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# VARIATIONS ON TWO-PARAMETER FAMILIES OF ADVANCED TIME SERIES FORECASTING FUNCTIONS IV: SEASONAL MODELS AND APPLICATION TO UNIVERSITY ENROLLMENT

NABIL KAHOUADJI

**ABSTRACT.** Using the twenty four two-parameter families of advanced time series forecasting functions along with their associated seasonal models introduced in [6], we perform a series of seasonal forecasts (sixteen years times 3 trimesters) on eighteen university enrollment time series, and compare their performance to both Holt–Winters and ARIMA models. For each of these eighteen university enrollment time series, we show that our twenty four seasonal models outperform both the Holt–Winters and ARIMA models with respect to the sum of absolute errors, the number of closest forecasts, and the number of closest yearly forecasts.

**Keywords:** Time series forecasting, seasonal models, enrollment forecasting.  
MSC 2020: 90C10, 90C15, 62G99, 62P20, 62P99, 90C90, 60C99.

## 1. INTRODUCTION

This fourth paper is a continuation of a research program initiated in [6] in which we introduced twenty four two-parameter families of advanced time series forecasting functions, defined the concept of powering, derived nonseasonal and seasonal models with examples in education, sales, finance and economy, compared the performance of the latter models to both the Holt–Winters and ARIMA models, and applied and compared the performance of those twenty four models using five-year stock market data of 467 companies of the S&P500. We continue our research program with an application to university enrollment forecasting. Indeed, using eighteen university enrollment times series for the same university, from fall 2004 to summer 2022, we perform a series of seasonal forecasting using the seasonal per-period SAE stochastic latest-optimized  $\tilde{\alpha}$ -power balanced forecasting models [6], and compare their performance to both Holt–Winters and ARIMA models, using the sum of absolute error, the number of closest forecasts, and the number of closest yearly forecasts as metrics.

Time series forecasting is an important, required and a common task in business to help inform future decision making and planning. There is a wide range of time series forecasting methods, often developed within specific disciplines for specific purposes, each of which has its own level of performance and cost. Exponential smoothing [2, 4, 8] and ARIMA (Auto Regressive Integrated Moving Average) [1] methods are the two most widely used approaches to time series forecasting, and provide complementary approaches to the problem. While exponential smoothing models are based on a description of the trend and seasonality in the data, ARIMA models aim to describe the autocorrelations in the data [5]. For seasonal time series forecasting, both the exponential smoothing and ARIMA models require a decomposition of the seasonal time series into a trend, seasonal and random components. In [6], we introduced a new and different approach/method to advanced time series forecasting: given a time series  $X$  and a time series  $Y$  of the same size  $n$ , we defined twenty four two-parameter functions that output the forecasted/predicted  $(n + 1)$  data entry. These functions are then used as building blocs for various models for both nonseasonal and seasonal time series. Our approach is nonparametric, i.e., assumes no particular probability

distribution, and rely on the computations of estimates via three methods, and then choosing one of the estimate via a series of optimization processes using eight different optimization criteria. In section 2, we recall both the rate of interest and the rate of discount times series of a given time series  $Y$ , both of which are used to define  $\sharp$ - and  $\flat$ - mean and median forecast estimates of a given order for a given time series  $X$ . We then recall eight optimization criteria  $\kappa = 1, \dots, 8$ , which allows us to choose a preferred maximum order and a preferred optimization length, both of which are used to forecast the next (unknown) observation for the time series  $X$ . Combining both the mean and median forecasts enables us to define the  $\sharp$ - and  $\flat$ -forecasts, and then combining the  $\sharp$  and  $\flat$  estimates enables us to define the  $\natural$ -forecast, leading to twenty four new two-parameter families of forecasting functions. In Section 3, we recall the seasonal functionally balanced forecasting models, and recall the seasonal power time series mapping. We then recall the seasonal per-period sum of absolute error stochastic latest-optimized  $\alpha$ -power model. In Section 4, we apply the seasonal per-period sum of absolute error stochastic latest-optimized  $\alpha$ -power model to eighteen university enrollment (head counts and credit hours) time series (Examples 1–18) by computing the sequence of three-step-ahead forecasts for the last sixteen years (48 trimesters). We also compare our forecasts to both Holt–Winters and ARIMA Models. In particular, we compute the sum of absolute errors, and we count the number of times (out of 48 trimesterly forecasts) that each of the models lead to the best trimesterly and yearly forecasts. For all eighteen university enrollment time series, we find that each of our twenty four model variations outperform both Holt–Winters and ARIMA models.

## 2. TWO-PARAMETER FAMILIES OF FORECASTING FUNCTIONS

Let  $X = (x_1, x_2, \dots, x_n)$  and  $Y = (y_1, y_2, \dots, y_n)$  be two time series with  $n \geq 2$  observations each. We define the rate of interest time series  $R_Y$  associated to  $Y$  as the time series  $R_Y = (r_1, r_2, \dots, r_{n-1})$  of size  $n - 1$  defined by

$$r_i := \frac{y_{i+1} - y_i}{y_i} \quad \text{for } i = 1, \dots, n - 1 \quad (1)$$

We define also the rate of discount time series  $D_Y$  associated to  $Y$  as the time series  $D_Y = (d_1, d_2, \dots, d_{n-1})$  of size  $n - 1$  defined by

$$d_i := \frac{y_{i+1} - y_i}{y_{i+1}} \quad \text{for } i = 1, \dots, n - 1 \quad (2)$$

Given the rate of interest time series  $R_Y$ , we define the mean (resp., median)  $i$ th rate of interest  $\bar{r}_{i,\lambda}$  (resp.,  $\tilde{r}_{i,\lambda}$ ) of order  $\lambda$  as follows:

$$\bar{r}_{i,\lambda} = \text{mean}(r_{i-\lambda+1}, \dots, r_i) \quad \text{where } i = 1, \dots, n - 1 \text{ and } \lambda = 1, \dots, i - 1 \quad (3)$$

$$\tilde{r}_{i,\lambda} = \text{median}(r_{i-\lambda+1}, \dots, r_i) \quad \text{where } i = 1, \dots, n - 1 \text{ and } \lambda = 1, \dots, i - 1 \quad (4)$$

Similarly, given the rate of discount time series  $D_Y$ , we define the mean (resp., median)  $i$ th rate of discount  $\bar{d}_{i,\lambda}$  (resp.,  $\tilde{d}_{i,\lambda}$ ) of order  $\lambda$  as follows:

$$\bar{d}_{i,\lambda} = \text{mean}(d_{i-\lambda+1}, \dots, d_i) \quad \text{where } i = 1, \dots, n - 1 \text{ and } \lambda = 1, \dots, i - 1 \quad (5)$$

$$\tilde{d}_{i,\lambda} = \text{median}(d_{i-\lambda+1}, \dots, d_i) \quad \text{where } i = 1, \dots, n - 1 \text{ and } \lambda = 1, \dots, i - 1 \quad (6)$$

Given a time series  $X$  and a rate of interest times series  $R_Y$ , we define two types of forecast estimates of order  $\lambda$  for  $x_{i+1}$ . The  $\sharp$ -mean (resp.,  $\sharp$ -median) forecast estimate of  $x_{i+1}$  of order  $\lambda$ , denoted  $\sharp\hat{x}_{i+1,\lambda}$  (resp.,  $\sharp\widetilde{\hat{x}}_{i+1,\lambda}$ ), is defined as

$$\sharp\hat{x}_{i+1,\lambda} = x_i \cdot (1 + \bar{r}_{i-1,\lambda}) \quad \text{where } i = 2, \dots, n \text{ and } \lambda = 1, \dots, i \quad (7)$$

$$\sharp\widetilde{\hat{x}}_{i+1,\lambda} = x_i \cdot (1 + \tilde{r}_{i-1,\lambda}) \quad \text{where } i = 2, \dots, n \text{ and } \lambda = 1, \dots, i \quad (8)$$

Similarly, given a time series  $X$  and a rate of discount times series  $D_Y$ , we define another two types of forecast estimates of order  $\lambda$  for  $x_{i+1}$ . The  $\flat$ -mean (resp.,  $\flat$ -median) forecast estimate of  $x_{i+1}$  of order  $\lambda$ , denoted  ${}^{\flat}\widehat{x}_{i+1,\lambda}$  (resp.,  ${}^{\flat}\widetilde{x}_{i+1,\lambda}$ ), is defined as

$${}^{\flat}\widehat{x}_{i+1,\lambda} = x_i \div (1 - \bar{d}_{i-1,\lambda}) \quad \text{where } i = 2, \dots, n \text{ and } \lambda = 1, \dots, i \quad (9)$$

$${}^{\flat}\widetilde{x}_{i+1,\lambda} = x_i \div (1 - \widetilde{d}_{i-1,\lambda}) \quad \text{where } i = 2, \dots, n \text{ and } \lambda = 1, \dots, i \quad (10)$$

In what follows, the symbol  $\star$  stands for either  $\sharp$  or  $\flat$ . Given the  $\star$ -mean (resp.,  $\star$ -median) estimate forecasts  ${}^{\star}\widehat{x}_{i,\lambda}$  (resp.,  ${}^{\star}\widetilde{x}_{i,\lambda}$ ) of order  $\lambda$  for  $i = 3, \dots, n$  and  $\lambda = 1, \dots, i - 1$ , we define eight optimization criteria of sum of  $\star$ -mean (resp.,  $\star$ -median) estimate forecast errors to forecast  $x_i$  of length  $\nu$ , denoted  ${}^{\star}\Sigma_{i,\lambda,\nu}^{\kappa}$  (resp.,  ${}^{\star}\widetilde{\Sigma}_{i,\lambda,\nu}^{\kappa}$ ), where  $\nu = 1, \dots, n - 1$ ,  $\lambda + \nu < n$  and  $\kappa = 1, \dots, 8$ , by :

$${}^{\star}\Sigma_{i,\lambda,\nu}^1 := \sum_{j=i-\nu+1}^i \left( \frac{{}^{\star}\widehat{x}_{j,\lambda} - x_j}{{}^{\star}\widehat{x}_{j,\lambda}} \right)^2, \quad {}^{\star}\widetilde{\Sigma}_{i,\lambda,\nu}^1 := \sum_{j=i-\nu+1}^i \left( \frac{{}^{\star}\widetilde{x}_{j,\lambda} - x_j}{{}^{\star}\widetilde{x}_{j,\lambda}} \right)^2, \quad (11)$$

$${}^{\star}\Sigma_{i,\lambda,\nu}^2 := \sum_{j=i-\nu+1}^i \left| \frac{{}^{\star}\widehat{x}_{j,\lambda} - x_j}{{}^{\star}\widehat{x}_{j,\lambda}} \right|, \quad {}^{\star}\widetilde{\Sigma}_{i,\lambda,\nu}^2 := \sum_{j=i-\nu+1}^i \left| \frac{{}^{\star}\widetilde{x}_{j,\lambda} - x_j}{{}^{\star}\widetilde{x}_{j,\lambda}} \right|, \quad (12)$$

$${}^{\star}\Sigma_{i,\lambda,\nu}^3 := \sum_{j=i-\nu+1}^i \frac{({}^{\star}\widehat{x}_{j,\lambda} - x_j)^2}{|{}^{\star}\widehat{x}_{j,\lambda}|}, \quad {}^{\star}\widetilde{\Sigma}_{i,\lambda,\nu}^3 := \sum_{j=i-\nu+1}^i \frac{({}^{\star}\widetilde{x}_{j,\lambda} - x_j)^2}{|{}^{\star}\widetilde{x}_{j,\lambda}|}, \quad (13)$$

$${}^{\star}\Sigma_{i,\lambda,\nu}^4 := \sum_{j=i-\nu+1}^i \left( \frac{{}^{\star}\widehat{x}_{j,\lambda} - x_j}{x_j} \right)^2, \quad {}^{\star}\widetilde{\Sigma}_{i,\lambda,\nu}^4 := \sum_{j=i-\nu+1}^i \left( \frac{{}^{\star}\widetilde{x}_{j,\lambda} - x_j}{x_j} \right)^2, \quad (14)$$

$${}^{\star}\Sigma_{i,\lambda,\nu}^5 := \sum_{j=i-\nu+1}^i \left| \frac{{}^{\star}\widehat{x}_{j,\lambda} - x_j}{x_j} \right|, \quad {}^{\star}\widetilde{\Sigma}_{i,\lambda,\nu}^5 := \sum_{j=i-\nu+1}^i \left| \frac{{}^{\star}\widetilde{x}_{j,\lambda} - x_j}{x_j} \right|, \quad (15)$$

$${}^{\star}\Sigma_{i,\lambda,\nu}^6 := \sum_{j=i-\nu+1}^i \frac{({}^{\star}\widehat{x}_{j,\lambda} - x_j)^2}{|x_j|}, \quad {}^{\star}\widetilde{\Sigma}_{i,\lambda,\nu}^6 := \sum_{j=i-\nu+1}^i \frac{({}^{\star}\widetilde{x}_{j,\lambda} - x_j)^2}{|x_j|}, \quad (16)$$

$${}^{\star}\Sigma_{i,\lambda,\nu}^7 := \sum_{j=i-\nu+1}^i ({}^{\star}\widehat{x}_{j,\lambda} - x_j)^2, \quad {}^{\star}\widetilde{\Sigma}_{i,\lambda,\nu}^7 := \sum_{j=i-\nu+1}^i ({}^{\star}\widetilde{x}_{j,\lambda} - x_j)^2, \quad (17)$$

$${}^{\star}\Sigma_{i,\lambda,\nu}^8 := \sum_{j=i-\nu+1}^i |{}^{\star}\widehat{x}_{j,\lambda} - x_j|, \quad {}^{\star}\widetilde{\Sigma}_{i,\lambda,\nu}^8 := \sum_{j=i-\nu+1}^i |{}^{\star}\widetilde{x}_{j,\lambda} - x_j|. \quad (18)$$

For a given optimization criteria  $\kappa = 1, \dots, 8$ , a chosen maximum allowed order  $\lambda_{\max}$ , and a fixed length  $\nu$  such that  $\lambda_{\max} + \nu < n$ , the least sum of  $\star$ -mean (resp.,  $\star$ -median) estimate forecast errors  ${}^{\star}\Sigma_{\bar{n},\lambda_{\max},\nu}^{\kappa,*}$  (resp.,  ${}^{\star}\widetilde{\Sigma}_{\bar{n},\lambda_{\max},\nu}^{\kappa,*}$ ), to forecast  $x_n$ , of order  $\lambda_{\max}$  and length  $\nu$ , are defined as follows:

$${}^{\star}\Sigma_{\bar{n},\lambda_{\max},\nu}^{\kappa,*} = \min_{\lambda=1,\dots,\lambda_{\max}} {}^{\star}\Sigma_{\bar{n},\lambda,\nu}^{\kappa} \quad \text{and} \quad {}^{\star}\widetilde{\Sigma}_{\bar{n},\lambda_{\max},\nu}^{\kappa,*} = \min_{\lambda=1,\dots,\lambda_{\max}} {}^{\star}\widetilde{\Sigma}_{\bar{n},\lambda,\nu}^{\kappa} \quad (19)$$

From the latter, and for a given optimization criteria  $\kappa = 1, \dots, 8$ , a chosen maximum order  $\lambda_{\max} = 1, \dots, n - 1$  and fixed length  $\nu = 1, \dots, n - 1$  such that  $\lambda_{\max} + \nu < n$ , we extract the optimal  $\star$ -mean (resp.,  $\star$ -median) order  ${}^{\star}\lambda_{\lambda_{\max},\nu}^{k,*}$  (resp.,  ${}^{\star}\widetilde{\lambda}_{\lambda_{\max},\nu}^{k,*}$ ) from 1 to  $\lambda_{\max}$  to forecast  $x_n$  for a given length  $\nu$  as follows:

$${}^{\star}\lambda_{\lambda_{\max},\nu}^{k,*} = \arg \left( \min_{\lambda=1,\dots,\lambda_{\max}} {}^{\star}\Sigma_{\bar{n},\lambda_{\max},\nu}^{\kappa,*} \right) \quad \text{and} \quad {}^{\star}\widetilde{\lambda}_{\lambda_{\max},\nu}^{k,*} = \arg \left( \min_{\lambda=1,\dots,\lambda_{\max}} {}^{\star}\widetilde{\Sigma}_{\bar{n},\lambda_{\max},\nu}^{\kappa,*} \right) \quad (20)$$

These latter four optimal orders  $\star \lambda_{\lambda_{max}, \nu}^{k, \star}$  and  $\star \widetilde{\lambda}_{\lambda_{max}, \nu}^{k, \star}$ , where  $\star$  is either  $\sharp$  or  $\flat$ , are then used as preferred orders to forecast  $x_{n+1}$  from the  $n$  observations  $x_1, \dots, x_n$  and either rate of interest times series  $R(Y)$  or rate of discount time series  $D(Y)$ . Finally, we propose the following thirty two 2-parameter families of forecasting functions, that is for each of the four type of forecast estimate ( $\sharp$ -mean,  $\sharp$ -median,  $\flat$ -mean, and  $\flat$ -median) with each one of the eight optimization criteria  $\kappa = 1, \dots, 8$ .

**Definition 1.** Let  $X = (x_1, x_2, \dots, x_n)$  and  $Y = (y_1, y_2, \dots, y_n)$  be two time series with  $n \geq 2$  observations. Let  $R_Y$  be the rate of interest time series associated to  $Y$ . For each optimization criteria  $\kappa = 1, \dots, 8$ , a chosen maximum order  $\lambda_{max} = 1, \dots, n - 2$ , and a fixed length  $\nu$  in  $1, \dots, n - 2$  such that  $\lambda_{max} + \nu < n$ , the  $\sharp$ -mean (resp.,  $\sharp$ -median) forecasting function  $\sharp \Psi_{\lambda_{max}, \nu}^{\kappa}$  (resp.,  $\sharp \widetilde{\Psi}_{\lambda_{max}, \nu}^{\kappa}$ ) of order  $\lambda_{max}$  and length  $\nu$  maps the time series  $X$  and the rate of interest  $R_Y$  to a  $\sharp$ -mean (resp.,  $\sharp$ -median) forecast of  $x_{n+1}$ , denoted  $\sharp \widehat{x}_{n+1}^{\kappa}$  (resp.,  $\sharp \widetilde{\widehat{x}}_{n+1}^{\kappa}$ ), as follows:

$$\sharp \widehat{x}_{n+1}^{\kappa} = \sharp \Psi_{\lambda_{max}, \nu}^{\kappa}(X, R_Y) := x_n \cdot \left(1 + \bar{r}_{n-1, \sharp \lambda_{\lambda_{max}, \nu}^{\kappa, \star}}\right) \quad (21)$$

resp.,

$$\sharp \widetilde{\widehat{x}}_{n+1}^{\kappa} = \sharp \widetilde{\Psi}_{\lambda_{max}, \nu}^{\kappa}(X, R_Y) := x_n \cdot \left(1 + \widetilde{r}_{n-1, \sharp \widetilde{\lambda}_{\lambda_{max}, \nu}^{\kappa, \star}}\right) \quad (22)$$

where the  $\sharp$ -mean (resp.,  $\sharp$ -median) optimal order  $\sharp \lambda_{\lambda_{max}, \nu}^{k, \star}$  (resp.,  $\sharp \widetilde{\lambda}_{\lambda_{max}, \nu}^{k, \star}$ ) is as in (20).

**Definition 2.** Let  $X = (x_1, x_2, \dots, x_n)$  and  $Y = (y_1, y_2, \dots, y_n)$  be two time series with  $n \geq 2$  observations. Let  $D_Y$  be the rate of discount time series associated to  $Y$ . For each optimization criteria  $\kappa = 1, \dots, 8$ , a chosen maximum order  $\lambda_{max} = 1, \dots, n - 2$ , and a fixed length  $\nu$  in  $1, \dots, n - 2$  such that  $\lambda_{max} + \nu < n$ , the  $\flat$ -mean (resp.,  $\flat$ -median) forecasting function  $\flat \Psi_{\lambda_{max}, \nu}^{\kappa}$  (resp.,  $\flat \widetilde{\Psi}_{\lambda_{max}, \nu}^{\kappa}$ ) of order  $\lambda_{max}$  and length  $\nu$  maps the time series  $X$  and the rate of discount  $D_Y$  to a  $\flat$ -mean (resp.,  $\flat$ -median) forecast of  $x_{n+1}$ , denoted  $\flat \widehat{x}_{n+1}^{\kappa}$  (resp.,  $\flat \widetilde{\widehat{x}}_{n+1}^{\kappa}$ ), as follows:

$$\flat \widehat{x}_{n+1}^{\kappa} = \flat \Psi_{\lambda_{max}, \nu}^{\kappa}(X, D_Y) := x_n \div \left(1 - \bar{d}_{n-1, \flat \lambda_{\lambda_{max}, \nu}^{\kappa, \star}}\right) \quad (23)$$

resp.,

$$\flat \widetilde{\widehat{x}}_{n+1}^{\kappa} = \flat \widetilde{\Psi}_{\lambda_{max}, \nu}^{\kappa}(X, D_Y) := x_n \div \left(1 + \widetilde{d}_{n-1, \flat \widetilde{\lambda}_{\lambda_{max}, \nu}^{\kappa, \star}}\right) \quad (24)$$

where the  $\flat$ -mean (resp.,  $\flat$ -median) optimal order  $\flat \lambda_{\lambda_{max}, \nu}^{k, \star}$  (resp.,  $\flat \widetilde{\lambda}_{\lambda_{max}, \nu}^{k, \star}$ ) is as in (20).

One can optimize within a pair of forecast estimates, that is, choosing two forecast estimate types among the four. There are six possible pairs, and thus one can define another forty eight 2-parameter families of forecasting functions. We propose in the following only two pairs (among the six) leading to a  $\sharp$ - (resp.,  $\flat$ -) forecasting function, which combines the  $\sharp$ -mean (resp.,  $\flat$ -mean) and  $\sharp$ -median (resp.,  $\flat$ -median) forecasting functions.

**Definition 3.** Let  $X = (x_1, x_2, \dots, x_n)$  and  $Y = (y_1, y_2, \dots, y_n)$  be two time series with  $n \geq 2$  observations. Let  $R_Y$  (resp.,  $D_Y$ ) be the rate of interest (resp., discount) time series associated to  $Y$ . For each optimization criteria  $\kappa = 1, \dots, 8$ , a chosen maximum order  $\lambda_{max} = 1, \dots, n - 2$ , and a fixed length  $\nu$  in  $1, \dots, n - 2$  such that  $\lambda_{max} + \nu < n$ , the  $\sharp$ - (resp.,  $\flat$ -) forecasting function  $\sharp \Psi_{\lambda_{max}, \nu}^{\kappa}$  (resp.,  $\flat \Psi_{\lambda_{max}, \nu}^{\kappa}$ ) of order  $\lambda_{max}$  and length  $\nu$  maps the time series  $X$  and the rate of interest (resp., discount)  $R_Y$  (resp.,  $D_Y$ ) to a  $\sharp$ - (resp.,  $\flat$ -) forecast of  $x_{n+1}$ , denoted  $\sharp \widehat{x}_{n+1}^{\kappa}$  (resp.,  $\flat \widehat{x}_{n+1}^{\kappa}$ ), as follows:

$$\sharp \widehat{x}_{n+1}^{\kappa} = \sharp \Psi_{\lambda_{max}, \nu}^{\kappa}(X, R_Y) := \begin{cases} \sharp \widehat{x}_{n+1}^{\kappa} & \text{if } \sharp \sum_{\bar{n}, \lambda_{max}, \nu}^{\kappa, \star} \leq \sharp \sum_{\bar{n}, \lambda_{max}, \nu}^{\kappa, \star} \\ \flat \widehat{x}_{n+1}^{\kappa} & \text{if } \sharp \sum_{\bar{n}, \lambda_{max}, \nu}^{\kappa, \star} > \sharp \sum_{\bar{n}, \lambda_{max}, \nu}^{\kappa, \star} \end{cases} \quad (25)$$

resp.,

$$\widehat{x}_{n+1}^\kappa = {}^b\Psi_{\lambda_{max},\nu}^\kappa(X, D_Y) := \begin{cases} {}^b\widehat{x}_{n+1}^\kappa & \text{if } {}^b\Sigma_{\bar{n},\lambda_{max},\nu}^{\kappa,*} \leq {}^b\Sigma_{\bar{n},\lambda_{max},\nu}^{\kappa,*} \\ {}^b\widehat{x}_{n+1}^\kappa & \text{if } {}^b\Sigma_{\bar{n},\lambda_{max},\nu}^{\kappa,*} > {}^b\Sigma_{\bar{n},\lambda_{max},\nu}^{\kappa,*} \end{cases} \quad (26)$$

where the least sum of  $\#$ - (resp.,  $b$ -) mean estimate forecast error  $\# \Sigma_{\bar{n},\lambda_{max},\nu}^{\kappa,*}$  (resp.,  ${}^b \Sigma_{\bar{n},\lambda_{max},\nu}^{\kappa,*}$ ) and the least sum of  $\#$ - (resp.,  $b$ ) median estimate forecast error  $\# \Sigma_{\bar{n},\lambda_{max},\nu}^{\kappa,*}$  (resp.,  ${}^b \Sigma_{\bar{n},\lambda_{max},\nu}^{\kappa,*}$ ) are as in (19).

Similarly, one can optimize within a triple of forecast estimates. There are four possible triples, and thus one can add another thirty two 2-parameter families of forecasting functions. Lastly, one can optimize across the four forecast estimates, and thus add another eight 2-parameter families of forecasting functions, as shown in the following proposition.

