



# Scale-Shift and Harmonic analysis approach to the Mellin transform for Discrete-time signals

Mor Ngom, Daniel Alpay, Mamadou Mboup

## ► To cite this version:

Mor Ngom, Daniel Alpay, Mamadou Mboup. Scale-Shift and Harmonic analysis approach to the Mellin transform for Discrete-time signals. Signal Processing, 2022, 204, pp.108830. <10.1016/j.sigpro.2022.108830>. <hal-03835066>

**HAL Id: hal-03835066**

**<https://hal.science/hal-03835066v1>**

Submitted on 31 Oct 2022

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



HAL Authorization

# Scale-Shift and Harmonic analysis approach to the Mellin transform for Discrete-time signals

Mor Ngom<sup>a</sup>, Daniel Alpay<sup>b</sup>, Mamadou Mboup<sup>a,\*</sup>

<sup>a</sup>*Université de Reims Champagne Ardenne, CReSTIC EA 3804, UFR des Sciences Exactes et Naturelles, Moulin de la Housse BP 1039, 51687 Reims cedex 2, France*

<sup>b</sup>*Faculty of Mathematics, Physics and Computation Schmid College of Science and Technology, Chapman University, CA, USA*

---

## Abstract

We investigate the scale-shift operator for discrete-time signals via the action of the hyperbolic Blaschke group. Practical implementation issues are discussed and given for any arbitrary scale, in the framework of very classical discrete-time linear filtering. Our group theoretical standpoint leads to a purely harmonic analysis definition of the Mellin transform for discrete-time signals. Explicit analytical expressions of the atoms of the discrete-time Fourier-Mellin decomposition are provided along with a simple algorithm for their computation. The so-defined scale-shift operator also allows us to establish a mathematical equivalence in between the discrete-time wavelet coefficients of a given discrete-time signal and the corresponding Voice-transform generated by a well-chosen unitary representation of the Hyperbolic Blaschke group, in the classical Hardy space of the unit disc.

**Keywords:** Discrete-time Scale-Shift, Hyperbolic Blaschke group, Mellin transform, Voice-Transform, Discrete Wavelet Transform

---

---

\*Corresponding author

*Email addresses:* Mor.Ngom@univ-reims.fr (Mor Ngom), alpay@chapman.edu (Daniel Alpay), Mamadou.Mboup@univ-reims.fr (Mamadou Mboup )

## 1. Introduction

Shift operators are very basic ingredients of many fundamental concepts in signal processing, such as periodicity, stationarity, linear filtering, etc. These operators also play a key role in several chapters of mathematics including functional analysis and harmonic analysis which fuel the development of linear systems theory and provide most of the signal processing tools for signal representation, signal analysis and signal synthesis.

The two most common shifts, namely translation (lag operator) and scaling, are combined in the definition of the wavelet transform [1, 2]. These two shift operators are linked by an anamorphosis, called the Lamperti transformation [3], which corresponds to a logarithmic deformation of the time axis (see also [4] where the logarithmic distortion appears as well in the defined scale operator). This anamorphosis defines a map which allows one to express almost any concept related to time-shift to its scale-shift counterpart. For instance, the Mellin transform appears as the image of the classical Fourier transform and the self-similarity property can be derived from the classical stationarity, up to some multiplicative factor [5], [6], [7]. Unfortunately, this map applies only in the continuous-time setting. Although scaling is used in many applications, the scale-shift operator does not admit a unanimous definition in the discrete-time case. In this case, scaling is generally restricted to some particular and favourable setting, as for instance the dyadic scales in discrete wavelet transform [8] or approximated, as with the principle of aggregation in the study of self-similarity [9]. Yet, a clear-cut mathematical definition was given in [10] in terms of the action of an Abelian subgroup of the group of automorphisms of the unit disc [11] (also called the hyperbolic Blaschke group). Below, we use this definition to revisit the wavelet transform in the discrete-time setting. However, we mention that the same operation has already been used for the purpose of frequency warping. Indeed, scaling in time also corresponds to scaling in frequency. Thus, the wavelet transform in discrete time has been revisited by [12], [13] in the frequency domain, using the frequency warping in associ-

ation with filterbanks. Surprisingly, only the frequency warping effect of the operator has been considered so far while the interpretation of the scale-shift in the time domain did not seem to hold the interest of the authors of [12], [13] (see also [14] and [15]). Now, the group action viewpoint of the scale-shift enlarges the perspective (see [16] and [17]) and leads to an exact and purely harmonic analysis approach of the Mellin transform for discrete-time signals. The proposed approach is in contrast with the principle of discretizing integral operators stemming from the continuous-time setting, using some adequate (geometric) grids (see e.g. [18], [19]). The presented viewpoint also leads us to consider the unitary representations of the hyperbolic Blaschke group and to find that the so-defined discrete-time scale-shift operator is the very basic ingredient of the Voice transform for this group (see Definition 4.1 and [20], [21]). Note that strong analogies between the wavelet transform and the Voice transform for the hyperbolic Blaschke group have already been observed by several studies [22], [23]. We make this link explicit. Indeed, Theorem 4.2 establishes that the discrete wavelet transform is mathematically equivalent to the Voice transform for an Abelian subgroup of the hyperbolic Blaschke group.

Besides the introduction, this paper is divided into four sections. In section 2, we describe briefly the long path leading to the rigorous definition of the scale-shift operator on discrete-time signals then, we give its implementation and illustrate it with some examples. The Mellin transform for a purely discrete-time signal is defined in section 3. Explicit analytical expressions are given for the atoms of the transform, along with a simple algorithm to compute them recursively. Section 4 starts by basically lightening up the voice transform theory. Then, the equivalence between discrete time wavelets and voice transform is established. We conclude our development in section 5.

## 2. The scale-shift in discrete-time

### 2.1. The scale-shift operator for discrete-time signals

In discrete-time, scaling has been first well defined in [10] using a parallelism with the continuous-time case and an algebraic viewpoint. Without loss of generality, all signals considered in this work are real valued. To begin, recall that, given a signal  $f(t)$ ,  $t \in \mathbb{R}$ , its scale-shift at scale  $\alpha \in \mathbb{R}_*^+$  reads as  $f(\alpha t)$ . Also, recall that if  $F(s)$  is the Laplace transform (LT) of  $f(t)$ ,  $t \in \mathbb{R}^+$ , then the LT of its scale-shifted version  $f(\alpha t)$ ,  $\alpha > 0$ , is  $\frac{1}{\alpha} F\left(\frac{s}{\alpha}\right)$ . Scaling is thus expressed, via the operator  $S_\alpha : u \mapsto S_\alpha(u) = \alpha u$ , in the same way in the two domains of continuous-time ( $u \in \mathbb{R}^+$ ) and frequency ( $u \in \mathbb{C}^+$ ).

In discrete-time world however, scaling a given signal  $\mathbf{f} = \{f_n\}_{n \geq 0}$  does not come directly as it is in continuous-time. The key step leading to its clear mathematical definition, given in [10], was the observation that the operator  $S_\alpha$  is equivalent to the action of a subgroup of the hyperbolic Blaschke group on the unit disc  $\mathbb{D}$  of the complex plane:

$$\mathcal{G} = \left\{ \gamma(z) = \frac{\gamma_1 z + \gamma_2}{\bar{\gamma}_2 z + \bar{\gamma}_1}, \quad |\gamma_1|^2 - |\gamma_2|^2 = 1 \right\} \quad (1)$$

Each element of  $\mathcal{G}$  maps  $\mathbb{D}$ , the unit circle  $\mathbb{T}$  and the exterior of the closed disc  $\bar{\mathbb{D}} = \mathbb{D} \cup \mathbb{T}$  into itself respectively. For  $\mathbf{f} = \{f_n\}_{n \geq 0} \in \ell^2$ , consider its  $Z$ -transform

$$F(z) = \sum_{n \geq 0} f_n z^n, \quad (2)$$

which converges to an element of the Hardy space of the disc  $H^2(\mathbb{D})$ . Now, for a given fixed  $\theta$  with  $|\theta| < \frac{\pi}{2}$ , it is well known that the Möbius function  $G_\theta(s) = \frac{e^{i\theta} - s}{e^{-i\theta} + s}$  maps  $\mathbb{C}^+$  into  $\mathbb{D}$  conformally [11]. Then, for any  $z \in \mathbb{D}$ , there exists a unique  $s \in \mathbb{C}^+$  such that  $z = G_\theta(s)$ . Hence, from  $F(z)$ , one considers the map defined in  $\mathbb{C}^+$  by

$$X(s) := F(G_\theta(s)) \quad (3)$$

The map  $X$  is well defined and it coincides with the LT of some causal continuous-time signal  $x(t)$ ,  $t \geq 0$  (this is by one of the fundamental results in Fourier-

Laplace analysis [24]). In the frequency domain, the scale-shifted signal  $x\left(\frac{t}{\alpha}\right)$ ,  $\alpha > 0$ , is represented by  $\alpha X(\alpha s) = X(S_\alpha(s)) = F(G_\theta \circ S_\alpha(s))$ .

