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FLUCTUATIONS OF THE LOCAL TIMES OF THE SELF-REPELLING RANDOM WALK WITH DIRECTED EDGES

LAURE MARÊCHÉ

ABSTRACT. In 2008, Tóth and Vető defined the self-repelling random walk with directed edges as a *non-Markovian* random walk on \mathbb{Z} : in this model, the probability that the walk moves from a point of \mathbb{Z} to a given neighbor depends on the number of previous crossings of the directed edge from the initial point to the target, called the local time of the edge. They found this model had a very peculiar behavior, as the process formed by the local times of all the edges, evaluated at a stopping time of a certain type and suitably renormalized, converges to a deterministic process, instead of a random one as in similar models. In this work, we study the fluctuations of the local times process around its deterministic limit, about which nothing was previously known. We prove that these fluctuations converge in the Skorohod M_1 topology, as well as in the uniform topology away from the discontinuities of the limit, but *not* in the most classical Skorohod topology. We also prove the convergence of the fluctuations of the aforementioned stopping times.

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Keywords: Self-interacting random walks, self-repelling random walk with directed edges, local times, functional limit theorems, fluctuations.

1. INTRODUCTION AND RESULTS

1.1. Self-interacting random walks. The study of self-interacting random walks began in 1983 in an article of Amit et al. [1]. Before [1], the expression “self-avoiding random walk” referred to paths on graphs that do not intersect themselves. However, these are not easy to construct step by step, hence one would consider the set of all possible paths of a given length. Since one does not follow a single path as it grows with time, it is not really a random walk model. In order to work with an actual random walk model with a self-avoiding behavior, the authors of [1] introduced the “*true*” *self-avoiding random walk*. It is a random walk on \mathbb{Z}^d for which, at each step, the position of the process at the next step is chosen randomly among the neighbors of the current position depending on the number of the previous visits to said neighbors, with lower probabilities for those that have been visited the most. This process is a random walk in the sense that it is constructed step by step, but contrary to most random walks in the literature, it is *non-Markovian*: at each step, the law of the next step depends on the whole past of the process.

It turns out that the “true” self-avoiding random walk is hard to study. This led to the introduction by Tóth [13, 14, 15] of non-Markovian random walks *with bond repulsion*, for which the probability to go from one site to another, instead of depending of the number of previous visits to the target, depends on the number of previous crossings of the undirected edge between the two sites, which is called the *local time* of the edge, with lower probabilities for the edges that were crossed the most in the past. These walks are much easier to study, at least on \mathbb{Z} , because one can apply the *Ray-Knight approach* to them. This approach was introduced by Ray and Knight in [11, 2], and used for the first time for non-Markovian random walks by Tóth in [13, 14, 15]. Since then, it was applied to many other non-Markovian random walks, such as a continuous-time version of the “true” self-avoiding random walk in [18], *edge-reinforced random walks* (see the corresponding part of the review [9] and references therein) and *excited random walks* (see [3] and references therein). The Ray-Knight approach works as follows: though the random walk itself is

not Markovian, if we stop it when the local time at a given edge has reached a certain threshold, then the local times on the edges will form a Markov chain, which allows their analysis. Thanks to this approach, Tóth was able to prove scaling limits for the local times process for many different random walks with bond repulsion in his works [13, 14, 15]. The law of the limit depends on the random walk model, but it is always a random process¹.

1.2. The self-repelling random walk with directed edges. In 2008, Tóth and Vető [17] introduced a process seemingly very similar to the aforementioned random walks with bond repulsion, in which the probability to go from one site to another depends on the number of crossings of the *directed* edge between them instead of the crossings of the undirected edge. This process, called *self-repelling random walk with directed edges*, is a nearest-neighbor random walk on \mathbb{Z} defined as follows. For any set A , we denote by $|A|$ the cardinal of A . Let $w : \mathbb{Z} \mapsto (0, +\infty)$ be a non-decreasing and non-constant function. We will denote the walk by $(X_n)_{n \in \mathbb{N}}$. We set $X_0 = 0$, and for any $n \in \mathbb{N}$, $i \in \mathbb{Z}$, we denote $\ell^\pm(n, i) = |\{0 \leq m \leq n-1 \mid (X_m, X_{m+1}) = (i, i \pm 1)\}|$ the number of crossings of the directed edge $(i, i \pm 1)$ before time n , that is the *local time* of the directed edge at time n . Then

$$\mathbb{P}(X_{n+1} = X_n \pm 1) = \frac{w(\pm(\ell^-(n, X_n) - \ell^+(n, X_n)))}{w(\ell^+(n, X_n) - \ell^-(n, X_n)) + w(\ell^-(n, X_n) - \ell^+(n, X_n))}.$$

Using the local time of directed edges instead of that of undirected edges may seem like a very small change in the definition of the process, but the behavior of the self-repelling random walk with directed edges is actually very different from that of classical random walks with bond repulsion. Indeed, Tóth and Vető [17] were able to prove that the local times process has a *deterministic* scaling limit, which is in sharp contrast with the random limit processes obtained for the random walks with bond repulsion on undirected edges [13, 14, 15] and even for the simple random walk [2].

The result of [17] is as follows. For any $a \in \mathbb{R}$, we denote $a_+ = \max(a, 0)$. If for any $n \in \mathbb{N}$, $i \in \mathbb{Z}$, we denote by $T_{n,i}^\pm$ the stopping time defined by $T_{n,i}^\pm = \min\{m \in \mathbb{N} \mid \ell^\pm(m, i) = n\}$, then $T_{n,i}^\pm$ is almost-surely finite by Proposition 1 of [17] and we have the following.

Theorem (Theorem 1 of [17]). *For any $\theta > 0$, $x \in \mathbb{R}$, then $\sup_{y \in \mathbb{R}} \left| \frac{1}{N} \ell^+(T_{\lfloor N\theta \rfloor, \lfloor Nx \rfloor}^\pm, \lfloor Ny \rfloor) - \left(\frac{|x| - |y|}{2} + \theta \right)_+ \right|$ converges in probability to 0 when N tends to $+\infty$.*

Thus the local times process of the self-repelling random walk with directed edges admits the deterministic scaling limit : $y \mapsto \left(\frac{|x| - |y|}{2} + \theta \right)_+$, which has the shape of a triangle. This also implies the following convergence result to a deterministic limit for the $T_{\lfloor N\theta \rfloor, \lfloor Nx \rfloor}^\pm$.

Proposition (Corollary 1 of [17]). *For any $\theta > 0$, $x \in \mathbb{R}$, then $\frac{1}{N^2} T_{\lfloor N\theta \rfloor, \lfloor Nx \rfloor}^\pm$ converges in probability to $(|x| + 2\theta)^2$ when N tends to $+\infty$.*

The deterministic character of these limits makes the behavior of the self-repelling random walk with directed edges very unusual, hence worthy of study. In particular, it is natural to consider the possible fluctuations of the local times process and of the $T_{\lfloor N\theta \rfloor, \lfloor Nx \rfloor}^\pm$ around their deterministic limits. However, prior to this paper, nothing was known about these fluctuations. In this work, we prove convergence in distribution of the fluctuations of the local times process and of the $T_{\lfloor N\theta \rfloor, \lfloor Nx \rfloor}^\pm$. It happens that the limit of the fluctuations of the local times process is discontinuous, hence before stating the results, we have to be careful of the topology in which it may converge.

¹The model studied by Tóth in [16] has a deterministic limit, but it is not a random walk with bond repulsion, as it is *self-attracting*: the more an edge was crossed in the past, the more likely it is to be crossed in the future.

1.3. Topologies for the convergence of the local times process. For any interval $I \subset \mathbb{R}$, let DI be the space of càdlàg functions on I , that is the set of functions $: I \mapsto \mathbb{R}$ that are right-continuous and have left limits everywhere in I . For any function $Z : I \mapsto \mathbb{R}$, we denote by $\|Z\|_\infty = \sup_{y \in I} |Z(y)|$ the uniform norm of Z on I . The uniform norm on I gives a topology on DI , but it is often too strong to deal with discontinuous functions.

For discontinuous càdlàg functions, the most widely used topology is the *Skorohod J_1 topology*, introduced by Skorohod in [12] (see chapter VI of [10] for a course), which is often called “the” Skorohod topology. Intuitively, two functions are close in this topology if they are close for the uniform norm after allowing some small perturbation of time. Rigorously, for $a < b$ in \mathbb{R} the Skorohod J_1 topology on $D[a, b]$ is defined as follows. We call $\Lambda_{a,b}$ the set of functions $\lambda : [a, b] \mapsto [a, b]$ that are bijective, strictly increasing and continuous (they correspond to the possible perturbations of time), and we denote by $\text{Id}_{a,b} : [a, b] \mapsto [a, b]$ the identity map, defined by $\text{Id}_{a,b}(y) = y$ for all $y \in [a, b]$. The Skorohod J_1 topology on $D[a, b]$ is defined through the following metric: for any $Z_1, Z_2 \in D[a, b]$, we set $d_{J_1, a, b}(Z_1, Z_2) = \inf_{\lambda \in \Lambda_{a,b}} \max(\|Z_1 \circ \lambda - Z_2\|_\infty, \|\lambda - \text{Id}_{a,b}\|_\infty)$. It can be proven rather easily that this is indeed a metric. We can then define the Skorohod J_1 topology in $D(-\infty, \infty)$ with the following metric: if for any sets $A_1 \subset A_2$ and A_3 and any function $Z : A_2 \mapsto A_3$, we denote $Z|_{A_1}$ the restriction of Z to A_1 , then for $Z_1, Z_2 \in D(-\infty, \infty)$, we set $d_{J_1}(Z_1, Z_2) = \int_0^{+\infty} e^{-a} (d_{J_1, -a, a}(Z_1|_{[-a, a]}, Z_2|_{[-a, a]}) \wedge 1) da$. The Skorohod J_1 topology is widely used to study the convergence of càdlàg functions. However, when the limit function has a jump, which will be the case here, convergence in the Skorohod J_1 topology requires the converging functions to have a single big jump approximating the jump of the limit process. To account for other cases, like having the jump of the limit functions approximated by several smaller jumps in quick succession or by a very steep continuous slope, one has to use a less restrictive topology, like the *Skorohod M_1 topology*.

The Skorohod M_1 topology was also introduced by Skorohod in [12] (see Section 3.3 of [19] for an overview). For any $a < b$ in \mathbb{R} , the Skorohod M_1 distance on $D[a, b]$ is defined as follows: the distance between two functions will be roughly “the distance between the completed graphs of the functions”. More rigorously, if $Z \in D[a, b]$, we denote $Z(a^-) = Z(a)$ and for any $y \in (a, b)$, we denote $Z(y^-) = \lim_{y' \rightarrow y, y' < y} Z(y')$. Then the *completed graph of Z* is $\Gamma_Z = \{(y, z) \mid y \in [a, b], \exists \varepsilon \in [0, 1] \text{ so that } z = \varepsilon Z(y^-) + (1 - \varepsilon)Z(y)\}$. To express the “distance between two such completed graphs”, we need to define the *parametric representations* of Γ_Z (by abuse of notation, we will often write “the parametric representations of Z ”). We define an order on Γ_Z as follows: for $(y_1, z_1), (y_2, z_2) \in \Gamma_Z$, we have $(y_1, z_1) \leq (y_2, z_2)$ when $y_1 < y_2$ or when $y_1 = y_2$ and $|Z(y_1^-) - z_1| \leq |Z(y_1^-) - z_2|$. A *parametric representation of Γ_Z* is a continuous, surjective function $(u, r) : [0, 1] \mapsto \Gamma_Z$ that is non-decreasing with respect to this order, thus intuitively, when t goes from 0 to 1, $(u(t), r(t))$ “travels through the completed graph of Z from its beginning to its end”. A parametric representation of Z always exists (see Remark 12.3.3 in [19]). For $Z_1, Z_2 \in D[a, b]$, the Skorohod M_1 distance between Z_1 and Z_2 , denoted by $d_{M_1, a, b}(Z_1, Z_2)$, is $\inf\{\max(\|u_1 - u_2\|_\infty, \|r_1 - r_2\|_\infty)\}$ where the infimum is on the parametric representations (u_1, r_1) of Z_1 and (u_2, r_2) of Z_2 . It can be proven that this indeed gives a metric (see Theorem 12.3.1 of [19]), and this metric defines the Skorohod M_1 topology on $D[a, b]$. For any $a > 0$, we will denote $d_{M_1, -a, a}$ by $d_{M_1, a}$ for short. We can now define the Skorohod M_1 topology in $D(-\infty, \infty)$ through the following metric: for $Z_1, Z_2 \in D(-\infty, \infty)$, we set $d_{M_1}(Z_1, Z_2) = \int_0^{+\infty} e^{-a} (d_{M_1, a}(Z_1|_{[-a, a]}, Z_2|_{[-a, a]}) \wedge 1) da$. It can be seen that the Skorohod M_1 topology is weaker than the Skorohod J_1 topology (see Theorem 12.3.2 of [19]), thus less restrictive. Indeed, since the distance between two functions is roughly “the distance between the completed graphs of the functions”, the Skorohod M_1 topology will allow a function with a jump to be the limit of functions with steep slopes or with several smaller jumps. For this reason, the Skorohod M_1 topology is often more adapted when considering convergence to a discontinuous function.

1.4. Results. We are now ready to state our results on the convergence of the fluctuations of the local times process. For any $\theta > 0$, $x \in \mathbb{R}$, $\iota \in \{+, -\}$, for any $N \in \mathbb{N}^*$, we define functions Y_N^-, Y_N^+ as follows: for any $y \in \mathbb{R}$, we set

$$Y_N^\pm(y) = \frac{1}{\sqrt{N}} \left(\ell^\pm(T_{\lfloor N\theta \rfloor, \lfloor Nx \rfloor}^\iota, \lfloor Ny \rfloor) - N \left(\frac{|x| - |y|}{2} + \theta \right)_+ \right).$$

Y_N^\pm actually depends on ι , but we do not write this dependency in the notation to make it lighter. Moreover, $(B_y^x)_{y \in \mathbb{R}}$ will denote a two-sided Brownian motion with $B_x^x = 0$ and variance $\text{Var}(\rho_-)$, where ρ_- is the distribution on \mathbb{Z} defined later in (3). We proved the following convergence for the fluctuations of the local times process of the self-repelling random walk with directed edges.

Theorem 1. *For any $\theta > 0$, $x \in \mathbb{R}$, $\iota \in \{+, -\}$, the process Y_N^\pm converges in distribution to $(B_y^x \mathbf{1}_{\{y \in [-|x| - 2\theta, |x| + 2\theta]\}})_{y \in \mathbb{R}}$ in the Skorohod M_1 topology on $D(-\infty, +\infty)$ when N tends to $+\infty$.*

Therefore the fluctuations of the local times process have a diffusive limit behavior. However, it is necessary to use the Skorohod M_1 topology here, as the following result states the convergence does not occur in the stronger Skorohod J_1 topology.

Proposition 2. *For any $\theta > 0$, $x \in \mathbb{R}$, $\iota \in \{+, -\}$, the process Y_N^\pm does not converge in distribution in the Skorohod J_1 topology on $D(-\infty, +\infty)$ when N tends to $+\infty$.*

We stress the fact that the use of the Skorohod M_1 topology is only required to deal with the discontinuities of the limit process at $-|x| - 2\theta$ and $|x| + 2\theta$. Indeed, if we consider the convergence of the process on an interval that does not include $-|x| - 2\theta$ or $|x| + 2\theta$, it converges in the much stronger topology given by the uniform norm, which is the following result.

Proposition 3. *For any $\theta > 0$, $x \in \mathbb{R}$, $\iota \in \{+, -\}$, for any closed interval $I \in \mathbb{R}$ that does not contain $-|x| - 2\theta$ or $|x| + 2\theta$, the process $(Y_N^\pm(y))_{y \in I}$ converges in distribution to $(B_y^x \mathbf{1}_{\{y \in [-|x| - 2\theta, |x| + 2\theta]\}})_{y \in I}$ in the topology on DI given by the uniform norm when N tends to $+\infty$.*

Finally, we also proved the convergence of the fluctuations of $T_{\lfloor N\theta \rfloor, \lfloor Nx \rfloor}^\pm$. For any $\sigma^2 > 0$, we denote by $\mathcal{N}(0, \sigma^2)$ the Gaussian distribution with mean 0 and variance σ^2 , and we recall that ρ_- will be defined in (3). We then have the following.

Proposition 4. *For any $\theta > 0$, $x \in \mathbb{R}$, $\iota \in \{+, -\}$, we have that $\frac{1}{N^{3/2}}(T_{\lfloor N\theta \rfloor, \lfloor Nx \rfloor}^\iota - N^2(|x| + 2\theta)^2)$ converges in distribution to $\mathcal{N}(0, \frac{32}{3}\text{Var}(\rho_-)((|x| + \theta)^3 + \theta^3))$ when N tends to $+\infty$.*

Remark 5. Instead of studying the fluctuations of $\ell^\pm(T_{\lfloor N\theta \rfloor, \lfloor Nx \rfloor}^\iota, \cdot)$, it would seem more natural to consider those of $\ell^\pm(N^2, \cdot)$. However, the Ray-Knight arguments that allow to study $\ell^\pm(T_{\lfloor N\theta \rfloor, \lfloor Nx \rfloor}^\iota, \cdot)$ completely break down for $\ell^\pm(N^2, \cdot)$, and it is not even clear whether these two processes should have the same behavior.

Remark 6. Besides the article of Tóth and Vető [17] that introduced the self-repelling random walk with directed edges, there have been few other works on this model. These works were motivated by another important question, that of the existence of a scaling limit for $(X_n)_{n \in \mathbb{N}}$, which means the convergence in distribution of the process $(\frac{1}{N^\alpha} X_{\lfloor Nt \rfloor})_{t \geq 0}$ for some α . Obtaining such a scaling limit for the trajectory of the random walk is harder than obtaining scaling limits for the local times. Indeed, for the random walks with bond repulsion with undirected edges introduced by Tóth in [13, 14, 15], the scaling limits for the local times are known since the introduction of the models, but the scaling

limits for the trajectories are not. Some results were proven by Kosygina, Mountford and Peterson in [4], but they do not cover all models. For the self-repelling random walk with directed edges, the behavior of the scaling limit of the trajectory turns out to be surprising. Indeed, Mountford, Pimentel, and Valle proved in [7] that $\frac{1}{\sqrt{N}}X_N$ converges in distribution, but Mountford and the author showed in [6] that $(\frac{1}{\sqrt{N}}X_{\lfloor Nt \rfloor})_{t \geq 0}$ does *not* converge in distribution, and that the trajectories of the walk satisfy a more complex limit theorem, of a new kind.

1.5. Proof ideas. We begin by explaining why the limit of the local times process Y_N^\pm is $(B_y^x \mathbf{1}_{\{y \in [-|x| - 2\theta, |x| + 2\theta]\}})_{y \in \mathbb{R}}$ and the ideas behind the proofs of Theorem 1 and Proposition 3. To show the convergence of the local times process, we use a Ray-Knight argument, that is we notice that $(\ell^-(T_{\lfloor N\theta \rfloor, \lfloor Nx \rfloor}^\nu, i))_i$ is a Markov chain. Moreover, as long as $\ell^-(T_{\lfloor N\theta \rfloor, \lfloor Nx \rfloor}^\nu, i)$ is not too low, the $\ell^-(T_{\lfloor N\theta \rfloor, \lfloor Nx \rfloor}^\nu, i+1) - \ell^-(T_{\lfloor N\theta \rfloor, \lfloor Nx \rfloor}^\nu, i)$ will roughly be i.i.d. random variables in the sense that they can be coupled with i.i.d. random variables with a high probability to be equal to them. This coupling was already used in [17] to prove the convergence of $\frac{1}{N}\ell^+(T_{\lfloor N\theta \rfloor, \lfloor Nx \rfloor}^\pm, \lfloor Ny \rfloor)$ to its deterministic limit (for a given y , the coupling makes this convergence a law of large numbers). However, when $\ell^-(T_{\lfloor N\theta \rfloor, \lfloor Nx \rfloor}^\nu, \lfloor Ny \rfloor)$ is too low, the coupling fails and the $\ell^-(T_{\lfloor N\theta \rfloor, \lfloor Nx \rfloor}^\nu, \lfloor Ny \rfloor + 1) - \ell^-(T_{\lfloor N\theta \rfloor, \lfloor Nx \rfloor}^\nu, \lfloor Ny \rfloor)$ are no longer i.i.d. We have to prove that this occurs only around $|x| + 2\theta$ and $-|x| - 2\theta$, and most of our work is dealing with what happens there. To show it occurs only around $|x| + 2\theta$ and $-|x| - 2\theta$, we control the amplitude of the fluctuations to prove the local times are close to their deterministic limit. This limit is large inside $(-|x| - 2\theta, |x| + 2\theta)$, so we can use the coupling inside this interval, thus the $\ell^-(T_{\lfloor N\theta \rfloor, \lfloor Nx \rfloor}^\nu, \lfloor Ny \rfloor + 1) - \ell^-(T_{\lfloor N\theta \rfloor, \lfloor Nx \rfloor}^\nu, \lfloor Ny \rfloor)$ are roughly i.i.d. there, hence the fluctuations will converge to a Brownian motion by Donsker's Invariance Principle. When we are close to $|x| + 2\theta$ (the same reasoning works for $-|x| - 2\theta$) the deterministic limit will be small hence the local times too, and tools of [17] allow to prove that they reach 0 quickly. Once they reach 0, we notice that for $y \geq |x| + 2\theta$, if $\ell^-(T_{\lfloor N\theta \rfloor, \lfloor Nx \rfloor}^\nu, \lfloor Ny \rfloor) = 0$, the walk X did not go from $\lfloor Ny \rfloor$ to $\lfloor Ny \rfloor + 1$ before time $T_{\lfloor N\theta \rfloor, \lfloor Nx \rfloor}^\nu$, so it did not go to $\lfloor Ny \rfloor + 1$ before time $T_{\lfloor N\theta \rfloor, \lfloor Nx \rfloor}^\nu$, hence $\ell^-(T_{\lfloor N\theta \rfloor, \lfloor Nx \rfloor}^\nu, j) = 0$ for any $j \geq \lfloor Ny \rfloor$. Therefore, once the local times process reaches 0, it stays there. Consequently, we expect $\ell^-(T_{\lfloor N\theta \rfloor, \lfloor Nx \rfloor}^\nu, \lfloor Ny \rfloor)$ to be 0 when $y > |x| + 2\theta$, and thus to have no fluctuations when $y > |x| + 2\theta$, and similarly when $y < -|x| - 2\theta$. This is why our limit is $(B_y^x \mathbf{1}_{\{y \in [-|x| - 2\theta, |x| + 2\theta]\}})_{y \in \mathbb{R}}$. Since Proposition 3 only describes convergence away from $-|x| - 2\theta$ and $|x| + 2\theta$, the previous arguments are enough to prove it. To prove the convergence in the Skorohod M_1 topology on $D(-\infty, +\infty)$ stated in Theorem 1, we need to handle what happens around $-|x| - 2\theta$ and $|x| + 2\theta$ with more precision. We first have to bound the difference between the local times and the i.i.d. random variables of the coupling even where the coupling fails. Afterwards comes the most important part of the paper: defining parametric representations of Y_N^\pm and of the sum of the i.i.d. random variables of the coupling, properly renormalized and set to 0 outside of $(-|x| - 2\theta, |x| + 2\theta)$, and then proving that they are close to each other. That allows to prove Y_N^\pm is close in the Skorohod M_1 distance to a process that will converge in distribution to $(B_y^x \mathbf{1}_{\{y \in [-|x| - 2\theta, |x| + 2\theta]\}})_{y \in \mathbb{R}}$ in the Skorohod M_1 topology and to complete the proof of Theorem 1.

