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Super-hedging an arbitrary number of European options with integer-valued strategies

Dorsaf Cherif¹ Meriam EL MANSOUR,²
Emmanuel LEPINETTE³

¹ *Faculty of Sciences of Tunis, Tunisia.
Email: dorsaf-cherif@hotmail.fr*

² *Ceremade, UMR CNRS 7534, Paris Dauphine University, PSL National Research,
Place du Maréchal De Lattre De Tassigny,
75775 Paris cedex 16, France and
Faculty of Sciences of Tunis, Tunisia.
Email: elmansour@ceremade.dauphine.fr*

³ *Ceremade, UMR CNRS 7534, Paris Dauphine University, PSL National Research,
Place du Maréchal De Lattre De Tassigny,
75775 Paris cedex 16, France and
Gosaef, Faculty of Sciences of Tunis, Tunisia.
Email: emmanuel.lepinette@ceremade.dauphine.fr*

Abstract: The usual theory of asset pricing in finance assumes that the financial strategies, i.e. the quantity of risky assets to invest, are real-valued so that they are not integer-valued in general, see the Black and Scholes model for instance. This is clearly contrary to what it is possible to do in the real world. Surprisingly, it seems that there is no contribution in that direction in the literature. In this paper, we show that, in discrete-time, it is possible to evaluate the minimal super-hedging price when we restrict ourselves to integer-valued strategies. To do so, we only consider terminal claims that are continuous piecewise affine functions of the underlying asset. We formulate a dynamic programming principle that can be directly implemented on an historical data and which also provides the optimal integer-valued strategy.

Keywords and phrases: Super-hedging prices–European options–Integer-valued strategies–Optimal super-hedging prices–Dynamic programming principle.

1. Introduction

The problem of super-hedging a European claim, such as a Call option, is very classical in mathematical finance but has only been solved for real-valued strategies so that the optimal strategy, corresponding to the minimal hedging or super-hedging price, is not integer-valued contrary to what it is allowed to do in the real life. This is why we propose to solve the problem of super-hedging a European claim with integer-valued financial strategies.

Let us recall that the usual approach of pricing assumes that the financial market model satisfies a no-arbitrage condition NA, which is equivalent to the existence of a risk-neutral probability measure Q under which the discounted asset prices are martingales, see the Dalang-Morton-Willinger theorem [4]. Under NA, we may show that there exists a minimal super-hedging price $P_0^*(\xi_T)$ for the European claim $\xi_T \geq 0$ given by

$$P_0^*(\xi_T) = \sup_{Q \in \mathcal{M}(P)} E_Q(\xi_T), \quad (1.1)$$

where $\mathcal{M}(P)$ is the set of all risk-neutral probability measures equivalent to the initial probability measure P of the model. Here, we suppose that the risk-free interest rate of the model is $r = 0$. Recall that the formula above holds in discrete time but also in continuous time with extra-conditions on the model. Indeed, the no-arbitrage condition needs to be strenghtened and it is only equivalent to the existence of $Q \sim P$ under which discounted asset prices are local martingales, see [5], [6], [7].

In any case, the optimal strategy that achieves the minimal super-hedging price (1.1) is not, in general, integer-valued. The typical example is the continuous-time Black and Scholes model where the so-called delta-hedging strategy for the European call is explicit and lies in the set $[0, 1] \setminus \{0, 1\}$, see [2].

Clearly, a new approach is necessary to compute the super-hedging prices for only integer-valued financial strategies. We follow the ideas developed in [3] where the problem is initially solved without any no-arbitrage conditions. Then, a no-arbitrage condition AIP naturally appears and means that the infimum price of the zero claim (non negative claims more generally) is not $-\infty$. This condition is clearly necessary for numerical purposes. Actually, it is shown that AIP is equivalent to the property that the infimum super-hedging price of any non negative claim is non negative, as observed in the real markets. In our paper, we do not explicitly suppose such a no-arbitrage

condition but the form of the conditional supports of the asset price we assume implies this condition for the model with integer-valued strategies.