**Definition 4.** Let  $X = (x_1, x_2, \dots, x_n)$  and  $Y = (y_1, y_2, \dots, y_n)$  be two time series with  $n \geq 2$  observations. Let  $R_Y$  (resp.,  $D_Y$ ) be the rate of interest (resp., discount) times series associated to  $Y$ . For each optimization criteria  $\kappa = 1, \dots, 8$ , a chosen maximum order  $\lambda_{max} = 1, \dots, n - 2$ , and fixed length  $\nu$  in  $1, \dots, n - 2$  such that  $\lambda_{max} + \nu < n$ , the forecasting function  ${}^b\Psi_{\lambda_{max},\nu}^\kappa$  of order  $\lambda_{max}$  and length  $\nu$  maps the time series  $X$ , the rate of interest  $R_Y$ , and the rate of discount  $D_Y$  to a forecast of  $x_{n+1}$ , denoted  ${}^b\widehat{x}_{n+1}^\kappa$ , as follows:

$${}^b\widehat{x}_{n+1}^\kappa = {}^b\Psi_{\lambda_{max},\nu}^\kappa(X, R_Y, D_Y) := \begin{cases} {}^b\widehat{x}_{n+1}^\kappa & \text{if } \# \Sigma_{n,\lambda_{max},\nu}^{\kappa,*} \leq {}^b \Sigma_{n,\lambda_{max},\nu}^{\kappa,*} \\ {}^b\widehat{x}_{n+1}^\kappa & \text{if } \# \Sigma_{n,\lambda_{max},\nu}^{\kappa,*} > {}^b \Sigma_{n,\lambda_{max},\nu}^{\kappa,*} \end{cases} \quad (27)$$

where the least  $\#$ -estimate forecast error  $\# \Sigma_{n,\lambda_{max},\nu}^{\kappa,*}$  is

$$\# \Sigma_{n,\lambda_{max},\nu}^{\kappa,*} := \min(\# \Sigma_{\bar{n},\lambda_{max},\nu}^{\kappa,*}, \# \Sigma_{\bar{n},\lambda_{max},\nu}^{\kappa,*}), \quad (28)$$

and the least  $b$ -estimate forecast error  ${}^b \Sigma_{n,\lambda_{max},\nu}^{\kappa,*}$  is

$${}^b \Sigma_{n,\lambda_{max},\nu}^{\kappa,*} := \min({}^b \Sigma_{\bar{n},\lambda_{max},\nu}^{\kappa,*}, {}^b \Sigma_{\bar{n},\lambda_{max},\nu}^{\kappa,*}) \quad (29)$$

Note that given a times series  $X$ , one can use its associated rate of interest  $R_X$  and/or rate of discount  $D_X$ , i.e., set  $Y = X$  in for forecasting functions. Note also that one can define a total of one hundred twenty forecasting functions. However, we focus our attention on only twenty four of them, i.e.,  $\# \Psi^\kappa$ ,  ${}^b \Psi^\kappa$  and  ${}^b \Psi^\kappa$  for  $\kappa = 1, \dots, 8$ .

### 3. SEASONAL FUNCTIONALLY BALANCED FORECASTING MODELS

Let  $X$  be a time series with a frequency  $f$  and size  $nf$ , i.e.,

$$X = (x_1, \dots, x_f, x_{f+1}, \dots, x_{2f}, \dots, x_{(n-1)f+1}, \dots, x_{nf}) \quad (30)$$

We denote by  $X_{s/f}$  the sub time series of size  $n$  defined as follows:

$$X_{s/f} = (x_s, x_{f+s}, x_{2f+s}, \dots, x_{(n-1)f+s}), \quad \text{where } s = 1, \dots, f \quad (31)$$

For instance, if  $X$  is a monthly time series starting in January, then the frequency is  $f = 12$  and  $X_{1/12}$  is the January sub time series,  $X_{7/12}$  is the July sub time series, and  $X_{12/12}$  is the December sub time series. Similarly, if  $X$  is a quarterly time series starting in the first quarter of the year (from January 1<sup>st</sup> to March 31<sup>st</sup>), then the frequency is  $f = 4$ ,  $X_{1/4}$  is the first quarter sub time series,  $X_{2/4}$  is the second quarter sub time series,  $X_{3/4}$  is the third quarter sub time series and  $X_{4/4}$  is the last quarter sub time series.

**Proposition 1** (Seasonal functionally balanced forecasting models). *Let  $X = (x_1, \dots, x_{nf})$  and  $Y = (y_1, \dots, y_{nf})$  be two time series with frequency  $f$  and with  $nf$  observations, where  $n \geq 2$ . For each optimization criteria  $\kappa = 1, \dots, 8$ , the  $\sharp$ - (resp.,  $\flat$ -,  $\natural$ -) functionally balanced forecasting function  $\sharp\Psi_f^\kappa$  (resp.,  $\flat\Psi_f^\kappa$ ,  $\natural\Psi_f^\kappa$ ) maps the seasonal time series  $X$  and the rate of interest (resp., discount) time series  $R_{\Phi(Y)}$  (resp.,  $D_{\Phi(Y)}$ ) to a  $\sharp$ - (resp.,  $\flat$ -,  $\natural$ -) balanced forecast of  $x_{nf+s}$ , denoted  $\sharp\hat{x}_{nf+s}^\kappa$  (resp.,  $\flat\hat{x}_{nf+s}^\kappa$ ,  $\natural\hat{x}_{nf+s}^\kappa$ ), where  $s = 1, \dots, f$ , and  $\Phi$  is a function that maps  $Y$  to a time series of the same size, as follows:*

$$\sharp\hat{x}_{nf+s}^\kappa = \sharp\Psi_f^\kappa(X, R_{\Phi(Y)}) := \sharp\Psi_{\lceil \frac{n-1}{2} \rceil, \lfloor \frac{n-1}{2} \rfloor}^\kappa(X_{s/f}, R_{\Phi(Y)_{s/f}}) \quad (32)$$

$$\flat\hat{x}_{nf+s}^\kappa = \flat\Psi_f^\kappa(X, D_{\Phi(Y)}) := \flat\Psi_{\lceil \frac{n-1}{2} \rceil, \lfloor \frac{n-1}{2} \rfloor}^\kappa(X_{s/f}, D_{\Phi(Y)_{s/f}}) \quad (33)$$

$$\natural\hat{x}_{nf+s}^\kappa = \natural\Psi_f^\kappa(X, R_{\Phi(Y)}, D_{\Phi(Y)}) := \natural\Psi_{\lceil \frac{n-1}{2} \rceil, \lfloor \frac{n-1}{2} \rfloor}^\kappa(X_{s/f}, R_{\Phi(Y)_{s/f}}, D_{\Phi(Y)_{s/f}}) \quad (34)$$

Therefore, given a seasonal time series of size  $nf$ , one can forecast the next  $f$  values of the time series, i.e.,  $\bullet\hat{x}_{nf+1}^\kappa, \bullet\hat{x}_{nf+2}^\kappa, \dots, \bullet\hat{x}_{(n+1)f}^\kappa$ , where  $\bullet$  is either  $\sharp$ ,  $\flat$  or  $\natural$ . In order to compare these seasonal functionally balanced forecasting functions for a given seasonal time series  $X = (x_1, \dots, x_{nf})$  with frequency  $f$ , where  $n \geq 2$ , let's compute the sequence of forecasts of  $x_{2f+1}$ ,  $x_{2f+2}$ ,  $\dots$ ,  $x_{nf}$ . For  $s = 1, \dots, f$ , and  $i = 2, \dots, n-1$ , the sequence of  $f$ -step-ahead forecasts for  $X$  are then

$$\left(\sharp\hat{x}_{if+s}^\kappa\right)_{i,s} := \left(\sharp\Psi_{\lceil \frac{i-1}{2} \rceil, \lfloor \frac{i-1}{2} \rfloor}^\kappa(X_{[1:i]_{s/f}}, R_{\Phi(Y_{[1:i]_{s/f}})})\right)_{i,s} \quad (35)$$

$$\left(\flat\hat{x}_{if+s}^\kappa\right)_{i,s} := \left(\flat\Psi_{\lceil \frac{i-1}{2} \rceil, \lfloor \frac{i-1}{2} \rfloor}^\kappa(X_{[1:i]_{s/f}}, D_{\Phi(Y_{[1:i]_{s/f}})})\right)_{i,s} \quad (36)$$

$$\left(\natural\hat{x}_{if+s}^\kappa\right)_{i,s} := \left(\natural\Psi_{\lceil \frac{i-1}{2} \rceil, \lfloor \frac{i-1}{2} \rfloor}^\kappa(X_{[1:i]_{s/f}}, R_{\Phi(Y_{[1:i]_{s/f}})}, D_{\Phi(Y_{[1:i]_{s/f}})})\right)_{i,s} \quad (37)$$

To compare these balanced forecasting functions, one may use two types of sum of absolute errors, or the sum of square errors: per period and global. Indeed, for each optimization criteria  $\kappa = 1, \dots, 8$ , the  $\sharp$ - (resp.,  $\flat$ -,  $\natural$ -) sum of absolute errors for the  $i^{\text{th}}$  period and the  $\sharp$ - (resp.,  $\flat$ -,  $\natural$ -) sum of square errors for the  $i^{\text{th}}$  period, where  $i = 3, \dots, n$ , are defined as follows:

$$\sharp\text{SAE}_i^\kappa := \sum_{s=1}^f \left| \sharp\hat{x}_{(i-1)f+s}^\kappa - x_{(i-1)f+s} \right| \quad \text{and} \quad \sharp\text{SSE}_i^\kappa := \sum_{s=3}^f \left( \sharp\hat{x}_{(i-1)f+j}^\kappa - x_{(i-1)f+s} \right)^2 \quad (38)$$

$$\flat\text{SAE}_i^\kappa := \sum_{s=1}^f \left| \flat\hat{x}_{(i-1)f+s}^\kappa - x_{(i-1)f+s} \right| \quad \text{and} \quad \flat\text{SSE}_i^\kappa := \sum_{s=3}^f \left( \flat\hat{x}_{(i-1)f+j}^\kappa - x_{(i-1)f+s} \right)^2 \quad (39)$$

$$\natural\text{SAE}_i^\kappa := \sum_{s=1}^f \left| \natural\hat{x}_{(i-1)f+s}^\kappa - x_{(i-1)f+s} \right| \quad \text{and} \quad \natural\text{SSE}_i^\kappa := \sum_{s=3}^f \left( \natural\hat{x}_{(i-1)f+j}^\kappa - x_{(i-1)f+s} \right)^2 \quad (40)$$

and the  $\sharp$ - (resp.,  $\flat$ -,  $\natural$ -) sum of absolute errors and the  $\sharp$ - (resp.,  $\flat$ -,  $\natural$ -) sum of square errors are defined as follows:

$$\sharp\text{SAE}^\kappa := \sum_{i=3}^n \sharp\text{SAE}_i^\kappa \quad \text{and} \quad \sharp\text{SSE}^\kappa := \sum_{i=3}^n \sharp\text{SSE}_i^\kappa \quad (41)$$

$$\flat\text{SAE}^\kappa := \sum_{i=3}^n \flat\text{SAE}_i^\kappa \quad \text{and} \quad \flat\text{SSE}^\kappa := \sum_{i=3}^n \flat\text{SSE}_i^\kappa \quad (42)$$

$$\natural\text{SAE}^\kappa := \sum_{i=3}^n \natural\text{SAE}_i^\kappa \quad \text{and} \quad \natural\text{SSE}^\kappa := \sum_{i=3}^n \natural\text{SSE}_i^\kappa \quad (43)$$

The model that minimizes either the sum of absolute errors or the sum of square errors (depending on the forecaster's interests) is said to be optimal for  $X$ . One can also count the number of optimal per-period sum of absolute or square errors. We illustrate the latter in the following two examples using one seasonal monthly time series and one seasonal quarterly time series using the following function  $\Phi$ .

**Definition 5** (Seasonal power time series mapping). *Let  $Y = (y_1, \dots, y_{nf})$  be a time series with frequency  $f$  and with  $nf$  observations, where  $n \geq 2$ . Let  $\alpha = (\alpha_1, \dots, \alpha_f)$  be an  $f$ -tuple where  $\alpha_s \in [0, 1]$  for  $s = 1, \dots, f$ . The seasonal power times series mapping  $\Phi$  maps  $Y$  and  $\alpha$  to a seasonal time series  $\Phi(Y)$  of the same size and frequency such that for each  $s = 1, \dots, f$ ,*

$$\Phi(Y)_{s/f} := Y_{s/f}^{\alpha_1} + Y_{(s+1)/f}^{\alpha_2} + \dots + Y_{f/f}^{\alpha_{f-s+1}} + Y_{1/f}^{\alpha_{f-s+2}} + Y_{2/f}^{\alpha_{f-s+3}} + \dots + Y_{(s-1)/f}^{\alpha_f}. \quad (44)$$

In Example 6 in [6], we used three fixed  $\alpha$  power to compute the seasonally balanced forecasts. While the sum of absolute errors of our  $\sharp$ -,  $\flat$ - and  $\natural$ -forecasts were consistently lower than both Holt–Winters and ARIMA models, the number of best monthly and/or yearly forecasts were fairly higher. Therefore, we introduced in [6] an optimized and dynamic method to select the  $\alpha$ -powers for the seasonally balanced forecasts. The latter leads to a substantially improved forecasts for seasonal time series.

**Definition 6** (Seasonal per-period SAE latest-optimized  $\alpha$ -power). *Let  $X = (x_1, \dots, x_{nf})$  and  $Y = (y_1, \dots, y_{nf})$  be two time series with frequency  $f$  and with  $nf$  observations, where  $n \geq 2$ . For each optimization criteria  $\kappa = 1, \dots, 8$ , the seasonal  $\sharp$  (resp.,  $\flat$ ,  $\natural$ ) SAE latest-optimized  $\alpha$ -power for the period  $i = 3, \dots, n$ , denoted  $\sharp\alpha_i^\kappa$  (resp.,  $\flat\alpha_i^\kappa$ ,  $\natural\alpha_i^\kappa$ ), are defined as the  $f$ -tuples*

$$\sharp\alpha_3^\kappa = \flat\alpha_3^\kappa = \natural\alpha_3^\kappa = (1, 1, \dots, 1) \quad (45)$$

and

$$\sharp\alpha_i^\kappa = (\sharp\alpha_{1,i}^\kappa, \dots, \sharp\alpha_{f,i}^\kappa) := \arg \left( \min_{\beta_1, \dots, \beta_f \in [0,1]} \sharp SAE_{i-1}^\kappa \right) \quad \text{for } i = 4, \dots, n \quad (46)$$

$$\flat\alpha_i^\kappa = (\flat\alpha_{1,i}^\kappa, \dots, \flat\alpha_{f,i}^\kappa) := \arg \left( \min_{\beta_1, \dots, \beta_f \in [0,1]} \flat SAE_{i-1}^\kappa \right) \quad \text{for } i = 4, \dots, n \quad (47)$$

$$\natural\alpha_i^\kappa = (\natural\alpha_{1,i}^\kappa, \dots, \natural\alpha_{f,i}^\kappa) := \arg \left( \min_{\beta_1, \dots, \beta_f \in [0,1]} \natural SAE_{i-1}^\kappa \right) \quad \text{for } i = 4, \dots, n \quad (48)$$

where  $\sharp SAE_i^\kappa$ ,  $\flat SAE_i^\kappa$  and  $\natural SAE_i^\kappa$  are as in (38), (39) and (40), and  $\Phi(Y)$  as in (44).

We define the sequence of seasonal per-period SSE latest-optimized  $\alpha$ -power in a similar way. While an analytic derivation of the sequence of these seasonal per-period SAE latest-optimized  $\alpha$ -powers can be (in practice) tedious, we introduce in the following stochastic (and more pragmatic) method to obtain an approximate of the sequence of these seasonal per-period SAE latest-optimized  $\alpha$ -powers, which we use for the forecasting.

**Definition 7** (Seasonal per-period SAE stochastic latest-optimized  $\alpha$ -power). *Let  $X = (x_1, \dots, x_{nf})$  and  $Y = (y_1, \dots, y_{nf})$  be two time series with frequency  $f$  and with  $nf$  observations, where  $n \geq 2$ . For each optimization criteria  $\kappa = 1, \dots, 8$ , the random  $\sharp$  (resp.,  $\flat$ ,  $\natural$ ) SAE  $\beta$ -power for the period  $i = 3, \dots, n$ , denoted  $\sharp_{\beta} \widetilde{SAE}_i^\kappa$  is the sum of absolute error for the  $i$ th period where the power  $\beta = (\beta_1, \dots, \beta_f)$  in  $\Phi(Y)$  are such that each of the  $\beta_i$  are drawn from a uniform distribution on the interval  $[0, 1]$ . Thus, for each optimization criteria  $\kappa = 1, \dots, 8$ , the seasonal  $\sharp$  (resp.,  $\flat$ ,  $\natural$ ) SAE stochastic latest-optimized  $\alpha$ -power for the period  $i = 3, \dots, n$ , denoted  $\sharp\widetilde{\alpha}_i^\kappa$  (resp.,  $\flat\widetilde{\alpha}_i^\kappa$ ,  $\natural\widetilde{\alpha}_i^\kappa$ ), are defined as the  $f$ -tuples*

$$\sharp\widetilde{\alpha}_3^\kappa = \flat\widetilde{\alpha}_3^\kappa = \natural\widetilde{\alpha}_3^\kappa = (1, 1, \dots, 1) \quad (49)$$

and

$$\# \tilde{\alpha}_i^\kappa = (\# \tilde{\alpha}_{1,i}^\kappa, \dots, \# \tilde{\alpha}_{f,i}^\kappa) := \arg_{\beta_1, \dots, \beta_f} \left( \min_{j=1, \dots, N} \# \beta \widetilde{SAE}_{i-1}^\kappa \right) \quad \text{for } i = 4, \dots, n \quad (50)$$

$$^b \tilde{\alpha}_i^\kappa = (^b \tilde{\alpha}_{1,i}^\kappa, \dots, ^b \tilde{\alpha}_{f,i}^\kappa) := \arg_{\beta_1, \dots, \beta_f} \left( \min_{j=1, \dots, N} ^b \beta \widetilde{SAE}_{i-1}^\kappa \right) \quad \text{for } i = 4, \dots, n \quad (51)$$

$$\natural \tilde{\alpha}_i^\kappa = (\natural \tilde{\alpha}_{1,i}^\kappa, \dots, \natural \tilde{\alpha}_{f,i}^\kappa) := \arg_{\beta_1, \dots, \beta_f} \left( \min_{j=1, \dots, N} \natural \beta \widetilde{SAE}_{i-1}^\kappa \right) \quad \text{for } i = 4, \dots, n \quad (52)$$

where  $N$  is a large number (for instance,  $N = 1000$ ).

**Proposition 2** (Seasonal per-period SAE stochastic latest-optimized  $\tilde{\alpha}$ -power balanced forecasting models). *Let  $X = (x_1, \dots, x_{nf})$  and  $Y = (y_1, \dots, y_{nf})$  be two time series with frequency  $f$  and with  $nf$  observations, where  $n \geq 2$ . For each optimization criteria  $\kappa = 1, \dots, 8$ , the  $\#$ - (resp.,  $^b$ -,  $\natural$ -) seasonal per-period SAE stochastic latest-optimized  $\tilde{\alpha}$ -power balanced forecasting function  $\# \Psi_f^\kappa$  (resp.,  $^b \Psi_f^\kappa$ ,  $\natural \Psi_f^\kappa$ ) maps the seasonal time series  $X$  and the rate of interest (resp., discount) time series  $R_{\tilde{\Phi}(Y)}$  (resp.,  $D_{\tilde{\Phi}(Y)}$ ) to a  $\#$ - (resp.,  $^b$ -,  $\natural$ -) balanced forecast of  $x_{nf+s}$ , denoted  $\# \hat{x}_{nf+s}^\kappa$  (resp.,  $^b \hat{x}_{nf+s}^\kappa$ ,  $\natural \hat{x}_{nf+s}^\kappa$ ), where  $s = 1, \dots, f$ , and  $\tilde{\Phi}$  is as in (44) with the  $\#$ -power  $\# \tilde{\alpha}^\kappa$  (resp.,  $^b \tilde{\alpha}^\kappa$ ,  $\natural \tilde{\alpha}^\kappa$ ) is as in (50) (resp., (51), (52)), as follows:*

$$\# \hat{x}_{nf+s}^\kappa = \# \Psi_f^\kappa(X, R_{\tilde{\Phi}(Y)}) := \# \Psi_{\lceil \frac{n-1}{2} \rceil, \lfloor \frac{n-1}{2} \rfloor}^\kappa(X_{s/f}, R_{\tilde{\Phi}(Y)_{s/f}}) \quad (53)$$

$$^b \hat{x}_{nf+s}^\kappa = ^b \Psi_f^\kappa(X, D_{\tilde{\Phi}(Y)}) := ^b \Psi_{\lceil \frac{n-1}{2} \rceil, \lfloor \frac{n-1}{2} \rfloor}^\kappa(X_{s/f}, D_{\tilde{\Phi}(Y)_{s/f}}) \quad (54)$$

$$\natural \hat{x}_{nf+s}^\kappa = \natural \Psi_f^\kappa(X, R_{\tilde{\Phi}(Y)}, D_{\tilde{\Phi}(Y)}) := \natural \Psi_{\lceil \frac{n-1}{2} \rceil, \lfloor \frac{n-1}{2} \rfloor}^\kappa(X_{s/f}, R_{\tilde{\Phi}(Y)_{s/f}}, D_{\tilde{\Phi}(Y)_{s/f}}) \quad (55)$$

Thus, for  $s = 1, \dots, f$ , and  $i = 2, \dots, n-1$ , the sequence of  $f$ -step-ahead forecasts for  $X$  are then

$$\left( \# \hat{x}_{if+s}^\kappa \right)_{i,s} := \left( \# \Psi_{\lceil \frac{i-1}{2} \rceil, \lfloor \frac{i-1}{2} \rfloor}^\kappa(X_{[1:i]_{s/f}}, R_{\tilde{\Phi}(Y_{[1:i]_{s/f}})}) \right)_{i,s} \quad (56)$$

$$\left( ^b \hat{x}_{if+s}^\kappa \right)_{i,s} := \left( ^b \Psi_{\lceil \frac{i-1}{2} \rceil, \lfloor \frac{i-1}{2} \rfloor}^\kappa(X_{[1:i]_{s/f}}, D_{\tilde{\Phi}(Y_{[1:i]_{s/f}})}) \right)_{i,s} \quad (57)$$

$$\left( \natural \hat{x}_{if+s}^\kappa \right)_{i,s} := \left( \natural \Psi_{\lceil \frac{i-1}{2} \rceil, \lfloor \frac{i-1}{2} \rfloor}^\kappa(X_{[1:i]_{s/f}}, R_{\tilde{\Phi}(Y_{[1:i]_{s/f}})}, D_{\tilde{\Phi}(Y_{[1:i]_{s/f}})}) \right)_{i,s} \quad (58)$$

#### 4. APPLICATION TO UNIVERSITY ENROLLMENT

Let's consider eighteen Northeastern Illinois University time series of enrollment and credit hours from fall 2004 to summer 2022. Table 1 contains eight of these times series. The remaining ten time series are derived by addition of such time series.

- $X_1$  (resp.,  $X_{10}$ ) represent the time series of total undergraduate enrollment (resp., credit hours), and is defined as  $X_1 = X_2 + X_3$  (resp.,  $X_{10} = X_{11} + X_{12}$ ).
- $X_2$  (resp.,  $X_{11}$ ) represent the time series of continuing undergraduate enrollment (resp., credit hours).
- $X_3$  (resp.,  $X_{12}$ ) represent the time series of new undergraduate enrollment (resp., credit hours).
- $X_4$  (resp.,  $X_{13}$ ) represent the time series of total graduate enrollment (resp., credit hours), and is defined as  $X_4 = X_5 + X_6$  (resp.,  $X_{13} = X_{14} + X_{15}$ ).
- $X_5$  (resp.,  $X_{14}$ ) represent the time series of continuing graduate enrollment (resp., credit hours).
- $X_6$  (resp.,  $X_{15}$ ) represent the time series of new graduate enrollment (resp., credit hours).
- $X_7$  (resp.,  $X_{16}$ ) represent the time series of total undergraduate + graduate enrollment (resp., credit hours), and is defined as  $X_7 = X_1 + X_4$  (resp.,  $X_{16} = X_{10} + X_{13}$ ).

