Fluctuations of the local times of the self-repelling random walk with directed edges

Laure Marêché

FLUCTUATIONS OF THE LOCAL TIMES OF THE SELF-REPELLING RANDOM WALK WITH DIRECTED EDGES

LAURE MARÊCHÉ

Abstract. In 2008, Tóth and Vető defined the self-repelling random walk with directed edges as a *non-Markovian* random walk on $\mathbb{Z}$: in this model, the probability that the walk moves from a point of $\mathbb{Z}$ to a given neighbor depends on the number of previous crossings of the directed edge from the initial point to the target, called the local time of the edge. They found this model had a very peculiar behavior. Indeed, for the non-Markovian random walks most closely related to it, which are defined with undirected edges replacing directed edges, Tóth proved in his works of 1994, 1995 and 1996 that the process formed by the local times of all the edges, evaluated at a stopping time of a certain type and suitably renormalized, converges to a random process. The behavior of the self-repelling random walk with directed edges is very different and unusual, as Tóth and Vető proved in 2008 that the limit process is deterministic. In this work, we study the fluctuations of the local times process around this deterministic limit, about which nothing was previously known. We prove that these fluctuations converge in the Skorohod M_1 topology, as well as in the uniform topology away from the discontinuities of the limit, but *not* in the most classical Skorohod topology. We also prove the convergence of the fluctuations of the aforementioned stopping times.

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1. Introduction and results

1.1. **Self-interacting random walks.** The study of self-interacting random walks began in 1983 in an article of Amit et al [1]. Before [1], the expression "self-avoiding random walk" referred to paths on graphs that do not intersect themselves. However, these are not easy to construct step by step, hence one would consider the set of all possible paths of a given length. Since one does not follow a single path as it grows with time, it is not really a random walk model. In order to work with an actual random walk model with a self-avoiding behavior, the authors of [1] introduced the *"true" self-avoiding random walk*. It is a random walk on $\mathbb{Z}^d$ for which, at each step, the position of the process at the next step is chosen randomly among the neighbors of the current position depending on the number of the previous visits to said neighbors, wtih lower probabilities for those that have been visited the most. This process is a random walk in the sense that it is constructed step by step, but contrary to most random walks in the literature, it is *non-Markovian*: at each step, the law of the next step depends on the whole past of the process.

It turns out that the "true" self-avoiding random walk is hard to study. This led to the introduction by Tóth [5, 6, 7] of non-Markovian random walks *with bond repulsion*, for which the probability to go from one site to another, instead of depending of the number of previous visits to the target, depends on the number of previous crossings of the undirected edge between the two sites, which is called the *local time* of the edge. These walks are much easier to study, at least on $\mathbb{Z}$, because one can apply them the *Ray-Knight approach*: though the random walk itself is not Markovian, if we stop it when the local time at a given edge has reached a certain threshold, then the local times on the edges will form a Markov chain, which allows their analysis. Thanks to this Ray-Knight approach, Tóth was able

to prove scaling limits for the local times process for many different random walks with bond repulsion in his works [5, 6, 7]. The law of the limit depends on the random walk model, but it is always a random process.

1.2. **The self-repelling random walk with directed edges.** In 2008, Tóth and Vető [8] introduced a process seemingly very similar to the aforementioned random walks with bond repulsion, in which the probability to go from one site to another depends on the number of crossings of the *directed* edge between them instead of the crossings of the undirected edge. This process, called *self-repelling random walk with directed edges*, is a nearest-neighbor random walk on $\mathbb{Z}$ defined as follows. For any set A, we denote by $|A|$ the cardinal of A. Let $w : \mathbb{Z} \mapsto (0, +\infty)$ be a non-decreasing and non-constant function. We will denote the walk by $(X_n)_{n\in\mathbb{N}}$. We set $X_0 = 0$, and for any $n \in \mathbb{N}$, $i \in \mathbb{Z}$, we denote $\ell^{\pm}(n, i) = |\{0 \leq m \leq n-1 \,|\, (X_m, X_{m+1}) = (i, i \pm 1)\}|$ the number of crossings of the directed edge $(i, i \pm 1)$ before time n, that is the *local time* of the directed edge at time n. Then

$$\mathbb{P}(X_{n+1} = X_n \pm 1) = \frac{w(\pm(\ell^-(n, X_n) - \ell^+(n, X_n)))}{w(\ell^+(n, X_n) - \ell^-(n, X_n)) + w(\ell^-(n, X_n) - \ell^+(n, X_n))}.$$

Using the local time of directed edges instead of that of undirected edges may seem like a very small change in the definition of the process, but the behavior of the self-repelling random walk with directed edges is actually very different from that of classical random walks with bond repulsion. Indeed, Tóth and Vető [8] were able to prove that the local times process has a *deterministic* scaling limit, which is in sharp contrast with the random limit processes obtained for the random walks with bond repulsion on undirected edges [5, 6, 7] and even for the simple random walk [2].

The result of [8] is as follows. For any $a \in \mathbb{R}$, we denote $a_+ = \max(a, 0)$. If for any $n \in \mathbb{N}$, $i \in \mathbb{Z}$, we denote by $T^{\pm}_{n,i}$ the stopping time defined by $T^{\pm}_{n,i} = \min\{m \in \mathbb{N} \,|\, \ell^{\pm}(m, i) = n\}$, then $T^{\pm}_{n,i}$ is almost-surely finite by Proposition 1 of [8] and we have the following.

Theorem (Theorem 1 of [8])**.** *For any $\theta > 0$, $x \in \mathbb{R}$, then $\sup_{y\in\mathbb{R}} |\frac{1}{N}\ell^+(T^{\pm}_{\lfloor N\theta\rfloor,\lfloor Nx\rfloor}, \lfloor Ny\rfloor) - (\frac{|x|-|y|}{2} + \theta)_+|$ converges in probability to 0 when N tends to $+\infty$.*

Thus the local times process of the self-repelling random walk with directed edges admits the deterministic scaling limit : $y \mapsto (\frac{|x|-|y|}{2} + \theta)_+$, which has the shape of a triangle. This also implies the following convergence result to a deterministic limit for the $T^{\pm}_{\lfloor N\theta\rfloor,\lfloor Nx\rfloor}$.

Proposition (Corollary 1 of [8])**.** *For any $\theta > 0$, $x \in \mathbb{R}$, then $\frac{1}{N^2}T^{\pm}_{\lfloor N\theta\rfloor,\lfloor Nx\rfloor}$ converges in probability to $(|x| + 2\theta)^2$ when N tends to $+\infty$.*

The deterministic character of these limits makes the behavior of the self-repelling random walk with directed edges very unusual, hence worthy of study. In particular, it is natural to consider the possible fluctuations of the local times process and of the $T^{\pm}_{\lfloor N\theta\rfloor,\lfloor Nx\rfloor}$ around their deterministic limits. However, prior to this paper, nothing was known about these fluctuations. In this work, we prove convergence in distribution of the fluctuations of the local times process and of the $T^{\pm}_{\lfloor N\theta\rfloor,\lfloor Nx\rfloor}$. It happens that the limit of the fluctuations of the local times process is discontinuous, hence before stating the results, we have to be careful of the topology in which it may converge.

1.3. **Topologies for the convergence of the local times process.** For any interval $I \subset R$, let DI be the space of *càdlàg* functions on I, that is the set of functions : $I \mapsto \mathbb{R}$ that are right-continuous and have left limits everywhere in I. For any function $Z : I \mapsto \mathbb{R}$, we denote by $\|Z\|_\infty = \sup_{y\in I} |Z(y)|$ the uniform norm of Z on I. The uniform norm on I gives a topology on DI, but it is often too strong to deal with discontinuous functions.

For discontinuous càdlàg functions, the most widely used topology is the *Skorohod J_1 topology*, introduced by Skorohod in [4] (see chapter VI of [3] for a course), which is often called "the" Skorohod topology. Intuitively, two functions are close in this topology if they are close for the uniform norm after allowing some small perturbation of time. Rigorously, for $a < b$ in $\mathbb{R}$ the Skorohod J_1 topology on $D[a,b]$ is defined as follows. We call $\Lambda_{a,b}$ the set of functions $\lambda : [a,b] \mapsto [a,b]$ that are bijective, strictly increasing and continuous (they correspond to the possible perturbations of time), and we denote by $\mathrm{Id}_{a,b} : [a,b] \mapsto [a,b]$ the identity map, defined by $\mathrm{Id}_{a,b}(y) = y$ for all $y \in [a,b]$. The Skorohod J_1 topology on $D[a,b]$ is defined through the following metric: for any $Z_1, Z_2 \in D[a,b]$, we set $d_{J_1,a,b}(Z_1,Z_2) = \inf_{\lambda\in\Lambda} \max(\|Z_1 \circ \lambda - Z_2\|_\infty, \|\lambda - \mathrm{Id}_{a,b}\|_\infty)$. It can be proven rather easily that this is indeed a metric. We can then define the Skorohod J_1 topology in $D(-\infty,\infty)$ with the following metric: if for any sets $A_1 \subset A_2$ and A_3 and any function $Z : A_2 \mapsto A_3$, we denote $Z|_{A_1}$ the restriction of Z to A_1, then for $Z_1, Z_2 \in D(-\infty,\infty)$, we set $d_{J_1}(Z_1,Z_2) = \int_0^{+\infty} e^{-a}(d_{J_1,-a,a}(Z_1|_{[-a,a]}, Z_2|_{[-a,a]}) \wedge 1)\mathrm{d}a$. The Skorohod J_1 topology is widely used to study the convergence of càdlàg functions. However, when the limit function has a jump, which will be the case here, convergence in the Skorohod J_1 topology requires the converging functions to have a single big jump approximating the jump of the limit process. To account for other cases, like having the jump of the limit functions approximated by several smaller jumps in quick succession or by a very steep continuous slope, one has to use a less restrictive topology, like the *Skorohod M_1 topology*.

The Skorohod M_1 topology was also introduced by Skorohod in [4] (see Section 3.3 of [9] for an overview). For any $a < b$ in $\mathbb{R}$, the Skorohod M_1 distance on $D[a,b]$ is defined as follows: the distance between two functions will be roughly "the distance between the completed graphs of the functions". More rigorously, if $Z \in D[a,b]$, we denote $Z(a^-) = Z(a)$ and for any $y \in (a,b]$, we denote $Z(y^-) = \lim_{y'\to y, y'<y} Z(y')$. Then the *completed graph of Z* is $\Gamma_Z = \{(y,z) \,|\, y \in [a,b], \exists\, \varepsilon \in [0,1] \text{ so that } z = \varepsilon Z(y^-) + (1-\varepsilon)Z(y)\}$. To express the "distance between two such completed graphs", we need to define the *parametric representations* of Γ_Z (by abuse of notation, we will often write "the parametric representations of Z"). We define an order on Γ_Z as follows: for $(y_1,z_1),(y_2,z_2) \in \Gamma_Z$, we have $(y_1,z_1) \leq (y_2,z_2)$ when $y_1 < y_2$ or when $y_1 = y_2$ and $|Z(y_1^-) - z_1| \leq |Z(y_1^-) - z_2|$. A *parametric representation* of Γ_Z is a continuous, surjective function $(u,r) : [0,1] \mapsto \Gamma_Z$ that is non-decreasing with respect to this order, thus intuitively, when t goes from 0 to 1, $(u(t),r(t))$ "travels through the completed graph of Z from its beginning to its end". A parametric representation of Z always exists (see Remark 12.3.3 in [9]). For $Z_1, Z_2 \in D[a,b]$, the Skorohod M_1 distance between Z_1 and Z_2, denoted by $d_{M_1,a,b}(Z_1,Z_2)$, is $\inf\{\max(\|u_1-u_2\|_\infty, \|r_1-r_2\|_\infty)\}$ where the infimum is on the parametric representations (u_1,r_1) of Z_1 and (u_2,r_2) of Z_2. It can be proven that this indeed gives a metric (see Theorem 12.3.1 of [9]), and this metric defines the Skorohod M_1 topology on $D[a,b]$. For any $a > 0$, we will denote $d_{M_1,-a,a}$ by $d_{M_1,a}$ for short. We can now define the Skorohod M_1 topology in $D(-\infty,\infty)$ through the following metric: for $Z_1, Z_2 \in D(-\infty,\infty)$, we set $d_{M_1}(Z_1,Z_2) = \int_0^{+\infty} e^{-a}(d_{M_1,a}(Z_1|_{[-a,a]}, Z_2|_{[-a,a]}) \wedge 1)\mathrm{d}a$. It can be seen that the Skorohod M_1 topology is weaker than the Skorohod J_1 topology (see Theorem 12.3.2 of [9]), thus less restrictive. Indeed, since the distance between two functions is roughly "the distance between the completed graphs of the functions", the Skorohod M_1 topology will allow a function with a jump to be the limit of functions with steep slopes or with several smaller jumps. For this reason, the Skorohod M_1 topology is often more adapted when considering convergence to a discontinuous function.

1.4. **Results.** We are now ready to state our results on the convergence of the fluctuations of the local times process. For any $\theta > 0$, $x \in \mathbb{R}$, $\iota \in \{+,-\}$, for any $N \in \mathbb{N}^*$, we define functions Y_N^-, Y_N^+ as follows: for any $y \in \mathbb{R}$, we set

$$Y_N^\pm(y) = \frac{1}{\sqrt{N}} \left(\ell^\pm(T^\iota_{\lfloor N\theta\rfloor, \lfloor Nx\rfloor}, \lfloor Ny\rfloor) - N \left(\frac{|x|-|y|}{2} + \theta \right)_+ \right).$$

Moreover, $(B_y^x)_{y\in\mathbb{R}}$ will denote a two-sided Brownian motion with $B_x^x = 0$ and variance $\mathrm{Var}(\rho_-)$, where ρ_- is the distribution on $\mathbb{Z}$ defined later in (3). We proved the following convergence for the fluctuations of the local times process of the self-repelling random walk with directed edges.

Theorem 1. *For any $\theta > 0$, $x \in \mathbb{R}$, $\iota \in \{+,-\}$, the process $Y_N^{\pm}$ converges in distribution to $(B_y^x \mathbb{1}_{\{y\in[-|x|-2\theta,|x|+2\theta)\}})_{y\in\mathbb{R}}$ in the Skorohod M_1 topology on $D(-\infty,+\infty)$ when N tends to $+\infty$.*

Therefore the fluctuations of the local times process have a diffusive limit behavior. However, it is necessary to use the Skorohod M_1 topology here, as the following result states the convergence does not occur in the stronger Skorohod J_1 topology.

Proposition 2. *For any $\theta > 0$, $x \in \mathbb{R}$, $\iota \in \{+,-\}$, the process $Y_N^{\pm}$ does not converge in distribution in the Skorohod J_1 topology on $D(-\infty,+\infty)$ when N tends to $+\infty$.*

We stress the fact that the use of the Skorohod M_1 topology is only required to deal with the discontinuities of the limit process in $-|x|-2\theta$ and $|x|+2\theta$. Indeed, if we consider the convergence of the process on an interval that does not include $-|x|-2\theta$ or $|x|+2\theta$, it converges in the much stronger topology given by the uniform norm, which is the following result.

Proposition 3. *For any $\theta > 0$, $x \in \mathbb{R}$, $\iota \in \{+,-\}$, for any closed interval $I \in \mathbb{R}$ that does not contain $-|x|-2\theta$ or $|x|+2\theta$, the process $(Y_N^{\pm}(y))_{y\in I}$ converges in distribution to $(B_y^x \mathbb{1}_{\{y\in[-|x|-2\theta,|x|+2\theta)\}})_{y\in I}$ in the topology on DI given by the uniform norm when N tends to $+\infty$.*

Finally, we also proved the convergence of the fluctuations of $T^{\pm}_{\lfloor N\theta\rfloor,\lfloor Nx\rfloor}$. For any $\sigma^2 > 0$, we denote by $\mathcal{N}(0,\sigma^2)$ the Gaussian distribution with mean 0 and variance σ^2, and we recall that ρ_- will be defined in (3). We then have the following.

Proposition 4. *For any $\theta > 0$, $x \in \mathbb{R}$, $\iota \in \{+,-\}$, we have that $\frac{1}{N^{3/2}}(T^{\iota}_{\lfloor N\theta\rfloor,\lfloor Nx\rfloor} - N^2(|x|+2\theta)^2)$ converges in distribution to $\mathcal{N}(0, \frac{32}{3}\mathrm{Var}(\rho_-)((|x|+\theta)^3+\theta^3))$ when N tends to $+\infty$.*

1.5. **Proof ideas.** We begin by explaining why the limit of the local times process $Y_N^{\pm}$ is $(B_y^x \mathbb{1}_{\{y\in[-|x|-2\theta,|x|+2\theta)\}})_{y\in\mathbb{R}}$ and the ideas behind the proofs of Theorem 1 and Proposition 3. To show the convergence of the local times process, we use a Ray-Knight argument, that is we notice that $(\ell^-(T^{\iota}_{\lfloor N\theta\rfloor,\lfloor Nx\rfloor}, i))_i$ is a Markov chain. Moreover, as long as $\ell^-(T^{\iota}_{\lfloor N\theta\rfloor,\lfloor Nx\rfloor}, i)$ is not too low, the $\ell^-(T^{\iota}_{\lfloor N\theta\rfloor,\lfloor Nx\rfloor}, i+1) - \ell^-(T^{\iota}_{\lfloor N\theta\rfloor,\lfloor Nx\rfloor}, i)$ will roughly be i.i.d. random variables in the sense that they can be coupled with i.i.d. random variables with a high probability to be equal to them. This coupling was already used in [8] to prove the convergence of $\frac{1}{N}\ell^+(T^{\pm}_{\lfloor N\theta\rfloor,\lfloor Nx\rfloor}, \lfloor Ny\rfloor)$ to its deterministic limit (for a given y, the coupling makes this convergence a law of large numbers). However, when $\ell^-(T^{\iota}_{\lfloor N\theta\rfloor,\lfloor Nx\rfloor}, \lfloor Ny\rfloor)$ is too low, the coupling fails and the $\ell^-(T^{\iota}_{\lfloor N\theta\rfloor,\lfloor Nx\rfloor}, \lfloor Ny\rfloor+1) - \ell^-(T^{\iota}_{\lfloor N\theta\rfloor,\lfloor Nx\rfloor}, \lfloor Ny\rfloor)$ are no longer i.i.d. We have to prove that this occurs only around $|x|+2\theta$ and $-|x|-2\theta$, and most of our work is dealing with what happens there. To show it occurs only around $|x|+2\theta$ and $-|x|-2\theta$, we control the amplitude of the fluctuations to prove the local times are close to their deterministic limit. This limit is large inside $(-|x|-2\theta, |x|+2\theta)$, so we can use the coupling inside this interval, thus the $\ell^-(T^{\iota}_{\lfloor N\theta\rfloor,\lfloor Nx\rfloor}, \lfloor Ny\rfloor+1) - \ell^-(T^{\iota}_{\lfloor N\theta\rfloor,\lfloor Nx\rfloor}, \lfloor Ny\rfloor)$ are roughly i.i.d. there, hence the fluctuations will converge to a Brownian motion by Donsker's Invariance Principle. When we are close to $|x|+2\theta$ (the same reasoning works for $-|x|-2\theta$) the deterministic limit will be small hence the local times too, and tools of [8] allow to prove that they reach 0 quickly. Once they reach 0, we notice that for $y \geq |x|+2\theta$, if $\ell^-(T^{\iota}_{\lfloor N\theta\rfloor,\lfloor Nx\rfloor}, \lfloor Ny\rfloor) = 0$,

the walk X did not go from $\lfloor Ny \rfloor$ to $\lfloor Ny \rfloor + 1$ before time $T^{\iota}_{\lfloor N\theta \rfloor, \lfloor Nx \rfloor}$, so it did not go to $\lfloor Ny \rfloor + 1$ before time $T^{\iota}_{\lfloor N\theta \rfloor, \lfloor Nx \rfloor}$, hence $\ell^-(T^{\iota}_{\lfloor N\theta \rfloor, \lfloor Nx \rfloor}, j) = 0$ for any $j \geq \lfloor Ny \rfloor$. Therefore, once the local times process reaches 0, it stays there. Consequently, we expect $\ell^-(T^{\iota}_{\lfloor N\theta \rfloor, \lfloor Nx \rfloor}, \lfloor Ny \rfloor)$ to be 0 when $y > |x| + 2\theta$, and thus to have no fluctuations when $y > |x| + 2\theta$, and similarly when $y < -|x| - 2\theta$. This is why our limit is $(B^x_y \mathbb{1}_{\{y \in [-|x|-2\theta, |x|+2\theta)\}})_{y \in \mathbb{R}}$. Since Proposition 3 only describes convergence away from $-|x| - 2\theta$ and $|x| + 2\theta$, the previous arguments are enough to prove it. To prove the convergence in the Skorohod M_1 topology on $D(-\infty, +\infty)$ stated in Theorem 1, we need to handle what happens around $-|x| - 2\theta$ and $|x| + 2\theta$ with more precision. We first have to bound the difference between the local times and the i.i.d. random variables of the coupling even where the coupling fails. Afterwards, we have to tackle the the main challenge of the paper: writing explicit parametric representations of $Y^{\pm}_N$ and of the sum of the i.i.d. random variables of the coupling, properly renormalized and set to 0 outside of $[-|x| - 2\theta, |x| + 2\theta)$, and then proving that they are close to each other. That allows to prove $Y^{\pm}_N$ is close in the Skorohod M_1 distance to a process that will converge in distribution to $(B^x_y \mathbb{1}_{\{y \in [-|x|-2\theta, |x|+2\theta)\}})_{y \in \mathbb{R}}$ in the Skorohod M_1 topology and to complete the proof of Theorem 1.

To prove Proposition 2, that is that $Y^{\pm}_N$ does not converge in the J_1 topology, we first notice that since the J_1 topology is stronger than the M_1 topology, if $Y^{\pm}_N$ did converge in the J_1 topology its limit would be $(B^x_y \mathbb{1}_{\{y \in [-|x|-2\theta, |x|+2\theta)\}})_{y \in \mathbb{R}}$. However, it is not possible, as $(B^x_y \mathbb{1}_{\{y \in [-|x|-2\theta, |x|+2\theta)\}})_{y \in \mathbb{R}}$ has a jump at $|x| + 2\theta$, while the jumps of $Y^{\pm}_N$ have size of order $\frac{1}{\sqrt{N}}$, so the jump in $(B^x_y \mathbb{1}_{\{y \in [-|x|-2\theta, |x|+2\theta)\}})_{y \in \mathbb{R}}$ is approximated in $Y^{\pm}_N$ by either a sequence of small jumps or a continuous slope, which prevents the convergence in the Skorohod J_1 topology.

Finally, to prove Proposition 4 on the fluctuations of $T^{\iota}_{\lfloor N\theta \rfloor, \lfloor Nx \rfloor}$, we use the fact that we have $T^{\iota}_{\lfloor N\theta \rfloor, \lfloor Nx \rfloor} = \sum_{i \in \mathbb{Z}} (\ell^+(T^{\iota}_{\lfloor N\theta \rfloor, \lfloor Nx \rfloor}, i) + \ell^-(T^{\iota}_{\lfloor N\theta \rfloor, \lfloor Nx \rfloor}, i))$. By using the coupling for the $\ell^-(T^{\iota}_{\lfloor N\theta \rfloor, \lfloor Nx \rfloor}, i+1) - \ell^-(T^{\iota}_{\lfloor N\theta \rfloor, \lfloor Nx \rfloor}, i)$ when $\ell^-(T^{\iota}_{\lfloor N\theta \rfloor, \lfloor Nx \rfloor}, i)$ is high enough and our estimates on the size of the window in which $\ell^-(T^{\iota}_{\lfloor N\theta \rfloor, \lfloor Nx \rfloor}, i)$ is neither high enough nor 0, we can prove that $T^{\iota}_{\lfloor N\theta \rfloor, \lfloor Nx \rfloor}$ is close to the integral of the sum of the i.i.d. random variables of the coupling, which will yield the convergence.

1.6. **Organization of the paper.** In Section 2, we define the coupling between the increments of the local time and i.i.d. random variables and prove some of its properties. In Section 3, we control where the local times hit 0, as well as where the local times are too low for the coupling of Section 2 to be useful. In Section 4, we prove a bound on the Skorohod M_1 distance between $Y^{\pm}_N$ and the renormalized sum of the i.i.d. random variables of the coupling set to 0 outside of $[-|x| - 2\theta, |x| + 2\theta)$ by writing explicit parametric representations of the two functions. In Section 5, we complete the proof of the convergence of $Y^{\pm}_N$ stated in Theorem 1 and Proposition 3. In Section 6, we prove that as claimed in Proposition 2, $Y^{\pm}_N$ does not converge in the J_1 topology. Finally, in Section 7, we prove the convergence of the fluctuations of $T^{\pm}_{\lfloor N\theta \rfloor, \lfloor Nx \rfloor}$ stated in Proposition 4.

In what follows, we set $\theta > 0$, $\iota \in \{+, -\}$ and $x > 0$ (the cases $x < 0$ and $x = 0$ can be dealt with in the same way). To shorten the notation, we denote $T_N = T^{\iota}_{\lfloor N\theta \rfloor, \lfloor Nx \rfloor}$. Moreover, for any $a, b \in \mathbb{R}$, we denote $a \vee b = \max(a, b)$ and $a \wedge b = \min(a, b)$.

2. Coupling of the local times increments with i.i.d. random variables

Our goal in this section will be to couple the $\ell^{\pm}(T_N, i+1) - \ell^{\pm}(T_N, i)$ with i.i.d. random variables and to prove some properties of this coupling. This part of the work is not very different from what was done in [8], but we still recall their concepts and definitions. If we fix $i \in \mathbb{Z}$ and observe the evolution of $(\ell^-(n, i) - \ell^+(n, i))_{n \in \mathbb{N}}$, and if we ignore the steps at which $\ell^-(n, i) - \ell^+(n, i)$ does not move (i.e. those at which the random walk is not

at i), we obtain a Markov chain ξ_i whose distribution ξ has the following transition probabilities: for all $n \in \mathbb{N}$, $\mathbb{P}(\xi(n+1) = \xi(n) \pm 1) = \frac{w(\mp\xi(n))}{w(\xi(n))+w(-\xi(n))}$, and so that $\xi_i(0) = 0$. Now, we denote $\tau_{i,\pm}(0) = 0$ and for any $n \in \mathbb{N}$, we denote $\tau_{i,\pm}(n+1) = \inf\{m > \tau_{i,\pm}(n) \,|\, \xi_i(m) = \xi_i(m-1) \pm 1\}$, so that $\tau_{i,+}(n)$ is the time of the n-th upwards step of ξ_i and $\tau_{i,-}(n)$ is the time of the n-th downwards step of ξ_i. Then since the distribution of ξ is symmetric, the processes $(\eta_{i,+}(n))_{n\in\mathbb{N}} = (-\xi_i(\tau_{i,+}(n)))_{n\in\mathbb{N}}$ and $(\eta_{i,-}(n))_{n\in\mathbb{N}} = (\xi_i(\tau_{i,-}(n)))_{n\in\mathbb{N}}$ have the same distribution, called η, and it can be checked that η is a Markov chain.

We are going to give an expression of $\ell^{\pm}(T_N, i+1) - \ell^{\pm}(T_N, i)$ depending on the $\eta_{i,-}$, $\eta_{i,+}$. We assume N large enough (so that $\lfloor Nx \rfloor - 1 > 0$). By definition of T_N we have $X_{T_N} = \lfloor Nx \rfloor \iota 1$. If $i \leq 0$ we thus have $X_{T_N} > i$, so the last step of the walk at i before T_N was going to the right, so the last step of ξ_i was a downwards step, and by definition of $\ell^+(T_N, i)$ we have that ξ_i made $\ell^+(T_N, i)$ downwards steps, hence $\ell^-(T_N, i) - \ell^+(T_N, i) = \xi_i(\tau_{i,-}(\ell^+(T_N, i))) = \eta_{i,-}(\ell^+(T_N, i))$, which yields $\ell^-(T_N, i) - \ell^+(T_N, i) = \eta_{i,-}(\ell^+(T_N, i))$. In addition, $\ell^-(T_N, i) = \ell^+(T_N, i-1)$, hence $\ell^+(T_N, i-1) = \ell^+(T_N, i) + \eta_{i,-}(\ell^+(T_N, i))$. If $0 < i < \lfloor Nx \rfloor$ (for $\iota = -$) or $0 < i \leq \lfloor Nx \rfloor$ (for $\iota = +$), the last step of the walk at i was also going to the right, so we also have $\ell^-(T_N, i) - \ell^+(T_N, i) = \eta_{i,-}(\ell^+(T_N, i))$. However, $\ell^-(T_N, i) = \ell^+(T_N, i-1) - 1$, so $\ell^+(T_N, i-1) = \ell^+(T_N, i) + \eta_{i,-}(\ell^+(T_N, i)) + 1$. Finally, if $i \geq \lfloor Nx \rfloor$ (for $\iota = -$) or $i > \lfloor Nx \rfloor$ (for $\iota = +$), then the last step of the walk at i was going to the left, so the last step of ξ_i was an upwards step, and ξ_i made $\ell^-(T_N, i)$ upwards steps, therefore $\ell^-(T_N, i) - \ell^+(T_N, i) = \xi_i(\tau_{i,+}(\ell^-(T_N, i))) = -\eta_{i,+}(\ell^-(T_N, i))$, which yields $\ell^-(T_N, i) - \ell^+(T_N, i) = -\eta_{i,+}(\ell^-(T_N, i))$. Moreover, $\ell^+(T_N, i) = \ell^-(T_N, i+1)$, hence $\ell^-(T_N, i+1) = \ell^-(T_N, i) + \eta_{i,+}(\ell^-(T_N, i))$.

