider a cohomological degree \( i \) for \( H^*(\Gamma_{g,1}; M(g)) \) where \( M(g) = \tilde{H}(g) \) or \( H\tilde{g}(g) \), for the homological stability bound of Theorem 2.4 to be reached. Also, we denote by \( H^{\text{odd}}(M) \) and \( H^{\text{even}}(M) \) the \( \mathbb{N} \)-graded submodules of \( H^*(M) \) defined by \( \{H^{2i+1}(M), i \in \mathbb{N}\} \) and \( \{H^{2i}(M), i \in \mathbb{N}\} \) respectively.

### 3.1. First cohomology group system

We start by studying the stable cohomology groups of the mapping class groups \( \Gamma_{g,1} \) with twisted coefficient given by \( H\tilde{g}(g) \). Let \( \delta^i : H^i(\Gamma_{g,1}; Z) \to H^{i+1}(\Gamma_{g,1}; H(g)) \) be the \( i \)th connecting morphism of the cohomology long exact sequence associated with the extension (1.4).

**Lemma 3.2** The connecting morphism of the cohomology long exact sequence associated (1.4) is equal to \( m_{1,1} \cdot - \).
Proof. Let \([z]\) be a cohomology class of \(H^i(\Gamma_{g,1}; \mathbb{Z})\). We use the normalized cochain complex and generically denote by \(\partial\) its differentials. We recall from §1.2 that \(h \in \check{H}(g)\) denotes the homology class of the fiber of the unit tangent bundle, and from Definition 1.1 that \(\xi\) denotes the homotopy class of a framing of the tangent bundle \(UT\Sigma_{g,1}\). We deduce from the action of \(\Gamma_{g,1}\) on \(\check{H}^\vee(g)\) (see dual of the matrix \((1.5)\)) that the map \(s = s_\xi: \mathbb{Z} \hookrightarrow \check{H}^\vee(g)\) defined by \(u \mapsto u + k_\xi(g, -)^v\) defines a splitting (as an abelian group morphism) to the surjection \(\check{H}^\vee(g) \twoheadrightarrow \mathbb{Z}, v \mapsto v(h)\). Since \(z\) is a cocycle, we have \(\partial z = 0\) and therefore that

\[-s\varphi_0(z([\varphi_1 | \cdots | \varphi_i])) = \sum_{j=1}^i (-1)^j s(z([\varphi_0 | \cdots | \varphi_j \varphi_{j+1} | \cdots | \varphi_i])) + (-1)^i s(z([\varphi_0 | \cdots | \varphi_{i-1}])).\]

Since \(\delta^i[z] = [\partial(s(z))]\) by the formal definition of \(\delta^i\), it follows from \((3.1)\) and from \((2.2)\) on the extension classes that

\[\delta^i([z])([\varphi_0 | \cdots | \varphi_i]) = (\varphi_0 \varphi_0^{-1} - s)\varphi_0(z([\varphi_1 | \cdots | \varphi_i])) = k_\xi(g, \varphi_0)\varphi_0(z([\varphi_1 | \cdots | \varphi_i])) = (m_{1,1} \cdot z)([\varphi_0 | \cdots | \varphi_i]),\]

which ends the proof. \(\boxdot\)

We consider the long exact sequence in cohomology applied to the rational version of \((1.4)\) to make further computations. We thus obtain the exact sequence for all natural numbers \(i \geq 0:\)

\[
\begin{array}{ccccccccc}
H^{2i}(\check{H}^\vee_Q(g)) & \longrightarrow & H^{2i}(\mathbb{Q}) & \stackrel{m_{1,1} -}{\longrightarrow} & H^{2i+1}(H_Q(g)) & \longrightarrow & H^{2i+1}(\check{H}^\vee_Q(g)),
\end{array}
\]

since \(H^{2i}(\mathbb{Q}) = 0\) by \((0.1)\) and \(H^{2i+1}(H_Q(g)) = 0\) by \((2.1)\). Since \(m_{1,1} \cdot e_\alpha = e_\alpha m_{1,1}\) for all \(\alpha \geq 1\), we deduce that:

**Theorem 3.3** The map \(m_{1,1} - \) is injective. Therefore, the \(\text{Sym}_Q(\mathcal{E})\)-module \(H^{2i}(\check{H}^\vee_Q(g))\) is null and \(H^{2i+1}(\check{H}^\vee_Q(g))\) is isomorphic to \(\text{Sym}_Q(\mathcal{E})\{m_{a,1}, a \geq 2\}\).

