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# Explicit construction of reproducing kernels

O. Lafitte<sup>‡</sup>

## Abstract

In this short Note, we calculate explicitly reproducing kernels for simple Sobolev spaces. In particular, we prove that for each  $m$  the reproducing kernel of  $H^{m+1}(\mathbb{R}^{2m})$  and of  $H^{m+1}(\mathbb{R}^{2m+1})$  are proportional to the ones of  $H^2(\mathbb{R}^2)$  and of  $H^2(\mathbb{R}^3)$ , the former being characterized by the modified Hankel function  $K_1$ , the latter by the modified Hankel function  $K_{\frac{1}{2}}$ .

## 1 Introduction and main results

This short Note gives and recalls the exact construction of the reproducing kernels of classical Sobolev spaces, in particular under the idea that, instead of describing first the reproducing kernel on a basis of examples  $\{x_i, 1 \leq i \leq N\}$  and then of inferring the Hilbert space obtained by density of the associated generating family  $\{K_{x_i}\}_{1 \leq i \leq N}$ , we start from the Hilbert space itself, in the line of L. Schwartz [4], and following closely the results of E. De Vito, N. Mucke, and L. Rozasco [1].

We begin this Note with a simple calculation of the reproducing kernel(s) of  $H^1([0, L])$  with the dependency of the chosen norm. In a second part, one of the extremely interesting results of [1] is that the reproducing kernel for the usual space  $H^s(\mathbb{R}^d)$  using the classical Sobolev norm (1) depends only, in terms of  $|x - y|_2$  of  $s - \frac{d}{2} > 0$ . In particular, when dealing with the case  $s \in \mathbb{N}$  (where one can also deduce the reproducing kernel with an usual variational formulation with finite elements), only two cases appear (using the largest possible Sobolev space of the form  $H^n(\mathbb{R}^d)$  for which the trace is continuous):  $d$  odd and  $d$  even, with the modified Bessel functions  $K_1$  and  $K_{\frac{1}{2}}$ . We slightly add a complement to this paper by showing another reproducing kernel of  $H^2(\mathbb{R}^2)$  associated with a norm equivalent to the classical norm.

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## 2 The reproducing kernels on $H^1([0, L])$ , $H_0^1([0, L])$ , $H^1(\mathbb{R})$

It is an exercise to obtain the reproducing kernels of these spaces, and the result is as follows.

**Lemma 1.** 1. The reproducing kernel of  $H^1(\mathbb{R})$  with the usual norm  $\sqrt{\int_{\mathbb{R}} (u^2 + (u')^2) dx}$  is  $K_1(x, y) = \frac{1}{2} e^{-|x-y|}$ .

2. The reproducing kernel of  $H^1([0, L])$  is  $K_2(x, y) = \frac{\cosh x \cosh y 1_{y < x} + \cosh(L-x) \cosh(L-y) 1_{y > x}}{\cosh L}$ .

3. The reproducing kernel of  $H_0^1([0, L])$  with the usual norm  $\sqrt{\int_{\mathbb{R}} (u^2 + (u')^2) dx}$  is  $K_3(x, y) = \frac{\cosh x \sinh y 1_{y < x} + \cosh(L-x) \sinh(L-y) 1_{y > x}}{\cosh L}$ .

4. The reproducing kernel of  $H_0^1([0, L])$  with the equivalent norm  $\sqrt{\int_{\mathbb{R}} (u')^2 dx}$  is  $K_4(x, y) = \frac{(L-x)y 1_{y < x} + x(L-y) 1_{y > x}}{L}$ .

*Proof.* We start with  $H^1(\mathbb{R})$  with the usual norm

$$\sqrt{\int_{\mathbb{R}} (u^2 + (u')^2) dx}.$$

One notes that the RK associated with this norm is given by

$$\forall w \in H^1(\mathbb{R}) \quad \langle K_x, w \rangle = w(x) \Leftrightarrow \forall w \in H^1(\mathbb{R}), \int_{\mathbb{R}} (K_x w + K'_x w') dy = w(x).$$

One then resumes to the ODE  $K_x - K''_x = 0$  on  $\mathbb{R} - \{x\}$ , which solution is  $y \rightarrow Ae^{-|y-x|}$ , and one has  $A = \frac{1}{2}$  thanks to the jump formula and Item 1 of Lemma 1 is proven.