We are going to use these results to deduce an expression of the $\ell^{\pm}(T_N, i)$ which will be very useful throughout this work. Denoting $\chi(N) = \lfloor Nx \rfloor$ if $\iota = -$ et $\chi(N) = \lfloor Nx \rfloor + 1$ if $\iota = +$, for $i \geq \chi(N)$ we have $\ell^-(T_N, i) = \ell^-(T_N, \chi(N)) + \sum_{j=\chi(N)}^{i-1} \eta_{j,+}(\ell^-(T_N, j))$, and for $i < \chi(N)$ we have $\ell^+(T_N, i) = \ell^+(T_N, \chi(N)-1) + \sum_{j=i+1}^{\chi(N)-1} (\eta_{j,-}(\ell^+(T_N, j)) + \mathbb{1}_{\{j>0\}})$. Now, we remember that the definition of T_N implies $\ell^\iota(T_N, \lfloor Nx \rfloor) = \lfloor N\theta \rfloor$, so if $\iota = -$ we have $\ell^-(T_N, \chi(N)) = \lfloor N\theta \rfloor$ and $\ell^+(T_N, \chi(N)-1) = \ell^-(T_N, \chi(N)) = \lfloor N\theta \rfloor$, and if $\iota = +$ we have $\ell^+(T_N, \chi(N)-1) = \lfloor N\theta \rfloor$ and $\ell^-(T_N, \chi(N)) = \ell^+(T_N, \chi(N)-1) - 1 = \lfloor N\theta \rfloor - 1$. Consequently, we have the following.

$$
\begin{aligned}
&\text{If } i \geq \chi(N), \quad \ell^-(T_N, i) = \lfloor N\theta \rfloor - \mathbb{1}_{\{\iota=+\}} + \sum_{j=\chi(N)}^{i-1} \eta_{j,+}(\ell^-(T_N, j)).\\
&\text{If } i < \chi(N), \quad \ell^+(T_N, i) = \lfloor N\theta \rfloor + \sum_{j=i+1}^{\chi(N)-1} (\eta_{j,-}(\ell^+(T_N, j)) + \mathbb{1}_{\{j>0\}}).
\end{aligned} \tag{1}
$$

We will also need to remember the following.

$$
\text{(2) If } i \geq \chi(N), \quad \ell^-(T_N, i) - \ell^+(T_N, i) = -\eta_{i,+}(\ell^-(T_N, i)). \quad \text{If } i < \chi(N), \quad \ell^-(T_N, i) - \ell^+(T_N, i) = \eta_{i,-}(\ell^+(T_N, i)).
$$

To couple the $\ell^{\pm}(T_N, i+1) - \ell^{\pm}(T_N, i)$ with i.i.d. random variables, we need to understand the $\eta_{i,+}(\ell^-(T_N, i))$ and the $\eta_{i,-}(\ell^+(T_N, i))$. [8] proved that the following measure ρ_- is the unique invariant probability distribution of the Markov chain η:

$$
\forall i \in \mathbb{Z}, \quad \rho_-(i) = \frac{1}{R} \prod_{j=1}^{\lfloor |2i+1|/2 \rfloor} \frac{w(-j)}{w(j)} \quad \text{with} \quad R = \sum_{i\in\mathbb{Z}} \prod_{j=1}^{\lfloor |2i+1|/2 \rfloor} \frac{w(-j)}{w(j)}. \tag{3}
$$

We also denote ρ_0 the measure on $\frac{1}{2} + \mathbb{Z}$ defined by $\rho_0(\cdot) = \rho_-(\cdot - \frac{1}{2})$.

We are now in position to construct the coupling of the $\ell^{\pm}(T_N,i+1)-\ell^{\pm}(T_N,i)$ with i.i.d. random variables $(\zeta_i)_{i\in\mathbb{Z}}$. The idea is that η can be expected to converge to its invariant distribution ρ_-, hence when $\ell^{\pm}(T_N,i)$ is large, $\eta_{i,\mp}(\ell^{\pm}(T_N,i))$ will be close to a random variable of law ρ_-. More rigorously, we begin by defining an i.i.d. sequence $(r_i)_{i\in\mathbb{Z}}$ of random variables of distribution ρ_- so that for $i\geq\chi(N)$ then $\mathbb{P}(r_i\neq\eta_{i,+}(\lfloor N^{1/6}\rfloor))$ is minimal, and for $i<\chi(N)$ then $\mathbb{P}(r_i\neq\eta_{i,-}(\lfloor N^{1/6}\rfloor))$ is minimal. We can then define i.i.d. Markov chains $(\bar\eta_{i,+}(n))_{n\geq\lfloor N^{1/6}\rfloor}$ for $i\geq\chi(N)$ and $(\bar\eta_{i,-}(n))_{n\geq\lfloor N^{1/6}\rfloor}$ for $i<\chi(N)$ so that $\bar\eta_{i,\pm}(\lfloor N^{1/6}\rfloor)=r_i$, $\bar\eta_{i,\pm}$ is a Markov chain of distribution that of η, and if $\bar\eta_{i,\pm}(\lfloor N^{1/6}\rfloor)=\eta_{i,\pm}(\lfloor N^{1/6}\rfloor)$ then $\bar\eta_{i,\pm}(n)=\eta_{i,\pm}(n)$ for any $n\geq\lfloor N^{1/6}\rfloor$. Since ρ_- is invariant for η, if $n\geq\lfloor N^{1/6}\rfloor$, the $\bar\eta_{i,+}(n)$ for $i\geq\chi(N)$ and $\bar\eta_{i,-}(n)$ for $i<\chi(N)$ have distribution ρ_-. We define the random variables $(\zeta_i)_{i\in\mathbb{Z}}$ as follows: for $i\geq\chi(N)$ we set $\zeta_i=\bar\eta_{i,+}(\ell^-(T_N,i)\vee\lfloor N^{1/6}\rfloor)+\frac12$, and for $i<\chi(N)$ we set $\zeta_i=\bar\eta_{i,-}(\ell^+(T_N,i)\vee\lfloor N^{1/6}\rfloor)+\frac12$. For $i\geq\chi(N)$, (1) implies that $\ell^-(T_N,i)$ depends only on the $\eta_{j,+}$, $\chi(N)\leq j\leq i-1$, hence is independent from $\bar\eta_{i,+}$, which implies ζ_i has distribution ρ_0 and is independent from the ζ_j, $\chi(N)\leq j\leq i-1$. This and a similar argument for $i<\chi(N)$ implies the $(\zeta_i)_{i\in\mathbb{Z}}$ are i.i.d. with distribution ρ_0.

We will prove several properties of $(\zeta_i)_{i\in\mathbb{Z}}$ that we will use in the remainder of the proof. In order to do that, we need the following lemma of [8].

Lemma 5 (Lemma 1 of [8])**.** *There exist two constants $\tilde c=\tilde c(w)>0$ and $\tilde C=\tilde C(w)<+\infty$ so that for any $n\in\mathbb{N}$,*

$$\mathbb{P}(\eta(n)=i|\eta(0)=0)\leq\tilde Ce^{-\tilde c|i|}\quad\text{and}\quad\sum_{i\in\mathbb{Z}}|\mathbb{P}(\eta(n)=i|\eta(0)=0)-\rho_-(i)|\leq\tilde Ce^{-\tilde cn}.$$

Firstly, we want to prove that our coupling is actually useful: that the ζ_i are close to the $\ell^{\pm}(T_N,i+1)-\ell^{\pm}(T_N,i)$. More precisely, we will show that except on an event of probability tending to 0, if $\ell^{\pm}(T_N,i)$ is large then $\zeta_i=\eta_{i,\mp}(\ell^{\pm}(T_N,i))+1/2$, which (1) relates to $\ell^{\pm}(T_N,i+1)-\ell^{\pm}(T_N,i)$. We denote

$$\begin{aligned}\mathcal{B}_1^-&=\{\exists\, i\in\{-\lceil2(|x|+2\theta)N\rceil,....,\chi(N)-1\},\ell^+(T_N,i)\geq\lfloor N^{1/6}\rfloor\text{ and }\zeta_i\neq\eta_{i,-}(\ell^+(T_N,i))+1/2\},\\ \mathcal{B}_1^+&=\{\exists\, i\in\{\chi(N),....,\lceil2(|x|+2\theta)N\rceil\},\ell^-(T_N,i)\geq\lfloor N^{1/6}\rfloor\text{ and }\zeta_i\neq\eta_{i,+}(\ell^-(T_N,i))+1/2\}.\end{aligned}\tag{4}$$

Lemma 5 will allow us to prove the following.

Lemma 6. *$\mathbb{P}(\mathcal{B}_1^-)$ and $\mathbb{P}(\mathcal{B}_1^+)$ tend to 0 when $N\to+\infty$.*

Proof. By definition, for any $i\in\{-\lceil2(|x|+2\theta)N\rceil,....,\chi(N)-1\}$ we have $\zeta_i=\bar\eta_{i,-}(\ell^+(T_N,i)\vee\lfloor N^{1/6}\rfloor)+\frac12$, which is $\bar\eta_{i,-}(\ell^+(T_N,i))+\frac12$ when $\ell^+(T_N,i)\geq\lfloor N^{1/6}\rfloor$. Now, $\bar\eta_{i,-}=\eta_{i,-}$ if $\bar\eta_{i,-}(\lfloor N^{1/6}\rfloor)=\eta_{i,-}(\lfloor N^{1/6}\rfloor)$, that is $r_i=\eta_{i,-}(\lfloor N^{1/6}\rfloor)$. We deduce $\mathbb{P}(\mathcal{B}_1^-)\leq\mathbb{P}(\exists\, i\in\{-\lceil2(|x|+2\theta)N\rceil,....,\chi(N)-1\},r_i\neq\eta_{i,-}(\lfloor N^{1/6}\rfloor))$. Now, for any $i<\chi(N)$, we have $\mathbb{P}(r_i\neq\eta_{i,-}(\lfloor N^{1/6}\rfloor))$ minimal, thus smaller than $\tilde Ce^{-\tilde c\lfloor N^{1/6}\rfloor}$ by Lemma 5. Consequently, when N is large enough, $\mathbb{P}(\mathcal{B}_1^-)\leq3(|x|+2\theta)N\tilde Ce^{-\tilde c\lfloor N^{1/6}\rfloor}$, which tends to 0 when $N\to+\infty$. The proof for $\mathbb{P}(\mathcal{B}_1^+)$ is the same. □

Unfortunately, the previous lemma does not allow to control the local times when $\ell^{\pm}(T_N,i)$ is small. In order to do that, we show several additional properties. We have to control the probability of

$$\begin{aligned}\mathcal{B}_2=&\{\exists\, i\in\{-\lceil2(|x|+2\theta)N\rceil,....,\lceil2(|x|+2\theta)N\rceil\},|\zeta_i|\geq N^{1/16}\}\\ &\cup\{\exists\, i\in\{-\lceil2(|x|+2\theta)N\rceil,....,\chi(N)-1\},|\eta_{i,-}(\ell^+(T_N,i))+1/2|\geq N^{1/16}\}\\ &\cup\{\exists\, i\in\{\chi(N),....,\lceil2(|x|+2\theta)N\rceil\},|\eta_{i,+}(\ell^-(T_N,i))+1/2|\geq N^{1/16}\}.\end{aligned}$$

Lemma 7. *$\mathbb{P}(\mathcal{B}_2)$ tends to 0 when N tends to $+\infty$.*

Proof. It is enough to find some constants $c > 0$ and $C < +\infty$ so that for any $i \in \{-\lceil 2(|x|+2\theta)N\rceil, ..., \lceil 2(|x|+2\theta)N\rceil\}$ we have $\mathbb{P}(|\zeta_i| \geq N^{1/16}) \leq Ce^{-cN^{1/16}}$, for any $i \in \{-\lceil 2(|x|+2\theta)N\rceil, ..., \chi(N)-1\}$ we have $\mathbb{P}(|\eta_{i,-}(\ell^+(T_N,i))+1/2| \geq N^{1/16}) \leq Ce^{-cN^{1/16}}$, and for all $i \in \{\chi(N), ..., \lceil 2(|x|+2\theta)N\rceil\}$ we have $\mathbb{P}(|\eta_{i,+}(\ell^-(T_N,i))+1/2| \geq N^{1/16}) \leq Ce^{-cN^{1/16}}$. For all $i \in \mathbb{Z}$, ζ_i has distribution ρ_0, which has exponential tails, hence there exists constants $c' = c'(w) > 0$ and $C' = C'(w) < +\infty$ so that for $i \in \{-\lceil 2(|x|+2\theta)N\rceil, ..., \lceil 2(|x|+2\theta)N\rceil\}$ we have $\mathbb{P}(|\zeta_i| \geq N^{1/16}) \leq C'e^{-c'N^{1/16}}$. We now consider $i \in \{-\lceil 2(|x|+2\theta)N\rceil, ..., \chi(N)-1\}$ and $\mathbb{P}(|\eta_{i,-}(\ell^+(T_N,i))+1/2| \geq N^{1/16})$ (the $\mathbb{P}(|\eta_{i,+}(\ell^-(T_N,i))+1/2| \geq N^{1/16})$ can be dealt with in the same way). Equation (1) implies $\ell^+(T_N,i)$ depends only on the $\eta_{j,-}$ for $j > i$, hence is independent of $\eta_{i,-}$. This implies $\mathbb{P}(|\eta_{i,-}(\ell^+(T_N,i))+1/2| \geq N^{1/16}) = \sum_{k\in\mathbb{N}} \mathbb{P}(|\eta_{i,-}(k)+1/2| \geq N^{1/16})\mathbb{P}(\ell^+(T_N,i)=k)$. Therefore the first part of Lemma 5 implies $\mathbb{P}(|\eta_{i,-}(\ell^+(T_N,i))+1/2| \geq N^{1/16}) \leq \sum_{k\in\mathbb{N}} \frac{2\tilde{C}e^{\tilde{c}/2}}{1-e^{-\tilde{c}}} e^{-\tilde{c}N^{1/16}} \mathbb{P}(\ell^+(T_N,i)=k) = \frac{2\tilde{C}e^{\tilde{c}/2}}{1-e^{-\tilde{c}}} e^{-\tilde{c}N^{1/16}}$, which is enough. □

We will also need the following, which is a rather standard result of large deviations.

Lemma 8. *For any $\alpha > 0$, $\varepsilon > 0$, $\mathbb{P}(\max_{0\leq i_1\leq i_2\leq \lceil N^\alpha\rceil} |\sum_{i=i_1}^{i_2} \zeta_i| \geq N^{\alpha/2+\varepsilon})$ tends to 0 when $N \to +\infty$.*

Proof. Let $0 \leq i_1 \leq i_2 \leq \lceil N^\alpha\rceil$, let us study $\mathbb{P}(|\sum_{i=i_1}^{i_2} \zeta_i| \geq N^{\alpha/2+\varepsilon})$. We know the ζ_i, $i \in \mathbb{Z}$ are i.i.d. with distribution ρ_0, and it can be checked that ρ_0 is symmetric with respect to 0, so from that and the Markov inequality we get

$$\begin{aligned}\mathbb{P}\left(\left|\sum_{i=i_1}^{i_2}\zeta_i\right| \geq N^{\alpha/2+\varepsilon}\right) &\leq 2\mathbb{P}\left(\sum_{i=i_1}^{i_2}\zeta_i \geq N^{\alpha/2+\varepsilon}\right) = 2\mathbb{P}\left(\exp\left(\frac{1}{N^{\alpha/2}}\sum_{i=i_1}^{i_2}\zeta_i\right) \geq \exp(N^\varepsilon)\right)\\ &\leq 2e^{-N^\varepsilon}\mathbb{E}\left(\exp\left(\frac{1}{N^{\alpha/2}}\sum_{i=i_1}^{i_2}\zeta_i\right)\right) \leq 2e^{-N^\varepsilon}\prod_{i=i_1}^{i_2}\mathbb{E}\left(\exp\left(\frac{1}{N^{\alpha/2}}\zeta_i\right)\right).\end{aligned} \tag{5}$$

Now, if ζ has distribution ρ_0, we can write $\exp(\frac{1}{N^{\alpha/2}}\zeta) = 1 + \frac{1}{N^{\alpha/2}}\zeta + \frac{1}{2}(\frac{1}{N^{\alpha/2}}\zeta)^2\exp(\frac{1}{N^{\alpha/2}}\zeta')$ with $|\zeta'| \leq |\zeta|$. Since ρ_0 is symmetric with respect to 0, we have $\mathbb{E}(\zeta) = 0$, therefore

$$\mathbb{E}\left(\exp\left(\frac{1}{N^{\alpha/2}}\zeta\right)\right) = 1 + \mathbb{E}\left(\frac{1}{2}\left(\frac{1}{N^{\alpha/2}}\zeta\right)^2\exp\left(\frac{1}{N^{\alpha/2}}\zeta'\right)\right) \leq 1 + \frac{1}{2N^\alpha}\mathbb{E}\left(\zeta^2\exp\left(\frac{1}{N^{\alpha/2}}|\zeta|\right)\right).$$

Moreover, ρ_0 has exponential tails, hence there exists constants $C < +\infty$ and $c > 0$ so that $\mathbb{E}(\zeta^2 e^{c|\zeta|}) \leq C$. When N is large enough, $\frac{1}{N^{\alpha/2}} \leq c$, therefore $\mathbb{E}(\exp(\frac{1}{N^{\alpha/2}}\zeta)) \leq 1 + \frac{C}{2N^\alpha} \leq \exp(\frac{C}{2N^\alpha})$. Together with (5), this yields $\mathbb{P}(|\sum_{i=i_1}^{i_2}\zeta_i| \geq N^{\alpha/2+\varepsilon}) \leq 2e^{-N^\varepsilon}e^{(i_2-i_1+1)\frac{C}{2N^\alpha}} \leq 2e^{-N^\varepsilon}e^{(\lceil N^\alpha\rceil+1)\frac{C}{2N^\alpha}} \leq 2e^Ce^{-N^\varepsilon}$ when N is large enough. We deduce that when N is large enough, $\mathbb{P}(\max_{0\leq i_1\leq i_2\leq\lceil N^\alpha\rceil}|\sum_{i=i_1}^{i_2}\zeta_i| \geq N^{\alpha/2+\varepsilon}) \leq (\lceil N^\alpha\rceil+1)^2 2e^Ce^{-N^\varepsilon}$, which tends to 0 when N tends to $+\infty$. □

We also prove an immediate application of Lemma 8, which we will use several times. If we define

$$\mathcal{B}_3^- = \left\{\max_{-\lfloor(|x|+2\theta)N\rfloor-N^{3/4}\leq i_1\leq i_2\leq-\lfloor(|x|+2\theta)N\rfloor+N^{3/4}}\left|\sum_{i=i_1}^{i_2}\zeta_i\right| \geq N^{19/48}\right\},$$

$$\mathcal{B}_3^+ = \left\{\max_{\lfloor(|x|+2\theta)N\rfloor-N^{3/4}\leq i_1\leq i_2\leq\lfloor(|x|+2\theta)N\rfloor+N^{3/4}}\left|\sum_{i=i_1}^{i_2}\zeta_i\right| \geq N^{19/48}\right\},$$

we have the following lemma.

Lemma 9. *$\mathbb{P}(\mathcal{B}_3^-)$ and $\mathbb{P}(\mathcal{B}_3^+)$ tend to 0 when N tends to $+\infty$.*

Proof. Since the $(\zeta_i)_{i\in\mathbb{Z}}$ are i.i.d., $\mathbb{P}(\mathcal{B}_3^+) = \mathbb{P}(\mathcal{B}_3^-) = \mathbb{P}(\max_{0\le i_1\le i_2\le 2\lceil N^{3/4}\rceil} |\sum_{i=i_1}^{i_2}\zeta_i| \ge N^{19/48})$, which is smaller than $\mathbb{P}(\max_{0\le i_1\le i_2\le \lceil N^{37/48}\rceil} |\sum_{i=i_1}^{i_2}\zeta_i| \ge N^{19/48})$ when N is large enough. Moreover, Lemma 8, used with $\alpha = 37/48$ and $\varepsilon = 1/96$, yields that the latter probability tends to 0 when N tends to $+\infty$. □

3. Where the local times approach 0

The aim of this section is twofold. Firstly, we need to control the place where $\ell^-(T_N,i)$ hits 0 when i is at the right of 0, as well as the place where $\ell^+(T_N,i)$ hits 0 when i is at the left of 0. Secondly, we have to show that even when $\ell^\pm(T_N,i)$ is close to 0, the local times do not stray too far away from the coupling. For any $N\in\mathbb{N}$, we denote $I^+ = \inf\{i\ge\chi(N)\,|\,\ell^-(T_N,i)=0\}$ and $I^- = \sup\{i<\chi(N)\,|\,\ell^+(T_N,i)=0\}$. We notice that $\ell^+(T_N,I^-)=0$, and from the definition of T_N we have $\ell^+(T_N,i)>0$ for any $0\le i\le\chi(N)-1$, hence $I^-<0$. We will also need the auxiliary random variables $\tilde{I}^+ = \inf\{i\ge\chi(N)\,|\,\ell^-(T_N,i)\le\lfloor N^{1/6}\rfloor\}$ and $\tilde{I}^- = \sup\{i<\chi(N)\,|\,\ell^+(T_N,i)\le\lfloor N^{1/6}\rfloor\}$.

3.1. Place where we hit 0.

We have the following result of control on I^+ and I^-.

Lemma 10. *For any $\delta>0$, $\mathbb{P}(|I^- + (|x|+2\theta)N| \ge N^{\delta+1/2})$ and $\mathbb{P}(|I^+ - (|x|+2\theta)N| \ge N^{\delta+1/2})$ tend to 0 when N tends to $+\infty$.*

Proof. The idea is to control the fluctuations of the local times around their deterministic limit: as long as $\ell^\pm(T_N,i)$ is large, the $\ell^\pm(T_N,i+1)-\ell^\pm(T_N,i)$ will be close to the i.i.d. random variables of the coupling, so the fluctuations of $\ell^\pm(T_N,i)$ around its deterministic limit are bounded and $\ell^\pm(T_N,i)$ can be small only when the deterministic limit is small, that is around $-(|x|+2\theta)N$ and $(|x|+2\theta)N$. We only spell out the proof for I^-, as the argument for I^+ is similar. The fact that $\mathbb{P}(I^- + (|x|+2\theta)N \le -N^{\delta+1/2})$ tends to 0 when N tends to $+\infty$ comes from inequalities (51) and (53) of [8], so we only have to prove that $\mathbb{P}(I^- + (|x|+2\theta)N \ge N^{\delta+1/2})$ tends to 0 when N tends to $+\infty$. Since $I^-\le\tilde{I}^-$, it is enough to prove that $\mathbb{P}(\tilde{I}^- + (|x|+2\theta)N \ge N^{\delta+1/2})$ tends to 0 when N tends to $+\infty$. Since by Lemma 6 we have that $\mathbb{P}(\mathcal{B}_1^-)$ tends to 0 when N tends to $+\infty$, it is enough to prove $\mathbb{P}(\tilde{I}^- + (|x|+2\theta)N \ge N^{\delta+1/2}, (\mathcal{B}_1^-)^c)$ tends to 0 when N tends to $+\infty$. We now assume N is large enough, $\tilde{I}^- + (|x|+2\theta)N \ge N^{\delta+1/2}$ and $(\mathcal{B}_1^-)^c$. Then there exists $i\in\{\lceil -(|x|+2\theta)N + N^{\delta+1/2}\rceil, ..., \chi(N)-1\}$ so that $\ell^+(T_N,i)\le\lfloor N^{1/6}\rfloor$ and $\ell^+(T_N,j)>\lfloor N^{1/6}\rfloor$ for all $j\in\{i+1,...,\chi(N)-1\}$. Thus, by (1) we get $\lfloor N\theta\rfloor + \sum_{j=i+1}^{\chi(N)-1}(\eta_{j,-}(\ell^+(T_N,j)) + \mathbb{1}_{\{j>0\}}) = \ell^+(T_N,i)\le\lfloor N^{1/6}\rfloor$. Furthermore, for all $j\in\{i+1,...,\chi(N)-1\}$, since $(\mathcal{B}_1^-)^c$ occurs and $\ell^+(T_N,j)>\lfloor N^{1/6}\rfloor$, we have $\eta_{j,-}(\ell^+(T_N,j)) + 1/2 = \zeta_j$. We deduce $\lfloor N\theta\rfloor + \sum_{j=i+1}^{\chi(N)-1}(\zeta_j + (\mathbb{1}_{\{j>0\}} - \mathbb{1}_{\{j\le 0\}})/2) \le \lfloor N^{1/6}\rfloor$, thus $\sum_{j=i+1}^{\chi(N)-1}\zeta_j + \lfloor N\theta\rfloor + \sum_{j=i+1}^{\chi(N)-1}(\mathbb{1}_{\{j>0\}} - \mathbb{1}_{\{j\le 0\}})/2 \le \lfloor N^{1/6}\rfloor$. Moreover, since $i\in\{\lceil -(|x|+2\theta)N + N^{\delta+1/2}\rceil, ..., \chi(N)-1\}$ we have $\sum_{j=i+1}^{\chi(N)-1}(\mathbb{1}_{\{j>0\}} - \mathbb{1}_{\{j\le 0\}})/2 = \frac{1}{2}(\chi(N)-1+i) \ge \frac{1}{2}(Nx - 2 - (|x|+2\theta)N + N^{\delta+1/2}) = -\theta N + \frac{1}{2}N^{\delta+1/2} - 1$. This yields $\sum_{j=i+1}^{\chi(N)-1}\zeta_j + \lfloor N\theta\rfloor - \theta N + \frac{1}{2}N^{\delta+1/2} - 1 \le \lfloor N^{1/6}\rfloor$, hence $\sum_{j=i+1}^{\chi(N)-1}\zeta_j \le -\frac{1}{2}N^{\delta+1/2} + \lfloor N^{1/6}\rfloor + 2 \le -N^{(1+\delta)/2}$ since N is large enough. Consequently, when N is large enough, $\mathbb{P}(\tilde{I}^- + (|x|+2\theta)N \ge N^{\delta+1/2}, (\mathcal{B}_1^-)^c) \le \mathbb{P}(\exists\, i\in\{\lceil -(|x|+2\theta)N + N^{\delta+1/2}\rceil, ..., \chi(N)-1\}, \sum_{j=i+1}^{\chi(N)-1}\zeta_j \le -N^{(1+\delta)/2})$. Since the ζ_i, $i\in\mathbb{Z}$ are i.i.d., when N is large enough this yields $\mathbb{P}(\tilde{I}^- + (|x|+2\theta)N \ge N^{\delta+1/2}, (\mathcal{B}_1^-)^c) \le \mathbb{P}(\max_{0\le i_1\le i_2\le\lceil N^{1+\delta/2}\rceil} |\sum_{i=i_1}^{i_2}\zeta_i| \ge N^{(1+\delta)/2})$, which tends to 0 when N tends to $+\infty$ by Lemma 8 (applied with $\alpha = 1+\delta/2$ and $\varepsilon=\delta/4$). This shows that $\mathbb{P}(I^- + (|x|+2\theta)N \ge N^{\delta+1/2})$ converges to 0 when N tends to $+\infty$, which ends the proof of Lemma 10. □

3.2. **Control of low local times.** We have to show that even when $\ell^{\pm}(T_N, i)$ is small, the local times are not too far from the random variables of the coupling. In order to do that, we first prove that the window where $\ell^{\pm}(T_N, i)$ is small but not zero, that is between $\tilde{I}^+$ and I^+ and between I^- and $\tilde{I}^-$, is small. Afterwards, we will give bounds on what happens inside. We begin by showing the following easy result.

Lemma 11. $\mathbb{P}(\tilde{I}^- \geq 0)$ *tends to 0 when* $N \to +\infty$.

Proof. Let N be large enough. If $\tilde{I}^- \geq 0$, there exists $i \in \{0, ..., \lfloor Nx \rfloor\}$ so that $\ell^+(T_N, i) \leq \lfloor N^{1/6} \rfloor$. Since N is large enough, this implies $\ell^+(T_N, i) \leq N\theta/2$, therefore $\sup_{y \in \mathbb{R}} |\frac{1}{N}\ell^+(T_N, \lfloor Ny \rfloor) - (\frac{|x|-|y|}{2} + \theta)_+| \geq \theta/2$. Moreover, by Theorem 1 of [8], $\sup_{y \in \mathbb{R}} |\frac{1}{N}\ell^+(T_N, \lfloor Ny \rfloor) - (\frac{|x|-|y|}{2} + \theta)_+|$ converges in probability to 0 when N tends to $+\infty$, hence we deduce that $\mathbb{P}(\sup_{y \in \mathbb{R}} |\frac{1}{N}\ell^+(T_N, \lfloor Ny \rfloor) - (\frac{|x|-|y|}{2} + \theta)_+| \geq \theta/2)$ tends to 0 when $N \to +\infty$. Therefore $\mathbb{P}(\tilde{I}^- \geq 0)$ tends to 0 when $N \to +\infty$. □

In order to control I^+, I^-, $\tilde{I}^+$ and $\tilde{I}^-$, we will use the fact the local times behave as the Markov chain L from [8], defined as follows. We consider i.i.d. copies of the Markov chain η starting at 0, called $(\eta_m)_{m \in \mathbb{N}}$. For any $m \in \mathbb{N}$, we then set $L(m+1) = L(m) + \eta_m(L(m))$. We denote $\tau = \inf\{m \in \mathbb{N} \,|\, L(m) \leq 0\}$. The following was proven in [8].

Lemma 12 (Lemma 2 of [8])**.** *There exists a constant* $K < +\infty$ *so that for any* $k \in \mathbb{N}$ *we have* $\mathbb{E}(\tau | L(0) = k) \leq 3k + K$.

