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► **To cite this version:**

Salim Zekraoui, Nicolás Espitia, Wilfrid Perruquetti. Finite-time estimation of second-order LTI systems in the presence of delayed measurement. *International Journal of Robust and Nonlinear Control*, In press, 10.1002/rnc.6224 . hal-03835025

**HAL Id: hal-03835025**

**<https://hal.science/hal-03835025>**

Submitted on 31 Oct 2022

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## ARTICLE TYPE

# Finite-time estimation of second-order LTI systems in the presence of delayed measurement.

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## Summary

In this paper, we design a nonlinear observer for second-order linear time-invariant (LTI) systems in the observable form with measurement delay. The design guarantees the convergence of the error state to the origin within a finite-time that depends on the initial conditions. To accomplish this, we reformulate the original system into a cascade ODE-PDE system where the PDE part is a transport equation that models the effect of the delay on the output. We construct the nonlinear gains in a way that ensures the error system to be finite-time stable (FTS). To prove this, we use an invertible backstepping transformation to convert the error system into a target system which is shown to be finite-time stable using Lyapunov-based analysis and homogeneity tools. We use the inverse transformation to transfer this property to the error system.

## KEYWORDS:

Finite-time convergence, Observer design, LTI systems, Delayed measurement.

## 1 | INTRODUCTION

State estimation is one of the most important topics in control theory. Indeed, usually the full state is not available and/or sensors cost are prohibitive. Moreover, in engineering applications, delay is often in different parts of the system. In particular, the delay can be caused by transmitting the state's measurements via a communication network. Furthermore, finite-time observation becomes very desired in several applications that require the transient process to finish in a finite amount of time. For instance, in Teleoperation, the surgery is performed using a robotic hand controlled by the doctor using a remote control. The robotic hand consists of a number of rigid links connected with joints. Using the Euler-Lagrange equation the dynamic model of the robotic hand is given by

$$M(q)\ddot{q} + C(q, \dot{q}) + G(q) = \tau, \quad (1)$$

where  $\tau$  is the applied torque generated by the actuators,  $q \in \mathbb{R}^n$ ,  $\dot{q} \in \mathbb{R}^n$  and  $\ddot{q} \in \mathbb{R}^n$  are respectively vectors of joint positions, velocities and accelerations,  $M(q)$  is the inertia matrix,  $C(q, \dot{q})$  the vector of Coriolis and centrifugal forces, and  $G(q)$  the vector of gravitational forces. In this situation, transmitting the measurements through a communication channel causes a time delay in reception and response. In addition, using a finite-time observer in this situation is critical, as the surgery ends in a finite period.

Concerning asymptotic estimation, a vast literature is available in both the cases of the absence of delay see, e.g.,<sup>1,2,3,4</sup>, and the presence of delay<sup>5,6,7,8,9</sup>. In contrast, less results are found in the framework of estimation in a finite time. However, in the absence of delay, we can highlight some worth mentioning techniques: two (Luenberger) observers coupled<sup>10,11,12,13,14</sup>, homogeneity tools<sup>15</sup>, high order sliding mode observer/differentiator<sup>16</sup>, sliding mode approach for nonlinear systems<sup>17,18</sup>, implicit Lyapunov function approach<sup>19,20</sup>.

<sup>0</sup>**Abbreviations:** LTI systems, linear time invariant systems; FTS, Finite-time stable.

On the other hand, using the old techniques becomes more challenging when the delay is present. As a consequence, new approaches are adopted to design finite-time observers. For instance, in<sup>21</sup>, an approach that makes use of homogeneity techniques and the ring theory is used for linear time-delay systems with commensurate delay. Similarly, in<sup>22</sup>, an approach based on higher-order sliding mode techniques is used for a class of multi-input multi-output nonlinear time-delay systems. In the same direction, for time-varying nonlinear systems with delays in the outputs, an approach based on a dynamic extension that computes fundamental matrices is introduced in<sup>23</sup>. In<sup>24</sup>, a finite-time observer is designed for time-delay systems with unknown inputs in based on the algebraic framework proposed in<sup>25</sup>. In<sup>26</sup>, another finite-time observer is designed employing Lyapunov-based techniques, for a class of nonlinear time-varying delay systems in the p-normal form. In the same context of finite-time observation, in<sup>27</sup> a stochastic Lyapunov-Krasovskii functional approach and linear matrix inequality technique were used for a class of uncertain discrete-time neural networks with Markovian jumps and time-varying delays.

Furthermore, in<sup>28</sup>, the developed observer not only estimates the state of a class of nonlinear systems in a finite-time but also guarantees that the time of convergence is prescribed independently of the initial condition (i.e., in a prescribed time), employing Lyapunov-based techniques. Additionally, in<sup>29,30,31</sup>, a prescribed-time observer is designed using two coupled Luenberger-like observers and tools from the ring theory for different classes of linear systems with commensurate delays. In<sup>32</sup>, the prescribed-time observer is presented with time-varying output gains that make use of polynomial functions blowing-up in a prescribed time. The prescribed-time convergence of the error states is then proved using the Backstepping techniques introduced in<sup>33</sup>.

However, in all the previous papers, the problem of robustness and the numerical challenges in implementation are difficult to overcome. In fact, saturating the gains may alleviate some of these problems in some cases.

In contrast to these papers, we design a robust nonlinear observer which may be quite easy to implement and that is able to also estimate the sensor dynamics (infinite-dimensional). Indeed, we study the problem of finite-time observation for second-order LTI systems in the observable form with a delay in the output measurement. To solve this problem, we first rewrite the original system into an ODE-PDE form, where the PDE part models the effect of the delay on the output. Next, We propose an observer with nonlinear injection terms, that reconstruct the system's states in a finite time. To achieve this, we use an invertible backstepping transformation to transform the error system into a finite-time stable target system. Finally, we use the inverse transformation to transfer this property to the error system.

This paper is organized as follows: In Section II, we recall some preliminary definitions and technical tools to be used in the rest of the paper. In Section III, we introduce the second-order linear time invariant system with delayed output that we are interested in, and we rewrite it into a PDE-ODE setting. Next, we present the nonlinear finite-time observer. Then, we use a suitable choice of the transformation to link the error system to the target system. In Section IV, we make use of the transformation and the fact that the target system is FTS to prove the main result of this paper. In Section V we consider a numerical example to illustrate the results. Finally, conclusions and perspectives are given in Section VI.

## Notations:

We denote by  $\mathbb{R}_+$  the set of nonnegative real numbers. For any  $x \in \mathbb{R}$ ,  $|x|$  denotes the absolute value of  $x$  and  $sgn(x)$  denotes its sign. For any  $a > 0$ , We denote by  $\{.\}^a$  the function  $x \mapsto sgn(x)|x|^a$ . Let  $I_2$  be the identity matrix of dimension 2. For  $1 \leq p \leq +\infty$ , the induced norm of a matrix  $M$  is defined as  $\|M\|_p = \sup \{\|MX\|_p : X \in \mathbb{R}^2 \text{ with } \|X\|_p = 1\}$ , where  $\|X\|_p$  denotes the  $p$ -vector norm of  $X$ . In particular, when  $p = 2$  we denote the norm  $\|X\|_2$  by  $\|X\|$  (respectively the norm  $\|M\|_2$  by  $\|M\|$ ). When  $p = 1$  (respectively  $p = +\infty$ ),  $\|M\|_p$  corresponds to the maximum absolute row (respectively column) sum norm.

The set of all functions  $g : [0, h] \rightarrow \mathbb{R}^2$  satisfying  $\int_0^h \|g(x)\|^2 dx < +\infty$  is denoted by  $L^2$ .

## 2 | PRELIMINARY DEFINITIONS AND TECHNICAL TOOLS

In this section, we present first some relevant preliminary definitions on notions of Finite-time stability and weighted homogeneity. Then, we give a useful proposition to be used in the rest of the paper.

Let us consider the domain  $\Omega \subset \mathbb{R}^n$  as an open connected set containing the origin, with  $n \in \mathbb{N}^*$ , and  $\partial\Omega$  its boundary. Consider the following system

$$\dot{Z} = f(Z), \quad Z \in \Omega, \quad (2)$$

where  $f : \Omega \rightarrow \mathbb{R}^n$  is such that  $f(0) = 0$ . and (2) has the property of existence and uniqueness of solutions in forward time outside the origin.

**Definition 1.**<sup>34</sup> The origin of the system (2) is said to be **Finite-time stable (FTS)**, if it is asymptotically stable and there exists a locally bounded function  $T_{max}$  such that  $Z(t) = 0$  for all  $t \geq T_{max}(Z_0)$ .

**Definition 2.**<sup>35</sup> For any positive numbers  $\lambda$  and  $\bar{r}_i$ ,  $i = 1, \dots, n$  (called weights), one can define:

- the vector of weights  $\bar{r} = (\bar{r}_1, \dots, \bar{r}_n)$ , with  $\bar{r}_{max} = \max_{1 \leq j \leq n} \bar{r}_j$  and  $\bar{r}_{min} = \min_{1 \leq j \leq n} \bar{r}_j$ .
- The matrix  $\Lambda_{\bar{r}}(\lambda) = \text{diag} \{ \lambda^{\bar{r}_1}, \dots, \lambda^{\bar{r}_n} \}$  is called the dilation matrix associated to the vector of weights  $\bar{r}$ . Note that for any  $Z = (z_1, \dots, z_n) \in \mathbb{R}^n$ ,

$$\Lambda_{\bar{r}}(\lambda)Z = (\lambda^{\bar{r}_1} z_1, \dots, \lambda^{\bar{r}_i} z_i, \dots, \lambda^{\bar{r}_n} z_n)^T.$$

**Definition 3.**<sup>35</sup> A function  $V : \mathbb{R}^n \rightarrow \mathbb{R}$  is said to be  $\bar{r}$ -homogeneous of degree  $k \in \mathbb{R}$  if

$$V(\Lambda_{\bar{r}}(\lambda)Z) = \lambda^k V(Z), \quad \forall Z \in \mathbb{R}^n, \forall \lambda > 0.$$

A vector field  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is said to be  $\bar{r}$ -homogeneous of degree  $\nu \in \mathbb{R}$  if

$$f(\Lambda_{\bar{r}}(\lambda)Z) = \lambda^\nu \Lambda_{\bar{r}}(\lambda)f(Z), \quad \forall Z \in \mathbb{R}^n, \forall \lambda > 0,$$

in other words,  $f_i$  are  $\bar{r}$ -homogeneous of degree  $\nu + \bar{r}_i$ , for each  $i \in \{1, \dots, n\}$ .