Replacing  $s$  by  $G_\theta^{-1}(z)$  gives

$$X(S_\alpha \circ G_\theta^{-1}(z)) = F(G_\theta \circ S_\alpha \circ G_\theta^{-1}(z)) \quad (4)$$

This establishes the relation between scaling a discrete-time signal, in the frequency domain, and its continuous-time counterpart. It remains now to identify exactly the discrete-time signal whose frequency representation corresponds to (4) (up to normalization). This leads us to consider the family

$$\mathcal{G}_\theta = \left\{ \gamma_\alpha(z) = (G_\theta \circ S_\alpha \circ G_\theta^{-1})(z), \quad |\theta| < \frac{\pi}{2} \right\}. \quad (5)$$

Now we have:

**Lemma 1** ([10, 17]). *Endowed with the composition law  $\circ$ ,  $\mathcal{G}_\theta$ , forms a group which is a subgroup  $\mathcal{G}$ . Each  $\gamma_\alpha \in \mathcal{G}_\theta$  can be written as*

$$\gamma_\alpha(z) = G_\theta \circ S_\alpha \circ G_\theta^{-1}(z) = \frac{\gamma_1 z + \gamma_2}{\bar{\gamma}_2 z + \bar{\gamma}_1}, \quad (6)$$

where

$$\gamma_1 = e^{i\theta} + \alpha e^{-i\theta} \quad \text{and} \quad \gamma_2 = e^{i\theta}(1 - \alpha). \quad (7)$$

Conversely, every element  $\gamma(z) \in \mathcal{G}$  is in this form for some  $\theta = \theta_\gamma$  and some  $\alpha = \alpha_\gamma$ .

Fixing  $\theta$  makes  $\mathcal{G}_\theta$  an Abelian group. In all the sequel we fix  $\theta$ , and we note  $\alpha_\gamma$  for the scale associated by (6) to the element  $\gamma$  of  $\mathcal{G}_\theta$  (also called the multiplier of  $\gamma$ , see [11]). A direct computation shows that for all  $\gamma, \varphi \in \mathcal{G}_\theta$ , we have the relation

$$\alpha_\gamma \alpha_\varphi = \alpha_{\gamma \circ \varphi} \quad (8)$$

which establishes a group homomorphism between  $\mathcal{G}_\theta$  and the multiplicative group of  $\mathbb{R}_*^+$ . From now on, the coefficients of  $\gamma_\alpha(z)$  in (7) are normalized such

that  $|\gamma_1|^2 - |\gamma_2|^2 = 1$ , as in (1):

$$\gamma_1 = \frac{e^{i\theta} + \alpha e^{-i\theta}}{2\sqrt{\alpha} \cos \theta} \quad (9)$$

$$\gamma_2 = \frac{e^{i\theta}(1 - \alpha)}{2\sqrt{\alpha} \cos \theta}. \quad (10)$$

Following [10], we introduce the Hardy space unitary operator  $\mathcal{T}_{\alpha_\gamma} : H^2(\mathbb{D}) \rightarrow H^2(\mathbb{D})$  defined by

$$\mathcal{T}_{\alpha_\gamma} F(z) := \frac{F(\gamma(z))}{\bar{\gamma}_2 z + \bar{\gamma}_1} = \sum_{n \geq 0} f_n \Phi_n^{(\gamma)}(z) \quad (11)$$

where

$$\Phi_n^{(\gamma)}(z) = \frac{[\gamma(z)]^n}{\bar{\gamma}_2 z + \bar{\gamma}_1}, \quad n = 0, 1, \dots \quad (12)$$

The transform (11) is just a normalization of (4). For each  $n \geq 0$ , it can be readily seen that  $\Phi_n^{(\gamma)}(z)$  is the image of  $z^n$  under  $\mathcal{T}_{\alpha_\gamma}$ . Being unitary [17],  $\mathcal{T}_{\alpha_\gamma}$  preserves the scalar product and therefore the family  $\{\Phi_n^{(\gamma)}(z)\}_{n \geq 0}$  is orthogonal and complete in the Hardy space  $H^2(\mathbb{D})$  of the unit disc. Hence, expressing each  $\Phi_n^{(\gamma)}(z)$  in the standard basis of  $H^2(\mathbb{D})$ , we can rewrite  $\mathcal{T}_{\alpha_\gamma} F(z)$  as

$$\mathcal{T}_{\alpha_\gamma} F(z) = \sum_{n \geq 0} f_n^{(\gamma)} z^n \quad (13)$$

where  $\forall n \geq 0$ ,

$$f_n^{(\gamma)} = \langle z^n, \mathcal{T}_{\alpha_\gamma} F(z) \rangle_{H^2(\mathbb{D})} = \sum_m f_m \phi_{n,m}^{(\gamma)}, \quad (14)$$

with

$$\phi_{n,m}^{(\gamma)} = \langle z^m, \Phi_n^{(\gamma)}(z) \rangle_{H^2(\mathbb{D})}. \quad (15)$$

Therefore, we have the following

**Definition.** *The discrete-time scale shift operator is defined as the mapping*

$$\mathbf{f} = \{f_n\}_{n \geq 0} \longrightarrow \mathbf{f}^{(\gamma)} = \{f_n^{(\gamma)}\}_{n \geq 0, \gamma \in \mathcal{G}_\theta}. \quad (16)$$

*described in (11)-(15). In matrix form, this reads as*

$$\mathbf{f}^{(\gamma)} = \Phi^{(\gamma)} \mathbf{f} \quad (17)$$

*where the matrix  $\Phi^{(\gamma)} = \left( \phi_{n,m}^{(\gamma)} \right)_{n,m \geq 0}$  and the column vectors  $\mathbf{f}$  and  $\mathbf{f}^{(\gamma)}$  contain the samples  $f_n, n \geq 0$  and their scale shifted counterparts  $f_n^{(\gamma)}$ , respectively.*

This operator allows one to make a direct zoom in any given discrete-time signal  $\mathbf{f} = \{f_n\}_{n \geq 0}$  for any given scale  $\frac{1}{\alpha_\gamma}$ . Now that the role of  $\mathcal{G}_\theta$  in this operator is clear, we invoke the homomorphism (8), (9) and (10) to replace all functions of  $\gamma \in \mathcal{G}_\theta$  by functions of  $\alpha \in \mathbb{R}_*^+$ . Thus, if  $\{f_n\}$  is the discrete-time analog of  $f(t)$  then we will denote by  $\{f_n^{(\alpha)}\}$  the discrete-time analog of  $f(t/\alpha)$ , keeping in mind that  $\alpha = \alpha_\gamma$  for some  $\gamma(z) \in \mathcal{G}_\theta$  as in Lemma 1.

## 2.2. Implementation of the discrete-time scale shift

The implementation is based on a digital filtering process. Note that a related implementation appeared in [25] in the context of system identification with Laguerre basis (see also [26] and [27]). We start by bringing back the functions  $\Phi_n^{(\gamma)}(z)$  in (12) and observe that they satisfy the recurrence relation

$$\Phi_0^{(\gamma)}(z) = \frac{1}{\bar{\gamma}_2 z + \bar{\gamma}_1} \quad (18)$$

$$\Phi_n^{(\gamma)}(z) = \gamma(z) \Phi_{n-1}^{(\gamma)}(z), \quad n \geq 1. \quad (19)$$

Let  $\{\delta_n\}$  denote the unit impulse sequence. Given a discrete-time signal  $\mathbf{f} = \{f_n\}$ , its scale-shift at a scale  $\frac{1}{\alpha_\gamma} > 0$ , as given in (14), can be implemented by the scheme below:

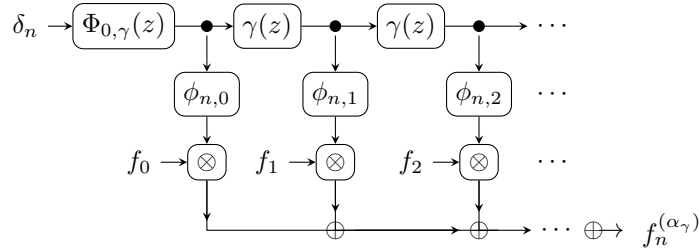


Figure 1: Implantation diagram obtained from (11), (18) and (19)

## 2.3. Finite support and Gibbs phenomenon

Without loss of generality, we consider the example below to fix the ideas. Let  $\{f_n\}$  be the discrete-time signal defined for all  $n \geq 0$  by  $f_n = f(nT_s)$  where



$T_s = 0.01$  and

$$f(t) = \left[0.66(t - 1.87)^2 + 0.1\right] e^{-0.66(t-2)^2}. \quad (20)$$

Below, we consider the signal  $\{f_n\}_{n=0}^N$  and investigate the effect of the finite support  $N$  on its scale-shifts  $\{f_n^{(\alpha)}\}_{n=0}^{N_\alpha}$ , for various  $N$  and  $\alpha$ . As one may readily observe, the support of the scale shifted signal is given by  $N_\alpha = \lfloor N\alpha \rfloor$ , where the notation  $\lfloor \cdot \rfloor$  stands for the integer part of the argument.