To prove Proposition 2, that is that Y_N^\pm does not converge in the J_1 topology, we first notice that since the J_1 topology is stronger than the M_1 topology, if Y_N^\pm did converge in the J_1 topology its limit would be $(B_y^x \mathbf{1}_{\{y \in [-|x| - 2\theta, |x| + 2\theta]\}})_{y \in \mathbb{R}}$. However, it is not possible, as $(B_y^x \mathbf{1}_{\{y \in [-|x| - 2\theta, |x| + 2\theta]\}})_{y \in \mathbb{R}}$ has a jump at $|x| + 2\theta$, while the jumps of Y_N^\pm have typical size of order $\frac{1}{\sqrt{N}}$, so the jump in $(B_y^x \mathbf{1}_{\{y \in [-|x| - 2\theta, |x| + 2\theta]\}})_{y \in \mathbb{R}}$ is approximated in Y_N^\pm by either a sequence of small jumps or a continuous slope, which prevents the convergence in the Skorohod J_1 topology.

Finally, to prove Proposition 4 on the fluctuations of $T_{\lfloor N\theta \rfloor, \lfloor Nx \rfloor}^\nu$, we use the fact that we have $T_{\lfloor N\theta \rfloor, \lfloor Nx \rfloor}^\nu = \sum_{i \in \mathbb{Z}} (\ell^+(T_{\lfloor N\theta \rfloor, \lfloor Nx \rfloor}^\nu, i) + \ell^-(T_{\lfloor N\theta \rfloor, \lfloor Nx \rfloor}^\nu, i))$. It can be checked that $|\ell^+(T_{\lfloor N\theta \rfloor, \lfloor Nx \rfloor}^\nu, i) - \ell^-(T_{\lfloor N\theta \rfloor, \lfloor Nx \rfloor}^\nu, i+1)| = 0$ or 1,

hence controlling the $\ell^-(T_{\lfloor N\theta \rfloor, \lfloor Nx \rfloor}^\iota, i)$ is enough. By using the coupling for the $\ell^-(T_{\lfloor N\theta \rfloor, \lfloor Nx \rfloor}^\iota, i+1) - \ell^-(T_{\lfloor N\theta \rfloor, \lfloor Nx \rfloor}^\iota, i)$ when $\ell^-(T_{\lfloor N\theta \rfloor, \lfloor Nx \rfloor}^\iota, i)$ is high enough and our estimates on the size of the window in which $\ell^-(T_{\lfloor N\theta \rfloor, \lfloor Nx \rfloor}^\iota, i)$ is neither high enough nor 0, we can prove that $T_{\lfloor N\theta \rfloor, \lfloor Nx \rfloor}^\iota$ is close to the integral of the sum of the i.i.d. random variables of the coupling, which will yield the convergence.

1.6. Organization of the paper. In Section 2, we define the coupling between the increments of the local time and i.i.d. random variables and prove some of its properties. In Section 3, we control where the local times hit 0, as well as where the local times are too low for the coupling of Section 2 to be useful. In Section 4, we prove a bound on the Skorohod M_1 distance between Y_N^\pm and the renormalized sum of the i.i.d. random variables of the coupling set to 0 outside of $[-|x| - 2\theta, |x| + 2\theta)$ by writing explicit parametric representations of the two functions. In Section 5, we complete the proof of the convergence of Y_N^\pm stated in Theorem 1 and Proposition 3. In Section 6, we prove that as claimed in Proposition 2, Y_N^\pm does not converge in the J_1 topology. Finally, in Section 7, we prove the convergence of the fluctuations of $T_{\lfloor N\theta \rfloor, \lfloor Nx \rfloor}^\pm$ stated in Proposition 4.

In what follows, we set $\theta > 0$, $\iota \in \{+, -\}$ and $x > 0$ (the cases $x < 0$ and $x = 0$ can be dealt with in the same way). To shorten the notation, we denote $T_N = T_{\lfloor N\theta \rfloor, \lfloor Nx \rfloor}^\iota$. Moreover, for any $a, b \in \mathbb{R}$, we denote $a \vee b = \max(a, b)$ and $a \wedge b = \min(a, b)$.

2. COUPLING OF THE LOCAL TIMES INCREMENTS WITH I.I.D. RANDOM VARIABLES

Our goal in this section will be to couple the $\ell^\pm(T_N, i+1) - \ell^\pm(T_N, i)$ with i.i.d. random variables and to prove some properties of this coupling. This part of the work is not very different from what was done in [17], but we still recall their concepts and definitions. If we fix $i \in \mathbb{Z}$ and observe the evolution of $(\ell^-(n, i) - \ell^+(n, i))_{n \in \mathbb{N}}$, and if we ignore the steps at which $\ell^-(n, i) - \ell^+(n, i)$ does not move (i.e. those at which the random walk is not at i), we obtain a Markov chain ξ_i whose distribution ξ has the following transition probabilities: for all $n \in \mathbb{N}$, $\mathbb{P}(\xi(n+1) = \xi(n) \pm 1) = \frac{w(\mp \xi(n))}{w(\xi(n)) + w(-\xi(n))}$, and so that $\xi_i(0) = 0$. Now, we denote $\tau_{i, \pm}(0) = 0$ and for any $n \in \mathbb{N}$, we denote $\tau_{i, \pm}(n+1) = \inf\{m > \tau_{i, \pm}(n) \mid \xi_i(m) = \xi_i(m-1) \pm 1\}$, so that $\tau_{i, +}(n)$ is the time of the n -th upwards step of ξ_i and $\tau_{i, -}(n)$ is the time of the n -th downwards step of ξ_i . Then since the distribution of ξ is symmetric, the processes $(\eta_{i, +}(n))_{n \in \mathbb{N}} = (-\xi_i(\tau_{i, +}(n)))_{n \in \mathbb{N}}$ and $(\eta_{i, -}(n))_{n \in \mathbb{N}} = (\xi_i(\tau_{i, -}(n)))_{n \in \mathbb{N}}$ have the same distribution, called η , and it can be checked that η is a Markov chain.

We are going to give an expression of $\ell^\pm(T_N, i+1) - \ell^\pm(T_N, i)$ depending on the $\eta_{i, -}$, $\eta_{i, +}$. We assume N large enough (so that $\lfloor Nx \rfloor - 1 > 0$). By definition of T_N we have $X_{T_N} = \lfloor Nx \rfloor \iota 1$. If $i \leq 0$ we thus have $X_{T_N} > i$, so the last step of the walk at i before T_N was going to the right, so the last step of ξ_i was a downwards step, and by definition of $\ell^+(T_N, i)$ we have that ξ_i made $\ell^+(T_N, i)$ downwards steps, hence $\ell^-(T_N, i) - \ell^+(T_N, i) = \xi_i(\tau_{i, -}(\ell^+(T_N, i))) = \eta_{i, -}(\ell^+(T_N, i))$, which yields $\ell^-(T_N, i) - \ell^+(T_N, i) = \eta_{i, -}(\ell^+(T_N, i))$. In addition, $\ell^-(T_N, i) = \ell^+(T_N, i-1)$, hence $\ell^+(T_N, i-1) = \ell^+(T_N, i) + \eta_{i, -}(\ell^+(T_N, i))$. If $0 < i < \lfloor Nx \rfloor$ (for $\iota = -$) or $0 < i \leq \lfloor Nx \rfloor$ (for $\iota = +$), the last step of the walk at i was also going to the right, so we also have $\ell^-(T_N, i) - \ell^+(T_N, i) = \eta_{i, -}(\ell^+(T_N, i))$. However, $\ell^-(T_N, i) = \ell^+(T_N, i-1) - 1$, so $\ell^+(T_N, i-1) = \ell^+(T_N, i) + \eta_{i, -}(\ell^+(T_N, i)) + 1$. Finally, if $i \geq \lfloor Nx \rfloor$ (for $\iota = -$) or $i > \lfloor Nx \rfloor$ (for $\iota = +$), then the last step of the walk at i was going to the left, so the last step of ξ_i was an upwards step, and ξ_i made $\ell^-(T_N, i)$ upwards steps, therefore $\ell^-(T_N, i) - \ell^+(T_N, i) = \xi_i(\tau_{i, +}(\ell^-(T_N, i))) = -\eta_{i, +}(\ell^-(T_N, i))$, which yields $\ell^-(T_N, i) - \ell^+(T_N, i) = -\eta_{i, +}(\ell^-(T_N, i))$. Moreover, $\ell^+(T_N, i) = \ell^-(T_N, i+1)$, hence $\ell^-(T_N, i+1) = \ell^-(T_N, i) + \eta_{i, +}(\ell^-(T_N, i))$.

We are going to use these results to deduce an expression of the $\ell^\pm(T_N, i)$ which will be very useful throughout this work. Denoting $\chi(N) = \lfloor Nx \rfloor$ if $\iota = -$ and $\chi(N) = \lfloor Nx \rfloor + 1$ if $\iota = +$, for $i \geq \chi(N)$ we have

$\ell^-(T_N, i) = \ell^-(T_N, \chi(N)) + \sum_{j=\chi(N)}^{i-1} \eta_{j,+}(\ell^-(T_N, j))$, and for $i < \chi(N)$ we have $\ell^+(T_N, i) = \ell^+(T_N, \chi(N) - 1) + \sum_{j=i+1}^{\chi(N)-1} (\eta_{j,-}(\ell^+(T_N, j)) + \mathbf{1}_{\{j>0\}})$. Now, we remember that the definition of T_N implies $\ell^\iota(T_N, \lfloor Nx \rfloor) = \lfloor N\theta \rfloor$, so if $\iota = -$ we have $\ell^-(T_N, \chi(N)) = \lfloor N\theta \rfloor$ and $\ell^+(T_N, \chi(N) - 1) = \ell^-(T_N, \chi(N)) = \lfloor N\theta \rfloor$, and if $\iota = +$ we have $\ell^+(T_N, \chi(N) - 1) = \lfloor N\theta \rfloor$ and $\ell^-(T_N, \chi(N)) = \ell^+(T_N, \chi(N) - 1) - 1 = \lfloor N\theta \rfloor - 1$. Consequently, we have the following.

$$(1) \quad \begin{aligned} \text{If } i \geq \chi(N), \quad \ell^-(T_N, i) &= \lfloor N\theta \rfloor - \mathbf{1}_{\{\iota=+\}} + \sum_{j=\chi(N)}^{i-1} \eta_{j,+}(\ell^-(T_N, j)). \\ \text{If } i < \chi(N), \quad \ell^+(T_N, i) &= \lfloor N\theta \rfloor + \sum_{j=i+1}^{\chi(N)-1} (\eta_{j,-}(\ell^+(T_N, j)) + \mathbf{1}_{\{j>0\}}). \end{aligned}$$

We will also need to remember the following.

$$(2) \quad \text{If } i \geq \chi(N), \quad \ell^-(T_N, i) - \ell^+(T_N, i) = -\eta_{i,+}(\ell^-(T_N, i)). \quad \text{If } i < \chi(N), \quad \ell^-(T_N, i) - \ell^+(T_N, i) = \eta_{i,-}(\ell^+(T_N, i)).$$

To couple the $\ell^\pm(T_N, i+1) - \ell^\pm(T_N, i)$ with i.i.d. random variables, we need to understand the $\eta_{i,+}(\ell^-(T_N, i))$ and the $\eta_{i,-}(\ell^+(T_N, i))$. [17] proved that the following measure ρ_- is the unique invariant probability distribution of the Markov chain η :

$$(3) \quad \forall i \in \mathbb{Z}, \quad \rho_-(i) = \frac{1}{R} \prod_{j=1}^{\lfloor 2i+1 \rfloor / 2} \frac{w(-j)}{w(j)} \quad \text{with} \quad R = \sum_{i \in \mathbb{Z}} \prod_{j=1}^{\lfloor 2i+1 \rfloor / 2} \frac{w(-j)}{w(j)}.$$

We also denote ρ_0 the measure on $\frac{1}{2} + \mathbb{Z}$ defined by $\rho_0(\cdot) = \rho_-(\cdot - \frac{1}{2})$.

We are now in position to construct the coupling of the $\ell^\pm(T_N, i+1) - \ell^\pm(T_N, i)$ with i.i.d. random variables $(\zeta_i)_{i \in \mathbb{Z}}$. The idea is that η can be expected to converge to its invariant distribution ρ_- , hence when $\ell^\pm(T_N, i)$ is large, $\eta_{i,\mp}(\ell^\pm(T_N, i))$ will be close to a random variable of law ρ_- . More rigorously, we begin by defining an i.i.d. sequence $(r_i)_{i \in \mathbb{Z}}$ of random variables of distribution ρ_- so that for $i \geq \chi(N)$ then $\mathbb{P}(r_i \neq \eta_{i,+}(\lfloor N^{1/6} \rfloor))$ is minimal, and for $i < \chi(N)$ then $\mathbb{P}(r_i \neq \eta_{i,-}(\lfloor N^{1/6} \rfloor))$ is minimal. We can then define i.i.d. Markov chains $(\bar{\eta}_{i,+}(n))_{n \geq \lfloor N^{1/6} \rfloor}$ for $i \geq \chi(N)$ and $(\bar{\eta}_{i,-}(n))_{n \geq \lfloor N^{1/6} \rfloor}$ for $i < \chi(N)$ so that $\bar{\eta}_{i,\pm}(\lfloor N^{1/6} \rfloor) = r_i$, $\bar{\eta}_{i,\pm}$ is a Markov chain of distribution that of η , and if $\bar{\eta}_{i,\pm}(\lfloor N^{1/6} \rfloor) = \eta_{i,\pm}(\lfloor N^{1/6} \rfloor)$ then $\bar{\eta}_{i,\pm}(n) = \eta_{i,\pm}(n)$ for any $n \geq \lfloor N^{1/6} \rfloor$. Since ρ_- is invariant for η , if $n \geq \lfloor N^{1/6} \rfloor$, the $\bar{\eta}_{i,+}(n)$ for $i \geq \chi(N)$ and $\bar{\eta}_{i,-}(n)$ for $i < \chi(N)$ have distribution ρ_- . We define the random variables $(\zeta_i)_{i \in \mathbb{Z}}$ as follows: for $i \geq \chi(N)$ we set $\zeta_i = \bar{\eta}_{i,+}(\ell^-(T_N, i) \vee \lfloor N^{1/6} \rfloor) + \frac{1}{2}$, and for $i < \chi(N)$ we set $\zeta_i = \bar{\eta}_{i,-}(\ell^+(T_N, i) \vee \lfloor N^{1/6} \rfloor) + \frac{1}{2}$. For $i \geq \chi(N)$, (1) implies that $\ell^-(T_N, i)$ depends only on the $\eta_{j,+}$, $\chi(N) \leq j \leq i-1$, hence is independent from $\bar{\eta}_{i,+}$, which implies ζ_i has distribution ρ_0 and is independent from the ζ_j , $\chi(N) \leq j \leq i-1$. This and a similar argument for $i < \chi(N)$ implies the $(\zeta_i)_{i \in \mathbb{Z}}$ are i.i.d. with distribution ρ_0 .

We will prove several properties of $(\zeta_i)_{i \in \mathbb{Z}}$ that we will use in the remainder of the proof. In order to do that, we need the following lemma of [17].

Lemma 7 (Lemma 1 of [17]). *There exist two constants $\tilde{c} = \tilde{c}(w) > 0$ and $\tilde{C} = \tilde{C}(w) < +\infty$ so that for any $n \in \mathbb{N}$,*

$$\mathbb{P}(\eta(n) = i | \eta(0) = 0) \leq \tilde{C} e^{-\tilde{c}|i|} \quad \text{and} \quad \sum_{i \in \mathbb{Z}} |\mathbb{P}(\eta(n) = i | \eta(0) = 0) - \rho_-(i)| \leq \tilde{C} e^{-\tilde{c}n}.$$

Firstly, we want to prove that our coupling is actually useful: that the ζ_i are close to the $\ell^\pm(T_N, i+1) - \ell^\pm(T_N, i)$. More precisely, we will show that except on an event of probability tending to 0, if $\ell^\pm(T_N, i)$ is large then $\zeta_i =$

$\eta_{i,\mp}(\ell^\pm(T_N, i)) + 1/2$, which (1) relates to $\ell^\pm(T_N, i+1) - \ell^\pm(T_N, i)$. We denote

$$(4) \quad \begin{aligned} \mathcal{B}_1^- &= \{\exists i \in \{-\lceil 2(|x| + 2\theta)N \rceil, \dots, \chi(N) - 1\}, \ell^+(T_N, i) \geq \lfloor N^{1/6} \rfloor \text{ and } \zeta_i \neq \eta_{i,-}(\ell^+(T_N, i)) + 1/2\}, \\ \mathcal{B}_1^+ &= \{\exists i \in \{\chi(N), \dots, \lceil 2(|x| + 2\theta)N \rceil\}, \ell^-(T_N, i) \geq \lfloor N^{1/6} \rfloor \text{ and } \zeta_i \neq \eta_{i,+}(\ell^-(T_N, i)) + 1/2\}. \end{aligned}$$

Lemma 7 will allow us to prove the following.

Lemma 8. $\mathbb{P}(\mathcal{B}_1^-)$ and $\mathbb{P}(\mathcal{B}_1^+)$ tend to 0 when $N \rightarrow +\infty$.

Proof. By definition, for any $i \in \{-\lceil 2(|x| + 2\theta)N \rceil, \dots, \chi(N) - 1\}$ we have $\zeta_i = \bar{\eta}_{i,-}(\ell^+(T_N, i) \vee \lfloor N^{1/6} \rfloor) + \frac{1}{2}$, which is $\bar{\eta}_{i,-}(\ell^+(T_N, i)) + \frac{1}{2}$ when $\ell^+(T_N, i) \geq \lfloor N^{1/6} \rfloor$. Now, $\bar{\eta}_{i,-} = \eta_{i,-}$ if $\bar{\eta}_{i,-}(\lfloor N^{1/6} \rfloor) = \eta_{i,-}(\lfloor N^{1/6} \rfloor)$, that is $r_i = \eta_{i,-}(\lfloor N^{1/6} \rfloor)$. We deduce $\mathbb{P}(\mathcal{B}_1^-) \leq \mathbb{P}(\exists i \in \{-\lceil 2(|x| + 2\theta)N \rceil, \dots, \chi(N) - 1\}, r_i \neq \eta_{i,-}(\lfloor N^{1/6} \rfloor))$. Now, for any $i < \chi(N)$, we have $\mathbb{P}(r_i \neq \eta_{i,-}(\lfloor N^{1/6} \rfloor))$ minimal, thus smaller than $\tilde{C}e^{-\tilde{c}\lfloor N^{1/6} \rfloor}$ by Lemma 7. Consequently, when N is large enough, $\mathbb{P}(\mathcal{B}_1^-) \leq 3(|x| + 2\theta)N\tilde{C}e^{-\tilde{c}\lfloor N^{1/6} \rfloor}$, which tends to 0 when $N \rightarrow +\infty$. The proof for $\mathbb{P}(\mathcal{B}_1^+)$ is the same. \square

Unfortunately, the previous lemma does not allow to control the local times when $\ell^\pm(T_N, i)$ is small. In order to do that, we show several additional properties. We have to control the probability of

$$\begin{aligned} \mathcal{B}_2 &= \{\exists i \in \{-\lceil 2(|x| + 2\theta)N \rceil, \dots, \lceil 2(|x| + 2\theta)N \rceil\}, |\zeta_i| \geq N^{1/16}\} \\ &\cup \{\exists i \in \{-\lceil 2(|x| + 2\theta)N \rceil, \dots, \chi(N) - 1\}, |\eta_{i,-}(\ell^+(T_N, i)) + 1/2| \geq N^{1/16}\} \\ &\cup \{\exists i \in \{\chi(N), \dots, \lceil 2(|x| + 2\theta)N \rceil\}, |\eta_{i,+}(\ell^-(T_N, i)) + 1/2| \geq N^{1/16}\}. \end{aligned}$$

Lemma 9. $\mathbb{P}(\mathcal{B}_2)$ tends to 0 when N tends to $+\infty$.

Proof. It is enough to find some constants $c > 0$ and $C < +\infty$ so that for any $i \in \{-\lceil 2(|x| + 2\theta)N \rceil, \dots, \lceil 2(|x| + 2\theta)N \rceil\}$ we have $\mathbb{P}(|\zeta_i| \geq N^{1/16}) \leq Ce^{-cN^{1/16}}$, for any $i \in \{-\lceil 2(|x| + 2\theta)N \rceil, \dots, \chi(N) - 1\}$ we have $\mathbb{P}(|\eta_{i,-}(\ell^+(T_N, i)) + 1/2| \geq N^{1/16}) \leq Ce^{-cN^{1/16}}$, and for all $i \in \{\chi(N), \dots, \lceil 2(|x| + 2\theta)N \rceil\}$ we have $\mathbb{P}(|\eta_{i,+}(\ell^-(T_N, i)) + 1/2| \geq N^{1/16}) \leq Ce^{-cN^{1/16}}$. For all $i \in \mathbb{Z}$, ζ_i has distribution ρ_0 , which has exponential tails, hence there exists constants $c' = c'(w) > 0$ and $C' = C'(w) < +\infty$ so that for $i \in \{-\lceil 2(|x| + 2\theta)N \rceil, \dots, \lceil 2(|x| + 2\theta)N \rceil\}$ we have $\mathbb{P}(|\zeta_i| \geq N^{1/16}) \leq C'e^{-c'N^{1/16}}$. We now consider $i \in \{-\lceil 2(|x| + 2\theta)N \rceil, \dots, \chi(N) - 1\}$ and $\mathbb{P}(|\eta_{i,-}(\ell^+(T_N, i)) + 1/2| \geq N^{1/16})$ (the $\mathbb{P}(|\eta_{i,+}(\ell^-(T_N, i)) + 1/2| \geq N^{1/16})$ can be dealt with in the same way). Equation (1) implies $\ell^+(T_N, i)$ depends only on the $\eta_{j,-}$ for $j > i$, hence is independent of $\eta_{i,-}$. This implies $\mathbb{P}(|\eta_{i,-}(\ell^+(T_N, i)) + 1/2| \geq N^{1/16}) = \sum_{k \in \mathbb{N}} \mathbb{P}(|\eta_{i,-}(k) + 1/2| \geq N^{1/16}) \mathbb{P}(\ell^+(T_N, i) = k)$. Therefore the first part of Lemma 7 implies $\mathbb{P}(|\eta_{i,-}(\ell^+(T_N, i)) + 1/2| \geq N^{1/16}) \leq \sum_{k \in \mathbb{N}} \frac{2\tilde{C}e^{\tilde{c}/2}}{1-e^{-\tilde{c}}} e^{-\tilde{c}N^{1/16}} \mathbb{P}(\ell^+(T_N, i) = k) = \frac{2\tilde{C}e^{\tilde{c}/2}}{1-e^{-\tilde{c}}} e^{-\tilde{c}N^{1/16}}$, which is enough. \square

We will also need the following, which is a rather standard result of large deviations.

Lemma 10. For any $\alpha > 0$, $\varepsilon > 0$, $\mathbb{P}(\max_{0 \leq i_1 \leq i_2 \leq \lfloor N^\alpha \rfloor} |\sum_{i=i_1}^{i_2} \zeta_i| \geq N^{\alpha/2+\varepsilon})$ tends to 0 when $N \rightarrow +\infty$.