Now, let us recall that Finite-time stability is linked to the notion of Homogeneity.

**Lemma 1.**<sup>35,36</sup> If the system (2) is  $\bar{r}$ -homogeneous of degree  $\nu$  and asymptotically stable at the origin, then it is globally FTS at the origin if  $\nu < 0$ .

Next, using the Young's inequality, we obtain the following Proposition (to be used in the sequel).

**Proposition 1.** Let  $a$  and  $b$  be positive real numbers. Let  $\beta$  be a real number in  $(0, 1)$ . Then, the following inequalities hold

$$a^\beta \leq \beta a + (1 - \beta), \quad (3)$$

and for any  $\varepsilon \in (0, 1)$ ,

$$a^\beta \leq \varepsilon a + (1 - \beta) \left( \frac{\beta}{\varepsilon} \right)^{\frac{\beta}{1-\beta}}, \quad (4)$$

and

$$(a + b)^\beta \leq a^\beta + b^\beta. \quad (5)$$

### 3 | PROBLEM STATEMENT

Let us consider a second-order linear time invariant system with delayed output. Assume that this system, when the output is not delayed, is observable, then one can rewrite the system in the form

$$\begin{aligned} \dot{Z}(t) &= AZ(t) + BU(t), \\ y(t) &= CZ(t - h), \end{aligned} \quad (6)$$

where

$$A = \begin{pmatrix} a_1 & 1 \\ a_0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \text{and} \quad C = (1 \quad 0),$$

and  $t \geq t_0 \geq 0$ ,  $Z = (z_1, z_2)^T \in \mathbb{R}^2$  with initial condition  $Z(\theta) = Z_0(\theta) = (z_{1,0}(\theta), z_{2,0}(\theta))^T$  for all  $\theta \in [t_0 - h, t_0]$ , where  $t_0$  is the initialization time,  $U(t) \in \mathbb{R}$  is the input signal, and  $y(t) \in \mathbb{R}$  is the measurement which is delayed by  $h > 0$  units of time. Our goal is to design a finite-time observer for (6): this means that the error state converges in finite-time to zero.

The system (6) can be rewritten in an ODE-PDE cascade setting from<sup>33, Chapter 3</sup> as follows:

$$\begin{aligned} \dot{Z}(t) &= AZ(t) + BU(t), \\ u_t(t, x) &= u_x(t, x), \\ u(t, h) &= CZ(t), \\ y(t) &= u(t, 0), \end{aligned} \quad (7)$$

where  $u(t_0, x) = CZ(t_0 - h + x)$  for all  $t_0 \geq 0$ .

Now, we propose the following observer system for (7) :

$$\begin{aligned}\dot{\hat{Z}}(t) &= A\hat{Z}(t) + BU(t) - \exp(Ah)\mathcal{L}(y(t) - \hat{u}(t, 0)), \\ \hat{u}_t(t, x) &= \hat{u}_x(t, x) - C \exp(Ax)\mathcal{L}(y(t) - \hat{u}(t, 0)), \\ \hat{u}(t, h) &= C\hat{Z}(t),\end{aligned}\quad (8)$$

where the initial condition is given by  $\hat{Z}(\theta) = \hat{Z}_0(\theta) = (\hat{z}_{1,0}(\theta), \hat{z}_{2,0}(\theta))^\top$  for all  $\theta \in [t_0 - h, t_0]$  and the function  $\mathcal{L}(\cdot) = (\mathcal{L}_1(\cdot), \mathcal{L}_2(\cdot))^\top$  will be designed later. Note that usually in practice, we take  $\hat{Z}(\theta) = 0$ .

*Remark 1.* Note that, by using classical predictor techniques, we can replace our observer (8) by the following observer:

$$\dot{\hat{Z}}(t) = A\hat{Z}(t) + BU(t) - \exp(Ah)\bar{\mathcal{L}}\left(y(t) - C \exp(-Ah)\hat{Z}(t) + C \int_{t-h}^t \exp[A(t-h-\theta)] BU(\theta) d\theta\right), \quad (9)$$

where the nonlinear function  $\bar{\mathcal{L}} = (\bar{\mathcal{L}}_1(\cdot), \bar{\mathcal{L}}_2(\cdot))^\top$  can be designed in the same way as in (8). Such classical predictor-based technique have the following drawbacks:

1. a predictor-based observer is a reduced-order observer (in the sense of<sup>33, Chapter 3, page 46.</sup>) because it does not estimate the sensor state  $y(t+x)$ ,  $x \in [0, h]$ , with  $h$  is the delay, whereas our observer (8) ensures finite-time estimation of the sensor state  $y(t+x)$ ,  $x \in [0, h]$ : this is an important disadvantage because it is well known that in contrast to full-order observers, reduced-order observers are overly sensitive to measurement noise.<sup>33, Chapter 3, page 46.</sup>
2. predictor-based observers do not have straightforward extension to systems with time-varying delays, state dependent delays, or distributed delays, whereas our formulation opens a path for extending the obtained results to this kind of systems and more general ones.
3. lastly, predictor-based observers do not provide a construction of the Lyapunov Krassovki functional, which allows us to asses directly the desired FTS property of the error system and to estimate the time of convergence (the settling time), whereas our observer helps in establishing an explicit Lyapunov-based stability for the error system.

Our goal is to prove that, for a well designed vector function  $\mathcal{L}$ , for a well designed vector function  $\mathcal{L}(\hat{Z}, \hat{u})$  for (8) converge in a finite-time to  $(Z, u)$  for (7). To achieve this, we consider the following error variables:

$$\begin{aligned}\tilde{Z} &= \exp(-Ah)(Z - \hat{Z}), \\ \tilde{u} &= u - \hat{u}.\end{aligned}\quad (10)$$

The error system is then given by

$$\begin{aligned}\dot{\tilde{Z}}(t) &= A\tilde{Z}(t) + \mathcal{L}(\tilde{u}(t, 0)), \\ \tilde{u}_t(t, x) &= \tilde{u}_x(t, x) + C \exp(Ax)\mathcal{L}(\tilde{u}(t, 0)), \\ \tilde{u}(t, h) &= C \exp(Ah)\tilde{Z}(t),\end{aligned}\quad (11)$$

with initial condition  $\tilde{Z}(\theta) = \tilde{Z}_0(\theta) = (\tilde{z}_{1,0}(\theta), \tilde{z}_{2,0}(\theta))^\top$  for all  $\theta \in [t_0 - h, t_0]$ .

Next, we consider the following transformation:

$$\tilde{\omega}(t, x) = \tilde{u}(t, x) - C \exp(Ax)\tilde{Z}(t), \quad (12)$$

from<sup>33, Chapter 3</sup> which transforms (11) into the following target system:

$$\begin{aligned}\dot{\tilde{Z}}(t) &= A\tilde{Z}(t) + \mathcal{L}(\tilde{\omega}(t, 0) + C\tilde{Z}(t)), \\ \tilde{\omega}_t(t, x) &= \tilde{\omega}_x(t, x), \\ \tilde{\omega}(t, h) &= 0,\end{aligned}\quad (13)$$

which is equivalent to the following system:

$$\begin{aligned}\dot{\tilde{z}}_1(t) &= a_1 \tilde{z}_1(t) + \tilde{z}_2(t) + \mathcal{L}_1(\tilde{\omega}(t, 0) + \tilde{z}_1(t)), \\ \dot{\tilde{z}}_2(t) &= a_0 \tilde{z}_1(t) + \mathcal{L}_2(\tilde{\omega}(t, 0) + \tilde{z}_1(t)), \\ \tilde{\omega}_t(t, x) &= \tilde{\omega}_x(t, x), \\ \tilde{\omega}(t, h) &= 0.\end{aligned}\tag{14}$$

Choosing  $\mathcal{L}_1$  and  $\mathcal{L}_2$  respectively as follows:

$$\mathcal{L}_1(\tilde{\omega}(t, 0) + \tilde{z}_1(t)) = -k_1 \{\tilde{\omega}(t, 0) + \tilde{z}_1(t)\}^{\alpha_1} - a_1(\tilde{\omega}(t, 0) + \tilde{z}_1(t)),\tag{15}$$

and

$$\mathcal{L}_2(\tilde{\omega}(t, 0) + \tilde{z}_1(t)) = -k_2 \{\tilde{\omega}(t, 0) + \tilde{z}_1(t)\}^{\alpha_2} - a_0(\tilde{\omega}(t, 0) + \tilde{z}_1(t)),\tag{16}$$

gives us

$$\begin{aligned}\dot{\tilde{z}}_1(t) &= \tilde{z}_2(t) - k_1 \{\tilde{\omega}(t, 0) + \tilde{z}_1(t)\}^{\alpha_1} - a_1 \tilde{\omega}(t, 0), \\ \dot{\tilde{z}}_2(t) &= -k_2 \{\tilde{\omega}(t, 0) + \tilde{z}_1(t)\}^{\alpha_2} - a_0 \tilde{\omega}(t, 0), \\ \tilde{\omega}_t(t, x) &= \tilde{\omega}_x(t, x), \\ \tilde{\omega}(t, h) &= 0,\end{aligned}\tag{17}$$

where  $k_1 > 0, k_2 > 0, \alpha_1 \in (\frac{1}{2}, 1)$ , and  $\alpha_2 \in (0, 1)$ . Note that  $\tilde{\omega}(t, 0) + \tilde{z}_1(t) = y(t)$  thus  $\mathcal{L}_1(\cdot)$  and  $\mathcal{L}_2(\cdot)$  are only functions of  $y(t)$ . In order to characterize transformation (12), let us calculate its time and spatial derivatives and the value of  $\tilde{\omega}(t, h)$ .