Since the local times will behave as L, Lemma 12 implies that if the local time starts small, then the time at which it reaches 0 has small expectation hence is not too large. This will help us to prove the following control on the window where $\ell^{\pm}(T_N, i)$ is small but not zero.

Lemma 13. $\mathbb{P}(I^+ - \tilde{I}^+ \geq N^{1/4})$ *and* $\mathbb{P}(\tilde{I}^- - I^- \geq N^{1/4})$ *tend to 0 when* $N \to +\infty$.

Proof. Let N be large enough. We deal only with $\mathbb{P}(\tilde{I}^- - I^- \geq N^{1/4})$, since $\mathbb{P}(I^+ - \tilde{I}^+ \geq N^{1/4})$ can be dealt with in the same way and with simpler arguments. Thanks to Lemma 11, it is enough to prove that $\mathbb{P}(\tilde{I}^- - I^- \geq N^{1/4}, \tilde{I}^- < 0)$ tends to 0 when $N \to +\infty$. Moreover, if $\tilde{I}^- < 0$, thanks to (1), for any $i < \tilde{I}^-$ we get $\ell^+(T_N, i) = \ell^+(T_N, \tilde{I}^-) + \sum_{j=i+1}^{\tilde{I}^-} \eta_{j,-}(\ell^+(T_N, j))$, which allows to prove that $(\ell^+(T_N, \tilde{I}^- - i))_{i \in \mathbb{N}}$ is a Markov chain with the transition probabilities of L. Therefore

$$\mathbb{P}\left(\tilde{I}^- - I^- \geq N^{1/4}, \tilde{I}^- < 0\right) = \sum_{k=0}^{\lfloor N^{1/6} \rfloor} \mathbb{P}\left(\tilde{I}^- - I^- \geq N^{1/4}, \tilde{I}^- < 0 \,\middle|\, \ell^+(T_N, \tilde{I}^-) = k\right) \mathbb{P}\left(\ell^+(T_N, \tilde{I}^-) = k\right)$$

$$= \sum_{k=0}^{\lfloor N^{1/6} \rfloor} \mathbb{P}\left(\tau \geq N^{1/4} \,\middle|\, L(0) = k\right) \mathbb{P}\left(\ell^+(T_N, \tilde{I}^-) = k\right) \leq \sum_{k=0}^{\lfloor N^{1/6} \rfloor} \frac{1}{N^{1/4}} \mathbb{E}(\tau | L(0) = k) \mathbb{P}(\ell^+(T_N, \tilde{I}^-) = k).$$

By Lemma 12 we deduce

$$\mathbb{P}\left(\tilde{I}^- - I^- \geq N^{1/4}, \tilde{I}^- < 0\right) \leq \frac{1}{N^{1/4}} \sum_{k=0}^{\lfloor N^{1/6} \rfloor} (3k + K) \mathbb{P}(\ell^+(T_N, \tilde{I}^-) = k) \leq \frac{3N^{1/6} + K}{N^{1/4}} \leq 4N^{-1/12}$$

since N is large enough, hence $\mathbb{P}(\tilde{I}^- - I^- \geq N^{1/4}, \tilde{I}^- < 0)$ tends to 0 when $N \to +\infty$, which ends the proof. □

We are now going to prove that even when $\ell^{\pm}(T_N, i)$ is small, the local times are not too far from the random variables of the coupling. More precisely, for any $n \in \mathbb{N}$, we define the following events.

$$\mathcal{B}_4^- = \left\{ \exists\, i \in \{I^-, ..., \chi(N)-1\}, \left| \sum_{j=i+1}^{\chi(N)-1} (\eta_{j,-}(\ell^+(T_N, j)) + 1/2) - \sum_{j=i+1}^{\chi(N)-1} \zeta_j \right| \geq N^{1/3} \right\},$$

$$\mathcal{B}_4^+ = \left\{ \exists\, i \in \{\chi(N), ..., I^+\}, \left| \sum_{j=\chi(N)}^{i-1} (\eta_{j,+}(\ell^-(T_N, j)) + 1/2) - \sum_{j=\chi(N)}^{i-1} \zeta_j \right| \geq N^{1/3} \right\}.$$

Lemma 14. *$\mathbb{P}(\mathcal{B}_4^-)$ and $\mathbb{P}(\mathcal{B}_4^+)$ tend to 0 when N tend to $+\infty$.*

Proof. The idea of the argument is that when $\ell^{\pm}(T_N, i)$ is large, $\eta_{i,\mp}(\ell^{\pm}(T_N, i)) + 1/2 = \zeta_i$ thanks to Lemma 6, that the window where $\ell^{\pm}(T_N, i)$ is small is bounded by Lemma 13, and that inside this window the $\eta_{i,\mp}(\ell^{\pm}(T_N, i)) + 1/2$, ζ_i are also bounded by Lemma 7. We only spell out the proof for $\mathbb{P}(\mathcal{B}_4^-)$, since the proof for $\mathbb{P}(\mathcal{B}_4^+)$ is the same. By Lemma 10, we have that $\mathbb{P}(I^- \leq -2(|x|+\theta)N)$ tends to 0 when N tends to $+\infty$. Furthermore, Lemma 13 implies that $\mathbb{P}(\tilde{I}^- - I^- \geq N^{1/4})$ tends to 0 when N tends to $+\infty$. In addition, by Lemmas 6 and 7 we have that $\mathbb{P}(\mathcal{B}_1^-)$ and $\mathbb{P}(\mathcal{B}_2)$ tend to 0 when N tends to $+\infty$. Consequently, it is enough to prove that for N large enough, if $(\mathcal{B}_1^-)^c$, $(\mathcal{B}_2)^c$ occur, if $\tilde{I}^- - I^- < N^{1/4}$ and if $I^- > -2(|x|+\theta)N$, then $(\mathcal{B}_4^-)^c$ occurs. We assume $(\mathcal{B}_1^-)^c$, $(\mathcal{B}_2)^c$, $\tilde{I}^- - I^- < N^{1/4}$ and $I^- > -2(|x|+\theta)N$. Since $(\mathcal{B}_1^-)^c$ occurs and $\tilde{I}^- \geq I^- > -2(|x|+\theta)N$, we get $\zeta_i = \eta_{j,-}(\ell^+(T_N, j)) + 1/2$ for any $i \in \{\tilde{I}^- + 1, ..., \chi(N)-1\}$. Therefore, if $i \in \{\tilde{I}^-, ..., \chi(N)-1\}$ we get $\sum_{j=i+1}^{\chi(N)-1}(\eta_{j,-}(\ell^+(T_N, j)) + 1/2) - \sum_{j=i+1}^{\chi(N)-1} \zeta_j = 0$, and for $i \in \{I^-, ..., \tilde{I}^- - 1\}$ we have

$$\left| \sum_{j=i+1}^{\chi(N)-1} (\eta_{j,-}(\ell^+(T_N, j)) + 1/2) - \sum_{j=i+1}^{\chi(N)-1} \zeta_j \right| = \left| \sum_{j=i+1}^{\tilde{I}^-} (\eta_{j,-}(\ell^+(T_N, j)) + 1/2) - \sum_{j=i+1}^{\tilde{I}^-} \zeta_j \right|$$

$$\leq \sum_{j=i+1}^{\tilde{I}^-} \left(|\eta_{j,-}(\ell^+(T_N, j)) + 1/2| + |\zeta_j| \right) \leq 2(\tilde{I}^- - I^-)N^{1/16}$$

since $(\mathcal{B}_2^-)^c$ occurs, $i+1 \geq I^- > -2(|x|+\theta)N$ and by definition $\tilde{I}^- \leq \chi(N) - 1 \leq 2(|x|+\theta)N$. Moreover, we assumed $\tilde{I}^- - I^- < N^{1/4}$, which implies $|\sum_{j=i+1}^{\chi(N)-1}(\eta_{j,-}(\ell^+(T_N, j)) + 1/2) - \sum_{j=i+1}^{\chi(N)-1} \zeta_j| \leq 2N^{1/4}N^{1/16} = 2N^{5/16} < N^{1/3}$ when N is large enough. Consequently, for any $i \in \{I^-, ..., \chi(N)-1\}$ we have $|\sum_{j=i+1}^{\chi(N)-1}(\eta_{j,-}(\ell^+(T_N, j)) + 1/2) - \sum_{j=i+1}^{\chi(N)-1} \zeta_j| < N^{1/3}$, therefore $(\mathcal{B}_4^-)^c$ occurs, which ends the proof. □

4. Skorohod M_1 distance

The goal of this section is to prove that when N is large, $Y_N^{\pm}$ is close in the Skorohod M_1 distance to the function Y_N defined as follows. For any N large enough, for $y \in \mathbb{R}$, we set $Y_N(y) = \frac{1}{\sqrt{N}} \sum_{i=\lfloor Ny \rfloor + 1}^{\chi(N)-1} \zeta_i$ if $y \in [-|x| - 2\theta, \frac{\chi(N)}{N})$, $Y_N(y) = \frac{1}{\sqrt{N}} \sum_{i=\chi(N)}^{\lfloor Ny \rfloor - 1} \zeta_i$ if $y \in [\frac{\chi(N)}{N}, |x| + 2\theta)$, and $Y_N(y) = 0$ otherwise. We want to prove the following proposition.

Proposition 15. *$\mathbb{P}(d_{M_1}(Y_N^{\pm}, Y_N) > 3N^{-1/12})$ tends to 0 when N tends to $+\infty$.*

If we denote

$$\mathcal{B} = \mathcal{B}_1^- \cup \mathcal{B}_1^+ \cup \mathcal{B}_2 \cup \mathcal{B}_3^- \cup \mathcal{B}_3^+ \cup \mathcal{B}_4^- \cup \mathcal{B}_4^+ \cup \{|I^- + (|x|+2\theta)N| \geq N^{3/4}\} \cup \{|I^+ - (|x|+2\theta)N| \geq N^{3/4}\},$$

it will be enough to prove the following proposition.

Proposition 16. *When N is large enough, for all $a > 0$ with $|(|x|+2\theta) - a| > N^{-1/8}$, we have that $\mathcal{B}^c \subset \{d_{M_1,a}(Y_N^\pm|_{[-a,a]}, Y_N|_{[-a,a]}) \leq 2N^{-1/12}\}$.*

Proof of Proposition 15 given Proposition 16. We assume Proposition 16 holds. Then, when N is large enough, if $\mathcal{B}^c$ occurs, for all $a > 0$ with $|(|x|+2\theta) - a| > N^{-1/8}$ we have $d_{M_1,a}(Y_N^\pm|_{[-a,a]}, Y_N|_{[-a,a]}) \leq 2N^{-1/12}$, which yields $d_{M_1}(Y_N^\pm, Y_N) = \int_0^{+\infty} e^{-a}(d_{M_1,a}(Y_N^\pm|_{[-a,a]}, Y_N|_{[-a,a]}) \wedge 1)\mathrm{d}a \leq \int_0^{+\infty} e^{-a} 2N^{-1/12}\mathrm{d}a + 2N^{-1/8} = 2N^{-1/12} + 2N^{-1/8} \leq 3N^{-1/12}$. This implies $\mathbb{P}(d_{M_1}(Y_N^\pm, Y_N) > 3N^{-1/12}) \leq \mathbb{P}(\mathcal{B})$ when N is large enough. In addition,

$$\begin{aligned}\mathbb{P}(\mathcal{B}) \leq & \mathbb{P}(\mathcal{B}_1^-) + \mathbb{P}(\mathcal{B}_1^+) + \mathbb{P}(\mathcal{B}_2) + \mathbb{P}(\mathcal{B}_3^-) + \mathbb{P}(\mathcal{B}_3^+) + \mathbb{P}(\mathcal{B}_4^-) + \mathbb{P}(\mathcal{B}_4^+) \\ & + \mathbb{P}(|I^- + (|x|+2\theta)N| \geq N^{3/4}) + \mathbb{P}(|I^+ - (|x|+2\theta)N| \geq N^{3/4}).\end{aligned}$$

Furthermore, Lemma 6 implies $\mathbb{P}(\mathcal{B}_1^-)$ and $\mathbb{P}(\mathcal{B}_1^+)$ tend to 0 when N tends to $+\infty$, Lemma 7 implies $\mathbb{P}(\mathcal{B}_2)$ tends to 0 when N tends to $+\infty$, Lemma 9 implies $\mathbb{P}(\mathcal{B}_3^-)$ and $\mathbb{P}(\mathcal{B}_3^+)$ tend to 0 when N tends to $+\infty$, Lemma 14 implies $\mathbb{P}(\mathcal{B}_4^-)$ and $\mathbb{P}(\mathcal{B}_4^+)$ tend to 0 when N tends to $+\infty$, and Lemma 10 implies $\mathbb{P}(|I^- + (|x|+2\theta)N| \geq N^{3/4})$ and $\mathbb{P}(|I^+ - (|x|+2\theta)N| \geq N^{3/4})$ tend to 0 when N tends to $+\infty$. Therefore $\mathbb{P}(\mathcal{B})$ tends to 0 when N tends to $+\infty$, hence $\mathbb{P}(d_{M_1}(Y_N^\pm, Y_N) > 3N^{-1/12})$ tends to 0 when N tends to $+\infty$, which is Proposition 15.

□

The remainder of this section is devoted to the proof of Proposition 16. The first thing we do is showing that between $\frac{(-(|x|+2\theta)N)\vee I^-}{N}$ and $\frac{((|x|+2\theta)N)\wedge I^+}{N}$, the functions $Y_N^\pm$ and Y_N are close in uniform distance, which is the following lemma.

Lemma 17. *When N is large enough, if $(\mathcal{B}_2)^c$, $(\mathcal{B}_4^-)^c$ and $(\mathcal{B}_4^+)^c$ occur, then if $I^+ < (|x|+2\theta)N$ then for any $y \in [\frac{(-(|x|+2\theta)N)\vee I^-}{N}, \frac{((|x|+2\theta)N)\wedge I^+}{N}]$ we have $|Y_N^\pm(y) - Y_N(y)| \leq N^{-1/12}$, while if $I^+ \geq (|x|+2\theta)N$ we have $|Y_N^\pm(y) - Y_N(y)| \leq N^{-1/12}$ for $y \in [\frac{(-(|x|+2\theta)N)\vee I^-}{N}, \frac{((|x|+2\theta)N)\wedge I^+}{N})$.*

Proof of Lemma 17. Writing down the proof is only a technical matter, as the meaning of $(\mathcal{B}_4^\pm)^c$ is that the local times are close to the process formed from the random variables of the coupling. $(\mathcal{B}_2)^c$ is there to ensure that the difference terms that appear will be small. We only spell out the proof for Y^-, as the proof for Y^+ is similar. We assume $(\mathcal{B}_2)^c$, $(\mathcal{B}_4^-)^c$ and $(\mathcal{B}_4^+)^c$. Then if $y \in [\frac{\chi(N)}{N}, \frac{((|x|+2\theta)N)\wedge I^+}{N}]$ (if $I^+ \geq (|x|+2\theta)N$ we exclude the case $y = \frac{((|x|+2\theta)N)\wedge I^+}{N}$) we have $y \in [\frac{\chi(N)}{N}, |x|+2\theta)$, so $|Y_N^-(y) - Y_N(y)| = \frac{1}{\sqrt{N}}|\ell^-(T_N, \lfloor Ny \rfloor) - N(\frac{|x|-|y|}{2} + \theta)_+ - \sum_{i=\chi(N)}^{\lfloor Ny \rfloor - 1} \zeta_i|$, thus by (1) we obtain the following: $|Y_N^-(y) - Y_N(y)| = \frac{1}{\sqrt{N}}|\lfloor N\theta \rfloor - \mathbb{1}_{\{\iota=+\}} + \sum_{i=\chi(N)}^{\lfloor Ny \rfloor - 1} \eta_{i,+}(\ell^-(T_N, i)) - N(\frac{|x|-|y|}{2} + \theta)_+ - \sum_{i=\chi(N)}^{\lfloor Ny \rfloor - 1} \zeta_i| \leq \frac{1}{\sqrt{N}}|\sum_{i=\chi(N)}^{\lfloor Ny \rfloor - 1} \eta_{i,+}(\ell^-(T_N, i)) + \frac{\lfloor Ny \rfloor - \chi(N)}{2} - \sum_{i=\chi(N)}^{\lfloor Ny \rfloor - 1} \zeta_i| + \frac{3}{\sqrt{N}} = \frac{1}{\sqrt{N}}|\sum_{i=\chi(N)}^{\lfloor Ny \rfloor - 1} (\eta_{i,+}(\ell^-(T_N, i)) + 1/2) - \sum_{i=\chi(N)}^{\lfloor Ny \rfloor - 1} \zeta_i| + \frac{3}{\sqrt{N}}$. Now, $y \in [\frac{\chi(N)}{N}, \frac{((|x|+2\theta)N)\wedge I^+}{N}]$ implies $\lfloor Ny \rfloor \in \{\chi(N), ..., I^+\}$, thus $(\mathcal{B}_4^+)^c$ yields $|Y_N^-(y) - Y_N(y)| \leq \frac{1}{\sqrt{N}} N^{1/3} + \frac{3}{\sqrt{N}} \leq N^{-1/12}$ when N is large enough. We now consider the case $y \in [\frac{(-(|x|+2\theta)N)\vee I^-}{N}, \frac{\chi(N)}{N})$. Then $y \in [-|x| - 2\theta, \frac{\chi(N)}{N})$, hence $|Y_N^-(y) - Y_N(y)| = \frac{1}{\sqrt{N}}|\ell^-(T_N, \lfloor Ny \rfloor) - N(\frac{|x|-|y|}{2} + \theta)_+) - \sum_{i=\lfloor Ny \rfloor + 1}^{\chi(N)-1} \zeta_i|$. Now, (2) yields $|\ell^-(T_N, \lfloor Ny \rfloor) - \ell^+(T_N, \lfloor Ny \rfloor)| = |\eta_{\lfloor Ny \rfloor, -}(\ell^+(T_N, \lfloor Ny \rfloor))|$, which is smaller than $N^{1/16} + 1/2$ thanks to $(\mathcal{B}_2)^c$. We

deduce that $|Y_N^-(y) - Y_N(y)| \leq \frac{1}{\sqrt{N}}|\ell^+(T_N, \lfloor Ny \rfloor) - N(\frac{|x|-|y|}{2} + \theta)_+) - \sum_{i=\lfloor Ny \rfloor+1}^{\chi(N)-1} \zeta_i| + \frac{N^{1/16}+1/2}{\sqrt{N}}$, thus (1) implies $|Y_N^-(y) - Y_N(y)| \leq \frac{1}{\sqrt{N}}|\lfloor N\theta \rfloor + \sum_{i=\lfloor Ny \rfloor+1}^{\chi(N)-1}(\eta_{i,-}(\ell^+(T_N,i)) + \mathbb{1}_{\{i>0\}}) - N(\frac{|x|-|y|}{2} + \theta)_+) - \sum_{i=\lfloor Ny \rfloor+1}^{\chi(N)-1} \zeta_i| + \frac{N^{1/16}+1/2}{\sqrt{N}} \leq \frac{1}{\sqrt{N}}|\sum_{i=\lfloor Ny \rfloor+1}^{\chi(N)-1}(\eta_{i,-}(\ell^+(T_N,i)) + \mathbb{1}_{\{i>0\}}) + \frac{\lfloor Ny \rfloor+1-\chi(N)}{2} - \sum_{i=\lfloor Ny \rfloor+1}^{\chi(N)-1} \zeta_i| + \frac{N^{1/16}+3}{\sqrt{N}} \leq \frac{1}{\sqrt{N}}|\sum_{i=\lfloor Ny \rfloor+1}^{\chi(N)-1}(\eta_{i,-}(\ell^+(T_N,i)) + 1/2) - \sum_{i=\lfloor Ny \rfloor+1}^{\chi(N)-1} \zeta_i| + \frac{N^{1/16}+3}{\sqrt{N}}$. Furthermore, $y \in [\frac{(-(|x|+2\theta)N)\vee I^-}{N}, \frac{\chi(N)}{N})$ implies $\lfloor Ny \rfloor \in \{I^-, ..., \chi(N)-1\}$, hence $(\mathcal{B}_4^-)^c$ yields $|Y_N^-(y) - Y_N(y)| \leq \frac{1}{\sqrt{N}}N^{1/3} + \frac{N^{1/16}+3}{\sqrt{N}} \leq N^{-1/12}$ when N is large enough. Consequently, for any $y \in [\frac{(-(|x|+2\theta)N)\vee I^-}{N}, \frac{((|x|+2\theta)N)\wedge I^+}{N}]$ we have $|Y_N^-(y) - Y_N(y)| \leq N^{-1/12}$, which ends the proof of Lemma 17. □

We now prove Proposition 16. Let $a > 0$ so that $|(|x| + 2\theta) - a| > N^{-1/8}$, we will prove that when N is large enough, $\mathcal{B}^c \subset \{d_{M_1,a}(Y_N^\pm|_{[-a,a]}, Y_N|_{[-a,a]}) \leq 2N^{-1/12}\}$, and the threshold for N given by the proof will not depend on the value of a. In order to bound the Skorohod M_1 distance between $Y_N^\pm|_{[-a,a]}$ and $Y_N|_{[-a,a]}$, we will write down parametric representations of these two functions and prove that they are close. There will be two cases depending on if a is smaller than $|x| + 2\theta$ or not.

4.1. **Case $\mathbf{a \in (0, |x| + 2\theta - N^{-1/8})}$.** This is the easier case. Indeed, the interval $[-a, a]$ will then be contained into $[\frac{(-(|x|+2\theta)N)\vee I^-}{N}, \frac{((|x|+2\theta)N)\wedge I^+}{N})$, inside which $Y_N^\pm$ and Y_N are close for the uniform norm by Lemma 17. We only write down the proof for Y_N^-, since the proof for Y_N^+ is similar. Let us define (u_N^-, r_N^-) and (u_N, r_N), which will be parametric representations of $Y_N^-|_{[-a,a]}$ and $Y_N|_{[-a,a]}$. The idea is to follow the completed graphs of the two functions in parallel. We set $\phi : [0,1] \mapsto [-a - \frac{\lfloor Na \rfloor+1}{N}, a + \frac{\lfloor Na \rfloor}{N}]$ the affine function mapping 0 to $-a - \frac{\lfloor Na \rfloor+1}{N}$ and 1 to $a + \frac{\lfloor Na \rfloor}{N}$ (if Na is an integer, we replace $-a - \frac{\lfloor Na \rfloor+1}{N}$ with $-a - \frac{\lfloor Na \rfloor}{N}$). Y_N^- and Y_N have discontinuities at the $\frac{i}{N}$ where i is an integer, and only there; for t so that $\phi(t) \in [\frac{2i}{N}, \frac{2i+1}{N}]$, the parametric representations will follow the graphs of the functions between $\frac{i}{N}$ and $\frac{i+1}{N}$, and when t satisfies $\phi(t) \in [\frac{2i+1}{N}, \frac{2i+2}{N}]$, the representations will follow the part of the completed graphs corresponding to the discontinuity at $\frac{i+1}{N}$. More rigorously, for any $t \in [0,1]$, if $\phi(t)$ belongs to some $[\frac{2i}{N}, \frac{2i+1}{N})$ for $i \in \{-\lfloor Na \rfloor, ..., \lfloor Na \rfloor - 1\}$, we set $(u_N^-(t), r_N^-(t)) = (\phi(t) - \frac{i}{N}, Y_N^-(\phi(t) - \frac{i}{N}))$, while if $\phi(t)$ belongs to some $[\frac{2i+1}{N}, \frac{2i+2}{N}]$ for $i \in \{-\lfloor Na \rfloor - 1, ..., \lfloor Na \rfloor - 1\}$ ($\{-\lfloor Na \rfloor, ..., \lfloor Na \rfloor - 1\}$ if Na is an integer), we set $(u_N^-(t), r_N^-(t)) = (\frac{i+1}{N}, (-N\phi(t) + 2i + 2)Y_N^-((\frac{i+1}{N})^-) + (N\phi(t) - 2i - 1)Y_N^-(\frac{i+1}{N}))$. Moreover, if $\phi(t) \in [-a - \frac{\lfloor Na \rfloor+1}{N}, -\frac{2\lfloor Na \rfloor+1}{N})$, we set $(u_N^-(t), r_N^-(t)) = (\phi(t) + \frac{\lfloor Na \rfloor+1}{N}, Y_N^-(\phi(t) + \frac{\lfloor Na \rfloor+1}{N}))$ and if $\phi(t) \in [\frac{2\lfloor Na \rfloor}{N}, a + \frac{\lfloor Na \rfloor}{N}]$, we set $(u_N^-(t), r_N^-(t)) = (\phi(t) - \frac{\lfloor Na \rfloor}{N}, Y_N^-(\phi(t) - \frac{\lfloor Na \rfloor}{N}))$. (u_N, r_N) is defined in the same way. Since the discontinuities of Y_N^- and Y_N in $[-a, a]$ can only be at the $\frac{i}{N}$, $i \in \mathbb{Z}$, this indeed defines parametric representations of $Y_N^-|_{[-a,a]}$ and $Y_N|_{[-a,a]}$.

We now use (u_N^-, r_N^-) and (u_N, r_N) to bound $d_{M_1,a}(Y_N^-|_{[-a,a]}, Y_N|_{[-a,a]})$. By definition of the Skorohod M_1 distance, we have $d_{M_1,a}(Y_N^-|_{[-a,a]}, Y_N|_{[-a,a]}) \leq \max(\|u_N^- - u_N\|_\infty, \|r_N^- - r_N\|_\infty)$. With the parametric representations chosen, we have $\|u_N^- - u_N\|_\infty = 0$ and $\|r_N^- - r_N\|_\infty \leq \sup_{y\in[-a,a]} |Y_N^-(y) - Y_N(y)|$, so we only have to prove $\mathcal{B}^c \subset \{\sup_{y\in[-a,a]} |Y_N^-(y) - Y_N(y)| \leq N^{-1/12}\}$. We now assume $\mathcal{B}^c$ occurs. Since $a \in (0, |x| + 2\theta - N^{-1/8})$, for any $y \in [-a, a]$ we have $y \in (-|x| - 2\theta + N^{-1/8}, |x| + 2\theta - N^{-1/8})$ thus $-(|x| + 2\theta)N + N^{3/4} \leq Ny \leq (|x| + 2\theta)N - N^{3/4}$, hence $I^- < Ny < I^+$, hence $y \in (\frac{(-(|x|+2\theta)N)\vee I^-}{N}, \frac{((|x|+2\theta)N)\wedge I^+}{N})$, so by Lemma 17 we have $|Y_N^-(y) - Y_N(y)| \leq N^{-1/12}$. We thus have $\mathcal{B}^c \subset \{\sup_{y\in[-a,a]} |Y_N^-(y) - Y_N(y)| \leq N^{-1/12}\}$, which ends the proof.

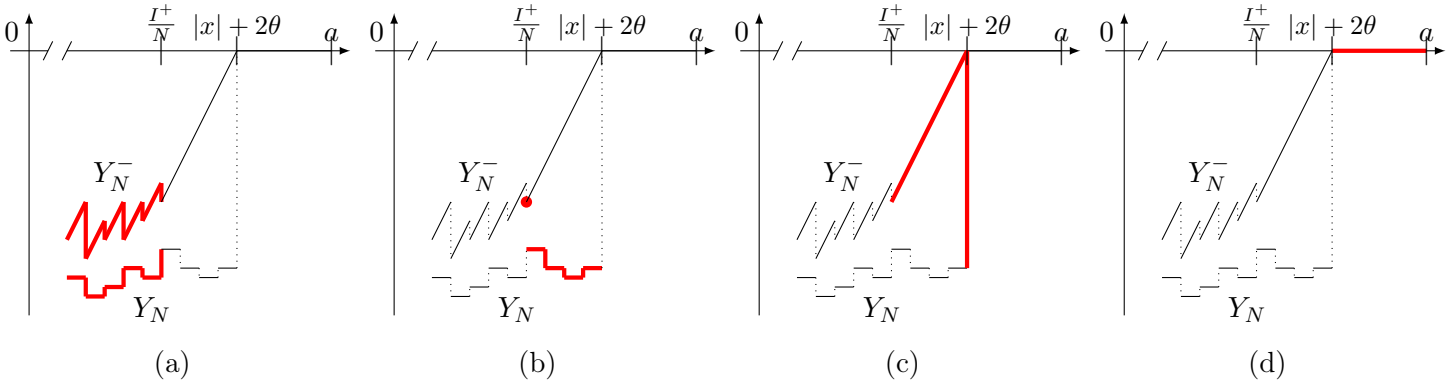

FIGURE 1. The successive steps of the parametric representations of $Y_N^-|_{[-a,a]}$ and $Y_N|_{[-a,a]}$ if $I^+ \leq \lfloor (|x|+2\theta)N \rfloor$. At each step, the parts of the graphs the parametric representations travel through are thickened.

4.2. **Case $a > |x| + 2\theta + N^{-1/8}$.** This is the harder case, as we have to deal with what happens around $|x|+2\theta$ and $-|x|-2\theta$. We only write down the proof for Y_N^-, since the proof for Y_N^+ is similar (one may remember that (2) allows to bound the $\ell^-(T_N,i) - \ell^+(T_N,i)$ when $(\mathcal{B}_2)^c$ occurs, hence when $\mathcal{B}^c$ occurs). Once again, we will define parametric representations (u_N^-, r_N^-) and (u_N, r_N) of $Y_N^-|_{[-a,a]}$ and $Y_N|_{[-a,a]}$.