**Remark 3.4** (Interpretation in terms of functor homology.) The modules \(\check{H}^\vee_Q(g)\) define a contravariant twisted coefficient system; see Proposition 2.2. It follows from Lemma 2.6 and Theorem 2.7 that \(H^*(\check{H}^\vee_Q(g)) \cong \text{Sym}_Q(\mathcal{E}) \otimes_Q H_*(\mathcal{U}(M_2; \check{H}_Q(g)))\). In particular, this explains why the stable cohomology is a free \(\text{Sym}_Q(\mathcal{E})\)-module. Then the long exact sequence for the homology of categories (see [FP03, §2] for instance) associated with the short exact sequence \((1.4)\) directly gives that \(H_i(\mathcal{U}(M_2; \check{H}_Q(g))) \cong H_i(\mathcal{U}(M_2; H_0(g)))\) if \(i \geq 2\). However, we need the above reasoning to compute that \(H_0(\mathcal{U}(M_2; \check{H}_Q(g))) = H_0(\mathcal{U}(M_2; H_0(g))) = 0\).

### 3.2. First homology group system

Constraining with the situation of §3.1, we now study the stable cohomology groups of the mapping class groups \(\Gamma_{g,1}\) with twisted coefficient in the first cohomology group of the unit tangent bundle of the surface.

#### 3.2.1. Determination of the connecting homomorphism

Let \(\delta^i: H^i(\Gamma_{g,1}; H(g)) \rightarrow H^{i+1}(\Gamma_{g,1}; \mathbb{Z})\) be the \(i\)th connecting homomorphism of the cohomology long exact sequence associated with the extension \((1.3)\).

**Lemma 3.5** The morphism \(\delta^i\) is equal to \(\mu(m_{1,1}, -)\).

*Proof.* Let \([z]\) be a cohomology class of \(H^i(\Gamma_{g,1}; H(g))\). We use the normalized cochain complex and generically denote by \(\partial\) its differentials. We deduce from the action of \(\Gamma_{g,1}\) on \(\check{H}(g)\) (see the matrix \((1.5)\)) that the homotopy class of a framing \(\xi\) of the tangent bundle \(UT\Sigma_{g,1}\) induces a map.
We may thus compute the homology of that algebra: \( \text{Sym} \) is stable under the action of the algebra \( T \) or \( \text{Sym} \). Since \( \delta \) deduce that:

\[
- s' \varphi_0(z([\varphi_1 | \cdots | \varphi_i])) = \sum_{j=1}^{i} (-1)^{i} s' (z([\varphi_0 | \cdots | \varphi_j \varphi_{j+1} | \cdots | \varphi_i])) + (-1)^{i+1} s' (z([\varphi_0 | \cdots | \varphi_{i-1}])).
\]

Since \( \delta'[z] = [\partial(s'(z))] \) by the formal definition of \( \delta' \), it follows from (3.3) that

\[
\delta'([z])([\varphi_0 | \cdots | \varphi_i]) = (\varphi_0 s' - s' \varphi_0)(z([\varphi_1 | \cdots | \varphi_i])).
\]