Note that we recover also this kernel thanks to Fourier transform and the fact that the norm is equal to  $\sqrt{\frac{1}{2\pi} \int_{\mathbb{R}} (1 + \xi^2) |\hat{u}|^2 d\xi}$ .

As for  $H^1([0, L])$ , we have

$$\forall w \in H^1([0, L]), \int_0^L (K_x w + K'_x w') dy = w(x)$$

that is  $K_x(y) = A_{\pm} \cosh(y-x) + B_{\pm} \sinh(y-x)$ ,  $\pm(y-x) > 0$ . The boundary condition  $K'_x(0) = K'_x(L) = 0$  yield  $A_- \sinh(-x) + B_- \cosh(-x) = 0$ ,  $A_+ \sinh(L-x) + B_+ \cosh(L-x) = 0$ , hence  $K_x(y) = C_- \cosh y 1_{y < x} + C_+ \cosh(L-y) 1_{y > x}$ . One observes then that  $K'_x = C_- \sinh y 1_{y < x} - C_+ \sinh(L-y) 1_{y > x} + (C_+ \cosh(L-x) - C_- \cosh x) \delta_x$ , inferring the equality  $C_+ \cosh(L-x) - C_- \cosh x = 0$ , then

$$K''_x = C_- \cosh y 1_{y < x} + C_+ \cosh(L-y) 1_{y > x} - (C_+ \sinh(L-x) + C_- \sinh x) \delta_x$$

and one obtains the second equality  $C_+ \sinh(L-x) + C_- \sinh x = 1$ , hence  $C_+ = \frac{\cosh x}{\cosh L}$ ,  $C_- = \frac{\cosh(L-x)}{\cosh L}$ , which proves Item 2 of Lemma 1.

The boundary conditions change for the third case, one has  $K_3(x, y) = D_- \sinh y 1_{y < x} + D_+ \sinh(L - y) 1_{y > x}$ , with  $D_- \sinh x = D_+ \sinh(L - x)$  and  $D_- \cosh x + D_+ \cosh(L - x) = 1$ , which yields the result of Item 3 of Lemma 1.

Finally, using the  $H_0^1$  norm, one has

$$\forall w \in H_0^1([0, L]), \int_0^L K_4'(y) w'(y) dy = w(x) \Rightarrow -K_4'' = \delta_x$$

hence  $K_4(x, y) = Ay 1_{y < x} + B(L - y) 1_{y > x}$ , and one uses  $K_4$  continuous at  $x = y$  to obtain the result of Item 4 of Lemma 1.  $\square$

### 3 The reproducing kernels on $H^2(\mathbb{R}^2)$

#### 3.1 Case of the usual Sobolev norm in $H^s(\mathbb{R}^d)$ , $s > \frac{d}{2}$

Under the same analysis as we did in Section 2, the smallest  $s$  such that, for  $\Omega \in \mathbb{R}^2$ ,  $H^s(\Omega) \subset C^0(\Omega)$ , is given by the Sobolev inequality  $H^s(\Omega) \subset C^0(\Omega)$  for  $s > \frac{2}{2} = 1$ . One then considers  $s = 2$ .

**Proposition 1.** *1. The reproducing kernel of  $H^s(\mathbb{R}^d)$  with the usual Sobolev norm (1) is given, in the case  $s = [\frac{d}{2}] + 1$*

$$C_d |x - y|^{[\frac{d}{2}] + 1 - \frac{d}{2}} K_{\frac{d}{2} - [\frac{d}{2}] - 1}(|x - y|) = C_d |x - y|^{[\frac{d}{2}] + 1 - \frac{d}{2}} K_{[\frac{d}{2}] + 1 - \frac{d}{2}}(|x - y|),$$

where  $K_\nu$  is the modified Bessel function of the second type, solution of  $z^2 w'' + z w' - (z^2 + \nu^2) w = 0$ .