Proof. Let $0 \leq i_1 \leq i_2 \leq \lceil N^\alpha \rceil$, let us study $\mathbb{P}(|\sum_{i=i_1}^{i_2} \zeta_i| \geq N^{\alpha/2+\varepsilon})$. We know the $\zeta_i, i \in \mathbb{Z}$ are i.i.d. with distribution ρ_0 , and it can be checked that ρ_0 is symmetric with respect to 0, so from that and the Markov inequality we get

$$(5) \quad \begin{aligned} \mathbb{P}\left(\left|\sum_{i=i_1}^{i_2} \zeta_i\right| \geq N^{\alpha/2+\varepsilon}\right) &\leq 2\mathbb{P}\left(\sum_{i=i_1}^{i_2} \zeta_i \geq N^{\alpha/2+\varepsilon}\right) = 2\mathbb{P}\left(\exp\left(\frac{1}{N^{\alpha/2}} \sum_{i=i_1}^{i_2} \zeta_i\right) \geq \exp(N^\varepsilon)\right) \\ &\leq 2e^{-N^\varepsilon} \mathbb{E}\left(\exp\left(\frac{1}{N^{\alpha/2}} \sum_{i=i_1}^{i_2} \zeta_i\right)\right) \leq 2e^{-N^\varepsilon} \prod_{i=i_1}^{i_2} \mathbb{E}\left(\exp\left(\frac{1}{N^{\alpha/2}} \zeta_i\right)\right). \end{aligned}$$

Now, if ζ has distribution ρ_0 , we can write $\exp(\frac{1}{N^{\alpha/2}}\zeta) = 1 + \frac{1}{N^{\alpha/2}}\zeta + \frac{1}{2}(\frac{1}{N^{\alpha/2}}\zeta)^2 \exp(\frac{1}{N^{\alpha/2}}\zeta')$ with $|\zeta'| \leq |\zeta|$. Since ρ_0 is symmetric with respect to 0, we have $\mathbb{E}(\zeta) = 0$, therefore

$$\mathbb{E}\left(\exp\left(\frac{1}{N^{\alpha/2}}\zeta\right)\right) = 1 + \mathbb{E}\left(\frac{1}{2}\left(\frac{1}{N^{\alpha/2}}\zeta\right)^2 \exp\left(\frac{1}{N^{\alpha/2}}\zeta'\right)\right) \leq 1 + \frac{1}{2N^\alpha} \mathbb{E}\left(\zeta^2 \exp\left(\frac{1}{N^{\alpha/2}}|\zeta|\right)\right).$$

Moreover, ρ_0 has exponential tails, hence there exists constants $C < +\infty$ and $c > 0$ so that $\mathbb{E}(\zeta^2 e^{c|\zeta|}) \leq C$. When N is large enough, $\frac{1}{N^{\alpha/2}} \leq c$, therefore $\mathbb{E}(\exp(\frac{1}{N^{\alpha/2}}\zeta)) \leq 1 + \frac{C}{2N^\alpha} \leq \exp(\frac{C}{2N^\alpha})$. Together with (5), this yields $\mathbb{P}(|\sum_{i=i_1}^{i_2} \zeta_i| \geq N^{\alpha/2+\varepsilon}) \leq 2e^{-N^\varepsilon} e^{(i_2-i_1+1)\frac{C}{2N^\alpha}} \leq 2e^{-N^\varepsilon} e^{(\lceil N^\alpha \rceil+1)\frac{C}{2N^\alpha}} \leq 2e^C e^{-N^\varepsilon}$ when N is large enough. We deduce that when N is large enough, $\mathbb{P}(\max_{0 \leq i_1 \leq i_2 \leq \lceil N^\alpha \rceil} |\sum_{i=i_1}^{i_2} \zeta_i| \geq N^{\alpha/2+\varepsilon}) \leq (\lceil N^\alpha \rceil + 1)^2 2e^C e^{-N^\varepsilon}$, which tends to 0 when N tends to $+\infty$. \square

We also prove an immediate application of Lemma 10, which we will use several times. If we define

$$\begin{aligned} \mathcal{B}_3^- &= \left\{ \max_{-\lfloor(|x|+2\theta)N\rfloor - N^{3/4} \leq i_1 \leq i_2 \leq -\lfloor(|x|+2\theta)N\rfloor + N^{3/4}} \left| \sum_{i=i_1}^{i_2} \zeta_i \right| \geq N^{19/48} \right\}, \\ \mathcal{B}_3^+ &= \left\{ \max_{\lfloor(|x|+2\theta)N\rfloor - N^{3/4} \leq i_1 \leq i_2 \leq \lfloor(|x|+2\theta)N\rfloor + N^{3/4}} \left| \sum_{i=i_1}^{i_2} \zeta_i \right| \geq N^{19/48} \right\}, \end{aligned}$$

we have the following lemma.

Lemma 11. $\mathbb{P}(\mathcal{B}_3^-)$ and $\mathbb{P}(\mathcal{B}_3^+)$ tend to 0 when N tends to $+\infty$.

Proof. Since the $(\zeta_i)_{i \in \mathbb{Z}}$ are i.i.d., $\mathbb{P}(\mathcal{B}_3^+) = \mathbb{P}(\mathcal{B}_3^-) = \mathbb{P}(\max_{0 \leq i_1 \leq i_2 \leq 2\lceil N^{3/4} \rceil} |\sum_{i=i_1}^{i_2} \zeta_i| \geq N^{19/48})$, which is smaller than $\mathbb{P}(\max_{0 \leq i_1 \leq i_2 \leq \lceil N^{37/48} \rceil} |\sum_{i=i_1}^{i_2} \zeta_i| \geq N^{19/48})$ when N is large enough. Moreover, Lemma 10, used with $\alpha = 37/48$ and $\varepsilon = 1/96$, yields that the latter probability tends to 0 when N tends to $+\infty$. \square

3. WHERE THE LOCAL TIMES APPROACH 0

The aim of this section is twofold. Firstly, we need to control the place where $\ell^-(T_N, i)$ hits 0 when i is at the right of 0, as well as the place where $\ell^+(T_N, i)$ hits 0 when i is at the left of 0. Secondly, we have to show that even when $\ell^\pm(T_N, i)$ is close to 0, the local times do not stray too far away from the coupling. For any $N \in \mathbb{N}$, we denote $I^+ = \inf\{i \geq \chi(N) \mid \ell^-(T_N, i) = 0\}$ and $I^- = \sup\{i < \chi(N) \mid \ell^+(T_N, i) = 0\}$. We notice that $\ell^+(T_N, I^-) = 0$, and from the definition of T_N we have $\ell^+(T_N, i) > 0$ for any $0 \leq i \leq \chi(N) - 1$, hence $I^- < 0$. We first state an elementary result that we will use many times in this work.

Lemma 12. *For any $i \geq I^+$ or $i \leq I^-$ we have $\ell^\pm(T_N, i) = 0$.*

Proof. Since $\ell^+(T_N, I^-) = 0$ and the random walk is at $\lfloor Nx \rfloor \mathbf{1} > 0$ at time T_N , the random walk did not reach I^- before time T_N , thus $\ell^\pm(T_N, i) = 0$ for any $i \leq I^-$. Moreover, $\ell^-(T_N, \chi(N)) > 0$ by definition of T_N , hence $I^+ > \chi(N)$ thus $X_{T_N} < I^+$ hence $\ell^-(T_N, I^+) = 0$ implies the random walk did not reach I^+ before time T_N , thus $\ell^\pm(T_N, i) = 0$ for any $i \geq I^+$. \square

We will also need the auxiliary random variables $\tilde{I}^+ = \inf\{i \geq \chi(N) \mid \ell^-(T_N, i) \leq \lfloor N^{1/6} \rfloor\}$ and $\tilde{I}^- = \sup\{i < \chi(N) \mid \ell^+(T_N, i) \leq \lfloor N^{1/6} \rfloor\}$.

3.1. Place where we hit 0. We have the following result of control on I^+ and I^- .

Lemma 13. *For any $\delta > 0$, $\mathbb{P}(|I^- + (|x| + 2\theta)N| \geq N^{\delta+1/2})$ and $\mathbb{P}(|I^+ - (|x| + 2\theta)N| \geq N^{\delta+1/2})$ tend to 0 when N tends to $+\infty$.*

Proof. The idea is to control the fluctuations of the local times around their deterministic limit: as long as $\ell^\pm(T_N, i)$ is large, the $\ell^\pm(T_N, i+1) - \ell^\pm(T_N, i)$ will be close to the i.i.d. random variables of the coupling, so the fluctuations of $\ell^\pm(T_N, i)$ around its deterministic limit are bounded and $\ell^\pm(T_N, i)$ can be small only when the deterministic limit is small, that is around $-(|x| + 2\theta)N$ and $(|x| + 2\theta)N$. We only spell out the proof for I^- , as the argument for I^+ is similar. The fact that $\mathbb{P}(I^- + (|x| + 2\theta)N \leq -N^{\delta+1/2})$ tends to 0 when N tends to $+\infty$ comes from inequalities (51) and (53) of [17], so we only have to prove that $\mathbb{P}(I^- + (|x| + 2\theta)N \geq N^{\delta+1/2})$ tends to 0 when N tends to $+\infty$. Since $I^- \leq \tilde{I}^-$, it is enough to prove that $\mathbb{P}(\tilde{I}^- + (|x| + 2\theta)N \geq N^{\delta+1/2})$ tends to 0 when N tends to $+\infty$. Since by Lemma 8 we have that $\mathbb{P}(\mathcal{B}_1^-)$ tends to 0 when N tends to $+\infty$, it is enough to prove $\mathbb{P}(\tilde{I}^- + (|x| + 2\theta)N \geq N^{\delta+1/2}, (\mathcal{B}_1^-)^c)$ tends to 0 when N tends to $+\infty$. We now assume N is large enough, $\tilde{I}^- + (|x| + 2\theta)N \geq N^{\delta+1/2}$ and $(\mathcal{B}_1^-)^c$. Then there exists $i \in \{[-(|x| + 2\theta)N + N^{\delta+1/2}], \dots, \chi(N) - 1\}$ so that $\ell^+(T_N, i) \leq \lfloor N^{1/6} \rfloor$ and $\ell^+(T_N, j) > \lfloor N^{1/6} \rfloor$ for all $j \in \{i+1, \dots, \chi(N) - 1\}$. Thus, by (1) we get $\lfloor N\theta \rfloor + \sum_{j=i+1}^{\chi(N)-1} (\eta_{j,-}(\ell^+(T_N, j)) + \mathbf{1}_{\{j>0\}}) = \ell^+(T_N, i) \leq \lfloor N^{1/6} \rfloor$. Furthermore, for all $j \in \{i+1, \dots, \chi(N) - 1\}$, since $(\mathcal{B}_1^-)^c$ occurs and $\ell^+(T_N, j) > \lfloor N^{1/6} \rfloor$, we have $\eta_{j,-}(\ell^+(T_N, j)) + 1/2 = \zeta_j$. We deduce $\lfloor N\theta \rfloor + \sum_{j=i+1}^{\chi(N)-1} (\zeta_j + (\mathbf{1}_{\{j>0\}} - \mathbf{1}_{\{j \leq 0\}})/2) \leq \lfloor N^{1/6} \rfloor$, thus $\sum_{j=i+1}^{\chi(N)-1} \zeta_j + \lfloor N\theta \rfloor + \sum_{j=i+1}^{\chi(N)-1} (\mathbf{1}_{\{j>0\}} - \mathbf{1}_{\{j \leq 0\}})/2 \leq \lfloor N^{1/6} \rfloor$. Moreover, since $i \in \{[-(|x| + 2\theta)N + N^{\delta+1/2}], \dots, \chi(N) - 1\}$ we have $\sum_{j=i+1}^{\chi(N)-1} (\mathbf{1}_{\{j>0\}} - \mathbf{1}_{\{j \leq 0\}})/2 = \frac{1}{2}(\chi(N) - 1 + i) \geq \frac{1}{2}(Nx - 2 - (|x| + 2\theta)N + N^{\delta+1/2}) = -\theta N + \frac{1}{2}N^{\delta+1/2} - 1$. This yields $\sum_{j=i+1}^{\chi(N)-1} \zeta_j + \lfloor N\theta \rfloor - \theta N + \frac{1}{2}N^{\delta+1/2} - 1 \leq \lfloor N^{1/6} \rfloor$, hence $\sum_{j=i+1}^{\chi(N)-1} \zeta_j \leq -\frac{1}{2}N^{\delta+1/2} + \lfloor N^{1/6} \rfloor + 2 \leq -N^{(1+\delta)/2}$ since N is large enough. Consequently, when N is large enough, $\mathbb{P}(\tilde{I}^- + (|x| + 2\theta)N \geq N^{\delta+1/2}, (\mathcal{B}_1^-)^c) \leq \mathbb{P}(\exists i \in \{[-(|x| + 2\theta)N + N^{\delta+1/2}], \dots, \chi(N) - 1\}, \sum_{j=i+1}^{\chi(N)-1} \zeta_j \leq -N^{(1+\delta)/2})$. Since the ζ_i , $i \in \mathbb{Z}$ are i.i.d., when N is large enough this yields $\mathbb{P}(\tilde{I}^- + (|x| + 2\theta)N \geq N^{\delta+1/2}, (\mathcal{B}_1^-)^c) \leq \mathbb{P}(\max_{0 \leq i_1 \leq i_2 \leq \lfloor N^{1+\delta/2} \rfloor} |\sum_{i=i_1}^{i_2} \zeta_i| \geq N^{(1+\delta)/2})$, which tends to 0 when N tends to $+\infty$ by Lemma 10 (applied with $\alpha = 1 + \delta/2$ and $\varepsilon = \delta/4$). This shows that $\mathbb{P}(I^- + (|x| + 2\theta)N \geq N^{\delta+1/2})$ converges to 0 when N tends to $+\infty$, which ends the proof of Lemma 13. \square

3.2. Control of low local times. We have to show that even when $\ell^\pm(T_N, i)$ is small, the local times are not too far from the random variables of the coupling. In order to do that, we first prove that the window where $\ell^\pm(T_N, i)$ is small but not zero, that is between \tilde{I}^+ and I^+ and between I^- and \tilde{I}^- , is small. Afterwards, we will give bounds on what happens inside. We begin by showing the following easy result.

Lemma 14. *$\mathbb{P}(\tilde{I}^- \geq 0)$ tends to 0 when $N \rightarrow +\infty$.*

Proof. Let N be large enough. If $\tilde{I}^- \geq 0$, there exists $i \in \{0, \dots, \lfloor Nx \rfloor\}$ so that $\ell^+(T_N, i) \leq \lfloor N^{1/6} \rfloor$. Since N is large enough, this implies $\ell^+(T_N, i) \leq N\theta/2$, therefore $\sup_{y \in \mathbb{R}} |\frac{1}{N} \ell^+(T_N, \lfloor Ny \rfloor) - (\frac{|x|-|y|}{2} + \theta)_+| \geq \theta/2$. Moreover, by Theorem 1 of [17], $\sup_{y \in \mathbb{R}} |\frac{1}{N} \ell^+(T_N, \lfloor Ny \rfloor) - (\frac{|x|-|y|}{2} + \theta)_+|$ converges in probability to 0 when N tends to $+\infty$, hence we deduce that $\mathbb{P}(\sup_{y \in \mathbb{R}} |\frac{1}{N} \ell^+(T_N, \lfloor Ny \rfloor) - (\frac{|x|-|y|}{2} + \theta)_+| \geq \theta/2)$ tends to 0 when $N \rightarrow +\infty$. Therefore $\mathbb{P}(\tilde{I}^- \geq 0)$ tends to 0 when $N \rightarrow +\infty$. \square

In order to control I^+ , I^- , \tilde{I}^+ and \tilde{I}^- , we will use the fact the local times behave as the Markov chain L from [17], defined as follows. We consider i.i.d. copies of the Markov chain η starting at 0, called $(\eta_m)_{m \in \mathbb{N}}$. For any $m \in \mathbb{N}$, we then set $L(m+1) = L(m) + \eta_m(L(m))$. We denote $\tau = \inf\{m \in \mathbb{N} \mid L(m) \leq 0\}$. The following was proven in [17].

Lemma 15 (Lemma 2 of [17]). *There exists a constant $K < +\infty$ so that for any $k \in \mathbb{N}$ we have $\mathbb{E}(\tau \mid L(0) = k) \leq 3k + K$.*

Since the local times will behave as L , Lemma 15 implies that if the local time starts small, then the time at which it reaches 0 has small expectation hence is not too large. This will help us to prove the following control on the window where $\ell^\pm(T_N, i)$ is small but not zero.

Lemma 16. $\mathbb{P}(I^+ - \tilde{I}^+ \geq N^{1/4})$ and $\mathbb{P}(\tilde{I}^- - I^- \geq N^{1/4})$ tend to 0 when $N \rightarrow +\infty$.

Proof. Let N be large enough. We deal only with $\mathbb{P}(\tilde{I}^- - I^- \geq N^{1/4})$, since $\mathbb{P}(I^+ - \tilde{I}^+ \geq N^{1/4})$ can be dealt with in the same way and with simpler arguments. Thanks to Lemma 14, it is enough to prove that $\mathbb{P}(\tilde{I}^- - I^- \geq N^{1/4}, \tilde{I}^- < 0)$ tends to 0 when $N \rightarrow +\infty$. Moreover, if $\tilde{I}^- < 0$, thanks to (1), for any $i < \tilde{I}^-$ we get $\ell^+(T_N, i) = \ell^+(T_N, \tilde{I}^-) + \sum_{j=i+1}^{\tilde{I}^-} \eta_{j,-}(\ell^+(T_N, j))$, which allows to prove that $(\ell^+(T_N, \tilde{I}^- - i))_{i \in \mathbb{N}}$ is a Markov chain with the transition probabilities of L . Therefore we have (recalling the notations just before Lemma 15)

$$\begin{aligned} \mathbb{P}(\tilde{I}^- - I^- \geq N^{1/4}, \tilde{I}^- < 0) &= \sum_{k=0}^{\lfloor N^{1/6} \rfloor} \mathbb{P}(\tilde{I}^- - I^- \geq N^{1/4}, \tilde{I}^- < 0 \mid \ell^+(T_N, \tilde{I}^-) = k) \mathbb{P}(\ell^+(T_N, \tilde{I}^-) = k) \\ &= \sum_{k=0}^{\lfloor N^{1/6} \rfloor} \mathbb{P}(\tau \geq N^{1/4} \mid L(0) = k) \mathbb{P}(\ell^+(T_N, \tilde{I}^-) = k) \leq \sum_{k=0}^{\lfloor N^{1/6} \rfloor} \frac{1}{N^{1/4}} \mathbb{E}(\tau \mid L(0) = k) \mathbb{P}(\ell^+(T_N, \tilde{I}^-) = k). \end{aligned}$$

By Lemma 15 we deduce

$$\mathbb{P}(\tilde{I}^- - I^- \geq N^{1/4}, \tilde{I}^- < 0) \leq \frac{1}{N^{1/4}} \sum_{k=0}^{\lfloor N^{1/6} \rfloor} (3k + K) \mathbb{P}(\ell^+(T_N, \tilde{I}^-) = k) \leq \frac{3N^{1/6} + K}{N^{1/4}} \leq 4N^{-1/12}$$

since N is large enough, hence $\mathbb{P}(\tilde{I}^- - I^- \geq N^{1/4}, \tilde{I}^- < 0)$ tends to 0 when $N \rightarrow +\infty$, which ends the proof. \square

We are now going to prove that even when $\ell^\pm(T_N, i)$ is small, the local times are not too far from the random variables of the coupling. More precisely, for any $n \in \mathbb{N}$, we define the following events.

$$\mathcal{B}_4^- = \left\{ \exists i \in \{I^-, \dots, \chi(N) - 1\}, \left| \sum_{j=i+1}^{\chi(N)-1} (\eta_{j,-}(\ell^+(T_N, j)) + 1/2) - \sum_{j=i+1}^{\chi(N)-1} \zeta_j \right| \geq N^{1/3} \right\},$$

$$\mathcal{B}_4^+ = \left\{ \exists i \in \{\chi(N), \dots, I^+\}, \left| \sum_{j=\chi(N)}^{i-1} (\eta_{j,+}(\ell^-(T_N, j)) + 1/2) - \sum_{j=\chi(N)}^{i-1} \zeta_j \right| \geq N^{1/3} \right\}.$$

Lemma 17. $\mathbb{P}(\mathcal{B}_4^-)$ and $\mathbb{P}(\mathcal{B}_4^+)$ tend to 0 when N tend to $+\infty$.

Proof. The idea of the argument is that when $\ell^\pm(T_N, i)$ is large, $\eta_{i,\mp}(\ell^\pm(T_N, i)) + 1/2 = \zeta_i$ thanks to Lemma 8, that the window where $\ell^\pm(T_N, i)$ is small is bounded by Lemma 16, and that inside this window the $\eta_{i,\mp}(\ell^\pm(T_N, i)) + 1/2$, ζ_i are also bounded by Lemma 9. We only spell out the proof for $\mathbb{P}(\mathcal{B}_4^-)$, since the proof for $\mathbb{P}(\mathcal{B}_4^+)$ is the same. By Lemma 13, we have that $\mathbb{P}(I^- \leq -2(|x| + \theta)N)$ tends to 0 when N tends to $+\infty$. Furthermore, Lemma 16 implies that $\mathbb{P}(\tilde{I}^- - I^- \geq N^{1/4})$ tends to 0 when N tends to $+\infty$. In addition, by Lemmas 8 and 9 we have that $\mathbb{P}(\mathcal{B}_1^-)$ and $\mathbb{P}(\mathcal{B}_2)$ tend to 0 when N tends to $+\infty$. Consequently, it is enough to prove that for N large enough, if $(\mathcal{B}_1^-)^c$, $(\mathcal{B}_2)^c$ occur, if $\tilde{I}^- - I^- < N^{1/4}$ and if $I^- > -2(|x| + \theta)N$, then $(\mathcal{B}_4^-)^c$ occurs. We assume $(\mathcal{B}_1^-)^c$, $(\mathcal{B}_2)^c$, $\tilde{I}^- - I^- < N^{1/4}$ and $I^- > -2(|x| + \theta)N$. Since $(\mathcal{B}_1^-)^c$ occurs and $\tilde{I}^- \geq I^- > -2(|x| + \theta)N$, we get $\zeta_i = \eta_{j,-}(\ell^+(T_N, j)) + 1/2$ for any $i \in \{\tilde{I}^- + 1, \dots, \chi(N) - 1\}$. Therefore, if $i \in \{\tilde{I}^-, \dots, \chi(N) - 1\}$ we get $\sum_{j=i+1}^{\chi(N)-1} (\eta_{j,-}(\ell^+(T_N, j)) + 1/2) - \sum_{j=i+1}^{\chi(N)-1} \zeta_j = 0$, and for $i \in \{I^-, \dots, \tilde{I}^- - 1\}$ we have

$$\begin{aligned} & \left| \sum_{j=i+1}^{\chi(N)-1} (\eta_{j,-}(\ell^+(T_N, j)) + 1/2) - \sum_{j=i+1}^{\chi(N)-1} \zeta_j \right| = \left| \sum_{j=i+1}^{\tilde{I}^-} (\eta_{j,-}(\ell^+(T_N, j)) + 1/2) - \sum_{j=i+1}^{\tilde{I}^-} \zeta_j \right| \\ & \leq \sum_{j=i+1}^{\tilde{I}^-} (|\eta_{j,-}(\ell^+(T_N, j)) + 1/2| + |\zeta_j|) \leq 2(\tilde{I}^- - I^-)N^{1/16} \end{aligned}$$

since $(\mathcal{B}_2)^c$ occurs, $i + 1 \geq I^- > -2(|x| + \theta)N$ and by definition $\tilde{I}^- \leq \chi(N) - 1 \leq 2(|x| + \theta)N$. Moreover, we assumed $\tilde{I}^- - I^- < N^{1/4}$, which implies $|\sum_{j=i+1}^{\chi(N)-1} (\eta_{j,-}(\ell^+(T_N, j)) + 1/2) - \sum_{j=i+1}^{\chi(N)-1} \zeta_j| \leq 2N^{1/4}N^{1/16} = 2N^{5/16} < N^{1/3}$ when N is large enough. Consequently, for any $i \in \{I^-, \dots, \chi(N) - 1\}$ we have $|\sum_{j=i+1}^{\chi(N)-1} (\eta_{j,-}(\ell^+(T_N, j)) + 1/2) - \sum_{j=i+1}^{\chi(N)-1} \zeta_j| < N^{1/3}$, therefore $(\mathcal{B}_4^-)^c$ occurs, which ends the proof. \square

4. SKOROHOD M_1 DISTANCE

The goal of this section is to prove that when N is large, Y_N^\pm is close in the Skorohod M_1 distance to the function Y_N defined as follows. For any N large enough, for $y \in \mathbb{R}$, we set $Y_N(y) = \frac{1}{\sqrt{N}} \sum_{i=\lfloor Ny \rfloor + 1}^{\chi(N)-1} \zeta_i$ if $y \in [-|x| - 2\theta, \frac{\chi(N)}{N})$, $Y_N(y) = \frac{1}{\sqrt{N}} \sum_{i=\chi(N)}^{\lfloor Ny \rfloor - 1} \zeta_i$ if $y \in [\frac{\chi(N)}{N}, |x| + 2\theta)$, and $Y_N(y) = 0$ otherwise. We want to prove the following proposition.