### 2.3.1. Smooth signal

In this first example, we set  $N = 400$ . This is large enough to consider that the whole support of  $f(t)$  is covered. The top plot in figure 2 represents the original signal  $\{f_n\}_{n=0}^N$  (corresponding to  $\alpha = 1$ , black curve) together with its shifts  $\{f_n^{(\alpha)}\}_{n=0}^{N_\alpha}$  at scales  $\alpha = \sqrt{2}$  (blue curve) and  $\alpha = 1/\sqrt{2}$  (red curve). The implementation is easy and effective.

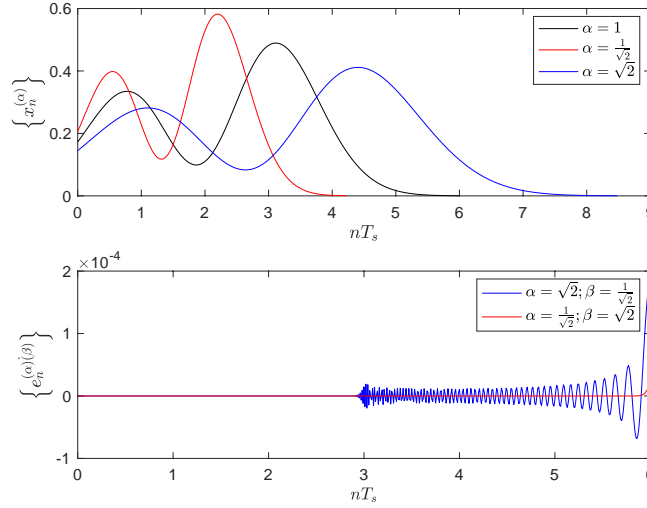


Figure 2: Effectiveness of the scale shift operator even for irrational scales (top plot) – Reversibility of the scaling (bottom plot)

The bottom plot in this figure provides an illustration of the composition property (8):  $\alpha_{\gamma^{-1}} = \frac{1}{\alpha_\gamma}$  and  $\alpha_{id} = 1$ . The plot displays the error signals

$\{e_n^{(\alpha) \cdot (\beta)}\}_{n=0}^N$  defined by

$$h_n = f_n^{(\alpha)} \quad (21)$$

$$e_n^{(\alpha) \cdot (\beta)} = f_n - h_n^{(\beta)} \quad (22)$$

with  $\alpha = \frac{1}{\beta} = \frac{1}{\sqrt{2}}$  for the red curve and  $\alpha = \frac{1}{\beta} = \sqrt{2}$  for the blue curve. The error signal  $\left\{e_n^{(\frac{1}{\sqrt{2}}) \cdot (\sqrt{2})}\right\}_{n \geq 0}$  is negligible, which thus shows an almost exact reversibility of the discrete-time scale-shift operation. The same conclusion can be drawn for the other error signal,  $\left\{e_n^{(\sqrt{2}) \cdot (\frac{1}{\sqrt{2}})}\right\}_{n \geq 0}$ , although it shows some fluctuations with gradual amplitudes in the second half of its support. As explained below, these fluctuations come from the Gibbs phenomenon, induced by the fact that  $f_N > 0$  corresponds to a jump, even though the amplitude is small.

### 2.3.2. Nonsmooth signals and Gibbs oscillations

Recall that, as a transfer function,  $\gamma(z)$  in figure 1 has an infinite impulse response. Strictly speaking, the matrix  $\Phi^{(\gamma)}$  in (17) is therefore doubly infinite even though the implementation considers a limited row dimension of  $N_\alpha$  as given above, to keep only the meaningful part of the scaled signal. The  $\alpha$ -scale shift of a finite support signal  $\{f_n\}_{n=0}^N$  then reads in matrix form as

$$\begin{bmatrix} f_0^{(\alpha)} \\ \vdots \\ f_{N_\alpha}^{(\alpha)} \\ * \\ * \\ \vdots \end{bmatrix} = \Phi^{(\gamma)} \begin{bmatrix} f_0 \\ \vdots \\ f_N \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix} \quad (23)$$

where the “\*” are meaningless values since only the first  $N_\alpha + 1$  components are of interest. This equation shows that the scale shift operator sees a jump in the finite support original signal, by considering its extension of infinite duration by

zero padding. The jump is always present unless the last samples smoothly decay to zero, that is:  $f_n \rightarrow 0$ ,  $n = N_0, N_0 + 1, \dots, N$  for some range  $N_0 < N$ . As is well known, low-pass filtering a signal with jump induces a Gibbs phenomenon. Now, we claim that these Gibbs oscillations start to occur, in the scaled version of the signal, from the time index

$$n_\alpha = \left\lfloor \frac{N}{\alpha} \right\rfloor. \quad (24)$$

When  $\alpha < 1$  (scale contraction) we have  $n_\alpha > N_\alpha$  and thus the Gibbs phenomenon does not appear in the finite support of the scaled signal. However, it happens in the meaningless part of the scaled signal, marked by the symbol “\*” in the left-hand side of (23).

For  $\alpha > 1$  however, the oscillations become visible in the whole time index interval  $[n_\alpha, N_\alpha]$ .

These conclusions are illustrated in the next example.

In this second example, we reconsider the discrete-time signal  $\{f_n\}_{n=0}^N$  sampled from  $f(t)$  in (20), with a finite duration  $N$  such that  $f((N + m)T_s)$  is significantly different from 0, for all  $m = 0, 1, \dots, M$ , for some  $M > 0$ . For example,  $N = 345$  satisfies this condition, with  $M = 55$ . The jump in the original signal, displayed by the black colored curve in figure 3, induces Gibbs oscillations for all scales  $\alpha_1 = \sqrt{3}$  and  $\alpha_2 = \frac{1}{\sqrt{3}}$ . With  $\alpha_1 > 1$ , the oscillations occur in the interval  $[n_{\alpha_1}, N_{\alpha_1}]$ , as claimed above (see the blue colored curve in figure 3).

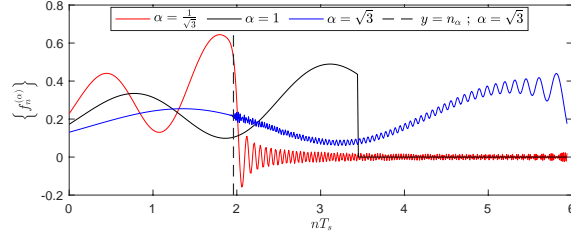


Figure 3: Gibbs oscillations and scale shift of a nonsmooth signal.

In the red colored curve of the same figure, corresponding to  $\alpha_2 < 1$ , the Gibbs phenomenon is still present but the oscillations start after the time index  $N_{\alpha_2}$  since in this experiment, we have  $n_{\alpha_2} > N_{\alpha_2}$ . Observe also that as  $\alpha_1 = 1/\alpha_2$ , we have a coincidence of  $n_{\alpha_1}$  and  $N_{\alpha_2}$ . These results confirm the previous analysis on the effect of a jump on the scale operator. Note also that the analysis already suggests a solution to make the scale shift operator safe from the Gibbs phenomenon.

### 2.3.3. Mitigating the Gibbs phenomenon

To begin, recall that for  $\alpha < 1$ , the Gibbs oscillations are unseen because they happen after the time  $N_\alpha$ , that is in the meaningless part of the scaled signal (see equation (23)). Now, observe that one can get back to the same situation with  $\alpha > 1$  by pushing (artificially) the jump of the signal far enough. More precisely, consider the computation of the  $\alpha$ -scale shift of the signal  $\{f_n\}_{n=0}^N$  (with jump at the end, as explained before), with  $\alpha > 1$ . First we define the signal  $\{\tilde{f}_n\}_{n=0}^{\tilde{N}}$  of duration  $\tilde{N} = N + K_\alpha$  such that

$$\tilde{f}_n = \begin{cases} f_n, & n = 0, 1, \dots, N \\ f_N & n = N + 1, \dots, \tilde{N} = N + K_\alpha \end{cases} \quad (25)$$

where  $K_\alpha$  is some constant depending on  $\alpha$ . According to eq.(23) and setting  $\tilde{N}_\alpha = \lfloor \tilde{N}\alpha \rfloor$ , the  $\alpha$ -scale shift of  $\{\tilde{f}_n\}_{n=0}^{\tilde{N}}$  then reads as

$$\begin{bmatrix} \tilde{f}_0^{(\alpha)} \\ \vdots \\ \tilde{f}_{N_\alpha}^{(\alpha)} \\ \hline \tilde{f}_{N_\alpha+1}^{(\alpha)} \\ \vdots \\ \tilde{f}_{\tilde{N}_\alpha}^{(\alpha)} \\ \hline * \\ \vdots \end{bmatrix} = \Phi^{(\gamma)} \left\{ \begin{bmatrix} f_0 \\ \vdots \\ f_N \\ \hline f_N \\ \vdots \\ f_N \\ \hline 0 \\ \vdots \end{bmatrix} \right\} \quad K_\alpha \text{ lines} \quad (26)$$

We thus get back to the setting in eq.(23) where the original jump, at time index  $N$ , is now carried over to the time index  $\tilde{N} = N + K_\alpha$ . Accordingly, the Gibbs oscillations start to occur at the time index  $\tilde{n}_\alpha = \lfloor \frac{\tilde{N}}{\alpha} \rfloor$ .