- $X_8$  (resp.,  $X_{17}$ ) represent the time series of continuing undergraduate + graduate enrollment (resp., credit hours), and is defined as  $X_8 = X_2 + X_5$  (resp.,  $X_{17} = X_{11} + X_{14}$ ).
- $X_9$  (resp.,  $X_{18}$ ) represent the time series of new undergraduate + graduate enrollment (resp., credit hours), and is defined as  $X_9 = X_3 + X_6$  (resp.,  $X_{18} = X_{12} + X_{15}$ ).

| Term        | $X_2$ | $X_3$ | $X_5$ | $X_6$ | $X_{11}$ | $X_{12}$ | $X_{14}$ | $X_{15}$ |
|-------------|-------|-------|-------|-------|----------|----------|----------|----------|
| Fall 2004   | 6723  | 2582  | 2174  | 685   | 66952    | 27802    | 10970    | 3665     |
| Spring 2005 | 7791  | 853   | 2392  | 368   | 79796    | 8080.5   | 12345    | 1847.5   |
| Summer 2005 | 3742  | 601   | 1889  | 412   | 21773    | 3207     | 8812     | 1813     |
| Fall 2005   | 6812  | 2606  | 2065  | 744   | 68614.5  | 27934.5  | 10485.5  | 4242     |
| Spring 2006 | 7890  | 801   | 2402  | 435   | 80561    | 7499     | 12785.5  | 2262.5   |
| Summer 2006 | 3757  | 568   | 1839  | 303   | 21708    | 2994     | 8956     | 1303     |
| Fall 2006   | 6710  | 2547  | 2148  | 651   | 66656.5  | 27819    | 11121    | 3710.5   |
| Spring 2007 | 7787  | 831   | 2353  | 296   | 78701    | 7840.5   | 12420    | 1472     |
| Summer 2007 | 3682  | 527   | 1716  | 230   | 21528.5  | 2820     | 8002     | 1014.5   |
| Fall 2007   | 6635  | 2480  | 2015  | 514   | 66213.5  | 26682    | 10221    | 2767     |
| Spring 2008 | 7658  | 748   | 2133  | 267   | 77788    | 7103     | 11022    | 1402     |
| Summer 2008 | 3667  | 444   | 1501  | 261   | 22097    | 2474     | 7020.5   | 1106.5   |
| Fall 2008   | 6537  | 2450  | 1652  | 554   | 65679.5  | 26822    | 8649     | 2933.5   |
| Spring 2009 | 7654  | 682   | 1810  | 441   | 79042.5  | 6536     | 9464     | 2367     |
| Summer 2009 | 3549  | 573   | 1411  | 403   | 21155    | 3218     | 6970     | 1920     |
| Fall 2009   | 6422  | 2769  | 1816  | 624   | 65681    | 30651    | 9445     | 3609     |
| Spring 2010 | 7852  | 908   | 1830  | 476   | 82308    | 8950.5   | 9845.5   | 2687     |
| Summer 2010 | 3580  | 581   | 1342  | 350   | 21866    | 3292     | 6483     | 1664     |
| Fall 2010   | 6757  | 2741  | 1533  | 715   | 70106    | 30859    | 7964     | 4358     |
| Spring 2011 | 8038  | 902   | 1767  | 459   | 84263.5  | 8671     | 9401.5   | 2753     |
| Summer 2011 | 3607  | 458   | 1281  | 292   | 22395    | 2717     | 6274.5   | 1421     |
| Fall 2011   | 6819  | 2602  | 1501  | 658   | 70515    | 29115    | 7876     | 4158     |
| Spring 2012 | 7964  | 792   | 1630  | 415   | 82889.5  | 7566     | 9004     | 2463.5   |
| Summer 2012 | 3147  | 594   | 1087  | 244   | 18303    | 3241     | 5123.5   | 1257     |
| Fall 2012   | 6516  | 2624  | 1342  | 667   | 67272    | 29258.5  | 7196.5   | 4007     |
| Spring 2013 | 7648  | 790   | 1465  | 373   | 79634    | 7758.5   | 7942     | 2182     |
| Summer 2013 | 3036  | 553   | 1009  | 173   | 17427    | 2949     | 4694     | 813      |
| Fall 2013   | 6478  | 2472  | 1304  | 567   | 65500    | 26139.5  | 6940     | 3298     |
| Spring 2014 | 7235  | 722   | 1478  | 268   | 74053.5  | 6517     | 8058     | 1558     |
| Summer 2014 | 2888  | 548   | 988   | 165   | 16219    | 2938     | 4750     | 800      |
| Fall 2014   | 6086  | 2326  | 1299  | 564   | 61502.5  | 24647    | 6870.5   | 3447     |
| Spring 2015 | 6847  | 680   | 1497  | 219   | 69983    | 6360     | 8360     | 1305     |
| Summer 2015 | 2830  | 526   | 1023  | 197   | 15695    | 2902     | 4954     | 1008     |
| Fall 2015   | 5776  | 2319  | 1191  | 605   | 58270    | 24562    | 6518     | 3607     |
| Spring 2016 | 6520  | 751   | 1494  | 325   | 66691    | 7061     | 8389     | 1624     |
| Summer 2016 | 2637  | 511   | 1116  | 132   | 14943    | 2742     | 5717     | 606      |
| Fall 2016   | 5345  | 2331  | 1266  | 596   | 53850.5  | 24524.5  | 6750     | 3761     |
| Spring 2017 | 6139  | 717   | 1567  | 273   | 62371.5  | 6595     | 8600     | 1435     |
| Summer 2017 | 2395  | 426   | 1149  | 205   | 13661    | 2824     | 5664     | 1195     |
| Fall 2017   | 4842  | 2276  | 1263  | 603   | 49268    | 24287    | 6820     | 3658     |
| Spring 2018 | 5697  | 595   | 1517  | 242   | 59103.5  | 5599     | 8662     | 1278.5   |
| Summer 2018 | 2297  | 350   | 1102  | 176   | 13304    | 2250     | 5551     | 925      |
| Fall 2018   | 4646  | 1754  | 1168  | 535   | 47271    | 18472    | 6340     | 3474     |
| Spring 2019 | 5115  | 490   | 1373  | 248   | 52945.5  | 4643     | 8208     | 1428     |
| Summer 2019 | 2189  | 315   | 977   | 174   | 12621    | 1839     | 5006     | 880      |
| Fall 2019   | 4037  | 1664  | 1113  | 609   | 41859.5  | 17758    | 6360     | 4049     |
| Spring 2020 | 4556  | 482   | 1333  | 392   | 48376    | 4507     | 8184     | 2362     |
| Summer 2020 | 2224  | 275   | 1014  | 263   | 14185    | 1626     | 5170     | 1463     |
| Fall 2020   | 3745  | 1475  | 1287  | 612   | 39523.5  | 15432    | 8023.5   | 4071     |
| Spring 2021 | 4251  | 468   | 1579  | 319   | 44765    | 4421     | 10096    | 1815     |
| Summer 2021 | 1901  | 265   | 1178  | 212   | 11978    | 1512     | 5859     | 1075     |
| Fall 2021   | 3311  | 1297  | 1303  | 529   | 34305.5  | 13687    | 7787     | 3681     |
| Spring 2022 | 3637  | 413   | 1467  | 245   | 38621    | 3765     | 9147.5   | 1573.5   |
| Summer 2022 | 1511  | 220   | 962   | 151   | 8734     | 1081     | 4577     | 776      |

TABLE 1. Northeastern Illinois University Enrollment and Credit Hours

*Example 1.* Let's consider undergraduate total enrollment (fall, spring and summer) over 18 years (fall 2004 to summer 2022). Thus,  $f = 3$  and  $X = (x_1, \dots, x_{54})$ . For  $s = 1, \dots, 3$  (trimesters), and  $i = 2, \dots, 17$  (years), we compute the sequence of three-step-ahead forecasts for  $X$  using both Holt–Winters and ARIMA models. The sum of absolute errors for each model are

$$SAE^{HW} = \sum_{j=7}^{54} |\hat{x}_j^{HW} - x_j| = 6,329 \text{ and } SAE^{ARIMA} = \sum_{j=7}^{54} |\hat{x}_j^{ARIMA} - x_j| = 12,556 \quad (59)$$

Comparing these 16 years times 3 trimesters forecasts, i.e., 48 trimester forecasts, the Holt–winters model lead to better trimesterly forecast (smallest absolute residual) 28 times as opposed

to 20 times for the ARIMA model, i.e.,  $HW/AR = 28/20$ . Moreover, if one compares the sum of absolute error per year, the Holt–Winters model lead to a better yearly forecast nine times as opposed to seven times for ARIMA, i.e.,  $HW/AR = 9/7$ .

For  $s = 1, \dots, 3$  (trimesters), and  $i = 2, \dots, 17$  (years), we compute the sequence of three-step-ahead  $\sharp, b$  and  $\natural$  forecasts for  $X$  as in (56), (57) and (58) for each optimization criteria  $\kappa = 1, \dots, 8$  using the seasonal per-period SAE stochastic latest-optimized  $\tilde{\alpha}$ -power balanced forecasting models (with  $N = 1000$  to find the optimal latest-optimized alpha power). Moreover, we compute the (global) sum of absolute error, and we compare our forecasts to both Holt–Winters and ARIMA models by counting the number of best monthly and yearly forecasts. The results are summarized in Table 2.

| $\bullet \kappa$ | SAE   | HW / AR / K. | HW / K. | AR / K. | HW / AR / K. | HW / K. | AR / K. |
|------------------|-------|--------------|---------|---------|--------------|---------|---------|
| #1               | 4,936 | 12 / 9 / 27  | 17 / 31 | 13 / 35 | 4 / 2 / 10   | 5 / 11  | 4 / 12  |
| #2               | 4,952 | 13 / 12 / 23 | 18 / 30 | 16 / 32 | 4 / 2 / 10   | 5 / 11  | 4 / 12  |
| #3               | 4,921 | 13 / 9 / 26  | 18 / 30 | 13 / 35 | 5 / 2 / 9    | 6 / 10  | 4 / 12  |
| #4               | 4,868 | 13 / 9 / 26  | 18 / 30 | 13 / 35 | 4 / 2 / 10   | 5 / 11  | 4 / 12  |
| #5               | 4,882 | 13 / 10 / 25 | 18 / 30 | 14 / 34 | 4 / 2 / 10   | 5 / 11  | 4 / 12  |
| #6               | 4,915 | 13 / 10 / 25 | 18 / 30 | 14 / 34 | 4 / 2 / 10   | 5 / 11  | 4 / 12  |
| #7               | 4,900 | 12 / 9 / 27  | 17 / 31 | 13 / 35 | 5 / 2 / 9    | 6 / 10  | 4 / 12  |
| #8               | 4,917 | 11 / 11 / 26 | 16 / 32 | 15 / 33 | 4 / 2 / 10   | 5 / 11  | 4 / 12  |
| b1               | 4,984 | 12 / 10 / 26 | 17 / 31 | 14 / 34 | 4 / 2 / 10   | 5 / 11  | 4 / 12  |
| b2               | 4,926 | 12 / 10 / 26 | 17 / 31 | 14 / 34 | 4 / 2 / 10   | 5 / 11  | 4 / 12  |
| b3               | 4,960 | 11 / 11 / 26 | 16 / 32 | 15 / 33 | 4 / 2 / 10   | 5 / 11  | 4 / 12  |
| b4               | 4,961 | 13 / 11 / 24 | 18 / 30 | 16 / 32 | 4 / 2 / 10   | 5 / 11  | 4 / 12  |
| b5               | 4,886 | 13 / 12 / 23 | 18 / 30 | 16 / 32 | 4 / 2 / 10   | 5 / 11  | 4 / 12  |
| b6               | 4,944 | 12 / 11 / 25 | 17 / 31 | 15 / 33 | 4 / 2 / 10   | 5 / 11  | 4 / 12  |
| b7               | 4,945 | 11 / 10 / 27 | 16 / 32 | 15 / 33 | 4 / 2 / 10   | 5 / 11  | 4 / 12  |
| b8               | 4,964 | 12 / 9 / 27  | 17 / 31 | 12 / 36 | 4 / 2 / 10   | 5 / 11  | 4 / 12  |
| ‡1               | 4,891 | 13 / 10 / 25 | 18 / 30 | 15 / 33 | 4 / 2 / 10   | 5 / 11  | 4 / 12  |
| ‡2               | 4,932 | 12 / 10 / 26 | 17 / 31 | 14 / 34 | 4 / 2 / 10   | 5 / 11  | 4 / 12  |
| ‡3               | 4,979 | 10 / 11 / 27 | 16 / 32 | 14 / 34 | 4 / 2 / 10   | 5 / 11  | 4 / 12  |
| ‡4               | 4,905 | 12 / 12 / 24 | 17 / 31 | 16 / 32 | 4 / 2 / 10   | 5 / 11  | 4 / 12  |
| ‡5               | 4,849 | 12 / 10 / 26 | 17 / 31 | 14 / 34 | 4 / 2 / 10   | 5 / 11  | 4 / 12  |
| ‡6               | 4,902 | 12 / 11 / 25 | 18 / 30 | 15 / 33 | 4 / 2 / 10   | 5 / 11  | 4 / 12  |
| ‡7               | 4,944 | 13 / 10 / 25 | 18 / 30 | 14 / 34 | 4 / 2 / 10   | 5 / 11  | 4 / 12  |
| ‡8               | 4,945 | 12 / 10 / 26 | 17 / 31 | 14 / 34 | 4 / 2 / 10   | 5 / 11  | 4 / 12  |

TABLE 2. Undergraduate Total Enrollment: Comparison of the seasonal per-period SAE stochastic latest-optimized  $\tilde{\alpha}$ -power balanced forecasting models to the seasonal Holt–Winters and ARIMA models

*Example 2.* Let's consider undergraduate continuing enrollment (fall, spring and summer) over 18 years (fall 2004 to summer 2022). Thus,  $f = 3$  and  $X = (x_1, \dots, x_{54})$ . For  $s = 1, \dots, 3$  (trimesters), and  $i = 2, \dots, 17$  (years), we compute the sequence of three-step-ahead forecasts for  $X$  using both Holt–Winters and ARIMA models. The sum of absolute errors for each model are

$$\text{SAE}^{\text{HW}} = \sum_{j=7}^{54} \left| \hat{x}_j^{\text{HW}} - x_j \right| = 6,115 \quad \text{and} \quad \text{SAE}^{\text{ARIMA}} = \sum_{j=7}^{54} \left| \hat{x}_j^{\text{ARIMA}} - x_j \right| = 11,884 \quad (60)$$

Comparing these 16 years times 3 trimesters forecasts, i.e., 48 trimester forecasts, the Holt–Winters model lead to better trimesterly forecast (smallest absolute residual) 30 times as opposed to 18 times for the ARIMA model, i.e.,  $HW/AR = 30/18$ . Moreover, if one compares the sum of absolute error per year, the Holt–Winters model lead to a better yearly forecast twelve times as opposed to four times for ARIMA, i.e.,  $HW/AR = 12/4$ .

For  $s = 1, \dots, 3$  (trimesters), and  $i = 2, \dots, 17$  (years), we compute the sequence of three-step-ahead  $\sharp, b$  and  $\natural$  forecasts for  $X$  as in (56), (57) and (58) for each optimization criteria  $\kappa = 1, \dots, 8$  using the seasonal per-period SAE stochastic latest-optimized  $\tilde{\alpha}$ -power balanced forecasting models (with  $N = 1000$  to find the optimal latest-optimized alpha power). Moreover,

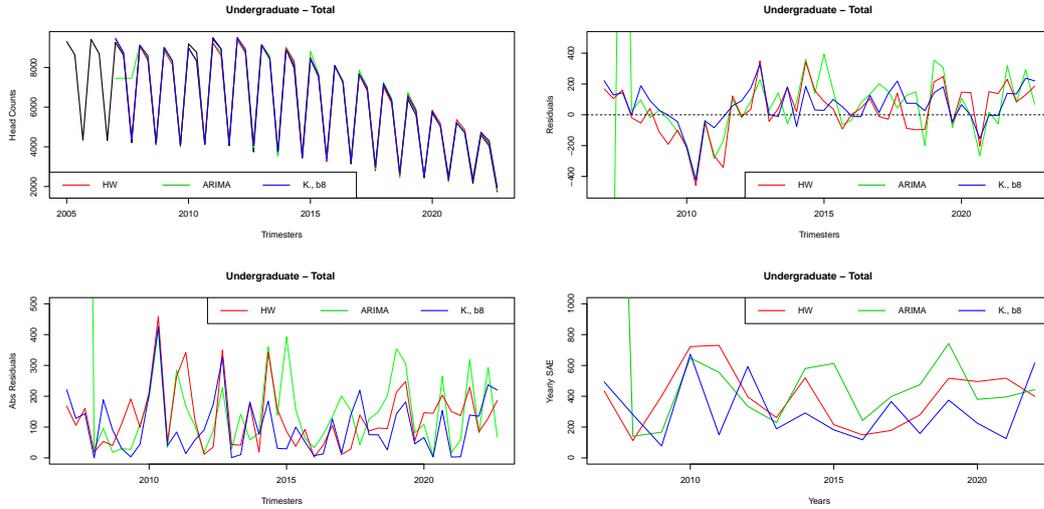


FIGURE 1. Undergraduate Total Enrollment: Forecasts, residuals, absolute residuals and yearly sum of absolute errors

we compute the (global) sum of absolute error, and we compare our forecasts to both Holt–Winters and ARIMA models by counting the number of best monthly and yearly forecasts. The results are summarized in Table 3.

| • $f_c$ | SAE   | HW / AR / K. | HW / K. | AR / K. | HW / AR / K. | HW / K. | AR / K. |
|---------|-------|--------------|---------|---------|--------------|---------|---------|
| #1      | 4,408 | 13 / 6 / 29  | 16 / 32 | 8 / 40  | 6 / 0 / 10   | 6 / 10  | 2 / 14  |
| #2      | 4,476 | 13 / 8 / 27  | 16 / 32 | 10 / 38 | 5 / 0 / 11   | 5 / 11  | 2 / 14  |
| #3      | 4,380 | 11 / 6 / 31  | 15 / 33 | 8 / 40  | 6 / 0 / 10   | 6 / 10  | 2 / 14  |
| #4      | 4,425 | 12 / 7 / 29  | 16 / 32 | 9 / 39  | 6 / 0 / 10   | 6 / 10  | 2 / 14  |
| #5      | 4,475 | 13 / 8 / 27  | 18 / 30 | 11 / 37 | 5 / 0 / 11   | 5 / 11  | 2 / 14  |
| #6      | 4,416 | 12 / 9 / 27  | 17 / 31 | 11 / 37 | 6 / 0 / 10   | 6 / 10  | 2 / 14  |
| #7      | 4,362 | 11 / 7 / 30  | 15 / 33 | 9 / 39  | 6 / 0 / 10   | 6 / 10  | 1 / 15  |
| #8      | 4,521 | 12 / 9 / 27  | 16 / 32 | 12 / 36 | 5 / 0 / 11   | 5 / 11  | 2 / 14  |
| b1      | 4,325 | 11 / 6 / 31  | 15 / 33 | 8 / 40  | 6 / 0 / 10   | 6 / 10  | 2 / 14  |
| b2      | 4,461 | 12 / 8 / 28  | 16 / 32 | 11 / 37 | 5 / 0 / 11   | 5 / 11  | 2 / 14  |
| b3      | 4,381 | 11 / 7 / 30  | 15 / 33 | 9 / 39  | 6 / 0 / 10   | 6 / 10  | 2 / 14  |
| b4      | 4,382 | 11 / 9 / 26  | 16 / 32 | 11 / 37 | 6 / 0 / 10   | 6 / 10  | 2 / 14  |
| b5      | 4,425 | 12 / 8 / 28  | 17 / 31 | 10 / 38 | 5 / 0 / 11   | 5 / 11  | 2 / 14  |
| b6      | 4,417 | 12 / 8 / 28  | 17 / 31 | 10 / 38 | 6 / 0 / 10   | 6 / 10  | 1 / 15  |
| b7      | 4,420 | 11 / 7 / 30  | 15 / 33 | 9 / 39  | 6 / 0 / 10   | 6 / 10  | 2 / 14  |
| b8      | 4,468 | 12 / 8 / 28  | 16 / 32 | 10 / 38 | 5 / 0 / 11   | 5 / 11  | 2 / 14  |
| h1      | 4,331 | 12 / 8 / 28  | 16 / 32 | 10 / 38 | 6 / 0 / 10   | 6 / 10  | 2 / 14  |
| h2      | 4,505 | 13 / 8 / 27  | 18 / 30 | 10 / 38 | 5 / 0 / 11   | 5 / 11  | 2 / 14  |
| h3      | 4,438 | 11 / 7 / 30  | 17 / 31 | 10 / 38 | 6 / 0 / 10   | 6 / 10  | 2 / 14  |
| h4      | 4,445 | 12 / 7 / 29  | 16 / 32 | 9 / 39  | 6 / 0 / 10   | 6 / 10  | 2 / 14  |
| h5      | 4,412 | 13 / 8 / 27  | 17 / 31 | 10 / 38 | 5 / 0 / 10   | 5 / 11  | 2 / 14  |
| h6      | 4,426 | 12 / 9 / 27  | 17 / 31 | 11 / 37 | 6 / 0 / 10   | 6 / 10  | 2 / 14  |
| h7      | 4,499 | 12 / 8 / 28  | 16 / 32 | 11 / 37 | 6 / 0 / 10   | 6 / 10  | 2 / 14  |
| h8      | 4,515 | 12 / 8 / 28  | 17 / 31 | 11 / 37 | 5 / 0 / 11   | 5 / 11  | 2 / 14  |

TABLE 3. Undergraduate Continuing Enrollment: Comparison of the seasonal per-period SAE stochastic latest-optimized  $\tilde{\alpha}$ -power balanced forecasting models to the seasonal Holt–Winters and ARIMA models

*Example 3.* Let’s consider undergraduate new enrollment (fall, spring and summer) over 18 years (fall 2004 to summer 2022). Thus,  $f = 3$  and  $X = (x_1, \dots, x_{54})$ . For  $s = 1, \dots, 3$  (trimesters), and  $i = 2, \dots, 17$  (years), we compute the sequence of three-step-ahead forecasts for  $X$  using both Holt–Winters and ARIMA models. The sum of absolute errors for each model are

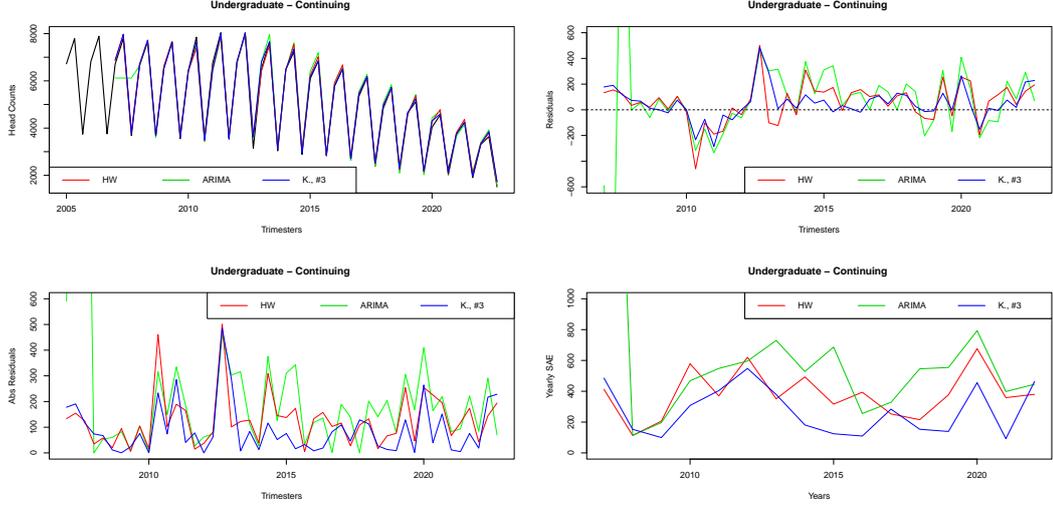


FIGURE 2. Undergraduate Continuing Enrollment: Forecasts, residuals, absolute residuals and yearly sum of absolute errors

$$\text{SAE}^{\text{HW}} = \sum_{j=7}^{54} \left| \hat{x}_j^{\text{HW}} - x_j \right| = 3,719 \text{ and } \text{SAE}^{\text{ARIMA}} = \sum_{j=7}^{54} \left| \hat{x}_j^{\text{ARIMA}} - x_j \right| = 6,289 \quad (61)$$

Comparing these 16 years times 3 trimesters forecasts, i.e., 48 trimester forecasts, the Holt–Winters model lead to better trimesterly forecast (smallest absolute residual) 32 times as opposed to 16 times for the ARIMA model, i.e.,  $HW/AR = 32/16$ . Moreover, if one compares the sum of absolute error per year, the Holt–Winters model lead to a better yearly forecast nine times as opposed to seven times for ARIMA, i.e.,  $HW/AR = 9/7$ .

For  $s = 1, \dots, 3$  (trimesters), and  $i = 2, \dots, 17$  (years), we compute the sequence of three-step-ahead  $\hat{x}_j^{\text{HW}}$ ,  $\hat{x}_j^{\text{ARIMA}}$  and  $\hat{x}_j^{\text{K., #3}}$  forecasts for  $X$  as in (56), (57) and (58) for each optimization criteria  $\kappa = 1, \dots, 8$  using the seasonal per-period SAE stochastic latest-optimized  $\tilde{\alpha}$ -power balanced forecasting models (with  $N = 1000$  to find the optimal latest-optimized alpha power). Moreover, we compute the (global) sum of absolute error, and we compare our forecasts to both Holt–Winters and ARIMA models by counting the number of best monthly and yearly forecasts. The results are summarized in Table 4..