Proposition 18. $\mathbb{P}(d_{M_1}(Y_N^\pm, Y_N) > 3N^{-1/12})$ tends to 0 when N tends to $+\infty$.

If we denote

$$\mathcal{B} = \mathcal{B}_1^- \cup \mathcal{B}_1^+ \cup \mathcal{B}_2 \cup \mathcal{B}_3^- \cup \mathcal{B}_3^+ \cup \mathcal{B}_4^- \cup \mathcal{B}_4^+ \cup \{|I^- + (|x| + 2\theta)N| \geq N^{3/4}\} \cup \{|I^+ - (|x| + 2\theta)N| \geq N^{3/4}\},$$

it will be enough to prove the following proposition.

Proposition 19. When N is large enough, for all $a > 0$ with $|(|x| + 2\theta) - a| > N^{-1/8}$, we have that $\mathcal{B}^c \subset \{d_{M_1,a}(Y_N^\pm|_{[-a,a]}, Y_N|_{[-a,a]}) \leq 2N^{-1/12}\}$.

Proof of Proposition 18 given Proposition 19. We assume Proposition 19 holds. Then, when N is large enough, if \mathcal{B}^c occurs, for all $a > 0$ with $||x| + 2\theta) - a| > N^{-1/8}$ we have $d_{M_1,a}(Y_N^\pm|_{[-a,a]}, Y_N|_{[-a,a]}) \leq 2N^{-1/12}$, which yields $d_{M_1}(Y_N^\pm, Y_N) = \int_0^{+\infty} e^{-a}(d_{M_1,a}(Y_N^\pm|_{[-a,a]}, Y_N|_{[-a,a]}) \wedge 1) da \leq \int_0^{+\infty} e^{-a} 2N^{-1/12} da + 2N^{-1/8} = 2N^{-1/12} + 2N^{-1/8} \leq 3N^{-1/12}$. This implies $\mathbb{P}(d_{M_1}(Y_N^\pm, Y_N) > 3N^{-1/12}) \leq \mathbb{P}(\mathcal{B})$ when N is large enough. In addition,

$$\begin{aligned} \mathbb{P}(\mathcal{B}) &\leq \mathbb{P}(\mathcal{B}_1^-) + \mathbb{P}(\mathcal{B}_1^+) + \mathbb{P}(\mathcal{B}_2) + \mathbb{P}(\mathcal{B}_3^-) + \mathbb{P}(\mathcal{B}_3^+) + \mathbb{P}(\mathcal{B}_4^-) + \mathbb{P}(\mathcal{B}_4^+) \\ &\quad + \mathbb{P}(|I^- + (|x| + 2\theta)N| \geq N^{3/4}) + \mathbb{P}(|I^+ - (|x| + 2\theta)N| \geq N^{3/4}). \end{aligned}$$

Applying Lemmas 8, 9, 11, 13 and 17 implies $\mathbb{P}(\mathcal{B})$ tends to 0 when N tends to $+\infty$, hence $\mathbb{P}(d_{M_1}(Y_N^\pm, Y_N) > 3N^{-1/12})$ tends to 0 when N tends to $+\infty$, which is Proposition 18. \square

The remainder of this section is devoted to the proof of Proposition 19. The first thing we do is showing that between $\frac{-(|x|+2\theta)N \vee I^-}{N}$ and $\frac{(|x|+2\theta)N \wedge I^+}{N}$, the functions Y_N^\pm and Y_N are close in uniform distance, which is the following lemma.

Lemma 20. *When N is large enough, if $(\mathcal{B}_2)^c$, $(\mathcal{B}_4^-)^c$ and $(\mathcal{B}_4^+)^c$ occur, then if $I^+ < (|x| + 2\theta)N$ then for any $y \in [\frac{-(|x|+2\theta)N \vee I^-}{N}, \frac{(|x|+2\theta)N \wedge I^+}{N}]$ we have $|Y_N^\pm(y) - Y_N(y)| \leq N^{-1/12}$, while if $I^+ \geq (|x| + 2\theta)N$ we have $|Y_N^\pm(y) - Y_N(y)| \leq N^{-1/12}$ for $y \in [\frac{-(|x|+2\theta)N \vee I^-}{N}, \frac{(|x|+2\theta)N \wedge I^+}{N}]$.*

Proof of Lemma 20. Writing down the proof is only a technical matter, as the meaning of $(\mathcal{B}_4^\pm)^c$ is that the local times are close to the process formed from the random variables of the coupling. $(\mathcal{B}_2)^c$ is there to ensure that the difference terms that appear will be small. We only spell out the proof for Y^- , as the proof for Y^+ is similar. We assume $(\mathcal{B}_2)^c$, $(\mathcal{B}_4^-)^c$ and $(\mathcal{B}_4^+)^c$. Then if $y \in [\frac{\chi(N)}{N}, \frac{(|x|+2\theta)N \wedge I^+}{N}]$ (if $I^+ \geq (|x| + 2\theta)N$ we exclude the case $y = \frac{(|x|+2\theta)N \wedge I^+}{N}$) we have $y \in [\frac{\chi(N)}{N}, |x| + 2\theta)$, so $|Y_N^-(y) - Y_N(y)| = \frac{1}{\sqrt{N}} |\ell^-(T_N, \lfloor Ny \rfloor) - N(\frac{|x|-|y|}{2} + \theta)_+ - \sum_{i=\chi(N)}^{\lfloor Ny \rfloor - 1} \zeta_i|$, thus by (1) we obtain the following:

$$\begin{aligned} |Y_N^-(y) - Y_N(y)| &= \frac{1}{\sqrt{N}} \left| \lfloor N\theta \rfloor - \mathbf{1}_{\{\ell=+\}} + \sum_{i=\chi(N)}^{\lfloor Ny \rfloor - 1} \eta_{i,+}(\ell^-(T_N, i)) - N \left(\frac{|x|-|y|}{2} + \theta \right)_+ - \sum_{i=\chi(N)}^{\lfloor Ny \rfloor - 1} \zeta_i \right| \\ &\leq \frac{1}{\sqrt{N}} \left| \sum_{i=\chi(N)}^{\lfloor Ny \rfloor - 1} \eta_{i,+}(\ell^-(T_N, i)) + \frac{\lfloor Ny \rfloor - \chi(N)}{2} - \sum_{i=\chi(N)}^{\lfloor Ny \rfloor - 1} \zeta_i \right| + \frac{3}{\sqrt{N}} \\ &= \frac{1}{\sqrt{N}} \left| \sum_{i=\chi(N)}^{\lfloor Ny \rfloor - 1} (\eta_{i,+}(\ell^-(T_N, i)) + 1/2) - \sum_{i=\chi(N)}^{\lfloor Ny \rfloor - 1} \zeta_i \right| + \frac{3}{\sqrt{N}}. \end{aligned}$$

Now, $y \in [\frac{\chi(N)}{N}, \frac{(|x|+2\theta)N \wedge I^+}{N}]$ implies $\lfloor Ny \rfloor \in \{\chi(N), \dots, I^+\}$, thus $(\mathcal{B}_4^+)^c$ yields $|Y_N^-(y) - Y_N(y)| \leq \frac{1}{\sqrt{N}} N^{1/3} + \frac{3}{\sqrt{N}} \leq N^{-1/12}$ when N is large enough. We now consider the case $y \in [\frac{-(|x|+2\theta)N \vee I^-}{N}, \frac{\chi(N)}{N}]$. Then $y \in [-|x| - 2\theta, \frac{\chi(N)}{N}]$, hence $|Y_N^-(y) - Y_N(y)| = \frac{1}{\sqrt{N}} |\ell^-(T_N, \lfloor Ny \rfloor) - N(\frac{|x|-|y|}{2} + \theta)_+ - \sum_{i=\lfloor Ny \rfloor + 1}^{\chi(N) - 1} \zeta_i|$. Now, (2) yields $|\ell^-(T_N, \lfloor Ny \rfloor) -$

$\ell^+(T_N, \lfloor Ny \rfloor) = |\eta_{\lfloor Ny \rfloor, -}(\ell^+(T_N, \lfloor Ny \rfloor))|$, which is smaller than $N^{1/16} + 1/2$ thanks to $(\mathcal{B}_2)^c$. We deduce that $|Y_N^-(y) - Y_N(y)| \leq \frac{1}{\sqrt{N}} |\ell^+(T_N, \lfloor Ny \rfloor) - N(\frac{|x|-|y|}{2} + \theta)_+ - \sum_{i=\lfloor Ny \rfloor+1}^{\chi(N)-1} \zeta_i| + \frac{N^{1/16}+1/2}{\sqrt{N}}$, thus (1) implies

$$\begin{aligned} |Y_N^-(y) - Y_N(y)| &\leq \frac{1}{\sqrt{N}} \left| \lfloor N\theta \rfloor + \sum_{i=\lfloor Ny \rfloor+1}^{\chi(N)-1} (\eta_{i,-}(\ell^+(T_N, i)) + \mathbf{1}_{\{i>0\}}) - N \left(\frac{|x|-|y|}{2} + \theta \right)_+ - \sum_{i=\lfloor Ny \rfloor+1}^{\chi(N)-1} \zeta_i \right| + \frac{N^{1/16} + 1/2}{\sqrt{N}} \\ &\leq \frac{1}{\sqrt{N}} \left| \sum_{i=\lfloor Ny \rfloor+1}^{\chi(N)-1} (\eta_{i,-}(\ell^+(T_N, i)) + \mathbf{1}_{\{i>0\}}) + \frac{\lfloor Ny \rfloor + 1 - \chi(N)}{2} - \sum_{i=\lfloor Ny \rfloor+1}^{\chi(N)-1} \zeta_i \right| + \frac{N^{1/16} + 3}{\sqrt{N}} \\ &\leq \frac{1}{\sqrt{N}} \left| \sum_{i=\lfloor Ny \rfloor+1}^{\chi(N)-1} (\eta_{i,-}(\ell^+(T_N, i)) + 1/2) - \sum_{i=\lfloor Ny \rfloor+1}^{\chi(N)-1} \zeta_i \right| + \frac{N^{1/16} + 3}{\sqrt{N}}. \end{aligned}$$

Furthermore, $y \in [\frac{-(|x|+2\theta)N \vee I^-}{N}, \frac{\chi(N)}{N})$ implies $\lfloor Ny \rfloor \in \{I^-, \dots, \chi(N) - 1\}$, hence $(\mathcal{B}_4^-)^c$ yields $|Y_N^-(y) - Y_N(y)| \leq \frac{1}{\sqrt{N}} N^{1/3} + \frac{N^{1/16}+3}{\sqrt{N}} \leq N^{-1/12}$ when N is large enough. Consequently, for any $y \in [\frac{-(|x|+2\theta)N \vee I^-}{N}, \frac{(|x|+2\theta)N \wedge I^+}{N}]$ we have $|Y_N^-(y) - Y_N(y)| \leq N^{-1/12}$, which ends the proof of Lemma 20. \square

We now prove Proposition 19. Let $a > 0$ so that $||x| + 2\theta - a| > N^{-1/8}$, we will prove that when N is large enough, $\mathcal{B}^c \subset \{d_{M_1, a}(Y_N^\pm|_{[-a, a]}, Y_N|_{[-a, a]}) \leq 2N^{-1/12}\}$, and the threshold for N given by the proof will not depend on the value of a . There will be two cases depending on if a is smaller than $|x| + 2\theta$ or not.

4.1. Case $a \in (0, |x| + 2\theta - N^{-1/8})$. This is the easier case. Indeed, the interval $[-a, a]$ will then be contained in $[\frac{-(|x|+2\theta)N \vee I^-}{N}, \frac{(|x|+2\theta)N \wedge I^+}{N})$, inside which Y_N^\pm and Y_N are close for the uniform norm by Lemma 20. We may then define parametric representations (u_N^\pm, r_N^\pm) and (u_N, r_N) of $Y_N^\pm|_{[-a, a]}$ and $Y_N|_{[-a, a]}$ “following the graphs of $Y_N^\pm|_{[-a, a]}$ and $Y_N|_{[-a, a]}$ together” so that $u_N^\pm(t) = u_N(t)$ for all $t \in [0, 1]$, and $\|r_N^\pm - r_N\|_\infty \leq \sup_{y \in [-a, a]} |Y_N^\pm(y) - Y_N(y)|$ (an explicit construction of these representations can be found in the first arXiv version of this paper [5]). We deduce $d_{M_1, a}(Y_N^\pm|_{[-a, a]}, Y_N|_{[-a, a]}) \leq \sup_{y \in [-a, a]} |Y_N^\pm(y) - Y_N(y)|$. Moreover, if \mathcal{B}^c occurs, since $a \in (0, |x| + 2\theta - N^{-1/8})$, for any $y \in [-a, a]$ we have $y \in (-|x| - 2\theta + N^{-1/8}, |x| + 2\theta - N^{-1/8})$ thus $-(|x| + 2\theta)N + N^{3/4} \leq Ny \leq (|x| + 2\theta)N - N^{3/4}$, hence $I^- < Ny < I^+$, hence $y \in (\frac{-(|x|+2\theta)N \vee I^-}{N}, \frac{(|x|+2\theta)N \wedge I^+}{N})$, so by Lemma 20 we have $|Y_N^\pm(y) - Y_N(y)| \leq N^{-1/12}$. Consequently, if \mathcal{B}^c occurs, $d_{M_1, a}(Y_N^\pm|_{[-a, a]}, Y_N|_{[-a, a]}) \leq N^{-1/12}$.

4.2. Case $a > |x| + 2\theta + N^{-1/8}$. This is the harder case, as we have to deal with what happens around $|x| + 2\theta$ and $-|x| - 2\theta$. We only write down the proof for Y_N^- , since the proof for Y_N^+ is similar (one may remember that (2) allows to bound the $\ell^-(T_N, i) - \ell^+(T_N, i)$ when $(\mathcal{B}_2)^c$ occurs, hence when \mathcal{B}^c occurs). Once again, we will define parametric representations (u_N^-, r_N^-) and (u_N, r_N) of $Y_N^-|_{[-a, a]}$ and $Y_N|_{[-a, a]}$. The definition will depend on whether $I^+ \leq \lfloor (|x| + 2\theta)N \rfloor$ or not, and also on whether $I^- \geq -\lfloor (|x| + 2\theta)N \rfloor$ or not. We explain it for abscissas in $[0, a]$ depending on whether $I^+ \leq \lfloor (|x| + 2\theta)N \rfloor$ or not; the construction for abscissas in $[-a, 0]$ are similar depending on whether $I^- \geq -\lfloor (|x| + 2\theta)N \rfloor$ or not. We first assume $I^+ \leq \lfloor (|x| + 2\theta)N \rfloor$. Between 0 and $\frac{I^+}{N}$, the parametric representations will be, as in the case $a \in (0, |x| + 2\theta - N^{-1/8})$, following the completed graphs of Y_N^- and Y_N in parallel (see Figure 1(a)). The next step, once (u_N^-, r_N^-) reached $(\frac{I^+}{N}, Y_N^-(\frac{I^+}{N}))$, is to freeze it there while (u_N, r_N) follows the graph of Y_N from $(\frac{I^+}{N}, Y_N(\frac{I^+}{N}))$ to $(|x| + 2\theta, Y_N((|x| + 2\theta))^-)$ (see Figure 1(b)). For $y \geq \frac{I^+}{N}$ we have $\ell^-(T_N, \lfloor Ny \rfloor) = 0$

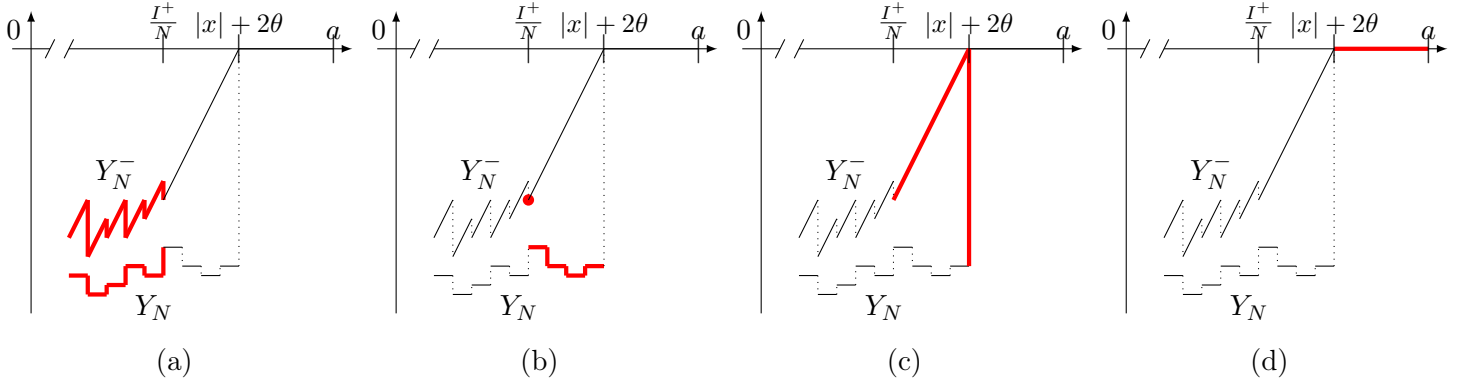


FIGURE 1. The successive steps of the parametric representations of $Y_N^-|_{[-a,a]}$ and $Y_N|_{[-a,a]}$ if $I^+ \leq \lfloor (|x| + 2\theta)N \rfloor$. At each step, the parts of the graphs the parametric representations travel through are thickened.

(see Lemma 12) thus $Y_N(y) = -N(\frac{|x|-|y|}{2} + \theta)_+$, hence $Y_N^- : [\frac{I^+}{N}, |x| + 2\theta] \mapsto \mathbb{R}$ is affine. Therefore, the following step is to move at the same time (u_N^-, r_N^-) from $(\frac{I^+}{N}, Y_N^-(\frac{I^+}{N}))$ to $(|x| + 2\theta, Y_N^- (|x| + 2\theta)) = (|x| + 2\theta, 0)$ and (u_N, r_N) from $(|x| + 2\theta, Y_N(|x| + 2\theta^-))$ to $(|x| + 2\theta, 0)$ (see Figure 1(c)), and the two parametric representations will remain close. After this step, both parametric representations are at $(|x| + 2\theta, 0)$, and they will go together to $(a, 0)$ (see Figure 1(d)). We now assume $I^+ > \lfloor (|x| + 2\theta)N \rfloor$. We also assume $\frac{I^+}{N} \leq a$ (if $\frac{I^+}{N} > a$, we may choose anything for (u_N^-, r_N^-) , (u_N, r_N) ; it will not happen if \mathcal{B}^c occurs). Between 0 and $|x| + 2\theta$, the parametric representations will follow the completed graphs of Y_N^- and Y_N in parallel (see Figure 2(a)). Once abscissa $|x| + 2\theta$ is reached, the next step is to move (u_N^-, r_N^-) from $(|x| + 2\theta, Y_N^- (|x| + 2\theta))$ to $(\frac{I^+}{N}, Y_N^-(\frac{I^+}{N}))$, which is $(\frac{I^+}{N}, 0)$, and to move at the same time (u_N, r_N) from $(|x| + 2\theta, Y_N(|x| + 2\theta))$ to $(|x| + 2\theta, 0)$ (see Figure 2(b)). We will prove the two representations are close by controlling the local times. At the next step we freeze (u_N^-, r_N^-) at $(\frac{I^+}{N}, 0)$ while (u_N, r_N) goes from $(|x| + 2\theta, 0)$ to $(\frac{I^+}{N}, 0)$ (see Figure 2(c)). After this step, both parametric representations are at $(\frac{I^+}{N}, 0)$, and they will go together from $(\frac{I^+}{N}, 0)$ to $(a, 0)$ (see Figure 2(d)). Again, a more rigorous definition of the parametric representations is available in the first arXiv version of this paper [5].