Our paper is a first attempt to compute super-hedging prices with only integer-valued financial strategies. We restrict ourselves to payoffs which are piecewise affine functions of the underlying asset and we assume specific conditional supports for the asset prices. Problems such as characterizations of the no-arbitrage condition NA with only integer-valued strategies or generalization of our work to arbitrary conditional supports of the asset prices remains open.

2. The super-hedging problem:

We consider $n \geq 1$ options that we want to super-replicate in discrete time $t = \{0, \dots, T\}$. Let $(\Omega, (\mathcal{F}_t)_{0 \leq t \leq T})$ be a stochastic basis where $(\mathcal{F}_t)_{0 \leq t \leq T}$ is supposed to be complete. We consider a financial market model composed of two assets. We suppose, without loss of generality, that the risk-free asset is $S_t^0 = 1$ for all $t \in \{0, \dots, T\}$, while the risky asset price is described by a stochastic process $S = (S_t)_{0 \leq t \leq T}$. Recall that a self-financing portfolio process $(V_t)_{0 \leq t \leq T}$ satisfies by definition:

$$\Delta V_t = V_t - V_{t-1} = \theta_{t-1} \Delta S_t, \quad t = 1, \dots, T.$$

In this paper, we consider European options whose payoffs are of the form $\xi_T = g_T(S) \in \mathbb{L}^0(\mathbf{R}, \mathcal{F}_T)$, where g_T is a continuous piecewise affine function. The typical example is the European call option, i.e. $g(x) = (x - K)^+$, $K > 0$. Our goal is to compute the set of all super-hedging prices of ξ_T , i.e the set of all V_0 , initial values of self-financing portfolio processes $(V_t)_{0 \leq t \leq T}$, such that $V_T \geq \xi_T$ almost surely. Contrarily to what it is usual to do in the literature, we restrict ourselves to the case of integer-valued strategies, i.e $\theta_t \in \mathbf{Z}$ almost surely, for all $t \in \{0, \dots, T\}$, where $\mathbf{Z} = \mathbb{N} \cup (-\mathbb{N})$ and \mathbb{N} is the set of all non negative integers. In the case of super-hedging an arbitrary number of options $n \geq 1$, the problem reads as $V_T \geq n\xi_T$, a.s. and it is clearly interesting to analyse the impact of n on the strategies and the infimum prices, as linearity is not necessarily preserved with respect to the quantity n of claims.

To solve this problem, we follow the approach of [3], [1] that we adapt to integer-valued strategies. To do so, we first solve backwardly the super-hedging problem between two dates $t - 1$ and t , and we show that the procedure may be propagated backwardly as the minimal super-hedging price we

obtain at time $t - 1$ is still a continuous piecewise affine payoff function of the underlying asset. It is then possible to deduce the minimal super-hedging price at time $t = 0$ by induction.

3. The super-hedging problem.

Let $t \leq T$ and g_t be a continuous piecewise affine function, i.e. there exists a subdivision $0 = a_0 < a_1 < \dots < a_{n-1} < a_n = \infty$ of $[0, \infty]$ such that g_t is an affine function for all $x \in [a_i, a_{i+1}), \forall i \in \{0, \dots, n-1\}$. As the asset prices are non negative, we suppose without loss of generality that $a_0 = 0$. We first solve the one step problem: find V_{t-1} and θ_{t-1} such that:

$$V_{t-1} + \theta_{t-1} \Delta S_t \geq g_t(S_t), \text{ a.s.}$$

This is equivalent to:

$$\begin{aligned} V_{t-1} &\geq g_t(S_t) - \theta_{t-1} \Delta S_t, \\ \Leftrightarrow V_{t-1} &\geq g_t(S_t) - \theta_{t-1} S_t + \theta_{t-1} S_{t-1}, \\ \Leftrightarrow V_{t-1} &\geq \text{ess sup}_{\mathcal{F}_{t-1}} (g_t(S_t) - \theta_{t-1} S_t) + \theta_{t-1} S_{t-1}. \end{aligned}$$