*Example 4.* Let's consider graduate total enrollment (fall, spring and summer) over 18 years (fall 2004 to summer 2022). Thus,  $f = 3$  and  $X = (x_1, \dots, x_{54})$ . For  $s = 1, \dots, 3$  (trimesters), and  $i = 2, \dots, 17$  (years), we compute the sequence of three-step-ahead forecasts for  $X$  using both Holt–Winters and ARIMA models. The sum of absolute errors for each model are

$$\text{SAE}^{\text{HW}} = \sum_{j=7}^{54} \left| \hat{x}_j^{\text{HW}} - x_j \right| = 5,013 \text{ and } \text{SAE}^{\text{ARIMA}} = \sum_{j=7}^{54} \left| \hat{x}_j^{\text{ARIMA}} - x_j \right| = 6,572 \quad (62)$$

Comparing these 16 years times 3 trimesters forecasts, i.e., 48 trimester forecasts, the Holt–Winters model lead to better trimesterly forecast (smallest absolute residual) 32 times as opposed to 16 times for the ARIMA model, i.e.,  $HW/AR = 32/16$ . Moreover, if one compares the sum of

| $\bullet\kappa$ | SAE   | HW / AR / K. | HW / K. | AR / K. | HW / AR / K. | HW / K. | AR / K. |
|-----------------|-------|--------------|---------|---------|--------------|---------|---------|
| #1              | 2,276 | 10 / 8 / 30  | 10 / 38 | 10 / 38 | 2 / 0 / 14   | 2 / 14  | 0 / 16  |
| #2              | 2,274 | 10 / 6 / 32  | 10 / 38 | 8 / 40  | 3 / 0 / 13   | 3 / 13  | 0 / 16  |
| #3              | 2,207 | 11 / 9 / 28  | 13 / 35 | 11 / 37 | 2 / 0 / 14   | 2 / 14  | 0 / 16  |
| #4              | 2,186 | 8 / 9 / 31   | 9 / 39  | 10 / 38 | 1 / 0 / 15   | 1 / 15  | 0 / 16  |
| #5              | 2,213 | 9 / 7 / 32   | 9 / 39  | 8 / 40  | 2 / 0 / 14   | 2 / 14  | 0 / 16  |
| #6              | 2,162 | 8 / 8 / 32   | 9 / 39  | 9 / 39  | 1 / 0 / 15   | 1 / 15  | 0 / 16  |
| #7              | 2,146 | 9 / 7 / 32   | 12 / 36 | 8 / 40  | 2 / 0 / 14   | 2 / 14  | 0 / 16  |
| #8              | 2,205 | 9 / 6 / 33   | 9 / 39  | 6 / 42  | 3 / 0 / 13   | 3 / 13  | 0 / 16  |
| b1              | 2,219 | 11 / 5 / 32  | 11 / 37 | 8 / 40  | 1 / 0 / 15   | 1 / 15  | 0 / 16  |
| b2              | 2,260 | 11 / 7 / 30  | 12 / 36 | 10 / 38 | 0 / 0 / 16   | 0 / 16  | 0 / 16  |
| b3              | 2,213 | 10 / 8 / 30  | 11 / 37 | 9 / 39  | 1 / 0 / 15   | 1 / 15  | 0 / 16  |
| b4              | 2,175 | 9 / 8 / 31   | 10 / 38 | 9 / 39  | 1 / 0 / 15   | 1 / 15  | 0 / 16  |
| b5              | 2,212 | 11 / 7 / 30  | 13 / 35 | 8 / 40  | 1 / 0 / 15   | 1 / 15  | 0 / 16  |
| b6              | 2,158 | 10 / 9 / 29  | 11 / 37 | 10 / 38 | 1 / 0 / 15   | 1 / 15  | 0 / 16  |
| b7              | 2,176 | 8 / 8 / 32   | 8 / 40  | 9 / 39  | 0 / 0 / 16   | 0 / 16  | 0 / 16  |
| b8              | 2,171 | 8 / 6 / 34   | 9 / 39  | 6 / 42  | 1 / 0 / 15   | 1 / 15  | 0 / 16  |
| ‡1              | 2,282 | 13 / 9 / 26  | 13 / 35 | 11 / 37 | 1 / 0 / 15   | 1 / 15  | 0 / 16  |
| ‡2              | 2,305 | 11 / 7 / 30  | 12 / 36 | 10 / 38 | 2 / 0 / 14   | 2 / 14  | 0 / 16  |
| ‡3              | 2,211 | 11 / 7 / 30  | 12 / 36 | 9 / 39  | 1 / 0 / 15   | 1 / 15  | 0 / 16  |
| ‡4              | 2,197 | 10 / 8 / 30  | 10 / 38 | 9 / 39  | 1 / 0 / 15   | 1 / 15  | 0 / 16  |
| ‡5              | 2,245 | 11 / 6 / 31  | 11 / 37 | 7 / 41  | 2 / 0 / 14   | 2 / 14  | 0 / 16  |
| ‡6              | 2,296 | 10 / 8 / 30  | 10 / 38 | 9 / 39  | 2 / 0 / 14   | 2 / 14  | 0 / 16  |
| ‡7              | 2,201 | 9 / 8 / 31   | 9 / 39  | 9 / 39  | 1 / 0 / 15   | 1 / 15  | 0 / 16  |
| ‡8              | 2,190 | 9 / 5 / 34   | 10 / 38 | 10 / 38 | 2 / 0 / 14   | 2 / 14  | 0 / 16  |

TABLE 4. Undergraduate New Enrollment: Comparison of the seasonal per-period SAE stochastic latest-optimized  $\tilde{\alpha}$ -power balanced forecasting models to the seasonal Holt–Winters and ARIMA models

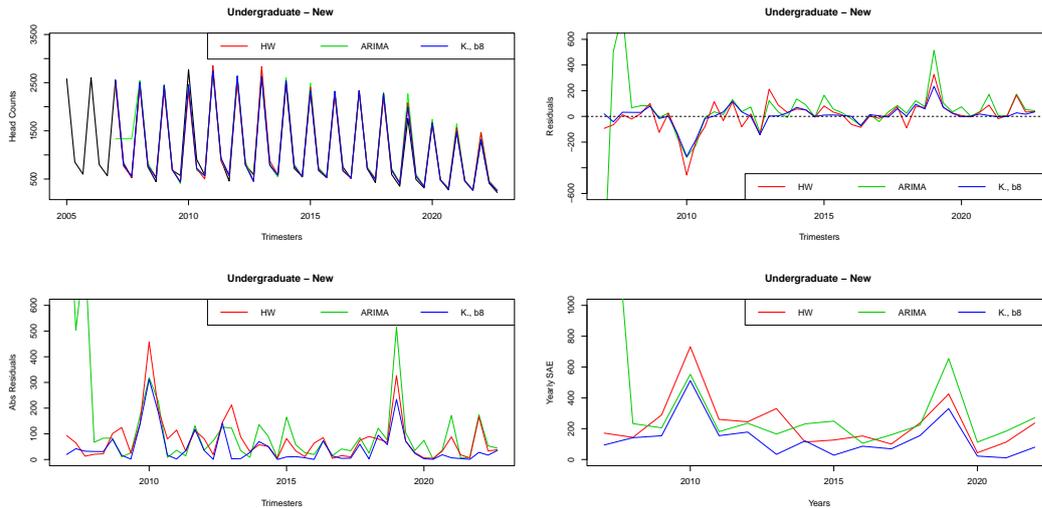


FIGURE 3. Undergraduate New Enrollment: Forecasts, residuals, absolute residuals and yearly sum of absolute errors

absolute error per year, the Holt–Winters model lead to a better yearly forecast thirteen times as opposed to three times for ARIMA, i.e.,  $HW/AR = 13/3$ .

For  $s = 1, \dots, 3$  (trimesters), and  $i = 2, \dots, 17$  (years), we compute the sequence of three-step-ahead  $\ddagger, b$  and  $\ddagger$  forecasts for  $X$  as in (56), (57) and (58) for each optimization criteria  $\kappa = 1, \dots, 8$  using the seasonal per-period SAE stochastic latest-optimized  $\tilde{\alpha}$ -power balanced forecasting models (with  $N = 1000$  to find the optimal latest-optimized alpha power). Moreover,

we compute the (global) sum of absolute error, and we compare our forecasts to both Holt–Winters and ARIMA models by counting the number of best monthly and yearly forecasts. The results are summarized in Table 5..

| ● <sub>FC</sub> | SAE   | HW / AR / K. | HW / K. | AR / K. | HW / AR / K. | HW / K. | AR / K. |
|-----------------|-------|--------------|---------|---------|--------------|---------|---------|
| #1              | 3,602 | 11 / 3 / 34  | 14 / 34 | 4 / 44  | 1 / 1 / 14   | 1 / 15  | 1 / 15  |
| #2              | 3,525 | 10 / 5 / 33  | 13 / 35 | 6 / 42  | 1 / 1 / 14   | 1 / 15  | 1 / 15  |
| #3              | 3,619 | 11 / 3 / 34  | 14 / 34 | 4 / 44  | 1 / 1 / 14   | 1 / 15  | 1 / 15  |
| #4              | 3,657 | 12 / 4 / 32  | 14 / 34 | 5 / 43  | 2 / 1 / 13   | 2 / 14  | 2 / 14  |
| #5              | 3,545 | 11 / 5 / 32  | 13 / 35 | 6 / 42  | 1 / 1 / 14   | 1 / 15  | 1 / 15  |
| #6              | 3,640 | 11 / 5 / 32  | 13 / 35 | 6 / 42  | 2 / 1 / 13   | 2 / 14  | 2 / 14  |
| #7              | 3,637 | 13 / 4 / 31  | 16 / 32 | 5 / 43  | 2 / 1 / 13   | 2 / 14  | 1 / 15  |
| #8              | 3,559 | 11 / 5 / 32  | 14 / 34 | 6 / 42  | 1 / 1 / 14   | 1 / 15  | 1 / 15  |
| b1              | 3,618 | 12 / 4 / 32  | 15 / 33 | 5 / 43  | 1 / 0 / 15   | 1 / 15  | 0 / 16  |
| b2              | 3,537 | 9 / 7 / 32   | 11 / 37 | 8 / 40  | 1 / 1 / 14   | 1 / 15  | 1 / 15  |
| b3              | 3,622 | 11 / 5 / 32  | 14 / 34 | 6 / 42  | 1 / 1 / 14   | 1 / 15  | 1 / 15  |
| b4              | 3,680 | 11 / 6 / 31  | 14 / 34 | 7 / 41  | 2 / 1 / 13   | 2 / 14  | 1 / 15  |
| b5              | 3,563 | 12 / 4 / 32  | 14 / 34 | 5 / 43  | 1 / 1 / 14   | 1 / 15  | 1 / 15  |
| b6              | 3,707 | 12 / 5 / 31  | 15 / 33 | 7 / 41  | 2 / 1 / 13   | 2 / 14  | 2 / 14  |
| b7              | 3,655 | 14 / 5 / 29  | 17 / 31 | 6 / 42  | 2 / 1 / 13   | 2 / 14  | 1 / 15  |
| b8              | 3,567 | 12 / 4 / 32  | 14 / 34 | 5 / 43  | 1 / 1 / 14   | 1 / 15  | 1 / 15  |
| t1              | 3,605 | 12 / 4 / 32  | 15 / 33 | 5 / 43  | 1 / 0 / 15   | 1 / 15  | 0 / 16  |
| t2              | 3,512 | 11 / 5 / 32  | 14 / 34 | 6 / 42  | 1 / 1 / 14   | 1 / 15  | 1 / 15  |
| t3              | 3,809 | 11 / 5 / 32  | 14 / 34 | 7 / 41  | 1 / 1 / 14   | 1 / 15  | 1 / 15  |
| t4              | 3,636 | 12 / 4 / 32  | 13 / 35 | 6 / 42  | 2 / 0 / 14   | 2 / 14  | 0 / 16  |
| t5              | 3,523 | 12 / 5 / 31  | 14 / 34 | 6 / 42  | 1 / 1 / 14   | 1 / 15  | 1 / 15  |
| t6              | 3,654 | 14 / 5 / 29  | 16 / 32 | 7 / 41  | 2 / 1 / 13   | 2 / 14  | 2 / 14  |
| t7              | 3,687 | 13 / 4 / 31  | 15 / 33 | 5 / 43  | 2 / 1 / 13   | 2 / 14  | 1 / 15  |
| t8              | 3,552 | 12 / 5 / 31  | 14 / 34 | 6 / 42  | 1 / 1 / 14   | 1 / 15  | 1 / 15  |

TABLE 5. Graduate Total Enrollment: Comparison of the seasonal per-period SAE stochastic latest-optimized  $\hat{\alpha}$ -power balanced forecasting models to the seasonal Holt–Winters and ARIMA models

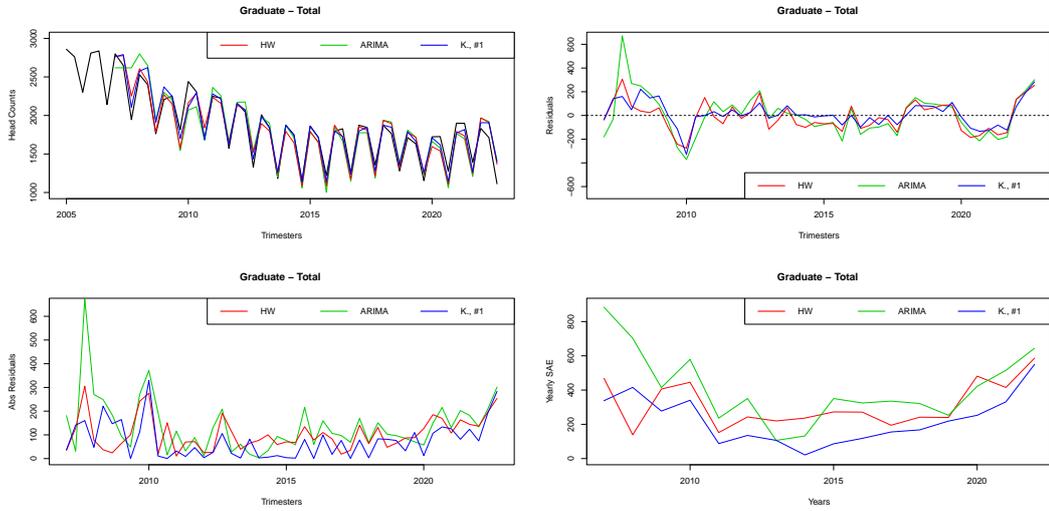


FIGURE 4. Graduate Total Enrollment: Forecasts, residuals, absolute residuals and yearly sum of absolute errors

*Example 5.* Let’s consider graduate continuing enrollment (fall, spring and summer) over 18 years (fall 2004 to summer 2022). Thus,  $f = 3$  and  $X = (x_1, \dots, x_{54})$ . For  $s = 1, \dots, 3$  (trimesters),

and  $i = 2, \dots, 17$  (years), we compute the sequence of three-step-ahead forecasts for  $X$  using both Holt–Winters and ARIMA models. The sum of absolute errors for each model are

$$\text{SAE}^{\text{HW}} = \sum_{j=7}^{54} \left| \hat{x}_j^{\text{HW}} - x_j \right| = 4,288 \text{ and } \text{SAE}^{\text{ARIMA}} = \sum_{j=7}^{54} \left| \hat{x}_j^{\text{ARIMA}} - x_j \right| = 5,781 \quad (63)$$

Comparing these 16 years times 3 trimesters forecasts, i.e., 48 trimester forecasts, the Holt–winters model lead to better trimesterly forecast (smallest absolute residual) 34 times as opposed to 14 times for the ARIMA model, i.e.,  $HW/AR = 34/14$ . Moreover, if one compares the sum of absolute error per year, the Holt–Winters model lead to a better yearly forecast fourthree times as opposed to two times for ARIMA, i.e.,  $HW/AR = 14/2$ .

For  $s = 1, \dots, 3$  (trimesters), and  $i = 2, \dots, 17$  (years), we compute the sequence of three-step-ahead  $\sharp, \flat$  and  $\natural$  forecasts for  $X$  as in (56), (57) and (58) for each optimization criteria  $\kappa = 1, \dots, 8$  using the seasonal per-period SAE stochastic latest-optimized  $\tilde{\alpha}$ -power balanced forecasting models (with  $N = 1000$  to find the optimal latest-optimized alpha power). Moreover, we compute the (global) sum of absolute error, and we compare our forecasts to both Holt–Winters and ARIMA models by counting the number of best monthly and yearly forecasts. The results are summarized in Table 6..

| $\bullet \kappa$ | SAE   | HW / AR / K. | HW / K. | AR / K. | HW / AR / K. | HW / K. | AR / K. |
|------------------|-------|--------------|---------|---------|--------------|---------|---------|
| $\sharp 1$       | 3,327 | 13 / 4 / 31  | 15 / 33 | 8 / 40  | 4 / 1 / 11   | 5 / 11  | 1 / 15  |
| $\sharp 2$       | 3,286 | 14 / 4 / 30  | 17 / 31 | 7 / 41  | 4 / 1 / 11   | 5 / 11  | 1 / 15  |
| $\sharp 3$       | 3,317 | 13 / 5 / 30  | 15 / 33 | 9 / 39  | 4 / 1 / 11   | 5 / 11  | 1 / 15  |
| $\sharp 4$       | 3,310 | 14 / 5 / 29  | 17 / 31 | 10 / 38 | 5 / 1 / 10   | 6 / 10  | 1 / 15  |
| $\sharp 5$       | 3,317 | 14 / 4 / 30  | 16 / 32 | 8 / 40  | 5 / 1 / 10   | 6 / 10  | 1 / 15  |
| $\sharp 6$       | 3,344 | 12 / 5 / 31  | 15 / 33 | 9 / 39  | 5 / 1 / 10   | 6 / 10  | 1 / 15  |
| $\sharp 7$       | 3,319 | 16 / 5 / 27  | 18 / 30 | 10 / 38 | 5 / 1 / 10   | 6 / 10  | 1 / 15  |
| $\sharp 8$       | 3,308 | 14 / 3 / 31  | 16 / 32 | 7 / 41  | 4 / 1 / 11   | 5 / 11  | 1 / 15  |
| $\flat 1$        | 3,327 | 13 / 4 / 31  | 15 / 33 | 8 / 40  | 5 / 1 / 10   | 6 / 10  | 1 / 15  |
| $\flat 2$        | 3,299 | 12 / 5 / 31  | 15 / 33 | 8 / 40  | 3 / 1 / 12   | 4 / 12  | 1 / 15  |
| $\flat 3$        | 3,287 | 14 / 3 / 31  | 16 / 32 | 7 / 41  | 4 / 1 / 11   | 5 / 11  | 1 / 15  |
| $\flat 4$        | 3,254 | 14 / 4 / 30  | 16 / 32 | 9 / 39  | 4 / 1 / 11   | 5 / 11  | 1 / 15  |
| $\flat 5$        | 3,300 | 13 / 3 / 32  | 15 / 33 | 7 / 41  | 4 / 1 / 11   | 5 / 11  | 1 / 15  |
| $\flat 6$        | 3,307 | 14 / 6 / 28  | 17 / 31 | 9 / 39  | 4 / 1 / 11   | 5 / 11  | 1 / 15  |
| $\flat 7$        | 3,310 | 14 / 6 / 28  | 15 / 33 | 11 / 37 | 3 / 1 / 12   | 4 / 12  | 1 / 15  |
| $\flat 8$        | 3,307 | 13 / 4 / 31  | 14 / 34 | 8 / 40  | 4 / 1 / 11   | 5 / 11  | 1 / 15  |
| $\natural 1$     | 3,309 | 14 / 4 / 30  | 17 / 31 | 7 / 41  | 4 / 1 / 11   | 5 / 11  | 1 / 15  |
| $\natural 2$     | 3,310 | 15 / 3 / 30  | 17 / 31 | 7 / 41  | 5 / 1 / 10   | 6 / 10  | 1 / 15  |
| $\natural 3$     | 3,443 | 15 / 6 / 27  | 18 / 30 | 10 / 38 | 5 / 1 / 10   | 6 / 10  | 2 / 14  |
| $\natural 4$     | 3,322 | 13 / 5 / 30  | 15 / 33 | 10 / 38 | 5 / 1 / 10   | 6 / 10  | 1 / 15  |
| $\natural 5$     | 3,323 | 14 / 4 / 30  | 16 / 32 | 8 / 40  | 4 / 1 / 11   | 5 / 11  | 1 / 15  |
| $\natural 6$     | 3,341 | 12 / 6 / 30  | 15 / 33 | 9 / 39  | 5 / 1 / 10   | 6 / 10  | 1 / 15  |
| $\natural 7$     | 3,318 | 13 / 4 / 31  | 14 / 34 | 9 / 39  | 5 / 1 / 10   | 6 / 10  | 1 / 15  |
| $\natural 8$     | 3,309 | 15 / 4 / 29  | 17 / 31 | 8 / 40  | 5 / 1 / 10   | 6 / 10  | 1 / 15  |

TABLE 6. Graduate Continuing Enrollment: Comparison of the seasonal per-period SAE stochastic latest-optimized  $\tilde{\alpha}$ -power balanced forecasting models to the seasonal Holt–Winters and ARIMA models

*Example 6.* Let’s consider graduate new enrollment (fall, spring and summer) over 18 years (fall 2004 to summer 2022). Thus,  $f = 3$  and  $X = (x_1, \dots, x_{54})$ . For  $s = 1, \dots, 3$  (trimesters), and  $i = 2, \dots, 17$  (years), we compute the sequence of three-step-ahead forecasts for  $X$  using both Holt–Winters and ARIMA models. The sum of absolute errors for each model are

$$\text{SAE}^{\text{HW}} = \sum_{j=7}^{54} \left| \hat{x}_j^{\text{HW}} - x_j \right| = 3,219 \text{ and } \text{SAE}^{\text{ARIMA}} = \sum_{j=7}^{54} \left| \hat{x}_j^{\text{ARIMA}} - x_j \right| = 3,274 \quad (64)$$

Comparing these 16 years times 3 trimesters forecasts, i.e., 48 trimester forecasts, the Holt–winters model lead to slightly worse trimesterly forecast (smallest absolute residual) 23 times as opposed to 25 times for the ARIMA model, i.e.,  $HW/AR = 23/25$ . Moreover, if one compares

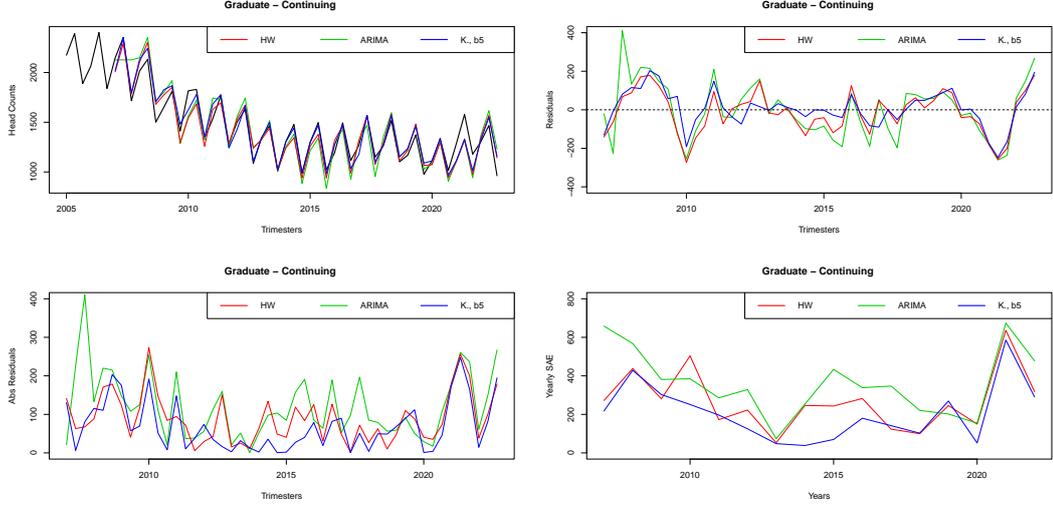


FIGURE 5. Graduate Continuing Enrollment: Forecasts, residuals, absolute residuals and yearly sum of absolute errors

the sum of absolute error per year, the Holt–Winters model lead to the same number of best yearly forecast as ARIMA, i.e.,  $HW/AR = 8/8$ .

For  $s = 1, \dots, 3$  (trimesters), and  $i = 2, \dots, 17$  (years), we compute the sequence of three-step-ahead  $\hat{x}_{t,b}$  and  $\hat{x}_{t,b}$  forecasts for  $X$  as in (56), (57) and (58) for each optimization criteria  $\kappa = 1, \dots, 8$  using the seasonal per-period SAE stochastic latest-optimized  $\tilde{\alpha}$ -power balanced forecasting models (with  $N = 1000$  to find the optimal latest-optimized alpha power). Moreover, we compute the (global) sum of absolute error, and we compare our forecasts to both Holt–Winters and ARIMA models by counting the number of best monthly and yearly forecasts. The results are summarized in Table 7. .

*Example 7.* Let’s consider student (undergraduate + graduate) total enrollment (fall, spring and summer) over 18 years (fall 2004 to summer 2022). Thus,  $f = 3$  and  $X = (x_1, \dots, x_{54})$ . For  $s = 1, \dots, 3$  (trimesters), and  $i = 2, \dots, 17$  (years), we compute the sequence of three-step-ahead forecasts for  $X$  using both Holt–Winters and ARIMA models. The sum of absolute errors for each model are

$$SAE^{HW} = \sum_{j=7}^{54} |\hat{x}_j^{HW} - x_j| = 9,653 \text{ and } SAE^{ARIMA} = \sum_{j=7}^{54} |\hat{x}_j^{ARIMA} - x_j| = 17,491 \quad (65)$$

Comparing these 16 years times 3 trimesters forecasts, i.e., 48 trimester forecasts, the Holt–winters model lead to better trimesterly forecast (smallest absolute residual) 31 times as opposed to 17 times for the ARIMA model, i.e.,  $HW/AR = 31/17$ . Moreover, if one compares the sum of absolute error per year, the Holt–Winters model lead to a better yearly forecast eleven times as opposed to five times for ARIMA, i.e.,  $HW/AR = 11/5$ .