We can now bound the Skorohod M_1 distance between $Y_N^-|_{[-a,a]}$ and $Y_N|_{[-a,a]}$. From its definition, we have $d_{M_1,a}(Y_N^-|_{[-a,a]}, Y_N|_{[-a,a]}) \leq \max(\|u_N^- - u_N\|_\infty, \|r_N^- - r_N\|_\infty)$, hence we only have to prove $\mathcal{B}^c \subset \{\max(\|u_N^- - u_N\|_\infty, \|r_N^- - r_N\|_\infty) \leq 2N^{-1/12}\}$ when N is large enough. We are going to break down $\{\max(\|u_N^- - u_N\|_\infty, \|r_N^- - r_N\|_\infty) \leq 2N^{-1/12}\}$ into several events. We may write

$$\{\max(\|u_N^- - u_N\|_\infty, \|r_N^- - r_N\|_\infty) \leq 2N^{-1/12}\}$$

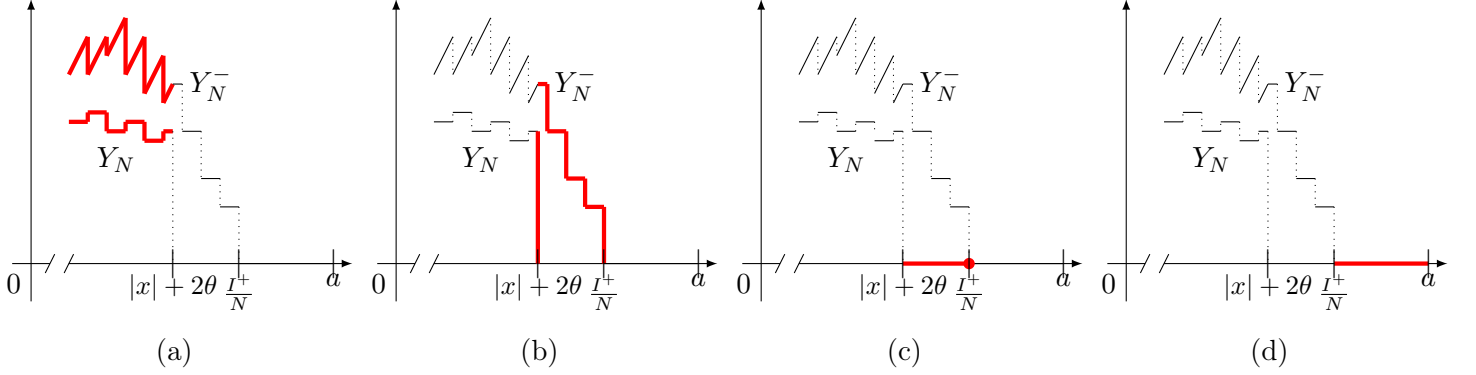


FIGURE 2. The successive steps of the parametric representations of $Y_N^-|_{[-a,a]}$ and $Y_N|_{[-a,a]}$ if $I^+ > \lfloor (|x| + 2\theta)N \rfloor$. At each step, the parts of the graphs the parametric representations travel through are thickened.

$$\begin{aligned}
&= \left\{ \text{between } \frac{-(|x| + 2\theta)N \vee I^-}{N} \text{ and } \frac{((|x| + 2\theta)N) \wedge I^+}{N}, \|u_N^- - u_N\|_\infty, \|r_N^- - r_N\|_\infty \leq 2N^{-1/12} \right\} \\
&\cap \left\{ \text{between } \frac{((|x| + 2\theta)N) \wedge I^+}{N} \text{ and } a, \|u_N^- - u_N\|_\infty, \|r_N^- - r_N\|_\infty \leq 2N^{-1/12} \right\} \\
&\cap \left\{ \text{between } -a \text{ and } \frac{-(|x| + 2\theta)N \vee I^-}{N}, \|u_N^- - u_N\|_\infty, \|r_N^- - r_N\|_\infty \leq 2N^{-1/12} \right\}.
\end{aligned}$$

Consequently, to prove that $\mathcal{B}^c \subset \{\max(\|u_N^- - u_N\|_\infty, \|r_N^- - r_N\|_\infty) \leq 2N^{-1/12}\}$ when N is large enough and thus end the proof of Proposition 19, we only have to prove the following claims.

Claim 21. $\mathcal{B}^c \subset \{\text{between } \frac{-(|x| + 2\theta)N \vee I^-}{N} \text{ and } \frac{((|x| + 2\theta)N) \wedge I^+}{N}, \|u_N^- - u_N\|_\infty, \|r_N^- - r_N\|_\infty \leq 2N^{-1/12}\}$ when N is large enough.

Claim 22. $\mathcal{B}^c \cap \{I^+ \leq \lfloor (|x| + 2\theta)N \rfloor\} \subset \{\text{between } \frac{((|x| + 2\theta)N) \wedge I^+}{N} \text{ and } a, \|u_N^- - u_N\|_\infty, \|r_N^- - r_N\|_\infty \leq 2N^{-1/12}\}$ and $\mathcal{B}^c \cap \{I^- \geq -\lfloor (|x| + 2\theta)N \rfloor\} \subset \{\text{between } -a \text{ and } \frac{-(|x| + 2\theta)N \vee I^-}{N}, \|u_N^- - u_N\|_\infty, \|r_N^- - r_N\|_\infty \leq 2N^{-1/12}\}$ when N is large enough.

Claim 23. $\mathcal{B}^c \cap \{I^+ > \lfloor (|x| + 2\theta)N \rfloor\} \subset \{\text{between } \frac{((|x| + 2\theta)N) \wedge I^+}{N} \text{ and } a, \|u_N^- - u_N\|_\infty, \|r_N^- - r_N\|_\infty \leq 2N^{-1/12}\}$ and $\mathcal{B}^c \cap \{I^- < -\lfloor (|x| + 2\theta)N \rfloor\} \subset \{\text{between } -a \text{ and } \frac{-(|x| + 2\theta)N \vee I^-}{N}, \|u_N^- - u_N\|_\infty, \|r_N^- - r_N\|_\infty \leq 2N^{-1/12}\}$ when N is large enough.

We now prove Claims 21, 22 and 23.

Proof of Claim 21. We assume \mathcal{B}^c occurs. In the part of the parametric representations between $\frac{-(|x| + 2\theta)N \vee I^-}{N}$ and $\frac{((|x| + 2\theta)N) \wedge I^+}{N}$, corresponding to Figures 1(a) and 2(a), we follow the completed graphs of Y_N^- and Y_N in parallel. Therefore we have $u_N^-(t) = u_N(t)$ and $|r_N^-(t) - r_N(t)| \leq \sup\{|Y_N^-(y) - Y_N(y)| : y \in [\frac{-(|x| + 2\theta)N \vee I^-}{N}, \frac{((|x| + 2\theta)N) \wedge I^+}{N}]\}$. If $(|x| + 2\theta)N$ is not an integer or $I^+ < (|x| + 2\theta)N$, this is smaller than N^{-12} when N is large enough by Lemma 20,

and we are done. If $(|x| + 2\theta)N$ is an integer and $I^+ \geq (|x| + 2\theta)N$, there is a small complication, since the parametric representations follow the graph of Y_N until $Y_N((|x| + 2\theta)^-)$, but should follow the graph of Y_N^- until $Y_N^- (|x| + 2\theta)$. The solution is to freeze the representation of Y_N at $Y_N((|x| + 2\theta)^-)$ while that of Y_N^- goes from $Y_N^-((|x| + 2\theta)^-)$ to $Y_N^- (|x| + 2\theta)$. Then, between $\frac{(-(|x|+2\theta)N)\vee I^-}{N}$ and $(|x| + 2\theta)^-$ we have $|r_N^-(t) - r_N(t)| \leq \sup\{|Y_N^-(y) - Y_N(y)| : y \in [\frac{(-(|x|+2\theta)N)\vee I^-}{N}, (|x| + 2\theta)N]\} \leq N^{-1/12}$ by Lemma 20 when N is large enough. Furthermore, when going from $Y_N^-((|x|+2\theta)^-)$ to $Y_N^- (|x|+2\theta)$, we have $|r_N^-(t) - r_N(t)| \leq |Y_N^-((|x|+2\theta)^-) - Y_N((|x|+2\theta)^-)| + |Y_N^-((|x|+2\theta)^-) - Y_N^- (|x|+2\theta)| \leq N^{-1/12} + |Y_N^-((|x|+2\theta)^-) - Y_N^- (|x|+2\theta)|$ when N is large enough. In addition, when N is large enough (1) yields $|Y_N^- (|x|+2\theta) - Y_N^-((|x|+2\theta)^-)| = \frac{1}{\sqrt{N}} |\ell^-(T_N, (|x|+2\theta)N) - \ell^-(T_N, (|x|+2\theta)N - 1)| = \frac{1}{\sqrt{N}} |\eta_{(|x|+2\theta)N-1,+}(\ell^-(T_N, (|x|+2\theta)N - 1))| \leq \frac{N^{1/16}+1/2}{\sqrt{N}}$ since $(\mathcal{B}_2)^c$ occurs. This yields $|r_N^-(t) - r_N(t)| \leq N^{-1/12} + \frac{N^{1/16}+1/2}{\sqrt{N}} \leq 2N^{-1/12}$ when N is large enough, which ends the proof. \square

Proof of Claim 22. This claim deals with the ‘‘right part’’ of the parametric representations in the case $I^+ \leq \lfloor (|x| + 2\theta)N \rfloor$, and with the ‘‘left part’’ in the case $I^- \geq -\lfloor (|x| + 2\theta)N \rfloor$, corresponding to Figure 1(b), (c) and (d). The idea of the argument is that in the step of Figure 1(b), the representation of Y_N does not move much horizontally as $\frac{I^+}{N}$ is close to $|x| + 2\theta$ by Lemma 13, so it does not have time to move too much vertically. In the step of Figure 1(c), the representations of Y_N^- and Y_N will thus start from points that are close and go to the same point, hence stay close to each other. We now give the rigorous argument. We only spell out the proof for $\mathcal{B}^c \cap \{I^+ \leq \lfloor (|x| + 2\theta)N \rfloor\}$, as the other case is similar. Let us assume \mathcal{B}^c occurs and $I^+ \leq \lfloor (|x| + 2\theta)N \rfloor$. Firstly, we notice that in the part of the parametric representations corresponding to Figure 1(d) we have $(u_N^-(t), r_N^-(t)) = (u_N(t), r_N(t))$, so we only consider the parts corresponding to Figure 1(b) and Figure 1(c). We first consider the case in which $(|x| + 2\theta)N$ is not an integer or $I^+ < \lfloor (|x| + 2\theta)N \rfloor$. We begin by dealing with $|u_N^-(t) - u_N(t)|$. By the definition of our parametric representations, $|u_N^-(t) - u_N(t)| \leq ||x| + 2\theta - \frac{I^+}{N}|$. Furthermore, \mathcal{B}^c occurs, thus we have $|I^+ - (|x| + 2\theta)N| < N^{3/4}$, hence $|u_N^-(t) - u_N(t)| \leq N^{-1/4}$. We now deal with $|r_N^-(t) - r_N(t)|$. Remembering the definition of our parametric representations, we notice that in the part corresponding to Figure 1(c), r_N^- and r_N are affine functions, so the maximum value of $|r_N^-(t) - r_N(t)|$ on this part is reached either at the beginning or at the end of the part. Moreover, at the end of the part we have $r_N^-(t) = r_N(t) = 0$, so the maximum is reached at the beginning. Therefore, if $|r_N^-(t) - r_N(t)| \leq 2N^{-1/12}$ in the part corresponding to Figure 1(b), then $|r_N^-(t) - r_N(t)| \leq 2N^{-1/12}$ in the part corresponding to Figure 1(c), and this ends the proof when $(|x| + 2\theta)N$ is not an integer or $I^+ < \lfloor (|x| + 2\theta)N \rfloor$.

We thus have to study the part corresponding to Figure 1(b). By the definition of our parametric representations, $|r_N^-(t) - r_N(t)| \leq \sup\{|Y_N^- (\frac{I^+}{N}) - Y_N(y)| : y \in [\frac{I^+}{N}, |x| + 2\theta]\}$, so it is enough to prove that when N is large enough, $\sup\{|Y_N^- (\frac{I^+}{N}) - Y_N(y)| : y \in [\frac{I^+}{N}, |x| + 2\theta]\} \leq 2N^{-1/12}$. Moreover, for any $y \in [\frac{I^+}{N}, |x| + 2\theta]$, we have $|Y_N^- (\frac{I^+}{N}) - Y_N(y)| \leq |Y_N^- (\frac{I^+}{N}) - Y_N(\frac{I^+}{N})| + |Y_N(\frac{I^+}{N}) - Y_N(y)|$. Since \mathcal{B}^c occurs, we have that $(\mathcal{B}_2)^c$, $(\mathcal{B}_4^-)^c$ and $(\mathcal{B}_4^+)^c$ occur, hence Lemma 20 implies $|Y_N^- (\frac{I^+}{N}) - Y_N(\frac{I^+}{N})| \leq N^{-1/12}$ when N is large enough, thus $|Y_N^- (\frac{I^+}{N}) - Y_N(y)| \leq |Y_N(\frac{I^+}{N}) - Y_N(y)| + N^{-1/12} \leq \frac{1}{\sqrt{N}} |\sum_{i=I^+}^{\lfloor Ny \rfloor} \zeta_i| + N^{-1/12}$. We deduce $\sup\{|Y_N^- (\frac{I^+}{N}) - Y_N(y)| : y \in [\frac{I^+}{N}, |x| + 2\theta]\} \leq \sup\{\frac{1}{\sqrt{N}} |\sum_{i=I^+}^{\lfloor Ny \rfloor} \zeta_i| : y \in [\frac{I^+}{N}, |x| + 2\theta]\} + N^{-1/12}$. Furthermore, \mathcal{B}^c occurs hence $|I^+ - (|x| + 2\theta)N| < N^{3/4}$, thus $\sup\{|Y_N^- (\frac{I^+}{N}) - Y_N(y)| : y \in [\frac{I^+}{N}, |x| + 2\theta]\} \leq \frac{1}{\sqrt{N}} \max_{\lfloor (|x|+2\theta)N \rfloor - N^{3/4} \leq i_1 \leq i_2 \leq \lfloor (|x|+2\theta)N \rfloor} \frac{1}{\sqrt{N}} |\sum_{i=i_1}^{i_2} \zeta_i| + N^{-1/12}$. Since \mathcal{B}^c occurs, $(\mathcal{B}_3^+)^c$ occurs, hence $\sup\{|Y_N^- (\frac{I^+}{N}) - Y_N(y)| : y \in [\frac{I^+}{N}, |x| + 2\theta]\} \leq \frac{N^{19/48}}{\sqrt{N}} + N^{-1/12} = N^{-5/48} + N^{-1/12} \leq 2N^{-1/12}$, which is enough.

We now consider the case in which $(|x| + 2\theta)N$ is an integer and $I^+ = \lfloor (|x| + 2\theta)N \rfloor$. Then the step of Figure 1(b) does not exist, we only have to deal with that of Figure 1(c), which comes mostly from Lemma 20 as this lemma ensures $Y_N^-((|x| + 2\theta)^-)$ and $Y_N((|x| + 2\theta)^-)$ are close (we will actually prove they are both close to 0). Since $I^+ = \lfloor (|x| + 2\theta)N \rfloor$, we have $u_N^-(t) = u_N(t)$. Moreover, $\ell^-(T_N, \lfloor N(|x| + 2\theta) \rfloor) = \ell^-(T_N, I^+) = 0$, so $Y_N^-(|x| + 2\theta) = 0$, hence $r_N^-(t) = 0$. Furthermore, $|r_N(t)| \leq |Y_N((|x| + 2\theta)^-)|$. Therefore we only have to prove that $|Y_N((|x| + 2\theta)^-)| \leq 2N^{-1/12}$ when N is large enough. In addition, \mathcal{B}^c occurs, thus by Lemma 20 we have $|Y_N((|x| + 2\theta)^-) - Y_N^-((|x| + 2\theta)^-)| \leq N^{-1/12}$ when N is large enough. Moreover by the definition of Y_N^- and by (1), we have $Y_N^-(|x| + 2\theta) = Y_N^-((|x| + 2\theta)^-) + \frac{1}{\sqrt{N}}\eta_{(|x|+2\theta)N-1,+}(\ell^-(T_N, (|x| + 2\theta)N - 1))$, and since \mathcal{B} occurs, $(\mathcal{B}_2)^c$ occurs, hence we get $|Y_N^-(|x| + 2\theta) - Y_N^-((|x| + 2\theta)^-)| \leq \frac{1}{\sqrt{N}}(N^{1/16} + 1/2) \leq N^{-1/4}$. Since $Y_N^-(|x| + 2\theta) = 0$, this yields $|Y_N^-((|x| + 2\theta)^-)| \leq N^{-1/4}$, which yields $|Y_N((|x| + 2\theta)^-)| \leq N^{-1/12} + N^{-1/4} < 2N^{-1/12}$, which is enough and ends the proof of Claim 22. \square

Proof of Claim 23. This claim deals with the ‘‘right part’’ of the parametric representations in the case $I^+ > \lfloor (|x| + 2\theta)N \rfloor$, and with the ‘‘left part’’ in the case $I^- < -\lfloor (|x| + 2\theta)N \rfloor$, corresponding to Figure 2(b), (c) and (d). We first give an idea of the argument. The most important part of the proof is to deal with the step corresponding to Figure 2(b). In this step, the function $Y_N^-(y) = \frac{1}{\sqrt{N}}\ell^-(T_N, \lfloor Ny \rfloor)$ evolves as a sum of $\frac{1}{\sqrt{N}}\eta_{j,+}(\ell^-(T_N, j))$ by (1), which is close to the sum of $\frac{1}{\sqrt{N}}(\zeta_j - \frac{1}{2})$ as $(\mathcal{B}_4^+)^c$ occurs. Since the ζ_j are i.i.d. with mean 0, the sum of $\frac{1}{\sqrt{N}}\zeta_j$ will be small, and the evolution of Y_N^- will be close to that of a deterministic sum of $-\frac{1}{2\sqrt{N}}$, thus it reaches 0 at constant speed, which is also what our parametric representation of Y_N does. We now give the proof, beginning with the detail of the argument to deal with $\mathcal{B}^c \cap \{I^- < -\lfloor (|x| + 2\theta)N \rfloor\}$. Let us assume \mathcal{B}^c occurs and $I^- < -\lfloor (|x| + 2\theta)N \rfloor$. We first see that $\frac{I^-}{N} \geq -a$, as since \mathcal{B}^c occurs we have $|I^- + (|x| + 2\theta)N| < N^{3/4}$, hence $\frac{I^-}{N} > -|x| - 2\theta - N^{-1/4}$, and by assumption $a > |x| + 2\theta + N^{-1/8}$, so $-a < -|x| - 2\theta - N^{-1/8} < \frac{I^-}{N}$, hence $\frac{I^-}{N} \geq -a$. Moreover, in the part of the parametric representations corresponding to Figure 2(d), we have $(u_N^-(t), r_N^-(t)) = (u_N(t), r_N(t))$. We now consider the equivalent of Figure 2(c). Then $r_N^-(t) = r_N(t) = 0$, and $|u_N^-(t) - u_N(t)| \leq |\frac{I^-}{N} + (|x| + 2\theta)|$, which is strictly smaller than $2N^{-1/12}$ since $|I^- + (|x| + 2\theta)N| < N^{3/4}$. It remains to consider the equivalent of Figure 2(b). Then $|u_N^-(t) - u_N(t)| \leq |\frac{I^-}{N} + (|x| + 2\theta)|$, which is strictly smaller than $2N^{-1/12}$, so we only have to prove $|r_N^-(t) - r_N(t)| \leq 2N^{-1/12}$.

We are going to study $\sup_{y \in [\frac{I^-}{N}, -|x| - 2\theta]} |Y_N^-(y) - Y_N^-(-|x| - 2\theta) + \frac{\lfloor (|x| + 2\theta)N \rfloor - \lfloor Ny \rfloor}{2\sqrt{N}}|$. Let $y \in [\frac{I^-}{N}, -|x| - 2\theta]$. By the definition of Y_N^- we have $Y_N^-(y) - Y_N^-(-|x| - 2\theta) = \frac{1}{\sqrt{N}}(\ell^-(T_N, \lfloor Ny \rfloor) - \ell^-(T_N, \lfloor -(|x| + 2\theta)N \rfloor))$. By (2) and since $(\mathcal{B}_2)^c$ occurs (remembering $\lfloor Ny \rfloor \geq I^- \geq -(|x| + 2\theta)N - N^{3/4} \geq -2\lfloor (|x| + 2\theta)N \rfloor$), we deduce

$$\begin{aligned} & \left| Y_N^-(y) - Y_N^-(-|x| - 2\theta) - \frac{1}{\sqrt{N}}(\ell^+(T_N, \lfloor Ny \rfloor) - \ell^+(T_N, \lfloor -(|x| + 2\theta)N \rfloor)) \right| \\ &= \left| \frac{\eta_{\lfloor Ny \rfloor, -}(\ell^+(T_N, \lfloor Ny \rfloor)) - \eta_{\lfloor -(|x| + 2\theta)N \rfloor, -}(\ell^+(T_N, \lfloor -(|x| + 2\theta)N \rfloor))}{\sqrt{N}} \right| \leq \frac{2N^{1/16}}{\sqrt{N}}. \end{aligned}$$

In addition, (1) yields the following:

$$\ell^+(T_N, \lfloor Ny \rfloor) - \ell^+(T_N, \lfloor -(|x| + 2\theta)N \rfloor) = \sum_{i=\lfloor Ny \rfloor + 1}^{\chi(N) - 1} (\eta_{i, -}(\ell^+(T_N, i)) + \mathbf{1}_{\{i > 0\}}) - \sum_{i=\lfloor -(|x| + 2\theta)N \rfloor + 1}^{\chi(N) - 1} (\eta_{i, -}(\ell^+(T_N, i)) + \mathbf{1}_{\{i > 0\}})$$

$$= \sum_{i=\lfloor Ny \rfloor + 1}^{\chi(N)-1} (\eta_{i,-}(\ell^+(T_N, i)) + 1/2) - \sum_{i=\lfloor -(|x|+2\theta)N \rfloor + 1}^{\chi(N)-1} (\eta_{i,-}(\ell^+(T_N, i)) + 1/2) - \frac{\lfloor -(|x|+2\theta)N \rfloor - \lfloor Ny \rfloor}{2}.$$

Since $(\mathcal{B}_4^-)^c$ occurs, this yields $|\ell^+(T_N, \lfloor Ny \rfloor) - \ell^+(T_N, \lfloor -(|x|+2\theta)N \rfloor) + \frac{\lfloor -(|x|+2\theta)N \rfloor - \lfloor Ny \rfloor}{2}| \leq |\sum_{i=\lfloor Ny \rfloor + 1}^{\chi(N)-1} \zeta_i - \sum_{i=\lfloor -(|x|+2\theta)N \rfloor + 1}^{\chi(N)-1} \zeta_i| + 2N^{1/3} = |\sum_{i=\lfloor Ny \rfloor + 1}^{\lfloor -(|x|+2\theta)N \rfloor} \zeta_i| + 2N^{1/3}$. As we also have $|Y_N^-(y) - Y_N^-(-|x| - 2\theta) - \frac{1}{\sqrt{N}}(\ell^+(T_N, \lfloor Ny \rfloor) - \ell^+(T_N, \lfloor -(|x|+2\theta)N \rfloor))| \leq \frac{2N^{1/16}}{\sqrt{N}}$, this implies $\sup_{y \in [\frac{I^-}{N}, -|x| - 2\theta]} |Y_N^-(y) - Y_N^-(-|x| - 2\theta) + \frac{\lfloor -(|x|+2\theta)N \rfloor - \lfloor Ny \rfloor}{2\sqrt{N}}| \leq \max_{I^-+1 \leq i \leq \lfloor -(|x|+2\theta)N \rfloor} \frac{1}{\sqrt{N}} |\sum_{j=i}^{\lfloor -(|x|+2\theta)N \rfloor} \zeta_j| + \frac{2N^{1/16}}{\sqrt{N}} + \frac{2N^{1/3}}{\sqrt{N}}$. Moreover, \mathcal{B}^c occurs, hence $|I^- + (|x| + 2\theta)N| < N^{3/4}$ and $(\mathcal{B}_3^-)^c$ occurs, therefore we obtain that $\sup_{y \in [\frac{I^-}{N}, -|x| - 2\theta]} |Y_N^-(y) - Y_N^-(-|x| - 2\theta) + \frac{\lfloor -(|x|+2\theta)N \rfloor - \lfloor Ny \rfloor}{2\sqrt{N}}| \leq \max_{\lfloor -(|x|+2\theta)N \rfloor - N^{3/4} \leq i \leq \lfloor -(|x|+2\theta)N \rfloor} \frac{1}{\sqrt{N}} |\sum_{j=i}^{\lfloor -(|x|+2\theta)N \rfloor} \zeta_j| + \frac{2N^{1/16}}{\sqrt{N}} + \frac{2N^{1/3}}{\sqrt{N}} \leq \frac{N^{19/48}}{\sqrt{N}} + \frac{2N^{1/16}}{\sqrt{N}} + \frac{2N^{1/3}}{\sqrt{N}} \leq 2N^{-5/48}$ when N is large enough. This yields $\sup_{y \in [\frac{I^-}{N}, -|x| - 2\theta]} |Y_N^-(y) - Y_N^-(-|x| - 2\theta) + \frac{\lfloor -(|x|+2\theta)N \rfloor - \lfloor Ny \rfloor}{2\sqrt{N}}| \leq 2N^{-5/48}$ when N is large enough.