On the other hand, we recall that the extension class of the short exact sequences (1.3) is Earle’s cohomology class \( m_{1,1} = k(g) = [k_{\xi}(g, -)] \); see (2.2). We thus compute from the definition of the contraction morphism that

\[
\mu(m_{1,1}, z)([\varphi_0 | \cdots | \varphi_i]) = \mu(k_{\xi}(g, \varphi_0) \varphi_0 \otimes z([\varphi_1 | \cdots | \varphi_i])) = (\varphi_0 s' - s' \varphi_0)(z([\varphi_1 | \cdots | \varphi_i]))
\]

which ends the proof. \( \square \)

### 3.2.2. Computation of the stable twisted cohomology

To make further computations, we must restrict to considering the rational homology representations. We recall from (0.1) that \( H^{odd}(\mathbb{Q}) = 0 \), and from (2.1) that thus \( H^{even}(\mathbb{Q}) = 0 \). Then we deduce from the long exact sequence in cohomology applied to (1.3) that \( H^0(\mathbb{Q}) \cong H^0(\mathbb{H}) \cong \mathbb{Q} \theta \) and obtain the exact sequence for all natural numbers \( i \geq 0 \):

\[
H^{2i+1}(\mathbb{H}) \longrightarrow H^{2i+2}(\mathbb{Q}) \longrightarrow H^{2i+2}(\mathbb{H}). \tag{3.4}
\]

From the contraction formula (2.3), we then compute that \( \mu(m_{1,1}, -) = -\theta \) for all \( \alpha \geq 1 \), and deduce that:

**Proposition 3.6** The map \( \mu(m_{1,1}, -) \) is surjective. Therefore, the \( \text{Sym}_\mathbb{Q}(\mathcal{E}) \)-module \( H^{even}(\mathbb{H}) \) is isomorphic to the trivial \( \text{Sym}_\mathbb{Q}(\mathcal{E}) \)-module \( H^0(\mathbb{H}) \cong \mathbb{Q} \theta \) and \( H^{odd}(\mathbb{H}) \) is isomorphic to the kernel of the graded morphism \( \mu^{odd}(m_{1,1}, -) \): \( H^{odd}(\mathbb{H}) \rightarrow \text{Sym}_\mathbb{Q}(\mathcal{E}) \).

Therefore, we have the following exact sequence of \( \text{Sym}_\mathbb{Q}(\mathcal{E}) \)-modules

\[
H^{odd}(\mathbb{H}) \longrightarrow H^{odd}(\mathbb{H}) \overset{\mu^{odd}(m_{1,1}, -)}{\longrightarrow} \text{Sym}_\mathbb{Q}(\mathcal{E}) \overset{\text{aug}}{\longrightarrow} \mathbb{Q} \longrightarrow 0. \tag{3.5}
\]

Moreover, using the injection \( H^{odd}(\mathbb{H}) \hookrightarrow H^{odd}(\mathbb{H}) \) and the surjection \( \text{Sym}_\mathbb{Q}(\mathcal{E}) \twoheadrightarrow \mathbb{Q} \) and noting that there is a compatibility with respect to the grading because \( H^{odd}(\mathbb{H}; \mathbb{Q}) = 0 \), we deduce that:

**Lemma 3.7** The cohomology groups \( H^*(\mathbb{H}) \) inherits a \( \text{Sym}_\mathbb{Q}(\mathcal{E}) \)-module structure from \( H^*(\mathbb{H}) \) and \( \text{Sym}_\mathbb{Q}(\mathcal{E}) \). Moreover, the decomposition

\[
H^*(\mathbb{H}) = H^{even}(\mathbb{H}) \oplus H^{odd}(\mathbb{H}) \tag{3.6}
\]

is stable under the action of the algebra \( \text{Sym}_\mathbb{Q}(\mathcal{E}) \).

### Computations of Tor-groups.

In order to give some qualitative properties of the stable twisted cohomology groups that we study, we may consider these twisted cohomology groups as forming an algebra \( H^*(\mathbb{H}) \) over the stable cohomology algebra \( \text{Sym}_\mathbb{Q}(\mathcal{E}) \) as follows.