For  $s = 1, \dots, 3$  (trimesters), and  $i = 2, \dots, 17$  (years), we compute the sequence of three-step-ahead  $\hat{x}_{t,b}$  and  $\hat{x}_{t,b}$  forecasts for  $X$  as in (56), (57) and (58) for each optimization criteria  $\kappa = 1, \dots, 8$  using the seasonal per-period SAE stochastic latest-optimized  $\tilde{\alpha}$ -power balanced

| $\bullet_{\alpha}$ | SAE   | HW / AR / K. | HW / K. | AR / K. | HW / AR / K. | HW / K. | AR / K. |
|--------------------|-------|--------------|---------|---------|--------------|---------|---------|
| #1                 | 2,215 | 11 / 15 / 22 | 16 / 32 | 21 / 27 | 2 / 3 / 11   | 4 / 12  | 4 / 12  |
| #2                 | 2,161 | 10 / 13 / 25 | 13 / 35 | 18 / 30 | 2 / 2 / 12   | 4 / 12  | 3 / 13  |
| #3                 | 2,177 | 11 / 13 / 24 | 14 / 34 | 20 / 28 | 2 / 3 / 11   | 4 / 12  | 4 / 12  |
| #4                 | 2,122 | 10 / 12 / 26 | 15 / 33 | 17 / 31 | 3 / 2 / 11   | 4 / 12  | 3 / 13  |
| #5                 | 2,190 | 11 / 13 / 24 | 15 / 33 | 18 / 30 | 2 / 2 / 12   | 4 / 12  | 3 / 13  |
| #6                 | 2089  | 10 / 12 / 26 | 15 / 33 | 18 / 30 | 2 / 2 / 12   | 3 / 13  | 3 / 13  |
| #7                 | 2,149 | 11 / 13 / 24 | 15 / 33 | 17 / 31 | 3 / 2 / 11   | 5 / 11  | 3 / 13  |
| #8                 | 2,139 | 11 / 13 / 24 | 15 / 33 | 19 / 29 | 2 / 2 / 12   | 4 / 12  | 3 / 13  |
| b1                 | 2,174 | 11 / 14 / 23 | 16 / 32 | 17 / 31 | 3 / 2 / 11   | 4 / 12  | 3 / 13  |
| b2                 | 2,174 | 10 / 13 / 25 | 15 / 33 | 16 / 32 | 2 / 2 / 12   | 4 / 12  | 3 / 13  |
| b3                 | 2,187 | 11 / 11 / 26 | 16 / 32 | 16 / 32 | 2 / 2 / 12   | 3 / 13  | 3 / 13  |
| b4                 | 2,172 | 10 / 14 / 24 | 14 / 34 | 19 / 29 | 3 / 2 / 11   | 4 / 12  | 4 / 12  |
| b5                 | 2,152 | 11 / 14 / 23 | 16 / 32 | 19 / 29 | 2 / 2 / 12   | 4 / 12  | 3 / 13  |
| b6                 | 2,134 | 10 / 14 / 24 | 16 / 32 | 19 / 29 | 1 / 2 / 13   | 2 / 14  | 3 / 13  |
| b7                 | 2,156 | 10 / 12 / 26 | 14 / 34 | 17 / 31 | 2 / 2 / 12   | 3 / 13  | 3 / 13  |
| b8                 | 2,176 | 12 / 14 / 22 | 17 / 31 | 19 / 29 | 2 / 2 / 12   | 4 / 12  | 3 / 13  |
| q1                 | 2,190 | 10 / 14 / 24 | 15 / 33 | 19 / 29 | 2 / 2 / 12   | 4 / 12  | 3 / 13  |
| q2                 | 2,161 | 9 / 14 / 25  | 14 / 34 | 18 / 30 | 2 / 2 / 12   | 4 / 12  | 3 / 13  |
| q3                 | 2,156 | 11 / 12 / 25 | 15 / 33 | 18 / 30 | 3 / 3 / 10   | 4 / 12  | 5 / 11  |
| q4                 | 2,156 | 9 / 13 / 26  | 14 / 34 | 16 / 32 | 2 / 2 / 12   | 3 / 13  | 3 / 13  |
| q5                 | 2,176 | 9 / 14 / 25  | 15 / 33 | 18 / 30 | 2 / 2 / 12   | 4 / 12  | 3 / 13  |
| q6                 | 2,131 | 9 / 12 / 27  | 14 / 34 | 17 / 31 | 2 / 2 / 12   | 3 / 13  | 3 / 13  |
| q7                 | 2,157 | 10 / 12 / 26 | 14 / 34 | 16 / 32 | 2 / 2 / 12   | 4 / 12  | 3 / 13  |
| q8                 | 2,159 | 10 / 14 / 24 | 15 / 33 | 18 / 30 | 2 / 2 / 12   | 4 / 12  | 3 / 13  |

TABLE 7. Graduate New Enrollment: Comparison of the seasonal per-period SAE stochastic latest-optimized  $\tilde{\alpha}$ -power balanced forecasting models to the seasonal Holt–Winters and ARIMA models

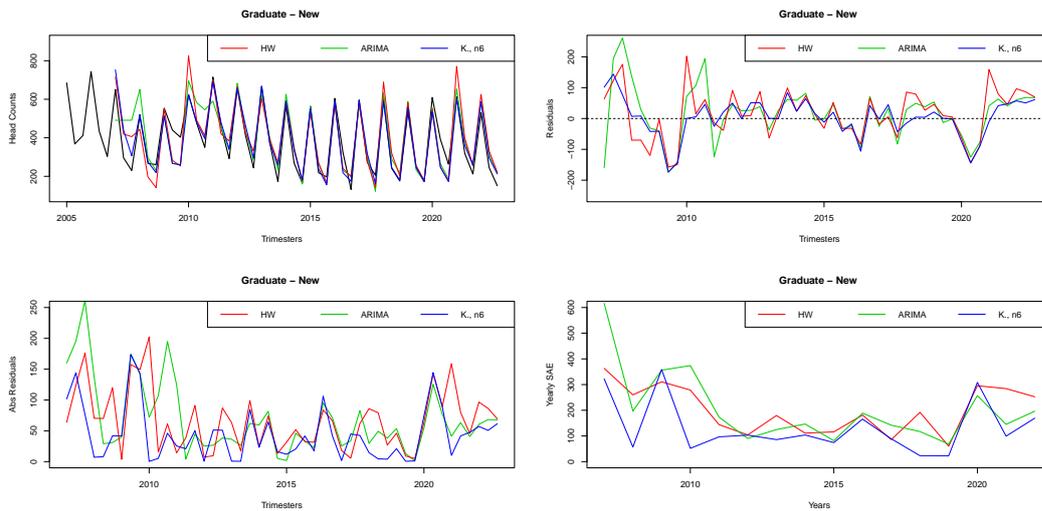


FIGURE 6. Graduate New Enrollment: Forecasts, residuals, absolute residuals and yearly sum of absolute errors

forecasting models (with  $N = 1000$  to find the optimal latest-optimized alpha power). Moreover, we compute the (global) sum of absolute error, and we compare our forecasts to both Holt–Winters and ARIMA models by counting the number of best monthly and yearly forecasts. The results are summarized in Table 8..

*Example 8.* Let’s consider student (undergraduate + graduate) continuing enrollment (fall, spring and summer) over 18 years (fall 2004 to summer 2022). Thus,  $f = 3$  and  $X = (x_1, \dots, x_{54})$ . For  $s = 1, \dots, 3$  (trimesters), and  $i = 2, \dots, 17$  (years), we compute the sequence of three-step-ahead forecasts for  $X$  using both Holt–Winters and ARIMA models. The sum of absolute errors for

| ● <sub>FC</sub> | SAE   | HW / AR / K. | HW / K. | AR / K. | HW / AR / K. | HW / K. | AR / K. |
|-----------------|-------|--------------|---------|---------|--------------|---------|---------|
| #1              | 7,487 | 14 / 9 / 25  | 18 / 30 | 13 / 35 | 3 / 2 / 11   | 3 / 13  | 3 / 13  |
| #2              | 7,387 | 13 / 8 / 27  | 15 / 33 | 12 / 36 | 4 / 1 / 11   | 5 / 11  | 2 / 14  |
| #3              | 7,469 | 13 / 7 / 28  | 15 / 33 | 11 / 37 | 3 / 2 / 11   | 4 / 12  | 3 / 13  |
| #4              | 7,410 | 14 / 7 / 27  | 16 / 32 | 11 / 37 | 3 / 2 / 11   | 3 / 13  | 3 / 13  |
| #5              | 7,540 | 13 / 8 / 27  | 17 / 31 | 12 / 36 | 4 / 2 / 10   | 4 / 12  | 4 / 12  |
| #6              | 7,500 | 14 / 9 / 25  | 16 / 32 | 13 / 35 | 3 / 2 / 11   | 4 / 12  | 3 / 13  |
| #7              | 7,623 | 15 / 9 / 24  | 18 / 30 | 15 / 33 | 3 / 2 / 11   | 5 / 11  | 3 / 13  |
| #8              | 7,336 | 13 / 7 / 28  | 15 / 33 | 11 / 37 | 4 / 2 / 10   | 4 / 12  | 3 / 13  |
| b1              | 7,567 | 16 / 9 / 23  | 18 / 30 | 14 / 34 | 3 / 2 / 11   | 3 / 13  | 3 / 13  |
| b2              | 7,396 | 13 / 8 / 27  | 15 / 33 | 12 / 36 | 4 / 1 / 11   | 4 / 12  | 2 / 14  |
| b3              | 7,562 | 15 / 9 / 24  | 18 / 30 | 15 / 33 | 3 / 3 / 10   | 4 / 12  | 4 / 12  |
| b4              | 7,513 | 13 / 10 / 25 | 16 / 32 | 15 / 33 | 3 / 2 / 11   | 4 / 12  | 3 / 13  |
| b5              | 7,546 | 12 / 10 / 26 | 16 / 32 | 15 / 33 | 4 / 2 / 10   | 5 / 11  | 4 / 12  |
| b6              | 7,523 | 14 / 9 / 25  | 16 / 32 | 14 / 34 | 3 / 3 / 10   | 5 / 11  | 4 / 12  |
| b7              | 7,579 | 16 / 9 / 23  | 18 / 30 | 15 / 33 | 3 / 3 / 10   | 4 / 12  | 4 / 12  |
| b8              | 7,268 | 14 / 8 / 26  | 17 / 31 | 12 / 36 | 4 / 2 / 10   | 5 / 11  | 3 / 13  |
| t1              | 7,540 | 13 / 8 / 27  | 17 / 31 | 12 / 36 | 3 / 3 / 10   | 3 / 13  | 4 / 12  |
| t2              | 7,434 | 12 / 7 / 29  | 15 / 33 | 11 / 37 | 4 / 2 / 10   | 4 / 12  | 4 / 12  |
| t3              | 7,742 | 15 / 9 / 24  | 20 / 28 | 14 / 34 | 4 / 3 / 9    | 5 / 11  | 4 / 12  |
| t4              | 7,432 | 15 / 8 / 25  | 18 / 30 | 12 / 36 | 3 / 2 / 11   | 3 / 13  | 3 / 13  |
| t5              | 7,631 | 13 / 9 / 26  | 18 / 30 | 13 / 35 | 4 / 2 / 10   | 4 / 12  | 4 / 12  |
| t6              | 7,519 | 14 / 9 / 25  | 16 / 32 | 13 / 35 | 3 / 3 / 10   | 5 / 11  | 4 / 12  |
| t7              | 7,640 | 16 / 8 / 24  | 20 / 28 | 14 / 34 | 3 / 2 / 11   | 5 / 11  | 3 / 13  |
| t8              | 7,381 | 13 / 7 / 28  | 16 / 32 | 11 / 37 | 4 / 2 / 10   | 4 / 12  | 3 / 13  |

TABLE 8. Undergraduate + Graduate Total Enrollment: Comparison of the seasonal per-period SAE stochastic latest-optimized  $\tilde{\alpha}$ -power balanced forecasting models to the seasonal Holt–Winters and ARIMA models

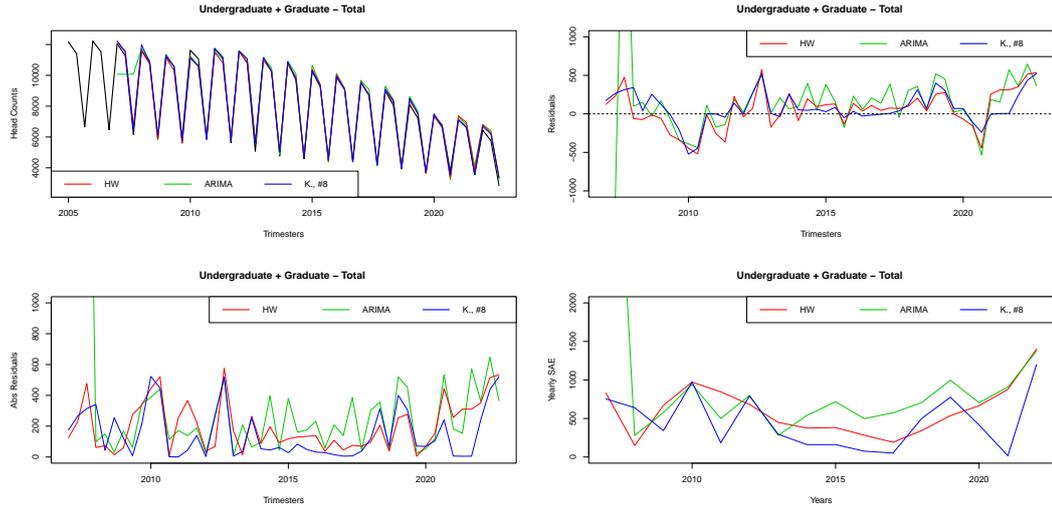


FIGURE 7. Undergraduate + Graduate Total Enrollment: Forecasts, residuals, absolute residuals and yearly sum of absolute errors

each model are

$$SAE^{HW} = \sum_{j=7}^{54} |\hat{x}_j^{HW} - x_j| = 7,843 \text{ and } SAE^{ARIMA} = \sum_{j=7}^{54} |\hat{x}_j^{ARIMA} - x_j| = 14,796 \quad (66)$$

Comparing these 16 years times 3 trimesters forecasts, i.e., 48 trimester forecasts, the Holt–Winters model lead to better trimesterly forecast (smallest absolute residual) 32 times as opposed to 16 times for the ARIMA model, i.e.,  $HW/AR = 32/16$ . Moreover, if one compares the sum of absolute error per year, the Holt–Winters model lead to a better yearly forecast twelve times

as opposed to four times for ARIMA, i.e.,  $HW/AR = 12/4$ .

For  $s = 1, \dots, 3$  (trimesters), and  $i = 2, \dots, 17$  (years), we compute the sequence of three-step-ahead  $\sharp, \flat$  and  $\natural$  forecasts for  $X$  as in (56), (57) and (58) for each optimization criteria  $\kappa = 1, \dots, 8$  using the seasonal per-period SAE stochastic latest-optimized  $\tilde{\alpha}$ -power balanced forecasting models (with  $N = 1000$  to find the optimal latest-optimized alpha power). Moreover, we compute the (global) sum of absolute error, and we compare our forecasts to both Holt–Winters and ARIMA models by counting the number of best monthly and yearly forecasts. The results are summarized in Table 9..

| $\bullet\kappa$ | SAE   | HW / AR / K. | HW / K. | AR / K. | HW / AR / K. | HW / K. | AR / K. |
|-----------------|-------|--------------|---------|---------|--------------|---------|---------|
| $\sharp 1$      | 5,484 | 13 / 7 / 28  | 18 / 30 | 9 / 39  | 2 / 1 / 13   | 2 / 14  | 1 / 15  |
| $\sharp 2$      | 5,418 | 11 / 6 / 31  | 15 / 33 | 8 / 40  | 2 / 1 / 13   | 2 / 14  | 1 / 15  |
| $\sharp 3$      | 5,483 | 12 / 7 / 29  | 16 / 32 | 9 / 39  | 2 / 1 / 13   | 2 / 14  | 1 / 15  |
| $\sharp 4$      | 5,472 | 13 / 6 / 29  | 16 / 32 | 7 / 41  | 2 / 1 / 13   | 2 / 14  | 1 / 15  |
| $\sharp 5$      | 5,356 | 14 / 7 / 27  | 18 / 30 | 8 / 40  | 3 / 1 / 12   | 3 / 13  | 1 / 15  |
| $\sharp 6$      | 5,430 | 12 / 7 / 29  | 16 / 32 | 8 / 40  | 2 / 1 / 13   | 2 / 14  | 1 / 15  |
| $\sharp 7$      | 5,504 | 13 / 7 / 28  | 17 / 31 | 9 / 39  | 2 / 1 / 13   | 2 / 14  | 1 / 15  |
| $\sharp 8$      | 5,384 | 12 / 7 / 29  | 16 / 32 | 8 / 40  | 3 / 1 / 12   | 3 / 13  | 1 / 15  |
| $\flat 1$       | 5,487 | 13 / 7 / 28  | 17 / 31 | 9 / 39  | 2 / 1 / 13   | 2 / 14  | 1 / 15  |
| $\flat 2$       | 5,487 | 13 / 7 / 28  | 17 / 31 | 9 / 39  | 2 / 1 / 13   | 2 / 14  | 1 / 15  |
| $\flat 3$       | 5,478 | 13 / 7 / 28  | 18 / 30 | 9 / 39  | 2 / 1 / 13   | 2 / 14  | 1 / 15  |
| $\flat 4$       | 5,452 | 12 / 6 / 30  | 16 / 32 | 8 / 40  | 2 / 1 / 13   | 2 / 14  | 1 / 15  |
| $\flat 5$       | 5,422 | 13 / 7 / 28  | 18 / 30 | 8 / 40  | 3 / 1 / 12   | 3 / 13  | 1 / 15  |
| $\flat 6$       | 5,435 | 13 / 5 / 30  | 16 / 32 | 6 / 42  | 2 / 1 / 13   | 2 / 14  | 1 / 15  |
| $\flat 7$       | 5,513 | 12 / 7 / 29  | 17 / 31 | 8 / 40  | 2 / 1 / 13   | 2 / 14  | 1 / 15  |
| $\flat 8$       | 5,391 | 14 / 6 / 28  | 18 / 30 | 7 / 41  | 3 / 1 / 12   | 3 / 13  | 1 / 15  |
| $\natural 1$    | 5,532 | 17 / 7 / 28  | 18 / 30 | 8 / 40  | 2 / 1 / 13   | 3 / 13  | 1 / 15  |
| $\natural 2$    | 5,487 | 14 / 7 / 27  | 19 / 29 | 9 / 39  | 2 / 1 / 13   | 2 / 14  | 1 / 15  |
| $\natural 3$    | 5,462 | 13 / 6 / 29  | 18 / 30 | 7 / 41  | 3 / 1 / 12   | 3 / 13  | 1 / 15  |
| $\natural 4$    | 5,508 | 13 / 6 / 29  | 17 / 31 | 8 / 40  | 2 / 1 / 13   | 2 / 14  | 1 / 15  |
| $\natural 5$    | 5,452 | 15 / 7 / 26  | 19 / 29 | 8 / 40  | 3 / 1 / 12   | 3 / 13  | 1 / 15  |
| $\natural 6$    | 5,478 | 13 / 7 / 28  | 16 / 32 | 8 / 40  | 3 / 1 / 12   | 3 / 13  | 1 / 15  |
| $\natural 7$    | 5,444 | 12 / 7 / 29  | 16 / 32 | 9 / 39  | 2 / 1 / 13   | 2 / 14  | 1 / 15  |
| $\natural 8$    | 5,449 | 15 / 7 / 26  | 19 / 29 | 8 / 40  | 3 / 1 / 12   | 3 / 13  | 1 / 15  |

TABLE 9. Undergraduate + Graduate Continuing Enrollment: Comparison of the seasonal per-period SAE stochastic latest-optimized  $\tilde{\alpha}$ -power balanced forecasting models to the seasonal Holt–Winters and ARIMA models

*Example 9.* Let’s consider student (undergraduate + graduate) new enrollment (fall, spring and summer) over 18 years (fall 2004 to summer 2022). Thus,  $f = 3$  and  $X = (x_1, \dots, x_{54})$ . For  $s = 1, \dots, 3$  (trimesters), and  $i = 2, \dots, 17$  (years), we compute the sequence of three-step-ahead forecasts for  $X$  using both Holt–Winters and ARIMA models. The sum of absolute errors for each model are

$$SAE^{HW} = \sum_{j=7}^{54} \left| \hat{x}_j^{HW} - x_j \right| = 4,784 \text{ and } SAE^{ARIMA} = \sum_{j=7}^{54} \left| \hat{x}_j^{ARIMA} - x_j \right| = 8,938 \quad (67)$$

Comparing these 16 years times 3 trimesters forecasts, i.e., 48 trimester forecasts, the Holt–winters model lead to better trimesterly forecast (smallest absolute residual) 34 times as opposed to 14 times for the ARIMA model, i.e.,  $HW/AR = 34/14$ . Moreover, if one compares the sum of absolute error per year, the Holt–Winters model lead to a better yearly forecast twelve times as opposed to four times for ARIMA, i.e.,  $HW/AR = 12/4$ .

For  $s = 1, \dots, 3$  (trimesters), and  $i = 2, \dots, 17$  (years), we compute the sequence of three-step-ahead  $\sharp, \flat$  and  $\natural$  forecasts for  $X$  as in (56), (57) and (58) for each optimization criteria  $\kappa = 1, \dots, 8$  using the seasonal per-period SAE stochastic latest-optimized  $\tilde{\alpha}$ -power balanced forecasting models (with  $N = 1000$  to find the optimal latest-optimized alpha power). Moreover,

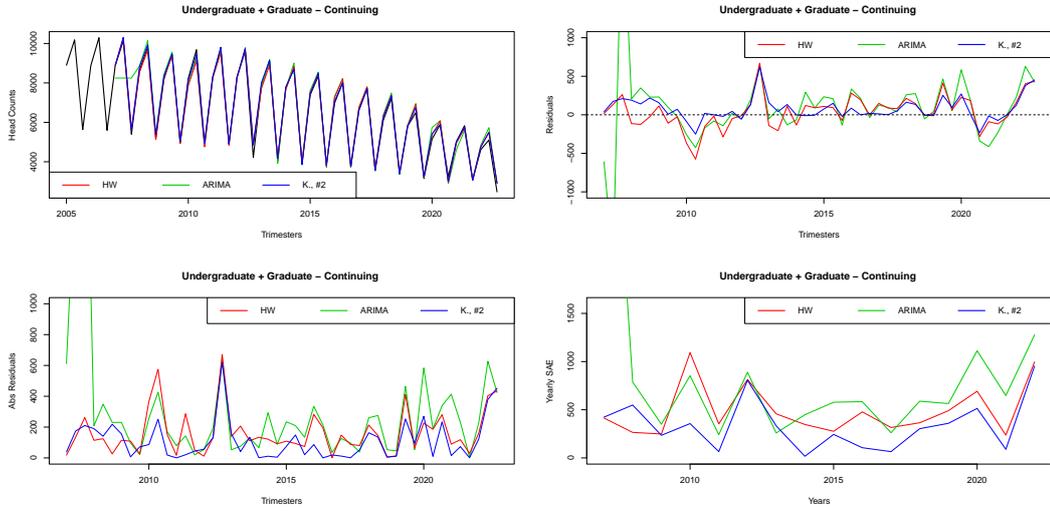


FIGURE 8. Undergraduate + Graduate Continuing Enrollment: Forecasts, residuals, absolute residuals and yearly sum of absolute errors

we compute the (global) sum of absolute error, and we compare our forecasts to both Holt–Winters and ARIMA models by counting the number of best monthly and yearly forecasts. The results are summarized in Table 10..

| ● <sub>k</sub> | SAE   | HW / AR / K. | HW / K. | AR / K. | HW / AR / K. | HW / K. | AR / K. |
|----------------|-------|--------------|---------|---------|--------------|---------|---------|
| #1             | 3,146 | 7 / 8 / 33   | 12 / 36 | 10 / 38 | 1 / 1 / 14   | 1 / 15  | 1 / 15  |
| #2             | 3,062 | 9 / 6 / 33   | 13 / 35 | 7 / 41  | 2 / 1 / 13   | 2 / 14  | 1 / 15  |
| #3             | 3,114 | 11 / 8 / 29  | 17 / 31 | 10 / 38 | 2 / 1 / 13   | 2 / 14  | 1 / 15  |
| #4             | 3,112 | 8 / 10 / 30  | 14 / 34 | 12 / 36 | 2 / 2 / 12   | 2 / 14  | 2 / 14  |
| #5             | 3,096 | 10 / 7 / 31  | 15 / 33 | 9 / 39  | 1 / 1 / 14   | 1 / 15  | 1 / 15  |
| #6             | 3,090 | 8 / 9 / 31   | 13 / 35 | 11 / 37 | 2 / 2 / 12   | 2 / 14  | 2 / 14  |
| #7             | 3,044 | 8 / 8 / 32   | 13 / 35 | 10 / 38 | 1 / 0 / 15   | 1 / 15  | 0 / 16  |
| #8             | 3,106 | 11 / 8 / 29  | 15 / 33 | 10 / 38 | 1 / 1 / 14   | 1 / 15  | 1 / 15  |
| b1             | 3,179 | 9 / 8 / 31   | 14 / 34 | 9 / 39  | 3 / 1 / 12   | 3 / 13  | 1 / 15  |
| b2             | 3,069 | 10 / 8 / 30  | 16 / 32 | 10 / 38 | 2 / 1 / 13   | 2 / 14  | 1 / 15  |
| b3             | 3,128 | 9 / 7 / 32   | 15 / 33 | 8 / 40  | 2 / 1 / 13   | 2 / 14  | 1 / 15  |
| b4             | 3,054 | 9 / 8 / 31   | 15 / 33 | 10 / 38 | 3 / 1 / 12   | 3 / 13  | 1 / 15  |
| b5             | 3,065 | 11 / 7 / 30  | 15 / 33 | 9 / 39  | 1 / 1 / 14   | 1 / 15  | 1 / 15  |
| b6             | 3,107 | 9 / 8 / 31   | 15 / 33 | 10 / 38 | 2 / 1 / 13   | 2 / 14  | 1 / 15  |
| b7             | 3,039 | 9 / 10 / 29  | 14 / 34 | 12 / 36 | 2 / 0 / 14   | 2 / 14  | 0 / 16  |
| b8             | 3,04  | 10 / 6 / 32  | 14 / 34 | 8 / 40  | 1 / 1 / 14   | 1 / 15  | 1 / 15  |
| †1             | 3,130 | 9 / 9 / 30   | 15 / 33 | 10 / 38 | 2 / 1 / 13   | 2 / 14  | 1 / 15  |
| †2             | 3,103 | 9 / 10 / 29  | 13 / 35 | 12 / 36 | 1 / 2 / 13   | 1 / 15  | 2 / 14  |
| †3             | 3,342 | 9 / 7 / 32   | 15 / 33 | 8 / 40  | 3 / 1 / 12   | 3 / 13  | 1 / 15  |
| †4             | 3,043 | 9 / 9 / 30   | 15 / 33 | 10 / 38 | 1 / 1 / 14   | 1 / 15  | 1 / 15  |
| †5             | 3,073 | 11 / 7 / 30  | 15 / 33 | 9 / 39  | 1 / 1 / 14   | 1 / 15  | 1 / 15  |
| †6             | 3,087 | 9 / 9 / 30   | 14 / 34 | 11 / 37 | 2 / 0 / 14   | 2 / 14  | 0 / 16  |
| †7             | 3,052 | 11 / 7 / 30  | 15 / 33 | 9 / 39  | 1 / 1 / 14   | 1 / 15  | 1 / 15  |
| †8             | 3,052 | 11 / 7 / 30  | 15 / 33 | 9 / 39  | 1 / 1 / 14   | 1 / 15  | 1 / 15  |

TABLE 10. Undergraduate + Graduate New Enrollment: Comparison of the seasonal per-period SAE stochastic latest-optimized  $\tilde{\alpha}$ -power balanced forecasting models to the seasonal Holt–Winters and ARIMA models

*Example 10.* Let’s consider undergraduate total credit hours (fall, spring and summer) over 18 years (fall 2004 to summer 2022). Thus,  $f = 3$  and  $X = (x_1, \dots, x_{54})$ . For  $s = 1, \dots, 3$  (trimesters), and  $i = 2, \dots, 17$  (years), we compute the sequence of three-step-ahead forecasts for  $X$  using both

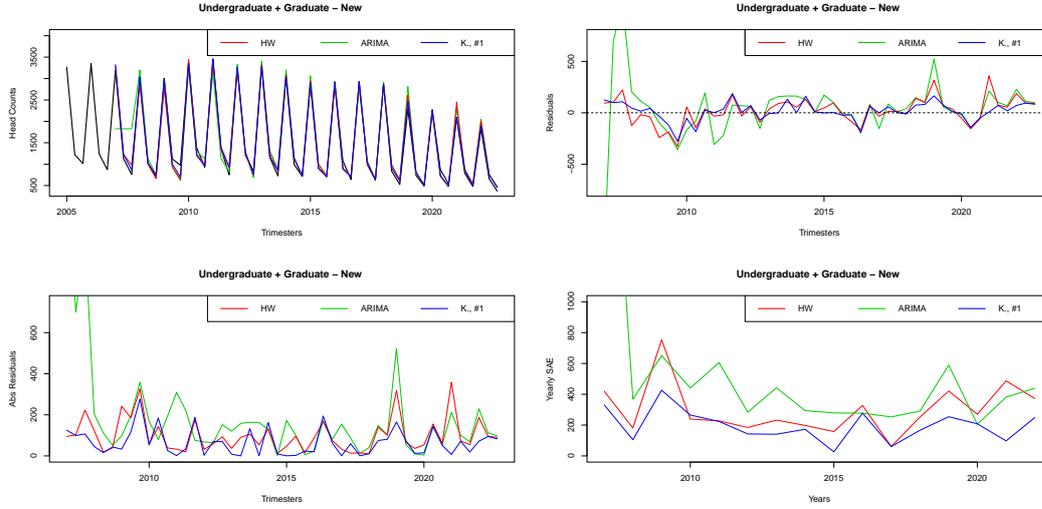


FIGURE 9. Undergraduate + Graduate New Enrollment: Forecasts, residuals, absolute residuals and yearly sum of absolute errors

Holt–Winters and ARIMA models. The sum of absolute errors for each model are

$$\text{SAE}^{\text{HW}} = \sum_{j=7}^{54} \left| \hat{x}_j^{\text{HW}} - x_j \right| = 93,088 \quad \text{and} \quad \text{SAE}^{\text{ARIMA}} = \sum_{j=7}^{54} \left| \hat{x}_j^{\text{ARIMA}} - x_j \right| = 171,033 \quad (68)$$

Comparing these 16 years times 3 trimesters forecasts, i.e., 48 trimester forecasts, the Holt–Winters model lead to better trimesterly forecast (smallest absolute residual) 32 times as opposed to 16 times for the ARIMA model, i.e.,  $HW/AR = 32/16$ . Moreover, if one compares the sum of absolute error per year, the Holt–Winters model lead to a better yearly forecast nine times as opposed to seven times for ARIMA, i.e.,  $HW/AR = 9/7$ .