*2. The reproducing kernel of  $H^2(\mathbb{R}^2)$  with the usual Sobolev norm (1) is  $\frac{|x-y|}{4\pi} K_{-1}(|x-y|) = \frac{|x-y|}{4\pi} K_1(|x-y|)$ .*

*3. The reproducing kernel of  $H^{m+1}(\mathbb{R}^{2m})$  is equal, up to a constant, to the reproducing kernel of  $H^2(\mathbb{R}^2)$ .*

*4. The reproducing kernel of  $H^{m+1}(\mathbb{R}^{2m+1})$  is equal, up to a constant, to the reproducing kernel of  $H^2(\mathbb{R}^3)$ .*

*Proof.* First of all, following [1], one can obtain readily the general expression of the RK of the Hilbert space  $H^s(\mathbb{R}^d)$  associated with the usual norm

$$\|u\|_{H^s(\mathbb{R}^d)}^2 = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^d} (1 + |\xi|^2)^s |\hat{u}|^2 d\xi. \quad (1)$$

We quickly reproduce the proof of [1]. The RK associated with  $\mathcal{H}_{s,d} = \{H^s(\mathbb{R}^d), \|u\|_{H^s(\mathbb{R}^d)}\}$  satisfies  $K(x, x_0) = u$ , with

$$\forall w \in H^s, \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} (1 + |\xi|^2)^s \hat{u}(\xi) \hat{w}(\xi) d\xi = w(x_0).$$

Using  $w(x_0) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \hat{w}(\xi) e^{ix_0\xi} d\xi$ , and as  $K(x, x_0) = K(x - x_0, 0)$  we can infer  $x_0 = 0$  and we obtain

$$(1 + |\xi|^2)^s \hat{u}(\xi) = 1 \Leftrightarrow \hat{u}(\xi) = \frac{1}{(1 + |\xi|^2)^s}.$$

Hence

$$u(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{e^{ix\xi} d\xi}{(1 + |\xi|^2)^s} = \frac{1}{(2\pi)^d} \int_0^{+\infty} \int_{S^{d-1}} e^{ir\omega \cdot x} d\omega \frac{r^{d-1} dr}{(1 + r^2)^s}$$

The integration on the unit sphere yields a function depending only on  $r|x|$ , and reads

$$\begin{aligned} u(x) &= \frac{1}{(2\pi)^d} \int_0^{+\infty} J_{\frac{d}{2}-1}(r|x|) (2\pi)^{\frac{d}{2}} (r|x|)^{1-\frac{d}{2}} \frac{r^{d-1} dr}{(1+r^2)^s} \\ &= \frac{1}{(2\pi)^{\frac{d}{2}}} |x|^{1-\frac{d}{2}} \int_0^{+\infty} J_{\frac{d}{2}-1}(r|x|) \frac{r^{\frac{d}{2}} dr}{(1+r^2)^s} \\ &= \frac{1}{(2\pi)^{\frac{d}{2}}} |x|^{\frac{1}{2}-\frac{d}{2}} \int_0^{+\infty} \sqrt{r|x|} J_{\frac{d}{2}-1}(r|x|) \frac{r^{\frac{d}{2}-\frac{1}{2}} dr}{(1+r^2)^s}. \end{aligned}$$

Define the Hankel transform of a function  $f$ :

$$\mathcal{L}_{Hankel}(f)(y) = \int_0^{+\infty} f(x) J_\nu(xy) (xy)^{\frac{1}{2}} dx = \frac{y^{\mu+\frac{1}{2}} K_{\nu-\mu}(y)}{2^\mu \Gamma(\mu+1)} \quad (2)$$

One then uses the formula from Erdelyi, Tome 2 [3] (expression (20) in Hankel transforms), for  $f(x) = x^{\nu+\frac{1}{2}}(1+x^2)^{-\mu-1}$ , where one applies  $\mu = s-1$ ,  $\nu = \frac{d}{2}-1$

We have finally, using the Bessel modified function of the second kind  $K_\nu$

$$u(x) = \frac{1}{(2\pi)^{\frac{d}{2}}} |x|^{\frac{1}{2}-\frac{d}{2}} \frac{|x|^{s-\frac{1}{2}} K_{\frac{d}{2}-s}(|x|)}{2^{s-1} \Gamma(s)} = \frac{|x|^{s-\frac{d}{2}}}{(2\pi)^{\frac{d}{2}} 2^{s-1} \Gamma(s)} K_{\frac{d}{2}-s}(|x|).$$

The application of this result to the case  $d = 2$ ,  $s = 2$  yields thus (correcting a typo of [1] and adding  $K_\nu = K_{-\nu}$  9.66 of [2])

$$K_{\mathcal{H}_{2,2}}(x, y) = \frac{|x-y|}{4\pi} K_{-1}(|x-y|) = \frac{|x-y|}{4\pi} K_1(|x-y|) \quad (3)$$

where we observe that  $z^\nu K_\nu(z)$  bounded when  $\Re \nu > 0$  (9.6.9 of [2]), this kernel is thus bounded.