$$\begin{aligned}\tilde{\omega}_t(t, x) &= \tilde{u}_t(t, x) - C \exp(Ax) \tilde{Z}(t), \\ &= \tilde{u}_x(t, x) + C \exp(Ax) \mathcal{L}(\tilde{u}(t, 0)) - C \exp(Ax) A \tilde{Z}(t) - C \exp(Ax) \mathcal{L}(\tilde{u}(t, 0)), \\ &= \tilde{u}_x(t, x) - C \exp(Ax) A \tilde{Z}(t).\end{aligned}\tag{18}$$

On the other hand,

$$\begin{aligned}\tilde{\omega}_x(t, x) &= \tilde{u}_x(t, x) - C A \exp(Ax) \tilde{Z}(t), \\ &= \tilde{\omega}_t(t, x),\end{aligned}\tag{19}$$

where we have used the fact that  $A \exp(Ax) = \exp(Ax)A$ . Using  $\tilde{u}(t, h) = C \exp(Ah) \tilde{Z}(t)$  in (12) at  $x = h$ , we obtain

$$\begin{aligned}\tilde{\omega}(t, h) &= \tilde{u}(t, h) - C \exp(Ah) \tilde{Z}(t), \\ &= C \exp(Ah) \tilde{Z}(t) - C \exp(Ah) \tilde{Z}(t), \\ &= 0.\end{aligned}\tag{20}$$

## 4 | STABILITY ANALYSIS

In this section, we present and prove the main result of this paper, i.e., we prove that the error state of (11) goes to zero within a finite time. To prove this result we need to establish some intermediate results. First, we prove that the solutions of the target system are bounded for all  $t \in [t_0, t_0 + h]$ . Next, we estimate in two different ways the solution of the  $\tilde{Z}$ -dynamics of the target system at time  $t_0 + h$  by a function that depends on the initial condition of the target system. Then, we prove that these solutions converge to zero within a finite time which can be estimated.

**Lemma 2.** Let  $t_0 \geq 0, \tilde{\omega}_0 \in L^2$  and bounded, and  $\tilde{Z}_0 = (\tilde{z}_{1,0}, \tilde{z}_{2,0})^\top \in \mathbb{R}^2$ . Then for any  $t \in [t_0, t_0 + h]$  and any  $x \in [0, h]$ ,  $(\tilde{z}_1, \tilde{z}_2, \tilde{\omega})$  the solution of the target system (17) is bounded. Moreover, the solution of the  $\tilde{Z}$ -dynamics satisfies the following inequalities:

$$\begin{aligned}\sup_{t \in [t_0, t_0 + h]} |\tilde{z}_1(t)|^2 &\leq M_1 \|\tilde{\omega}_0\|_{L^2}^2 + M_2(\|\tilde{Z}_0\|), \\ \sup_{t \in [t_0, t_0 + h]} |\tilde{z}_2(t)|^2 &\leq N_1 \|\tilde{\omega}_0\|_{L^2}^2 + N_2 \sup_{t \in [t_0, t_0 + h]} |\tilde{z}_1(t)|^2 + N_3(\|\tilde{Z}_0\|),\end{aligned}\tag{21}$$

where

$$\begin{aligned}
M_1 &= \frac{4h [2h^2(k_2 + |a_0|)^2 + (k_1 + |a_1|)^2]}{1 - 4h^2\varepsilon^2 (4h^2k_2^2 + k_1^2)}, \\
M_2(\|\tilde{Z}_0\|) &= \frac{8h^2 \left[ 4h^2k_2^2 \left( 1 + \varepsilon^{-\frac{\alpha_2}{1-\alpha_2}} \right)^2 + k_1^2 \left( 1 + \varepsilon^{-\frac{\alpha_1}{1-\alpha_1}} \right)^2 \right] + 8 [4h^2 + 1] \|\tilde{Z}_0\|^2}{1 - 4h^2\varepsilon^2 (4h^2k_2^2 + k_1^2)}, \\
N_1 &= 2h(k_2 + |a_0|)^2, \\
N_2 &= 4h^2k_2^2\varepsilon^2, \\
N_3(\|\tilde{Z}_0\|) &= 8 \left[ h^2k_2^2 \left( 1 + \varepsilon^{-\frac{\alpha_2}{1-\alpha_2}} \right)^2 + \|\tilde{Z}_0\|^2 \right],
\end{aligned} \tag{22}$$

with  $\varepsilon \in (0, 1)$  is chosen small enough so that  $(1 - 4h^2\varepsilon^2 (4h^2k_2^2 + k_1^2))$  is positive.

*Proof.* Using the initial condition  $\tilde{\omega}(t_0, x) = \tilde{\omega}_0(x)$  and by the method of characteristics, we have that  $\tilde{\omega}(t, x) = \tilde{\omega}_0(x + t - t_0)$  for any  $t \leq t_0 + h - x$  and  $\tilde{\omega}(t, x) = 0$  for any  $t \geq t_0 + h - x$ . Thus  $\tilde{\omega}$  is bounded for all  $t \in [t_0, t_0 + h]$ .

Next, integrating the second equation of (17), gives us

$$\tilde{z}_2(t) = -k_2 \int_{t_0}^t \{ \tilde{\omega}(s, 0) + \tilde{z}_1(s) \}^{\alpha_2} ds - a_0 \int_{t_0}^t \tilde{\omega}(s, 0) ds + \tilde{z}_{2,0}. \tag{23}$$

Using the fact that  $t \leq t_0 + h$ , we get

$$|\tilde{z}_2(t)| \leq k_2 \int_{t_0}^{t_0+h} |\tilde{\omega}(s, 0) + \tilde{z}_1(s)|^{\alpha_2} ds + |a_0| \int_{t_0}^{t_0+h} |\tilde{\omega}(s, 0)| ds + |\tilde{z}_{2,0}|. \tag{24}$$

Using the inequality (5) and since  $\alpha_2 \in (0, 1)$ , we obtain

$$|\tilde{z}_2(t)| \leq k_2 \int_{t_0}^{t_0+h} |\tilde{\omega}(s, 0)|^{\alpha_2} ds + k_2 \int_{t_0}^{t_0+h} |\tilde{z}_1(s)|^{\alpha_2} ds + |a_0| \int_{t_0}^{t_0+h} |\tilde{\omega}(s, 0)| ds + |\tilde{z}_{2,0}|. \tag{25}$$

Now, employing the inequality (3) on the first term and the inequality (4) on the second term, we get

$$\begin{aligned}
|\tilde{z}_2(t)| &\leq \alpha_2 k_2 \int_{t_0}^{t_0+h} |\tilde{\omega}(s, 0)| ds + (1 - \alpha_2) h k_2 + |a_0| \int_{t_0}^{t_0+h} |\tilde{\omega}(s, 0)| ds + k_2 \varepsilon \int_{t_0}^{t_0+h} |\tilde{z}_1(s)| ds \\
&\quad + k_2 (1 - \alpha_2) h \left( \frac{\alpha_2}{\varepsilon} \right)^{\frac{\alpha_2}{1-\alpha_2}} + |\tilde{z}_{2,0}|, \\
&\leq k_2 \int_{t_0}^{t_0+h} |\tilde{\omega}(s, 0)| ds + h k_2 + |a_0| \int_{t_0}^{t_0+h} |\tilde{\omega}(s, 0)| ds + k_2 \varepsilon \int_{t_0}^{t_0+h} |\tilde{z}_1(s)| ds + k_2 h \varepsilon^{-\frac{\alpha_2}{1-\alpha_2}} + |\tilde{z}_{2,0}|, \\
&\leq (k_2 + |a_0|) \int_{t_0}^{t_0+h} |\tilde{\omega}(s, 0)| ds + k_2 \varepsilon \int_{t_0}^{t_0+h} |\tilde{z}_1(s)| ds + \left[ h k_2 \left( 1 + \varepsilon^{-\frac{\alpha_2}{1-\alpha_2}} \right) + |\tilde{z}_{2,0}| \right].
\end{aligned} \tag{26}$$

By squaring this inequality, we find

$$|\tilde{z}_2(t)|^2 \leq 2(k_2 + |a_0|)^2 \left( \int_{t_0}^{t_0+h} |\tilde{\omega}(s, 0)| ds \right)^2 + 4k_2^2 \varepsilon^2 \left( \int_{t_0}^{t_0+h} |\tilde{z}_1(s)| ds \right)^2 + 4 \left[ h k_2 \left( 1 + \varepsilon^{-\frac{\alpha_2}{1-\alpha_2}} \right) + |\tilde{z}_{2,0}| \right]^2. \tag{27}$$

Finally, using the Jensen's inequality on each integral and the Young's inequality on the last term, we get

$$|\tilde{z}_2(t)|^2 \leq 2h(k_2 + |a_0|)^2 \int_{t_0}^{t_0+h} |\tilde{\omega}(s, 0)|^2 ds + 4hk_2^2 \varepsilon^2 \int_{t_0}^{t_0+h} |\tilde{z}_1(s)|^2 ds + 8 \left[ h^2 k_2^2 \left( 1 + \varepsilon^{-\frac{\alpha_2}{1-\alpha_2}} \right)^2 + |\tilde{z}_{2,0}|^2 \right]. \quad (28)$$