For  $s = 1, \dots, 3$  (trimesters), and  $i = 2, \dots, 17$  (years), we compute the sequence of three-step-ahead  $\sharp, \flat$  and  $\natural$  forecasts for  $X$  as in (56), (57) and (58) for each optimization criteria  $\kappa = 1, \dots, 8$  using the seasonal per-period SAE stochastic latest-optimized  $\tilde{\alpha}$ -power balanced forecasting models (with  $N = 1000$  to find the optimal latest-optimized alpha power). Moreover, we compute the (global) sum of absolute error, and we compare our forecasts to both Holt–Winters and ARIMA models by counting the number of best monthly and yearly forecasts. The results are summarized in Table 11..

*Example 11.* Let’s consider undergraduate continuing credit hours (fall, spring and summer) over 18 years (fall 2004 to summer 2022). Thus,  $f = 3$  and  $X = (x_1, \dots, x_{54})$ . For  $s = 1, \dots, 3$  (trimesters), and  $i = 2, \dots, 17$  (years), we compute the sequence of three-step-ahead forecasts for  $X$  using both Holt–Winters and ARIMA models. The sum of absolute errors for each model are

$$\text{SAE}^{\text{HW}} = \sum_{j=7}^{54} \left| \hat{x}_j^{\text{HW}} - x_j \right| = 89,007 \quad \text{and} \quad \text{SAE}^{\text{ARIMA}} = \sum_{j=7}^{54} \left| \hat{x}_j^{\text{ARIMA}} - x_j \right| = 147,859 \quad (69)$$

Comparing these 16 years times 3 trimesters forecasts, i.e., 48 trimester forecasts, the Holt–Winters model lead to better trimesterly forecast (smallest absolute residual) 28 times as opposed to 20 times for the ARIMA model, i.e.,  $HW/AR = 28/20$ . Moreover, if one compares the sum

| •/κ | SAE    | HW / AR / K. | HW / K. | AR / K. | HW / AR / K. | HW / K. | AR / K. |
|-----|--------|--------------|---------|---------|--------------|---------|---------|
| #1  | 55,490 | 16 / 7 / 25  | 19 / 29 | 10 / 38 | 4 / 2 / 10   | 6 / 10  | 2 / 14  |
| #2  | 54,052 | 14 / 6 / 28  | 16 / 32 | 9 / 39  | 4 / 2 / 10   | 6 / 10  | 2 / 14  |
| #3  | 55,601 | 18 / 7 / 23  | 10 / 29 | 9 / 39  | 3 / 2 / 11   | 5 / 11  | 2 / 14  |
| #4  | 55,949 | 16 / 7 / 25  | 18 / 30 | 10 / 38 | 4 / 2 / 10   | 6 / 10  | 2 / 14  |
| #5  | 54,621 | 14 / 7 / 27  | 17 / 31 | 10 / 38 | 4 / 2 / 10   | 6 / 10  | 2 / 14  |
| #6  | 56,656 | 17 / 7 / 24  | 19 / 29 | 10 / 38 | 4 / 2 / 10   | 6 / 10  | 2 / 14  |
| #7  | 56,264 | 14 / 7 / 27  | 15 / 33 | 9 / 39  | 5 / 2 / 9    | 7 / 9   | 2 / 14  |
| #8  | 53,830 | 15 / 7 / 26  | 16 / 32 | 11 / 37 | 4 / 2 / 10   | 6 / 10  | 2 / 14  |
| ‡1  | 55,714 | 16 / 7 / 25  | 17 / 31 | 11 / 37 | 4 / 2 / 10   | 6 / 10  | 2 / 14  |
| ‡2  | 54,597 | 15 / 7 / 26  | 17 / 31 | 11 / 37 | 4 / 2 / 10   | 6 / 10  | 2 / 14  |
| ‡3  | 56,149 | 17 / 6 / 25  | 19 / 29 | 9 / 39  | 4 / 2 / 10   | 6 / 10  | 2 / 14  |
| ‡4  | 55,733 | 14 / 7 / 27  | 16 / 32 | 9 / 39  | 4 / 2 / 10   | 6 / 10  | 2 / 14  |
| ‡5  | 54,500 | 14 / 6 / 28  | 16 / 32 | 10 / 38 | 4 / 2 / 10   | 6 / 10  | 2 / 14  |
| ‡6  | 56,058 | 16 / 7 / 25  | 18 / 30 | 9 / 39  | 3 / 2 / 11   | 5 / 11  | 2 / 14  |
| ‡7  | 55,918 | 15 / 7 / 26  | 17 / 31 | 9 / 39  | 5 / 2 / 9    | 7 / 9   | 2 / 14  |
| ‡8  | 54,360 | 14 / 7 / 27  | 17 / 31 | 10 / 38 | 4 / 2 / 10   | 6 / 10  | 2 / 14  |
| ‡1  | 56,484 | 15 / 6 / 27  | 17 / 31 | 9 / 39  | 4 / 2 / 10   | 6 / 10  | 2 / 14  |
| ‡2  | 53,906 | 14 / 6 / 28  | 17 / 31 | 9 / 39  | 4 / 2 / 10   | 6 / 10  | 2 / 14  |
| ‡3  | 55,252 | 16 / 6 / 26  | 18 / 30 | 9 / 39  | 4 / 2 / 10   | 6 / 10  | 2 / 14  |
| ‡4  | 56,108 | 16 / 7 / 25  | 19 / 29 | 10 / 38 | 4 / 2 / 10   | 6 / 10  | 2 / 14  |
| ‡5  | 54,387 | 13 / 6 / 29  | 15 / 33 | 10 / 38 | 4 / 2 / 10   | 6 / 10  | 2 / 14  |
| ‡6  | 56,531 | 15 / 7 / 29  | 16 / 32 | 9 / 39  | 4 / 2 / 10   | 6 / 10  | 2 / 14  |
| ‡7  | 56,770 | 15 / 7 / 26  | 17 / 31 | 9 / 39  | 5 / 2 / 9    | 7 / 9   | 2 / 14  |
| ‡8  | 54,332 | 14 / 6 / 28  | 15 / 33 | 10 / 38 | 4 / 2 / 10   | 6 / 10  | 2 / 14  |

TABLE 11. Undergraduate Total Credit Hours: Comparison of the seasonal per-period SAE stochastic latest-optimized  $\tilde{\alpha}$ -power balanced forecasting models to the seasonal Holt–Winters and ARIMA models

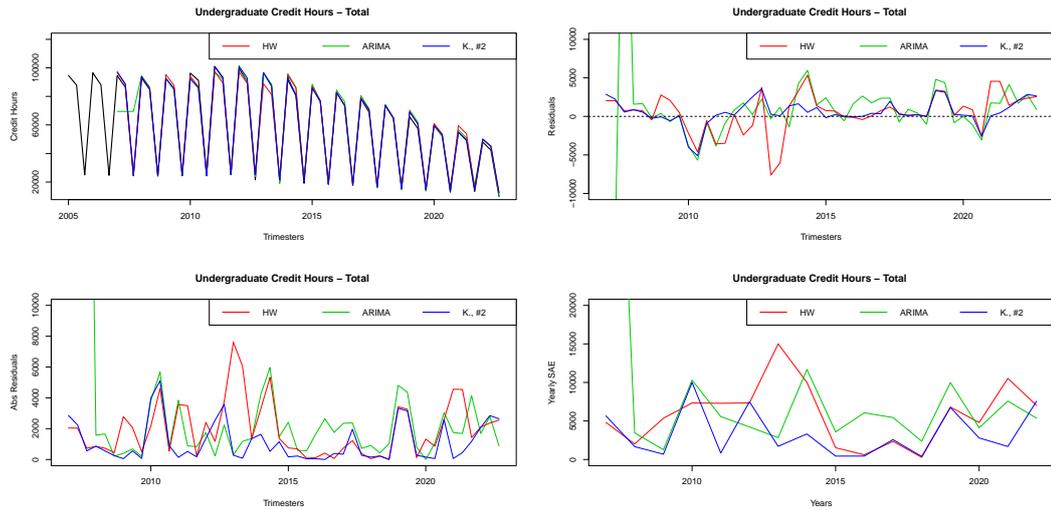


FIGURE 10. Undergraduate Total Credit Hours: Forecasts, residuals, absolute residuals and yearly sum of absolute errors

of absolute error per year, the Holt–Winters model lead to a better yearly forecast nine times as opposed to seven times for ARIMA, i.e.,  $HW/AR = 9/7$ .

For  $s = 1, \dots, 3$  (trimesters), and  $i = 2, \dots, 17$  (years), we compute the sequence of three-step-ahead  $\ddagger, \ddot{b}$  and  $\ddot{b}$  forecasts for  $X$  as in (56), (57) and (58) for each optimization criteria  $\kappa = 1, \dots, 8$  using the seasonal per-period SAE stochastic latest-optimized  $\tilde{\alpha}$ -power balanced forecasting models (with  $N = 1000$  to find the optimal latest-optimized alpha power). Moreover,

we compute the (global) sum of absolute error, and we compare our forecasts to both Holt–Winters and ARIMA models by counting the number of best monthly and yearly forecasts. The results are summarized in Table 12..

| •K | SAE    | HW / AR / K. | HW / K. | AR / K. | HW / AR / K. | HW / K. | AR / K. |
|----|--------|--------------|---------|---------|--------------|---------|---------|
| #1 | 46,307 | 9 / 6 / 33   | 10 / 38 | 9 / 39  | 2 / 1 / 13   | 2 / 14  | 1 / 15  |
| #2 | 45,926 | 9 / 8 / 31   | 12 / 36 | 11 / 37 | 1 / 1 / 14   | 1 / 15  | 1 / 15  |
| #3 | 46,317 | 8 / 9 / 31   | 9 / 39  | 12 / 36 | 2 / 1 / 13   | 2 / 14  | 1 / 15  |
| #4 | 45,977 | 8 / 7 / 33   | 10 / 38 | 9 / 39  | 2 / 1 / 13   | 2 / 14  | 1 / 15  |
| #5 | 45,678 | 8 / 8 / 32   | 11 / 37 | 10 / 38 | 1 / 1 / 14   | 1 / 15  | 1 / 15  |
| #6 | 46,389 | 7 / 7 / 34   | 9 / 39  | 8 / 40  | 2 / 1 / 13   | 2 / 14  | 1 / 15  |
| #7 | 46,323 | 10 / 6 / 32  | 11 / 37 | 7 / 41  | 2 / 1 / 13   | 2 / 14  | 1 / 15  |
| #8 | 46,229 | 8 / 8 / 32   | 11 / 37 | 11 / 37 | 1 / 1 / 14   | 1 / 15  | 1 / 15  |
| b1 | 46,694 | 9 / 8 / 31   | 12 / 36 | 10 / 38 | 2 / 1 / 13   | 2 / 14  | 1 / 15  |
| b2 | 45,117 | 6 / 8 / 34   | 9 / 39  | 9 / 39  | 1 / 1 / 14   | 1 / 15  | 1 / 15  |
| b3 | 45,663 | 9 / 8 / 31   | 11 / 37 | 10 / 38 | 2 / 1 / 13   | 2 / 14  | 1 / 15  |
| b4 | 45,736 | 9 / 8 / 31   | 10 / 38 | 10 / 38 | 2 / 1 / 13   | 2 / 14  | 1 / 15  |
| b5 | 45,927 | 7 / 9 / 32   | 11 / 37 | 11 / 37 | 1 / 1 / 14   | 1 / 15  | 1 / 15  |
| b6 | 46,173 | 7 / 8 / 33   | 10 / 38 | 9 / 39  | 2 / 1 / 13   | 2 / 14  | 1 / 15  |
| b7 | 46,161 | 7 / 9 / 32   | 8 / 40  | 11 / 37 | 2 / 1 / 13   | 2 / 14  | 1 / 15  |
| b8 | 45,662 | 7 / 9 / 32   | 11 / 37 | 11 / 37 | 1 / 1 / 14   | 1 / 15  | 1 / 15  |
| h1 | 46,610 | 8 / 10 / 30  | 10 / 38 | 13 / 35 | 2 / 1 / 13   | 2 / 14  | 1 / 15  |
| h2 | 45,659 | 8 / 7 / 33   | 12 / 36 | 8 / 40  | 1 / 1 / 14   | 1 / 15  | 1 / 15  |
| h3 | 46,403 | 9 / 8 / 31   | 12 / 36 | 10 / 38 | 2 / 1 / 13   | 2 / 14  | 1 / 15  |
| h4 | 45,971 | 8 / 9 / 31   | 11 / 37 | 11 / 37 | 2 / 1 / 13   | 2 / 14  | 1 / 15  |
| h5 | 45,730 | 8 / 8 / 32   | 12 / 36 | 11 / 37 | 1 / 1 / 14   | 1 / 15  | 1 / 15  |
| h6 | 45,865 | 8 / 9 / 31   | 10 / 38 | 10 / 38 | 2 / 1 / 13   | 2 / 14  | 1 / 15  |
| h7 | 45,799 | 6 / 8 / 34   | 8 / 40  | 10 / 38 | 2 / 1 / 13   | 2 / 14  | 1 / 15  |
| h8 | 45,506 | 8 / 8 / 32   | 12 / 36 | 9 / 39  | 1 / 1 / 14   | 1 / 15  | 1 / 15  |

TABLE 12. Undergraduate Continuing Credit Hours: Comparison of the seasonal per-period SAE stochastic latest-optimized  $\tilde{\alpha}$ -power balanced forecasting models to the seasonal Holt–Winters and ARIMA models

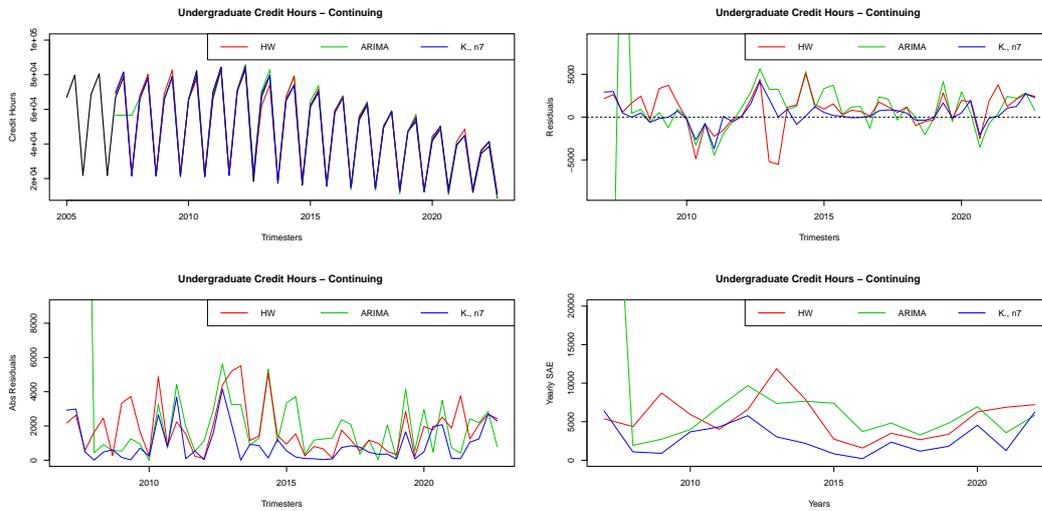


FIGURE 11. Undergraduate Continuing Credit Hours: Forecasts, residuals, absolute residuals and yearly sum of absolute errors

*Example 12.* Let’s consider undergraduate new credit hours (fall, spring and summer) over 18 years (fall 2004 to summer 2022). Thus,  $f = 3$  and  $X = (x_1, \dots, x_{54})$ . For  $s = 1, \dots, 3$  (trimesters),

and  $i = 2, \dots, 17$  (years), we compute the sequence of three-step-ahead forecasts for  $X$  using both Holt–Winters and ARIMA models. The sum of absolute errors for each model are

$$\text{SAE}^{\text{HW}} = \sum_{j=7}^{54} \left| \hat{x}_j^{\text{HW}} - x_j \right| = 34,265 \text{ and } \text{SAE}^{\text{ARIMA}} = \sum_{j=7}^{54} \left| \hat{x}_j^{\text{ARIMA}} - x_j \right| = 67,620 \quad (70)$$

Comparing these 16 years times 3 trimesters forecasts, i.e., 48 trimester forecasts, the Holt–winters model lead to better trimesterly forecast (smallest absolute residual) 32 times as opposed to 16 times for the ARIMA model, i.e.,  $HW/AR = 32/16$ . Moreover, if one compares the sum of absolute error per year, the Holt–Winters model lead to a better yearly forecast ten times as opposed to six times for ARIMA, i.e.,  $HW/AR = 10/6$ .

For  $s = 1, \dots, 3$  (trimesters), and  $i = 2, \dots, 17$  (years), we compute the sequence of three-step-ahead  $\sharp, \flat$  and  $\natural$  forecasts for  $X$  as in (56), (57) and (58) for each optimization criteria  $\kappa = 1, \dots, 8$  using the seasonal per-period SAE stochastic latest-optimized  $\tilde{\alpha}$ -power balanced forecasting models (with  $N = 1000$  to find the optimal latest-optimized alpha power). Moreover, we compute the (global) sum of absolute error, and we compare our forecasts to both Holt–Winters and ARIMA models by counting the number of best monthly and yearly forecasts. The results are summarized in Table 13..

| $\bullet \kappa$ | SAE    | HW / AR / K. | HW / K. | AR / K. | HW / AR / K. | HW / K. | AR / K. |
|------------------|--------|--------------|---------|---------|--------------|---------|---------|
| #1               | 22,391 | 13 / 6 / 29  | 14 / 34 | 11 / 37 | 1 / 0 / 15   | 1 / 15  | 0 / 16  |
| #2               | 22,958 | 14 / 6 / 28  | 16 / 32 | 10 / 38 | 2 / 0 / 14   | 2 / 14  | 0 / 16  |
| #3               | 22,093 | 13 / 6 / 29  | 16 / 32 | 9 / 39  | 1 / 0 / 15   | 1 / 15  | 0 / 16  |
| #4               | 21,986 | 13 / 6 / 29  | 15 / 33 | 9 / 39  | 1 / 0 / 15   | 1 / 15  | 0 / 16  |
| #5               | 22,551 | 14 / 7 / 27  | 17 / 31 | 11 / 37 | 3 / 0 / 13   | 3 / 13  | 0 / 16  |
| #6               | 22,319 | 13 / 5 / 30  | 14 / 34 | 9 / 39  | 2 / 0 / 14   | 2 / 14  | 0 / 16  |
| #7               | 22,507 | 12 / 8 / 28  | 13 / 35 | 12 / 36 | 1 / 0 / 15   | 1 / 15  | 0 / 16  |
| #8               | 22,294 | 12 / 9 / 27  | 13 / 35 | 12 / 36 | 2 / 0 / 14   | 2 / 14  | 0 / 16  |
| ♯1               | 21,708 | 14 / 4 / 30  | 15 / 33 | 8 / 40  | 0 / 0 / 16   | 0 / 16  | 0 / 16  |
| ♯2               | 23,246 | 14 / 6 / 28  | 16 / 32 | 11 / 37 | 2 / 0 / 14   | 2 / 14  | 0 / 16  |
| ♯3               | 22,004 | 12 / 7 / 29  | 13 / 35 | 11 / 37 | 1 / 0 / 15   | 1 / 15  | 0 / 16  |
| ♯4               | 22,265 | 12 / 8 / 28  | 13 / 35 | 11 / 37 | 2 / 0 / 14   | 2 / 14  | 0 / 16  |
| ♯5               | 22,945 | 12 / 7 / 29  | 14 / 34 | 10 / 38 | 2 / 0 / 14   | 2 / 14  | 0 / 16  |
| ♯6               | 22,206 | 14 / 7 / 27  | 15 / 33 | 11 / 37 | 2 / 0 / 14   | 2 / 14  | 0 / 16  |
| ♯7               | 22,145 | 14 / 5 / 29  | 16 / 32 | 8 / 40  | 2 / 0 / 14   | 2 / 14  | 0 / 16  |
| ♯8               | 22,537 | 11 / 7 / 30  | 13 / 35 | 11 / 37 | 3 / 0 / 13   | 3 / 13  | 0 / 16  |
| ♭1               | 21,412 | 13 / 7 / 28  | 15 / 33 | 11 / 37 | 0 / 0 / 16   | 0 / 16  | 0 / 16  |
| ♭2               | 23,530 | 14 / 7 / 27  | 16 / 32 | 12 / 36 | 2 / 0 / 14   | 2 / 14  | 0 / 16  |
| ♭3               | 21,654 | 11 / 7 / 30  | 13 / 35 | 10 / 38 | 0 / 0 / 16   | 0 / 16  | 0 / 16  |
| ♭4               | 22,104 | 14 / 5 / 29  | 14 / 34 | 9 / 39  | 2 / 0 / 14   | 2 / 14  | 0 / 16  |
| ♭5               | 22,657 | 11 / 8 / 29  | 12 / 36 | 11 / 37 | 3 / 0 / 13   | 3 / 13  | 0 / 16  |
| ♭6               | 22,311 | 13 / 7 / 28  | 14 / 34 | 10 / 38 | 2 / 0 / 14   | 2 / 14  | 0 / 16  |
| ♭7               | 22,058 | 15 / 7 / 26  | 17 / 31 | 9 / 39  | 2 / 0 / 14   | 2 / 14  | 0 / 16  |
| ♭8               | 22,373 | 12 / 8 / 28  | 14 / 34 | 12 / 36 | 3 / 0 / 13   | 3 / 13  | 0 / 16  |

TABLE 13. Undergraduate New Credit Hours: Comparison of the seasonal per-period SAE stochastic latest-optimized  $\tilde{\alpha}$ -power balanced forecasting models to the seasonal Holt–Winters and ARIMA models

*Example 13.* Let’s consider graduate total credit hours (fall, spring and summer) over 18 years (fall 2004 to summer 2022). Thus,  $f = 3$  and  $X = (x_1, \dots, x_{54})$ . For  $s = 1, \dots, 3$  (trimesters), and  $i = 2, \dots, 17$  (years), we compute the sequence of three-step-ahead forecasts for  $X$  using both Holt–Winters and ARIMA models. The sum of absolute errors for each model are

$$\text{SAE}^{\text{HW}} = \sum_{j=7}^{54} \left| \hat{x}_j^{\text{HW}} - x_j \right| = 33,070 \text{ and } \text{SAE}^{\text{ARIMA}} = \sum_{j=7}^{54} \left| \hat{x}_j^{\text{ARIMA}} - x_j \right| = 37,629 \quad (71)$$

Comparing these 16 years times 3 trimesters forecasts, i.e., 48 trimester forecasts, the Holt–winters model lead to better trimesterly forecast (smallest absolute residual) 26 times as opposed to 22 times for the ARIMA model, i.e.,  $HW/AR = 26/22$ . Moreover, if one compares the sum of

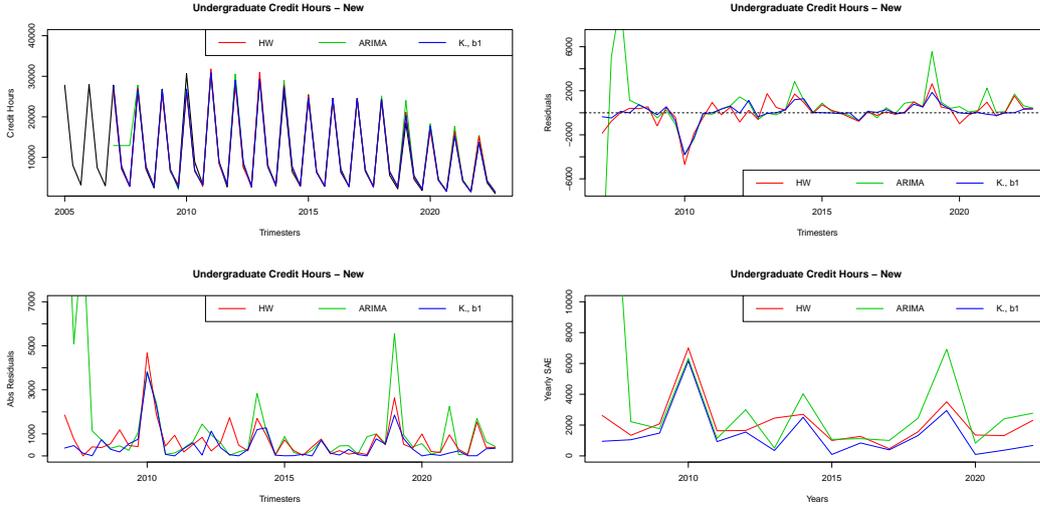


FIGURE 12. Undergraduate New Credit Hours: Forecasts, residuals, absolute residuals and yearly sum of absolute errors

absolute error per year, the Holt–Winters model lead to the same number of best yearly forecast as ARIMA, i.e.,  $HW/AR = 8/8$ .