We first give an idea of the definition. It will depend on whether $I^+ \leq \lfloor (|x|+2\theta)N \rfloor$ or not, and also on whether $I^- \geq -\lfloor (|x|+2\theta)N \rfloor$ or not. We are going to explain the ideas of the parametric representations of the completed graphs for abscissas in $[0,a]$ depending on whether $I^+ \leq \lfloor (|x|+2\theta)N \rfloor$ or not; the ideas for abscissas in $[-a,0]$ are similar depending on whether $I^- \geq -\lfloor (|x|+2\theta)N \rfloor$ or not. We first assume $I^+ \leq \lfloor (|x|+2\theta)N \rfloor$. Between 0 and $\frac{I^+}{N}$, the parametric representations will be, as in the case $a \in (0, |x|+2\theta - N^{-1/8})$, following the completed graphs of Y_N^- and Y_N in parallel (see Figure 1(a)). The next step, once (u_N^-, r_N^-) reached $(\frac{I^+}{N}, Y_N^-(\frac{I^+}{N}))$, is to freeze it there while (u_N, r_N) follows the graph of Y_N from $(\frac{I^+}{N}, Y_N(\frac{I^+}{N}))$ to $(|x|+2\theta, Y_N((|x|+2\theta))^-)$ (see Figure 1(b)). For $y \geq \frac{I^+}{N}$ we have $\ell^-(T_N, \lfloor Ny \rfloor) = 0$ thus $Y_N(y) = -N(\frac{|x|-|y|}{2} + \theta)_+$, hence $Y_N^- : [\frac{I^+}{N}, |x|+2\theta] \mapsto \mathbb{R}$ is affine. Therefore, the following step is to move at the same time (u_N^-, r_N^-) from $(\frac{I^+}{N}, Y_N^-(\frac{I^+}{N}))$ to $(|x|+2\theta, Y_N^-(|x|+2\theta)) = (|x|+2\theta, 0)$ and (u_N, r_N) from $(|x|+2\theta, Y_N((|x|+2\theta)^-))$ to $(|x|+2\theta, 0)$ (see Figure 1(c)), and the two parametric representations will remain close. After this step, both parametric representations are at $(|x|+2\theta, 0)$, and they will go together to $(a,0)$ (see Figure 1(d)). We now assume $I^+ > \lfloor (|x|+2\theta)N \rfloor$. Between 0 and $|x|+2\theta$, the parametric representations will follow the completed graphs of Y_N^- and Y_N in parallel (see Figure 2(a)). Once abscissa $|x|+2\theta$ is reached, the next step is to move (u_N^-, r_N^-) from $(|x|+2\theta, Y_N^-(|x|+2\theta))$ to $(\frac{I^+}{N}, Y_N^-(\frac{I^+}{N}))$, which is $(\frac{I^+}{N}, 0)$, and to move at the same time (u_N, r_N) from $(|x|+2\theta, Y_N(|x|+2\theta))$ to $(|x|+2\theta, 0)$ (see Figure 2(b)). We will prove the two representations are close by controlling the local times. At the next step we freeze (u_N^-, r_N^-) at $(\frac{I^+}{N}, 0)$ while (u_N, r_N) goes from $(|x|+2\theta, 0)$ to $(\frac{I^+}{N}, 0)$ (see Figure 2(c)). After this step, both parametric representations are at $(\frac{I^+}{N}, 0)$, and they will go together from $(\frac{I^+}{N}, 0)$ to $(a,0)$ (see Figure 2(d)).

We now give the rigorous definition of the parametric representations. We will set $a_N^+, a_N^- > 0$ depending on the cases. If $I^+ \leq \lfloor (|x|+2\theta)N \rfloor$ we set $a_N^+ = a + |x| + 2\theta + \frac{\lfloor (|x|+2\theta)N \rfloor - I^+}{N}$, except if $(|x|+2\theta)N$ is an integer, in

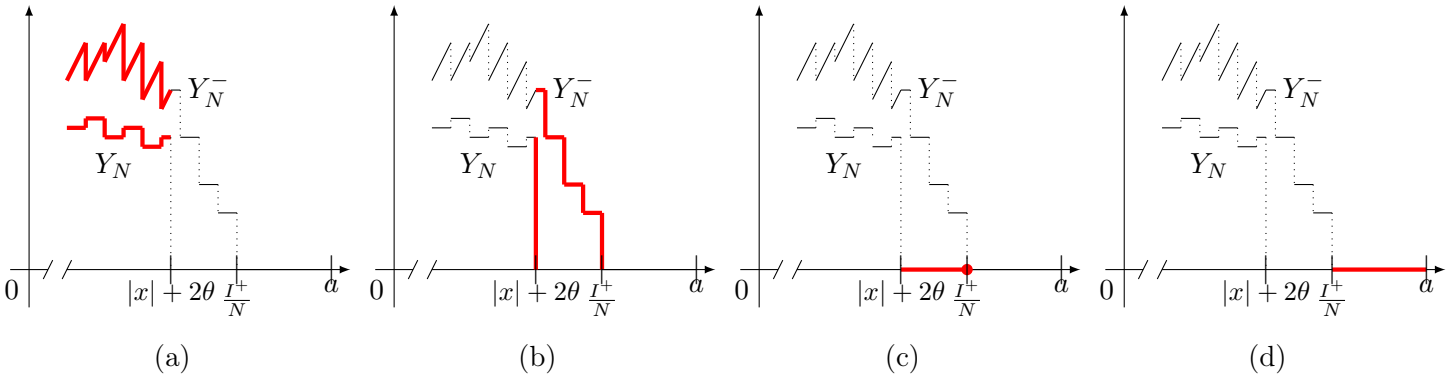

FIGURE 2. The successive steps of the parametric representations of $Y_N^-|_{[-a,a]}$ and $Y_N|_{[-a,a]}$ if $I^+ > \lfloor(|x|+2\theta)N\rfloor$. At each step, the parts of the graphs the parametric representations travel through are thickened.

which case we add $-\frac{1}{N}$ to a_N^+ if $I^+ \neq \lfloor(|x|+2\theta)N\rfloor$ and add 1 if $I^+ = \lfloor(|x|+2\theta)N\rfloor$. If $I^+ \geq \lfloor(|x|+2\theta)N\rfloor + 1$, we set $a_N^+ = a + \frac{2I^+}{N} - (|x|+2\theta)$. If $I^- \geq -\lfloor(|x|+2\theta)N\rfloor$, we set $a_N^- = a + (|x|+2\theta) + \frac{2\lfloor(|x|+2\theta)N\rfloor + 2I^- + 2}{N}$, to which we add $-\frac{2}{N}$ if $(|x|+2\theta)N$ is an integer and $-\frac{2}{N}+1$ if furthermore $I^- = -\lfloor(|x|+2\theta)N\rfloor$. If $I^- < -\lfloor(|x|+2\theta)N\rfloor$, we set $a_N^- = a + \frac{2I^-}{N} + |x| + 2\theta$. Let $\phi : [0,1] \mapsto [-a_N^-, a_N^+]$ be the affine function mapping 0 to $-a_N^-$ and 1 to a_N^+. Let $t \in [0,1]$. We begin with the "central part" of the interval $[-a_N^-, a_N^+]$, that is the one between $2\frac{(-\lfloor(|x|+2\theta)N\rfloor)\vee I^-}{N}$ and $2\frac{\lfloor(|x|+2\theta)N\rfloor \wedge I^+}{N}$, corresponding to Figures 1(a) and 2(a). If $\phi(t)$ belongs to some $[\frac{2i}{N}, \frac{2i+1}{N})$ for $i \in \{(-\lfloor(|x|+2\theta)N\rfloor) \vee I^-, ..., \lfloor(|x|+2\theta)N\rfloor \wedge I^+ - 1\}$, we set $(u_N^-(t), r_N^-(t)) = (\phi(t) - \frac{i}{N}, Y_N^-(\phi(t) - \frac{i}{N}))$ and $(u_N(t), r_N(t)) = (\phi(t) - \frac{i}{N}, Y_N(\phi(t) - \frac{i}{N}))$, while if $\phi(t)$ belongs to some $[\frac{2i+1}{N}, \frac{2i+2}{N}]$ for $i \in \{(-\lfloor(|x|+2\theta)N\rfloor) \vee I^-, ..., \lfloor(|x|+2\theta)N\rfloor \wedge I^+ - 1\}$, we set $(u_N^-(t), r_N^-(t)) = (\frac{i+1}{N}, (-N\phi(t)+2i+2)Y_N^-((\frac{i+1}{N})^-) + (N\phi(t)-2i-1)Y_N^-(\frac{i+1}{N}))$ and $(u_N(t), r_N(t)) = (\frac{i+1}{N}, (-N\phi(t)+2i+2)Y_N((\frac{i+1}{N})^-) + (N\phi(t)-2i-1)Y_N(\frac{i+1}{N}))$ (with the exception of the case $(|x|+2\theta)N$ integer, $I^+ \geq (|x|+2\theta)N$, $i = (|x|+2\theta)N - 1$, for which we set $(u_N(t), r_N(t)) = (\frac{i+1}{N}, Y_N((\frac{i+1}{N})^-)$, because then $Y_N(\frac{i+1}{N}) = Y_N(|x|+2\theta) = 0$, and we want to handle the move from $Y_N((|x|+2\theta)^-)$ to 0 later).

Assume $I^+ \leq \lfloor(|x|+2\theta)N\rfloor$. We first define the step of the parametric representations corresponding to Figure 1(b). If $\phi(t) \in [\frac{2I^+}{N}, |x|+2\theta+\frac{\lfloor(|x|+2\theta)N\rfloor}{N}]$, we set $(u_N^-(t), r_N^-(t)) = (\frac{I^+}{N}, Y_N^-(\frac{I^+}{N}))$. If $\phi(t)$ belongs to some $[\frac{2i}{N}, \frac{2i+1}{N})$ for $i \in \{I^+, ..., \lfloor(|x|+2\theta)N\rfloor - 1\}$, we set $(u_N(t), r_N(t)) = (\phi(t) - \frac{i}{N}, Y_N(\phi(t) - \frac{i}{N}))$, while if $\phi(t)$ belongs to some $[\frac{2i+1}{N}, \frac{2i+2}{N}]$ for $i \in \{I^+, ..., \lfloor(|x|+2\theta)N\rfloor - 1\}$, we set $(u_N(t), r_N(t)) = (\frac{i+1}{N}, (-N\phi(t)+2i+2)Y_N((\frac{i+1}{N})^-) + (N\phi(t)-2i-1)Y_N(\frac{i+1}{N}))$, and if $\phi(t) \in [\frac{2\lfloor(|x|+2\theta)N\rfloor}{N}, |x|+2\theta+\frac{\lfloor(|x|+2\theta)N\rfloor}{N})$ we set $(u_N(t), r_N(t)) = (\phi(t) - \frac{\lfloor(|x|+2\theta)N\rfloor}{N}, Y_N(\phi(t) - \frac{\lfloor(|x|+2\theta)N\rfloor}{N}))$. If $(|x|+2\theta)N$ is an integer and $I^+ < \lfloor(|x|+2\theta)N\rfloor$, we keep only the part of the definition corresponding to $\phi(t) \in [\frac{2I^+}{N}, |x|+2\theta+\frac{\lfloor(|x|+2\theta)N\rfloor - 1}{N})$, and read $\phi + \frac{1}{N}$ for ϕ in what follows. We now consider the next step of the parametric representations, corresponding to Figure 1(c). Let us study $Y_N^-(y)$ for $y \geq \frac{I^+}{N}$. For such y, if we had $\ell^-(T_N, \lfloor Ny\rfloor) \neq 0$, since at time T_N the random walk is at $\lfloor Nx\rfloor\iota 1$, it would have gone from $\lfloor Ny\rfloor$ to $\lfloor Nx\rfloor\iota 1$ before time T_N, thus would have gone from I^+ to $I^+ - 1$ before time T_N, which is impossible since by definition of I^+ we have $\ell^-(T_N, I^+) = 0$. We deduce that if $y \geq \frac{I^+}{N}$ we have $\ell^-(T_N, \lfloor Ny\rfloor) = 0$, which implies $Y_N^-(y) = -\sqrt{N}(\frac{|x|-|y|}{2} + \theta)_+$. In

particular, this is continuous, so we may set the following. For $\phi(t) \in [|x|+2\theta+\frac{\lfloor(|x|+2\theta)N\rfloor}{N}, 2(|x|+2\theta)+\frac{\lfloor(|x|+2\theta)N\rfloor-I^+}{N}]$, we set $(u_N^-(t), r_N^-(t)) = (\phi(t)-(|x|+2\theta)-\frac{\lfloor(|x|+2\theta)N\rfloor}{N}+\frac{I^+}{N}, Y_N^-(\phi(t)-(|x|+2\theta)-\frac{\lfloor(|x|+2\theta)N\rfloor}{N}+\frac{I^+}{N}))$, and $(u_N(t), r_N(t)) = (|x|+2\theta, \hat{\phi}(\phi(t)))$, where $\hat{\phi}$ is the affine function mapping $|x|+2\theta+\frac{\lfloor(|x|+2\theta)N\rfloor}{N}$ to $Y_N((|x|+2\theta)^-)$ and $2(|x|+2\theta)+\frac{\lfloor(|x|+2\theta)N\rfloor-I^+}{N}$ to 0. If $I^+ = \lfloor(|x|+2\theta)N\rfloor$ and $(|x|+2\theta)N$ is an integer, (in which case $|x|+2\theta+\frac{\lfloor(|x|+2\theta)N\rfloor}{N} = 2(|x|+2\theta)+\frac{\lfloor(|x|+2\theta)N\rfloor-I^+}{N} = 2(|x|+2\theta)$), we instead say that for $\phi(t) \in [2(|x|+2\theta), 2(|x|+2\theta)+1]$ we set $(u_N^-(t), r_N^-(t)) = (|x|+2\theta, Y_N^-(|x|+2\theta))$ and $(u_N(t), r_N(t)) = (|x|+2\theta, \hat{\phi}(\phi(t)))$, where $\hat{\phi}$ is the affine function mapping $2(|x|+2\theta)$ to $Y_N((|x|+2\theta)^-)$ and $2(|x|+2\theta)+1$ to 0, and in what follows we must read $\phi-1$ for ϕ. We now consider the rightmost part of the interval $[-a, a]$, corresponding to Figure 1(d). We already know that for $y \geq |x|+2\theta$, we have $y \geq \frac{I^+}{N}$, so $Y_N^-(y) = -\sqrt{N}(\frac{|x|-|y|}{2}+\theta)_+ = 0$. Consequently, if $\phi(t) \in [2(|x|+2\theta)+\frac{\lfloor(|x|+2\theta)N\rfloor-I^+}{N}, a+|x|+2\theta+\frac{\lfloor(|x|+2\theta)N\rfloor-I^+}{N}]$, we may define $(u_N^-(t), r_N^-(t)) = (u_N(t), r_N(t)) = (\phi(t)-(|x|+2\theta)-\frac{\lfloor(|x|+2\theta)N\rfloor-I^+}{N}, 0)$.

Assume $I^+ \geq \lfloor(|x|+2\theta)N\rfloor+1$ and $\frac{I^+}{N} \leq a$ (if $\frac{I^+}{N} > a$, we may choose anything for (u_N^-, r_N^-), (u_N, r_N); it will not happen if $\mathcal{B}^c$ occurs). Then if $\phi(t)$ belongs to some $[\frac{2i}{N}, \frac{2i+1}{N})$ with $i \in \{\lfloor(|x|+2\theta)N\rfloor, ..., I^+-1\}$, we set $(u_N^-(t), r_N^-(t)) = (\phi(t)-\frac{i}{N}, Y_N^-(\phi(t)-\frac{i}{N}))$, while if $\phi(t)$ belongs to some $[\frac{2i+1}{N}, \frac{2i+2}{N}]$ for $i \in \{\lfloor(|x|+2\theta)N\rfloor, ..., I^+-1\}$, we set $(u_N^-(t), r_N^-(t)) = (\frac{i+1}{N}, (-N\phi(t)+2i+2)Y_N^-((\frac{i+1}{N})^-)+(N\phi(t)-2i-1)Y_N^-(\frac{i+1}{N}))$. Now, if $\phi(t) \in [\frac{2\lfloor(|x|+2\theta)N\rfloor}{N}, |x|+2\theta+\frac{\lfloor(|x|+2\theta)N\rfloor}{N})$, we set $(u_N(t), r_N(t)) = (\phi(t)-\frac{\lfloor(|x|+2\theta)N\rfloor}{N}, Y_N(\phi(t)-\frac{\lfloor(|x|+2\theta)N\rfloor}{N}))$. This corresponds to a part of Figure 2(a) and more precisely to the parametric representations of the part of the completed graphs with abscissas in $[\frac{\lfloor(|x|+2\theta)N\rfloor}{N}, |x|+2\theta]$. Furthermore, if $\phi(t) \in [|x|+2\theta+\frac{\lfloor(|x|+2\theta)N\rfloor}{N}, \frac{2I^+}{N}]$, we set $(u_N(t), r_N(t)) = (|x|+2\theta, \hat{\phi}(\phi(t)))$, where $\hat{\phi}$ is the affine function mapping $|x|+2\theta+\frac{\lfloor(|x|+2\theta)N\rfloor}{N}$ to $Y_N((|x|+2\theta)^-)$ and $\frac{2I^+}{N}$ to 0. This corresponds to Figure 2(b). We now consider the next step of the parametric representations, corresponding to Figure 2(c). By the same arguments as previously, if $y \geq \frac{I^+}{N}$ we have $\ell^-(T_N, \lfloor Ny\rfloor) = 0$, so since $y \geq \frac{\lfloor(|x|+2\theta)N\rfloor+1}{N} \geq |x|+2\theta$ we get $Y_N^-(y) = 0$. For $\phi(t) \in [\frac{2I^+}{N}, \frac{3I^+}{N}-(|x|+2\theta)]$, we set $(u_N^-(t), r_N^-(t)) = (\frac{I^+}{N}, Y_N^-(\frac{I^+}{N})) = (\frac{I^+}{N}, 0)$ and $(u_N(t), r_N(t)) = (\phi(t)-\frac{2I^+}{N}+|x|+2\theta, 0)$. Finally, let us define the last step of the parametric representations, corresponding to Figure 2(d): if $\phi(t) \in [\frac{3I^+}{N}-(|x|+2\theta), a+\frac{2I^+}{N}-(|x|+2\theta)]$, we set $(u_N^-(t), r_N^-(t)) = (u_N(t), r_N(t)) = (\phi(t)-\frac{2I^+}{N}+|x|+2\theta, 0)$.

Assume $I^- \geq -\lfloor(|x|+2\theta)N\rfloor$. We first define the step of the parametric representations corresponding to the equivalent to Figure 1(b). If $\phi(t) \in [-|x|-2\theta-\frac{\lfloor(|x|+2\theta)N\rfloor+1}{N}, \frac{2I^-}{N}]$, we set $(u_N^-(t), r_N^-(t)) = (\frac{I^-}{N}, Y_N^-(\frac{I^-}{N}))$. If $\phi(t)$ belongs to some $[\frac{2i}{N}, \frac{2i+1}{N})$ for $i \in \{-\lfloor(|x|+2\theta)N\rfloor, ..., I^--1\}$, we set $(u_N(t), r_N(t)) = (\phi(t)-\frac{i}{N}, Y_N(\phi(t)-\frac{i}{N}))$, while if $\phi(t)$ belongs to some $[\frac{2i+1}{N}, \frac{2i+2}{N}]$ for $i \in \{-\lfloor(|x|+2\theta)N\rfloor-1, ..., I^--1\}$, we set $(u_N(t), r_N(t)) = (\frac{i+1}{N}, (-N\phi(t)+2i+2)Y_N((\frac{i+1}{N})^-)+(N\phi(t)-2i-1)Y_N(\frac{i+1}{N}))$, and if $\phi(t) \in [-|x|-2\theta-\frac{\lfloor(|x|+2\theta)N\rfloor+1}{N}, -\frac{2\lfloor(|x|+2\theta)N\rfloor+1}{N})$ we set $(u_N(t), r_N(t)) = (\phi(t)+\frac{\lfloor(|x|+2\theta)N\rfloor+1}{N}, Y_N(\phi(t)+\frac{\lfloor(|x|+2\theta)N\rfloor+1}{N}))$. If $(|x|+2\theta)N$ is an integer, we keep only the part of the previous definition for $\phi(t) \in [-2(|x|+2\theta), \frac{2I^-}{N}]$ and in the following one must read $\phi-\frac{1}{N}$ instead of ϕ. We now consider the second step of the parametric representations, corresponding to the equivalent of Figure 1(c). Let us study $Y_N^-(y)$ for $y \leq \frac{I^-}{N}$. We recall that by the definition of I^- we have $I^- < 0$. Since $\ell^+(T_N, I^-) = 0$ and the random walk is at $\lfloor Nx\rfloor\iota 1 > 0$ at time T_N, the random walk did not reach I^- before time T_N, thus did not reach $\lfloor Ny\rfloor$ before time T_N, which yields $\ell^-(T_N, \lfloor Ny\rfloor) = 0$. This yields that for $y \leq \frac{I^-}{N}$ we have $Y_N^-(y) = -\sqrt{N}(\frac{|x|-|y|}{2}+\theta)_+$, which is continuous. Consequently, we may choose the following definition. For $\phi(t) \in [-2(|x|+2\theta)+\frac{-\lfloor(|x|+2\theta)N\rfloor-I^--1}{N}, -|x|-2\theta+\frac{-\lfloor(|x|+2\theta)N\rfloor-1}{N}]$, we set $(u_N^-(t), r_N^-(t)) = (\phi(t)+\frac{I^-}{N}+|x|+2\theta+\frac{\lfloor(|x|+2\theta)N\rfloor+1}{N}, Y_N^-(\phi(t)+$

$\frac{I^-}{N} + |x| + 2\theta + \frac{\lfloor(|x|+2\theta)N\rfloor+1}{N}))$. Moreover, if $\phi(t) \in [-2(|x|+2\theta) + \frac{-\lfloor(|x|+2\theta)N\rfloor-I^--1}{N}, -|x| - 2\theta + \frac{-\lfloor(|x|+2\theta)N\rfloor-1}{N}]$, we set $(u_N(t), r_N(t)) = (-|x| - 2\theta, \hat{\phi}(\phi(t)))$, where $\hat{\phi}$ is the affine function mapping $-|x| - 2\theta + \frac{-\lfloor(|x|+2\theta)N\rfloor-1}{N}$ to $Y_N(-|x| - 2\theta)$ and $-2(|x| + 2\theta) + \frac{-\lfloor(|x|+2\theta)N\rfloor-I^--1}{N}$ to 0. If $(|x| + 2\theta)N$ is an integer and $I^- = -\lfloor(|x| + 2\theta)N\rfloor$, the second step of the parametric representations is instead defined thus: for $\phi(t) \in [-2(|x| + 2\theta) - 1, -2(|x| + 2\theta)]$ we set $(u_N^-(t), r_N^-(t)) = (-|x| - 2\theta, Y_N^-(-|x| - 2\theta))$ and $(u_N(t), r_N(t)) = (-|x| - 2\theta, \hat{\phi}(\phi(t)))$, where $\hat{\phi}$ is the affine function mapping $-2(|x|+2\theta)$ to $Y_N(-|x|-2\theta)$ and $-2(|x|+2\theta)-1$ to 0, and in what follows we must read $\phi+1-\frac{1}{N}$ for ϕ. We now consider the leftmost part of the interval $[-a, a]$, corresponding to the equivalent of Figure 1(d). Let $y \leq -|x|-2\theta$, then $y \leq \frac{I^-}{N}$, so $Y_N^-(y) = -\sqrt{N}(\frac{|x|-|y|}{2} - \theta)_+ = 0$. Consequently, for $\phi(t) \in [-a-(|x|+2\theta)+\frac{-\lfloor(|x|+2\theta)N\rfloor-I^--1}{N}, -2(|x|+2\theta) + \frac{-\lfloor(|x|+2\theta)N\rfloor-I^--1}{N}]$, we may set $(u_N^-(t), r_N^-(t)) = (u_N(t), r_N(t)) = (\phi(t) + (|x| + 2\theta) + \frac{\lfloor(|x|+2\theta)N\rfloor+I^-+1}{N}, 0)$.

Assume $I^- \leq -\lfloor(|x| + 2\theta)N\rfloor - 1$ and $\frac{I^-}{N} \geq -a$ (if $\frac{I^-}{N} < -a$, we may choose anything for (u_N^-, r_N^-), (u_N, r_N); this will not happen if $\mathcal{B}^c$ occurs). If $\phi(t)$ belongs to some $[\frac{2i}{N}, \frac{2i+1}{N})$ with $i \in \{I^-, ..., -\lfloor(|x| + 2\theta)N\rfloor - 1\}$, we set $(u_N^-(t), r_N^-(t)) = (\phi(t) - \frac{i}{N}, Y_N^-(\phi(t) - \frac{i}{N}))$, while if $\phi(t)$ belongs to some $[\frac{2i+1}{N}, \frac{2i+2}{N}]$ for $i \in \{I^-, ..., -\lfloor(|x| + 2\theta)N\rfloor - 1\}$, we set $(u_N^-(t), r_N^-(t)) = (\frac{i+1}{N}, (-N\phi(t) + 2i + 2)Y_N^-((\frac{i+1}{N})^-) + (N\phi(t) - 2i - 1)Y_N^-(\frac{i+1}{N}))$. Moreover, if $(|x|+2\theta)N$ is not an integer, if $\phi(t) \in [-\frac{2\lfloor(|x|+2\theta)N\rfloor+1}{N}, -\frac{2\lfloor(|x|+2\theta)N\rfloor}{N}]$ we set $(u_N(t), r_N(t)) = (\frac{-\lfloor(|x|+2\theta)N\rfloor}{N}, (-N\phi(t) - 2\lfloor(|x| + 2\theta)N\rfloor)Y_N((-\frac{\lfloor(|x|+2\theta)N\rfloor}{N})^-) + (N\phi(t) + 2\lfloor(|x| + 2\theta)N\rfloor + 1)Y_N(-\frac{\lfloor(|x|+2\theta)N\rfloor}{N}))$, while if $\phi(t) \in [-|x| - 2\theta - \frac{\lfloor(|x|+2\theta)N\rfloor+1}{N}, -\frac{2\lfloor(|x|+2\theta)N\rfloor+1}{N})$, we set $(u_N(t), r_N(t)) = (\phi(t)+\frac{\lfloor(|x|+2\theta)N\rfloor+1}{N}, Y_N(\phi(t)+\frac{\lfloor(|x|+2\theta)N\rfloor+1}{N}))$. This corresponds to a part of the equivalent of Figure 2(a), more precisely to abscissas in $[-|x| - 2\theta, -\frac{\lfloor(|x|+2\theta)N\rfloor}{N}]$. In addition, if $\phi(t) \in [\frac{2I^-}{N}, -|x|-2\theta-\frac{\lfloor(|x|+2\theta)N\rfloor+1}{N}]$, we set $(u_N(t), r_N(t)) = (-|x|-2\theta, \hat{\phi}(\phi(t)))$, where $\hat{\phi}$ is the affine function mapping $-|x|-2\theta-\frac{\lfloor(|x|+2\theta)N\rfloor+1}{N}$ to $Y_N(-|x|-2\theta)$ and $\frac{2I^-}{N}$ to 0. This corresponds to an equivalent of Figure 2(b). If $(|x|+2\theta)N$ is an integer, instead of the previous construction, if $\phi(t) \in [\frac{2I^-}{N}, -2(|x|+2\theta)]$ we set $(u_N(t), r_N(t)) = (-|x|-2\theta, \hat{\phi}(\phi(t)))$, where $\hat{\phi}$ is the affine function mapping $-2(|x| + 2\theta)$ to $Y_N(-|x| - 2\theta)$ and $\frac{2I^-}{N}$ to 0. We now consider the next step of the parametric representations, corresponding to an equivalent of Figure 2(c). By the same arguments as previously, if $y \leq \frac{I^-}{N}$ we have $\ell^-(T_N, \lfloor Ny\rfloor) = 0$, so since $y \leq \frac{-\lfloor(|x|+2\theta)N\rfloor-1}{N} \leq -|x| - 2\theta$ this implies $Y_N^-(y) = 0$. For $\phi(t) \in [\frac{3I^-}{N}+|x|+2\theta, \frac{2I^-}{N}]$, we set $(u_N^-(t), r_N^-(t)) = (\frac{I^-}{N}, Y_N^-(\frac{I^-}{N})) = (\frac{I^-}{N}, 0)$ and $(u_N(t), r_N(t)) = (\phi(t)-\frac{2I^-}{N}-|x|-2\theta, 0)$. Finally, we define the last step of the parametric representations, corresponding to an equivalent of Figure 2(d): if $\phi(t) \in [-a + \frac{2I^-}{N} + |x| + 2\theta, \frac{3I^-}{N} + |x| + 2\theta]$, we set $(u_N^-(t), r_N^-(t)) = (u_N(t), r_N(t)) = (\phi(t) - \frac{2I^-}{N} - |x| - 2\theta, 0)$.