Equivalently, we have:

$$V_{t-1} \geq V_{t-1}(\theta_{t-1}) := \sup_{x \in \text{supp}_{\mathcal{F}_{t-1}}(S_t)} (g_t(x) - \theta_{t-1} x) + \theta_{t-1} S_{t-1}, \quad (3.2)$$

where $\text{supp}_{\mathcal{F}_{t-1}}(S_t)$ is the conditional support of S_t knowing \mathcal{F}_{t-1} , see [3] and [8] for the definition.

In the following, we suppose that there exist two deterministic numbers $k_{t-1}^d \in (0, 1)$ and $k_{t-1}^u \in (1, \infty)$ such that $\text{supp}_{\mathcal{F}_{t-1}}(S_t) = [k_{t-1}^d S_{t-1}, k_{t-1}^u S_{t-1}]$. This model may be seen as a generalization of the Binomial model and the conditions imposed on the coefficients k_{t-1}^d and k_{t-1}^u are equivalent to a no-arbitrage condition, see [3]. In particular, we have:

$$V_{t-1}(\theta_{t-1}) = \sup_{x \in [k_{t-1}^d S_{t-1}, k_{t-1}^u S_{t-1}]} (g_t(x) - \theta_{t-1} x) + \theta_{t-1} S_{t-1} \in L^0(\mathbf{R}, \mathcal{F}_{t-1}).$$

We define V_{t-1}^* as the infimum of all the superhedging prices at time $t - 1$ over all integer-valued strategies in \mathbf{Z} , i.e.

$$V_{t-1}^* := \text{ess inf}_{\theta_{t-1} \in \mathbb{L}^0(\mathbf{Z}, \mathcal{F}_{t-1})} V_{t-1}(\theta_{t-1}).$$

Lemma 3.1. *We have*

$$V_{t-1}^* = \inf_{\theta \in \mathbf{Z}} V_{t-1}(\theta). \quad (3.3)$$

Proof. Let us define $\gamma = \inf_{\theta \in \mathbf{Z}} V_{t-1}(\theta) \in L^0(\mathbf{R}, \mathcal{F}_{t-1})$, see [3]. As $V_{t-1}^* \leq V_{t-1}(\theta)$, for all $\theta \in \mathbf{Z}$, we get that $V_{t-1}^* \leq \inf_{\theta \in \mathbf{Z}} V_{t-1}(\theta) = \gamma$.

On the other hand, if $\theta_{t-1} \in \mathbb{L}^0(\mathbf{Z}, \mathcal{F}_{t-1})$, then:

$$\begin{aligned} \theta_{t-1} &= \sum_{\theta \in \mathbf{Z}} \theta 1_{\{\theta_{t-1}=\theta\}}, \\ V_{t-1}(\theta_{t-1}) &= \sum_{\theta \in \mathbf{Z}} V_{t-1}(\theta) 1_{\{\theta_{t-1}=\theta\}} \geq \sum_{\theta \in \mathbf{Z}} \gamma 1_{\{\theta_{t-1}=\theta\}} = \gamma. \end{aligned}$$

We deduce that $V_{t-1}^* \geq \gamma$ and the conclusion follows. \square

Theorem 3.2 (One step problem). *Let us consider $t \in \{1, \dots, T\}$ and suppose that $\xi_t = g_t(S_t)$ where g_t is a continuous piecewise affine function. Moreover, we assume that there exists two deterministic numbers $k_{t-1}^d \in (0, 1)$ and $k_{t-1}^u \in (1, \infty)$ such that*

$$\text{supp}_{\mathcal{F}_{t-1}}(S_t) = [k_{t-1}^d S_{t-1}, k_{t-1}^u S_{t-1}].$$

Then, $V_{t-1}^ = g_{t-1}(S_{t-1})$ where g_{t-1} is a continuous piecewise linear function.*