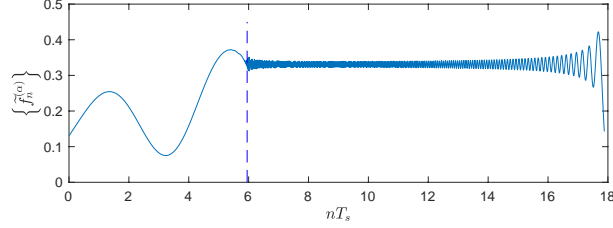


Figure 4: Making scale shift insensitive to Gibbs oscillations

This is shown in figure 4 above which displays  $\{\tilde{f}_n^{(\alpha)}\}_{n \geq 0}$  obtained from the previous signal  $\{f_n\}_{n=0}^N$ , with the same value of  $\alpha = \sqrt{3}$ . The value of  $K_\alpha$  is such that  $\tilde{N} = 3N$  and the corresponding value of  $\tilde{n}_\alpha$  is represented by the vertical dashed line in the figure. Note how this matches with the expected starting point of the Gibbs oscillations. Finally, to make the  $\alpha$ -scale shift operation immune from the Gibbs phenomenon, it suffices to select  $K_\alpha$  such that

$$N_\alpha = \lfloor N\alpha \rfloor \leq \tilde{n}_\alpha = \left\lfloor \frac{N + K_\alpha}{\alpha} \right\rfloor.$$

Now this is achieved when

$$K_\alpha \geq \lceil N(\alpha^2 - 1) \rceil.$$

This is confirmed by the results in figure 5, where the scaled versions  $\{f_n^{(\alpha)}\}_{n=0}^{N_\alpha}$  of the finite support signal  $\{f_n\}_{n=0}^N$  (black curve) are free from any Gibbs oscillations for all  $\alpha$ , be it lower or greater than 1.

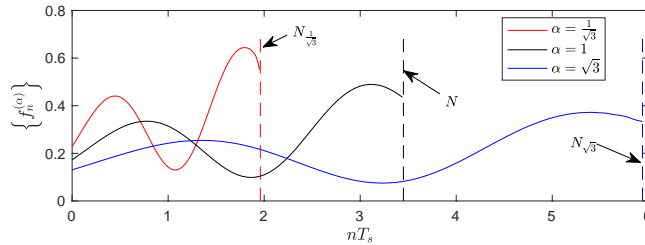


Figure 5: Scale-shift immune to Gibbs oscillations

The scaled version is obtained from the filtering scheme in figure 1 equation (23), for  $\alpha < 1$  (red curve in the plot). For  $\alpha > 1$ , it is defined by  $f_n^{(\alpha)} = \tilde{f}_n^{(\alpha)}$ ,  $n = 0, \dots, N_\alpha$  (blue curve).

### 3. Mellin transform for discrete-time signals

To outline the guiding ideas of our approach, we begin with a brief review of the basic elements of harmonic analysis that underlie the abstract Fourier transform. This subject is standard and is exposed in a number of classical (re-edited) textbooks such as e.g. [28], [29] and [30]. Nonetheless, we recall the basic bricks because the subject may not be familiar to some readers. Next, we show how easy and natural the definition of the Mellin transform for discrete-time signals falls within this general framework. The last part of this section will provide a simple algorithm to compute the atoms of the decomposition.

#### 3.1. Abstract Harmonic Analysis

##### 3.1.1. Basic definitions

Let  $(\mathcal{G}, \bullet)$  be a locally compact Abelian group, with Haar measure  $\mu$  and neutral element  $\iota$ . A function  $\sigma : \mathcal{G} \rightarrow \mathbb{T}$  satisfying

$$i) \quad \sigma(\iota) = 1 \tag{27}$$

$$ii) \quad \sigma(a \bullet b) = \sigma(a)\sigma(b), \quad \forall a, b \in \mathcal{G} \tag{28}$$

is called a character. The set of all continuous characters forms a group  $\widehat{\mathcal{G}}$  called the dual group of  $\mathcal{G}$ . By the Pontryagin duality,  $\widehat{\mathcal{G}}$  is also a locally compact Abelian group and we denote by  $\hat{\mu}$  its associated Haar measure. Now, for  $f : \mathcal{G} \rightarrow \mathbb{C}$  in  $L_1(\mathcal{G}, d\mu)$ , the Fourier transform  $\mathcal{F}$  and its inverse  $\mathcal{F}^{-1}$  are defined respectively by

$$\widehat{f}(\sigma) = \mathcal{F}\{f\}(\sigma) = \int_{\mathcal{G}} f(a) \overline{\sigma(a)} d\mu(a), \quad \sigma \in \widehat{\mathcal{G}} \tag{29}$$

$$f(a) = \mathcal{F}^{-1}\{\widehat{f}\}(a) = \int_{\widehat{\mathcal{G}}} \widehat{f}(\sigma) \sigma(a) d\hat{\mu}(\sigma), \quad a \in \mathcal{G}. \tag{30}$$

### 3.1.2. Group action

Consider the action of  $\mathcal{G}$  on a non-empty set  $E$ . This can be described via a morphism  $\eta : \mathcal{G} \times E \rightarrow E$  such that

- $\eta(\iota, t) = t$ , for all  $t \in E$
- $\eta(a, \eta(b, t)) = \eta(a \bullet b, t)$ ,  $\forall a, b \in \mathcal{G}$  and  $\forall t \in E$ .

To every  $f : E \rightarrow \mathbb{C}$ , one can thus associate a function  $h : \mathcal{G} \times E \rightarrow \mathbb{C}$ , defined for all  $(a, t) \in \mathcal{G} \times E$  by  $h(a, t) = f \circ \eta(a, t)$ . Now, for each fixed  $t \in E$ , one can compute by (29) the Fourier transform  $\widehat{h}_t(\sigma)$  of the function  $h_t(\cdot) = h(\cdot, t)$ , provided it belongs to  $L_1(\mathcal{G}, d\mu)$ . In general, one has to compute  $\widehat{h}_t(\sigma)$  via (29) for each different  $t$ . In some situations however, it becomes straightforward to determine  $\widehat{h}_t(\sigma)$  based on the knowledge of  $\widehat{h}_\tau(\sigma)$  for any given pair  $(\tau, t) \in E^2$ . This arises when the group action is transitive. Recall that the action of  $\mathcal{G}$  on  $E$  is said transitive if for any pair  $(\tau, t) \in E^2$ , there exists  $a \in \mathcal{G}$  such that  $\tau = \eta(a, t)$ . Let then  $t_0 \in E$  be fixed once for all and given  $t \in E$ , let  $b_t \in \mathcal{G}$  be defined such that  $\eta(b_t, t_0) = t$ . Then we get

$$\begin{aligned} h_t(a) &= h(a, t) = f(\eta(a, t)) = f(\eta(a, \eta(b_t, t_0))) \\ &= f(\eta(a \bullet b_t, t_0)) = h_{t_0}(a \bullet b_t), \end{aligned}$$

showing that the two variables merge into one and the function  $h(\cdot, \cdot)$  reduces to a function of a single variable. The Fourier transform of  $h_t(\cdot)$  becomes:

$$\begin{aligned} \widehat{h}_t(\sigma) &= \int_{\mathcal{G}} h_t(a) \overline{\sigma(a)} d\mu(a) = \int_{\mathcal{G}} h_{t_0}(a \bullet b_t) \overline{\sigma(a)} d\mu(a) \\ &= \int_{\mathcal{G}} h_{t_0}(a) \overline{\sigma(a \bullet b_t^{-1})} d\mu(a \bullet b_t^{-1}) \\ &= \sigma(b_t) \widehat{h}_{t_0}(\sigma), \end{aligned}$$

where the last equality is obtained using (27)-(28) and the translation-invariance property of the Haar measure. An example of such simple situation is the case when  $\mathcal{G}$  is the additive group of real (resp. integer) numbers, and  $E = \mathcal{G}$ : The Haar measure is the Lebesgue measure (resp. counting measure) and the

characters are the exponential functions  $\sigma_\omega(a) = e^{-i\omega a}$ . This corresponds to the classical Fourier transform. Another common example is obtained with the multiplicative group of positive real numbers, with again,  $E = \mathcal{G}$ : The Haar measure is  $d\mu(a) = \frac{da}{a}$ , the characters are the functions  $\sigma_\omega(a) = a^{-i\omega}$ ,  $\omega \in \mathbb{R}$  and we recover the Mellin transform.