We can now bound the Skorohod M_1 distance between $Y_N^-|_{[-a,a]}$ and $Y_N|_{[-a,a]}$. From its definition, we have $d_{M_1,a}(Y_N^-|_{[-a,a]}, Y_N|_{[-a,a]}) \leq \max(\|u_N^- - u_N\|_\infty, \|r_N^- - r_N\|_\infty)$, hence we only have to prove $\mathcal{B}^c \subset \{\max(\|u_N^- - u_N\|_\infty, \|r_N^- - r_N\|_\infty) \leq 2N^{-1/12}\}$ when N is large enough. We are going to break down $\{\max(\|u_N^- - u_N\|_\infty, \|r_N^- - r_N\|_\infty) \leq 2N^{-1/12}\}$ into several events. We may write

$$\{\max(\|u_N^- - u_N\|_\infty, \|r_N^- - r_N\|_\infty) \leq 2N^{-1/12}\}$$

$$= \left\{ \forall t \text{ so that } \phi(t) \in \left[2\frac{(-\lfloor (|x|+2\theta)N \rfloor) \vee I^-}{N}, 2\frac{\lfloor (|x|+2\theta)N \rfloor \wedge I^+}{N} \right], |u_N^-(t) - u_N(t)|, |r_N^-(t) - r_N(t)| \leq 2N^{-1/12} \right\}$$
$$\cap \left\{ \forall t \text{ so that } \phi(t) \in \left[2\frac{\lfloor (|x|+2\theta)N \rfloor \wedge I^+}{N}, a_N^+ \right], |u_N^-(t) - u_N(t)|, |r_N^-(t) - r_N(t)| \leq 2N^{-1/12} \right\}$$
$$\cap \left\{ \forall t \text{ so that } \phi(t) \in \left[-a_N^-, 2\frac{(-\lfloor (|x|+2\theta)N \rfloor) \vee I^-}{N} \right], |u_N^-(t) - u_N(t)|, |r_N^-(t) - r_N(t)| \leq 2N^{-1/12} \right\}.$$

Consequently, to prove that $\mathcal{B}^c \subset \{\max(\|u_N^- - u_N\|_\infty, \|r_N^- - r_N\|_\infty) \leq 2N^{-1/12}\}$ when N is large enough and thus end the proof of Proposition 16, we only have to prove the following claims.

Claim 18. *$\mathcal{B}^c \subset \{\forall t$ so that $\phi(t) \in [2\frac{(-\lfloor (|x|+2\theta)N \rfloor) \vee I^-}{N}, 2\frac{\lfloor (|x|+2\theta)N \rfloor \wedge I^+}{N}], |u_N^-(t) - u_N(t)|, |r_N^-(t) - r_N(t)| \leq 2N^{-1/12}\}$ when N is large enough.*

Claim 19. *$\mathcal{B}^c \cap \{I^+ \leq \lfloor (|x|+2\theta)N \rfloor\} \subset \{\forall t$ so that $\phi(t) \in [2\frac{\lfloor (|x|+2\theta)N \rfloor \wedge I^+}{N}, a_N^+], |u_N^-(t) - u_N(t)|, |r_N^-(t) - r_N(t)| \leq 2N^{-1/12}\}$ and $\mathcal{B}^c \cap \{I^- \geq -\lfloor (|x|+2\theta)N \rfloor\} \subset \{\forall t$ so that $\phi(t) \in [-a_N^-, 2\frac{(-\lfloor (|x|+2\theta)N \rfloor) \vee I^-}{N}], |u_N^-(t) - u_N(t)|, |r_N^-(t) - r_N(t)| \leq 2N^{-1/12}\})$ when N is large enough.*

Claim 20. *$\mathcal{B}^c \cap \{I^+ > \lfloor (|x|+2\theta)N \rfloor\} \subset \{\forall t$ so that $\phi(t) \in [2\frac{\lfloor (|x|+2\theta)N \rfloor \wedge I^+}{N}, a_N^+], |u_N^-(t) - u_N(t)|, |r_N^-(t) - r_N(t)| \leq 2N^{-1/12}\}$ and $\mathcal{B}^c \cap \{I^- < -\lfloor (|x|+2\theta)N \rfloor\} \subset \{\forall t$ so that $\phi(t) \in [-a_N^-, 2\frac{(-\lfloor (|x|+2\theta)N \rfloor) \vee I^-}{N}], |u_N^-(t) - u_N(t)|, |r_N^-(t) - r_N(t)| \leq 2N^{-1/12}\})$ when N is large enough.*

We now prove Claims 18, 19 and 20.

Proof of Claim 18. This claim deals with the "central part" of the parametric representations, between $\frac{(-\lfloor (|x|+2\theta)N \rfloor) \vee I^-}{N}$ and $\frac{\lfloor (|x|+2\theta)N \rfloor \wedge I^+}{N}$, corresponding to Figures 1(a) and 2(a). In this part of the parametric representations we follow the completed graphs of Y_N^- and Y_N in parallel, and the two functions are close by Lemma 17, hence the parametric representations will be close too. We only have to be careful in the case $(|x|+2\theta)N$ is an integer and $I^+ \geq (|x|+2\theta)N$, for which we follow the graph of Y_N^- until $Y_N^-(|x|+2\theta)$, but the graph of Y_N until $Y_N((|x|+2\theta)^-)$ only. By the definition of (u_N^-, r_N^-) and (u_N, r_N), if $\phi(t) \in [2\frac{(-\lfloor (|x|+2\theta)N \rfloor) \vee I^-}{N}, 2\frac{\lfloor (|x|+2\theta)N \rfloor \wedge I^+}{N}]$ we have $|u_N^-(t) - u_N(t)| = 0$, and we also have $|r_N^-(t) - r_N(t)| \leq \sup\{|Y_N^-(y) - Y_N(y)| : y \in [\frac{(-\lfloor (|x|+2\theta)N \rfloor) \vee I^-}{N}, \frac{\lfloor (|x|+2\theta)N \rfloor \wedge I^+}{N}]\}$, or $\max(\sup\{|Y_N^-(y) - Y_N(y)| : y \in [\frac{(-\lfloor (|x|+2\theta)N \rfloor) \vee I^-}{N}, \frac{\lfloor (|x|+2\theta)N \rfloor \wedge I^+}{N})\}, |Y_N^-(|x|+2\theta) - Y_N((|x|+2\theta)^-)|)$ if $(|x|+2\theta)N$ is an integer and $I^+ \geq (|x|+2\theta)N$. Therefore it is enough to prove that $\mathcal{B}^c \cap (\{(|x|+2\theta)N \notin \mathbb{N}\} \cup \{I^+ < (|x|+2\theta)N\}) \subset \{\sup\{|Y_N^-(y) - Y_N(y)| : y \in [\frac{(-\lfloor (|x|+2\theta)N \rfloor) \vee I^-}{N}, \frac{\lfloor (|x|+2\theta)N \rfloor \wedge I^+}{N}]\} \leq 2N^{-1/12}\}$ and $\mathcal{B}^c \cap \{(|x|+2\theta)N \in \mathbb{N}\} \cap \{I^+ \geq (|x|+2\theta)N\} \subset \{\max(\sup\{|Y_N^-(y) - Y_N(y)| : y \in [\frac{(-\lfloor (|x|+2\theta)N \rfloor) \vee I^-}{N}, \frac{\lfloor (|x|+2\theta)N \rfloor \wedge I^+}{N})\}, |Y_N^-(|x|+2\theta) - Y_N((|x|+2\theta)^-)|) \leq 2N^{-1/12}\}$. In addition, if $\mathcal{B}^c \cap (\{(|x|+2\theta)N \notin \mathbb{N}\} \cup \{I^+ < (|x|+2\theta)N\})$ occurs, then $(\mathcal{B}_2)^c$, $(\mathcal{B}_4^-)^c$ and $(\mathcal{B}_4^+)^c$ occur, hence Lemma 17 yields that when N is large enough, for any $y \in [\frac{(-\lfloor (|x|+2\theta)N \rfloor) \vee I^-}{N}, \frac{\lfloor (|x|+2\theta)N \rfloor \wedge I^+}{N}]$ we have $|Y_N^-(y) - Y_N(y)| \leq N^{-1/12}$. Therefore we have $\mathcal{B}^c \cap (\{(|x|+2\theta)N \notin \mathbb{N}\} \cup \{I^+ < (|x|+2\theta)N\}) \subset \{\sup\{|Y_N^-(y) - Y_N(y)| : y \in [\frac{(-\lfloor (|x|+2\theta)N \rfloor) \vee I^-}{N}, \frac{\lfloor (|x|+2\theta)N \rfloor \wedge I^+}{N}]\} \leq 2N^{-1/12}\}$. We now assume $\mathcal{B}^c$, $(|x|+2\theta)N \in \mathbb{N}$ and $I^+ \geq (|x|+2\theta)N$. Then Lemma 17 yields $|Y_N^-(y) - Y_N(y)| \leq N^{-1/12}$ for $y \in [\frac{(-(|x|+2\theta)N) \vee I^-}{N}, \frac{((|x|+2\theta)N) \wedge I^+}{N})$. Furthermore, we have $|Y_N^-(|x|+2\theta) - Y_N((|x|+2\theta)^-)| \leq |Y_N^-(|x|+2\theta) - Y_N^-((|x|+2\theta)^-)| + |Y_N^-((|x|+2\theta)^-) - Y_N((|x|+2\theta)^-)| \leq$

$|Y_N^-(|x|+2\theta)-Y_N^-((|x|+2\theta)^-)|+N^{-1/12}$ by what we just proved. In addition, since $(|x|+2\theta)N$ is an integer, when N is large enough (1) yields $|Y_N^-(|x|+2\theta)-Y_N^-((|x|+2\theta)^-)|=\frac{1}{\sqrt{N}}|\ell^-(T_N,(|x|+2\theta)N)-\ell^-(T_N,(|x|+2\theta)N-1)|=\frac{1}{\sqrt{N}}|\eta_{(|x|+2\theta)N-1,+}(\ell^-(T_N,(|x|+2\theta)N-1))|\leq\frac{N^{1/16}+1/2}{\sqrt{N}}$ since $(\mathcal{B}_2)^c$ occurs. We thus have $\max(\sup\{|Y_N^-(y)-Y_N(y)|:y\in[\frac{(-\lfloor(|x|+2\theta)N\rfloor)\vee I^-}{N},\frac{\lfloor(|x|+2\theta)N\rfloor\wedge I^+}{N})\},|Y_N^-(|x|+2\theta)-Y_N((|x|+2\theta)^-)|)\leq 2N^{-1/12}$, which ends the proof of Claim 18. □

Proof of Claim 19. This claim deals with the "right part" of the parametric representations in the case $I^+\leq\lfloor(|x|+2\theta)N\rfloor$, and with the "left part" in the case $I^-\geq-\lfloor(|x|+2\theta)N\rfloor$, corresponding to Figure 1(b),(c) and (d). The idea of the argument is that in the step of Figure 1(b), the representation of Y_N does not move much horizontally as $\frac{I^+}{N}$ is close to $|x|+2\theta$ by Lemma 10, so it does not have time to move too much vertically. In the step of Figure 1(c), the representations of Y_N^- and Y_N will thus start from points that are close and go to the same point, hence stay close to each other. We now give the rigorous argument. We only spell out the proof for $\mathcal{B}^c\cap\{I^+\leq\lfloor(|x|+2\theta)N\rfloor\}$, as the other case is similar. We first consider the case in which $(|x|+2\theta)N$ is not an integer. Let us assume $\mathcal{B}^c$ occurs and $I^+\leq\lfloor(|x|+2\theta)N\rfloor$. Firstly, we notice that for $\phi(t)\in[2(|x|+2\theta)+\frac{\lfloor(|x|+2\theta)N\rfloor-I^+}{N},a+|x|+2\theta+\frac{\lfloor(|x|+2\theta)N\rfloor-I^+}{N}]$ we have $(u_N^-(t),r_N^-(t))=(u_N(t),r_N(t))$, so it is enough to consider $\phi(t)\in[2\frac{\lfloor(|x|+2\theta)N\rfloor\wedge I^+}{N},2(|x|+2\theta)+\frac{\lfloor(|x|+2\theta)N\rfloor-I^+}{N}]=[\frac{2I^+}{N},2(|x|+2\theta)+\frac{\lfloor(|x|+2\theta)N\rfloor-I^+}{N}]$ since $I^+\leq\lfloor(|x|+2\theta)N\rfloor$. We deal with $|u_N^-(t)-u_N(t)|$ first. By the definition of our parametric representations, $|u_N^-(t)-u_N(t)|\leq||x|+2\theta-\frac{I^+}{N}|$. Furthermore, $\mathcal{B}^c$ occurs, thus we have $|I^+-(|x|+2\theta)N|<N^{3/4}$, hence $|u_N^-(t)-u_N(t)|\leq N^{-1/4}$. We now deal with $|r_N^-(t)-r_N(t)|$. Remembering the definition of our parametric representations, we notice that for $\phi(t)\in[|x|+2\theta+\frac{\lfloor(|x|+2\theta)N\rfloor}{N},2(|x|+2\theta)+\frac{\lfloor(|x|+2\theta)N\rfloor-I^+}{N}]$, r_N^- and r_N are affine functions of $\phi(t)$, so the maximum value of $|r_N^-(t)-r_N(t)|$ on this interval is reached either at $\phi(t)=|x|+2\theta+\frac{\lfloor(|x|+2\theta)N\rfloor}{N}$ or at $\phi(t)=2(|x|+2\theta)+\frac{\lfloor(|x|+2\theta)N\rfloor-I^+}{N}$. Moreover, for $\phi(t)=2(|x|+2\theta)+\frac{\lfloor(|x|+2\theta)N\rfloor-I^+}{N}$ we have $r_N^-(t)=r_N(t)=0$, so the maximum is reached at $\phi(t)=|x|+2\theta+\frac{\lfloor(|x|+2\theta)N\rfloor}{N}$. Therefore, if $|r_N^-(t)-r_N(t)|\leq 2N^{-1/12}$ for all $\phi(t)\in[\frac{2I^+}{N},|x|+2\theta+\frac{\lfloor(|x|+2\theta)N\rfloor}{N}]$, then $|r_N^-(t)-r_N(t)|\leq 2N^{-1/12}$ for all t with $\phi(t)\in[\frac{2I^+}{N},2(|x|+2\theta)+\frac{\lfloor(|x|+2\theta)N\rfloor-I^+}{N}]$. Consequently, it is enough to prove that when N is large enough, $|r_N^-(t)-r_N(t)|\leq 2N^{-1/12}$ for all $\phi(t)\in[\frac{2I^+}{N},|x|+2\theta+\frac{\lfloor(|x|+2\theta)N\rfloor}{N}]$.

Now, let us assume $\phi(t)\in[\frac{2I^+}{N},|x|+2\theta+\frac{\lfloor(|x|+2\theta)N\rfloor}{N}]$. By the definition of our parametric representations, $|r_N^-(t)-r_N(t)|\leq\sup\{|Y_N^-(\frac{I^+}{N})-Y_N(y)|:y\in[\frac{I^+}{N},|x|+2\theta)\}$, so it is enough to prove that when N is large enough, $\sup\{|Y_N^-(\frac{I^+}{N})-Y_N(y)|:y\in[\frac{I^+}{N},|x|+2\theta)\}\leq 2N^{-1/12}$. Moreover, for any $y\in[\frac{I^+}{N},|x|+2\theta)$, we have $|Y_N^-(\frac{I^+}{N})-Y_N(y)|\leq|Y_N^-(\frac{I^+}{N})-Y_N(\frac{I^+}{N})|+|Y_N(\frac{I^+}{N})-Y_N(y)|$. Since $\mathcal{B}^c$ occurs, we have that $(\mathcal{B}_2)^c$, $(\mathcal{B}_4^-)^c$ and $(\mathcal{B}_4^+)^c$ occur, hence Lemma 17 implies $|Y_N^-(\frac{I^+}{N})-Y_N(\frac{I^+}{N})|\leq N^{-1/12}$ when N is large enough, thus $|Y_N^-(\frac{I^+}{N})-Y_N(y)|\leq|Y_N(\frac{I^+}{N})-Y_N(y)|+N^{-1/12}\leq\frac{1}{\sqrt{N}}|\sum_{i=I^+}^{\lfloor Ny\rfloor-1}\zeta_i|+N^{-1/12}$. We deduce $\sup\{|Y_N^-(\frac{I^+}{N})-Y_N(y)|:y\in[\frac{I^+}{N},|x|+2\theta)\}\leq\sup\{\frac{1}{\sqrt{N}}|\sum_{i=I^+}^{\lfloor Ny\rfloor-1}\zeta_i|:y\in[\frac{I^+}{N},|x|+2\theta)\}+N^{-1/12}$. Furthermore, $\mathcal{B}^c$ occurs hence $|I^+-(|x|+2\theta)N|<N^{3/4}$, thus $\sup\{|Y_N^-(\frac{I^+}{N})-Y_N(y)|:y\in[\frac{I^+}{N},|x|+2\theta)\}\leq\frac{1}{\sqrt{N}}\max_{\lfloor(|x|+2\theta)N\rfloor-N^{3/4}\leq i_1\leq i_2\leq\lfloor(|x|+2\theta)N\rfloor}\frac{1}{\sqrt{N}}|\sum_{i=i_1}^{i_2}\zeta_i|+N^{-1/12}$. Since $\mathcal{B}^c$ occurs, $(\mathcal{B}_3^+)^c$ occurs, hence $\sup\{|Y_N^-(\frac{I^+}{N})-Y_N(y)|:y\in[\frac{I^+}{N},|x|+2\theta)\}\leq\frac{N^{19/48}}{\sqrt{N}}+N^{-1/12}=N^{-5/48}+N^{-1/12}\leq 2N^{-1/12}$, which is enough.

We now consider the case for which $(|x|+2\theta)N$ is an integer. Since $I^+\leq\lfloor(|x|+2\theta)N\rfloor$, we have either $I^+<\lfloor(|x|+2\theta)N\rfloor$ or $I^+=\lfloor(|x|+2\theta)N\rfloor$. If $I^+<\lfloor(|x|+2\theta)N\rfloor$, we can prove that for all t so that $\phi(t)\in[2\frac{\lfloor(|x|+2\theta)N\rfloor\wedge I^+}{N},a_N^+]$

we have $|u_N^-(t)-u_N(t)|, |r_N^-(t)-r_N(t)| \leq 2N^{-1/12}$ by the same argument as before, which we do not repeat. Assume now $I^+ = \lfloor(|x|+2\theta)N\rfloor$. In this case, the step of Figure 1(b) does not exist, we only have to deal with that of Figure 1(c), which comes mostly from Lemma 17 as this lemma ensures $Y_N^-((|x|+2\theta)^-)$ and $Y_N((|x|+2\theta)^-)$ are close (we will actually prove they are both close to 0). More rigorously, the definition of the parametric representations yields $(u_N^-(t), r_N^-(t)) = (u_N(t), r_N(t))$ if $\phi(t) \in [2(|x|+2\theta)+1, a+|x|+2\theta+1]$. Moreover, for $\phi(t) \in [2(|x|+2\theta), 2(|x|+2\theta)+1]$ we obtain $u_N^-(t) = u_N(t)$, $r_N^-(t) = Y_N^-(|x|+2\theta)$ and $|r_N(t)| \leq |Y_N((|x|+2\theta)^-)|$. Therefore we only have to prove that $|Y_N^-(|x|+2\theta)| + |Y_N((|x|+2\theta)^-)| \leq 2N^{-1/12}$ when N is large enough. Furthermore, since $I^+ = \lfloor(|x|+2\theta)N\rfloor$, we have $\ell^-(T_N, \lfloor N(|x|+2\theta)\rfloor) = \ell^-(T_N, I^+) = 0$, so $Y_N^-(|x|+2\theta) = 0$, thus we only have to prove $|Y_N((|x|+2\theta)^-)| \leq 2N^{-1/12}$ when N is large enough. Moreover by the definition of Y_N^- and by (1), we have $Y_N^-(|x|+2\theta) = Y_N^-((|x|+2\theta)^-) + \frac{1}{\sqrt{N}}\eta_{(|x|+2\theta)N-1,+}(\ell^-(T_N,(|x|+2\theta)N-1))$, and since $\mathcal{B}$ occurs, $(\mathcal{B}_2)^c$ occurs, hence we get $|Y_N^-(|x|+2\theta) - Y_N^-((|x|+2\theta)^-)| \leq \frac{1}{\sqrt{N}}(N^{1/16}+1/2) \leq N^{-1/4}$. Since $Y_N^-(|x|+2\theta) = 0$, this yields $|Y_N^-((|x|+2\theta)^-)| \leq N^{-1/4}$. In addition, $\mathcal{B}^c$ occurs hence $(\mathcal{B}_2)^c$, $(\mathcal{B}_4^-)^c$ and $(\mathcal{B}_4^-)^c$ occur, thus by Lemma 17 we have $|Y_N((|x|+2\theta)^-) - Y_N^-((|x|+2\theta)^-)| \leq N^{-1/12}$ when N is large enough, which yields $|Y_N((|x|+2\theta)^-)| \leq N^{-1/12}+N^{-1/4} < 2N^{-1/12}$, which is enough and ends the proof of Claim 19. □

Proof of Claim 20. This claim deals with the "right part" of the parametric representations in the case $I^+ > \lfloor(|x|+2\theta)N\rfloor$, and with the "left part" in the case $I^- < -\lfloor(|x|+2\theta)N\rfloor$, corresponding to Figure 2(b),(c) and (d). We first give an idea of the argument. The most important part of the proof is to deal with the step corresponding to Figure 2(b). In this step, the function $Y_N^-(y) = \frac{1}{\sqrt{N}}\ell^-(T_N, \lfloor Ny\rfloor)$ evolves as a sum of $\frac{1}{\sqrt{N}}\eta_{j,+}(\ell^-(T_N,j))$ by (1), which is close to the sum of $\frac{1}{\sqrt{N}}(\zeta_j - \frac{1}{2})$ as $(\mathcal{B}_4^+)^c$ occurs. Since the ζ_j are i.i.d. with mean 0, the sum of $\frac{1}{\sqrt{N}}\zeta_j$ will be small, and the evolution of Y_N^- will be close to that of a deterministic sum of $-\frac{1}{2\sqrt{N}}$, thus it reaches 0 at constant speed, which is also what our parametric representation of Y_N does. We now give the proof, beginning with the detail of the argument to deal with $\mathcal{B}^c \cap \{I^- < -\lfloor(|x|+2\theta)N\rfloor\}$. Let us assume $\mathcal{B}^c$ occurs and $I^- < -\lfloor(|x|+2\theta)N\rfloor$. Then $\frac{(-\lfloor(|x|+2\theta)N\rfloor)\vee I^-}{N} = -\frac{\lfloor(|x|+2\theta)N\rfloor}{N}$. Furthermore, since $\mathcal{B}^c$ occurs we have $|I^- + (|x|+2\theta)N| < N^{3/4}$, hence $\frac{I^-}{N} > -|x| - 2\theta - N^{-1/4}$. Moreover, by assumption $a > |x|+2\theta+N^{-1/8}$, so $-a < -|x|-2\theta-N^{-1/8} < \frac{I^-}{N}$, hence we are in the case $\frac{I^-}{N} \geq -a$. By the definition of the parametric representations, if $\phi(t) \in [-a-\frac{2I^+}{N}+|x|+2\theta, \frac{3I^+}{N}+|x|+2\theta]$ we have $(u_N^-(t), r_N^-(t)) = (u_N(t), r_N(t))$. If $(|x|+2\theta)N$ is not an integer, we now consider t so that $\phi(t) \in [-|x|-2\theta-\frac{\lfloor(|x|+2\theta)N\rfloor+1}{N}, -2\frac{\lfloor(|x|+2\theta)N\rfloor}{N}]$ (this corresponds to the small part of the equivalent of Figure 2(a) that we had to include in this part of the parametric representations). Then $u_N^-(t) = u_N(t)$, and $|r_N^-(t) - r_N(t)| \leq \sup_{y\in[-|x|-2\theta, -\frac{\lfloor(|x|+2\theta)N\rfloor}{N}]} |Y_N^-(y) - Y_N(y)|$, which is smaller than $N^{-1/12}$ by Lemma 17, which we can use since we assumed $\mathcal{B}^c$ so $(\mathcal{B}_2)^c$, $(\mathcal{B}_4^-)^c$ and $(\mathcal{B}_4^+)^c$ occur. We deduce $|r_N^-(t) - r_N(t)| < 2N^{-1/12}$. Now, we consider t such that $\phi(t) \in [\frac{3I^-}{N}+|x|+2\theta, \frac{2I^-}{N}]$ (that is, in the equivalent of Figure 2(c)). Then $r_N^-(t) = r_N(t) = 0$, and $|u_N^-(t) - u_N(t)| \leq |\frac{I^-}{N} + (|x|+2\theta)|$, which is strictly smaller than $2N^{-1/12}$ since $|I^- + (|x|+2\theta)N| < N^{3/4}$. We now consider $\phi(t) \in [\frac{2I^-}{N}, -|x|-2\theta-\frac{\lfloor(|x|+2\theta)N\rfloor+1}{N}]$, that is in the equivalent of Figure 2(b) (if $(|x|+2\theta)N$ is an integer, it is $[\frac{2I^-}{N}, -2(|x|+2\theta)]$ but the argument is the same). Then $|u_N^-(t) - u_N(t)| \leq |\frac{I^-}{N} + (|x|+2\theta)|$, which is strictly smaller than $2N^{-1/12}$, so we only have to prove $|r_N^-(t) - r_N(t)| \leq 2N^{-1/12}$.

We are going to study $\sup_{y\in[\frac{I^-}{N},-|x|-2\theta]}|Y_N^-(y)-Y_N^-(-|x|-2\theta)+\frac{\lfloor(|x|+2\theta)N\rfloor-\lfloor Ny\rfloor}{2\sqrt{N}}|$. Let $y\in[\frac{I^-}{N},-|x|-2\theta]$. By the definition of Y_N^- we have $Y_N^-(y)-Y_N^-(-|x|-2\theta)=\frac{1}{\sqrt{N}}(\ell^-(T_N,\lfloor Ny\rfloor)-\ell^-(T_N,\lfloor-(|x|+2\theta)N\rfloor))$. By (2) and since $(\mathcal{B}_2)^c$ occurs (remembering $\lfloor Ny\rfloor\geq I^-\geq-(|x|+2\theta)N-N^{3/4}\geq-\lceil 2(|x|+2\theta)N\rceil$), we deduce $|Y_N^-(y)-Y_N^-(-|x|-2\theta)-\frac{1}{\sqrt{N}}(\ell^+(T_N,\lfloor Ny\rfloor)-\ell^+(T_N,\lfloor-(|x|+2\theta)N\rfloor))|=|\frac{\eta_{\lfloor Ny\rfloor,-}(\ell^+(T_N,\lfloor Ny\rfloor))-\eta_{\lfloor-(|x|+2\theta)N\rfloor,-}(\ell^+(T_N,\lfloor-(|x|+2\theta)N\rfloor))}{\sqrt{N}}|\leq\frac{2N^{1/16}}{\sqrt{N}}$. In addition, (1) yields the following: $\ell^+(T_N,\lfloor Ny\rfloor)-\ell^+(T_N,\lfloor-(|x|+2\theta)N\rfloor)=\sum_{i=\lfloor Ny\rfloor+1}^{\chi(N)-1}(\eta_{i,-}(\ell^+(T_N,i))+\mathbb{1}_{\{i>0\}})-\sum_{i=\lfloor-(|x|+2\theta)N\rfloor+1}^{\chi(N)-1}(\eta_{i,-}(\ell^+(T_N,i))+\mathbb{1}_{\{i>0\}})=\sum_{i=\lfloor Ny\rfloor+1}^{\chi(N)-1}(\eta_{i,-}(\ell^+(T_N,i))+1/2)-\sum_{i=\lfloor-(|x|+2\theta)N\rfloor+1}^{\chi(N)-1}(\eta_{i,-}(\ell^+(T_N,i))+1/2)-(\lfloor-(|x|+2\theta)N\rfloor-\lfloor Ny\rfloor)/2$. Since $(\mathcal{B}_4^-)^c$ occurs, this yields $|\ell^+(T_N,\lfloor Ny\rfloor)-\ell^+(T_N,\lfloor-(|x|+2\theta)N\rfloor)+\frac{\lfloor-(|x|+2\theta)N\rfloor-\lfloor Ny\rfloor}{2}|\leq|\sum_{i=\lfloor Ny\rfloor+1}^{\chi(N)-1}\zeta_i-\sum_{i=\lfloor-(|x|+2\theta)N\rfloor+1}^{\chi(N)-1}\zeta_i|+2N^{1/3}=|\sum_{i=\lfloor Ny\rfloor+1}^{\lfloor-(|x|+2\theta)N\rfloor}\zeta_i|+2N^{1/3}$. As we also have $|Y_N^-(y)-Y_N^-(-|x|-2\theta)-\frac{1}{\sqrt{N}}(\ell^+(T_N,\lfloor Ny\rfloor)-\ell^+(T_N,\lfloor-(|x|+2\theta)N\rfloor))|\leq\frac{2N^{1/16}}{\sqrt{N}}$, this implies $\sup_{y\in[\frac{I^-}{N},-|x|-2\theta]}|Y_N^-(y)-Y_N^-(-|x|-2\theta)+\frac{\lfloor(|x|+2\theta)N\rfloor-\lfloor Ny\rfloor}{2\sqrt{N}}|\leq\max_{I^-+1\leq i\leq\lfloor-(|x|+2\theta)N\rfloor}\frac{1}{\sqrt{N}}|\sum_{j=i}^{\lfloor-(|x|+2\theta)N\rfloor}\zeta_j|+\frac{2N^{1/16}}{\sqrt{N}}+\frac{2N^{1/3}}{\sqrt{N}}$. Moreover, $\mathcal{B}^c$ occurs, hence $|I^-+(|x|+2\theta)N|<N^{3/4}$ and $(\mathcal{B}_3^-)^c$ occurs, therefore $\sup_{y\in[\frac{I^-}{N},-|x|-2\theta]}|Y_N^-(y)-Y_N^-(-|x|-2\theta)+\frac{\lfloor(|x|+2\theta)N\rfloor-\lfloor Ny\rfloor}{2\sqrt{N}}|\leq\max_{-\lfloor(|x|+2\theta)N\rfloor-N^{3/4}\leq i\leq\lfloor-(|x|+2\theta)N\rfloor}\frac{1}{\sqrt{N}}|\sum_{j=i}^{\lfloor-(|x|+2\theta)N\rfloor}\zeta_j|+\frac{2N^{1/16}}{\sqrt{N}}+\frac{2N^{1/3}}{\sqrt{N}}\leq\frac{N^{19/48}}{\sqrt{N}}+\frac{2N^{1/16}}{\sqrt{N}}+\frac{2N^{1/3}}{\sqrt{N}}\leq 2N^{-5/48}$ when N is large enough. This yields $\sup_{y\in[\frac{I^-}{N},-|x|-2\theta]}|Y_N^-(y)-Y_N^-(-|x|-2\theta)+\frac{\lfloor(|x|+2\theta)N\rfloor-\lfloor Ny\rfloor}{2\sqrt{N}}|\leq 2N^{-5/48}$ when N is large enough.