Proof. By assumption, there exist a subdivision $(a_i)_{i=0, \dots, n}$ of $[0, \infty]$, with $a_0 = 0 < a_1 < \dots < a_{n-1} < a_n = \infty$, such that g_{t-1} is an affine function on each interval. Let us define

$$x_i(S_{t-1}) = (k_{t-1}^d S_{t-1} \vee a_i) \wedge k_{t-1}^u S_{t-1}, \quad i = 0, \dots, n.$$

It is straightforward that

$$V_{t-1}(\theta_{t-1}) = \sup_{i=0, \dots, n} [g_t(x_i(S_{t-1})) - \theta_{t-1} x_i(S_{t-1})] + \theta_{t-1} S_{t-1}.$$

Note that $x_0(S_{t-1}) = k^d S_{t-1}$ and $x_n(S_{t-1}) = k^u S_{t-1}$ and some terms of the sequence $(x_i)_i$ may coincide. Let us define the functions

$$h^i(\theta_{t-1}, S_{t-1}) = g_t(x_i(S_{t-1})) + \theta_{t-1}(S_{t-1} - x_i(S_{t-1})).$$

The slopes of the affine functions $\theta_{t-1} \mapsto h^i(\theta_{t-1}, S_{t-1})$ are given by the non decreasing sequence $(S_{t-1} - x_i(S_{t-1}))_{i=n, n-1, \dots, 0}$ such that $S_{t-1} - x_n(S_{t-1}) < 0$ and $S_{t-1} - x_0(S_{t-1}) > 0$.

By ordering the indices in the decreasing order, we obtain $(n + 1)$ affine functions $\theta_{t-1} \mapsto h^i(\theta_{t-1})$ for $i \in \{n, n - 1, \dots, 1, 0\}$ with increasing slopes $(S_{t-1} - x_i)_{i \in \{n, n-1, \dots, 1, 0\}}$, such that: $S_{t-1} - x_n < 0$ and $S_{t-1} - x_0 > 0$. Therefore, the mapping $V_{t-1} : \theta_{t-1} \mapsto \sup_{i=n, \dots, 0} h^i(\theta_{t-1}, S_{t-1})$ is a piecewise affine function, i.e. there exists a subdivision:

$$-\infty = \alpha_0 < \alpha_1(S_{t-1}) \leq \dots \leq \alpha_{m-1}(S_{t-1}) < \alpha_m = \infty$$

such that V_{t-1} is an affine function of θ_{t-1} on each interval $[\alpha_i(S_{t-1}), \alpha_{i+1}(S_{t-1})]$, $i = 0, \dots, m - 1$. Note that the function V_{t-1} is convex in θ_{t-1} and the elements of the partition define the intersection points between two distinct and successive graphs of the affine functions $h^{i+1}(\theta_{t-1}), h^i(\theta_{t-1})$. So, there exists $\theta_{t-1}^* \in [\alpha_1(S_{t-1}) - 1, \dots, \alpha_{m-1}(S_{t-1}) + 1] \cap \mathbf{Z}$ such that:

$$\inf_{\theta_{t-1} \in \mathbf{Z}} V_{t-1}(\theta_{t-1}) = V_{t-1}(\theta_{t-1}^*).$$

It remains to evaluate $\alpha_1(S_{t-1})$ and $\alpha_{m-1}(S_{t-1})$. To do so, let us solve the equations $h^i(\alpha) = h^j(\alpha)$, $i, j = 0, \dots, m$ and $x_i \neq x_j$. Since we suppose that $x_i - x_j \neq 0$, we get that

$$\alpha = \frac{g_t(x_i) - g_t(x_j)}{x_i - x_j}.$$

We deduce that $|\alpha| \leq L_t$ where $L_t > 0$ is a Lipschitz constant of the piecewise affine function g_t . We deduce that $\theta_{t-1}^* \in [-L_t - 1, L_t + 1] \cap \mathbf{Z}$ and

$$g_{t-1}(S_{t-1}) = V_{t-1}(\theta_{t-1}^*(S_{t-1})) = \min_{\theta_{t-1} \in [-L_t - 1, L_t + 1] \cap \mathbf{Z}} \sup_{i=0, \dots, n} h^i(\theta_{t-1}, S_{t-1}).$$