### 3.2. Scaling and Mellin transform in discrete-time

As observed already, scale translation in discrete-time is expressed via the action of the group  $\mathcal{G}_\theta$  ( $\theta$  being fixed as before). We recall that by Lemma 1, (8) and Definition 2.1, this action is equivalent to that of the multiplicative group of positive reals, with  $E = \mathbb{N}$ . The associated morphism  $\eta(\alpha, n)$  is defined such that for a given discrete-time signal  $\mathbf{f} = \{f_n\}_{n \geq 0}$  with its  $Z$ -transform  $F(z) = \sum f_n z^n$ , we have

$$f_{\eta(\alpha, n)} := f_n^{(\alpha)} = \langle z^n, \mathcal{T}_\alpha F(z) \rangle_{H^2(\mathbb{D})}. \quad (31)$$

Clearly, this action is not transitive. Therefore, the two variables  $\alpha \in \mathbb{R}_*^+$  and  $n \in \mathbb{N}$  cannot be merged. For any fixed pair  $(n, m)$  of positive integers with  $m \neq n$ ,  $f_n^{(\alpha)}$  and  $f_m^{(\alpha)}$  are therefore two unrelated functions of  $\alpha$ . Hence a different Mellin transform needs to be computed for each different  $m$ .

**Theorem.** *Let  $\mathbf{f} = \{f_n\}_{n \geq 0} \in \ell_2$  be a causal discrete-time signal and let  $m \in \mathbb{N}$  be a given time index. Then*

1) - *The Mellin transform of  $\{f_n\}_{n \geq 0}$  with respect to the time index  $m$ , denoted by  $\widehat{\mathbf{f}}_m = \mathcal{M}_m\{\mathbf{f}\}$ , is given by*

$$\widehat{\mathbf{f}}_m(\omega) = \mathcal{M}_m\{\mathbf{f}\}(\omega) = \sum_{n \geq 0} f_n \mathcal{E}_{m,n}(\omega), \quad (32)$$

where the  $n^{\text{th}}$  atom of the transform is

$$\mathcal{E}_{m,n}(\omega) = (-1)^m \frac{2\pi}{\cosh(\pi\omega)} P_m(i\omega) P_n(i\omega), \quad n = 0, 1, 2, \dots \quad (33)$$

where  $P_n(s)$  is the polynomial of degree  $n$  which reads as

$$P_n(s) = \frac{1}{n!} \sum_{k=0}^n \binom{n}{k} \prod_{\ell=1}^n (\ell - k - \frac{1}{2} - s) \quad (34)$$



2) - The transform is reversible and the original signal  $\{f_n\}$  can be retrieved by

$$f_n = \left\langle \Phi_n^{(\alpha)}(z), \widehat{F}^{(\alpha)}(z) \right\rangle_{H^2(\mathbb{D})}, \quad (35)$$

where

$$\widehat{F}^{(\alpha)}(z) = \sum_m \mathcal{M}^{-1} \left\{ \widehat{\mathbf{f}}_m(s) \right\} z^m$$

and where we use the notation  $\mathcal{M}^{-1}$  for the classical inverse Mellin transform.

*Proof.* The proof is given in the appendix.  $\square$

In the sequel, we define the polynomials

$$p_{k,n}(s) = \prod_{\ell=1}^n (\ell - k - s - \frac{1}{2}), \quad n \in \mathbb{N}, \quad 0 \leq k \leq n \quad (36)$$

so that the polynomial in (34) reads as  $P_n(s) = \frac{1}{n!} \sum_{k=0}^n \binom{n}{k} p_{k,n}(s)$ .

### 3.3. Computation algorithm

The following lemma shows some relations on these polynomials that are useful for the computation of the atoms  $\mathcal{E}_{m,n}(s)$ .

**Lemma 2.** *The polynomials  $p_{k,n}$  satisfy the following recurrence relations*

$$p_{n-k,n}(s) = (-1)^n p_{k,n}(-s) \quad (37)$$

$$p_{k,n}(s) = \left( \frac{2(n-k)-1}{2} - s \right) p_{k,n-1}(s), \quad \forall n \geq 1 \quad (38)$$

$$p_{k,n}(s) = \frac{\frac{-2k+1}{2} - s}{\frac{2(n-k)+1}{2} - s} p_{k-1,n}(s), \quad 1 \leq k \leq n \quad (39)$$

*Proof.* See the appendix.  $\square$

**Lemma 3.** *On the imaginary axis,  $s = i\omega$ ,  $\omega \in \mathbb{R}$ , the polynomial  $P_n(s)$  is either real valued or purely imaginary and we have:*

1. For  $n$  even,  $n = 2q$ ,

$$P_{2q}(i\omega) = \frac{2}{(2q)!} \sum_{k=0}^{q-1} \binom{2q}{k} \Re \{p_{k,2q}(i\omega)\} + \frac{1}{(q!)^2} p_{q,2q}(i\omega) \quad (40)$$

and  $p_{q,2q}(i\omega)$  is real.

2. For  $n$  odd,  $n = 2q + 1$ ,

$$P_{2q+1}(i\omega) = \frac{2i}{(2q+1)!} \sum_{k=0}^q \binom{2q+1}{k} \Im \{p_{k,2q+1}(i\omega)\} \quad (41)$$

These lemmas show how one can compute the polynomials  $P_n(s)$  for all  $n$ , using simple recurrences initialized from  $p_{0,0}(s) = 1$ . Indeed, for each given  $n > 0$ , the polynomials  $p_{k,n}(s)$ ,  $k = 1, \dots, n$  can be easily obtained recursively from  $p_{0,n}(s)$  by following the recurrence relation (39). Now, the polynomial  $p_{0,n}(s)$ , in turn, easily derives from  $p_{0,0}(s)$  by a recursive application of the relation (38). Finally, the polynomials  $p_{k,n}(s)$ , and hence  $P_n(s)$ , can be computed using the diagram in figure 6 below, where the vertical arrow with the circle mark in the middle represents the relation (39) and the one with the square mark represents the relation (38).

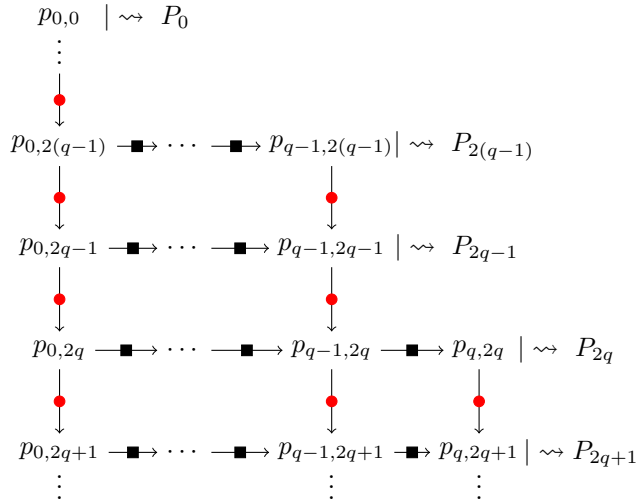


Figure 6: Computation diagram of the set of polynomials  $P_n(i\omega)$ .

Based on Lemma 3 and (33), it holds that  $\mathcal{E}_{m,n}(\omega)$  is real when  $n+m$  is even and purely imaginary otherwise. The supports and the shapes of the different atoms vary markedly with  $m$  and  $n$ . This is illustrated in figure 7 below, where the atoms or their imaginary part where appropriate, are represented for some values of  $n$  and  $m$ .

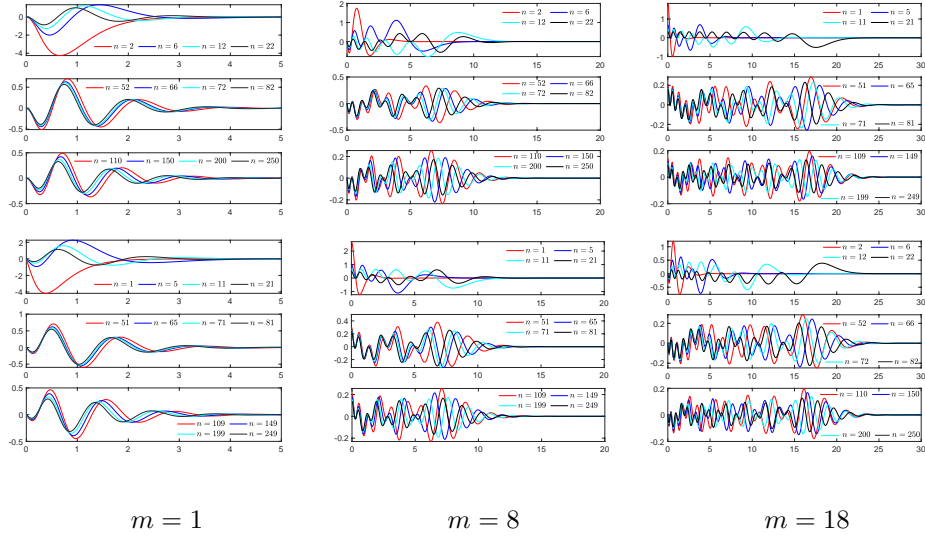


Figure 7: Some atoms  $\mathcal{E}_{m,n}$  of the Mellin transform

#### 4. Voice transform and wavelet coefficients

In this section, we use the scale-shift operator defined above to revisit two popular signal transforms: the discrete wavelet transform and the Voice transform for the Blaschke group. We establish a strict equivalence between these two transform.