We recall that it is enough to prove that for all t so that $\phi(t)\in[\frac{2I^-}{N},-|x|-2\theta-\frac{\lfloor(|x|+2\theta)N\rfloor+1}{N}]$, we have $|r_N^-(t)-r_N(t)|<2N^{-1/12}$. For such t, we are going to study the quantity $|r_N^-(t)-Y_N^-(-|x|-2\theta)+\frac{\sqrt{N}}{4}\phi(t)-\frac{\lfloor-(|x|+2\theta)N\rfloor}{2\sqrt{N}}|$. We first suppose that $\phi(t)\in[\frac{2i}{N},\frac{2i+1}{N})$ with $i\in\{I^-,...,-\lfloor(|x|+2\theta)N\rfloor-1\}$. In this case, $r_N^-(t)=Y_N^-(\phi(t)-\frac{i}{N})$ and $|\frac{\phi(t)}{2}-\frac{1}{N}\lfloor N(\phi(t)-\frac{i}{N})\rfloor|=|\frac{\phi(t)}{2}-\frac{i}{N}|\leq\frac{1}{2N}$, hence $|r_N^-(t)-Y_N^-(-|x|-2\theta)+\frac{\sqrt{N}}{4}\phi(t)-\frac{\lfloor-(|x|+2\theta)N\rfloor}{2\sqrt{N}}|=|Y_N^-(\phi(t)-\frac{i}{N})-Y_N^-(-|x|-2\theta)+\frac{\lfloor N(\phi(t)-\frac{i}{N})\rfloor-\lfloor-(|x|+2\theta)N\rfloor}{2\sqrt{N}}+\frac{\frac{N}{2}\phi(t)-\lfloor N(\phi(t)-\frac{i}{N})\rfloor}{2\sqrt{N}}|\leq|Y_N^-(\phi(t)-\frac{i}{N})-Y_N^-(-|x|-2\theta)+\frac{\lfloor N(\phi(t)-\frac{i}{N})\rfloor-\lfloor-(|x|+2\theta)N\rfloor}{2\sqrt{N}}|+\frac{\sqrt{N}}{2}|\frac{\phi(t)}{2}-\frac{1}{N}\lfloor N(\phi(t)-\frac{i}{N})\rfloor|\leq 2N^{-5/48}+\frac{1}{4\sqrt{N}}$, thus $|r_N^-(t)-Y_N^-(-|x|-2\theta)+\frac{\sqrt{N}}{4}\phi(t)-\frac{\lfloor-(|x|+2\theta)N\rfloor}{2\sqrt{N}}|\leq 2N^{-5/48}+\frac{1}{4\sqrt{N}}$. We now consider the case $\phi(t)\in[\frac{2i+1}{N},\frac{2i+2}{N}]$ with $i\in\{I^-,...,-\lfloor(|x|+2\theta)N\rfloor-1\}$. We temporarily denote $N\phi(t)-2i-1$ by ε for short, with $\varepsilon\in[0,1]$. Then we have $r_N^-(t)=(1-\varepsilon)Y_N^-((\frac{i+1}{N})^-)+\varepsilon Y_N^-(\frac{i+1}{N})$, $|\frac{\phi(t)}{2}-\frac{i}{N}|\leq\frac{1}{N}$ and $|\frac{\phi(t)}{2}-\frac{i+1}{N}|\leq\frac{1}{2N}$, therefore $|r_N^-(t)-Y_N^-(-|x|-2\theta)+\frac{\sqrt{N}}{4}\phi(t)-\frac{\lfloor-(|x|+2\theta)N\rfloor}{2\sqrt{N}}|=|(1-\varepsilon)(Y_N^-((\frac{i+1}{N})^-)-Y_N^-(-|x|-2\theta)+\frac{\sqrt{N}}{4}\phi(t)-\frac{\lfloor-(|x|+2\theta)N\rfloor}{2\sqrt{N}}))+\varepsilon(Y_N^-(\frac{i+1}{N})-Y_N^-(-|x|-2\theta)+\frac{\sqrt{N}}{4}\phi(t)-\frac{\lfloor-(|x|+2\theta)N\rfloor}{2\sqrt{N}})|\leq(1-\varepsilon)|Y_N^-((\frac{i+1}{N})^-)-Y_N^-(-|x|-2\theta)+\frac{\sqrt{N}}{4}\phi(t)-\frac{\lfloor-(|x|+2\theta)N\rfloor}{2\sqrt{N}})|+\varepsilon|Y_N^-(\frac{i+1}{N})-Y_N^-(-|x|-2\theta)+\frac{\sqrt{N}}{4}\phi(t)-\frac{\lfloor-(|x|+2\theta)N\rfloor}{2\sqrt{N}})|\leq(1-\varepsilon)|Y_N^-((\frac{i+1}{N})^-)-Y_N^-(-|x|-2\theta)+\frac{i-\lfloor-(|x|+2\theta)N\rfloor}{2\sqrt{N}})|+(1-\varepsilon)|\frac{\sqrt{N}}{4}\phi(t)-\frac{i}{2\sqrt{N}}|+\varepsilon|Y_N^-(\frac{i+1}{N})-Y_N^-(-|x|-2\theta)+\frac{i+1-\lfloor-(|x|+2\theta)N\rfloor}{2\sqrt{N}})|+\varepsilon|\frac{\sqrt{N}}{4}\phi(t)-\frac{i+1}{2\sqrt{N}}|\leq(1-\varepsilon)\sup_{y\in[\frac{I^-}{N},-|x|-2\theta]}|Y_N^-(y)-Y_N^-(-|x|-2\theta)+\frac{\lfloor Ny\rfloor-\lfloor-(|x|+2\theta)N\rfloor}{2\sqrt{N}}|+(1-\varepsilon)\frac{\sqrt{N}}{2}|\frac{\phi(t)}{2}-\frac{i}{N}|+\varepsilon\sup_{y\in[\frac{I^-}{N},-|x|-2\theta]}|Y_N^-(y)-Y_N^-(-|x|-2\theta)+\frac{\lfloor Ny\rfloor-\lfloor-(|x|+2\theta)N\rfloor}{2\sqrt{N}}|+\varepsilon\frac{\sqrt{N}}{2}|\frac{\phi(t)}{2}-\frac{i+1}{N}|\leq\sup_{y\in[\frac{I^-}{N},-|x|-2\theta]}|Y_N^-(y)-Y_N^-(-|x|-2\theta)+\frac{\lfloor Ny\rfloor-\lfloor-(|x|+2\theta)N\rfloor}{2\sqrt{N}}|+\frac{1}{2\sqrt{N}}\leq 2N^{-5/48}+\frac{1}{2\sqrt{N}}$ thanks to our bound on the sup. Since this was also true for $\phi(t)\in[\frac{2i}{N},\frac{2i+1}{N})$ with $i\in$

$\{I^-, ..., -\lfloor(|x|+2\theta)N\rfloor - 1\}$, we get that for all t such that $\phi(t) \in [\frac{2I^-}{N}, -|x| - 2\theta - \frac{\lfloor(|x|+2\theta)N\rfloor+1}{N}]$ we have $|r_N^-(t) - Y_N^-(-|x|-2\theta) + \frac{\sqrt{N}}{4}\phi(t) - \frac{\lfloor-(|x|+2\theta)N\rfloor}{2\sqrt{N}}| \leq 2N^{-5/48} + \frac{1}{2\sqrt{N}}$.

The latter expression yields that for any t such that $\phi(t) \in [\frac{2I^-}{N}, -|x|-2\theta - \frac{\lfloor(|x|+2\theta)N\rfloor+1}{N}]$, we have $|r_N^-(t) - r_N(t)| \leq |r_N(t) - Y_N^-(-|x|-2\theta) + \frac{\sqrt{N}}{4}\phi(t) - \frac{\lfloor-(|x|+2\theta)N\rfloor}{2\sqrt{N}}| + 2N^{-5/48} + \frac{1}{2\sqrt{N}} = |\hat{\phi}(\phi(t)) - Y_N^-(-|x|-2\theta) + \frac{\sqrt{N}}{4}\phi(t) - \frac{\lfloor-(|x|+2\theta)N\rfloor}{2\sqrt{N}}| + 2N^{-5/48} + \frac{1}{2\sqrt{N}}$, where $\hat{\phi}$ is the affine function mapping $-|x|-2\theta - \frac{\lfloor(|x|+2\theta)N\rfloor+1}{N}$ to $Y_N(-|x|-2\theta)$ and $\frac{2I^-}{N}$ to 0. Therefore it is enough to prove $|\hat{\phi}(\phi(t)) - Y_N^-(-|x| - 2\theta) + \frac{\sqrt{N}}{4}\phi(t) - \frac{\lfloor-(|x|+2\theta)N\rfloor}{2\sqrt{N}}| \leq N^{-1/12} + \frac{1}{2\sqrt{N}}$ to end the proof. Now, $\hat{\phi}(\phi(t)) - Y_N^-(-|x| - 2\theta) + \frac{\sqrt{N}}{4}\phi(t) - \frac{\lfloor-(|x|+2\theta)N\rfloor}{2\sqrt{N}}$ is an affine function of $\phi(t)$, so it is enough to prove the bound for $\phi(t) = -|x| - 2\theta - \frac{\lfloor(|x|+2\theta)N\rfloor+1}{N}$ and for $\phi(t) = \frac{2I^-}{N}$. We first consider $\phi(t) = \frac{2I^-}{N}$. We already saw in the construction of the parametric representations that $Y_N^-(\frac{I^-}{N}) = 0$. We deduce $|\hat{\phi}(\phi(t)) - Y_N^-(-|x| - 2\theta) + \frac{\sqrt{N}}{4}\phi(t) - \frac{\lfloor-(|x|+2\theta)N\rfloor}{2\sqrt{N}}| = |-Y_N^-(-|x|-2\theta) + \frac{\sqrt{N}}{4}\frac{2I^-}{N} - \frac{\lfloor-(|x|+2\theta)N\rfloor}{2\sqrt{N}}| = |Y_N^-(\frac{I^-}{N}) - Y_N^-(-|x|-2\theta) + \frac{I^- - \lfloor-(|x|+2\theta)N\rfloor}{2\sqrt{N}}| \leq \sup_{y\in[\frac{I^-}{N}, -|x|-2\theta]} |Y_N^-(y) - Y_N^-(-|x|-2\theta) + \frac{\lfloor Ny\rfloor - \lfloor-(|x|+2\theta)N\rfloor}{2\sqrt{N}}| \leq 2N^{-5/48}$, which is enough. We now consider $\phi(t) = -|x|-2\theta - \frac{\lfloor(|x|+2\theta)N\rfloor+1}{N}$. Then $|\hat{\phi}(\phi(t)) - Y_N^-(-|x|-2\theta) + \frac{\sqrt{N}}{4}\phi(t) - \frac{\lfloor-(|x|+2\theta)N\rfloor}{2\sqrt{N}}| = |Y_N(-|x|-2\theta) - Y_N^-(-|x|-2\theta) + \frac{\sqrt{N}}{4}(-|x|-2\theta - \frac{\lfloor(|x|+2\theta)N\rfloor+1}{N}) - \frac{\lfloor-(|x|+2\theta)N\rfloor}{2\sqrt{N}}| \leq |Y_N(-|x|-2\theta) - Y_N^-(-|x|-2\theta)| + |\frac{1}{4\sqrt{N}}(-(|x|+2\theta)N - \lfloor(|x|+2\theta)N\rfloor - 1 - 2\lfloor-(|x|+2\theta)N\rfloor| \leq |Y_N(-|x|-2\theta) - Y_N^-(-|x|-2\theta)| + \frac{1}{2\sqrt{N}} \leq N^{-1/12} + \frac{1}{2\sqrt{N}}$ by Lemma 17, which ends the proof for $\mathcal{B}^c \cap \{I^- < -\lfloor(|x| + 2\theta)N\rfloor\}$.

The argument to show $\mathcal{B}^c \cap \{I^+ > \lfloor(|x|+2\theta)N\rfloor\} \subset \{\forall\, t \text{ so that } \phi(t) \in [2\frac{\lfloor(|x|+2\theta)N\rfloor \wedge I^+}{N}, a_N^+], |u_N^-(t) - u_N(t)|, |r_N^-(t) - r_N(t)| \leq 2N^{-1/12}\}$ is similar and simpler, except for the end of the argument, which we give here. In a similar way as in the previous case, we must bound $|Y_N((|x| + 2\theta)^-) - Y_N^-(|x| + 2\theta) + \frac{\sqrt{N}}{4}(|x| + 2\theta + \frac{\lfloor(|x|+2\theta)N\rfloor}{N}) - \frac{\lfloor(|x|+2\theta)N\rfloor}{2\sqrt{N}}| \leq |Y_N((|x| + 2\theta)^-) - Y_N^-(|x| + 2\theta)| + |\frac{(|x|+2\theta)N - \lfloor(|x|+2\theta)N\rfloor}{4\sqrt{N}}| \leq |Y_N((|x| + 2\theta)^-) - Y_N^-((|x| + 2\theta)^-)| + |Y_N^-((|x| + 2\theta)^-) - Y_N^-(|x|+2\theta)| + \frac{1}{4\sqrt{N}}$, hence Lemma 17 yields $|Y_N((|x|+2\theta)^-) - Y_N^-(|x|+2\theta) + \frac{\sqrt{N}}{4}(|x|+2\theta + \frac{\lfloor(|x|+2\theta)N\rfloor}{N}) - \frac{\lfloor(|x|+2\theta)N\rfloor}{2\sqrt{N}}| \leq N^{-1/12} + |Y_N^-((|x|+2\theta)^-) - Y_N^-(|x|+2\theta)| + \frac{1}{4\sqrt{N}}$. In addition, the definition of Y_N^- and (1) yield that if $(|x|+2\theta)N$ is not an integer, then $Y_N^-((|x|+2\theta)^-) = Y_N^-(|x|+2\theta)$, while if $(|x|+2\theta)N$ is an integer then $|Y_N^-((|x|+2\theta)^-) - Y_N^-(|x|+2\theta)| = \frac{1}{\sqrt{N}}|\ell^-(T_N, (|x| + 2\theta)N - 1) - \ell^-(T_N, (|x| + 2\theta)N)| = \frac{1}{\sqrt{N}}|\eta_{(|x|+2\theta)N-1,+}(\ell^-(T_N, (|x| + 2\theta)N - 1))| \leq \frac{N^{1/16}+1/2}{\sqrt{N}}$ since $(\mathcal{B}_2)^c$ occurs. In all cases we obtain $|Y_N^-((|x|+2\theta)^-) - Y_N^-(|x|+2\theta)| \leq \frac{N^{1/16}+1/2}{\sqrt{N}}$, therefore $|Y_N((|x|+2\theta)^-) - Y_N^-(|x|+2\theta) + \frac{\sqrt{N}}{4}(|x| + 2\theta + \frac{\lfloor(|x|+2\theta)N\rfloor}{N}) - \frac{\lfloor(|x|+2\theta)N\rfloor}{2\sqrt{N}}| \leq N^{-1/12} + \frac{N^{1/16}+1/2}{\sqrt{N}} + \frac{1}{4\sqrt{N}}$, which is a bound small enough to end the proof of the claim. □

5. Convergence of the local times process: proof of Theorem 1 and Proposition 3

5.1. **Proof of Theorem 1.** Our aim is to prove that $Y_N^\pm$ converges in distribution to $(B_y^x \mathbb{1}_{\{y\in[-|x|-2\theta, |x|+2\theta)\}})_{y\in\mathbb{R}}$ in the Skorohod M_1 topology on $D(-\infty, +\infty)$ when N tends to $+\infty$. Proposition 15 yields that $Y_N^\pm$ is close to the function Y_N defined by $Y_N(y) = \frac{1}{\sqrt{N}}\sum_{i=\lfloor Ny\rfloor+1}^{\chi(N)-1}\zeta_i$ if $y \in [-|x| - 2\theta, \frac{\chi(N)}{N})$, $Y_N(y) = \frac{1}{\sqrt{N}}\sum_{i=\chi(N)}^{\lfloor Ny\rfloor-1}\zeta_i$ if $y \in [\frac{\chi(N)}{N}, |x| + 2\theta)$, and $Y_N(y) = 0$ otherwise. One has the feeling that by Donsker's Invariance Principle, Y_N should converge to $(B_y^x \mathbb{1}_{\{y\in[-|x|-2\theta, |x|+2\theta)\}})_{y\in\mathbb{R}}$ and so we should be able to conclude quickly, but proving rigorously the

convergence in the Skorohod M_1 topology on $D(-\infty,+\infty)$ is harder than it looks. We are instead going to use a similar argument with a new process Y''_N which will be "like Y_N, but continuous in $[-|x|-2\theta,|x|+2\theta)$". We will define it as follows. We first define a process Y'_N thus: if $Ny\in\mathbb{Z}$ then $Y'_N(y)=\frac{1}{\sqrt{N}}\sum_{i=Ny+1}^{\chi(N)-1}\zeta_i$ if $y\in(-\infty,\frac{\chi(N)}{N})$ and $Y'_N(y)=\frac{1}{\sqrt{N}}\sum_{i=\chi(N)}^{Ny-1}\zeta_i$ if $y\in[\frac{\chi(N)}{N},+\infty)$, and in-between Y'_N is linearly interpoled. We then define Y''_N by $Y''_N(y)=Y'_N(y)\mathbb{1}_{\{y\in[-|x|-2\theta,|x|+2\theta)\}}$ for any $y\in\mathbb{R}$. Then Y''_N will converge to $(B^x_y\mathbb{1}_{\{y\in[-|x|-2\theta,|x|+2\theta)\}})_{y\in\mathbb{R}}$ and be close to Y_N, which is stated in the two following lemmas.

Lemma 21. *Y''_N converges to $(B^x_y\mathbb{1}_{\{y\in[-|x|-2\theta,|x|+2\theta)\}})_{y\in\mathbb{R}}$ in distribution when N tends to $+\infty$ for the Skorohod M_1 topology in $D(-\infty,\infty)$.*

Lemma 22. *$\mathbb{P}(d_{M_1}(Y_N,Y''_N)>N^{-7/16})$ tends to 0 when N tends to $+\infty$.*

Armed with these two lemmas, we are now able to prove Theorem 1.

Proof of Theorem 1. The Portmanteau Theorem states that a sequence $(Z_k)_{k\in\mathbb{N}}$ of random variables converges in distribution to a random variable Z if and only if for any closed set Ξ we have $\limsup_{k\to+\infty}\mathbb{P}(Z_k\in\Xi)\leq\mathbb{P}(Z\in\Xi)$. We are going to prove it for the sequence of the $Y^{\pm}_N$ and the variable $(B^x_y\mathbb{1}_{\{y\in[-|x|-2\theta,|x|+2\theta)\}})_{y\in\mathbb{R}}$ in the Skorohod M_1 topology in $D(-\infty,\infty)$. Let Ξ be a closed set of $D(-\infty,+\infty)$ for the Skorohod M_1 topology, we study $\mathbb{P}(Y^{\pm}_N\in\Xi)$. For any $m\in\mathbb{N}^*$, we denote $\Xi_m=\{Z\in D(-\infty,+\infty)\,|\,\inf_{\tilde{Z}\in\Xi}d_{M_1}(Z,\tilde{Z})\leq 1/m\}$. For any N large enough, we then have $\mathbb{P}(Y^{\pm}_N\in\Xi)\leq\mathbb{P}(Y''_N\in\Xi_m)+\mathbb{P}(d_{M_1}(Y^{\pm}_N,Y''_N)>1/m)$. In addition, Ξ_m is closed in the Skorohod M_1 topology on $D(-\infty,+\infty)$, and Lemma 21 implies that Y''_N converges to $(B^x_y\mathbb{1}_{\{y\in[-|x|-2\theta,|x|+2\theta)\}})_{y\in\mathbb{R}}$ in distribution when N tends to $+\infty$ for the Skorohod M_1 topology in $D(-\infty,\infty)$, hence $\limsup_{N\to+\infty}\mathbb{P}(Y''_N\in\Xi_m)\leq\mathbb{P}((B^x_y\mathbb{1}_{\{y\in[-|x|-2\theta,|x|+2\theta)\}})_{y\in\mathbb{R}}\in\Xi_m)$. Furthermore, $\mathbb{P}(d_{M_1}(Y^{\pm}_N,Y''_N)>1/m)\leq\mathbb{P}(d_{M_1}(Y^{\pm}_N,Y_N)>1/(2m))+\mathbb{P}(d_{M_1}(Y_N,Y''_N)>1/(2m))\leq\mathbb{P}(d_{M_1}(Y^{\pm}_N,Y_N)>3N^{-1/12})+\mathbb{P}(d_{M_1}(Y_N,Y''_N)>N^{-7/16})$ when N is large enough, and the latter probability tends to 0 when N tends to $+\infty$ by Proposition 15 and Lemma 22. We deduce that we have $\limsup_{N\to+\infty}\mathbb{P}(Y^{\pm}_N\in\Xi)\leq\mathbb{P}((B^x_y\mathbb{1}_{\{y\in[-|x|-2\theta,|x|+2\theta)\}})_{y\in\mathbb{R}}\in\Xi_m)$. Moreover, we notice that we have $\lim_{m\to+\infty}\mathbb{P}((B^x_y\mathbb{1}_{\{y\in[-|x|-2\theta,|x|+2\theta)\}})_{y\in\mathbb{R}}\in\Xi_m)=\mathbb{P}((B^x_y\mathbb{1}_{\{y\in[-|x|-2\theta,|x|+2\theta)\}})_{y\in\mathbb{R}}\in\cap_{m\in\mathbb{N}^*}\Xi_m)$ which is equal to $\mathbb{P}((B^x_y\mathbb{1}_{\{y\in[-|x|-2\theta,|x|+2\theta)\}})_{y\in\mathbb{R}}\in\Xi)$ since Ξ is closed. Therefore, having m tend to $+\infty$ yields $\limsup_{N\to+\infty}\mathbb{P}(Y^{\pm}_N\in\Xi)\leq\mathbb{P}((B^x_y\mathbb{1}_{\{y\in[-|x|-2\theta,|x|+2\theta)\}})_{y\in\mathbb{R}}\in\Xi)$. Consequently $Y^{\pm}_N$ converges in distribution to $(B^x_y\mathbb{1}_{\{y\in[-|x|-2\theta,|x|+2\theta)\}})_{y\in\mathbb{R}}$ for the Skorohod M_1 topology on $D(-\infty,+\infty)$, which is Theorem 1. □

It now remains to prove Lemmas 21 and 22, that is the convergence of Y''_N to $(B^x_y\mathbb{1}_{\{y\in[-|x|-2\theta,|x|+2\theta)\}})_{y\in\mathbb{R}}$ and the fact Y''_N is close to Y_N in the Skorohod M_1 topology in $D(-\infty,\infty)$. We will first prove that it is true for the uniform norm on $[-|x|-2\theta,|x|+2\theta)$, then transfer the results to the Skorohod M_1 topology. The first part is rather easy. Indeed, if we denote $C[-|x|-2\theta,|x|+2\theta]$ the space of continuous functions : $[-|x|-2\theta,|x|+2\theta]\mapsto\mathbb{R}$, since the $(\zeta_i)_{i\in\mathbb{Z}}$ are i.i.d. with law ρ_0 which is symmetric so has zero mean, Donsker's Invariance Principle yields the following.

Lemma 23. *$Y'_N|_{[-|x|-2\theta,|x|+2\theta]}$ converges in distribution to $B^x|_{[-|x|-2\theta,|x|+2\theta]}$ when N tends to $+\infty$ for the topology defined on $C[-|x|-2\theta,|x|+2\theta]$ by the uniform norm.*

The following lemma is also easy to prove.

Lemma 24. *If $(\mathcal{B}_2)^c$ occurs, $\sup\{|Y_N(y)-Y''_N(y)|:y\in[-|x|-2\theta,|x|+2\theta)\}\leq N^{-7/16}$.*

Proof. By the definition of Y_N and Y''_N, we have $\sup\{|Y_N(y)-Y''_N(y)|:y\in[-|x|-2\theta,|x|+2\theta)\}\leq\frac{1}{\sqrt{N}}\sup\{|\zeta_i|:-(|x|+2\theta)N\leq i\leq(|x|+2\theta)N\}$, which is smaller than $\frac{N^{1/16}}{\sqrt{N}}=N^{-7/16}$ if $(\mathcal{B}_2)^c$ occurs. □

To transfer the results of Lemmas 23 and 24 to the Skorohod M_1 topology and thus prove Lemmas 21 and 23, we will need the following technical lemma.

Lemma 25. *Let $N > 0$ and $Z_1, Z_2 \in D(-\infty, +\infty)$ whose possible discontinuities belong to $\frac{1}{N}\mathbb{Z}$, then we have $d_{M_1}((Z_1(y)\mathbb{1}_{\{y\in[-|x|-2\theta,|x|+2\theta)\}})_{y\in\mathbb{R}}, (Z_2(y)\mathbb{1}_{\{y\in[-|x|-2\theta,|x|+2\theta)\}})_{y\in\mathbb{R}}) \leq \sup\{|Z_1(y) - Z_2(y)| : y \in [-|x| - 2\theta, |x| + 2\theta)\}$.*

Proof. In order to prove this, it is enough to prove that for any $a > 0$ with $a \neq |x| + 2\theta$, we have the following: $d_{M_1,a}((Z_1(y)\mathbb{1}_{\{y\in[-|x|-2\theta,|x|+2\theta)\}})_{y\in[-a,a]}, (Z_2(y)\mathbb{1}_{\{y\in[-|x|-2\theta,|x|+2\theta)\}})_{y\in[-a,a]}) \leq \sup\{|Z_1(y)-Z_2(y)| : y \in [-|x|-2\theta, |x|+2\theta)\}$.

We first set $a \in (0, |x|+2\theta)$, and write parametric representations of our functions $(Z_1(y)\mathbb{1}_{\{y\in[-|x|-2\theta,|x|+2\theta)\}})_{y\in[-a,a]}$ and $(Z_2(y)\mathbb{1}_{\{y\in[-|x|-2\theta,|x|+2\theta)\}})_{y\in[-a,a]}$, the idea being to "follow their completed graphs together". We set $\phi : [0,1] \mapsto [-a - \frac{\lfloor Na\rfloor+1}{N}, a + \frac{\lfloor Na\rfloor}{N}]$ the affine function mapping 0 to $-a - \frac{\lfloor Na\rfloor+1}{N}$ and 1 to $a + \frac{\lfloor Na\rfloor}{N}$ (if Na is an integer, we replace $-a - \frac{\lfloor Na\rfloor+1}{N}$ with $-a - \frac{\lfloor Na\rfloor}{N}$). For $\ell = 1$ or 2, we define the parametric representation (u_ℓ, r_ℓ) of $(Z_\ell(y)\mathbb{1}_{\{y\in[-|x|-2\theta,|x|+2\theta)\}})_{y\in[-a,a]}$ as follows. For any $t \in [0,1]$, if $\phi(t)$ belongs to an $[\frac{2i}{N}, \frac{2i+1}{N})$ for $i \in \{-\lfloor Na\rfloor, ..., \lfloor Na\rfloor - 1\}$, we set $(u_\ell(t), r_\ell(t)) = (\phi(t) - \frac{i}{N}, Z_\ell(\phi(t) - \frac{i}{N}))$, while if $\phi(t)$ belongs to some $[\frac{2i+1}{N}, \frac{2i+2}{N}]$ for $i \in \{-\lfloor Na\rfloor - 1, ..., \lfloor Na\rfloor - 1\}$ ($\{-\lfloor Na\rfloor, ..., \lfloor Na\rfloor - 1\}$ if Na is an integer), we set $(u_\ell(t), r_\ell(t)) = (\frac{i+1}{N}, (-N\phi(t)+2i+2)Z_\ell((\frac{i+1}{N})^-) + (N\phi(t)-2i-1)Z_\ell(\frac{i+1}{N}))$. Moreover, if $\phi(t) \in [-a-\frac{\lfloor Na\rfloor+1}{N}, -\frac{2\lfloor Na\rfloor+1}{N})$, we set $(u_\ell(t), r_\ell(t)) = (\phi(t)+\frac{\lfloor Na\rfloor+1}{N}, Z_\ell(\phi(t)+\frac{\lfloor Na\rfloor+1}{N}))$ and if $\phi(t) \in [\frac{2\lfloor Na\rfloor}{N}, a + \frac{\lfloor Na\rfloor}{N}]$, we set $(u_\ell(t), r_\ell(t)) = (\phi(t) - \frac{\lfloor Na\rfloor}{N}, Z_\ell(\phi(t) - \frac{\lfloor Na\rfloor}{N}))$. We then have that $\|u_1 - u_2\|_\infty = 0$ and that $\|r_1 - r_2\|_\infty \leq \sup\{|Z_1(y) - Z_2(y)| : y \in [-|x| - 2\theta, |x| + 2\theta)\}$, hence we obtain $d_{M_1,a}((Z_1(y)\mathbb{1}_{\{y\in[-|x|-2\theta,|x|+2\theta)\}})_{y\in[-a,a]}, (Z_2(y)\mathbb{1}_{\{y\in[-|x|-2\theta,|x|+2\theta)\}})_{y\in[-a,a]}) \leq \max(\|u_1 - u_2\|_\infty, \|r_1 - r_2\|_\infty) \leq \sup\{|Z_1(y) - Z_2(y)| : y \in [-|x| - 2\theta, |x| + 2\theta)\}$.