We may thus compute the homology of that algebra:
Theorem 3.8 For any \( j > 0 \), we have \( \text{Tor}_j^{\text{Sym}_Q(\mathcal{E})}(\mathbb{Q}, H^*(\tilde{H}_Q)) \cong \Lambda^j \mathcal{E} \oplus \Lambda^{j+2} \mathcal{E} \). In particular, the \( \text{Sym}_Q(\mathcal{E}) \)-module \( H^*(\tilde{H}_Q) \) is not free. Furthermore, we have \( \text{Tor}_{j+1}^{\text{Sym}_Q(\mathcal{E})}(\mathbb{Q}, H^*(\tilde{H}_Q)) \cong \Lambda^2 \mathcal{E} \oplus \mathbb{Q} \theta \).

Proof. Using the general decomposition (3.6), the results follow from the respective computations of the left derived functors \( \text{Tor}_j^{\text{Sym}_Q(\mathcal{E})}(\mathbb{Q}, -) \) of the \( \text{Sym}_Q(\mathcal{E}) \)-modules \( H^{\text{even}}(\tilde{H}_Q) \) and \( H^{\text{odd}}(\tilde{H}_Q) \). It straightforwardly follows from the Koszul resolution (see [Lod98, §3.4.6] for instance) that \( \text{Tor}_j^{\text{Sym}_Q(\mathcal{E})}(\mathbb{Q}, \theta) \cong \Lambda_j \mathcal{E} \) for \( j \geq 1 \) and \( \mathbb{Q} \) if \( j = 0 \).

We note that the projection map \( \text{Sym}_Q(\mathcal{E}) \to \mathbb{Q} \theta \) induces isomorphisms \( \text{Tor}_0^{\text{Sym}_Q(\mathcal{E})}(\mathbb{Q}, \text{Sym}_Q(\mathcal{E})) \cong \mathbb{Q} \cong \text{Tor}_0^{\text{Sym}_Q(\mathcal{E})}(\mathbb{Q}, \theta) \). We thus deduce from the long exact sequence for the left derived functor \( \text{Tor}_{\text{Sym}_Q(\mathcal{E})} \), associated with the short exact sequence \( \text{Im}(\mu^{\text{odd}}(m_{1,1}, -)) \hookrightarrow \text{Sym}_Q(\mathcal{E}) \to \mathbb{Q} \theta \), that for any \( j \geq 0 \)

\[
\text{Tor}_j^{\text{Sym}_Q(\mathcal{E})}(\mathbb{Q}, \text{Im}(\mu^{\text{odd}}(m_{1,1}, -))) \cong \text{Tor}_{j+1}^{\text{Sym}_Q(\mathcal{E})}(\mathbb{Q}, \theta).
\]

We recall that \( H^{\text{odd}}(\tilde{H}_Q) \) is a free \( \text{Sym}_Q(\mathcal{E}) \)-module by (2.1). Since \( \mu(m_{1,1}, m_{\alpha,1}) = -\epsilon_\alpha \) for all \( \alpha \geq 1 \), the map \( H^{\text{odd}}(\tilde{H}_Q) \to \text{Im}(\mu^{\text{odd}}(m_{1,1}, -)) \) induces an isomorphism

\[
\text{Tor}_0^{\text{Sym}_Q(\mathcal{E})}(\mathbb{Q}, H^{\text{odd}}(\tilde{H}_Q)) \cong \mathcal{E} \cong \text{Tor}_0^{\text{Sym}_Q(\mathcal{E})}(\mathbb{Q}, \text{Im}(\mu^{\text{odd}}(m_{1,1}, -)));
\]

because \( \text{Im}(\mu^{\text{odd}}(m_{1,1}, -)) \) equals the augmentation ideal of \( \text{Sym}_Q(\mathcal{E}) \). Therefore, the long exact sequence for the left derived functor \( \text{Tor}_{\text{Sym}_Q(\mathcal{E})} \) associated with the short exact sequence deduced from Proposition 3.6