##### 4.1. The Voice transform

The Voice transform was originally introduced in [31, 32] and serves as a unifying theory for lot of atomic representations (see also [21, 20, 22]). Generally

speaking, one starts by representing a given locally compact topological group  $\mathcal{G}$  in a given Hilbert space  $\mathcal{H}$  by unitary operators  $U_g, g \in \mathcal{G}$ , defined in  $\mathcal{H}$ . More precisely, the voice transform of an element  $f \in \mathcal{H}$  generated by  $U_g, g \in \mathcal{G}$ , and with respect to some parameter  $h \in \mathcal{H}$ , is defined as

$$[V_h f](g) = \langle f, U_g h \rangle_{\mathcal{H}} \quad (42)$$

where  $\langle \cdot, \cdot \rangle_{\mathcal{H}}$  is the inner product in the Hilbert space  $\mathcal{H}$ .

Related to the present context, let us consider the subgroup  $\mathcal{G}_\theta$  defined in (5), with an angle  $\theta \in (-\pi/2, \pi/2)$  fixed as before. Recall that the operators  $\mathcal{T}_{\alpha_\gamma} : H^2(\mathbb{D}) \rightarrow H^2(\mathbb{D})$  defined in (11) are unitary representations of  $\mathcal{G}_\theta$  in the Hilbert space of the disc,  $H^2(\mathbb{D})$ .

Therefore, we can define the Voice transform with respect to  $\mathcal{G}_\theta$ , for discrete-time signals:

**Definition.** Let  $X(z) = \sum_{n \geq 0} x_n z^n \in H^2(\mathbb{D})$  be given. For a causal discrete-time signal  $\mathbf{f} = \{f_n\}_{n \geq 0} \in \ell_2$ , we define its associated Voice transform, generated by  $\mathcal{T}_{\alpha_\gamma}$ , with parameter  $X$ , as:  $[V_{X(z)} \mathbf{f}] : \mathcal{G}_\theta \rightarrow \mathbb{C}$ , such that

$$[V_{X(z)} \mathbf{f}](\gamma) = \langle F(z), \mathcal{T}_{\alpha_\gamma} X(z) \rangle_{H^2(\mathbb{D})} = \sum_n f_n x_n^{(\alpha_\gamma)} \quad (43)$$

where  $F(z) = \sum_{n \geq 0} f_n z^n$

#### 4.2. Voice transform to Wavelets coefficients

We are now ready to state and prove the mathematical equivalence between the Voice transform and the wavelet coefficients of a given discrete-time signal. We mention that the relation in between these two transforms has been studied in [22] without touching an explicit and strict equivalence. The reason lies probably on the lack of a proper scale-shift operator. Indeed, because of the non commutativity of the whole Blaschke group  $\mathcal{G}$ , different elements of the group present different rotation angles and this precludes to interpret the action of the entire group in terms of scale-shift. Here, we get around this difficulty by

fixing the angle  $\theta$  and just consider the subgroup  $\mathcal{G}_\theta$  of  $SU(1)$  which, in turn, is Abelian.

Recall that the wavelet transform of a given continuous-time signal  $f(t) \in L^2(\mathbb{R})$  is defined as

$$[W_\psi f](\alpha, u) = \frac{1}{\sqrt{\alpha}} \int_{\mathbb{R}} f(t) \psi\left(\frac{t-u}{\alpha}\right) dt, \quad (44)$$

where  $\psi$  is the corresponding mother wavelet,  $\alpha$  and  $u$  are the scale and time-shift parameters, respectively.

In the context of this present work, signals are assumed to be known only at discrete-time instants i.e.  $f = \{f_n\}_{n \geq 0}$ . Let  $\{\psi_n\}$  be the discrete-time version of the mother wavelet, that may be obtained, for instance, by  $\psi_n = \psi(nT_s)$  where  $T_s$  is a sampling time. Accordingly, let  $\{\psi_n^{(\alpha)}\}$  denote the discrete-time version of  $\psi\left(\frac{t}{\alpha}\right)$  as defined by the discrete-time scale shift operator in Definition 2.1. Then, the discrete wavelet transform may be readily given by the discrete-time version of eq.(44) by

$$c_k(\alpha) = \sum_n f_n \psi_{n-k}^{(\alpha)}. \quad (45)$$

This expression leads to the following result:

**Theorem.** *The voice transform of a discrete-time signal  $\{f_n\}_{n \geq 0}$  is equivalent to its discrete wavelet transform as given in (45), that is:*

$$c_k(\alpha) = [V_{[z^k \Psi(z)]} f](\gamma_\alpha) \quad (46)$$

*Proof.* Let  $F(z) = \sum_{n \geq 0} f_n z^n$  and  $\Psi(z) = \sum_n \psi_n z^n$  be the  $Z$ -transforms of the sequences  $\{f_n\}_{n \geq 0}$  and  $\{\psi_n\}_{n \geq 0}$  respectively. The  $Z$ -transform of the discrete-time scale shifted signal  $\{\psi_n^{(\alpha)}\}$  then readily reads as

$$\Psi^{(\alpha)}(z) = \sum_n \psi_n^{(\alpha)} z^n = \mathcal{T}_\alpha \Psi(z).$$

Likewise, for any fixed  $k \in \mathbb{N}$ , we will use the notation  $\Psi_k^{(\alpha)}(z) = \sum_n \psi_{n-k}^{(\alpha)} z^n = \mathcal{T}_\alpha[z^k \Psi(z)]$ , to represent the discrete-time scale shift of  $\{\psi_{n-k}\}_{n \geq 0}$ . The proof

follows upon noting that the expression (45) also reads in a form of a scalar product

$$c_k(\alpha) = \sum_n f_n \psi_{n-k}^{(\alpha)} \quad (47)$$

$$= \langle F(z), \Psi_k^{(\alpha)}(z) \rangle_{H^2(\mathbb{D})} \quad (48)$$

$$= \langle F(z), \mathcal{T}_\alpha z^k \Psi(z) \rangle_{H^2(\mathbb{D})} \quad (49)$$

$$= [V_{[z^k \Psi(z)]} f](\gamma_\alpha), \quad (50)$$

where  $\gamma_\alpha(z)$  is the element of  $\mathcal{G}_\theta$  associated with the scale  $\alpha$  as in (5). This ends the proof.  $\square$

## 5. Conclusion

This paper has investigated a clear cut definition of the scale-shift operator in discrete-time. Its implementation has been given under a very simple and classical linear filtering setup. It has then been illustrated with some examples showing the presence of Gibbs phenomenon under the scale-shift of discrete-time signal with finite support. However, the Gibbs fluctuations can be fully avoided and a simple and effective solution is provided. Then, the scale-shift operator has been used to define rigorously the Mellin transform for discrete-time signals. The definition falls within the classical harmonic analysis framework. The atoms of the decomposition have been given by explicit expressions and a easy computational algorithm has also been devised. Finally, the definition of this operator has enabled us to establish an explicit equivalence between two popular signal transforms: the Voice transform associated to a specific Blaschke hyperbolic subgroup and the discrete wavelet transform.

## Appendix

### 5.1. Proof of Theorem 3.2

Following (31), we are interested in the calculation of the Mellin transform of  $f_{\eta(\alpha, m)}$  as a function of  $\alpha \in \mathbb{R}_*^+$ , for  $m \in \mathbb{N}$  fixed. By definition, this is given by

$$\begin{aligned}\widehat{\mathbf{f}}_m(s) &= \mathcal{M}_m\{\mathbf{f}\}(s) \\ &= \mathcal{M}\{f_{\eta(\alpha, m)}\}(s) = \int_0^\infty f_{\eta(\alpha, m)} \alpha^{s-1} d\alpha \\ &= \sum_{n=0}^\infty f_n \left\langle z^m, \mathcal{M}\left\{\Phi_n^{(\alpha)}(z)\right\}(s) \right\rangle_{H^2(\mathbb{D})}\end{aligned}$$

#### 5.1.1. Computation of $\mathcal{M}\left\{\Phi_n^{(\alpha)}(z)\right\}$

The first task towards the computation of  $\mathcal{M}\left\{\Phi_n^{(\alpha)}(z)\right\}$  is to express  $\Phi_n^{(\alpha)}(z)$  as a function of  $\alpha$ , regarding  $n$  and  $z$  as fixed parameters. Without loss of generality, we fix  $\theta = 0$  and consider the subgroup  $\mathcal{G}_0$ . The dependence of  $\Phi_n^{(\alpha)}(z)$  on  $\alpha$  can be made explicit by recalling (12), (9) and (10) with  $\theta = 0$ . We get

$$\begin{aligned}\Phi_n^{(\alpha)}(z) &= \frac{2\sqrt{\alpha}}{(1-\alpha)z + (1+\alpha)} \left[ \frac{(1+\alpha)z + (1-\alpha)}{(1-\alpha)z + (1+\alpha)} \right]^n \\ &= \frac{2\sqrt{\alpha}}{1+z} \frac{[1 - \lambda_z \alpha]^n}{[1 + \lambda_z \alpha]^{n+1}} \\ &= \frac{2}{1+z} \sum_{k=0}^n (-1)^k \binom{n}{k} \lambda_z^k \alpha^{k+1/2} \frac{1}{(1 + \lambda_z \alpha)^{n+1}}\end{aligned}$$

where we have set

$$\lambda_z = \frac{1-z}{1+z}.$$

and where we have used the well-known binomial formula in the last equality.