Using the value of  $\tilde{\omega}(t, 0)$  in the previous inequality, gives us

$$\begin{aligned} |\tilde{z}_2(t)|^2 &\leq 2h(k_2 + |a_0|)^2 \int_{t_0}^{t_0+h} |\tilde{\omega}_0(s - t_0)|^2 ds + 4hk_2^2 \varepsilon^2 \int_{t_0}^{t_0+h} |\tilde{z}_1(s)|^2 ds + 8 \left[ h^2 k_2^2 \left( 1 + \varepsilon^{-\frac{\alpha_2}{1-\alpha_2}} \right)^2 + |\tilde{z}_{2,0}|^2 \right], \\ &\leq 2h(k_2 + |a_0|)^2 \int_0^h |\tilde{\omega}_0(x)|^2 dx + 4hk_2^2 \varepsilon^2 \int_{t_0}^{t_0+h} |\tilde{z}_1(s)|^2 ds + 8 \left[ h^2 k_2^2 \left( 1 + \varepsilon^{-\frac{\alpha_2}{1-\alpha_2}} \right)^2 + |\tilde{z}_{2,0}|^2 \right], \\ &\leq 2h(k_2 + |a_0|)^2 \|\tilde{\omega}_0\|_{L^2}^2 + 4hk_2^2 \varepsilon^2 \int_{t_0}^{t_0+h} |\tilde{z}_1(s)|^2 ds + 8 \left[ h^2 k_2^2 \left( 1 + \varepsilon^{-\frac{\alpha_2}{1-\alpha_2}} \right)^2 + |\tilde{z}_{2,0}|^2 \right], \end{aligned} \quad (29)$$

which implies that

$$|\tilde{z}_2(t)|^2 \leq 2h(k_2 + |a_0|)^2 \|\tilde{\omega}_0\|_{L^2}^2 + 4h^2 k_2^2 \varepsilon^2 \sup_{t \in [t_0, t_0+h]} |\tilde{z}_1(t)|^2 + 8 \left[ h^2 k_2^2 \left( 1 + \varepsilon^{-\frac{\alpha_2}{1-\alpha_2}} \right)^2 + \|\tilde{Z}_0\|^2 \right]. \quad (30)$$

Then,

$$\sup_{t \in [t_0, t_0+h]} |\tilde{z}_2(t)|^2 \leq N_1 \|\tilde{\omega}_0\|_{L^2}^2 + N_2 \sup_{t \in [t_0, t_0+h]} |\tilde{z}_1(t)|^2 + N_3 (\|Z_0\|), \quad (31)$$

where

$$\begin{aligned} N_1 &= 2h(k_2 + |a_0|)^2, \\ N_2 &= 4h^2 k_2^2 \varepsilon^2, \\ N_3(\|\tilde{Z}_0\|) &= 8 \left[ h^2 k_2^2 \left( 1 + \varepsilon^{-\frac{\alpha_2}{1-\alpha_2}} \right)^2 + \|\tilde{Z}_0\|^2 \right]. \end{aligned} \quad (32)$$

On the other hand, integrating the first equation of the target system (17), gives us

$$\tilde{z}_1(t) = \int_{t_0}^t \tilde{z}_2(s) ds - k_1 \int_{t_0}^t \{ \tilde{\omega}(s, 0) + \tilde{z}_1(s) \}^{\alpha_1} ds - a_1 \int_{t_0}^t \tilde{\omega}(s, 0) ds + \tilde{z}_{1,0}. \quad (33)$$

Using the fact that  $t \leq t_0 + h$ , we get

$$|\tilde{z}_1(t)| \leq \int_{t_0}^{t_0+h} |\tilde{z}_2(s)| ds + k_1 \int_{t_0}^{t_0+h} |\tilde{\omega}(s, 0) + \tilde{z}_1(s)|^{\alpha_1} ds + |a_1| \int_{t_0}^{t_0+h} |\tilde{\omega}(s, 0)| ds + |\tilde{z}_{1,0}|. \quad (34)$$

Using the inequality (5) with  $\alpha_1 \in (0, 1)$ , we obtain

$$|\tilde{z}_1(t)| \leq \int_{t_0}^{t_0+h} |\tilde{z}_2(s)| ds + k_1 \int_{t_0}^{t_0+h} |\tilde{\omega}(s, 0)|^{\alpha_1} ds + k_1 \int_{t_0}^{t_0+h} |\tilde{z}_1(s)|^{\alpha_1} ds + |a_1| \int_{t_0}^{t_0+h} |\tilde{\omega}(s, 0)| ds + |\tilde{z}_{1,0}|. \quad (35)$$

Now, employing the inequality (3) on the second term and the inequality (4) on the third term of the previous inequality, we get

$$\begin{aligned}
|\tilde{z}_1(t)| &\leq \int_{t_0}^{t_0+h} |\tilde{z}_2(s)| ds + \alpha_1 k_1 \int_{t_0}^{t_0+h} |\tilde{\omega}(s, 0)| ds + h k_1 (1 - \alpha_1) + k_1 \varepsilon \int_{t_0}^{t_0+h} |\tilde{z}_1(s)| ds \\
&\quad + h k_1 (1 - \alpha_1) \left( \frac{\alpha_1}{\varepsilon} \right)^{\frac{\alpha_1}{1-\alpha_1}} + |a_1| \int_{t_0}^{t_0+h} |\tilde{\omega}(s, 0)| ds + |\tilde{z}_{1,0}|, \\
&\leq \int_{t_0}^{t_0+h} |\tilde{z}_2(s)| ds + k_1 \int_{t_0}^{t_0+h} |\tilde{\omega}(s, 0)| ds + h k_1 + k_1 \varepsilon \int_{t_0}^{t_0+h} |\tilde{z}_1(s)| ds \\
&\quad + h k_1 \varepsilon^{-\frac{\alpha_1}{1-\alpha_1}} + |a_1| \int_{t_0}^{t_0+h} |\tilde{\omega}(s, 0)| ds + |\tilde{z}_{1,0}|, \\
&\leq \int_{t_0}^{t_0+h} |\tilde{z}_2(s)| ds + (k_1 + |a_1|) \int_{t_0}^{t_0+h} |\tilde{\omega}(s, 0)| ds + k_1 \varepsilon \int_{t_0}^{t_0+h} |\tilde{z}_1(s)| ds \\
&\quad + \left[ h k_1 \left( 1 + \varepsilon^{-\frac{\alpha_1}{1-\alpha_1}} \right) + |\tilde{z}_{1,0}| \right].
\end{aligned} \tag{36}$$

Squaring this last inequality, gives us

$$\begin{aligned}
|\tilde{z}_1(t)|^2 &\leq 4 \left( \int_{t_0}^{t_0+h} |\tilde{z}_2(s)| ds \right)^2 + 4(k_1 + |a_1|)^2 \left( \int_{t_0}^{t_0+h} |\tilde{\omega}(s, 0)| ds \right)^2 + 4k_1^2 \varepsilon^2 \left( \int_{t_0}^{t_0+h} |\tilde{z}_1(s)| ds \right)^2 \\
&\quad + 4 \left[ h k_1 \left( 1 + \varepsilon^{-\frac{\alpha_1}{1-\alpha_1}} \right) + |\tilde{z}_{1,0}| \right]^2.
\end{aligned} \tag{37}$$

Applying the Jensen's inequality on each integral and the Young's inequality on the last term, we obtain

$$\begin{aligned}
|\tilde{z}_1(t)|^2 &\leq 4h \int_{t_0}^{t_0+h} |\tilde{z}_2(s)|^2 ds + 4h(k_1 + |a_1|)^2 \int_{t_0}^{t_0+h} |\tilde{\omega}(s, 0)|^2 ds + 4h k_1^2 \varepsilon^2 \int_{t_0}^{t_0+h} |\tilde{z}_1(s)|^2 ds \\
&\quad + 8 \left[ h^2 k_1^2 \left( 1 + \varepsilon^{-\frac{\alpha_1}{1-\alpha_1}} \right)^2 + |\tilde{z}_{1,0}|^2 \right].
\end{aligned} \tag{38}$$

Using the inequality (28) on the last inequality, gives us

$$\begin{aligned}
|\tilde{z}_1(t)|^2 &\leq 4h^2 \left[ 2h(k_2 + |a_0|)^2 \int_{t_0}^{t_0+h} |\tilde{\omega}(s, 0)|^2 ds + 4h k_2^2 \varepsilon^2 \int_{t_0}^{t_0+h} |\tilde{z}_1(s)|^2 ds \right. \\
&\quad \left. + 8 \left[ h^2 k_2^2 \left( 1 + \varepsilon^{-\frac{\alpha_2}{1-\alpha_2}} \right)^2 + \|\tilde{Z}_0\|^2 \right] \right] + 4h(k_1 + |a_1|)^2 \int_{t_0}^{t_0+h} |\tilde{\omega}(s, 0)|^2 ds \\
&\quad + 4h k_1^2 \varepsilon^2 \int_{t_0}^{t_0+h} |\tilde{z}_1(s)|^2 ds + 8 \left[ h^2 k_1^2 \left( 1 + \varepsilon^{-\frac{\alpha_1}{1-\alpha_1}} \right)^2 + \|\tilde{Z}_0\|^2 \right].
\end{aligned} \tag{39}$$

By developing this last inequality and the value of  $\tilde{\omega}(t, 0)$ , we get for all  $t \in [t_0, t_0 + h]$

$$\begin{aligned}
|\tilde{z}_1(t)|^2 &\leq 4h\varepsilon^2 (4h^2k_2^2 + k_1^2) \int_{t_0}^{t_0+h} |\tilde{z}_1(s)|^2 ds + 4h (2h^2(k_2 + |a_0|)^2 + (k_1 + |a_1|)^2) \int_{t_0}^{t_0+h} |\tilde{\omega}(s, 0)|^2 ds \\
&\quad + 8h^2 \left[ 4h^2k_2^2 \left( 1 + \varepsilon^{-\frac{\alpha_2}{1-\alpha_2}} \right)^2 + k_1^2 \left( 1 + \varepsilon^{-\frac{\alpha_1}{1-\alpha_1}} \right)^2 \right] + 8 [4h^2 + 1] \|\tilde{Z}_0\|^2, \\
&\leq 4h\varepsilon^2 (4h^2k_2^2 + k_1^2) \int_{t_0}^{t_0+h} |\tilde{z}_1(s)|^2 ds + 4h (2h^2(k_2 + |a_0|)^2 + (k_1 + |a_1|)^2) \int_{t_0}^{t_0+h} |\tilde{\omega}_0(s - t_0)|^2 ds \\
&\quad + 8h^2 \left[ 4h^2k_2^2 \left( 1 + \varepsilon^{-\frac{\alpha_2}{1-\alpha_2}} \right)^2 + k_1^2 \left( 1 + \varepsilon^{-\frac{\alpha_1}{1-\alpha_1}} \right)^2 \right] + 8 [4h^2 + 1] \|\tilde{Z}_0\|^2, \\
&\leq 4h\varepsilon^2 (4h^2k_2^2 + |k_1|^2) \int_{t_0}^{t_0+h} |\tilde{z}_1(s)|^2 ds + 4h (2h^2(k_2 + |a_0|)^2 + (k_1 + |a_1|)^2) \int_0^h |\tilde{\omega}_0(x)|^2 dx \\
&\quad + 8h^2 \left[ 4h^2k_2^2 \left( 1 + \varepsilon^{-\frac{\alpha_2}{1-\alpha_2}} \right)^2 + k_1^2 \left( 1 + \varepsilon^{-\frac{\alpha_1}{1-\alpha_1}} \right)^2 \right] + 8 [4h^2 + 1] \|\tilde{Z}_0\|^2, \\
&\leq 4h^2\varepsilon^2 (4h^2k_2^2 + k_1^2) \sup_{t \in [t_0, t_0+h]} |\tilde{z}_1(t)|^2 + 4h (2h^2(k_2 + |a_0|)^2 + (k_1 + |a_1|)^2) \|\tilde{\omega}_0\|_{L^2}^2 \\
&\quad + 8h^2 \left[ 4h^2k_2^2 \left( 1 + \varepsilon^{-\frac{\alpha_2}{1-\alpha_2}} \right)^2 + k_1^2 \left( 1 + \varepsilon^{-\frac{\alpha_1}{1-\alpha_1}} \right)^2 \right] + 8 [4h^2 + 1] \|\tilde{Z}_0\|^2,
\end{aligned} \tag{40}$$