We also need an explicit expression of the parametric representations. Assume the part of $[0, 1]$ devoted to the equivalent of Figure 2(b) in the parametric representations is $[a_N, a'_N]$. We set ϕ the affine function mapping a_N to $-\frac{2I^-}{N}$ and a'_N to $-(|x| + 2\theta)N$. Then, if $\phi(t)$ belongs to some $[\frac{2i}{N}, \frac{2i+1}{N}]$ with $i \in \{I^-, \dots, -\lfloor (|x| + 2\theta)N \rfloor - 1\}$, we set $(u_N^-(t), r_N^-(t)) = (\phi(t) - \frac{i}{N}, Y_N^-(\phi(t) - \frac{i}{N}))$, while if $\phi(t)$ belongs to some $[\frac{2i+1}{N}, \frac{2i+2}{N}]$ for $i \in \{I^-, \dots, -\lfloor (|x| + 2\theta)N \rfloor - 1\}$, we set $(u_N^-(t), r_N^-(t)) = (\frac{i+1}{N}, (-N\phi(t) + 2i + 2)Y_N^-((\frac{i+1}{N})^-) + (N\phi(t) - 2i - 1)Y_N^-(\frac{i+1}{N}))$. In addition, we set $(u_N(t), r_N(t)) = (-|x| - 2\theta, \hat{\phi}(\phi(t)))$, where $\hat{\phi}$ is the affine function mapping $-|x| - 2\theta - \frac{\lfloor -(|x|+2\theta)N \rfloor + 1}{N}$ to $Y_N^-(-|x| - 2\theta)$ and $\frac{2I^-}{N}$ to 0.

We recall that it is enough to prove $|r_N^-(t) - r_N(t)| < 2N^{-1/12}$. We are going to study $|r_N^-(t) - Y_N^-(-|x| - 2\theta) + \frac{\sqrt{N}}{4}\phi(t) - \frac{\lfloor -(|x|+2\theta)N \rfloor}{2\sqrt{N}}|$. We first suppose that $\phi(t) \in [\frac{2i}{N}, \frac{2i+1}{N}]$ with $i \in \{I^-, \dots, -\lfloor (|x| + 2\theta)N \rfloor - 1\}$. In this case, $r_N^-(t) = Y_N^-(\phi(t) - \frac{i}{N})$ and $|\frac{\phi(t)}{2} - \frac{1}{N} \lfloor N(\phi(t) - \frac{i}{N}) \rfloor| = |\frac{\phi(t)}{2} - \frac{i}{N}| \leq \frac{1}{2N}$, hence

$$\begin{aligned} & \left| r_N^-(t) - Y_N^-(-|x| - 2\theta) + \frac{\sqrt{N}}{4}\phi(t) - \frac{\lfloor -(|x|+2\theta)N \rfloor}{2\sqrt{N}} \right| \\ &= \left| Y_N^- \left(\phi(t) - \frac{i}{N} \right) - Y_N^-(-|x| - 2\theta) + \frac{\lfloor N(\phi(t) - \frac{i}{N}) \rfloor - \lfloor -(|x|+2\theta)N \rfloor}{2\sqrt{N}} + \frac{\frac{N}{2}\phi(t) - \lfloor N(\phi(t) - \frac{i}{N}) \rfloor}{2\sqrt{N}} \right| \\ &\leq \left| Y_N^- \left(\phi(t) - \frac{i}{N} \right) - Y_N^-(-|x| - 2\theta) + \frac{\lfloor N(\phi(t) - \frac{i}{N}) \rfloor - \lfloor -(|x|+2\theta)N \rfloor}{2\sqrt{N}} \right| + \frac{\sqrt{N}}{2} \left| \frac{\phi(t)}{2} - \frac{1}{N} \lfloor N \left(\phi(t) - \frac{i}{N} \right) \rfloor \right| \end{aligned}$$

is smaller than $2N^{-5/48} + \frac{1}{4\sqrt{N}}$, thus $|r_N^-(t) - Y_N^-(-|x| - 2\theta) + \frac{\sqrt{N}}{4}\phi(t) - \frac{\lfloor -(|x|+2\theta)N \rfloor}{2\sqrt{N}}| \leq 2N^{-5/48} + \frac{1}{4\sqrt{N}}$. We now consider the case $\phi(t) \in [\frac{2i+1}{N}, \frac{2i+2}{N}]$ with $i \in \{I^-, \dots, -\lfloor (|x| + 2\theta)N \rfloor - 1\}$. We temporarily denote $N\phi(t) - 2i - 1$ by ε for short, with $\varepsilon \in [0, 1]$. Then we have $r_N^-(t) = (1 - \varepsilon)Y_N^-((\frac{i+1}{N})^-) + \varepsilon Y_N^-(\frac{i+1}{N})$, $|\frac{\phi(t)}{2} - \frac{i}{N}| \leq \frac{1}{N}$ and $|\frac{\phi(t)}{2} - \frac{i+1}{N}| \leq \frac{1}{2N}$, therefore

$$\left| r_N^-(t) - Y_N^-(-|x| - 2\theta) + \frac{\sqrt{N}}{4}\phi(t) - \frac{\lfloor -(|x|+2\theta)N \rfloor}{2\sqrt{N}} \right|$$

$$\begin{aligned}
&= \left| (1 - \varepsilon) \left(Y_N^- \left(\left(\frac{i+1}{N} \right)^- \right) - Y_N^-(-|x| - 2\theta) + \frac{\sqrt{N}}{4} \phi(t) - \frac{\lfloor -(|x| + 2\theta)N \rfloor}{2\sqrt{N}} \right) \right. \\
&\quad \left. + \varepsilon \left(Y_N^- \left(\frac{i+1}{N} \right) - Y_N^-(-|x| - 2\theta) + \frac{\sqrt{N}}{4} \phi(t) - \frac{\lfloor -(|x| + 2\theta)N \rfloor}{2\sqrt{N}} \right) \right| \\
&\leq (1 - \varepsilon) \left| Y_N^- \left(\left(\frac{i+1}{N} \right)^- \right) - Y_N^-(-|x| - 2\theta) + \frac{\sqrt{N}}{4} \phi(t) - \frac{\lfloor -(|x| + 2\theta)N \rfloor}{2\sqrt{N}} \right| \\
&\quad + \varepsilon \left| Y_N^- \left(\frac{i+1}{N} \right) - Y_N^-(-|x| - 2\theta) + \frac{\sqrt{N}}{4} \phi(t) - \frac{\lfloor -(|x| + 2\theta)N \rfloor}{2\sqrt{N}} \right| \\
&\leq (1 - \varepsilon) \left| Y_N^- \left(\left(\frac{i+1}{N} \right)^- \right) - Y_N^-(-|x| - 2\theta) + \frac{i - \lfloor -(|x| + 2\theta)N \rfloor}{2\sqrt{N}} \right| + (1 - \varepsilon) \left| \frac{\sqrt{N}}{4} \phi(t) - \frac{i}{2\sqrt{N}} \right| \\
&\quad + \varepsilon \left| Y_N^- \left(\frac{i+1}{N} \right) - Y_N^-(-|x| - 2\theta) + \frac{i+1 - \lfloor -(|x| + 2\theta)N \rfloor}{2\sqrt{N}} \right| + \varepsilon \left| \frac{\sqrt{N}}{4} \phi(t) - \frac{i+1}{2\sqrt{N}} \right| \\
&\leq (1 - \varepsilon) \sup_{y \in [\frac{I^-}{N}, -|x| - 2\theta]} \left| Y_N^-(y) - Y_N^-(-|x| - 2\theta) + \frac{\lfloor Ny \rfloor - \lfloor -(|x| + 2\theta)N \rfloor}{2\sqrt{N}} \right| + (1 - \varepsilon) \frac{\sqrt{N}}{2} \left| \frac{\phi(t)}{2} - \frac{i}{N} \right| \\
&\quad + \varepsilon \sup_{y \in [\frac{I^-}{N}, -|x| - 2\theta]} \left| Y_N^-(y) - Y_N^-(-|x| - 2\theta) + \frac{\lfloor Ny \rfloor - \lfloor -(|x| + 2\theta)N \rfloor}{2\sqrt{N}} \right| + \varepsilon \frac{\sqrt{N}}{2} \left| \frac{\phi(t)}{2} - \frac{i+1}{N} \right| \\
&\leq \sup_{y \in [\frac{I^-}{N}, -|x| - 2\theta]} \left| Y_N^-(y) - Y_N^-(-|x| - 2\theta) + \frac{\lfloor Ny \rfloor - \lfloor -(|x| + 2\theta)N \rfloor}{2\sqrt{N}} \right| + \frac{1}{2\sqrt{N}} \leq 2N^{-5/48} + \frac{1}{2\sqrt{N}}
\end{aligned}$$

thanks to our bound on the sup. Since this was also true for $\phi(t) \in [\frac{2i}{N}, \frac{2i+1}{N}]$ with $i \in \{I^-, \dots, -\lfloor (|x| + 2\theta)N \rfloor - 1\}$, we have $|r_N^-(t) - Y_N^-(-|x| - 2\theta) + \frac{\sqrt{N}}{4} \phi(t) - \frac{\lfloor -(|x| + 2\theta)N \rfloor}{2\sqrt{N}}| \leq 2N^{-5/48} + \frac{1}{2\sqrt{N}}$.

The latter expression yields $|r_N^-(t) - r_N(t)| \leq |r_N(t) - Y_N^-(-|x| - 2\theta) + \frac{\sqrt{N}}{4} \phi(t) - \frac{\lfloor -(|x| + 2\theta)N \rfloor}{2\sqrt{N}}| + 2N^{-5/48} + \frac{1}{2\sqrt{N}} = |\hat{\phi}(\phi(t)) - Y_N^-(-|x| - 2\theta) + \frac{\sqrt{N}}{4} \phi(t) - \frac{\lfloor -(|x| + 2\theta)N \rfloor}{2\sqrt{N}}| + 2N^{-5/48} + \frac{1}{2\sqrt{N}}$, where $\hat{\phi}$ is the affine function mapping $-|x| - 2\theta - \frac{\lfloor (|x| + 2\theta)N \rfloor + 1}{N}$ to $Y_N^-(-|x| - 2\theta)$ and $\frac{2I^-}{N}$ to 0. Therefore it is enough to prove $|\hat{\phi}(\phi(t)) - Y_N^-(-|x| - 2\theta) + \frac{\sqrt{N}}{4} \phi(t) - \frac{\lfloor -(|x| + 2\theta)N \rfloor}{2\sqrt{N}}| \leq N^{-1/12} + \frac{1}{2\sqrt{N}}$ to end the proof. Now, $\hat{\phi}(\phi(t)) - Y_N^-(-|x| - 2\theta) + \frac{\sqrt{N}}{4} \phi(t) - \frac{\lfloor -(|x| + 2\theta)N \rfloor}{2\sqrt{N}}$ is an affine function of $\phi(t)$, so it is enough to prove the bound for $\phi(t) = -|x| - 2\theta - \frac{\lfloor (|x| + 2\theta)N \rfloor + 1}{N}$ and for $\phi(t) = \frac{2I^-}{N}$. We first consider $\phi(t) = \frac{2I^-}{N}$. By Lemma 12, $\ell^-(T_N, I^-) = 0$. Moreover, $I^- < -\lfloor (|x| + 2\theta)N \rfloor$, hence $Y_N^-(\frac{I^-}{N}) = 0$. We deduce

$$\begin{aligned}
&\left| \hat{\phi}(\phi(t)) - Y_N^-(-|x| - 2\theta) + \frac{\sqrt{N}}{4} \phi(t) - \frac{\lfloor -(|x| + 2\theta)N \rfloor}{2\sqrt{N}} \right| = \left| -Y_N^-(-|x| - 2\theta) + \frac{\sqrt{N}}{4} \frac{2I^-}{N} - \frac{\lfloor -(|x| + 2\theta)N \rfloor}{2\sqrt{N}} \right| \\
&= \left| Y_N^- \left(\frac{I^-}{N} \right) - Y_N^-(-|x| - 2\theta) + \frac{I^- - \lfloor -(|x| + 2\theta)N \rfloor}{2\sqrt{N}} \right|
\end{aligned}$$

$$\leq \sup_{y \in [\frac{x}{N}, -|x|-2\theta]} \left| Y_N^-(y) - Y_N^-(-|x|-2\theta) + \frac{\lfloor Ny \rfloor - \lfloor -(|x| + 2\theta)N \rfloor}{2\sqrt{N}} \right| \leq 2N^{-5/48},$$

which is enough. We now consider $\phi(t) = -|x| - 2\theta - \frac{\lfloor (|x|+2\theta)N \rfloor + 1}{N}$. Then $|\hat{\phi}(\phi(t)) - Y_N^-(-|x| - 2\theta) + \frac{\sqrt{N}}{4}\phi(t) - \frac{\lfloor -(|x| + 2\theta)N \rfloor}{2\sqrt{N}}|$ is equal to

$$\begin{aligned} & \left| Y_N^-(-|x| - 2\theta) - Y_N^-(-|x| - 2\theta) + \frac{\sqrt{N}}{4} \left(-|x| - 2\theta - \frac{\lfloor (|x| + 2\theta)N \rfloor + 1}{N} \right) - \frac{\lfloor -(|x| + 2\theta)N \rfloor}{2\sqrt{N}} \right| \\ & \leq |Y_N^-(-|x| - 2\theta) - Y_N^-(-|x| - 2\theta)| + \left| \frac{1}{4\sqrt{N}} (-(|x| + 2\theta)N - \lfloor (|x| + 2\theta)N \rfloor - 1 - 2\lfloor -(|x| + 2\theta)N \rfloor) \right| \\ & \leq |Y_N^-(-|x| - 2\theta) - Y_N^-(-|x| - 2\theta)| + \frac{1}{2\sqrt{N}} \leq N^{-1/12} + \frac{1}{2\sqrt{N}} \end{aligned}$$

by Lemma 20, which ends the proof for $\mathcal{B}^c \cap \{I^- < -\lfloor (|x| + 2\theta)N \rfloor\}$.

The argument to show $\mathcal{B}^c \cap \{I^+ > \lfloor (|x| + 2\theta)N \rfloor\} \subset \{\text{between } \frac{\lfloor (|x|+2\theta)N \rfloor \wedge I^+}{N} \text{ and } a, \|u_N^- - u_N\|_\infty, \|r_N^- - r_N\|_\infty \leq 2N^{-1/12}\}$ is similar and simpler, except for the end of the argument, which we give here. In a similar way as in the previous case, we must bound $|Y_N^-((|x| + 2\theta)^-) - Y_N^-(|x| + 2\theta) + \frac{\sqrt{N}}{4}(|x| + 2\theta + \frac{\lfloor (|x|+2\theta)N \rfloor}{N}) - \frac{\lfloor (|x|+2\theta)N \rfloor}{2\sqrt{N}}| \leq |Y_N^-((|x| + 2\theta)^-) - Y_N^-(|x| + 2\theta)| + |\frac{\lfloor (|x|+2\theta)N \rfloor - \lfloor (|x|+2\theta)N \rfloor}{4\sqrt{N}}| \leq |Y_N^-((|x| + 2\theta)^-) - Y_N^-((|x| + 2\theta)^-)| + |Y_N^-((|x| + 2\theta)^-) - Y_N^-(|x| + 2\theta)| + \frac{1}{4\sqrt{N}}$, hence Lemma 20 yields $|Y_N^-((|x| + 2\theta)^-) - Y_N^-(|x| + 2\theta) + \frac{\sqrt{N}}{4}(|x| + 2\theta + \frac{\lfloor (|x|+2\theta)N \rfloor}{N}) - \frac{\lfloor (|x|+2\theta)N \rfloor}{2\sqrt{N}}| \leq N^{-1/12} + |Y_N^-((|x| + 2\theta)^-) - Y_N^-(|x| + 2\theta)| + \frac{1}{4\sqrt{N}}$. In addition, the definition of Y_N^- and (1) yield that if $(|x| + 2\theta)N$ is not an integer, then $Y_N^-((|x| + 2\theta)^-) = Y_N^-(|x| + 2\theta)$, while if $(|x| + 2\theta)N$ is an integer then $|Y_N^-((|x| + 2\theta)^-) - Y_N^-(|x| + 2\theta)| = \frac{1}{\sqrt{N}} |\ell^-(T_N, (|x| + 2\theta)N - 1) - \ell^-(T_N, (|x| + 2\theta)N)| = \frac{1}{\sqrt{N}} |\eta_{(|x|+2\theta)N-1,+}(\ell^-(T_N, (|x| + 2\theta)N - 1))| \leq \frac{N^{1/16}+1/2}{\sqrt{N}}$ since $(\mathcal{B}_2)^c$ occurs. In all cases we obtain $|Y_N^-((|x| + 2\theta)^-) - Y_N^-(|x| + 2\theta)| \leq \frac{N^{1/16}+1/2}{\sqrt{N}}$, therefore $|Y_N^-((|x| + 2\theta)^-) - Y_N^-(|x| + 2\theta) + \frac{\sqrt{N}}{4}(|x| + 2\theta + \frac{\lfloor (|x|+2\theta)N \rfloor}{N}) - \frac{\lfloor (|x|+2\theta)N \rfloor}{2\sqrt{N}}| \leq N^{-1/12} + \frac{N^{1/16}+1/2}{\sqrt{N}} + \frac{1}{4\sqrt{N}}$, which is a bound small enough to end the proof of the claim. \square

5. CONVERGENCE OF THE LOCAL TIMES PROCESS: PROOF OF THEOREM 1 AND PROPOSITION 3

5.1. Proof of Theorem 1. Our aim is to prove that Y_N^\pm converges in distribution to $(B_y^x \mathbf{1}_{\{y \in [-|x|-2\theta, |x|+2\theta]\}})_{y \in \mathbb{R}}$ in the Skorohod M_1 topology on $D(-\infty, +\infty)$ when N tends to $+\infty$. Proposition 18 yields that Y_N^\pm is close to the function Y_N defined by $Y_N(y) = \frac{1}{\sqrt{N}} \sum_{i=\lfloor Ny \rfloor + 1}^{\chi(N)-1} \zeta_i$ if $y \in [-|x| - 2\theta, \frac{\chi(N)}{N})$, $Y_N(y) = \frac{1}{\sqrt{N}} \sum_{i=\chi(N)}^{\lfloor Ny \rfloor - 1} \zeta_i$ if $y \in [\frac{\chi(N)}{N}, |x| + 2\theta)$, and $Y_N(y) = 0$ otherwise. One has the feeling that by Donsker's Invariance Principle, Y_N should converge to $(B_y^x \mathbf{1}_{\{y \in [-|x|-2\theta, |x|+2\theta]\}})_{y \in \mathbb{R}}$ and so we should be able to conclude quickly, but proving rigorously the convergence in the Skorohod M_1 topology on $D(-\infty, +\infty)$ is harder than it looks. We are instead going to use a similar argument with a new process Y_N'' which will be "like Y_N , but continuous in $[-|x| - 2\theta, |x| + 2\theta]$ ". We will define it as follows. We first set a process Y_N' thus: if $Ny \in \mathbb{Z}$ then $Y_N'(y) = \frac{1}{\sqrt{N}} \sum_{i=Ny+1}^{\chi(N)-1} \zeta_i$ if $y \in (-\infty, \frac{\chi(N)}{N})$ and $Y_N'(y) = \frac{1}{\sqrt{N}} \sum_{i=\chi(N)}^{Ny-1} \zeta_i$ if $y \in [\frac{\chi(N)}{N}, +\infty)$, and in-between Y_N' is linearly interpolated. We then define Y_N'' by

$Y_N''(y) = Y_N'(y)\mathbf{1}_{\{y \in [-|x|-2\theta, |x|+2\theta]\}}$ for any $y \in \mathbb{R}$. Then Y_N'' will converge to $(B_y^x \mathbf{1}_{\{y \in [-|x|-2\theta, |x|+2\theta]\}})_{y \in \mathbb{R}}$ and be close to Y_N , which is stated in the two following lemmas.

Lemma 24. Y_N'' converges to $(B_y^x \mathbf{1}_{\{y \in [-|x|-2\theta, |x|+2\theta]\}})_{y \in \mathbb{R}}$ in distribution when N tends to $+\infty$ for the Skorohod M_1 topology in $D(-\infty, \infty)$.

Lemma 25. $\mathbb{P}(d_{M_1}(Y_N^\pm, Y_N'') > N^{-7/16})$ tends to 0 when N tends to $+\infty$.

Given these two lemmas, the proof of Theorem 1 is rather standard. One may for example look at the end of the proof of the Donsker invariance principle in [8] (here Y_N'' converges to the desired distribution instead of having it outright, but this convergence yields that the probability Y_N'' is in a closed set has the right limit). Thus we only have to prove Lemmas 24 and 25. In order to do this, we first need two easy lemmas which will also be used later in this work. If we denote $C[-|x|-2\theta, |x|+2\theta]$ the space of continuous functions $[-|x|-2\theta, |x|+2\theta] \mapsto \mathbb{R}$, since the $(\zeta_i)_{i \in \mathbb{Z}}$ are i.i.d. with law ρ_0 which is symmetric so has zero mean, Donsker's Invariance Principle yields the following.

Lemma 26. $Y_N'|_{[-|x|-2\theta, |x|+2\theta]}$ converges in distribution to $B^x|_{[-|x|-2\theta, |x|+2\theta]}$ when N tends to $+\infty$ for the topology defined on $C[-|x|-2\theta, |x|+2\theta]$ by the uniform norm.

The following lemma is also easy to prove.

Lemma 27. If $(\mathcal{B}_2)^c$ occurs, $\sup\{|Y_N(y) - Y_N''(y)| : y \in [-|x|-2\theta, |x|+2\theta]\} \leq N^{-7/16}$.

Proof. By the definition of Y_N and Y_N'' , we have $\sup\{|Y_N(y) - Y_N''(y)| : y \in [-|x|-2\theta, |x|+2\theta]\} \leq \frac{1}{\sqrt{N}} \sup\{|\zeta_i| : -(|x|+2\theta)N \leq i \leq (|x|+2\theta)N\}$, which is smaller than $\frac{N^{1/16}}{\sqrt{N}} = N^{-7/16}$ if $(\mathcal{B}_2)^c$ occurs. \square

We also need the following technical lemma in order to deduce results on the Skorohod M_1 topology from Lemmas 26 and 27.

Lemma 28. Let $N > 0$ and $Z_1, Z_2 \in D(-\infty, +\infty)$ whose possible discontinuities belong to $\frac{1}{N}\mathbb{Z}$, then we have $d_{M_1}((Z_1(y)\mathbf{1}_{\{y \in [-|x|-2\theta, |x|+2\theta]\}})_{y \in \mathbb{R}}, (Z_2(y)\mathbf{1}_{\{y \in [-|x|-2\theta, |x|+2\theta]\}})_{y \in \mathbb{R}}) \leq \sup\{|Z_1(y) - Z_2(y)| : y \in [-|x|-2\theta, |x|+2\theta]\}$.