For  $s = 1, \dots, 3$  (trimesters), and  $i = 2, \dots, 17$  (years), we compute the sequence of three-step-ahead  $\sharp, \flat$  and  $\natural$  forecasts for  $X$  as in (56), (57) and (58) for each optimization criteria  $\kappa = 1, \dots, 8$  using the seasonal per-period SAE stochastic latest-optimized  $\tilde{\alpha}$ -power balanced forecasting models (with  $N = 1000$  to find the optimal latest-optimized alpha power). Moreover, we compute the (global) sum of absolute error, and we compare our forecasts to both Holt–Winters and ARIMA models by counting the number of best monthly and yearly forecasts. The results are summarized in Table 14.

*Example 14.* Let’s consider graduate continuing credit hours (fall, spring and summer) over 18 years (fall 2004 to summer 2022). Thus,  $f = 3$  and  $X = (x_1, \dots, x_{54})$ . For  $s = 1, \dots, 3$  (trimesters), and  $i = 2, \dots, 17$  (years), we compute the sequence of three-step-ahead forecasts for  $X$  using both Holt–Winters and ARIMA models. The sum of absolute errors for each model are

$$SAE^{HW} = \sum_{j=7}^{54} \left| \hat{x}_j^{HW} - x_j \right| = 26,863 \text{ and } SAE^{ARIMA} = \sum_{j=7}^{54} \left| \hat{x}_j^{ARIMA} - x_j \right| = 31,672 \quad (72)$$

Comparing these 16 years times 3 trimesters forecasts, i.e., 48 trimester forecasts, the Holt–winters model lead to better trimesterly forecast (smallest absolute residual) 26 times as opposed to 22 times for the ARIMA model, i.e.,  $HW/AR = 26/22$ . Moreover, if one compares the sum of absolute error per year, the Holt–Winters model lead to a better yearly forecast twelve times as opposed to four times for ARIMA, i.e.,  $HW/AR = 12/4$ .

For  $s = 1, \dots, 3$  (trimesters), and  $i = 2, \dots, 17$  (years), we compute the sequence of three-step-ahead  $\sharp, \flat$  and  $\natural$  forecasts for  $X$  as in (56), (57) and (58) for each optimization criteria  $\kappa = 1, \dots, 8$  using the seasonal per-period SAE stochastic latest-optimized  $\tilde{\alpha}$ -power balanced forecasting models (with  $N = 1000$  to find the optimal latest-optimized alpha power). Moreover,

| •/s | SAE    | HW / AR / K. | HW / K. | AR / K. | HW / AR / K. | HW / K. | AR / K. |
|-----|--------|--------------|---------|---------|--------------|---------|---------|
| #1  | 22,292 | 7 / 8 / 33   | 10 / 38 | 11 / 37 | 2 / 1 / 13   | 2 / 14  | 1 / 15  |
| #2  | 22,853 | 8 / 10 / 30  | 10 / 38 | 14 / 34 | 2 / 1 / 13   | 2 / 14  | 1 / 15  |
| #3  | 22,591 | 9 / 8 / 31   | 12 / 36 | 11 / 37 | 2 / 1 / 13   | 2 / 14  | 1 / 15  |
| #4  | 22,617 | 8 / 7 / 33   | 10 / 38 | 10 / 38 | 2 / 1 / 13   | 2 / 14  | 1 / 15  |
| #5  | 22,498 | 8 / 8 / 32   | 11 / 37 | 12 / 36 | 2 / 1 / 13   | 2 / 14  | 1 / 15  |
| #6  | 22,957 | 9 / 9 / 30   | 12 / 36 | 12 / 36 | 2 / 1 / 13   | 2 / 14  | 1 / 15  |
| #7  | 22,613 | 9 / 8 / 31   | 12 / 36 | 11 / 37 | 2 / 1 / 13   | 2 / 14  | 1 / 15  |
| #8  | 22,577 | 9 / 9 / 30   | 11 / 37 | 13 / 35 | 2 / 1 / 13   | 2 / 14  | 1 / 15  |
| b1  | 22,603 | 9 / 9 / 30   | 12 / 36 | 13 / 35 | 2 / 1 / 13   | 2 / 14  | 1 / 15  |
| b2  | 22,739 | 9 / 10 / 29  | 11 / 37 | 14 / 34 | 2 / 1 / 13   | 2 / 14  | 1 / 15  |
| b3  | 22,433 | 8 / 9 / 31   | 10 / 38 | 13 / 35 | 2 / 1 / 13   | 2 / 14  | 1 / 15  |
| b4  | 22,750 | 8 / 11 / 29  | 10 / 38 | 15 / 22 | 2 / 1 / 13   | 2 / 14  | 1 / 15  |
| b5  | 22,441 | 7 / 8 / 33   | 9 / 39  | 11 / 37 | 2 / 1 / 13   | 2 / 14  | 1 / 15  |
| b6  | 22,999 | 9 / 10 / 29  | 12 / 36 | 14 / 34 | 2 / 1 / 13   | 2 / 14  | 1 / 15  |
| b7  | 22,714 | 8 / 8 / 32   | 11 / 37 | 11 / 37 | 2 / 1 / 13   | 2 / 14  | 1 / 15  |
| b8  | 22,594 | 7 / 10 / 31  | 9 / 39  | 12 / 36 | 2 / 1 / 13   | 2 / 14  | 1 / 15  |
| ‡1  | 22,435 | 7 / 9 / 32   | 10 / 38 | 12 / 36 | 2 / 1 / 13   | 2 / 14  | 1 / 15  |
| ‡2  | 22,801 | 7 / 10 / 31  | 9 / 39  | 14 / 34 | 2 / 1 / 13   | 2 / 14  | 1 / 15  |
| ‡3  | 23,498 | 9 / 9 / 30   | 11 / 37 | 13 / 35 | 2 / 1 / 13   | 2 / 14  | 1 / 15  |
| ‡4  | 22,758 | 8 / 9 / 31   | 10 / 38 | 12 / 36 | 2 / 1 / 13   | 2 / 14  | 1 / 15  |
| ‡5  | 22,631 | 8 / 8 / 32   | 10 / 38 | 12 / 36 | 2 / 1 / 13   | 2 / 14  | 1 / 15  |
| ‡6  | 22,710 | 9 / 10 / 29  | 12 / 36 | 13 / 35 | 2 / 1 / 13   | 2 / 14  | 1 / 15  |
| ‡7  | 22,796 | 8 / 8 / 32   | 11 / 37 | 11 / 37 | 2 / 1 / 13   | 2 / 14  | 1 / 15  |
| ‡8  | 22,913 | 8 / 9 / 31   | 11 / 37 | 13 / 35 | 2 / 1 / 13   | 2 / 14  | 1 / 15  |

TABLE 14. Graduate Total Credit Hours: Comparison of the seasonal per-period SAE stochastic latest-optimized  $\tilde{\alpha}$ -power balanced forecasting models to the seasonal Holt–Winters and ARIMA models

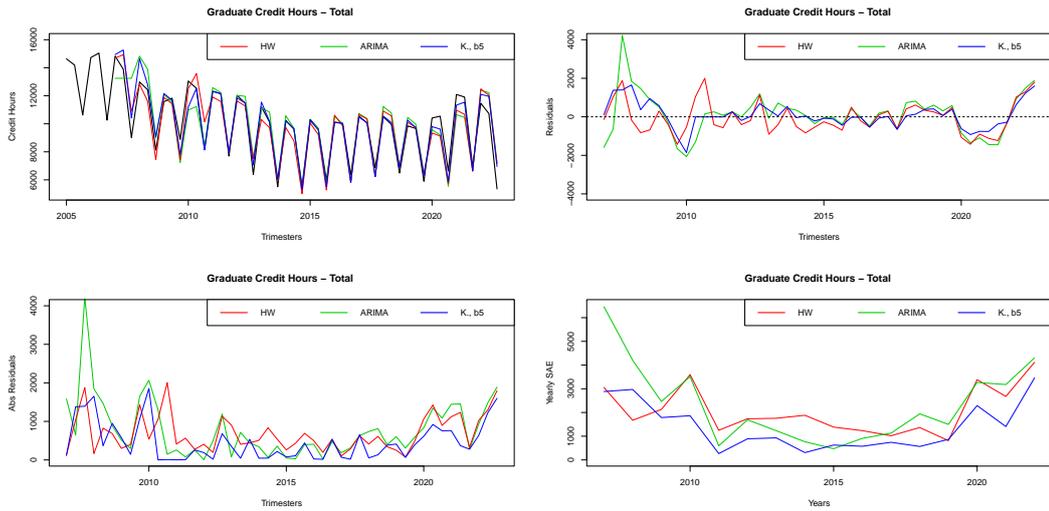


FIGURE 13. Graduate Total Credit Hours: Forecasts, residuals, absolute residuals and yearly sum of absolute errors

we compute the (global) sum of absolute error, and we compare our forecasts to both Holt–Winters and ARIMA models by counting the number of best monthly and yearly forecasts. The results are summarized in Table 15..

*Example 15.* Let’s consider graduate new credit hours (fall, spring and summer) over 18 years (fall 2004 to summer 2022). Thus,  $f = 3$  and  $X = (x_1, \dots, x_{54})$ . For  $s = 1, \dots, 3$  (trimesters), and  $i = 2, \dots, 17$  (years), we compute the sequence of three-step-ahead forecasts for  $X$  using both

| #/s | SAE    | HW / AR / K. | HW / K. | AR / K. | HW / AR / K. | HW / K. | AR / K. |
|-----|--------|--------------|---------|---------|--------------|---------|---------|
| #1  | 19,752 | 9 / 10 / 29  | 15 / 33 | 11 / 37 | 3 / 1 / 12   | 4 / 12  | 2 / 14  |
| #2  | 19,770 | 11 / 11 / 26 | 18 / 30 | 12 / 36 | 3 / 1 / 12   | 4 / 12  | 2 / 14  |
| #3  | 19,759 | 10 / 10 / 28 | 16 / 32 | 11 / 37 | 3 / 1 / 12   | 4 / 12  | 2 / 14  |
| #4  | 20,002 | 9 / 10 / 29  | 15 / 33 | 11 / 37 | 4 / 1 / 11   | 5 / 11  | 2 / 14  |
| #5  | 19,732 | 12 / 11 / 25 | 19 / 29 | 12 / 36 | 4 / 1 / 11   | 5 / 11  | 2 / 14  |
| #6  | 20,173 | 11 / 10 / 27 | 18 / 30 | 11 / 37 | 5 / 2 / 9    | 6 / 10  | 3 / 13  |
| #7  | 19,942 | 12 / 9 / 27  | 19 / 29 | 10 / 38 | 4 / 1 / 11   | 5 / 11  | 2 / 14  |
| #8  | 19,702 | 11 / 11 / 26 | 18 / 30 | 12 / 36 | 4 / 1 / 11   | 5 / 11  | 2 / 14  |
| b1  | 19,679 | 10 / 9 / 29  | 16 / 32 | 10 / 38 | 3 / 1 / 12   | 4 / 12  | 2 / 14  |
| b2  | 19,744 | 10 / 10 / 28 | 17 / 31 | 11 / 37 | 4 / 1 / 11   | 5 / 11  | 2 / 14  |
| b3  | 19,571 | 10 / 8 / 30  | 16 / 32 | 9 / 39  | 3 / 1 / 12   | 4 / 12  | 2 / 14  |
| b4  | 19,991 | 10 / 9 / 29  | 17 / 31 | 10 / 38 | 4 / 1 / 11   | 5 / 11  | 2 / 14  |
| b5  | 19,790 | 11 / 10 / 27 | 18 / 30 | 11 / 37 | 4 / 1 / 11   | 5 / 11  | 2 / 14  |
| b6  | 20,094 | 10 / 11 / 27 | 17 / 31 | 12 / 36 | 4 / 1 / 11   | 5 / 11  | 2 / 14  |
| b7  | 19,753 | 10 / 10 / 28 | 17 / 31 | 11 / 37 | 3 / 1 / 12   | 4 / 12  | 2 / 14  |
| b8  | 19,762 | 11 / 10 / 27 | 18 / 30 | 11 / 37 | 4 / 1 / 11   | 5 / 11  | 2 / 14  |
| ♯1  | 19,829 | 9 / 10 / 29  | 15 / 33 | 11 / 37 | 3 / 1 / 12   | 4 / 12  | 2 / 14  |
| ♯2  | 19,699 | 11 / 11 / 26 | 18 / 30 | 12 / 36 | 3 / 1 / 12   | 4 / 12  | 2 / 14  |
| ♯3  | 20,202 | 10 / 10 / 28 | 18 / 30 | 12 / 36 | 3 / 1 / 12   | 4 / 12  | 2 / 14  |
| ♯4  | 20,146 | 12 / 11 / 25 | 18 / 30 | 13 / 35 | 4 / 2 / 10   | 5 / 11  | 3 / 13  |
| ♯5  | 19,758 | 10 / 11 / 27 | 17 / 31 | 12 / 36 | 4 / 1 / 11   | 5 / 11  | 2 / 14  |
| ♯6  | 20,083 | 12 / 9 / 27  | 19 / 29 | 10 / 38 | 4 / 1 / 11   | 5 / 11  | 2 / 14  |
| ♯7  | 19,776 | 10 / 10 / 28 | 17 / 31 | 11 / 37 | 3 / 1 / 12   | 4 / 12  | 2 / 14  |
| ♯8  | 19,819 | 9 / 11 / 28  | 16 / 32 | 12 / 36 | 4 / 1 / 11   | 5 / 11  | 2 / 14  |

TABLE 15. Graduate Continuing Credit Hours: Comparison of the seasonal per-period SAE stochastic latest-optimized  $\tilde{\alpha}$ -power balanced forecasting models to the seasonal Holt–Winters and ARIMA models

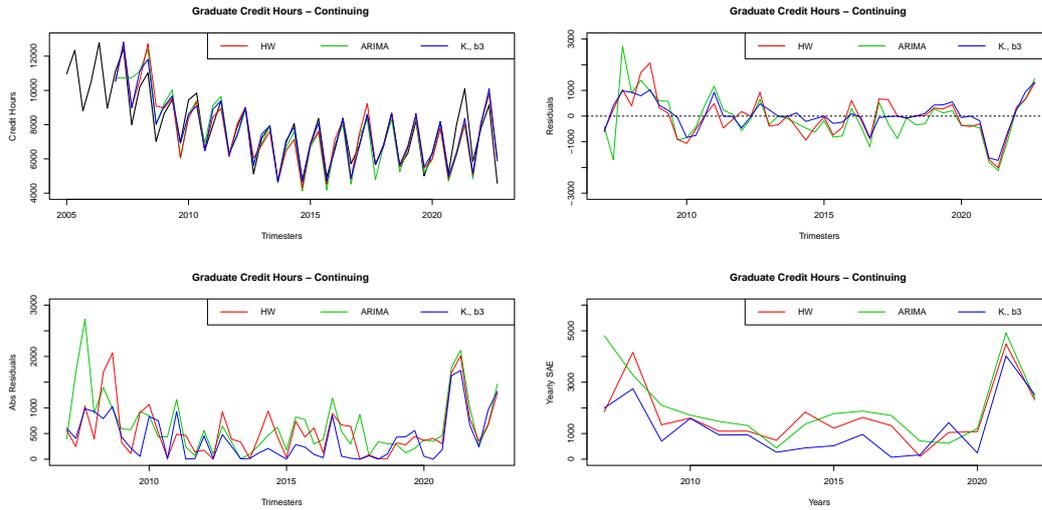


FIGURE 14. Graduate Continuing Credit Hours: Forecasts, residuals, absolute residuals and yearly sum of absolute errors

Holt–Winters and ARIMA models. The sum of absolute errors for each model are

$$SAE^{HW} = \sum_{j=7}^{54} \left| \hat{x}_j^{HW} - x_j \right| = 20,138 \text{ and } SAE^{ARIMA} = \sum_{j=7}^{54} \left| \hat{x}_j^{ARIMA} - x_j \right| = 19,258 \quad (73)$$

Comparing these 16 years times 3 trimesters forecasts, i.e., 48 trimester forecasts, the Holt–winters model lead to slightly worse trimesterly forecast (smallest absolute residual) 23 times as opposed to 25 times for the ARIMA model, i.e.,  $HW/AR = 23/25$ . Moreover, if one compares the sum of absolute error per year, the Holt–Winters model lead to a better yearly forecast nine

times as opposed to seven times for ARIMA, i.e.,  $HW/AR = 9/7$ .

For  $s = 1, \dots, 3$  (trimesters), and  $i = 2, \dots, 17$  (years), we compute the sequence of three-step-ahead  $\sharp, \flat$  and  $\natural$  forecasts for  $X$  as in (56), (57) and (58) for each optimization criteria  $\kappa = 1, \dots, 8$  using the seasonal per-period SAE stochastic latest-optimized  $\tilde{\alpha}$ -power balanced forecasting models (with  $N = 1000$  to find the optimal latest-optimized alpha power). Moreover, we compute the (global) sum of absolute error, and we compare our forecasts to both Holt–Winters and ARIMA models by counting the number of best monthly and yearly forecasts. The results are summarized in Table 16..

| $\bullet\kappa$ | SAE    | HW / AR / K. | HW / K. | AR / K. | HW / AR / K. | HW / K. | AR / K. |
|-----------------|--------|--------------|---------|---------|--------------|---------|---------|
| #1              | 12,829 | 12 / 12 / 24 | 13 / 35 | 16 / 32 | 4 / 2 / 10   | 4 / 12  | 3 / 13  |
| #2              | 12,950 | 12 / 12 / 24 | 13 / 35 | 16 / 32 | 4 / 1 / 11   | 4 / 12  | 2 / 14  |
| #3              | 12,820 | 11 / 13 / 24 | 12 / 36 | 16 / 32 | 4 / 2 / 10   | 4 / 12  | 3 / 13  |
| #4              | 12,316 | 9 / 12 / 27  | 9 / 39  | 15 / 33 | 4 / 1 / 11   | 4 / 12  | 2 / 14  |
| #5              | 12,606 | 9 / 13 / 26  | 10 / 38 | 15 / 33 | 4 / 1 / 11   | 4 / 12  | 2 / 14  |
| #6              | 12,639 | 12 / 13 / 23 | 13 / 35 | 18 / 30 | 4 / 2 / 10   | 4 / 12  | 3 / 13  |
| #7              | 12,779 | 12 / 14 / 22 | 13 / 35 | 17 / 31 | 4 / 2 / 10   | 4 / 12  | 3 / 13  |
| #8              | 12,779 | 10 / 14 / 24 | 11 / 37 | 18 / 30 | 4 / 1 / 11   | 4 / 12  | 2 / 14  |
| b1              | 12,550 | 13 / 11 / 24 | 14 / 34 | 15 / 33 | 2 / 1 / 13   | 2 / 14  | 2 / 14  |
| b2              | 12,605 | 12 / 11 / 25 | 12 / 36 | 17 / 31 | 3 / 1 / 12   | 3 / 13  | 2 / 14  |
| b3              | 12,834 | 13 / 12 / 23 | 14 / 34 | 15 / 33 | 4 / 1 / 11   | 4 / 12  | 2 / 14  |
| b4              | 12,560 | 12 / 11 / 25 | 12 / 36 | 15 / 33 | 3 / 1 / 12   | 3 / 13  | 2 / 14  |
| b5              | 12,346 | 11 / 12 / 25 | 12 / 36 | 17 / 31 | 3 / 1 / 12   | 3 / 13  | 2 / 14  |
| b6              | 12,733 | 12 / 12 / 24 | 12 / 36 | 15 / 33 | 4 / 1 / 11   | 4 / 12  | 2 / 14  |
| b7              | 12,883 | 14 / 13 / 21 | 15 / 33 | 18 / 30 | 4 / 2 / 10   | 4 / 12  | 3 / 13  |
| b8              | 12,607 | 13 / 13 / 22 | 14 / 34 | 17 / 31 | 3 / 1 / 12   | 3 / 13  | 2 / 14  |
| ‡1              | 12,751 | 12 / 12 / 24 | 13 / 35 | 16 / 32 | 4 / 2 / 10   | 4 / 12  | 3 / 13  |
| ‡2              | 12,895 | 11 / 12 / 25 | 12 / 36 | 16 / 32 | 4 / 1 / 11   | 4 / 12  | 2 / 14  |
| ‡3              | 12,492 | 13 / 12 / 23 | 13 / 35 | 15 / 33 | 3 / 2 / 11   | 3 / 13  | 3 / 13  |
| ‡4              | 12,350 | 10 / 13 / 25 | 10 / 38 | 17 / 31 | 4 / 1 / 11   | 4 / 12  | 2 / 14  |
| ‡5              | 12,565 | 11 / 13 / 24 | 12 / 36 | 17 / 31 | 4 / 1 / 11   | 4 / 12  | 2 / 14  |
| ‡6              | 12,719 | 12 / 11 / 25 | 12 / 36 | 14 / 34 | 4 / 1 / 11   | 4 / 12  | 2 / 14  |
| ‡7              | 12,817 | 12 / 12 / 24 | 13 / 35 | 16 / 32 | 4 / 2 / 10   | 4 / 12  | 3 / 13  |
| ‡8              | 12,834 | 11 / 13 / 24 | 12 / 36 | 16 / 32 | 3 / 1 / 12   | 3 / 13  | 2 / 14  |

TABLE 16. Graduate New Credit Hours: Comparison of the seasonal per-period SAE stochastic latest-optimized  $\tilde{\alpha}$ -power balanced forecasting models to the seasonal Holt–Winters and ARIMA models

*Example 16.* Let's consider student (undergraduate + graduate) total credit hours (fall, spring and summer) over 18 years (fall 2004 to summer 2022). Thus,  $f = 3$  and  $X = (x_1, \dots, x_{54})$ . For  $s = 1, \dots, 3$  (trimesters), and  $i = 2, \dots, 17$  (years), we compute the sequence of three-step-ahead forecasts for  $X$  using both Holt–Winters and ARIMA models. The sum of absolute errors for each model are

$$\text{SAE}^{\text{HW}} = \sum_{j=7}^{54} |\hat{x}_j^{\text{HW}} - x_j| = 105,985 \text{ and } \text{SAE}^{\text{ARIMA}} = \sum_{j=7}^{54} |\hat{x}_j^{\text{ARIMA}} - x_j| = 199,672 \quad (74)$$

Comparing these 16 years times 3 trimesters forecasts, i.e., 48 trimester forecasts, the Holt–winters model lead to better trimesterly forecast (smallest absolute residual) 29 times as opposed to 19 times for the ARIMA model, i.e.,  $HW/AR = 29/19$ . Moreover, if one compares the sum of absolute error per year, the Holt–Winters model lead to a better yearly forecast nine times as opposed to seven times for ARIMA, i.e.,  $HW/AR = 9/7$ .