which implies that

$$\begin{aligned}
\sup_{t \in [t_0, t_0+h]} |\tilde{z}_1(t)|^2 &\leq 4h^2\varepsilon^2 (4h^2k_2^2 + k_1^2) \sup_{t \in [t_0, t_0+h]} |\tilde{z}_1(t)|^2 + 4h (2h^2(k_2 + |a_0|)^2 + (k_1 + |a_1|)^2) \|\tilde{\omega}_0\|_{L^2}^2 \\
&\quad + 8h^2 \left[ 4h^2k_2^2 \left( 1 + \varepsilon^{-\frac{\alpha_2}{1-\alpha_2}} \right)^2 + k_1^2 \left( 1 + \varepsilon^{-\frac{\alpha_1}{1-\alpha_1}} \right)^2 \right] + 8 [4h^2 + 1] \|\tilde{Z}_0\|^2.
\end{aligned} \tag{41}$$

Then :

$$\begin{aligned}
(1 - 4h^2\varepsilon^2 (4h^2k_2^2 + k_1^2)) \sup_{t \in [t_0, t_0+h]} |\tilde{z}_1(t)|^2 &\leq 4h (2h^2(k_2 + |a_0|)^2 + (k_1 + |a_1|)^2) \|\tilde{\omega}_0\|_{L^2}^2 \\
&\quad + 8h^2 \left[ 4h^2k_2^2 \left( 1 + \varepsilon^{-\frac{\alpha_2}{1-\alpha_2}} \right)^2 + k_1^2 \left( 1 + \varepsilon^{-\frac{\alpha_1}{1-\alpha_1}} \right)^2 \right] \\
&\quad + 8 [4h^2 + 1] \|\tilde{Z}_0\|^2.
\end{aligned} \tag{42}$$

Finally, choosing  $\varepsilon$  small enough so that  $(1 - 4h^2\varepsilon^2 (4h^2k_2^2 + k_1^2))$  is positive, (for instance, we can take  $\varepsilon = [8h^2(4h^2k_2^2 + k_1^2)]^{-\frac{1}{2}}$ ), gives us

$$\sup_{t \in [t_0, t_0+h]} |\tilde{z}_1(t)|^2 \leq M_1 \sup_{t \in [t_0, t_0+h]} |\tilde{\omega}(t, 0)|^2 + M_2(\|\tilde{Z}_0\|), \tag{43}$$

where

$$\begin{aligned}
M_1 &= \frac{4h [2h^2(k_2 + |a_0|)^2 + (k_1 + |a_1|)^2]}{1 - 4h^2\varepsilon^2 (4h^2k_2^2 + k_1^2)}, \\
M_2(\|\tilde{Z}_0\|) &= \frac{8h^2 \left[ 4h^2k_2^2 \left( 1 + \varepsilon^{-\frac{\alpha_2}{1-\alpha_2}} \right)^2 + k_1^2 \left( 1 + \varepsilon^{-\frac{\alpha_1}{1-\alpha_1}} \right)^2 \right] + 8 [4h^2 + 1] \|\tilde{Z}_0\|^2}{1 - 4h^2\varepsilon^2 (4h^2k_2^2 + k_1^2)}.
\end{aligned} \tag{44}$$

Looking at (31) and (43), we can clearly see that the solution of the  $\tilde{Z}$ -dynamics is also bounded for all  $t \in [t_0, t_0 + h]$ .  $\square$

**Lemma 3.** Let  $\tilde{\omega}_0 \in L^2$  and bounded, and  $\tilde{Z}_0, Z_0 \in \mathbb{R}^2$ . For any  $t_0, h > 0$ , we have

$$\|\tilde{Z}(t_0 + h)\|^2 \leq G_1(\|\tilde{\omega}_0\|_{L^2}, \|Z_0 - \tilde{Z}_0\|), \tag{45}$$

where  $G_1$  is the function given by

$$G_1(s_1, s_2) = (N_1 + N_2 M_1) s_1^2 + [N_2 M_2 (\|\exp(-Ah)\|_{s_2}) + N_3 (\|\exp(-Ah)\|_{s_2})], \quad (46)$$

where  $M_1, M_2, N_1, N_2,$  and  $N_3$  are given in (22).

*Proof.* Using Lemma 2, we can clearly see that

$$\|\tilde{Z}(t)\|^2 \leq (N_1 + N_2 M_1) \|\tilde{\omega}_0\|_{L^2}^2 + [N_2 M_2 (\|\tilde{Z}_0\|) + N_3 (\|\tilde{Z}_0\|)] \text{ for all } t \in [t_0, t_0 + h], \quad (47)$$

where  $M_1, M_2, N_1, N_2,$  and  $N_3$  are given in (22).

Using (10), we obtain

$$\|\tilde{Z}(t)\|^2 \leq (N_1 + N_2 M_1) \|\tilde{\omega}_0\|_{L^2}^2 + [N_2 M_2 (\|\exp(-Ah)\| \|Z_0 - \hat{Z}_0\|) + N_3 (\|\exp(-Ah)\| \|Z_0 - \hat{Z}_0\|)]. \quad (48)$$

In particular for  $t = t_0 + h$ , we have

$$\|\tilde{Z}(t_0 + h)\|^2 \leq G_1 (\|\tilde{\omega}_0\|_{L^2}, \|Z_0 - \hat{Z}_0\|). \quad (49)$$

where  $G_1$  is the function given by (46). □

**Lemma 4.** For any  $t_0, h > 0$  and any  $\hat{Z}_0$  in  $\mathbb{R}^2$ , we have

$$\|\tilde{Z}(t_0 + h)\|^2 \leq G_2 (\|Z_0 - \hat{Z}_0\|), \quad (50)$$

where  $G_2$  is a function defined for any  $s \geq 0$  by

$$G_2(s) = 4 \exp [2h(|a_0| + |a_1| + 1)] [\|\exp(-Ah)\|^2 s^2 + h^2 (|a_0| + |a_1|)^2 \tilde{M}^2 + 2h^2 (k_1 + k_2)^2 (\tilde{M}^{2\alpha_1} + \tilde{M}^{2\alpha_2})], \quad (51)$$

with  $\tilde{M}$  is such that

$$\sup_{t \in [t_0, t_0 + h]} |\tilde{u}(t, 0)| \leq \tilde{M}. \quad (52)$$

*Proof.* By integrating the second equation of (17), we obtain

$$\tilde{z}_2(t) = -k_2 \int_{t_0}^t \{\tilde{\omega}(s, 0) + \tilde{z}_1(s)\}^{\alpha_2} ds - a_0 \int_{t_0}^t \tilde{\omega}(s, 0) ds + \tilde{z}_{2,0}. \quad (53)$$

Using the transformation (12) at  $x = 0$ , we get

$$|\tilde{z}_2(t)| \leq k_2 \int_{t_0}^t |\tilde{u}(s, 0)|^{\alpha_2} ds + |a_0| \int_{t_0}^t |\tilde{u}(s, 0)| ds + |a_0| \int_{t_0}^t |\tilde{z}_1(s)| ds + |\tilde{z}_{2,0}|. \quad (54)$$

On the other hand, integrating the first equation of (17), gives us

$$\tilde{z}_1(t) = \int_{t_0}^t \tilde{z}_2(s) ds - k_1 \int_{t_0}^t \{\tilde{\omega}(s, 0) + \tilde{z}_1(s)\}^{\alpha_1} ds - a_1 \int_{t_0}^t \tilde{\omega}(s, 0) ds + \tilde{z}_{1,0}. \quad (55)$$

Using the transformation (12) at  $x = 0$ , we get

$$|\tilde{z}_1(t)| \leq \int_{t_0}^t |\tilde{z}_2(s)| ds + k_1 \int_{t_0}^t |\tilde{u}(s, 0)|^{\alpha_1} ds + |a_1| \int_{t_0}^t |\tilde{u}(s, 0)| ds + |a_1| \int_{t_0}^t |\tilde{z}_1(s)| ds + |\tilde{z}_{1,0}|. \quad (56)$$

Summing the two inequalities (54) and (56), we obtain

$$\begin{aligned} |\tilde{z}_1(t)| + |\tilde{z}_2(t)| &\leq (|a_0| + |a_1| + 1) \int_{t_0}^t (|\tilde{z}_1(s)| + |\tilde{z}_2(s)|) ds + (|\tilde{z}_{1,0}| + |\tilde{z}_{2,0}|) \\ &\quad + (k_1 + k_2) \int_{t_0}^{t_0+h} |\tilde{u}(s, 0)|^{\alpha_2} + |\tilde{u}(s, 0)|^{\alpha_1} ds + (|a_0| + |a_1|) \int_{t_0}^{t_0+h} |\tilde{u}(s, 0)| ds. \end{aligned} \quad (57)$$