Moreover, if we consider the limiting case with for  $s \in \mathbb{N}$ , that is  $s = \frac{d+1}{2}$  if  $d$  is odd and  $s = \frac{d}{2} + 1$  if  $d$  is even, one has

$$K_{\mathcal{H}_{s,d}}(x, y) = \begin{cases} \frac{2^{1-s}}{(2\pi)^{\frac{d}{2}} \Gamma(s)} |x-y| K_{-1}(|x-y|) = \frac{2^{1-s}}{(2\pi)^{\frac{d}{2}} \Gamma(s)} |x-y| K_1(|x-y|), & d \text{ even} \\ \frac{2^{1-s}}{(2\pi)^{\frac{d}{2}} \Gamma(s)} |x-y|^{\frac{1}{2}} K_{-\frac{1}{2}}(|x|) = \frac{2^{1-s}}{(2\pi)^{\frac{d}{2}} \Gamma(s)} |x-y|^{\frac{1}{2}} K_{\frac{1}{2}}(|x|), & d \text{ odd} \end{cases}$$

□

### 3.2 Case of a modified Sobolev norm

With another norm of  $H^s(\mathbb{R}^d)$  for  $s > \frac{d}{2}$ , namely

$$\|u\|_{H^s(\mathbb{R}^d,*)} = \sqrt{\frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} (1 + |\xi|^{2s}) |\hat{u}(\xi)|^2 d\xi}, \quad (4)$$

we have

**Proposition 2.** 1. The norms  $\|u\|_{H^s(\mathbb{R}^d)}$  and (4) are equivalent

2. The reproducing kernel of  $H^s(\mathbb{R}^d)$  with the norm (4) is

$$K_{H^s(\mathbb{R}^d,*)}(x, y) = \frac{1}{(2\pi)^{\frac{d}{2}}} |x|^{\frac{1}{2} - \frac{d}{2}} \mathcal{L}_{Hankel}(g_{s,d})(|x - y|)$$

where

$$g_{s,d}(r) = \frac{r^{d-\frac{1}{2}-\frac{d}{2}}}{1+r^{2s}}$$

and the Hankel transform is given by (2).

*Proof.* We see that here exists two constants  $c_-$  and  $c_+$  such that

$$\forall u \in H^s(\mathbb{R}^d), c_- \|u\|_{H^s(\mathbb{R}^d)} \leq \|u\|_{H^s(\mathbb{R}^d,*)} \leq c_+ \|u\|_{H^s(\mathbb{R}^d)}.$$

These constants are respectively the maximum and the minimum of  $\frac{1+|\xi|^{2s}}{(1+|\xi|^2)^s}$ . This proves the item 1 of this proposition.

For the second one, one has, similarly

$$\forall w \in H^s(\mathbb{R}^d), \langle u_{x_0}, w \rangle = u(x_0) \Leftrightarrow u_{x_0}(y) = u_0(y-x_0), \text{ with } \hat{u}_0(\xi) = \frac{1}{1+|\xi|^{2s}}.$$

Hence

$$\begin{aligned} u_0(x) &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{e^{ix \cdot \xi}}{1+|\xi|^{2s}} d\xi \\ &= \frac{1}{(2\pi)^d} \int_0^{+\infty} \frac{r^{d-1} dr}{1+r^{2s}} \int_{S^{d-1}} e^{ix \cdot r\omega} d_{S^{d-1}}\omega \\ &= \frac{1}{(2\pi)^d} \int_0^{+\infty} J_{\frac{d}{2}-1}(r|x|) (2\pi)^{\frac{d}{2}} (r|x|)^{1-\frac{d}{2}} \frac{r^{d-1} dr}{1+r^{2s}} \\ &= \frac{1}{(2\pi)^{\frac{d}{2}}} |x|^{\frac{1}{2}-\frac{d}{2}} \int_0^{+\infty} J_{\frac{d}{2}-1}(r|x|) \sqrt{r|x|} \frac{r^{\frac{d}{2}-\frac{1}{2}} dr}{1+r^{2s}} \\ &= \frac{1}{(2\pi)^{\frac{d}{2}}} |x|^{\frac{1}{2}-\frac{d}{2}} \mathcal{L}_{Hankel}(g_{s,d})(|x|). \end{aligned}$$

This proves the second item of Proposition 2.  $\square$

## References

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