Proof. Lemma 28 can be shown by writing for each $a \neq |x|+2\theta$ parametric representations of the two processes on $[-a, a]$ "following their completed graphs together" (one can find an explicit construction of such representations in the first arXiv version of this paper [5]). \square

Lemma 28 will allow us to deduce Lemma 24 from Lemma 26, and Lemma 25 from Lemma 27 and Proposition 18, which will end the proof of Theorem 1.

Proof of Lemma 24. Let $f : D(-\infty, +\infty) \mapsto \mathbb{R}$ be bounded and continuous with respect to the Skorohod M_1 topology on $D(-\infty, +\infty)$, we need to prove that $\mathbb{E}(f(Y_N''))$ converges to $\mathbb{E}(f((B_y^x \mathbf{1}_{\{y \in [-|x|-2\theta, |x|+2\theta]\}})_{y \in \mathbb{R}}))$ when N tends to $+\infty$. We define $g : C[-|x|-2\theta, |x|+2\theta] \mapsto \mathbb{R}$ by $g(Z) = f((Z(y)\mathbf{1}_{\{y \in [-|x|-2\theta, |x|+2\theta]\}})_{y \in \mathbb{R}})$ for any $Z \in C[-|x|-2\theta, |x|+2\theta]$. We then have $\mathbb{E}(f(Y_N'')) = \mathbb{E}(g(Y_N'|_{[-|x|-2\theta, |x|+2\theta]}))$ and $\mathbb{E}(f((B_y^x \mathbf{1}_{\{y \in [-|x|-2\theta, |x|+2\theta]\}})_{y \in \mathbb{R}})) = \mathbb{E}(g(B^x|_{[-|x|-2\theta, |x|+2\theta]}))$, hence it is enough to prove $\mathbb{E}(g(Y_N'|_{[-|x|-2\theta, |x|+2\theta]}))$ converges to $\mathbb{E}(g(B^x|_{[-|x|-2\theta, |x|+2\theta]}))$ when N tends to $+\infty$. Furthermore, Lemma 26 yields that $Y_N'|_{[-|x|-2\theta, |x|+2\theta]}$ converges in distribution to $B^x|_{[-|x|-2\theta, |x|+2\theta]}$ when N tends to $+\infty$ for the topology defined on $C[-|x|-2\theta, |x|+2\theta]$ by the uniform norm. Consequently, we only have to prove that g is continuous for this topology.

Let $(Z_k)_{k \in \mathbb{N}}$ be a sequence in $C[-|x|-2\theta, |x|+2\theta]$ converging uniformly to $Z \in C[-|x|-2\theta, |x|+2\theta]$ when k tends to $+\infty$. Then Lemma 28 states that for all $k \in \mathbb{N}$, $d_{M_1}((Z_k(y)\mathbf{1}_{\{y \in [-|x|-2\theta, |x|+2\theta]\}})_{y \in \mathbb{R}}, (Z(y)\mathbf{1}_{\{y \in [-|x|-2\theta, |x|+2\theta]\}})_{y \in \mathbb{R}}) \leq$

$\sup\{|Z_k(y) - Z(y)| : y \in [-|x| - 2\theta, |x| + 2\theta]\} \leq \|Z_k - Z\|_\infty$. Since the latter tends to 0 when k tends to $+\infty$, we deduce $(Z_k(y)\mathbf{1}_{\{y \in [-|x| - 2\theta, |x| + 2\theta]\}})_{y \in \mathbb{R}}$ converges to $(Z(y)\mathbf{1}_{\{y \in [-|x| - 2\theta, |x| + 2\theta]\}})_{y \in \mathbb{R}}$ when k tends to $+\infty$ with respect to the Skorohod M_1 topology on $D(-\infty, +\infty)$. Since f is continuous with respect to this topology, $(g(Z_k))_{k \in \mathbb{N}}$ converges to $g(Z)$ when k tends to $+\infty$. Consequently g is continuous for the topology defined on $C[-|x| - 2\theta, |x| + 2\theta]$ by the uniform norm, which ends the proof. \square

Proof of Lemma 25. $\mathbb{P}(d_{M_1}(Y_N^\pm, Y_N'') > 4N^{-1/12}) \leq \mathbb{P}(d_{M_1}(Y_N^\pm, Y_N) > 3N^{-1/12}) + \mathbb{P}(d_{M_1}(Y_N, Y_N'') > N^{-7/16})$ when N is large enough. By Lemmas 27 and 28 $\mathbb{P}(d_{M_1}(Y_N, Y_N'') > N^{-7/16}) \leq \mathbb{P}(\mathcal{B}_2)$. Therefore $\mathbb{P}(d_{M_1}(Y_N^\pm, Y_N'') > 4N^{-1/12}) \leq \mathbb{P}(d_{M_1}(Y_N^\pm, Y_N) > 3N^{-1/12}) + \mathbb{P}(\mathcal{B}_2)$, which tends to 0 when N tends to $+\infty$ by Proposition 18 and Lemma 9. \square

5.2. Proof of Proposition 3. Our goal is to prove that for any closed interval $I \in \mathbb{R}$ that does not contain $-|x| - 2\theta$ or $|x| + 2\theta$, the process $(Y_N^\pm(y))_{y \in I}$ converges in distribution to $(B_y^x \mathbf{1}_{\{y \in [-|x| - 2\theta, |x| + 2\theta]\}})_{y \in I}$ in the topology on DI given by the uniform norm when N tends to $+\infty$. We first assume $I = [a, b]$ or $[a, +\infty)$ with $a > |x| + 2\theta$ (the case $I = [a, b]$ or $(-\infty, b]$ with $b < -|x| - 2\theta$ can be dealt with in the same way). We are going to prove that outside an event of small probability, $(Y_N^\pm(y))_{y \in I} = 0 = (B_y^x \mathbf{1}_{\{y \in [-|x| - 2\theta, |x| + 2\theta]\}})_{y \in I}$. For any $y \geq (|x| + 2\theta) \vee \frac{I^+}{N}$, by Lemma 12 we have $\ell^\pm(T_N, \lfloor Ny \rfloor) = 0$, thus $Y_N^\pm(y) = 0$. We deduce that as soon as $\frac{I^+}{N} \leq a$, we have $(Y_N^\pm(y))_{y \in I} = 0 = (B_y^x \mathbf{1}_{\{y \in [-|x| - 2\theta, |x| + 2\theta]\}})_{y \in I}$. In addition, when N is large enough we have $a \geq |x| + 2\theta + N^{-1/4}$. Therefore, when N is large enough, $\mathbb{P}((Y_N^\pm(y))_{y \in I} \neq (B_y^x \mathbf{1}_{\{y \in [-|x| - 2\theta, |x| + 2\theta]\}})_{y \in I}) \leq \mathbb{P}(|I^+ - (|x| + 2\theta)N| \geq N^{3/4})$, which tends to 0 when N tends to $+\infty$ by Lemma 13. This yields that $(Y_N^\pm(y))_{y \in I}$ converges in distribution to $(B_y^x \mathbf{1}_{\{y \in [-|x| - 2\theta, |x| + 2\theta]\}})_{y \in I}$ in the topology on DI given by the uniform norm.

We now deal with the case $I = [a, b]$ with $-|x| - 2\theta < a < b < |x| + 2\theta$. The idea is that we will be far from the problems at $-|x| - 2\theta$ and $|x| + 2\theta$, thus Y_N^\pm will be close to Y_N' in all I , and Y_N' converges to the right limit, hence Y_N^\pm too. We first prove the following lemma.

Lemma 29. *For any $-|x| - 2\theta < a < b < |x| + 2\theta$, we have that $\mathbb{P}(\|Y_N^\pm|_{[a,b]} - Y_N'|_{[a,b]}\|_\infty > 2N^{-1/12})$ tends to 0 when N tends to $+\infty$.*

Proof. We assume $(\mathcal{B}_2)^c$, $(\mathcal{B}_4^-)^c$, $(\mathcal{B}_4^+)^c$ occurs, as well as $|I^- + (|x| + 2\theta)N| < N^{3/4}$, $|I^+ - (|x| + 2\theta)N| < N^{3/4}$. When N is large enough, we have $a \geq -|x| - 2\theta + N^{-1/4} > \frac{I^-}{N}$ and $b \leq |x| + 2\theta - N^{-1/4} < \frac{I^+}{N}$, hence $[a, b] \subset (\frac{I^-}{N}, \frac{I^+}{N})$. Therefore, for any $y \in [a, b]$, Lemma 20 yields $|Y_N^\pm(y) - Y_N(y)| \leq N^{-1/12}$, and Lemma 27 gives $|Y_N(y) - Y_N'(y)| \leq N^{-7/16}$, hence we get $|Y_N^\pm(y) - Y_N'(y)| \leq 2N^{-1/12}$, and we deduce $\|Y_N^\pm|_{[a,b]} - Y_N'|_{[a,b]}\|_\infty \leq 2N^{-1/12}$. This implies $\mathbb{P}(\|Y_N^\pm|_{[a,b]} - Y_N'|_{[a,b]}\|_\infty > 2N^{-1/12}) \leq \mathbb{P}(\mathcal{B}_2 \cup \mathcal{B}_4^- \cup \mathcal{B}_4^+ \cup \{|I^- + (|x| + 2\theta)N| \geq N^{3/4}\} \cup \{|I^+ - (|x| + 2\theta)N| \geq N^{3/4}\})$, which tends to 0 when N tends to $+\infty$ thanks to Lemmas 9, 13 and 17. \square

Moreover, for any $-|x| - 2\theta < a < b < |x| + 2\theta$, by Donsker's Invariance Principle, $Y_N'|_{[a,b]}$ converges in distribution to $B^x|_{[a,b]}$ when N tends to $+\infty$ for the topology defined on $D[a, b]$ by the uniform norm. The proof of Proposition 3 from this is standard, as was the proof of Theorem 1 from Lemmas 24 and 25.

6. NO CONVERGENCE IN THE SKOROHOD J_1 TOPOLOGY: PROOF OF PROPOSITION 2

In this section, our aim is to prove that Y_N^\pm does not converge in distribution in the Skorohod J_1 topology on $D(-\infty, +\infty)$ when N tends to $+\infty$. We will first prove that if Y_N^\pm converges in the Skorohod J_1 topology, the limit has to be the same as in the Skorohod M_1 topology, that is $(B_y^x \mathbf{1}_{\{y \in [-|x| - 2\theta, |x| + 2\theta]\}})_{y \in \mathbb{R}}$ by Theorem

1 (this will be Lemma 30). Afterwards, we will prove that Y_N^\pm does not converge in distribution in the Skorohod J_1 topology to $(B_y^x \mathbf{1}_{\{y \in [-|x|-2\theta, |x|+2\theta]\}})_{y \in \mathbb{R}}$ by finding some closed set Ξ so that $\limsup_{N \rightarrow +\infty} \mathbb{P}(Y_N^\pm \in \Xi) > \mathbb{P}((B_y^x \mathbf{1}_{\{y \in [-|x|-2\theta, |x|+2\theta]\}})_{y \in \mathbb{R}} \in \Xi)$, which is enough by the Portmanteau Theorem.

Lemma 30. *If Y_N^\pm converges in distribution in the Skorohod J_1 topology on $D(-\infty, +\infty)$ when N tends to $+\infty$, the limit is $(B_y^x \mathbf{1}_{\{y \in [-|x|-2\theta, |x|+2\theta]\}})_{y \in \mathbb{R}}$.*

Proof. The idea is that the Skorohod J_1 topology is stronger than the Skorohod M_1 topology. We assume Y_N^\pm converges in distribution to some Z in the Skorohod J_1 topology on $D(-\infty, +\infty)$ when N tends to $+\infty$. It can be proven that for any $a > 0$ we have $d_{M_1, a} \leq d_{J_1, -a, a}$. Indeed, this is Theorem 12.3.2 of [19], whose proof is in the Internet supplement of that book (just replace the discontinuity points of x_1 with their image by λ^{-1}). This implies $d_{M_1} \leq d_{J_1}$. Therefore a function $g : D(-\infty, +\infty) \mapsto \mathbb{R}$ bounded and continuous for the Skorohod M_1 topology is also continuous for the Skorohod J_1 topology. We deduce that $\mathbb{E}(g(Y_N^\pm))$ converges to $\mathbb{E}(g(Z))$ when N tends to $+\infty$, thus Y_N^\pm converges in distribution to Z in the Skorohod M_1 topology when N tends to $+\infty$. By Theorem 1, the limit has to be $(B_y^x \mathbf{1}_{\{y \in [-|x|-2\theta, |x|+2\theta]\}})_{y \in \mathbb{R}}$. \square

We now define our closed set Ξ . The idea behind this definition is that with high probability, $B_{|x|+2\theta}^x$ is at some distance from 0, hence at some point around $|x| + 2\theta$, Y_N^\pm will be close to $B_{|x|+2\theta}^x$, thus at some distance from 0. Furthermore, at $|x| + 2\theta$ the process $(B_y^x \mathbf{1}_{\{y \in [-|x|-2\theta, |x|+2\theta]\}})_{y \in \mathbb{R}}$ will jump directly from $B_{|x|+2\theta}^x$ to 0, while Y_N^\pm , which can make only jumps of order $\frac{1}{\sqrt{N}}$, will have to cross the distance separating $B_{|x|+2\theta}^x$ from 0 without big jumps. Therefore if $\delta_1 > 0$ is much smaller than $B_{|x|+2\theta}^x$, then $Y_N^\pm(y)$ will enter the interval $[\delta_1, 2\delta_1]$ for y near $|x| + 2\theta$, while $(B_y^x \mathbf{1}_{\{y \in [-|x|-2\theta, |x|+2\theta]\}})_{y \in \mathbb{R}}$ will not. We thus set Ξ to be roughly “the function enters $[\delta_1, 2\delta_1]$ around $|x| + 2\theta$ ”. More rigorously, by the definition of B^x , the random variable $B_{|x|+2\theta}^x$ has distribution $\mathcal{N}(0, 2\theta)$, hence there exists $\delta_1 > 0$ so that $\mathbb{P}(|B_{|x|+2\theta}^x| \leq 4\delta_1) \leq 1/8$. Moreover, B^x is continuous, hence there exists $0 < \delta_2 < \theta$ so that $\mathbb{P}(\exists y \in [|x| + 2\theta - \delta_2, |x| + 2\theta], |B_y^x| \leq 3\delta_1) \leq 1/4$. We then define $\Xi = \{Z \in D(-\infty, +\infty) \mid \exists y \in [|x| + 2\theta - \delta_2, |x| + 2\theta + \delta_2], |Z(y)| \in [\delta_1, 2\delta_1] \text{ or } |Z(y^-)| \in [\delta_1, 2\delta_1]\}$ (the inclusion of $Z(y^-)$ was necessary for Ξ to be closed). Then $\mathbb{P}((B_y^x \mathbf{1}_{\{y \in [-|x|-2\theta, |x|+2\theta]\}})_{y \in \mathbb{R}} \in \Xi) \leq 1/4$. We will prove the two following lemmas.

Lemma 31. *When N is large enough, $\mathbb{P}(Y_N^\pm \in \Xi) \geq 1/2$.*

Lemma 32. *Ξ is closed in the Skorohod J_1 topology on $D(-\infty, +\infty)$.*

With these two lemmas, the proof of Proposition 2 becomes easy.

Proof of Proposition 2. Lemma 31 yields $\limsup_{N \rightarrow +\infty} \mathbb{P}(Y_N^\pm \in \Xi) \geq 1/2$, and the definition of Ξ ensures that $\mathbb{P}((B_y^x \mathbf{1}_{\{y \in [-|x|-2\theta, |x|+2\theta]\}})_{y \in \mathbb{R}} \in \Xi) \leq 1/4$, hence $\limsup_{N \rightarrow +\infty} \mathbb{P}(Y_N^\pm \in \Xi) > \mathbb{P}((B_y^x \mathbf{1}_{\{y \in [-|x|-2\theta, |x|+2\theta]\}})_{y \in \mathbb{R}} \in \Xi)$. Since Lemma 32 yields Ξ is closed in the Skorohod J_1 topology on $D(-\infty, +\infty)$, the Portmanteau Theorem implies Y_N^\pm does not converge in distribution in the Skorohod J_1 topology on $D(-\infty, +\infty)$ to $(B_y^x \mathbf{1}_{\{y \in [-|x|-2\theta, |x|+2\theta]\}})_{y \in \mathbb{R}}$ when N tends to $+\infty$. Hence Lemma 30 yields that Y_N^\pm does not converge in distribution in the Skorohod J_1 topology on $D(-\infty, +\infty)$ when N tends to $+\infty$, which is Proposition 2. \square

Thus it remains only to prove Lemmas 31 and 32.

Proof of Lemma 31. The idea is that with good probability, when y is a bit smaller than $|x| + 2\theta$, we have $Y_N^\pm(y)$ of the same order as $B_{|x|+2\theta}^x$, thus away from 0, while when y is a bit larger than $|x| + 2\theta$, we have $Y_N^\pm(y) = 0$, so

since Y_N can only make jumps of order $\frac{1}{\sqrt{N}}$, it will enter $[\delta_1, 2\delta_1]$. We now give the rigorous argument. We begin by assuming that $|Y_N^\pm(|x| + 2\theta - \delta_2)| > 3\delta_1$ (that is $Y_N(y)$ is indeed away from 0 when y is a bit smaller than $|x| + 2\theta$), $(\mathcal{B}_2)^c$ occurs and $|I^+ - (|x| + 2\theta)N| < N^{3/4}$, and proving that when N is large enough, $Y_N^\pm \in \Xi$. We first show $Y_N^\pm(|x| + 2\theta + \delta_2) = 0$. When N is large enough, $\frac{I^+}{N} \leq |x| + 2\theta + N^{-1/4} \leq |x| + 2\theta + \delta_2$. Moreover, Lemma 12 implies $\ell^\pm(T_N, \lfloor Ny \rfloor) = 0$ for any $y \geq \frac{I^+}{N}$, hence for $y = |x| + 2\theta + \delta_2$. This yields $Y_N^\pm(|x| + 2\theta + \delta_2) = 0$. Moreover, we assumed $|Y_N^\pm(|x| + 2\theta - \delta_2)| > 3\delta_1$. Furthermore, equations (1) and (2) yield that the jumps of Y_N^\pm in $[|x| + 2\theta - \delta_2, |x| + 2\theta + \delta_2]$ are either $\frac{1}{\sqrt{N}}\eta_{i,+}(\ell^-(T_N, i))$ (if we deal with Y_N^-) or $\frac{1}{\sqrt{N}}\eta_{i+1,+}(\ell^-(T_N, i+1))$ (if we deal with Y_N^+) with $i \in \{ \lfloor (|x| + 2\theta - \delta_2)N \rfloor, \dots, \lfloor (|x| + 2\theta + \delta_2)N \rfloor - 1 \}$. Since $(\mathcal{B}_2)^c$ occurs, the jumps of Y_N^\pm in $[|x| + 2\theta - \delta_2, |x| + 2\theta + \delta_2]$ have size at most $\frac{1}{\sqrt{N}}(N^{1/16} + 1/2)$, which tends to 0 when N tends to $+\infty$. Therefore, when N is large enough, there exists $y \in [|x| + 2\theta - \delta_2, |x| + 2\theta + \delta_2]$ so that $|Y_N^\pm(y)| \in [\delta_1, 2\delta_1]$, hence $Y_N^\pm \in \Xi$. Consequently, when N is large enough, if $|Y_N^\pm(|x| + 2\theta - \delta_2)| > 3\delta_1$, $(\mathcal{B}_2)^c$ and $|I^+ - (|x| + 2\theta)N| < N^{3/4}$ then $Y_N^\pm \in \Xi$. This implies $\mathbb{P}(Y_N^\pm \notin \Xi) \leq \mathbb{P}(|Y_N^\pm(|x| + 2\theta - \delta_2)| \leq 3\delta_1) + \mathbb{P}(\mathcal{B}_2) + \mathbb{P}(|I^+ - (|x| + 2\theta)N| \geq N^{3/4})$. In addition, Lemma 9 and Lemma 13 yield respectively that $\mathbb{P}(\mathcal{B}_2)$ and $\mathbb{P}(|I^+ - (|x| + 2\theta)N| \geq N^{3/4})$ tend to 0 when N tends to $+\infty$. Therefore it is enough to prove that $\mathbb{P}(|Y_N^\pm(|x| + 2\theta - \delta_2)| \leq 3\delta_1) \leq 3/8$ when N is large enough to deduce that $\mathbb{P}(Y_N^\pm \notin \Xi) \leq 1/2$ when N is large enough and end the proof of Lemma 31.