Next, recall that if  $Y(s)$  represents the Mellin transform of a function  $y(x)$  then the Mellin transform of  $x^\nu y(x)$  would read as  $Y(s + \nu)$ . So, let us set

$$y_z(\alpha) = \frac{1}{(1 + \lambda_z \alpha)^{n+1}}.$$

The corresponding Mellin transform is well-known as

$$Y_z(s) = \lambda_z^{-s} \frac{\Gamma(s)\Gamma(n+1-s)}{\Gamma(n+1)},$$

where  $\Gamma(\cdot)$  is the classical Gamma function. Therefore, we deduce the Mellin transform

$$\begin{aligned} \widehat{\Phi}_n(s, z) &:= \mathcal{M} \left\{ \Phi_n^{(\alpha)}(z) \right\} \\ &= \frac{2}{1+z} \sum_{k=0}^n (-1)^k \binom{n}{k} \lambda_z^k Y_z \left( s + k + \frac{1}{2} \right) \\ &= \frac{2}{1+z} \sum_{k=0}^n (-1)^k \binom{n}{k} \lambda_z^k \lambda_z^{-s-k-1/2} \\ &\quad \cdot \frac{\Gamma(s+k+1/2)\Gamma(n-k+1/2-s)}{\Gamma(n+1)}. \end{aligned}$$

Setting  $\tilde{s} = s + k + 1/2$ , we write

$$\Gamma(n-k+1/2-s) = \Gamma(n+1-\tilde{s}) = \prod_{\ell=1}^n (\ell - \tilde{s}) \Gamma(1-\tilde{s})$$

With the Euler's reflection formula

$$\Gamma(\zeta)\Gamma(1-\zeta) = \frac{\pi}{\sin \pi \zeta}, \quad \forall \zeta \in \mathbb{C} \setminus \mathbb{Z},$$

and recalling the expression (36), namely

$$p_{k,n}(s) = \prod_{\ell=1}^n (\ell - s_k) = \prod_{\ell=1}^n (\ell - k - 1/2 - s),$$

we obtain:

$$\begin{aligned} \widehat{\Phi}_n(s, z) &= \frac{2\lambda_z^{-s-1/2}}{1+z} \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{\pi p_{k,n}(s)}{n! \sin [\pi(s-k-1/2)]} \\ &= \frac{2\lambda_z^{-s-1/2}}{1+z} \frac{\pi}{\cos(\pi s)} \sum_{k=0}^n \binom{n}{k} \frac{p_{k,n}(s)}{n!} \\ &= \frac{2\lambda_z^{-s-1/2}}{1+z} \frac{\pi}{n! \cos(\pi s)} P_n(s) \end{aligned}$$

### 5.1.2. Computation of $\langle z^m, \widehat{\Phi}_n(s, z) \rangle$

To finish the proof, we need to express the development of  $\widehat{\Phi}_n(s, z)$  in a formal power series of the variable  $z$ ,  $s$  being a fixed parameter. To begin, we



set

$$A_s(z) := \frac{\lambda_z^{-s-1/2}}{1+z} = \frac{(1-z)^{-s-1/2}}{(1+z)^{-s+1/2}}.$$

The following formal power series

$$\begin{aligned} (1-z)^{-s-1/2} &= \sum_{\ell=0}^{\infty} \frac{\Gamma(s+1/2)}{\Gamma(s-\ell+1/2)\Gamma(\ell+1)} z^{\ell} \\ (1+z)^{-s+1/2} &= \sum_{\ell=0}^{\infty} (-1)^{\ell} \frac{\Gamma(-s+1/2)}{\Gamma(-s-\ell+1/2)\Gamma(\ell+1)} z^{\ell}, \end{aligned}$$

allow one to rewrite  $A_s(z)$  as

$$\begin{aligned} A_s(z) &= \sum_{k,\ell=0}^{\infty} \frac{(-1)^k \Gamma(-s+\frac{1}{2}) \Gamma(s+\frac{1}{2}) z^{k+\ell}}{k! \ell! \Gamma(-s+\frac{1}{2}-k) \Gamma(s+\frac{1}{2}-\ell)} \\ &= \sum_{m=0}^{\infty} \sum_{k=0}^m \frac{(-1)^k}{k!(m-k)!} \frac{\Gamma(-s+\frac{1}{2})}{\Gamma(-s+\frac{1}{2}-k)} \\ &\quad \cdot \frac{\Gamma(s+\frac{1}{2})}{\Gamma(s+\frac{1}{2}-m+k)} z^m \\ &= \sum_{m=0}^{\infty} a_m(s) z^m. \end{aligned}$$

where

$$\begin{aligned} a_m(s) &= \sum_{k=0}^m \frac{(-1)^k}{k!(m-k)!} \frac{\Gamma(-s+\frac{1}{2})}{\Gamma(-s+\frac{1}{2}-k)} \frac{\Gamma(s+\frac{1}{2})}{\Gamma(s+\frac{1}{2}-m+k)} \\ &= \frac{1}{m!} \sum_{k=0}^m (-1)^k \binom{m}{k} r_k(s) r_{m-k}(-s). \end{aligned}$$

with  $r_0(s) \equiv 1$  and

$$r_k(s) = \frac{\Gamma(-s+\frac{1}{2})}{\Gamma(-s+\frac{1}{2}-k)} \tag{51}$$

$$= \frac{\Gamma(-s-\frac{1}{2}+1)}{\Gamma(-s-\frac{1}{2}-(k-1))} \tag{52}$$

$$= \prod_{\ell=0}^{k-1} (-s-\frac{1}{2}-\ell) = \prod_{\ell=1}^k (-s-\frac{1}{2}-k+\ell) \tag{53}$$

Hence, replacing  $k$  by  $m-k$  and  $s$  by  $-s$ , leads to

$$r_{m-k}(-s) = \prod_{\ell=1}^{m-k} (s-\frac{1}{2}-m+k+\ell) = \prod_{\ell=k+1}^m (s+\frac{1}{2}+k-\ell).$$

If we split the product in the expression of  $p_{k,n}(s)$  in (36) as

$$p_{k,n}(s) = \prod_{\ell=1}^k (\ell - k - 1/2 - s) \prod_{\ell=k+1}^n (\ell - k - 1/2 - s)$$

then one immediately recognizes the equality

$$p_{k,n}(s) = (-1)^{n-k} r_k(s) r_{n-k}(-s).$$

This leads to the conclusion that

$$a_m(s) = \frac{1}{m!} \sum_{k=0}^m (-1)^k \binom{m}{k} (-1)^{m-k} p_{k,m}(s) = (-1)^m P_m(s).$$

The proof is now complete with

$$\mathcal{E}_{m,n}(s) = \langle z^m, \widehat{\Phi}_n(s, z) \rangle \quad (54)$$

$$= \frac{\pi}{n! \cos(\pi s)} P_n(s) \langle z^m, A_s(z) \rangle \quad (55)$$

$$= \frac{\pi}{n! \cos(\pi s)} (-1)^m P_m(s) P_n(s) \quad (56)$$

□

## 5.2. Proof of Lemma 2

### 5.2.1. Symmetry relation (37)

$$\begin{aligned} p_{n-k,n}(s) &= \prod_{\ell=1}^n (\ell - n + k - s - \frac{1}{2}) \\ &= \prod_{m=0}^{n-1} (-m + k - s - \frac{1}{2}) = \prod_{\ell=1}^n (-\ell + k - s + \frac{1}{2}) \\ &= (-1)^n \prod_{\ell=1}^n (\ell - k + s - \frac{1}{2}) \\ &= (-1)^n p_{k,n}(-s) \end{aligned}$$

5.2.2. Recurrence on  $n$  (38)

$$\begin{aligned}
p_{k,n}(s) &= \prod_{\ell=1}^n (\ell - k - s - \frac{1}{2}) \\
&= \prod_{\ell=1}^{n-1} (\ell - k - s - \frac{1}{2}) \cdot (n - k - s - \frac{1}{2}) \\
&= a_{k,n}(s) p_{k,n-1}(s)
\end{aligned}$$

where  $a_{k,n}(s) = n - k - s - \frac{1}{2} = \frac{2(n-k)-1}{2} - s$ .