\[
0 \to H^{\text{odd}}(\tilde{H}_Q) \to H^{\text{odd}}(\tilde{H}_Q) \to \text{Im}(\mu^{\text{odd}}(m_{1,1}, -)) \to 0
\]

provides that \( \text{Tor}_j^{\text{Sym}_Q(\mathcal{E})}(\mathbb{Q}, H^{\text{odd}}(\tilde{H}_Q)) \cong \text{Tor}_{j+1}^{\text{Sym}_Q(\mathcal{E})}(\mathbb{Q}, \theta) \) for all \( j \geq 0 \), which ends the proof. \( \square \)

In our forthcoming paper [KS], we will compute \( \text{Tor}_j^{\text{Sym}_Q(\mathcal{E})}(\mathbb{Q}, \Lambda^d H^*(\tilde{H}_Q)) \) for \( 2 \leq d \leq 5 \). In particular, this group is non-trivial for each degree \( j \geq 0 \), except for the case \( d = 2 \).

Computation of the stable cohomology. Actually, we may explicitly describe the generators and relations of the module \( H^{\text{odd}}(\tilde{H}_Q) \) as follows. We consider \( \Omega^n_{\text{Sym}_Q(\mathcal{E})|\mathbb{Q}} \cong \text{Sym}_Q(\mathcal{E}) \otimes \Lambda^n \mathcal{E} \) the \( \text{Sym}_Q(\mathcal{E}) \)-module of \( n \)-differential forms and the classical exterior derivative \( d : \Omega^n_{\text{Sym}_Q(\mathcal{E})|\mathbb{Q}} \to \Omega^{n+1}_{\text{Sym}_Q(\mathcal{E})|\mathbb{Q}} \); see [Lod98, pp.26-27]. We consider the Euler vector field

\[
D := \sum_{i=1}^{\infty} e_i \frac{\partial}{\partial e_i}.
\]

The interior product \( p_D : \Omega^n_{\text{Sym}_Q(\mathcal{E})|\mathbb{Q}} \to \Omega^{n-1}_{\text{Sym}_Q(\mathcal{E})|\mathbb{Q}} \) with respect to \( D \) is a \( \text{Sym}_Q(\mathcal{E}) \)-linear derivation satisfying \( p_D(m_{1,1}) = e_i \) for all \( i \geq 1 \). Here we identify each \( m_{1,1} \) with the 1-form \( de_i \). Hence we deduce from Proposition 3.6 that

\[
H^{\text{odd}}(\tilde{H}_Q) \cong \text{Ker}(p_D : \Omega^1_{\text{Sym}_Q(\mathcal{E})|\mathbb{Q}} \to \Omega^0_{\text{Sym}_Q(\mathcal{E})|\mathbb{Q}}).
\]

Using this description, we can prove:

Theorem 3.9 The \( \text{Sym}_Q(\mathcal{E}) \)-module \( H^{\text{odd}}(\tilde{H}_Q) \) is generated by \( M_{i,j} := e_i m_{j,1} - e_j m_{i,1} \) for \( i, j \geq 1 \), with relations

\[
e_i M_{j,k} + e_j M_{k,i} + e_k M_{i,j} \sim 0
\]

for all \( i, j, k \geq 1 \).
A generalization of the Morita-Mumford classes to extended mapping class groups.

The primary approximation to the cohomology of the moduli families of Riemann surfaces and Jacobi varieties.

The cohomology of the braid group and Theorem E.

The third homology group of the moduli space of curves and cocycles for the Mumford-Morita-Miller classes and of Johnson homomorphisms do not apply since the modules $\tilde{H}_Q$.

Remark 3.10 (Interpretation in terms of functor homology.) Contrary to the case of $\tilde{H}_Q(y)$ (see Remark 3.4), Lemma 2.6 and Theorem 2.7 do not apply since the modules $\tilde{H}_Q(y)$ define a covariant twisted coefficient system.

References