We now prove $\mathbb{P}(|Y_N^\pm(|x| + 2\theta - \delta_2)| \leq 3\delta_1) \leq 3/8$ when N is large enough, by noticing $Y_N^\pm(|x| + 2\theta - \delta_2)$ is close to $Y'_N(|x| + 2\theta - \delta_2)$, which will converge in distribution to $B_{|x|+2\theta-\delta_2}^x$ when N tends to $+\infty$. Lemma 29 implies $\mathbb{P}(\|Y_N^\pm|_{[0,|x|+2\theta-\delta_2]} - Y'_N|_{[0,|x|+2\theta-\delta_2]}\|_\infty > 2N^{-1/12})$ tends to 0 when N tends to $+\infty$, hence $\mathbb{P}(|Y_N^\pm(|x| + 2\theta - \delta_2) - Y'_N(|x| + 2\theta - \delta_2)| > 2N^{-1/12})$ tends to 0 when N tends to $+\infty$, which implies $Y_N^\pm(|x| + 2\theta - \delta_2) - Y'_N(|x| + 2\theta - \delta_2)$ converges in probability to 0 when N tends to $+\infty$. In addition, Lemma 26 states $Y'_N|_{[-|x|-2\theta, |x|+2\theta]}$ converges in distribution to $B^x|_{[-|x|-2\theta, |x|+2\theta]}$ when N tends to $+\infty$ for the topology defined on $C[-|x|-2\theta, |x|+2\theta]$ by the uniform norm, hence $Y'_N(|x| + 2\theta - \delta_2)$ converges in distribution to $B_{|x|+2\theta-\delta_2}^x$ when N tends to $+\infty$. Therefore Slutsky's Theorem yields that $Y_N^\pm(|x| + 2\theta - \delta_2)$ converges in distribution to $B_{|x|+2\theta-\delta_2}^x$ when N tends to $+\infty$. Moreover, we defined Ξ so that $\mathbb{P}(\exists y \in [|x| + 2\theta - \delta_2, |x| + 2\theta], |B_y^x| \leq 3\delta_1) \leq 1/4$, hence $\mathbb{P}(|B_{|x|+2\theta-\delta_2}^x| \leq 3\delta_1) \leq 1/4$. This implies that when N is large enough, $\mathbb{P}(|Y_N^\pm(|x| + 2\theta - \delta_2)| \leq 3\delta_1) \leq 3/8$. \square

Proof of Lemma 32. Let $(Z_N)_{N \in \mathbb{N}}$ be a sequence of elements of Ξ converging to Z in the Skorohod J_1 topology on $D(-\infty, +\infty)$, we will prove $Z \in \Xi$. By taking a subsequence, we may assume $d_{J_1}(Z, Z_N) < e^{-|x|-2\theta-\delta_2-1}/N$ for any $N \in \mathbb{N}^*$. Then for any $N \in \mathbb{N}^*$, some $a_N > |x| + 2\theta + \delta_2 + 1$ so that $d_{J_1, -a_N, a_N}(Z|_{[-a_N, a_N]}, Z_N|_{[-a_N, a_N]}) \leq 1/N$ will exist. Indeed, if it was not the case, for some N we would have $d_{J_1}(Z, Z_N) = \int_0^{+\infty} e^{-a}(d_{J_1, -a, a}(Z|_{[-a, a]}, Z_N|_{[-a, a]}) \wedge 1) da \geq \int_{|x|+2\theta+\delta_2+1}^{+\infty} e^{-a} \frac{1}{N} da = e^{-|x|-2\theta-\delta_2-1}/N$, which does not happen. For all $N \in \mathbb{N}^*$, the fact that we have $d_{J_1, -a_N, a_N}(Z|_{[-a_N, a_N]}, Z_N|_{[-a_N, a_N]}) \leq 1/N$ implies there exists $\lambda_N \in \Lambda_{-a_N, a_N}$ with $\|Z|_{[-a_N, a_N]} \circ \lambda_N - Z_N|_{[-a_N, a_N]}\|_\infty \leq 2/N$ and $\|\lambda_N - \text{Id}_{-a_N, a_N}\|_\infty \leq 2/N$. Moreover, $Z_N \in \Xi$, hence there exists $y_N \in [|x| + 2\theta - \delta_2, |x| + 2\theta + \delta_2]$, $|Z_N(y_N)| \in [\delta_1, 2\delta_1]$ or $|Z_N(y_N^-)| \in [\delta_1, 2\delta_1]$. We now define y'_N as follows: if $|Z_N(y_N)| \in [\delta_1, 2\delta_1]$ we set $y'_N = y_N$. Otherwise, since $|Z_N(y_N^-)| \in [\delta_1, 2\delta_1]$ we can take some y'_N in $[y_N - \frac{1}{N}, y_N]$ so that $|Z_N(y'_N)| \in [\delta_1 - 1/N, 2\delta_1 + 1/N]$. In both cases, we have $y'_N \in [|x| + 2\theta - \delta_2 - 1/N, |x| + 2\theta + \delta_2]$ and $|Z_N(y'_N)| \in [\delta_1 - 1/N, 2\delta_1 + 1/N]$. Furthermore, $\|\lambda_N - \text{Id}_{-a_N, a_N}\|_\infty \leq 2/N$, hence $|\lambda_N(y'_N) - y'_N| \leq 2/N$, thus $\lambda_N(y'_N) \in [|x| + 2\theta - \delta_2 - 3/N, |x| + 2\theta + \delta_2 + 2/N]$. In addition, $\|Z(\lambda_N(y'_N)) - Z_N(y'_N)\|_\infty \leq 2/N$, hence $|Z(\lambda_N(y'_N))| \in [\delta_1 - 3/N, 2\delta_1 + 3/N]$. By taking a subsequence, we may assume that $\lambda_N(y'_N)$ converges to some $y_\infty \in [|x| + 2\theta - \delta_2, |x| + 2\theta + \delta_2]$. In addition, Z is càdlàg, hence there is a

subsequence of $(Z(\lambda_N(y'_N)))_{N \in \mathbb{N}^*}$ that converges to either $Z(y_\infty)$ or $Z(y_\infty^-)$. Since $|Z(\lambda_N(y'_N))| \in [\delta_1 - 3/N, 2\delta_1 + 3/N]$, we have $|Z(y_\infty)|$ or $|Z(y_\infty^-)|$ in $[\delta_1, 2\delta_1]$. Therefore $Z \in \Xi$, which ends the proof. \square

7. CONVERGENCE OF THE STOPPING TIME: PROOF OF PROPOSITION 4

We want to prove Proposition 4, that is the convergence in distribution of $\frac{1}{N^{3/2}}(T_N - N^2(|x| + 2\theta)^2)$ to the law $\mathcal{N}(0, \frac{32}{3}\text{Var}(\rho_-)((|x| + \theta)^3 + \theta^3))$ when N tends to $+\infty$. In order to do that, we will prove that $\frac{1}{N^{3/2}}(T_N - N^2(|x| + 2\theta)^2)$ is close to $2 \int_{-|x|-2\theta}^{|x|+2\theta} Y'_N(y) dy$ (where Y'_N was defined at the beginning of Section 5.1), then show that $2 \int_{-|x|-2\theta}^{|x|+2\theta} Y'_N(y) dy$ converges to the desired distribution.

Proposition 33. $\mathbb{P}(|\frac{1}{N^{3/2}}(T_N - N^2(|x| + 2\theta)^2) - 2 \int_{-|x|-2\theta}^{|x|+2\theta} Y'_N(y) dy| > 5(|x| + 2\theta)N^{-1/12})$ tends to 0 when N tends to $+\infty$.

Proof. The result will come from the fact that T_N can be written as the sum of the local times, which is itself related to the integrals of Y_N^- and Y_N^+ , which are close to Y_N by Lemma 20 hence to Y'_N by Lemma 27. It is enough to prove that if $(\mathcal{B}_2)^c$, $(\mathcal{B}_4^-)^c$ and $(\mathcal{B}_4^+)^c$ occur and if $|I^- + (|x| + 2\theta)N| < N^{5/8}$, $|I^+ - (|x| + 2\theta)N| < N^{5/8}$, then $|\frac{1}{N^{3/2}}(T_N - N^2(|x| + 2\theta)^2) - 2 \int_{-|x|-2\theta}^{|x|+2\theta} Y'_N(y) dy| \leq 5(|x| + 2\theta)N^{-1/12}$, since Lemma 9 implies $\mathbb{P}(\mathcal{B}_2)$ tends to 0 when N tends to $+\infty$, Lemma 17 implies $\mathbb{P}(\mathcal{B}_4^-)$ and $\mathbb{P}(\mathcal{B}_4^+)$ tend to 0 when N tends to $+\infty$, and Lemma 13 implies $\mathbb{P}(|I^- + (|x| + 2\theta)N| \geq N^{5/8})$ and $\mathbb{P}(|I^+ - (|x| + 2\theta)N| \geq N^{5/8})$ tend to 0 when N tends to $+\infty$. We assume $(\mathcal{B}_2)^c$, $(\mathcal{B}_4^-)^c$ and $(\mathcal{B}_4^+)^c$ occur and $|I^- + (|x| + 2\theta)N| < N^{5/8}$, $|I^+ - (|x| + 2\theta)N| < N^{5/8}$, let us study T_N .

In order to do that, we first need to prove an auxiliary result, more precisely that the following holds when N is large enough:

$$(6) \quad \text{if } |i - I^+| \leq N^{5/8} + 1 \text{ or } |i - I^-| \leq N^{5/8} + 1 \text{ then } \ell^+(T_N, i) \leq 4N^{11/16} \text{ and } \ell^-(T_N, i) \leq 4N^{11/16}.$$

We prove (6) for the case $|i - I^-| \leq N^{5/8} + 1$, since the other is similar. Let $i \in \mathbb{Z}$ so that $|i - I^-| < N^{5/8} + 1$. We notice that since $|I^- + (|x| + 2\theta)N| < N^{5/8}$ we have $I^-, i < 0$ when N is large enough, so (1) yields $|\ell^+(T_N, i) - \ell^+(T_N, I^-)| \leq \sum_{|j - I^-| < N^{5/8} + 1} |\eta_{j,-}(\ell^+(T_N, j))|$, thus since $\ell^+(T_N, I^-) = 0$ we have $\ell^+(T_N, i) \leq \sum_{|j - I^-| < N^{5/8} + 1} |\eta_{j,-}(\ell^+(T_N, j))|$. In addition, we assumed $(\mathcal{B}_2)^c$, hence $\ell^+(T_N, i) \leq \sum_{|j - I^-| < N^{5/8} + 1} (N^{1/16} + 1/2) \leq 3N^{5/8}N^{1/16} = 3N^{11/16}$ when N is large enough. Furthermore, (2) implies $|\ell^-(T_N, i) - \ell^+(T_N, i)| = |\eta_{i,-}(\ell^+(T_N, i))| \leq N^{1/16} + 1/2$ thanks to $(\mathcal{B}_2)^c$, hence $\ell^-(T_N, i) \leq 3N^{11/16} + N^{1/16} + 1/2 \leq 4N^{11/16}$ when N is large enough, which ends the proof of (6).

We now write T_N as the sum of the local times and relate $\frac{1}{N^{3/2}}(T_N - N^2(|x| + 2\theta)^2)$ to the integral of Y^+ and Y^- . We have $T_N = \sum_{i \in \mathbb{Z}} (\ell^+(T_N, i) + \ell^-(T_N, i))$. Moreover, Lemma 12 implies that for all $i \geq I^+$ and $i \leq I^-$ we have $\ell^+(T_N, i) = \ell^-(T_N, i) = 0$. Consequently, $T_N = \sum_{i=I^- \wedge (-(|x|+2\theta)N)}^{I^+ \vee (|x|+2\theta)N} (\ell^+(T_N, i) + \ell^-(T_N, i))$. We thus have $|\frac{1}{N^{3/2}}(T_N - N^2(|x| + 2\theta)^2) - \int_{(I^- \wedge (-(|x|+2\theta)N))/N}^{(I^+ \vee (|x|+2\theta)N)/N} (Y_N^+(y) + Y_N^-(y)) dy| \leq \frac{1}{N^{3/2}}(\ell^+(T_N, I^+ \vee \lfloor (|x| + 2\theta)N \rfloor) + \ell^-(T_N, I^+ \vee \lfloor (|x| + 2\theta)N \rfloor) + \ell^+(T_N, -\lfloor (|x| + 2\theta)N \rfloor - 1) + \ell^-(T_N, -\lfloor (|x| + 2\theta)N \rfloor - 1))$. Since we assumed $|I^- + (|x| + 2\theta)N| < N^{5/8}$ and $|I^+ - (|x| + 2\theta)N| < N^{5/8}$, equation (6) yields

$$(7) \quad \left| \frac{1}{N^{3/2}}(T_N - N^2(|x| + 2\theta)^2) - \int_{(I^- \wedge (-(|x|+2\theta)N))/N}^{(I^+ \vee (|x|+2\theta)N)/N} (Y_N^+(y) + Y_N^-(y)) dy \right| \leq \frac{1}{N^{3/2}} 16N^{11/16} = 16N^{-13/16}.$$

We now prove that $\int_{(I^+ \vee (|x|+2\theta)N)/N}^{(I^+ \vee (|x|+2\theta)N)/N} (Y_N^+(y) + Y_N^-(y)) dy$ is close to $2 \int_{-(|x|+2\theta)}^{|x|+2\theta} Y_N(y) dy$. We begin by considering $\int_{\chi(N)/N}^{(I^+ \vee (|x|+2\theta)N)/N} (Y_N^+(y) + Y_N^-(y)) dy$. We first assume $I^+ \geq (|x| + 2\theta)N$. Since we assumed $(\mathcal{B}_2)^c$, $(\mathcal{B}_4^-)^c$ and $(\mathcal{B}_4^+)^c$ occur, Lemma 20 yields $|\int_{\chi(N)/N}^{(I^+ \vee (|x|+2\theta)N)/N} (Y_N^+(y) + Y_N^-(y)) dy - 2 \int_{\chi(N)/N}^{|x|+2\theta} Y_N(y) dy| \leq 2(|x| + 2\theta - \frac{\chi(N)}{N})N^{-1/12} + \int_{|x|+2\theta}^{I^+/N} |Y_N^+(y) + Y_N^-(y)| dy$. In addition, we know $I^+ - (|x| + 2\theta)N \leq N^{5/8}$ and (6), hence

$$\begin{aligned} \int_{|x|+2\theta}^{I^+/N} |Y_N^+(y) + Y_N^-(y)| dy &\leq N^{-3/8} \frac{1}{\sqrt{N}} \left(\max_{\lfloor (|x|+2\theta)N \rfloor \leq i \leq I^+} \ell^+(T_N, i) + \max_{\lfloor (|x|+2\theta)N \rfloor \leq i \leq I^+} \ell^-(T_N, i) \right) \\ &\leq N^{-3/8} N^{-1/2} 8N^{11/16} = 8N^{-3/16}. \end{aligned}$$

We deduce

$$\left| \int_{\chi(N)/N}^{(I^+ \vee (|x|+2\theta)N)/N} (Y_N^+(y) + Y_N^-(y)) dy - 2 \int_{\chi(N)/N}^{|x|+2\theta} Y_N(y) dy \right| \leq 2 \left(|x| + 2\theta - \frac{\chi(N)}{N} \right) N^{-1/12} + 8N^{-3/16}.$$

We now assume $I^+ < (|x| + 2\theta)N$. In this case, we have

$$\begin{aligned} &\left| \int_{\chi(N)/N}^{(I^+ \vee (|x|+2\theta)N)/N} (Y_N^+(y) + Y_N^-(y)) dy - 2 \int_{\chi(N)/N}^{|x|+2\theta} Y_N(y) dy \right| \\ &\leq \int_{\chi(N)/N}^{I^+/N} |Y_N^+(y) + Y_N^-(y) - 2Y_N(y)| dy + \int_{I^+/N}^{|x|+2\theta} |Y_N^+(y) + Y_N^-(y)| dy + \int_{I^+/N}^{|x|+2\theta} |2Y_N(y)| dy. \end{aligned}$$

Moreover, Lemma 20 yields $\int_{\chi(N)/N}^{I^+/N} |Y_N^+(y) + Y_N^-(y) - 2Y_N(y)| dy \leq 2(|x| + 2\theta - \frac{\chi(N)}{N})N^{-1/12}$. Furthermore, for $y \geq \frac{I^+}{N}$ we have $\ell^\pm(T_N, \lfloor Ny \rfloor) = 0$. Since $|I^+ - (|x| + 2\theta)N| < N^{5/8}$ this yields $|Y_N^\pm(y)| \leq \frac{1}{2}N^{1/8}$. Thus $\int_{I^+/N}^{|x|+2\theta} |Y_N^+(y) + Y_N^-(y)| dy \leq \int_{I^+/N}^{|x|+2\theta} N^{1/8} dy \leq N^{-3/8} N^{1/8} = N^{-1/4}$. We deduce

$$\begin{aligned} &\left| \int_{\chi(N)/N}^{(I^+ \vee (|x|+2\theta)N)/N} (Y_N^+(y) + Y_N^-(y)) dy - 2 \int_{\chi(N)/N}^{|x|+2\theta} Y_N(y) dy \right| \\ &\leq 2 \left(|x| + 2\theta - \frac{\chi(N)}{N} \right) N^{-1/12} + N^{-1/4} + \int_{I^+/N}^{|x|+2\theta} |2Y_N(y)| dy. \end{aligned}$$

In addition, for any $y \in [\frac{I^+}{N}, |x| + 2\theta]$, we have $|Y_N(y)| \leq |Y_N(y) - Y_N(\frac{I^+}{N})| + |Y_N(\frac{I^+}{N}) - Y_N^-(\frac{I^+}{N})| + |Y_N^-(\frac{I^+}{N})|$. Lemma 20 yields that $|Y_N(\frac{I^+}{N}) - Y_N^-(\frac{I^+}{N})| \leq N^{-1/12}$, and since $|I^+ - (|x| + 2\theta)N| < N^{5/8}$ we have $|Y_N^-(\frac{I^+}{N})| = |\frac{1}{\sqrt{N}}(\ell^\pm(T_N, I^+) - N(\frac{\lfloor |x| - \lfloor I^+/N \rfloor}{2} + \theta)_+)| \leq \frac{1}{2}N^{1/8}$, hence

$$\begin{aligned} |Y_N(y)| &\leq \left| Y_N(y) - Y_N\left(\frac{I^+}{N}\right) \right| + N^{-1/12} + \frac{1}{2}N^{1/8} = \frac{1}{\sqrt{N}} \left| \sum_{i=I^+}^{\lfloor Ny \rfloor - 1} \zeta_i \right| + N^{-1/12} + \frac{1}{2}N^{1/8} \\ &\leq \frac{1}{\sqrt{N}} \sum_{i=I^+}^{\lfloor (|x|+2\theta)N \rfloor - 1} |\zeta_i| + N^{-1/12} + \frac{1}{2}N^{1/8} \leq \frac{1}{\sqrt{N}} N^{5/8} N^{1/16} + N^{-1/12} + \frac{1}{2}N^{1/8} \leq 2N^{3/16} \end{aligned}$$

since $(\mathcal{B}_2)^c$ occurs. This implies $\int_{I^+/\!N}^{|x|+2\theta} |2Y_N(y)|dy \leq \int_{I^+/\!N}^{|x|+2\theta} 4N^{3/16}dy = N^{-3/8}4N^{3/16} = 4N^{-3/16}$. We deduce $|\int_{\chi(N)/N}^{(I^+\vee(|x|+2\theta)N)/N} (Y_N^+(y) + Y_N^-(y))dy - 2\int_{\chi(N)/N}^{|x|+2\theta} Y_N(y)dy| \leq 2(|x| + 2\theta - \frac{\chi(N)}{N})N^{-1/12} + 5N^{-3/16}$. Consequently, in all cases we have $|\int_{\chi(N)/N}^{(I^+\vee(|x|+2\theta)N)/N} (Y_N^+(y) + Y_N^-(y))dy - 2\int_{\chi(N)/N}^{|x|+2\theta} Y_N(y)dy| \leq 2(|x| + 2\theta - \frac{\chi(N)}{N})N^{-1/12} + 8N^{-3/16}$. One can prove similarly that $|\int_{(I^-\wedge(-(|x|+2\theta)N))/N}^{\chi(N)/N} (Y_N^+(y) + Y_N^-(y))dy - 2\int_{-(|x|+2\theta)}^{\chi(N)/N} Y_N(y)dy| \leq 2(|x| + 2\theta + \frac{\chi(N)}{N})N^{-1/12} + 8N^{-3/16}$. We conclude that $|\int_{(I^-\wedge(-(|x|+2\theta)N))/N}^{(I^+\vee(|x|+2\theta)N)/N} (Y_N^+(y) + Y_N^-(y))dy - 2\int_{-(|x|+2\theta)}^{|x|+2\theta} Y_N(y)dy| \leq 4(|x| + 2\theta)N^{-1/12} + 16N^{-3/16}$.

We are now in position to conclude. Indeed, the previous result and (7) imply that when N is large enough, $|\frac{1}{N^{3/2}}(T_N - N^2(|x| + 2\theta)^2) - 2\int_{-(|x|+2\theta)}^{|x|+2\theta} Y_N(y)dy| \leq 16N^{-13/16} + 4(|x| + 2\theta)N^{-1/12} + 16N^{-3/16}$. Moreover, $(\mathcal{B}_2)^c$ occurs, hence Lemma 27 yields $\sup\{|Y_N(y) - Y'_N(y)| : y \in [-|x| - 2\theta, |x| + 2\theta]\} \leq N^{-7/16}$, therefore $|\int_{-(|x|+2\theta)}^{|x|+2\theta} Y_N(y)dy - \int_{-(|x|+2\theta)}^{|x|+2\theta} Y'_N(y)dy| \leq 2(|x| + 2\theta)N^{-7/16}$. We deduce that when N is large enough, $|\frac{1}{N^{3/2}}(T_N - N^2(|x| + 2\theta)^2) - 2\int_{-(|x|+2\theta)}^{|x|+2\theta} Y'_N(y)dy| \leq 16N^{-13/16} + 4(|x| + 2\theta)N^{-1/12} + 16N^{-3/16} + 4(|x| + 2\theta)N^{-7/16} \leq 5(|x| + 2\theta)N^{-1/12}$, which ends the proof. \square

Now that we know $\frac{1}{N^{3/2}}(T_N - N^2(|x| + 2\theta)^2)$ is close to $2\int_{-|x|-2\theta}^{|x|+2\theta} Y'_N(y)dy$, we need to prove $2\int_{-|x|-2\theta}^{|x|+2\theta} Y'_N(y)dy$ converges to the desired distribution. In order to do that, we will use the convergence of Y'_N to a Brownian motion stated in Lemma 26, so $2\int_{-|x|-2\theta}^{|x|+2\theta} Y'_N(y)dy$ will converge to the integral of a Brownian motion, the law of the latter being characterized by the following lemma, where we denote by $(B_t)_{t \in \mathbb{R}^+}$ a standard Brownian motion with $B_0 = 0$. This lemma is quite standard (the interested reader can find a proof in the first arXiv version of this paper [5]).

Lemma 34. *For any $y > 0$, the integral $\int_0^y B_z dz$ has distribution $\mathcal{N}(0, \frac{y^3}{3})$.*

We are now able to prove Proposition 4.

Proof of Proposition 4. Proposition 33 implies $\frac{1}{N^{3/2}}(T_N - N^2(|x| + 2\theta)^2) - 2\int_{-|x|-2\theta}^{|x|+2\theta} Y'_N(y)dy$ converges in probability to 0 when N tends to $+\infty$. Hence by Slutsky's Theorem, it is enough to prove $2\int_{-|x|-2\theta}^{|x|+2\theta} Y'_N(y)dy$ converges in distribution to $\mathcal{N}(0, \text{Var}(\rho_-) \frac{32}{3} ((|x| + \theta)^3 + \theta^3))$ when N tends to $+\infty$ to prove Proposition 4. In addition, by Lemma 26, $Y'_N|_{[-|x|-2\theta, |x|+2\theta]}$ converges in distribution to $B^x|_{[-|x|-2\theta, |x|+2\theta]}$ when N tends to $+\infty$ for the topology defined on $C[-|x| - 2\theta, |x| + 2\theta]$ by the uniform norm. Moreover, the integral between $-|x| - 2\theta$ and $|x| + 2\theta$ is continuous for this topology, hence $\int_{-|x|-2\theta}^{|x|+2\theta} Y'_N(y)dy$ converges in distribution to $\int_{-|x|-2\theta}^{|x|+2\theta} B_y^x dy$ when N tends to $+\infty$. Furthermore, B^x is a two-sided Brownian motion with $B_x^x = 0$ and variance $\text{Var}(\rho_-)$, hence we can write $\int_{-|x|-2\theta}^{|x|+2\theta} B_y^x dy = \int_{-|x|-2\theta}^x B_y^x dy + \int_x^{|x|+2\theta} B_y^x dy$ where $\int_{-|x|-2\theta}^x B_y^x dy$ and $\int_x^{|x|+2\theta} B_y^x dy$ are independent. In addition, $\int_x^{|x|+2\theta} B_y^x dy$ has the distribution of $\sqrt{\text{Var}(\rho_-)} \int_0^{2\theta} B_y dy$, which is $\mathcal{N}(0, \text{Var}(\rho_-) \frac{(2\theta)^3}{3})$ by Lemma 34, and $\int_{-|x|-2\theta}^x B_y^x dy$ has the distribution of $\sqrt{\text{Var}(\rho_-)} \int_0^{2|x|+2\theta} B_y dy$, which is $\mathcal{N}(0, \text{Var}(\rho_-) \frac{(2|x|+2\theta)^3}{3})$ by Lemma 34. We obtain that $\int_{-|x|-2\theta}^{|x|+2\theta} B_y^x dy$ has the distribution $\mathcal{N}(0, \text{Var}(\rho_-) \frac{(2|x|+2\theta)^3}{3} + \text{Var}(\rho_-) \frac{(2\theta)^3}{3}) = \mathcal{N}(0, \text{Var}(\rho_-) \frac{8}{3} ((|x| + \theta)^3 + \theta^3))$. Consequently, $\int_{-|x|-2\theta}^{|x|+2\theta} Y'_N(y)dy$ converges in distribution to $\mathcal{N}(0, \text{Var}(\rho_-) \frac{8}{3} ((|x| + \theta)^3 + \theta^3))$ when N tends to $+\infty$, which ends the proof of Proposition 4. \square

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