We now set $a > |x|+2\theta$. We also define parametric representations of our functions "following their completed graphs together". We set s_N^- to be $\frac{\lfloor(|x|+2\theta)N\rfloor+1}{N}$ if $(|x|+2\theta)N$ is not an integer, and $\frac{\lfloor(|x|+2\theta)N\rfloor}{N}$ otherwise, and we set d_N^+ to be $\frac{\lfloor(|x|+2\theta)N\rfloor}{N}$ if $(|x|+2\theta)N$ is not an integer, and $\frac{\lfloor(|x|+2\theta)N\rfloor-1}{N}$ otherwise. We define $\phi : [0,1] \mapsto [-a - d_N^- - 1, a + d_N^+ + 1]$ as the affine function mapping 0 to $-a - d_N^- - 1$ and 1 to $a + d_N^+ + 1$. For $\ell = 1$ or 2, we define the parametric representation (u_ℓ, r_ℓ) of $(Z_\ell(y)\mathbb{1}_{\{y\in[-|x|-2\theta,|x|+2\theta)\}})_{y\in[-a,a]}$ as follows. For $\phi(t) \in [-|x| - 2\theta - d_N^-, |x| + 2\theta + d_N^+)$, we define $(u_\ell(t), r_\ell(t))$ as it would have been defined in the case $a < |x| + 2\theta$, with $|x| + 2\theta$ replacing a. For $\phi(t) \in [|x| + 2\theta + d_N^+, |x| + 2\theta + d_N^+ + 1]$ we set $(u_k(t), r_k(t)) = (|x| + 2\theta, \hat{\phi}_{\ell,+}(\phi(t)))$ where $\hat{\phi}_{\ell,+}$ is the affine function mapping $|x| + 2\theta + d_N^+$ to $Z_\ell((|x| + 2\theta)^-)$ and $|x| + 2\theta + d_N^+ + 1$ to 0. For $\phi(t) \in [|x| + 2\theta + d_N^+ + 1, a + d_N^+ + 1]$, we set $(u_\ell(t), r_\ell(t)) = (\phi(t) - d_N^+ - 1, 0)$. Similarly, for $\phi(t) \in [-|x| - 2\theta - d_N^- - 1, -|x| - 2\theta - d_N^-]$ we set $(u_\ell(t), r_\ell(t)) = (-|x|-2\theta, \hat{\phi}_{\ell,-}(\phi(t)))$ where $\hat{\phi}_{\ell,-}$ is the affine function mapping $-|x|-2\theta-d_N^-$ to $Z_\ell(-|x|-2\theta)$ and $-|x|-2\theta-d_N^--1$ to 0, and for $\phi(t) \in [-a-d_N^--1, -|x|-2\theta-d_N^--1]$, we set $(u_\ell(t), r_\ell(t)) = (\phi(t)+d_N^-+1, 0)$. We then have $\|u_1-u_2\|_\infty = 0$. Moreover, for $\phi(t) \in [|x|+2\theta+d_N^++1, a+d_N^++1] \cup [-a-d_N^--1, -|x|-2\theta-d_N^--1]$ we have $r_1(t) = r_2(t)$. Furthermore, for $\phi(t) \in [|x| + 2\theta + d_N^+, |x| + 2\theta + d_N^+ + 1)$ we have $|r_1(t) - r_2(t)| \leq \sup\{|Z_1(y) - Z_2(y)| : y \in [-|x| - 2\theta, |x| + 2\theta)\}$ by the definition of the parametric representations. We now consider $\phi(t) \in [|x| + 2\theta + d_N^+, |x| + 2\theta + d_N^+ + 1]$ (the argument for $\phi(t) \in [-|x| - 2\theta - d_N^- - 1, -|x| - 2\theta - d_N^-]$ is the same). Since $\phi_{1,+}$ and $\phi_{2,+}$ are affine functions, $|r_1(t)-r_2(t)| \leq \max(|\phi_{1,+}(|x|+2\theta+d_N^+)-\phi_{2,+}(|x|+2\theta+d_N^+)|, |\phi_{1,+}(|x|+2\theta+d_N^++1)-\phi_{2,+}(|x|+2\theta+d_N^++1)|) = |Z_1((|x|+2\theta)^-)-Z_2((|x|+2\theta)^-)| \leq \sup\{|Z_1(y)-Z_2(y)| : y \in [-|x|-2\theta, |x|+2\theta)\}$. We deduce $\|r_1-r_2\|_\infty \leq \sup\{|Z_1(y)-Z_2(y)| : y \in [-|x|-2\theta, |x|+2\theta)\}$. Consequently, $d_{M_1,a}((Z_1(y)\mathbb{1}_{\{y\in[-|x|-2\theta,|x|+2\theta)\}})_{y\in[-a,a]}, (Z_2(y)\mathbb{1}_{\{y\in[-|x|-2\theta,|x|+2\theta)\}})_{y\in[-a,a]}) \leq \max(\|u_1 - u_2\|_\infty, \|r_1 - r_2\|_\infty) \leq \sup\{|Z_1(y) - Z_2(y)| : y \in [-|x| - 2\theta, |x| + 2\theta)\}$, which ends the proof. □

We are now able to deduce Lemmas 21 and 22 from Lemmas 23 and 24, which will end the proof of Theorem 1.

Proof of Lemma 21. Let $f : D(-\infty,+\infty) \mapsto \mathbb{R}$ be bounded and continuous with respect to the Skorohod M_1 topology on $D(-\infty,+\infty)$, we need to prove that $\mathbb{E}(f(Y''_N))$ converges to $\mathbb{E}(f((B^x_y \mathbb{1}_{\{y\in[-|x|-2\theta,|x|+2\theta)\}})_{y\in\mathbb{R}}))$ when N tends to $+\infty$. We define $g : C[-|x|-2\theta,|x|+2\theta] \mapsto \mathbb{R}$ by $g(Z) = f((Z(y)\mathbb{1}_{\{y\in[-|x|-2\theta,|x|+2\theta)\}})_{y\in\mathbb{R}})$ for any $Z \in C[-|x|-2\theta,|x|+2\theta]$. We then have $\mathbb{E}(f(Y''_N)) = \mathbb{E}(g(Y'_N|_{[-|x|-2\theta,|x|+2\theta]}))$ and $\mathbb{E}(f((B^x_y \mathbb{1}_{\{y\in[-|x|-2\theta,|x|+2\theta)\}})_{y\in\mathbb{R}})) = \mathbb{E}(g(B^x|_{[-|x|-2\theta,|x|+2\theta]}))$, hence it is enough to prove $\mathbb{E}(g(Y'_N|_{[-|x|-2\theta,|x|+2\theta]}))$ converges to $\mathbb{E}(g(B^x|_{[-|x|-2\theta,|x|+2\theta]}))$ when N tends to $+\infty$. Furthermore, Lemma 23 yields that $Y'_N|_{[-|x|-2\theta,|x|+2\theta]}$ converges in distribution to $B^x|_{[-|x|-2\theta,|x|+2\theta]}$ when N tends to $+\infty$ for the topology defined on $C[-|x|-2\theta,|x|+2\theta]$ by the uniform norm. Consequently, we only have to prove that g is continuous for this topology.

Let $(Z_k)_{k\in\mathbb{N}}$ be a sequence in $C[-|x|-2\theta,|x|+2\theta]$ converging uniformly to $Z \in C[-|x|-2\theta,|x|+2\theta]$ when k tends to $+\infty$. Then Lemma 25 states that for all $k \in \mathbb{N}$, $d_{M_1}((Z_k(y)\mathbb{1}_{\{y\in[-|x|-2\theta,|x|+2\theta)\}})_{y\in\mathbb{R}}, (Z(y)\mathbb{1}_{\{y\in[-|x|-2\theta,|x|+2\theta)\}})_{y\in\mathbb{R}}) \leq \sup\{|Z_k(y)-Z(y)| : y \in [-|x|-2\theta,|x|+2\theta)\} \leq \|Z_k - Z\|_\infty$. Since the latter tends to 0 when k tends to $+\infty$, we deduce $(Z_k(y)\mathbb{1}_{\{y\in[-|x|-2\theta,|x|+2\theta)\}})_{y\in\mathbb{R}}$ converges to $(Z(y)\mathbb{1}_{\{y\in[-|x|-2\theta,|x|+2\theta)\}})_{y\in\mathbb{R}}$ when k tends to $+\infty$ with respect to the Skorohod M_1 topology on $D(-\infty,+\infty)$. Since f is continuous with respect to this topology, $(g(Z_k))_{k\in\mathbb{N}}$ converges to $g(Z)$ when k tends to $+\infty$. Consequently g is continuous for the topology defined on $C[-|x|-2\theta,|x|+2\theta]$ by the uniform norm, which ends the proof. □

Proof of Lemma 22. Lemma 24 yields that if $(\mathcal{B}_2)^c$ occurs, then $\sup\{|Y_N(y)-Y'_N(y)| : y \in [-|x|-2\theta,|x|+2\theta)\} \leq N^{-7/16}$. From Lemma 25 we deduce that if $(\mathcal{B}_2)^c$ occurs then $d_{M_1}(Y_N, Y''_N) \leq N^{-7/16}$. Therefore $\mathbb{P}(d_{M_1}(Y_N,Y''_N) > N^{-7/16}) \leq \mathbb{P}(\mathcal{B}_2)$, which tends to 0 when N tends to $+\infty$ by Lemma 7. □

5.2. **Proof of Proposition 3.** Our goal is to prove that for any closed interval $I \in \mathbb{R}$ that does not contain $-|x|-2\theta$ or $|x|+2\theta$, the process $(Y^{\pm}_N(y))_{y\in I}$ converges in distribution to $(B^x_y \mathbb{1}_{\{y\in[-|x|-2\theta,|x|+2\theta)\}})_{y\in I}$ in the topology on DI given by the uniform norm when N tends to $+\infty$. We first assume $I = [a,b]$ or $[a,+\infty)$ with $a > |x|+2\theta$ (the case $I=[a,b]$ or $(-\infty,b]$ with $b < -|x|-2\theta$ can be dealt with in the same way). We are going to prove that outside an event of small probability, $(Y^{\pm}_N(y))_{y\in I} = 0 = (B^x_y \mathbb{1}_{\{y\in[-|x|-2\theta,|x|+2\theta)\}})_{y\in I}$. Indeed, we notice that the random walk never reached I^+ before time T_N, as since it is below $\chi(N)$ at time T_N we would otherwise have $\ell^-(T_N, I^+) > 0$, which is not the case. Therefore, for any $y \geq (|x|+2\theta) \vee \frac{I^+}{N}$, the random walk did not reach $\lfloor Ny \rfloor$ before time T_N, hence $\ell^{\pm}(T_N, \lfloor Ny \rfloor) = 0$, thus $Y^{\pm}_N(y) = 0$. We deduce that as soon as $\frac{I^+}{N} \leq a$, we have $(Y^{\pm}_N(y))_{y\in I} = 0 = (B^x_y \mathbb{1}_{\{y\in[-|x|-2\theta,|x|+2\theta)\}})_{y\in I}$. In addition, when N is large enough we have $a \geq |x|+2\theta+N^{-1/4}$. Therefore, when N is large enough, $\mathbb{P}((Y^{\pm}_N(y))_{y\in I} \neq (B^x_y \mathbb{1}_{\{y\in[-|x|-2\theta,|x|+2\theta)\}})_{y\in I}) \leq \mathbb{P}(|I^+ - (|x|+2\theta)N| \geq N^{3/4})$, which tends to 0 when N tends to $+\infty$ by Lemma 10. This yields that $(Y^{\pm}_N(y))_{y\in I}$ converges in distribution to $(B^x_y \mathbb{1}_{\{y\in[-|x|-2\theta,|x|+2\theta)\}})_{y\in I})$ in the topology on DI given by the uniform norm.

We now deal with the case $I=[a,b]$ with $-|x|-2\theta < a < b < |x|+2\theta$. The idea is that we will be far from the problems at $-|x|-2\theta$ and $|x|+2\theta$, thus $Y^{\pm}_N$ will be close to Y'_N in all I, and Y'_N converges to the right limit, hence $Y^{\pm}_N$ too. We first prove the following lemma.

Lemma 26. *For any $-|x|-2\theta < a < b < |x|+2\theta$, we have that $\mathbb{P}(\|Y^{\pm}_N|_{[a,b]} - Y'_N|_{[a,b]}\|_\infty > 2N^{-1/12})$ tends to 0 when N tends to $+\infty$.*

Proof. We assume $(\mathcal{B}_2)^c$, $(\mathcal{B}^-_4)^c$, $(\mathcal{B}^+_4)^c$ occurs, as well as $|I^- + (|x|+2\theta)N| < N^{3/4}$, $|I^+ - (|x|+2\theta)N| < N^{3/4}$. When N is large enough, we have $a \geq -|x|-2\theta+N^{-1/4} > \frac{I^-}{N}$ and $b \leq |x|+2\theta-N^{-1/4} < \frac{I^+}{N}$, hence $[a,b] \subset (\frac{I^-}{N}, \frac{I^+}{N})$. Therefore, for any $y \in [a,b]$, Lemma 17 yields $|Y^{\pm}_N(y) - Y_N(y)| \leq N^{-1/12}$, and Lemma 24 gives $|Y_N(y) - Y'_N(y)| \leq N^{-7/16}$, hence we get $|Y^{\pm}_N(y) - Y'_N(y)| \leq 2N^{-1/12}$, and we deduce $\|Y^{\pm}_N|_{[a,b]} - Y'_N|_{[a,b]}\|_\infty \leq 2N^{-1/12}$. This implies

$\mathbb{P}(\|Y_N^{\pm}|_{[a,b]} - Y'_N|_{[a,b]}\|_\infty > 2N^{-1/12}) \leq \mathbb{P}(\mathcal{B}_2 \cup \mathcal{B}_4^- \cup \mathcal{B}_4^+ \cup \{|I^- + (|x|+2\theta)N| \geq N^{3/4}\} \cup \{|I^+ - (|x|+2\theta)N| \geq N^{3/4}\})$. In addition, the latter tends to 0 when N tends to $+\infty$ because Lemma 7 implies $\mathbb{P}(\mathcal{B}_2)$ tends to 0 when N tends to $+\infty$, Lemma 14 yields $\mathbb{P}(\mathcal{B}_4^-)$ and $\mathbb{P}(\mathcal{B}_4^+)$ tend to 0 when N tends to $+\infty$, and Lemma 10 implies $\mathbb{P}(|I^- + (|x|+2\theta)N| \geq N^{3/4})$ and $\mathbb{P}(|I^+ - (|x|+2\theta)N| \geq N^{3/4})$ tend to 0 when N tends to $+\infty$. This yields $\mathbb{P}(\|Y_N^{\pm}|_{[a,b]} - Y'_N|_{[a,b]}\|_\infty > 2N^{-1/12})$ tends to 0 when N tends to $+\infty$. □

We now set $-|x|-2\theta < a < b < |x|+2\theta$ and end the proof of Proposition 3. We begin by showing $Y'_N|_{[a,b]}$ converges in distribution to $B^x|_{[a,b]}$ when N tends to $+\infty$ for the topology defined on $D[a,b]$ by the uniform norm. For any function $f : D[a,b] \mapsto \mathbb{R}$ bounded and continuous for this topology, we may define $g : C[-|x|-2\theta, |x|+2\theta] \mapsto \mathbb{R}$ by $g(Z) = f(Z|_{[a,b]})$ for any $Z \in C[-|x|-2\theta, |x|+2\theta]$, and g is continuous for the topology of the uniform norm on $C[-|x|-2\theta, |x|+2\theta]$. Moreover, Lemma 23 states that $Y'_N|_{[-|x|-2\theta,|x|+2\theta]}$ converges in distribution to $B^x|_{[-|x|-2\theta,|x|+2\theta]}$ when N tends to $+\infty$ for the latter topology, hence $\mathbb{E}(g(Y'_N|_{[-|x|-2\theta,|x|+2\theta]}))$ converges to $\mathbb{E}(g(B^x|_{[-|x|-2\theta,|x|+2\theta]}))$ when N tends to $+\infty$. This means $\mathbb{E}(f(Y'_N|_{[a,b]}))$ converges to $\mathbb{E}(f(B^x|_{[a,b]}))$ when N tends to $+\infty$. Therefore $Y'_N|_{[a,b]}$ converges in distribution to $B^x|_{[a,b]}$ when N tends to $+\infty$ for the topology defined on $D[a,b]$ by the uniform norm. Redoing the argument with the Portmanteau Theorem used in the proof of Theorem 1, while replacing the Skorohod M_1 topology on $D(-\infty,\infty)$ by the topology of the uniform norm on $D[a,b]$, replacing $Y_N^{\pm}$ by $Y_N^{\pm}|_{[a,b]}$, replacing Y''_N by $Y'_N|_{[a,b]}$, and replacing Proposition 15 and Lemma 22 by Lemma 26 yields that $Y_N^{\pm}|_{[a,b]}$ converges to $B^x|_{[a,b]}$ in distribution for this topology when N tends to $+\infty$, which is what we needed to prove.

6. No convergence in the Skorohod J_1 topology: proof of Proposition 2

In this section, our aim is to prove that $Y_N^{\pm}$ does not converge in distribution in the Skorohod J_1 topology on $D(-\infty,+\infty)$ when N tends to $+\infty$. We will first prove that if $Y_N^{\pm}$ converges in the Skorohod J_1 topology, the limit has to be the same as in the Skorohod M_1 topology, that is $(B_y^x \mathbb{1}_{\{y\in[-|x|-2\theta,|x|+2\theta)\}})_{y\in\mathbb{R}}$ by Theorem 1 (this will be Lemma 27). Afterwards, we will prove that $Y_N^{\pm}$ does not converge in distribution in the Skorohod J_1 topology to $(B_y^x \mathbb{1}_{\{y\in[-|x|-2\theta,|x|+2\theta)\}})_{y\in\mathbb{R}}$ by finding some closed set Ξ so that $\limsup_{N\to+\infty} \mathbb{P}(Y_N^{\pm} \in \Xi) > \mathbb{P}((B_y^x \mathbb{1}_{\{y\in[-|x|-2\theta,|x|+2\theta)\}})_{y\in\mathbb{R}} \in \Xi)$, which is enough by the Portmanteau Theorem.

Lemma 27. *If $Y_N^{\pm}$ converges in distribution in the Skorohod J_1 topology on $D(-\infty,+\infty)$ when N tends to $+\infty$, the limit is $(B_y^x \mathbb{1}_{\{y\in[-|x|-2\theta,|x|+2\theta)\}})_{y\in\mathbb{R}}$.*

Proof. The idea is that the Skorohod J_1 topology is stronger than the Skorohod M_1 topology, hence if $Y_N^{\pm}$ converges in distribution in the Skorohod J_1 topology it will converge in distribution in the Skorohod M_1 topology to the same limit, and we already know by Theorem 1 that it converges in distribution to $(B_y^x \mathbb{1}_{\{y\in[-|x|-2\theta,|x|+2\theta)\}})_{y\in\mathbb{R}}$ in the Skorohod M_1 topology. More rigorously, it can be proven that for any $a > 0$ we have $d_{M_1,a} \leq d_{J_1,-a,a}$. Indeed, this is Theorem 12.3.2 of [9], whose proof is in the Internet supplement of that book (just replace the discontinuity points of x_1 with their image by λ^{-1}). This implies $d_{M_1} \leq d_{J_1}$. Therefore a converging sequence in the Skorohod J_1 topology on $D(-\infty,+\infty)$ also converges in the Skorohod M_1 topology on $D(-\infty,+\infty)$. Consequently, a function $g : D(-\infty,+\infty) \mapsto \mathbb{R}$ that is continuous for the Skorohod M_1 topology is also continuous for the Skorohod J_1 topology. We deduce that if $Y_N^{\pm}$ converges in distribution to some Z in the Skorohod J_1 topology when N tends to $+\infty$, then for any bounded function $g : D(-\infty,+\infty) \mapsto \mathbb{R}$ continuous for the Skorohod M_1 topology, g is also continuous for the Skorohod J_1 topology, hence $\mathbb{E}(g(Y_N^{\pm}))$ converges to $\mathbb{E}(g(Z))$ when N tends to $+\infty$, thus $Y_N^{\pm}$ converges in distribution to Z in the Skorohod M_1 topology when N tends to $+\infty$. Furthermore, Theorem 1 states $Y_N^{\pm}$ converges in distribution

to $(B_y^x \mathbb{1}_{\{y\in[-|x|-2\theta,|x|+2\theta)\}})_{y\in\mathbb{R}}$ in the Skorohod M_1 topology when N tends to $+\infty$. Therefore, if $Y_N^{\pm}$ converges in distribution in the Skorohod J_1 topology when N tends to $+\infty$, its limit is $(B_y^x \mathbb{1}_{\{y\in[-|x|-2\theta,|x|+2\theta)\}})_{y\in\mathbb{R}}$. □

We now define our closed set Ξ. The idea behind this definition is that with high probability, $B^x_{|x|+2\theta}$ is at some distance from 0, hence at some point around $|x|+2\theta$, $Y_N^{\pm}$ will be close to $B^x_{|x|+2\theta}$, thus at some distance from 0. Furthermore, at $|x|+2\theta$ the process $(B_y^x \mathbb{1}_{\{y\in[-|x|-2\theta,|x|+2\theta)\}})_{y\in\mathbb{R}}$ will jump directly from $B^x_{|x|+2\theta}$ to 0, while $Y_N^{\pm}$, which can make only jumps of order $\frac{1}{\sqrt{N}}$, will have to cross the distance separating $B^x_{|x|+2\theta}$ from 0 without bigs jumps. Therefore if $\delta_1 > 0$ is much smaller than $B^x_{|x|+2\theta}$, then $Y_N^{\pm}(y)$ will enter the interval $[\delta_1, 2\delta_1]$ for y near $|x|+2\theta$, while $(B_y^x \mathbb{1}_{\{y\in[-|x|-2\theta,|x|+2\theta)\}})_{y\in\mathbb{R}}$ will not. We thus set Ξ to be roughly "the function enters $[\delta_1, 2\delta_1]$ around $|x|+2\theta$". More rigorously, by the definition of B^x, the random variable $B^x_{|x|+2\theta}$ has distribution $\mathcal{N}(0, 2\theta)$, hence there exists $\delta_1 > 0$ so that $\mathbb{P}(|B^x_{|x|+2\theta}| \leq 4\delta_1) \leq 1/8$. Moreover, B^x is continuous, hence there exists $0 < \delta_2 < \theta$ so that $\mathbb{P}(\exists\, y \in [|x|+2\theta-\delta_2, |x|+2\theta], |B^x_y| \leq 3\delta_1) \leq 1/4$. We then define $\Xi = \{Z \in D(-\infty,+\infty) \,|\, \exists y \in [|x|+2\theta-\delta_2, |x|+2\theta+\delta_2], |Z(y)| \in [\delta_1, 2\delta_1] \text{ or } |Z(y^-)| \in [\delta_1, 2\delta_1]\}$ (the inclusion of $Z(y^-)$ was necessary for Ξ to be closed). Then $\mathbb{P}((B_y^x \mathbb{1}_{\{y\in[-|x|-2\theta,|x|+2\theta)\}})_{y\in\mathbb{R}} \in \Xi) \leq 1/4$. We will prove the two following lemmas.

Lemma 28. *When N is large enough, $\mathbb{P}(Y_N^{\pm} \in \Xi) \geq 1/2$.*

Lemma 29. *Ξ is closed in the Skorohod J_1 topology on $D(-\infty,+\infty)$.*

With these two lemmas, the proof of Proposition 2 becomes easy.

Proof of Proposition 2. Lemma 28 yields $\limsup_{N\to+\infty} \mathbb{P}(Y_N^{\pm} \in \Xi) \geq 1/2$, and the definition of Ξ ensures that $\mathbb{P}((B_y^x \mathbb{1}_{\{y\in[-|x|-2\theta,|x|+2\theta)\}})_{y\in\mathbb{R}} \in \Xi) \leq 1/4$, hence $\limsup_{N\to+\infty} \mathbb{P}(Y_N^{\pm} \in \Xi) > \mathbb{P}((B_y^x \mathbb{1}_{\{y\in[-|x|-2\theta,|x|+2\theta)\}})_{y\in\mathbb{R}} \in \Xi)$. Since Lemma 29 yields Ξ is closed in the Skorohod J_1 topology on $D(-\infty,+\infty)$, the Portmanteau Theorem implies $Y_N^{\pm}$ does not converge in distribution in the Skorohod J_1 topology on $D(-\infty,+\infty)$ to $(B_y^x \mathbb{1}_{\{y\in[-|x|-2\theta,|x|+2\theta)\}})_{y\in\mathbb{R}}$ when N tends to $+\infty$. Hence Lemma 27 yields that $Y_N^{\pm}$ does not converge in distribution in the Skorohod J_1 topology on $D(-\infty,+\infty)$ when N tends to $+\infty$, which is Proposition 2. □

Thus it remains only to prove Lemmas 28 and 29.

Proof of Lemma 28. The idea is that with good probability, when y is a bit smaller than $|x|+2\theta$, we have $Y_N^{\pm}(y)$ of the same order as $B^x_{|x|+2\theta}$, thus away from 0, while when y is a bit larger than $|x|+2\theta$, we have $Y_N^{\pm}(y) = 0$, so since Y_N can only make jumps of order $\frac{1}{\sqrt{N}}$, it will enter $[\delta_1, 2\delta_1]$. We now give the rigorous argument. We begin by assuming that $|Y_N^{\pm}(|x|+2\theta-\delta_2)| > 3\delta_1$ (that is $Y_N(y)$ is indeed away from 0 when y is a bit smaller than $|x|+2\theta$), $(\mathcal{B}_2)^c$ occurs and $|I^+ - (|x|+2\theta)N| < N^{3/4}$, and proving that when N is large enough, $Y_N^{\pm} \in \Xi$. We first show $Y_N^{\pm}(|x|+2\theta+\delta_2) = 0$. When N is large enough, $\frac{I^+}{N} \leq |x|+2\theta+N^{-1/4} \leq |x|+2\theta+\delta_2$. Moreover, for any $y \geq \frac{I^+}{N}$, if we had $\ell^{\pm}(T_N, \lfloor Ny \rfloor) \neq 0$, since at time T_N the random walk is at $\lfloor Nx \rfloor \iota 1$, it would have gone from $\lfloor Ny \rfloor$ to $\lfloor Nx \rfloor \iota 1$ before time T_N, thus would have gone from I^+ to $I^+ - 1$ before time T_N, which is impossible by the definition of I^+. We deduce $\ell^{\pm}(T_N, \lfloor Ny \rfloor) = 0$ for any $y \geq \frac{I^+}{N}$, hence for $y = |x|+2\theta+\delta_2$. This yields $Y_N^{\pm}(|x|+2\theta+\delta_2) = 0$. Moreover, we assumed $|Y_N^{\pm}(|x|+2\theta-\delta_2)| > 3\delta_1$. Furthermore, equations (1) and (2) yield that the jumps of $Y_N^{\pm}$ in $[|x|+2\theta-\delta_2, |x|+2\theta+\delta_2]$ are either $\frac{1}{\sqrt{N}}\eta_{i,+}(\ell^-(T_N,i))$ (if we deal with Y_N^-) or $\frac{1}{\sqrt{N}}\eta_{i+1,+}(\ell^-(T_N,i+1))$ (if we deal with Y_N^+) with $i \in \{\lfloor (|x|+2\theta-\delta_2)N \rfloor, ..., \lfloor (|x|+2\theta+\delta_2)N \rfloor - 1\}$. Since $(\mathcal{B}_2)^c$ occurs, the jumps of $Y_N^{\pm}$ in $[|x|+2\theta-\delta_2, |x|+2\theta+\delta_2]$ have size at most $\frac{1}{\sqrt{N}}(N^{1/16}+1/2)$, which tends to 0 when N tends to $+\infty$. Therefore,

when N is large enough, there exists $y \in [|x| + 2\theta - \delta_2, |x| + 2\theta + \delta_2]$ so that $|Y_N^{\pm}(y)| \in [\delta_1, 2\delta_1]$, hence $Y_N^{\pm} \in \Xi$. Consequently, when N is large enough, if $|Y_N^{\pm}(|x|+2\theta-\delta_2)| > 3\delta_1$, $(\mathcal{B}_2)^c$ and $|I^+ - (|x|+2\theta)N| < N^{3/4}$ then $Y_N^{\pm} \in \Xi$. This implies $\mathbb{P}(Y_N^{\pm} \notin \Xi) \leq \mathbb{P}(|Y_N^{\pm}(|x|+2\theta-\delta_2)| \leq 3\delta_1) + \mathbb{P}(\mathcal{B}_2) + \mathbb{P}(|I^+ - (|x|+2\theta)N| \geq N^{3/4})$. In addition, Lemma 7 and Lemma 10 yield respectively that $\mathbb{P}(\mathcal{B}_2)$ and $\mathbb{P}(|I^+ - (|x| + 2\theta)N| \geq N^{3/4})$ tend to 0 when N tends to $+\infty$. Therefore it is enough to prove that $\mathbb{P}(|Y_N^{\pm}(|x| + 2\theta - \delta_2)| \leq 3\delta_1) \leq 3/8$ when N is large enough to deduce that $\mathbb{P}(Y_N^{\pm} \notin \Xi) \leq 1/2$ when N is large enough and end the proof of Lemma 28.