Using the Grönwall's Lemma, we find

$$\begin{aligned}
|\tilde{z}_1(t)| + |\tilde{z}_2(t)| &\leq \exp \left[ (|a_0| + |a_1| + 1)(t - t_0) \right] \left[ (|\tilde{z}_{1,0}| + |\tilde{z}_{2,0}|) + (k_1 + k_2) \int_{t_0}^{t_0+h} |\tilde{u}(s, 0)|^{\alpha_2} + |\tilde{u}(s, 0)|^{\alpha_1} ds \right. \\
&\quad \left. + (|a_0| + |a_1|) \int_{t_0}^{t_0+h} |\tilde{u}(s, 0)| ds \right], \\
&\leq \exp \left[ (|a_0| + |a_1| + 1)h \right] \left[ (|\tilde{z}_{1,0}| + |\tilde{z}_{2,0}|) + (k_1 + k_2) \int_{t_0}^{t_0+h} |\tilde{u}(s, 0)|^{\alpha_2} + |\tilde{u}(s, 0)|^{\alpha_1} ds \right. \\
&\quad \left. + (|a_0| + |a_1|) \int_{t_0}^{t_0+h} |\tilde{u}(s, 0)| ds \right].
\end{aligned} \tag{58}$$

Squaring this inequality, gives us

$$\begin{aligned}
\|\tilde{Z}(t)\|^2 &\leq (|\tilde{z}_1(t)| + |\tilde{z}_2(t)|)^2 \leq \exp \left[ 2h(|a_0| + |a_1| + 1) \right] \left[ 2(|\tilde{z}_{1,0}| + |\tilde{z}_{2,0}|)^2 + 4(|a_0| + |a_1|)^2 \left( \int_{t_0}^{t_0+h} |\tilde{u}(s, 0)| ds \right)^2 \right. \\
&\quad \left. + 4(k_1 + k_2)^2 \left( \int_{t_0}^{t_0+h} |\tilde{u}(s, 0)|^{\alpha_2} + |\tilde{u}(s, 0)|^{\alpha_1} ds \right)^2 \right].
\end{aligned} \tag{59}$$

Using the Jensen's inequality and the Young's inequality, we find

$$\begin{aligned}
\|\tilde{Z}(t)\|^2 &\leq (|\tilde{z}_1(t)| + |\tilde{z}_2(t)|)^2 \leq \exp \left[ 2h(|a_0| + |a_1| + 1) \right] \left[ 4\|\tilde{Z}_0\|^2 + 4h(|a_0| + |a_1|)^2 \int_{t_0}^{t_0+h} |\tilde{u}(s, 0)|^2 ds \right. \\
&\quad \left. + 8h(k_1 + k_2)^2 \int_{t_0}^{t_0+h} |\tilde{u}(s, 0)|^{2\alpha_2} + |\tilde{u}(s, 0)|^{2\alpha_1} ds \right], \\
&\leq \exp \left[ 2h(|a_0| + |a_1| + 1) \right] \left[ 4\|\tilde{Z}_0\|^2 + 4h^2(|a_0| + |a_1|)^2 \sup_{t \in [t_0, t_0+h]} |\tilde{u}(t, 0)|^2 \right. \\
&\quad \left. + 8h^2(k_1 + k_2)^2 \sup_{t \in [t_0, t_0+h]} |\tilde{u}(t, 0)|^{2\alpha_2} + 8h^2(k_1 + k_2)^2 \sup_{t \in [t_0, t_0+h]} |\tilde{u}(t, 0)|^{2\alpha_1} \right].
\end{aligned} \tag{60}$$

Using the transformation (12) and Lemma 2, we have  $\tilde{u}$  is bounded for all  $t \in [t_0, t_0 + h]$ , then, it exists a positive constant  $\tilde{M}$  such that

$$\sup_{t \in [t_0, t_0+h]} |\tilde{u}(t, 0)| \leq \tilde{M}. \tag{61}$$

Using this property in the previous inequality, we get

$$\|\tilde{Z}(t)\|^2 \leq 4 \exp \left[ 2h(|a_0| + |a_1| + 1) \right] \left[ \|\tilde{Z}_0\|^2 + h^2(|a_0| + |a_1|)^2 \tilde{M}^2 + 2h^2(k_1 + k_2)^2 (\tilde{M}^{2\alpha_1} + \tilde{M}^{2\alpha_2}) \right]. \tag{62}$$

This inequality is true for all  $t \in [t_0, t_0 + h]$ , in particular for  $t = t_0 + h$ , we have :

$$\|\tilde{Z}(t_0 + h)\|^2 \leq 4 \exp \left[ 2h(|a_0| + |a_1| + 1) \right] \left[ \|\tilde{Z}_0\|^2 + h^2(|a_0| + |a_1|)^2 \tilde{M}^2 + 2h^2(k_1 + k_2)^2 (\tilde{M}^{2\alpha_1} + \tilde{M}^{2\alpha_2}) \right]. \tag{63}$$

Employing (10), we find

$$\|\tilde{Z}(t_0 + h)\|^2 \leq G_2(\|Z_0 - \hat{Z}_0\|). \tag{64}$$

where  $G_2$  is the function defined by (51). □

**Lemma 5.** For any  $t_0, h \geq 0$ , the target system (17) is FTS. Moreover, there exist positive scalar parameters  $\delta$ ,  $P_{min}$  and  $P_{max}$  such that for  $t \geq t_0 + h$

$$\|\tilde{Z}(t)\|^2 \leq F(t - t_0 - h, \|\tilde{Z}(t_0 + h)\|) + F(t - t_0 - h, \|\tilde{Z}(t_0 + h)\|)^{\frac{2}{1+\alpha}}, \quad (65)$$

where  $F$  is an increasing function with respect to the second variable defined for any  $s \geq 0$  by

$$F(t - t_0 - h, s) = \frac{1}{P_{min}} \left[ -\delta \left( 1 - \frac{v''}{v'} \right) (t - t_0 - h) + (P_{max})^{1 - \frac{v''}{v'}} (s^{\alpha+1} + s^2)^{1 - \frac{v''}{v'}} \right]^{\frac{v'}{v' - v''}}, \quad (66)$$

with  $\alpha \in (0, 1)$ ,  $v' = \alpha + 1$ ,  $v'' = \frac{3\alpha+1}{2}$ ,

In particular, when  $t$  tends to the settling time  $T_{max}(\|\tilde{Z}(t_0 + h)\|) + t_0 + h$ , the function  $F$  and the norm  $\|\tilde{Z}\|^2$  goes to zero, where  $T_{max}$  is given for any  $s \geq 0$  by

$$T_{max}(s) = \frac{v'}{\delta(v' - v'')} (P_{max})^{1 - \frac{v''}{v'}} (s^{\alpha+1} + s^2)^{1 - \frac{v''}{v'}}. \quad (67)$$

*Proof.* By the method of the characteristics, the solution of the  $\tilde{\omega}$ -dynamics in (17) for  $t \geq t_0 + h$  is equal to zero, then  $\tilde{\omega}(t, 0) = 0$  for all  $t \geq t_0 + h$ . Hence, the target system (17) becomes

$$\begin{aligned} \dot{\tilde{z}}_1(t) &= \tilde{z}_2(t) - k_1 \{\tilde{z}_1(t)\}^{\alpha_1}, \\ \dot{\tilde{z}}_2(t) &= -k_2 \{\tilde{z}_1(t)\}^{\alpha_2}. \end{aligned} \quad (68)$$

To prove that this system is FTS, we choose  $\alpha_1$  and  $\alpha_2$  so that the system (68) is  $r$ -homogeneous of degree  $\nu$ , where  $r = (r_1, r_2) \in \mathbb{R}$  and  $\nu < 0$  satisfy

$$\begin{aligned} r_1 + \nu &= r_2 = r_1 \alpha_1, \\ r_2 + \nu &= r_1 \alpha_2. \end{aligned} \quad (69)$$

Let  $\alpha_2 = \alpha \in (0, 1)$  and  $r_1 = 1$ . Then we find

$$\begin{aligned} \nu &= \frac{(\alpha - 1)}{2} < 0, \\ \alpha_1 = r_2 &= \frac{(\alpha + 1)}{2} \in \left( \frac{1}{2}, 1 \right). \end{aligned} \quad (70)$$

Next, we will prove that the system is asymptotically stable. In order to do that, let  $\varphi$  be a function defined by

$$\varphi(\tilde{z}_1, \tilde{z}_2) = \left( \{\tilde{z}_1\}^{\frac{\alpha+1}{2}}, \tilde{z}_2 \right). \quad (71)$$

We can clearly see that  $\varphi$  is  $\left( 1, \frac{\alpha+1}{2} \right)$ -homogeneous of degree  $\mu = \frac{\alpha+1}{2}$ .