We conclude that g_{t-1} is a continuous piecewise affine function as a finite minimum of continuous piecewise affine functions. \square

Corollary 3.3. *(The multi-period super-hedging problem) Suppose that, at time $T > 0$, the payoff is $\xi_T = g_T(S_T)$ where g_T is a continuous piecewise affine function. Moreover, we assume that there exists deterministic numbers $k_{t-1}^d \in (0, 1)$ and $k_{t-1}^u \in (1, \infty)$ for each $t = 1, \dots, T$ such that we have $\sup_{\mathcal{F}_{t-1}}(S_t) = [k_{t-1}^d S_{t-1}, k_{t-1}^u S_{t-1}]$. Then, there exists a minimal super-hedging portfolio process $(V_t^*)_{t=0, \dots, T}$ such that $V_T^* \geq \xi_T$. We have*

$V_t^* = g(t, S_t)$ where $g(t, \cdot)$ is a continuous piecewise affine function given by

$$g(t, s) = \min_{\theta \in [-L_{t+1}-1, L_{t+1}+1] \cap \mathbf{Z}} \sup_{i=0, \dots, n^{(t+1)}} \left(g(t+1, x_i^{(t+1)}(s)) + \theta(s - x_i^{(t+1)}(s)) \right),$$

$$x_i^{(t+1)}(s) = (k_t^d s \vee a_i^{(t+1)}) \wedge k_t^u s, \quad i = 0, \dots, n^{(t+1)},$$

where L_{t+1} is any Lipschitz constant of $g(t+1, \cdot)$ and $(a_i^{(t+1)})_{i=0, \dots, n^{(t+1)}}$ is any partition such that $a_0^{(t+1)} = 0 < a_1^{(t+1)} < \dots < a_{n^{(t+1)}-1}^{(t+1)} < a_{n^{(t+1)}}^{(t+1)} = \infty$ and $g(t+1, \cdot)$ is an affine function on $[a_i^{(t+1)}, a_{i+1}^{(t+1)}]$, $i \leq n^{(t+1)} - 1$. The associated super-hedging strategy θ^* is given by the argmin of the minimisation problem defining $g(t, \cdot)$ in the expression above.

3.1. Example in the one step problem: the case of the Call option

At time $t = T$, suppose that the payoff is $\xi_T^n = ng(S_T)$ where $n \geq 1$ and $g(x) = (x - K)^+$, $K = 500$. We suppose that $\text{supp}_{\mathcal{F}_{T-1}}(S_T) = [k^d S_{T-1}, k^u S_{T-1}]$ for some constants k^d, k^u such that $0 < k^d < 1 < k^u$. Precisely, we suppose that $k^d = 0.9$ and $k^u = 1.2$. Observe that the super-hedging problem $V_{T-1} + \theta_{T-1} \Delta S_T \geq ng(S_T)$ is equivalent to

$$V_{T-1} \geq V_{T-1}(\theta_{T-1}) = \max_{k \in \{k^d, k^u\}} [ng(kS_{T-1}) - \theta_{T-1} k S_{T-1}] + \theta_{T-1} S_{T-1}.$$

In the following we give the explicit expression of $V_{T-1}(\theta_{T-1}) = V_{T-1}^n(\theta_{T-1})$.

If $k^u \leq K/S_{T-1}$, i.e. $S_{T-1} \leq K/k^u$, then

$$V_{T-1}(\theta_{T-1}) = \begin{cases} \theta_{T-1} S_{T-1} (1 - k^u) & \text{if } \theta_{T-1} \leq 0, \\ \theta_{T-1} S_{T-1} (1 - k^d) & \text{if } \theta_{T-1} \geq 0. \end{cases}$$

Therefore, $\theta_{T-1}^{*,n}(S_{T-1}) = 0$, and $V_{T-1}^{*,n}(S_{T-1}) = V_{T-1}(\theta_{T-1}^{*,n}) = 0$.