We now prove $\mathbb{P}(|Y_N^{\pm}(|x| + 2\theta - \delta_2)| \leq 3\delta_1) \leq 3/8$ when N is large enough, by noticing $Y_N^{\pm}(|x| + 2\theta - \delta_2)$ is close to $Y'_N(|x| + 2\theta - \delta_2)$, which will converge in distribution to $B^x_{|x|+2\theta-\delta_2}$ when N tends to $+\infty$. Lemma 26 implies $\mathbb{P}(\|Y_N^{\pm}|_{[0,|x|+2\theta-\delta_2]} - Y'_N|_{[0,|x|+2\theta-\delta_2]}\|_\infty > 2N^{-1/12})$ tends to 0 when N tends to $+\infty$, hence $\mathbb{P}(|Y_N^{\pm}(|x| + 2\theta - \delta_2) - Y'_N(|x| + 2\theta - \delta_2)| > 2N^{-1/12})$ tends to 0 when N tends to $+\infty$, which implies $Y_N^{\pm}(|x| + 2\theta - \delta_2) - Y'_N(|x| + 2\theta - \delta_2)$ converges in probability to 0 when N tends to $+\infty$. In addition, Lemma 23 states $Y'_N|_{[-|x|-2\theta,|x|+2\theta]}$ converges in distribution to $B^x|_{[-|x|-2\theta,|x|+2\theta]}$ when N tends to $+\infty$ for the topology defined on $C[-|x| - 2\theta, |x| + 2\theta]$ by the uniform norm, hence $Y'_N(|x|+2\theta-\delta_2)$ converges in distribution to $B^x_{|x|+2\theta-\delta_2}$ when N tends to $+\infty$. Therefore Slutsky's Theorem yields that $Y_N^{\pm}(|x| + 2\theta - \delta_2)$ converges in distribution to $B^x_{|x|+2\theta-\delta_2}$ when N tends to $+\infty$. Moreover, we defined Ξ so that $\mathbb{P}(\exists\, y \in [|x| + 2\theta - \delta_2, |x| + 2\theta], |B^x_y| \leq 3\delta_1) \leq 1/4$, hence $\mathbb{P}(|B^x_{|x|+2\theta-\delta_2}| \leq 3\delta_1) \leq 1/4$. This implies that when N is large enough, $\mathbb{P}(|Y_N^{\pm}(|x| + 2\theta - \delta_2)| \leq 3\delta_1) \leq 3/8$. □

Proof of Lemma 29. Let $(Z_N)_{N\in\mathbb{N}}$ be a sequence of elements of Ξ converging to Z in the Skorohod J_1 topology on $D(-\infty, +\infty)$, we will prove $Z \in \Xi$. By taking an extraction, we may assume $d_{J_1}(Z, Z_N) < e^{-|x|-2\theta-\delta_2-1}/N$ for any $N \in \mathbb{N}^*$. Then for any $N \in \mathbb{N}^*$, some $a_N > |x| + 2\theta + \delta_2 + 1$ so that $d_{J_1,-a_N,a_N}(Z|_{[-a_N,a_N]}, Z_N|_{[-a_N,a_N]}) \leq 1/N$ will exist. Indeed, if it was not the case, for some N we would have $d_{J_1}(Z, Z_N) = \int_0^{+\infty} e^{-a}(d_{J_1,-a,a}(Z|_{[-a,a]}, Z_N|_{[-a,a]}) \wedge 1)\mathrm{d}a \geq \int_{|x|+2\theta+\delta_2+1}^{+\infty} e^{-a}\frac{1}{N}\mathrm{d}a = e^{-|x|-2\theta-\delta_2-1}/N$, which does not happen. For all $N \in \mathbb{N}^*$, the fact that we have $d_{J_1,-a_N,a_N}(Z|_{[-a_N,a_N]}, Z_N|_{[-a_N,a_N]}) \leq 1/N$ implies there exists $\lambda_N \in \Lambda_{-a_N,a_N}$ with $\|Z|_{[-a_N,a_N]} \circ \lambda_N - Z_N|_{[-a_N,a_N]}\|_\infty \leq 2/N$ and $\|\lambda_N - \mathrm{Id}_{-a_N,a_N}\|_\infty \leq 2/N$. Moreover, $Z_N \in \Xi$, hence there exists $y_N \in [|x|+2\theta-\delta_2, |x|+2\theta+\delta_2], |Z_N(y_N)| \in [\delta_1, 2\delta_1]$ or $|Z_N(y_N^-)| \in [\delta_1, 2\delta_1]$. We now define y'_N as follows: if $|Z_N(y_N)| \in [\delta_1, 2\delta_1]$ we set $y'_N = y_N$. Otherwise, since $|Z_N(y_N^-)| \in [\delta_1, 2\delta_1]$ we can take some y'_N in $[y_N - \frac{1}{N}, y'_N]$ so that $|Z_N(y'_N)| \in [\delta_1 - 1/N, 2\delta_1 + 1/N]$. In both cases, we have $y'_N \in [|x| + 2\theta - \delta_2 - 1/N, |x| + 2\theta + \delta_2]$ and $|Z_N(y'_N)| \in [\delta_1 - 1/N, 2\delta_1 + 1/N]$. Furthermore, $\|\lambda_N - \mathrm{Id}_{-a_N,a_N}\|_\infty \leq 2/N$, hence $|\lambda_N(y'_N) - y'_N| \leq 2/N$, thus $\lambda_N(y'_N) \in [|x| + 2\theta - \delta_2 - 3/N, |x| + 2\theta + \delta_2 + 2/N]$. In addition, $\|Z(\lambda_N(y'_N)) - Z_N(y'_N)\|_\infty \leq 2/N$, hence $|Z(\lambda_N(y'_N))| \in [\delta_1 - 3/N, 2\delta_1 + 3/N]$. By taking an extraction, we may assume that $\lambda_N(y'_N)$ converges to some $y_\infty \in [|x|+2\theta-\delta_2, |x|+2\theta+\delta_2]$. In addition, Z is càdlàg, hence there is an extraction of $(Z(\lambda_N(y'_N))_{N\in\mathbb{N}^*}$ that converges to either $Z(y_\infty)$ or $Z(y_\infty^-)$. Since $|Z(\lambda_N(y'_N))| \in [\delta_1 - 3/N, 2\delta_1 + 3/N]$, we have $|Z(y_\infty)|$ or $|Z(y_\infty^-)|$ in $[\delta_1, 2\delta_1]$. Therefore $Z \in \Xi$, which ends the proof. □

7. Convergence of the stopping time: proof of Proposition 4

We want to prove Proposition 4, that is the convergence in distribution of $\frac{1}{N^{3/2}}(T_N - N^2(|x| + 2\theta)^2)$ to the law $\mathcal{N}(0, \frac{32}{3}\mathrm{Var}(\rho_-)((|x|+\theta)^3+\theta^3))$ when N tends to $+\infty$. In order to do that, we will prove that $\frac{1}{N^{3/2}}(T_N - N^2(|x|+2\theta)^2)$ is close to $2\int_{-|x|-2\theta}^{|x|+2\theta} Y'_N(y)\mathrm{d}y$, then show that $2\int_{-|x|-2\theta}^{|x|+2\theta} Y'_N(y)\mathrm{d}y$ converges to the desired distribution.

Proposition 30. $\mathbb{P}(|\frac{1}{N^{3/2}}(T_N - N^2(|x| + 2\theta)^2) - 2\int_{-|x|-2\theta}^{|x|+2\theta} Y'_N(y)\mathrm{d}y| > 5(|x| + 2\theta)N^{-1/12})$ *tends to 0 when* N *tends to* $+\infty$.

Proof. The result will come from the fact that T_N can be written as the sum of the local times, which is itself related to the integrals of Y_N^- and Y_N^+, which are close to Y_N by Lemma 17 hence to Y_N' by Lemma 24. It is enough to prove that if $(\mathcal{B}_2)^c$, $(\mathcal{B}_4^-)^c$ and $(\mathcal{B}_4^+)^c$ occur and if $|I^- + (|x|+2\theta)N| < N^{5/8}$, $|I^+ - (|x|+2\theta)N| < N^{5/8}$, then $|\frac{1}{N^{3/2}}(T_N - N^2(|x|+2\theta)^2) - 2\int_{-|x|-2\theta}^{|x|+2\theta} Y_N'(y)\mathrm{d}y| \leq 5(|x|+2\theta)N^{-1/12}$, since Lemma 7 implies $\mathbb{P}(\mathcal{B}_2)$ tends to 0 when N tends to $+\infty$, Lemma 14 implies $\mathbb{P}(\mathcal{B}_4^-)$ and $\mathbb{P}(\mathcal{B}_4^+)$ tend to 0 when N tends to $+\infty$, and Lemma 10 implies $\mathbb{P}(|I^- + (|x|+2\theta)N| \geq N^{5/8})$ and $\mathbb{P}(|I^+ - (|x|+2\theta)N| \geq N^{5/8})$ tend to 0 when N tends to $+\infty$. We assume $(\mathcal{B}_2)^c$, $(\mathcal{B}_4^-)^c$ and $(\mathcal{B}_4^+)^c$ occur and $|I^- + (|x|+2\theta)N| < N^{5/8}$, $|I^+ - (|x|+2\theta)N| < N^{5/8}$, let us study T_N.

In order to do that, we first need to prove an auxiliary result, more precisely that the following holds when N is large enough:

$$\text{if } |i - I^+| \leq N^{5/8}+1 \text{ or } |i-I^-| \leq N^{5/8}+1 \text{ then } \ell^+(T_N,i) \leq 4N^{11/16} \text{ and } \ell^-(T_N,i) \leq 4N^{11/16}. \tag{6}$$

We prove (6) for the case $|i-I^-| \leq N^{5/8}+1$, since the other is similar. Let $i \in \mathbb{Z}$ so that $|i-I^-| < N^{5/8}+1$. We notice that since $|I^- + (|x|+2\theta)N| < N^{5/8}$ we have $I^-, i < 0$ when N is large enough, so (1) yields $|\ell^+(T_N,i) - \ell^+(T_N,I^-)| \leq \sum_{|j-I^-|<N^{5/8}+1} |\eta_{j,-}(\ell^+(T_N,j))|$, thus since $\ell^+(T_N,I^-)=0$ we have $\ell^+(T_N,i) \leq \sum_{|j-I^-|<N^{5/8}+1} |\eta_{j,-}(\ell^+(T_N,j))|$. In addition, we assumed $(\mathcal{B}_2)^c$, hence $\ell^+(T_N,i) \leq \sum_{|j-I^-|<N^{5/8}+1}(N^{1/16}+1/2) \leq 3N^{5/8}N^{1/16} = 3N^{11/16}$ when N is large enough. Furthermore, (2) implies $|\ell^-(T_N,i) - \ell^+(T_N,i)| = |\eta_{i,-}(\ell^+(T_N,i))| \leq N^{1/16}+1/2$ thanks to $(\mathcal{B}_2)^c$, hence $\ell^-(T_N,i) \leq 3N^{11/16} + N^{1/16} + 1/2 \leq 4N^{11/16}$ when N is large enough, which ends the proof of (6).

We now write T_N as the sum of the local times and relate $\frac{1}{N^{3/2}}(T_N - N^2(|x|+2\theta)^2)$ to the integral of Y^+ and Y^-. We have $T_N = \sum_{i\in\mathbb{Z}}(\ell^+(T_N,i) + \ell^-(T_N,i))$. Moreover, if the random walk $(X_n)_{n\in\mathbb{N}}$ had visited any $i \geq I^+$ before time T_N, since it is at $\lfloor Nx \rfloor \iota 1$ at time T_N, it would have gone from I^+ to I^+-1 before time T_N, which is impossible since the definition of I^+ yields $\ell^-(T_N,I^+)=0$. This implies $(X_n)_{n\in\mathbb{N}}$ did not visit any $i \geq I^+$, hence for all $i \geq I^+$ we have $\ell^+(T_N,i) = \ell^-(T_N,i) = 0$. Similarly, for any $i \leq I^-$ we have $\ell^+(T_N,i) = \ell^-(T_N,i) = 0$. Consequently, $T_N = \sum_{i=I^-\wedge(-\lfloor(|x|+2\theta)N\rfloor)}^{I^+\vee\lfloor(|x|+2\theta)N\rfloor}(\ell^+(T_N,i)+\ell^-(T_N,i))$. We thus have $|\frac{1}{N^{3/2}}(T_N - N^2(|x|+2\theta)^2) - \int_{(I^-\wedge(-(|x|+2\theta)N))/N}^{(I^+\vee(|x|+2\theta)N)/N}(Y_N^+(y) + Y_N^-(y))\mathrm{d}y| \leq \frac{1}{N^{3/2}}(\ell^+(T_N,I^+\vee\lfloor(|x|+2\theta)N\rfloor)+\ell^-(T_N,I^+\vee\lfloor(|x|+2\theta)N\rfloor)+\ell^+(T_N,-\lfloor(|x|+2\theta)N\rfloor-1)+\ell^-(T_N,-\lfloor(|x|+2\theta)N\rfloor-1))$. Since $|I^- + (|x|+2\theta)N| < N^{5/8}$ and $|I^+ - (|x|+2\theta)N| < N^{5/8}$, equation (6) yields

$$\left|\frac{1}{N^{3/2}}(T_N - N^2(|x|+2\theta)^2) - \int_{(I^-\wedge(-(|x|+2\theta)N))/N}^{(I^+\vee(|x|+2\theta)N)/N}(Y_N^+(y)+Y_N^-(y))\mathrm{d}y\right| \leq \frac{1}{N^{3/2}}16N^{11/16} = 16N^{-13/16}. \tag{7}$$

We now prove that $\int_{(I^-\wedge(-(|x|+2\theta)N))/N}^{(I^+\vee(|x|+2\theta)N)/N}(Y_N^+(y)+Y_N^-(y))\mathrm{d}y$ is close to $2\int_{-(|x|+2\theta)}^{|x|+2\theta} Y_N(y)\mathrm{d}y$. We begin by considering $\int_{\chi(N)/N}^{(I^+\vee(|x|+2\theta)N)/N}(Y_N^+(y)+Y_N^-(y))\mathrm{d}y$. We first assume $I^+ \geq (|x|+2\theta)N$. Since we assumed $(\mathcal{B}_2)^c$, $(\mathcal{B}_4^-)^c$ and $(\mathcal{B}_4^+)^c$ occur, Lemma 17 yields $|\int_{\chi(N)/N}^{(I^+\vee(|x|+2\theta)N)/N}(Y_N^+(y)+Y_N^-(y))\mathrm{d}y - 2\int_{\chi(N)/N}^{|x|+2\theta} Y_N(y)\mathrm{d}y| \leq 2(|x|+2\theta-\frac{\chi(N)}{N})N^{-1/12} + \int_{|x|+2\theta}^{I^+/N}|Y_N^+(y)+Y_N^-(y))|\mathrm{d}y$. In addition, we know $I^+-(|x|+2\theta)N \leq N^{5/8}$ and (6), hence $\int_{|x|+2\theta}^{I^+/N}|Y_N^+(y)+Y_N^-(y))|\mathrm{d}y \leq N^{-3/8}\frac{1}{\sqrt{N}}(\max_{\lfloor(|x|+2\theta)N\rfloor\leq i\leq I^+}\ell^+(T_N,i) + \max_{\lfloor(|x|+2\theta)N\rfloor\leq i\leq I^+}\ell^-(T_N,i)) \leq N^{-3/8}N^{-1/2}8N^{11/16} = 8N^{-3/16}$. We deduce $|\int_{\chi(N)/N}^{(I^+\vee(|x|+2\theta)N)/N}(Y_N^+(y)+Y_N^-(y))\mathrm{d}y - 2\int_{\chi(N)/N}^{|x|+2\theta} Y_N(y)\mathrm{d}y| \leq 2(|x|+2\theta-\frac{\chi(N)}{N})N^{-1/12}+8N^{-3/16}$. We now assume $I^+ < (|x|+2\theta)N$. In this case, we have $|\int_{\chi(N)/N}^{(I^+\vee(|x|+2\theta)N)/N}(Y_N^+(y)+Y_N^-(y))\mathrm{d}y - 2\int_{\chi(N)/N}^{|x|+2\theta} Y_N(y)\mathrm{d}y| \leq \int_{\chi(N)/N}^{I^+/N}|Y_N^+(y)+Y_N^-(y)-2Y_N(y)|\mathrm{d}y + \int_{I^+/N}^{|x|+2\theta}|Y_N^+(y)+Y_N^-(y)|\mathrm{d}y + \int_{I^+/N}^{|x|+2\theta}|2Y_N(y)|\mathrm{d}y$. Moreover, Lemma 17 yields $\int_{\chi(N)/N}^{I^+/N}|Y_N^+(y)+$

$Y_N^-(y) - 2Y_N(y)|\mathrm{d}y \leq 2(|x| + 2\theta - \frac{\chi(N)}{N})N^{-1/12}$. Furthermore, for $y \geq \frac{I^+}{N}$ we have $\ell^{\pm}(T_N, \lfloor Ny \rfloor) = 0$. Since $|I^+ - (|x| + 2\theta)N| < N^{5/8}$ this yields $|Y_N^{\pm}(y)| \leq \frac{1}{2}N^{1/8}$. Thus $\int_{I^+/N}^{|x|+2\theta} |Y_N^+(y) + Y_N^-(y)|\mathrm{d}y \leq \int_{I^+/N}^{|x|+2\theta} N^{1/8}\mathrm{d}y \leq N^{-3/8}N^{1/8} = N^{-1/4}$. We deduce $|\int_{\chi(N)/N}^{(I^+\vee(|x|+2\theta)N)/N}(Y_N^+(y) + Y_N^-(y))\mathrm{d}y - 2\int_{\chi(N)/N}^{|x|+2\theta} Y_N(y)\mathrm{d}y| \leq 2(|x| + 2\theta - \frac{\chi(N)}{N})N^{-1/12} + N^{-1/4} + \int_{I^+/N}^{|x|+2\theta} |2Y_N(y)|\mathrm{d}y$. In addition, for any $y \in [\frac{I^+}{N}, |x| + 2\theta]$, we have $|Y_N(y)| \leq |Y_N(y) - Y_N(\frac{I^+}{N})| + |Y_N(\frac{I^+}{N}) - Y_N^-(\frac{I^+}{N})| + |Y_N^-(\frac{I^+}{N})|$. Lemma 17 yields that $|Y_N(\frac{I^+}{N}) - Y_N^-(\frac{I^+}{N})| \leq N^{-1/12}$, and since $|I^+ - (|x| + 2\theta)N| < N^{5/8}$ we have $|Y_N^-(\frac{I^+}{N})| = |\frac{1}{\sqrt{N}}(\ell^{\pm}(T_N, I^+) - N(\frac{|x|-|I^+/N|}{2} + \theta)_+)| \leq \frac{1}{2}N^{1/8}$, hence $|Y_N(y)| \leq |Y_N(y) - Y_N(\frac{I^+}{N})| + N^{-1/12} + \frac{1}{2}N^{1/8} = \frac{1}{\sqrt{N}}|\sum_{i=I^+}^{\lfloor Ny \rfloor - 1}\zeta_i| + N^{-1/12} + \frac{1}{2}N^{1/8} \leq \frac{1}{\sqrt{N}}\sum_{i=I^+}^{\lfloor (|x|+2\theta)N \rfloor - 1}|\zeta_i| + N^{-1/12} + \frac{1}{2}N^{1/8} \leq \frac{1}{\sqrt{N}}N^{5/8}N^{1/16} + N^{-1/12} + \frac{1}{2}N^{1/8} \leq 2N^{3/16}$ since $(\mathcal{B}_2)^c$ occurs. This implies $\int_{I^+/N}^{|x|+2\theta} |2Y_N(y)|\mathrm{d}y \leq \int_{I^+/N}^{|x|+2\theta} 4N^{3/16}\mathrm{d}y = N^{-3/8}4N^{3/16} = 4N^{-3/16}$. We deduce $|\int_{\chi(N)/N}^{(I^+\vee(|x|+2\theta)N)/N}(Y_N^+(y) + Y_N^-(y))\mathrm{d}y - 2\int_{\chi(N)/N}^{|x|+2\theta} Y_N(y)\mathrm{d}y| \leq 2(|x| + 2\theta - \frac{\chi(N)}{N})N^{-1/12} + 5N^{-3/16}$. Consequently, in all cases we have $|\int_{\chi(N)/N}^{(I^+\vee(|x|+2\theta)N)/N}(Y_N^+(y) + Y_N^-(y))\mathrm{d}y - 2\int_{\chi(N)/N}^{|x|+2\theta} Y_N(y)\mathrm{d}y| \leq 2(|x| + 2\theta - \frac{\chi(N)}{N})N^{-1/12} + 8N^{-3/16}$. One can prove similarly that $|\int_{(I^-\wedge(-(|x|+2\theta)N))/N}^{\chi(N)/N}(Y_N^+(y) + Y_N^-(y))\mathrm{d}y - 2\int_{-(|x|+2\theta)}^{\chi(N)/N} Y_N(y)\mathrm{d}y| \leq 2(|x| + 2\theta + \frac{\chi(N)}{N})N^{-1/12} + 8N^{-3/16}$. We conclude that $|\int_{(I^-\wedge(-(|x|+2\theta)N))/N}^{(I^+\vee(|x|+2\theta)N)/N}(Y_N^+(y) + Y_N^-(y))\mathrm{d}y - 2\int_{-(|x|+2\theta)}^{|x|+2\theta} Y_N(y)\mathrm{d}y| \leq 4(|x| + 2\theta)N^{-1/12} + 16N^{-3/16}$.

We are now in position to conclude. Indeed, the previous result and (7) imply that when N is large enough, $|\frac{1}{N^{3/2}}(T_N - N^2(|x| + 2\theta)^2) - 2\int_{-(|x|+2\theta)}^{|x|+2\theta} Y_N(y)\mathrm{d}y| \leq 16N^{-13/16} + 4(|x| + 2\theta)N^{-1/12} + 16N^{-3/16}$. Moreover, $(\mathcal{B}_2)^c$ occurs, hence Lemma 24 yields $\sup\{|Y_N(y) - Y_N'(y)| : y \in [-|x| - 2\theta, |x| + 2\theta)\} \leq N^{-7/16}$, therefore $|\int_{-(|x|+2\theta)}^{|x|+2\theta} Y_N(y)\mathrm{d}y - \int_{-(|x|+2\theta)}^{|x|+2\theta} Y_N'(y)\mathrm{d}y| \leq 2(|x| + 2\theta)N^{-7/16}$. We deduce that when N is large enough, $|\frac{1}{N^{3/2}}(T_N - N^2(|x| + 2\theta)^2) - 2\int_{-(|x|+2\theta)}^{|x|+2\theta} Y_N'(y)\mathrm{d}y| \leq 16N^{-13/16} + 4(|x| + 2\theta)N^{-1/12} + 16N^{-3/16} + 4(|x| + 2\theta)N^{-7/16} \leq 5(|x| + 2\theta)N^{-1/12}$, which ends the proof. □

Now that we know $\frac{1}{N^{3/2}}(T_N - N^2(|x| + 2\theta)^2)$ is close to $2\int_{-|x|-2\theta}^{|x|+2\theta} Y_N'(y)\mathrm{d}y$, we need to prove $2\int_{-|x|-2\theta}^{|x|+2\theta} Y_N'(y)\mathrm{d}y$ converges to the desired distribution. In order to do that, we will use the convergence of Y_N' to a Brownian motion stated in Lemma 23, so $2\int_{-|x|-2\theta}^{|x|+2\theta} Y_N'(y)\mathrm{d}y$ will converge to the integral of a Brownian motion, the law of the latter being characterized by the following lemma, where we denote by $(B_t)_{t\in\mathbb{R}^+}$ a standard Brownian motion with $B_0 = 0$.

Lemma 31. *For any $y > 0$, the integral $\int_0^y B_z\mathrm{d}z$ has distribution $\mathcal{N}(0, \frac{y^3}{3})$.*

We are now able to prove Proposition 4.

Proof of Proposition 4. Proposition 30 implies $\frac{1}{N^{3/2}}(T_N - N^2(|x| + 2\theta)^2) - 2\int_{-|x|-2\theta}^{|x|+2\theta} Y_N'(y)\mathrm{d}y$ converges in probability to 0 when N tends to $+\infty$. Hence by Slutsky's Theorem, it is enough to prove $2\int_{-|x|-2\theta}^{|x|+2\theta} Y_N'(y)\mathrm{d}y$ converges in distribution to $\mathcal{N}(0, \mathrm{Var}(\rho_-)\frac{32}{3}((|x| + \theta)^3 + \theta^3))$ when N tends to $+\infty$ to prove Proposition 4. In addition, by Lemma 23, $Y_N'|_{[-|x|-2\theta, |x|+2\theta]}$ converges in distribution to $B^x|_{[-|x|-2\theta, |x|+2\theta]}$ when N tends to $+\infty$ for the topology defined on $C[-|x| - 2\theta, |x| + 2\theta]$ by the uniform norm. Moreover, the integral between $-|x| - 2\theta$ and $|x| + 2\theta$ is continuous for this topology, hence $\int_{-|x|-2\theta}^{|x|+2\theta} Y_N'(y)\mathrm{d}y$ converges in distribution to $\int_{-|x|-2\theta}^{|x|+2\theta} B_y^x\mathrm{d}y$ when N tends to $+\infty$. Furthermore, B^x is a two-sided Brownian motion with $B_x^x = 0$ and variance $\mathrm{Var}(\rho_-)$, hence we can write

$\int_{-|x|-2\theta}^{|x|+2\theta} B_y^x \mathrm{d}y = \int_{-|x|-2\theta}^{x} B_y^x \mathrm{d}y + \int_x^{|x|+2\theta} B_y^x \mathrm{d}y$ where $\int_{-|x|-2\theta}^{x} B_y^x \mathrm{d}y$ and $\int_x^{|x|+2\theta} B_y^x \mathrm{d}y$ are independent. In addition, $\int_x^{|x|+2\theta} B_y^x \mathrm{d}y$ has the distribution of $\sqrt{\mathrm{Var}(\rho_-)}\int_0^{2\theta} B_y \mathrm{d}y$, which is $\mathcal{N}(0, \mathrm{Var}(\rho_-)\frac{(2\theta)^3}{3})$ by Lemma 31, and $\int_{-|x|-2\theta}^{x} B_y^x \mathrm{d}y$ has the distribution of $\sqrt{\mathrm{Var}(\rho_-)}\int_0^{2|x|+2\theta} B_y \mathrm{d}y$, which is $\mathcal{N}(0, \mathrm{Var}(\rho_-)\frac{(2|x|+2\theta)^3}{3})$ by Lemma 31. We obtain that $\int_{-|x|-2\theta}^{|x|+2\theta} B_y^x \mathrm{d}y$ has the distribution $\mathcal{N}(0, \mathrm{Var}(\rho_-)\frac{(2|x|+2\theta)^3}{3} + \mathrm{Var}(\rho_-)\frac{(2\theta)^3}{3}) = \mathcal{N}(0, \mathrm{Var}(\rho_-)\frac{8}{3}((|x|+\theta)^3 + \theta^3))$. Consequently, $\int_{-|x|-2\theta}^{|x|+2\theta} Y'_N(y)\mathrm{d}y$ converges in distribution to $\mathcal{N}(0, \mathrm{Var}(\rho_-)\frac{8}{3}((|x|+\theta)^3 + \theta^3))$ when N tends to $+\infty$, which ends the proof of Proposition 4. □

Therefore we only have to prove Lemma 31 to end the proof of Proposition 4.

Proof of Lemma 31. Let $y > 0$. When k tends to $+\infty$, the random variable $\frac{1}{k}\sum_{\ell=0}^{\lfloor ky \rfloor} B_{\ell/k}$ converges almost-surely to $\int_0^y B_z \mathrm{d}z$. Moreover, $\frac{1}{k}\sum_{\ell=0}^{\lfloor ky \rfloor} B_{\ell/k} = \frac{1}{k}\sum_{\ell=1}^{\lfloor ky \rfloor}(\lfloor ky \rfloor + 1 - \ell)(B_{\ell/k} - B_{(\ell-1)/k})$ has distribution $\mathcal{N}(0, \frac{1}{k^2}\sum_{\ell=1}^{\lfloor ky \rfloor}(\lfloor ky \rfloor + 1 - \ell)^2 \frac{1}{k}) = \mathcal{N}(0, \frac{1}{k^3}\sum_{\ell=1}^{\lfloor ky \rfloor} \ell^2) = \mathcal{N}(0, \frac{1}{k^3}\frac{\lfloor ky \rfloor(\lfloor ky \rfloor+1)(2\lfloor ky \rfloor+1)}{6})$. Denoting by ϕ_k the characteristic function of $\frac{1}{k}\sum_{\ell=0}^{\lfloor ky \rfloor} B_{\ell/k}$, for all $u \in \mathbb{R}$ we have $\phi_k(u) = \exp(-\frac{u^2}{2}\frac{1}{k^3}\frac{\lfloor ky \rfloor(\lfloor ky \rfloor+1)(2\lfloor ky \rfloor+1)}{6})$, which converges to $\exp(-\frac{u^2}{2}\frac{y^3}{3})$ when k tends to $+\infty$. Therefore the characteristic function of $\frac{1}{k}\sum_{\ell=0}^{\lfloor ky \rfloor} B_{\ell/k}$ converges pointwise to the characteristic function of $\mathcal{N}(0, \frac{y^3}{3})$, which implies $\frac{1}{k}\sum_{\ell=0}^{\lfloor ky \rfloor} B_{\ell/k}$ converges in distribution to $\mathcal{N}(0, \frac{y^3}{3})$. As $\frac{1}{k}\sum_{\ell=0}^{\lfloor ky \rfloor} B_{\ell/k}$ converges almost-surely to $\int_0^y B_z \mathrm{d}z$, this implies $\int_0^y B_z \mathrm{d}z$ has distribution $\mathcal{N}(0, \frac{y^3}{3})$. □

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Email address: laure.mareche@math.unistra.fr

Institut de Recherche Mathématique Avancée, UMR 7501 Université de Strasbourg et CNRS, 7 rue René-Descartes, 67000 Strasbourg, France