Now, let  $V$  be a Lyapunov candidate function defined by (as in<sup>37</sup>)

$$V(\tilde{z}_1, \tilde{z}_2) = \frac{1}{2} \varphi(\tilde{z}_1, \tilde{z}_2) P \varphi(\tilde{z}_1, \tilde{z}_2)^\top, \quad (72)$$

with  $P = \begin{pmatrix} a & c \\ c & b \end{pmatrix}$ , where for any  $k_1, k_2 > 0$ , the parameters  $a$ ,  $b$ , and  $c$  are given as follows:

$$\begin{aligned} a &= \frac{(-2ck_2 + 1)}{(\alpha + 1)k_1} > 0, \\ b &= \frac{(\alpha + 1)(a - ck_1)}{2k_2} > 0, \\ c &= -1 < 0. \end{aligned} \quad (73)$$

The function  $V$  is  $\left(1, \frac{\alpha+1}{2}\right)$ -homogeneous of degree  $v' = \alpha + 1$  and definite positive because the matrix  $P$  is also positive definite. In fact, the principal minors of  $P$  are positive

$$\begin{aligned} a &= \frac{(-2ck_2 + 1)}{(\alpha + 1)k_1} > 0, \\ ab - c^2 &= \frac{(a - ck_1)(-2ck_2 + 1) - 2k_1k_2c^2}{2k_1k_2}, \\ &= \frac{(a + k_1)(1 + 2k_2) - 2k_1k_2}{2k_1k_2}, \\ &= \frac{a + k_1 + 2ak_2 + 2k_1k_2 - 2k_1k_2}{2k_1k_2}, \\ &= \frac{a + k_1 + 2ak_2}{2k_1k_2} > 0. \end{aligned} \tag{74}$$

On the other hand, the time derivative of  $V$  is given for all  $t \geq t_0 + h$  as follows:

$$\begin{aligned} \dot{V}(\tilde{z}_1, \tilde{z}_2) &= \left( \left\{ \tilde{z}_1 \right\}^{\frac{\alpha+1}{2}}, \tilde{z}_2 \right) \begin{pmatrix} a & c \\ c & b \end{pmatrix} \begin{pmatrix} \frac{\alpha+1}{2} \dot{\tilde{z}}_1 |\tilde{z}_1|^{\frac{\alpha-1}{2}} \\ \dot{\tilde{z}}_2 \end{pmatrix}, \\ &= \left( a \left\{ \tilde{z}_1 \right\}^{\frac{\alpha+1}{2}} + c\tilde{z}_2, \quad c \left\{ \tilde{z}_1 \right\}^{\frac{\alpha+1}{2}} + b\tilde{z}_2 \right) \begin{pmatrix} \frac{\alpha+1}{2} |\tilde{z}_1|^{\frac{\alpha-1}{2}} (\tilde{z}_2 - k_1 \left\{ \tilde{z}_1 \right\}^{\frac{\alpha+1}{2}}) \\ -k_2 \left\{ \tilde{z}_1 \right\}^\alpha \end{pmatrix}, \\ &= \frac{1}{2} [-2bk_2 + (\alpha + 1)(a - ck_1)] \left\{ \tilde{z}_1 \right\}^\alpha \tilde{z}_2 + \frac{1}{2} [-ak_1(\alpha + 1) - 2ck_2] |\tilde{z}_1|^{\frac{3\alpha+1}{2}} + \frac{1}{2} c(\alpha + 1) |\tilde{z}_1|^{\frac{\alpha-1}{2}} \tilde{z}_2^2, \\ &= \frac{1}{2} |\tilde{z}_1|^{\frac{\alpha-1}{2}} \tilde{z}_2^2 \left( [-2bk_2 + (\alpha + 1)(a - ck_1)] \frac{\left\{ \tilde{z}_1 \right\}^{\frac{\alpha+1}{2}}}{\tilde{z}_2} + [-ak_1(\alpha + 1) - 2ck_2] \left( \frac{\left\{ \tilde{z}_1 \right\}^{\frac{\alpha+1}{2}}}{\tilde{z}_2} \right)^2 + c(\alpha + 1) \right), \\ &= \frac{1}{2} |\tilde{z}_1|^{\frac{\alpha-1}{2}} \tilde{z}_2^2 ([-2bk_2 + (\alpha + 1)(a - ck_1)] E + [-ak_1(\alpha + 1) - 2ck_2] E^2 + c(\alpha + 1)), \\ &= \frac{1}{2} |\tilde{z}_1|^{\frac{3\alpha+1}{2}} E^{-2} ([-2bk_2 + (\alpha + 1)(a - ck_1)] E + [-ak_1(\alpha + 1) - 2ck_2] E^2 + c(\alpha + 1)), \end{aligned} \tag{75}$$

where  $E = \frac{\left\{ \tilde{z}_1 \right\}^{\frac{\alpha+1}{2}}}{\tilde{z}_2}$ . We can clearly see that  $\dot{V}$  is  $\left(1, \frac{\alpha+1}{2}\right)$ -homogeneous of degree  $v'' = \frac{3\alpha+1}{2}$ .

Let  $Q$  be the polynomial function given by

$$Q(E) = \bar{a}E^2 + \bar{b}E + \bar{c} = \bar{a} \left[ \left( E + \frac{\bar{b}}{2\bar{a}} \right)^2 - \frac{\bar{b}^2 - 4\bar{a}\bar{c}}{4\bar{a}^2} \right], \tag{76}$$

where  $\bar{a}$ ,  $\bar{b}$ , and  $\bar{c}$  are given as follows:

$$\begin{aligned} \bar{a} &= [-ak_1(\alpha + 1) - 2ck_2], \\ \bar{b} &= [-2bk_2 + (\alpha + 1)(a - ck_1)], \\ \bar{c} &= c(\alpha + 1). \end{aligned} \tag{77}$$

Our goal is to prove that  $\dot{V}$  is definite negative, which is equivalent to proving that  $Q$  is negative, or more precisely it is equivalent to prove that  $\bar{a}$  and  $\bar{b}^2 - 4\bar{a}\bar{c}$  are negative. In fact using the value of  $a$ ,  $b$ , and  $c$  from (73), we obtain

$$\begin{aligned} \bar{a} &= -ak_1(\alpha + 1) - 2ck_2, \\ &= \frac{-2k_2 - 1}{k_1(\alpha + 1)} k_1(\alpha + 1) + 2k_2, \\ &= -2k_2 - 1 + 2k_2, \\ &= -1 < 0, \end{aligned} \tag{78}$$

and

$$\begin{aligned} \bar{b}^2 - 4\bar{a}\bar{c} &= [-2bk_2 + (\alpha + 1)(a - ck_1)]^2 - 4c(\alpha + 1)[-ak_1(\alpha + 1) - 2ck_2], \\ &= \left[ -2k_2 \frac{(\alpha + 1)(a - ck_1)}{2k_2} + (\alpha + 1)(a - ck_1) \right]^2 - 4(\alpha + 1), \\ &= -4(\alpha + 1) < 0. \end{aligned} \quad (79)$$

Thus, we proved that  $\dot{V}$  is definite negative and that the system (68) is asymptotically stable. Combining this result with the homogeneity of the system (68) with degree  $\nu < 0$ , implies that the system (68) is FTS.

Now, let  $\mathcal{W}$  be a compact defined as follows: (see e.g.<sup>34</sup>)

$$\mathcal{W} := \{(\tilde{z}_1, \tilde{z}_2) \in \mathbb{R}^2 : V(\tilde{z}_1, \tilde{z}_2) = 1\}, \quad (80)$$

and set the constant  $\delta$  as follows:

$$\delta = - \max_{(\tilde{z}_1, \tilde{z}_2) \in \mathcal{W}} \{\dot{V}(\tilde{z}_1, \tilde{z}_2)\} = \min_{(\tilde{z}_1, \tilde{z}_2) \in \mathcal{W}} \{-\dot{V}(\tilde{z}_1, \tilde{z}_2)\} > 0. \quad (81)$$

Using the homogeneity of  $\dot{V}$  for  $\lambda = \left( V(\tilde{z}_1, \tilde{z}_2)^{\frac{-1}{\nu'}} \right)$ , we have

$$\dot{V}(\lambda^{r_1} \tilde{z}_1, \lambda^{r_2} \tilde{z}_2) = \frac{\dot{V}(\tilde{z}_1, \tilde{z}_2)}{V(\tilde{z}_1, \tilde{z}_2)^{\frac{\nu''}{\nu'}}}. \quad (82)$$

On the other hand, we have

$$\{(\lambda^{r_1} \tilde{z}_1, \lambda^{r_2} \tilde{z}_2) : (\tilde{z}_1, \tilde{z}_2) \in \mathbb{R}^2 \setminus \{(0, 0)\}\} = \mathcal{W}, \quad (83)$$

because

$$\begin{aligned} V(\lambda^{r_1} \tilde{z}_1, \lambda^{r_2} \tilde{z}_2) &= \left( V(\tilde{z}_1, \tilde{z}_2)^{\frac{-1}{\nu'}} \right)^{\nu'} V(\tilde{z}_1, \tilde{z}_2), \\ &= 1. \end{aligned} \quad (84)$$

Therefore,

$$\begin{aligned} \sup_{(\tilde{z}_1, \tilde{z}_2) \neq (0,0)} \frac{\dot{V}(\tilde{z}_1, \tilde{z}_2)}{[V(\tilde{z}_1, \tilde{z}_2)]^{\frac{\nu''}{\nu'}}} &= \sup_{(\tilde{z}_1, \tilde{z}_2) \neq (0,0)} \dot{V}(\lambda^{r_1} \tilde{z}_1, \lambda^{r_2} \tilde{z}_2), \\ &= \sup_{(\tilde{z}_1, \tilde{z}_2) \in \mathcal{W}} \dot{V}(\tilde{z}_1, \tilde{z}_2), \\ &= -\delta. \end{aligned} \quad (85)$$

Hence it follows that:

$$\dot{V}(\tilde{z}_1, \tilde{z}_2) \leq -\delta V(\tilde{z}_1, \tilde{z}_2)^{\frac{\nu''}{\nu'}}. \quad (86)$$

Integrating this inequality between  $t_0 + h$  and  $t$ , leads us to

$$V(\tilde{z}_1(t), \tilde{z}_2(t))^{1-\frac{\nu''}{\nu'}} \leq -\delta \left( 1 - \frac{\nu''}{\nu'} \right) (t - t_0 - h) + V(\tilde{z}_1(t_0 + h), \tilde{z}_2(t_0 + h))^{1-\frac{\nu''}{\nu'}}. \quad (87)$$

Then, we get

$$V(\tilde{z}_1(t), \tilde{z}_2(t)) \leq \left[ -\delta \left( 1 - \frac{\nu''}{\nu'} \right) (t - t_0 - h) + V(\tilde{z}_1(t_0 + h), \tilde{z}_2(t_0 + h))^{1-\frac{\nu''}{\nu'}} \right]^{\frac{\nu'}{\nu'-\nu''}}. \quad (88)$$