If $k^d \geq K/S_{T-1}$, i.e. $S_{T-1} \geq K/k^d$,

$$V_{T-1}(\theta_{T-1}) = \begin{cases} \theta_{T-1} S_{T-1} (1 - k^u) + nk^u S_{T-1} - nK & \text{if } \theta_{T-1} \leq n, \\ \theta_{T-1} S_{T-1} (1 - k^d) + nk^d S_{T-1} - nK & \text{if } \theta_{T-1} \geq n. \end{cases}$$

We conclude that $\theta_{T-1}^{*,n}(S_{T-1}) = n$, and $V_{T-1}^{*,n}(S_{T-1}) = n(S_{T-1} - K)$.

If $k^d \leq K/S_{T-1} \leq k^u$, i.e. $S_{T-1} \in [K/k^u, K/k^d]$,

$$V_{T-1}(\theta_{T-1}) = \begin{cases} \theta_{T-1} S_{T-1} (1 - k^u) + nk^u S_{T-1} - nK & \text{if } \theta_{T-1} \leq \frac{nk^u S_{T-1} - nK}{S_{T-1}(k^u - k^d)}, \\ \theta_{T-1} S_{T-1} (1 - k^d) & \text{if } \theta_{T-1} \geq \frac{nk^u S_{T-1} - nK}{S_{T-1}(k^u - k^d)}. \end{cases}$$

Let us define

$$\begin{aligned} \alpha_{T-1}^n(S_{T-1}) &:= \frac{nk^u S_{T-1} - nK}{S_{T-1}(k^u - k^d)}, \\ f^n(x, S_{T-1}) &:= x S_{T-1} (1 - k^d) \mathbf{1}_{\{x \geq \alpha_{T-1}^n(S_{T-1})\}} \\ &\quad + (x S_{T-1} (1 - k^u) + nk^u S_{T-1} - nK) \mathbf{1}_{\{x < \alpha_{T-1}^n(S_{T-1})\}}. \end{aligned}$$

We denote by $\lfloor \alpha_{T-1}^n(S_{T-1}) \rfloor$ the lower integer part of α_{T-1}^n . Then,

$$\theta_{T-1}^{*,n}(S_{T-1}) = \begin{cases} \lfloor \alpha_{T-1}^n(S_{T-1}) \rfloor & \text{if } f^n(\lfloor \alpha_{T-1}^n(S_{T-1}) \rfloor) \leq f^n(\lfloor \alpha_{T-1}^n(S_{T-1}) \rfloor + 1), \\ \lfloor \alpha_{T-1}^n(S_{T-1}) \rfloor + 1 & \text{otherwise.} \end{cases}$$

So,

$$\begin{aligned} V_{T-1}^{*,n}(S_{T-1}) &= (\lfloor \alpha_{T-1}^n(S_{T-1}) \rfloor S_{T-1} (1 - k^u) + nk^u S_{T-1} - nK) \mathbf{1}_{G_{T-1}^n}(S_{T-1}) \\ &\quad + (\lfloor \alpha_{T-1}^n(S_{T-1}) \rfloor + 1) S_{T-1} (1 - k^d) \mathbf{1}_{(G_{T-1}^n)^c}(S_{T-1}), \end{aligned}$$

where

$$\begin{aligned} G_{T-1}^n &:= \{S : f^n(\lfloor \alpha_{T-1}^n(S) \rfloor) \leq f^n(\lfloor \alpha_{T-1}^n(S) \rfloor + 1)\} = \{\lfloor \alpha_{T-1}^n(S) \rfloor \leq \beta_{T-1}^n(S)\}, \\ \beta_{T-1}^n(S) &:= \alpha_{T-1}^n(S) + \frac{1 - k^d}{k^d - k^u}. \end{aligned}$$

A graphic illustration of $V_{T-1}^{*,n}/n$ as a function of S_{T-1} is given in Figure 1.