5.2.3. Recurrence on  $k$  (39)

$$\begin{aligned}
p_{k,n}(s) &= \prod_{\ell=1}^n (\ell - k - s - \frac{1}{2}) \\
&= \prod_{\ell=1}^n \left( \ell - 1 - (k-1) - s - \frac{1}{2} \right) \\
&= \prod_{m=0}^{n-1} \left( m - (k-1) - s - \frac{1}{2} \right) \\
&= \frac{-k + \frac{1}{2} - s}{n - k + \frac{1}{2} - s} \prod_{m=1}^n \left( m - (k-1) - s - \frac{1}{2} \right) \\
&= b_{k,n}(s) p_{k-1,n}(s)
\end{aligned}$$

where  $b_{k,n}(s) = \frac{-k + \frac{1}{2} - s}{n - k + \frac{1}{2} - s} = \frac{\frac{-2k+1}{2} - s}{\frac{2(n-k)+1}{2} - s}$ .

□

5.3. Proof of Lemma 3

Observe first that the symmetry relation (37) implies that

$$p_{q,2q}(i\omega) = p_{q,2q}(-i\omega) = \overline{p_{q,2q}(i\omega)}$$

showing that  $p_{q,2q}(i\omega)$  is real. Next using the recurrence relation (37) and the identity  $\binom{n}{k} = \binom{n}{n-k}$ , we can simplify the computation of the polynomials  $P_n(s)$ , for  $n$  even/odd, by

For  $n = 2q$ ,

$$\begin{aligned} P_{2q}(s) &= \frac{1}{n!} \sum_{k=0}^{2q} \binom{2q}{k} p_{k,2q}(s) \\ &= \sum_{k=0}^{q-1} \frac{\binom{2q}{k}}{n!} (p_{k,2q}(s) + p_{k,2q}(-s)) + \frac{1}{(q!)^2} p_{q,2q}(s). \end{aligned}$$

For  $n = 2q + 1$ ,

$$\begin{aligned} P_{2q+1}(s) &= \frac{1}{n!} \sum_{k=0}^{2q+1} \binom{2q+1}{k} p_{k,2q+1}(s) \\ &= \sum_{k=0}^q \frac{\binom{2q+1}{k}}{n!} p_{k,n}(s) + \sum_{k=q+1}^{2q+1} \frac{\binom{2q+1}{k}}{n!} p_{k,n}(s) \\ &= \frac{1}{n!} \sum_{k=0}^q \binom{2q+1}{k} (p_{k,n}(s) + p_{n-k,n}(s)) \\ &= \frac{1}{n!} \sum_{k=0}^q \binom{2q+1}{k} (p_{k,n}(s) - p_{k,n}(-s)). \end{aligned}$$

The Lemma follows upon noting that for  $s = i\omega$ ,  $\omega \in \mathbb{R}$ , it holds that  $p_{k,n}(-i\omega) = \overline{p_{k,n}(i\omega)}$  since all coefficients of  $p_{k,n}$  are real.  $\square$

## References

- [1] Y. Meyer, Wavelets and operators, Advanced mathematics. Cambridge University Press, 1992.
- [2] I. Daubechies, Ten lectures on wavelets, Vol. 61, SIAM, Philadelphia, PA, USA, 1992.
- [3] J. Lamperti, Semi-stable stochastic processes, Trans. Amer. Math. Soc. 104 (1962) 62–78.
- [4] L. Cohen, The scale representation, IEEE Trans. Signal Processing 41 (12) (1993) 3275–3292.
- [5] B. Yazıcı, R. L. Kashyap, A class of second-order stationary self-similar processes for  $1/f$  phenomena, IEEE Trans. Signal Processing 45 (2) (1997) 396–410.

- [6] M. Mboup, On the structure of self-similar systems: A Hilbert space approach, in: *Operator Theory: Advances and applications*, Vol. OT-143, Birkhäuser-Verlag, 2003, pp. 273–302.
- [7] P. A. P. Borgnat, P. Flandrin, Scale invariances and Lamperti transformations for stochastic processes, *Journ. Phys. A: Math. Gen.* 38 (10) (2005) 2081–2101.
- [8] S. Mallat, *A Wavelet Tour of Signal Processing*, 2nd Edition, Academic Press, San Diego, 1999.
- [9] W. Leland, M. Taqqu, W. Willinger, D. Wilson, On the self-similar nature of Ethernet traffic (extended version), *IEEE/ACM Trans. Networking* (1994) 1–15.
- [10] M. Mboup, A character-automorphic Hardy spaces approach to discrete-time scale-invariant systems, in: *MTNS'06*, Kyoto, Japan, 2006, pp. 183–188.
- [11] L. R. Ford, *Automorphic functions*, Vol. 85, American Mathematical Soc., 2004.
- [12] G. Evangelista, S. Cavaliere, Discrete frequency-wrapped wavelets: theory and applications, *IEEE Trans. Signal Process.* 46 (4) (1998) 874–885.
- [13] G. Evangelista, S. Cavaliere, Audio effects based on biorthogonal time-varying frequency warping, *EURASIP Journal on Applied Signal Processing* 1 (2001) 27–35.
- [14] C.-J. Hsu, K.-S. Huang, C.-B. Yang, Y.-P. Guo, Flexible dynamic time warping for time series classification, *Procedia Computer Science* 51 (2015) 2838–2842, international Conference On Computational Science, ICCS 2015.
- [15] M. Schmidt, M. Baumert, H. Malberg, S. Zaunseder, Iterative two-dimensional signal warping—Towards a generalized approach for adaption

- of one-dimensional signals, *Biomedical Signal Processing and Control* 43 (2018) 311–319.
- [16] D. Alpay, M. Mboup, A characterization of Schur multipliers between character-automorphic Hardy spaces, *Integral Equations and Operator Theory* 62 (2008) 455–463.
  - [17] D. Alpay, M. Mboup, Discrete-time multi-scale systems, *Integral Equations and Operator Theory* 68 (2) (2010) 163–191.
  - [18] J. Bertrand, P. Bertrand, J. P. Ovarlez, Discrete Mellin transform for signal analysis, in: *International Conference on Acoustics, Speech, and Signal Processing*, IEEE, 1990, pp. 1603–1606.
  - [19] A. De Sena, D. Rocchesso, A fast mellin and scale transform, *EURASIP Journal on Advances in Signal Processing* 2007 (1) (2007) 089170. doi:10.1155/2007/89170.  
URL <https://doi.org/10.1155/2007/89170>
  - [20] M. Pap, F. Schipp, The voice transform on the Blaschke group II, *Annales Univ. Sci. Budapest., Sect. Comp.* 29 (2008) 157–173.
  - [21] M. Pap, Properties of the voice transform of the Blaschke group and connections with atomic decomposition results in the weighted Bergman spaces, *Journal of Mathematical Analysis and Applications* 389 (1) (2012) 340–350.
  - [22] A. Soumelidis, J. Bokor, F. Schipp, Applying hyperbolic wavelet constructions in the identification of signals and systems, *IFAC Proceedings Volumes* 42 (10) (2009) 1334–1339, 15th IFAC Symposium on System Identification.
  - [23] M. Pap, Hyperbolic wavelets and multiresolution in  $H_2(\mathbb{T})$ , *J Fourier Anal Appl* 17 (2011) 755–776.
  - [24] E. M. Stein, R. Shakarchi, *Fourier analysis: an introduction*, Vol. 1, Princeton University Press, 2011.

- [25] P. S. C. Heuberger, P. M. J. Van den Hof, O. H. Bosgra, A generalized orthonormal basis for linear dynamical systems, *IEEE Trans. Automatic Control* 40 (3) (1995) 451–465.
- [26] M. Manngård, H. T. Toivonen, Identification of low-order models using Laguerre basis function expansions, *IFAC-PapersOnLine* 51 (15) (2018) 72–77.
- [27] B. Wahlberg, P. Mäkilä, On approximation of stable linear dynamical systems using Laguerre and Kautz functions, *Automatica* 32 (5) (1996) 693–708.
- [28] W. Rudin, *Fourier analysis on groups*, Dover Publications, Inc, Mineola, New York, 2017.
- [29] R. R. Goldberg, *Fourier transforms*, 2009th Edition, Cambridge Tracts in Mathematics, Cambridge University Press, New York, 1961.
- [30] E. Hewitt, K. Ross, *Abstract Harmonic Analysis*, Vol. I/II, Springer, Berlin, Göttingen Heidelberg, 1963/1970.
- [31] H. G. Feichtinger, K. Gröchenig, Banach spaces related to integrable group representations and their atomic decompositions, I, *Journal on Functional Analysis* 86 (2) (1989) 307–340.
- [32] H. G. Feichtinger, K. Gröchenig, Banach spaces related to integrable group representations and their atomic decompositions. Part II, *Monatshefte für Mathematik* 108 (1989) 129–148.