For  $s = 1, \dots, 3$  (trimesters), and  $i = 2, \dots, 17$  (years), we compute the sequence of three-step-ahead  $\sharp, \flat$  and  $\natural$  forecasts for  $X$  as in (56), (57) and (58) for each optimization criteria  $\kappa = 1, \dots, 8$  using the seasonal per-period SAE stochastic latest-optimized  $\tilde{\alpha}$ -power balanced forecasting models (with  $N = 1000$  to find the optimal latest-optimized alpha power). Moreover,

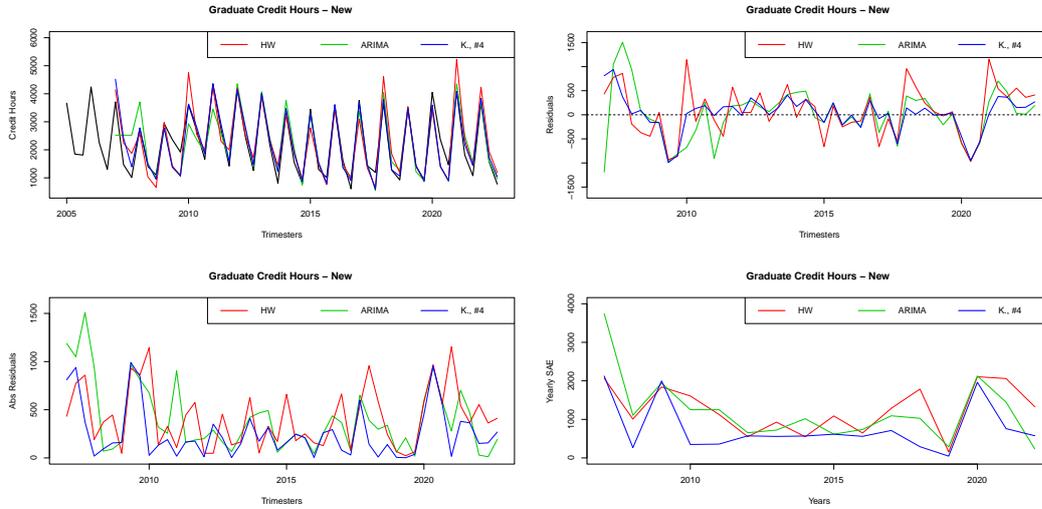


FIGURE 15. Graduate New Credit Hours: Forecasts, residuals, absolute residuals and yearly sum of absolute errors

we compute the (global) sum of absolute error, and we compare our forecasts to both Holt–Winters and ARIMA models by counting the number of best monthly and yearly forecasts. The results are summarized in Table 17..

| ● FC | SAE    | HW / AR / K. | HW / K. | AR / K. | HW / AR / K. | HW / K. | AR / K. |
|------|--------|--------------|---------|---------|--------------|---------|---------|
| #1   | 67,807 | 8 / 8 / 32   | 10 / 38 | 11 / 37 | 3 / 1 / 12   | 4 / 12  | 2 / 14  |
| #2   | 68,195 | 9 / 8 / 31   | 11 / 37 | 11 / 37 | 3 / 1 / 12   | 4 / 12  | 2 / 14  |
| #3   | 68,629 | 10 / 7 / 31  | 12 / 36 | 11 / 37 | 3 / 1 / 12   | 4 / 12  | 2 / 14  |
| #4   | 68,035 | 8 / 6 / 34   | 10 / 38 | 9 / 39  | 3 / 1 / 12   | 4 / 12  | 2 / 14  |
| #5   | 68,152 | 9 / 7 / 32   | 11 / 37 | 10 / 38 | 3 / 1 / 12   | 4 / 12  | 2 / 14  |
| #6   | 69,908 | 10 / 7 / 31  | 12 / 36 | 11 / 37 | 3 / 1 / 12   | 4 / 12  | 2 / 14  |
| #7   | 69,003 | 10 / 6 / 32  | 12 / 36 | 10 / 38 | 3 / 1 / 12   | 4 / 12  | 2 / 14  |
| #8   | 67,842 | 8 / 7 / 33   | 10 / 38 | 11 / 37 | 3 / 0 / 13   | 3 / 13  | 1 / 15  |
| b1   | 66,854 | 9 / 7 / 32   | 11 / 37 | 10 / 38 | 3 / 1 / 12   | 4 / 12  | 2 / 14  |
| b2   | 67,568 | 10 / 8 / 30  | 12 / 36 | 11 / 37 | 3 / 1 / 12   | 4 / 12  | 2 / 14  |
| b3   | 67,605 | 9 / 7 / 32   | 11 / 37 | 11 / 37 | 3 / 1 / 12   | 4 / 12  | 2 / 14  |
| b4   | 66,966 | 9 / 7 / 32   | 11 / 37 | 10 / 38 | 3 / 1 / 12   | 4 / 12  | 2 / 14  |
| b5   | 67,480 | 11 / 8 / 29  | 13 / 35 | 11 / 37 | 3 / 1 / 12   | 4 / 12  | 2 / 14  |
| b6   | 68,033 | 10 / 7 / 31  | 12 / 36 | 11 / 37 | 3 / 1 / 12   | 4 / 12  | 2 / 14  |
| b7   | 68,211 | 9 / 7 / 32   | 11 / 37 | 11 / 37 | 3 / 1 / 12   | 4 / 12  | 2 / 14  |
| b8   | 67,202 | 10 / 8 / 30  | 12 / 36 | 11 / 37 | 3 / 1 / 12   | 4 / 12  | 2 / 14  |
| ̣1   | 68,272 | 9 / 7 / 32   | 11 / 37 | 10 / 38 | 3 / 1 / 12   | 4 / 12  | 2 / 14  |
| ̣2   | 67,838 | 9 / 8 / 31   | 11 / 37 | 12 / 36 | 3 / 1 / 12   | 4 / 12  | 2 / 14  |
| ̣3   | 69,186 | 10 / 6 / 32  | 12 / 36 | 11 / 37 | 3 / 1 / 12   | 4 / 12  | 2 / 14  |
| ̣4   | 67,559 | 9 / 6 / 33   | 11 / 37 | 10 / 38 | 3 / 1 / 12   | 4 / 12  | 2 / 14  |
| ̣5   | 67,925 | 8 / 8 / 32   | 10 / 38 | 11 / 37 | 3 / 1 / 12   | 4 / 12  | 2 / 14  |
| ̣6   | 68,603 | 10 / 6 / 32  | 12 / 36 | 10 / 38 | 3 / 1 / 12   | 4 / 12  | 2 / 14  |
| ̣7   | 68,570 | 10 / 7 / 31  | 12 / 36 | 11 / 37 | 3 / 1 / 12   | 4 / 12  | 2 / 14  |
| ̣8   | 67,446 | 9 / 8 / 31   | 11 / 37 | 11 / 37 | 3 / 1 / 12   | 3 / 13  | 2 / 14  |

TABLE 17. Undergraduate + Graduate Total Credit Hours: Comparison of the seasonal per-period SAE stochastic latest-optimized  $\tilde{\alpha}$ -power balanced forecasting models to the seasonal Holt–Winters and ARIMA models

*Example 17.* Let’s consider student (undergraduate + graduate) continuing credit hours (fall, spring and summer) over 18 years (fall 2004 to summer 2022). Thus,  $f = 3$  and  $X = (x_1, \dots, x_{54})$ . For  $s = 1, \dots, 3$  (trimesters), and  $i = 2, \dots, 17$  (years), we compute the sequence of three-step-ahead forecasts for  $X$  using both Holt–Winters and ARIMA models. The sum of absolute errors

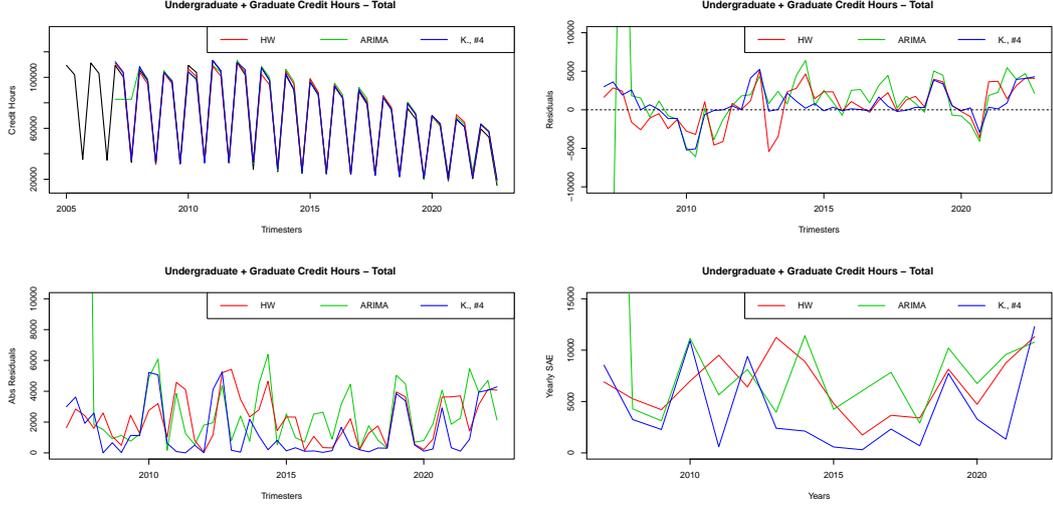


FIGURE 16. Undergraduate + Graduate Total Credit Hours: Forecasts, residuals, absolute residuals and yearly sum of absolute errors

for each model are

$$\text{SAE}^{\text{HW}} = \sum_{j=7}^{54} \left| \hat{x}_j^{\text{HW}} - x_j \right| = 92,711 \quad \text{and} \quad \text{SAE}^{\text{ARIMA}} = \sum_{j=7}^{54} \left| \hat{x}_j^{\text{ARIMA}} - x_j \right| = 156,508 \quad (75)$$

Comparing these 16 years times 3 trimesters forecasts, i.e., 48 trimester forecasts, the Holt–Winters model lead to better trimesterly forecast (smallest absolute residual) 28 times as opposed to 20 times for the ARIMA model, i.e.,  $\text{HW}/\text{AR} = 28/20$ . Moreover, if one compares the sum of absolute error per year, the Holt–Winters model lead to a better yearly forecast eleven times as opposed to five times for ARIMA, i.e.,  $\text{HW}/\text{AR} = 11/5$ .

For  $s = 1, \dots, 3$  (trimesters), and  $i = 2, \dots, 17$  (years), we compute the sequence of three-step-ahead  $\hat{x}_t^{\text{HW}}$ ,  $\hat{x}_t^{\text{ARIMA}}$  and  $\hat{x}_t^{\text{K, \#4}}$  forecasts for  $X$  as in (56), (57) and (58) for each optimization criteria  $\kappa = 1, \dots, 8$  using the seasonal per-period SAE stochastic latest-optimized  $\tilde{\alpha}$ -power balanced forecasting models (with  $N = 1000$  to find the optimal latest-optimized alpha power). Moreover, we compute the (global) sum of absolute error, and we compare our forecasts to both Holt–Winters and ARIMA models by counting the number of best monthly and yearly forecasts. The results are summarized in Table 18.

*Example 18.* Let's consider student (undergraduate + graduate) new credit hours (fall, spring and summer) over 18 years (fall 2004 to summer 2022). Thus,  $f = 3$  and  $X = (x_1, \dots, x_{54})$ . For  $s = 1, \dots, 3$  (trimesters), and  $i = 2, \dots, 17$  (years), we compute the sequence of three-step-ahead forecasts for  $X$  using both Holt–Winters and ARIMA models. The sum of absolute errors for each model are

$$\text{SAE}^{\text{HW}} = \sum_{j=7}^{54} \left| \hat{x}_j^{\text{HW}} - x_j \right| = 56,764 \quad \text{and} \quad \text{SAE}^{\text{ARIMA}} = \sum_{j=7}^{54} \left| \hat{x}_j^{\text{ARIMA}} - x_j \right| = 84,466 \quad (76)$$

Comparing these 16 years times 3 trimesters forecasts, i.e., 48 trimester forecasts, the Holt–Winters model lead to better trimesterly forecast (smallest absolute residual) 29 times as opposed

| •/s | SAE    | HW / AR / K. | HW / K. | AR / K. | HW / AR / K. | HW / K. | AR / K. |
|-----|--------|--------------|---------|---------|--------------|---------|---------|
| #1  | 51,659 | 9 / 10 / 29  | 11 / 37 | 11 / 37 | 2 / 1 / 13   | 2 / 14  | 1 / 15  |
| #2  | 51,644 | 7 / 10 / 31  | 12 / 36 | 11 / 37 | 1 / 1 / 14   | 1 / 15  | 1 / 15  |
| #3  | 52,501 | 10 / 9 / 29  | 12 / 36 | 10 / 38 | 2 / 1 / 13   | 2 / 14  | 1 / 15  |
| #4  | 51,888 | 10 / 10 / 28 | 13 / 35 | 11 / 37 | 2 / 1 / 13   | 2 / 14  | 1 / 15  |
| #5  | 50,919 | 7 / 10 / 31  | 11 / 37 | 11 / 37 | 1 / 1 / 14   | 1 / 15  | 1 / 15  |
| #6  | 52,325 | 10 / 9 / 29  | 12 / 36 | 11 / 37 | 2 / 1 / 13   | 2 / 14  | 1 / 15  |
| #7  | 53,028 | 10 / 10 / 28 | 12 / 36 | 12 / 36 | 2 / 1 / 13   | 2 / 14  | 1 / 15  |
| #8  | 51,262 | 9 / 8 / 31   | 11 / 37 | 0 / 39  | 1 / 1 / 14   | 1 / 15  | 1 / 15  |
| b1  | 52,007 | 9 / 10 / 29  | 12 / 36 | 12 / 36 | 2 / 0 / 14   | 2 / 14  | 0 / 16  |
| b2  | 51,214 | 6 / 10 / 32  | 10 / 38 | 11 / 37 | 1 / 1 / 14   | 1 / 15  | 1 / 15  |
| b3  | 52,543 | 10 / 9 / 29  | 12 / 36 | 11 / 37 | 2 / 1 / 13   | 2 / 14  | 1 / 15  |
| b4  | 52,097 | 10 / 10 / 28 | 13 / 35 | 12 / 36 | 2 / 1 / 13   | 2 / 14  | 1 / 15  |
| b5  | 50,952 | 6 / 10 / 32  | 11 / 37 | 11 / 37 | 1 / 1 / 14   | 1 / 15  | 1 / 15  |
| b6  | 52,142 | 10 / 10 / 28 | 12 / 36 | 12 / 36 | 2 / 1 / 13   | 2 / 14  | 1 / 15  |
| b7  | 52,575 | 9 / 9 / 30   | 11 / 37 | 10 / 38 | 2 / 1 / 13   | 2 / 14  | 1 / 15  |
| b8  | 50,263 | 6 / 10 / 32  | 10 / 38 | 11 / 37 | 1 / 0 / 15   | 1 / 15  | 0 / 16  |
| ‡1  | 52,284 | 11 / 10 / 27 | 13 / 35 | 12 / 36 | 2 / 1 / 13   | 2 / 14  | 1 / 15  |
| ‡2  | 51,293 | 7 / 10 / 31  | 11 / 37 | 11 / 37 | 1 / 1 / 14   | 1 / 15  | 1 / 15  |
| ‡3  | 51,545 | 8 / 10 / 30  | 9 / 39  | 11 / 37 | 2 / 0 / 14   | 2 / 14  | 0 / 16  |
| ‡4  | 52,308 | 11 / 9 / 28  | 13 / 35 | 11 / 37 | 2 / 1 / 13   | 2 / 14  | 1 / 15  |
| ‡5  | 51,587 | 8 / 10 / 30  | 13 / 35 | 11 / 37 | 1 / 1 / 14   | 1 / 15  | 1 / 15  |
| ‡6  | 53,126 | 11 / 10 / 27 | 14 / 34 | 12 / 36 | 2 / 1 / 13   | 2 / 14  | 1 / 15  |
| ‡7  | 53,635 | 10 / 8 / 30  | 12 / 36 | 9 / 39  | 2 / 1 / 13   | 2 / 14  | 1 / 15  |
| ‡8  | 50,883 | 6 / 10 / 32  | 11 / 37 | 11 / 37 | 1 / 1 / 14   | 1 / 15  | 1 / 15  |

TABLE 18. Undergraduate + Graduate Continuing Credit Hours: Comparison of the seasonal per-period SAE stochastic latest-optimized  $\tilde{\alpha}$ -power balanced forecasting models to the seasonal Holt–Winters and ARIMA models

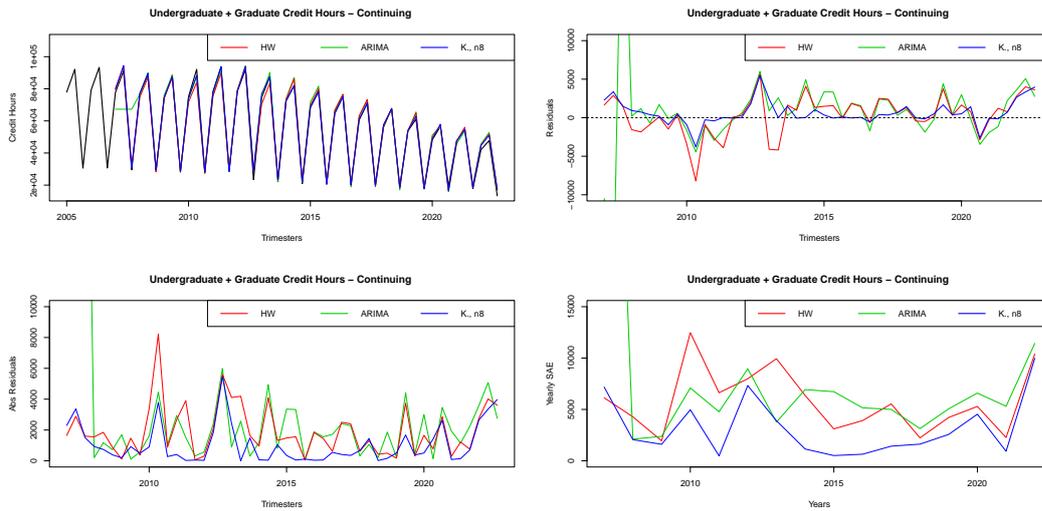


FIGURE 17. Undergraduate + Graduate Continuing Credit Hours: Forecasts, residuals, absolute residuals and yearly sum of absolute errors

to 19 times for the ARIMA model, i.e.,  $HW/AR = 29/19$ . Moreover, if one compares the sum of absolute error per year, the Holt–Winters model lead to a better yearly forecast ten times as opposed to six times for ARIMA, i.e.,  $HW/AR = 10/6$ .

For  $s = 1, \dots, 3$  (trimesters), and  $i = 2, \dots, 17$  (years), we compute the sequence of three-step-ahead  $\ddagger, b$  and  $\ddagger$  forecasts for  $X$  as in (56), (57) and (58) for each optimization criteria  $\kappa = 1, \dots, 8$  using the seasonal per-period SAE stochastic latest-optimized  $\tilde{\alpha}$ -power balanced forecasting models (with  $N = 1000$  to find the optimal latest-optimized alpha power). Moreover,

we compute the (global) sum of absolute error, and we compare our forecasts to both Holt–Winters and ARIMA models by counting the number of best monthly and yearly forecasts. The results are summarized in Table 19..

| ● <sub>FC</sub> | SAE    | HW / AR / K. | HW / K. | AR / K. | HW / AR / K. | HW / K. | AR / K. |
|-----------------|--------|--------------|---------|---------|--------------|---------|---------|
| #1              | 24,360 | 11 / 3 / 34  | 13 / 35 | 7 / 41  | 3 / 0 / 13   | 3 / 13  | 0 / 16  |
| #2              | 23,922 | 11 / 5 / 32  | 14 / 34 | 10 / 38 | 2 / 1 / 13   | 2 / 14  | 1 / 15  |
| #3              | 24,359 | 13 / 4 / 31  | 16 / 32 | 9 / 39  | 2 / 0 / 14   | 2 / 14  | 0 / 16  |
| #4              | 24,585 | 14 / 4 / 30  | 16 / 32 | 11 / 37 | 2 / 1 / 13   | 2 / 14  | 1 / 15  |
| #5              | 24,777 | 12 / 4 / 32  | 15 / 33 | 11 / 37 | 3 / 0 / 13   | 3 / 13  | 0 / 16  |
| #6              | 24,034 | 14 / 3 / 31  | 17 / 31 | 10 / 38 | 3 / 0 / 13   | 3 / 13  | 0 / 16  |
| #7              | 24,076 | 12 / 4 / 32  | 14 / 34 | 9 / 39  | 2 / 1 / 13   | 2 / 14  | 1 / 15  |
| #8              | 24,289 | 11 / 4 / 33  | 14 / 34 | 11 / 37 | 2 / 1 / 13   | 2 / 14  | 1 / 15  |
| b1              | 24,066 | 13 / 4 / 31  | 16 / 32 | 11 / 37 | 2 / 0 / 14   | 2 / 14  | 0 / 16  |
| b2              | 24,146 | 12 / 4 / 32  | 16 / 32 | 9 / 39  | 2 / 0 / 14   | 2 / 14  | 0 / 16  |
| b3              | 24,245 | 14 / 4 / 30  | 16 / 32 | 11 / 37 | 2 / 0 / 14   | 2 / 14  | 0 / 16  |
| b4              | 24,198 | 13 / 4 / 31  | 15 / 33 | 11 / 37 | 3 / 0 / 13   | 3 / 13  | 0 / 16  |
| b5              | 24,423 | 12 / 4 / 32  | 15 / 33 | 11 / 37 | 3 / 0 / 13   | 3 / 13  | 0 / 16  |
| b6              | 24,341 | 15 / 3 / 30  | 17 / 31 | 10 / 38 | 3 / 0 / 13   | 3 / 13  | 0 / 16  |
| b7              | 24,239 | 13 / 5 / 30  | 15 / 33 | 12 / 36 | 3 / 1 / 12   | 3 / 13  | 1 / 15  |
| b8              | 24,335 | 12 / 4 / 32  | 15 / 33 | 11 / 37 | 2 / 1 / 13   | 2 / 14  | 1 / 15  |
| t1              | 24,286 | 12 / 4 / 32  | 14 / 34 | 9 / 39  | 2 / 1 / 13   | 2 / 14  | 1 / 15  |
| t2              | 24,371 | 12 / 5 / 31  | 16 / 32 | 10 / 38 | 3 / 0 / 13   | 3 / 13  | 0 / 16  |
| t3              | 24,580 | 12 / 5 / 31  | 16 / 32 | 10 / 38 | 2 / 0 / 14   | 2 / 14  | 0 / 16  |
| t4              | 24,938 | 14 / 4 / 30  | 16 / 32 | 11 / 37 | 3 / 1 / 12   | 3 / 13  | 1 / 15  |
| t5              | 24,882 | 12 / 4 / 32  | 15 / 33 | 11 / 37 | 2 / 0 / 14   | 2 / 14  | 0 / 16  |
| t6              | 24,472 | b13 / 4 / 31 | 16 / 32 | 11 / 37 | 3 / 0 / 13   | 3 / 13  | 0 / 16  |
| t7              | 23,965 | 12 / 5 / 31  | 15 / 33 | 11 / 37 | 2 / 1 / 13   | 2 / 14  | 1 / 15  |
| t8              | 24,651 | 12 / 3 / 33  | 15 / 33 | 10 / 38 | 2 / 0 / 14   | 2 / 14  | 0 / 16  |

TABLE 19. Undergraduate + Graduate New Credit Hours: Comparison of the seasonal per-period SAE stochastic latest-optimized  $\tilde{\alpha}$ -power balanced forecasting models to the seasonal Holt–Winters and ARIMA models

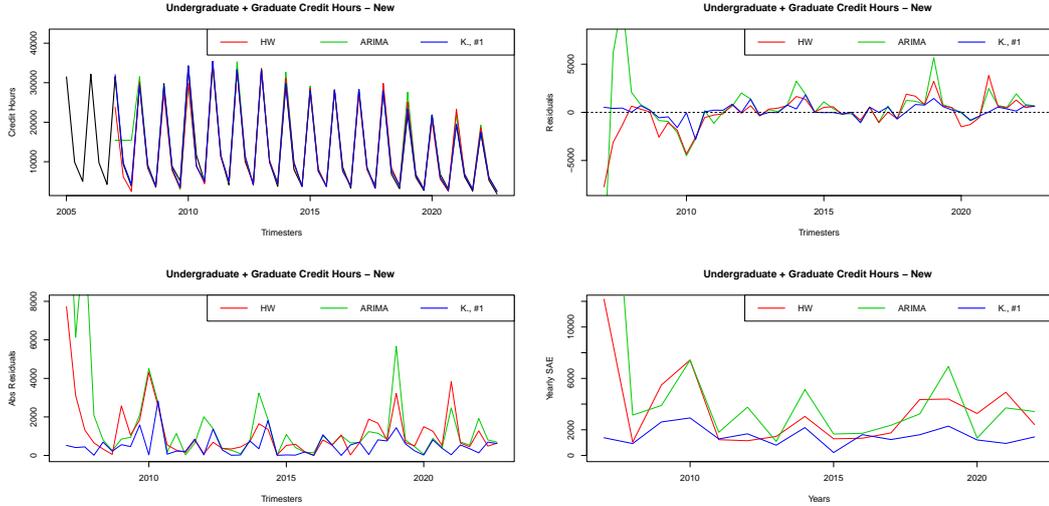


FIGURE 18. Undergraduate + Graduate New Credit Hours: Forecasts, residuals, absolute residuals and yearly sum of absolute errors

### 5. DISCUSSION

In light of Examples 2–19, several conclusions can be formed. First, for all eighteen times series, each of our twenty four seasonal per-period SAE stochastic latest-optimized  $\tilde{\alpha}$ -power balanced

forecasting models outperformed both Holt–Winters and ARIMA models with respect to the sum of absolute error, the number of closest trimesterly forecasts, and the number of closest yearly forecasts. Second, Holt–Winters model performed better than the ARIMA model for sixteen out of the eighteen time series. Indeed, for the new graduate enrollment time series, ARIMA had two more closest trimesterly forecast, i.e.,  $HW/AR = 23/25$ , but had the same number of closest yearly forecasts, i.e.,  $HW/AR = 8/8$ . For the new graduate credit hours, ARIMA also had two more closest trimesterly forecast, i.e.,  $HW/AR = 23/25$ , but the Holt–Winters model had two more closest yearly forecasts, i.e.,  $HW/AR = 9/7$ . Third, for almost all considered seasonal time series in [6] and in Example 2–19, ARIMA models leads to horrible first year forecast, that is, given the first two periods,  $(x_1, \dots, x_f, x_{f+1}, \dots, x_{2f})$ , the forecasted values  $\hat{x}_{2f+1}, \dots, \hat{x}_{3f}$  using the ARIMA models lead to a substantially larger absolute residual than Holt–Winters and our models. Thus, using ARIMA to predict the third sound of period using the the first two periods is not reliable.

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