We observe that $V_{T-1}^{*,n}$ is not a convex function of S_{T-1} even if the payoff function is and, moreover, $g(T-1, x, n) \neq ng(T-1, x, 1)$.

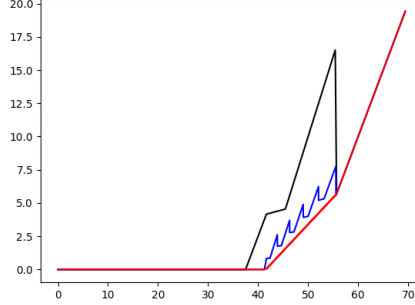


Fig 1: The function $x = S_{T-1} \mapsto g(T-1, x, n)/n = V_{T-1}^{*,n}/n, K = 500$, for $n = 1$ (black), $n = 5$ (blue), $n = 100$ (red).

4. Numerical illustration

In this section, we illustrate the method developed above when the underlying asset S is the french CAC 40 index and the European claim is the Call option. The historical data is composed of daily observations of the CAC 40 values between the 6th of June 2019 and the 16th of June 2021. We use the two first years of the data set to calibrate the model while we implement the model on the third year. Here, we suppose that $\text{supp}_{\mathcal{F}_t} S_{t+1} = [k_t^d S_t, k_t^u S_t]$ where k_t^d and k_t^u are estimated as follows:

$$k_t^d = \min_{i=j, \dots, N} S_{t+1}^{(j)} / S_t^{(j)},$$

$$k_t^u = \max_{i=j, \dots, N} S_{t+1}^{(j)} / S_t^{(j)},$$

where N is the number of training periods and $S_t^{(j)}$ are the observed values at time t during the j -th periods. The algorithms are written in Python. The main difficulty is to write a code whose execution time is reasonable. Indeed, recall that the price function $g(t, x)$ is computed backwardly from $g(t+1, x)$. If this function $g(t, x)$ is naively coded from $g(t+1, x)$, then the computation may take more than two weeks ! So it is better to approximate, at each step, the function $g(t, x)$ as a numpy array consisting of discretized values following a grid $(x_i)_{i=0}^{N_t}$ where $x_i = \text{step} * i$. Here, we choose $\text{step} = 0.1$

and N_t is chosen so that $x_{N_0} \leq S_0^{\max}$ where S_0^{\max} is the maximal value for S_0 that we observe in our data. At last, $x_{N_t} \leq S_0^{\max} * (\max_{r \leq t} k_r^u)^t$.

The relative hedging error is defined as $\epsilon_T = 100 * (V_T - g(T, S_T)) / S_T$ where $(V_t^*)_{t=0, \dots, T}$ is the optimal super-hedging portfolio process whose initial value is the minimal super-hedging price, as computed in the last section. We present in Figure 2.1 the distribution of ϵ_T when $n = 1$. Of course, we expect that $\epsilon_T \geq 0$ a.s. and this is confirmed on our test data set. Note that, we could have observed some negative values as the model is calibrated from data values anterior to the test data set.

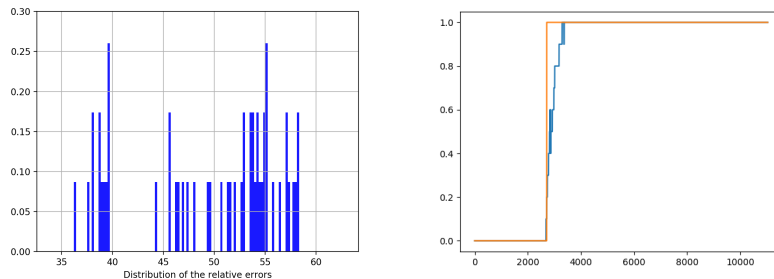


Fig 2: F1.1: Super-hedging errors with $n = 1$ and $K = 3000$. F1.2: Comparison of the optimal strategies per unit of claims for $n = 1$ and $n = 10$.

Let us denote by $g(t, x, n)$ the price function at time t of the optimal portfolio process, i.e. $V_t^{*,n} = g(t, S_t, n)$ such that $V_T^{*,n} \geq \xi_T^n$ a.s., when the European claim is $\xi_T^n := n * (S_T - K)^+$. The natural question is the following: Do we have $g(t, x, n) = ng(t, x, 1)$? The answer is yes when real-valued strategies are allowed since the hedging problem is then linear with respect to the number of claims.

In the case of integer-valued strategies, the answer is not trivial and is actually negative, see the first example above. By definition of the infimum super-hedging price, we have $g(t, x, n) \leq ng(t, x, 1)$. As a first step, we have computed the relative infimum super-hedging prices per unit of claims, i.e. $V_0^{*,n}/n$ at time 0, for different values of n on each period of the test data set. Then, computing the average of the $V_0^{*,n}/n$ values over all the periods, we get that the empirical average of $V_0^{*,n}/n$ is approximately equal to 49.48% for $n = 1, 5, 10, 15, 20$. Nevertheless, we observe that the price function per unit of unit of claims, i.e. $g(0, S_0, n)/n$ is non-increasing when n increases, see Figure 3. This implies that the equality $g(t, x, n) = ng(t, x, 1)$ does not

hold. We conjecture that $g(t, x, n)/n$ converges to the price function $\hat{g}(t, x)$ of the model where real-valued strategies are allowed, see [3]. This is an open question we suggest. In Figure 3, we clearly observe the convergence of $x \mapsto g(0, x, n)/n$ as $n \rightarrow \infty$.

The same question arises for the optimal strategy associated to $V^{*,n}$, i.e. do we have $\theta^*(t, S_t, n) = n\theta^*(t, S_t, 1)$? Intuitively, this is a priori not the case as $\theta^*(t, S_t, 1) = \theta^*(t, S_t, n)/n$ could be not integer-valued. This is confirmed at time 0 when we compute the optimal strategy $\theta^*(0, S_0, n)/n$ per unit of claims. This is illustrated by Figure 2.2 where we compare $\theta^*(0, S_0, n)/n$ for $n = 10$ to $\theta^*(0, S_0, 1)$. We may observe that the optimal strategy per unit of claims $\theta^*(0, S_0, n)/n$ (blue graph) is smaller than $\theta^*(0, S_0, 1)$ for $n = 10$.

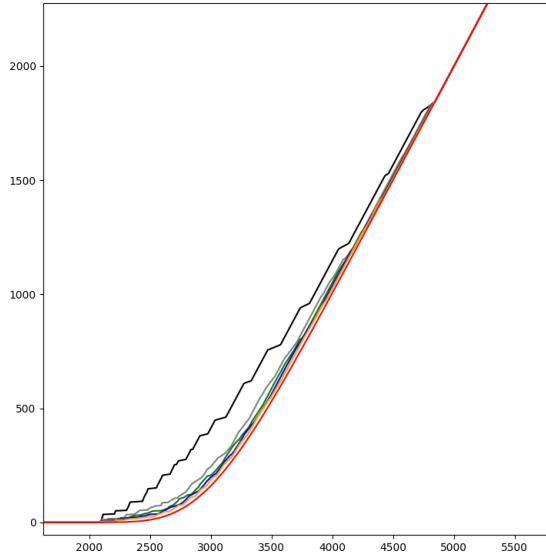


Fig 3: Super-hedging price mapping $x \mapsto g(0, x, n)/n$ of n units of Call option per unit of claims for different values of $n = 1$ (black), $n = 3$ (grey), $n = 5$ (green), $n = 7$ (blue), $n = 10$ (orange), $n = 100$ (red).

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