Moreover, we obtain

$$\begin{aligned} \left( |\tilde{z}_1(t)|^{\alpha+1} + |\tilde{z}_2(t)|^2 \right) &\leq \frac{1}{P_{min}} \left[ -\delta \left( 1 - \frac{\nu''}{\nu'} \right) (t - t_0 - h) + (P_{max})^{1-\frac{\nu''}{\nu'}} \left( |\tilde{z}_1(t_0 + h)|^{\alpha+1} + |\tilde{z}_2(t_0 + h)|^2 \right)^{1-\frac{\nu''}{\nu'}} \right]^{\frac{\nu'}{\nu'-\nu''}}, \\ &\leq \frac{1}{P_{min}} \left[ -\delta \left( 1 - \frac{\nu''}{\nu'} \right) (t - t_0 - h) + (P_{max})^{1-\frac{\nu''}{\nu'}} \left( \|\tilde{Z}(t_0 + h)\|^{\alpha+1} + \|\tilde{Z}(t_0 + h)\|^2 \right)^{1-\frac{\nu''}{\nu'}} \right]^{\frac{\nu'}{\nu'-\nu''}}, \end{aligned} \quad (89)$$

where we have used the following property:

$$P_{\min} \left( |\tilde{z}_1|^{\alpha+1} + |\tilde{z}_2|^2 \right) \leq V(\tilde{z}_1, \tilde{z}_2) \leq P_{\max} \left( |\tilde{z}_1|^{\alpha+1} + |\tilde{z}_2|^2 \right), \quad (90)$$

with  $P_{\min}$  and  $P_{\max}$  are the eigenvalues of  $P$  with

$$0 < P_{\min} \leq P_{\max}.$$

Finally, using the fact that

$$|\tilde{z}_1(t)|^2 \leq \left[ \frac{1}{P_{\min}} \left[ -\delta \left( 1 - \frac{v''}{v'} \right) (t - t_0 - h) + (P_{\max})^{1 - \frac{v''}{v'}} \left( \|\tilde{Z}(t_0 + h)\|^{\alpha+1} + \|\tilde{Z}(t_0 + h)\|^2 \right)^{1 - \frac{v''}{v'}} \right]^{\frac{v'}{v' - v''}} \right]^{\frac{2}{1+\alpha}}, \quad (91)$$

we get

$$\begin{aligned} \|\tilde{Z}(t)\|^2 &\leq \frac{1}{P_{\min}} \left[ -\delta \left( 1 - \frac{v''}{v'} \right) (t - t_0 - h) + (P_{\max})^{1 - \frac{v''}{v'}} \left( \|\tilde{Z}(t_0 + h)\|^{\alpha+1} + \|\tilde{Z}(t_0 + h)\|^2 \right)^{1 - \frac{v''}{v'}} \right]^{\frac{v'}{v' - v''}} \\ &+ \left( \frac{1}{P_{\min}} \left[ -\delta \left( 1 - \frac{v''}{v'} \right) (t - t_0 - h) + (P_{\max})^{1 - \frac{v''}{v'}} \left( \|\tilde{Z}(t_0 + h)\|^{\alpha+1} + \|\tilde{Z}(t_0 + h)\|^2 \right)^{1 - \frac{v''}{v'}} \right]^{\frac{v'}{v' - v''}} \right)^{\frac{2}{1+\alpha}}. \end{aligned} \quad (92)$$

Then,

$$\|\tilde{Z}(t)\|^2 \leq F(t - t_0 - h, \|\tilde{Z}(t_0 + h)\|) + F(t - t_0 - h, \|\tilde{Z}(t_0 + h)\|)^{\frac{2}{1+\alpha}}, \quad (93)$$

where  $F$  is the increasing function defined by (66).

In particular, when  $t$  tend to the settling time  $T_{\max}(\|\tilde{Z}(t_0 + h)\|) + t_0 + h$ , the function  $F$  and the norm  $\|\tilde{Z}(t)\|$  goes to zero, where  $T_{\max}$  is given by (67).  $\square$

*Remark 2.* Note that the problem of finite-time stability of the target system (68) is studied in the presence of some disturbances in<sup>20</sup> using an implicit Lyapunov-based approach and in<sup>38</sup> using a strict explicit Lyapunov-based approach. In the absence of the disturbance in<sup>20</sup>, the settling-time is then estimated by the following formulas:

$$T_{\max}(s_1, s_2) = \frac{3 + \alpha}{\bar{\delta}(1 - \alpha)} V_0(s_1, s_2)^{\frac{1-\alpha}{3+\alpha}}, \quad \text{for any } s_1, s_2 \in \mathbb{R}^2, \quad (94)$$

where  $\bar{\delta} = \min_{(\tilde{z}_1, \tilde{z}_2) \in \mathcal{W}_1} \left\{ l_1 |s_1|^{\frac{2}{2-\alpha}} + l_2 s_2^2 \right\}$ ,  $\mathcal{W}_1 = \{(s_1, s_2) \in \mathbb{R}^2 : V_0(s_1, s_2) = 1\}$ ,  $l_1 = C \left( \frac{(3+\alpha)k_1}{2(1+\alpha)^{\frac{1-\alpha}{2+2\alpha}}} - \sqrt{k_2} - \frac{k_1^2}{\sqrt{k_2}} \right)$ ,  $l_2 = \frac{1}{2\sqrt{k_2}}$ ,

and  $V_0(s_1, s_2) = C \left( \frac{|s_1|^{\alpha+1}}{\alpha+1} + \frac{s_2^2}{2k_2} \right)^{\frac{3+\alpha}{2+2\alpha}} - \frac{s_1 s_2}{\sqrt{k_2}}$ , with  $\alpha = \alpha_2$ ,  $k_1$ , and  $k_2$  are given in (17), and with some constant  $C > \left( \frac{2+2\alpha}{3+\alpha} \right)^{\frac{3+\alpha}{2+2\alpha}}$  such that  $l_1 > 0$  and  $V_0(s_1, s_2) > 0$ . This estimation is quite similar to our estimation (67), but may be less accurate than our estimation or even far from the real settling-time as we will see in the simulation later.

Now, let us state the main result of this paper.

**Theorem 1.** Let the matrices  $A$ ,  $B$ , and  $C$  be as in (6), the vector function  $\mathcal{L}(\cdot) = (\mathcal{L}_1(\cdot), \mathcal{L}_2(\cdot))$  be as in (15) and (16), the function  $F$  be as in (66), the function  $G_1$  be as in (46) and  $G_2$  as in (51),  $\delta$  be given as in (81),  $P_{\min}$  and  $P_{\max}$  be as in (90). Let  $t_0, h > 0$ . Then, for any initial conditions  $Z_0, \hat{Z}_0$  the observer system guarantees that  $(\hat{Z}, \hat{u})$  converges to  $(Z, u)$  within a finite time. Moreover the quantity  $\|u(t, \cdot) - \hat{u}(t, \cdot)\|_{L^2}^2 + \|Z(t) - \hat{Z}(t)\|^2$  is bounded for all  $t \in [t_0, t_0 + h]$ , and for all  $t \geq t_0 + h$

$$\|u(t, \cdot) - \hat{u}(t, \cdot)\|_{L^2}^2 + \|Z(t) - \hat{Z}(t)\|^2 \leq M \left[ F(t - t_0 - h, G(\|Z_0 - \hat{Z}_0\|)) + F(t - t_0 - h, G(\|Z_0 - \hat{Z}_0\|))^{\frac{2}{1+\alpha}} \right], \quad (95)$$

with  $M = \left( 2\|C \exp(A \cdot)\|_{L^2}^2 + 1 \right)$  and  $G = G_1$  (or  $G = G_2$ ).

In particular, the norm  $\|u(t, \cdot) - \hat{u}(t, \cdot)\|_{L^2}^2 + \|X(t) - \hat{X}(t)\|^2 \rightarrow 0$ , as  $t \rightarrow \bar{T}_{\max}(\|Z_0 - \hat{Z}_0\|) + t_0 + h$ , where  $\bar{T}_{\max}$  is given for any  $s \geq 0$  by

$$\bar{T}_{\max}(s) = \frac{v'}{\delta(v' - v'')} (P_{\max})^{1 - \frac{v''}{v'}} (G(s)^{\alpha+1} + G(s)^2)^{1 - \frac{v''}{v'}}, \quad (96)$$

with  $\alpha \in (0, 1)$ ,  $\nu' = \alpha + 1$ ,  $\nu'' = \frac{3\alpha+1}{2}$ .

*Proof.* Using the transformation (12) and The Cauchy-Schwartz inequality, we obtain

$$|\tilde{u}(t, x)| \leq |\tilde{\omega}(t, x)| + \|C \exp(Ax)\| \|\tilde{Z}(t)\|. \quad (97)$$

Squaring this inequality and integrating with respect to the variable  $x$  between 0 and  $h$ , gives us

$$\|\tilde{u}(t, \cdot)\|_{L^2}^2 \leq 2\|\tilde{\omega}(t, \cdot)\|_{L^2}^2 + 2\|C \exp(A \cdot)\|_{L^2}^2 \|\tilde{Z}(t)\|^2. \quad (98)$$

Adding the norm  $\|Z(t) - \hat{Z}(t)\|^2$  in both sides and using (10), we get

$$\begin{aligned} \|u(t, \cdot) - \hat{u}(t, \cdot)\|_{L^2}^2 + \|Z(t) - \hat{Z}(t)\|^2 &\leq 2\|\tilde{\omega}(t, \cdot)\|_{L^2}^2 + 2\|C \exp(A \cdot)\|_{L^2}^2 \|\tilde{Z}(t)\|^2 + \|Z(t) - \hat{Z}(t)\|^2, \\ &\leq 2\|\tilde{\omega}(t, \cdot)\|_{L^2}^2 + (2\|C \exp(A \cdot)\|_{L^2}^2 + \|\exp(Ah)\|^2) \|\tilde{Z}(t)\|^2. \end{aligned} \quad (99)$$

Using Lemma 2 and the last inequality, we see that  $\|\tilde{u}(t, \cdot)\|_{L^2}^2 + \|\tilde{Z}(t)\|^2$  is bounded for all  $t \in [t_0, t_0 + h]$ .

On the other hand, for all  $t \geq t_0 + h$ , employing lemma 5 and the fact that  $\tilde{\omega}(t, \cdot) \equiv 0$  for all  $t \geq t_0 + h$ , we get

$$\|u(t, \cdot) - \hat{u}(t, \cdot)\|_{L^2}^2 + \|Z(t) - \hat{Z}(t)\|^2 \leq M \left[ F(t - t_0 - h, \|\tilde{Z}(t_0 + h)\|) + F(t - t_0 - h, \|\tilde{Z}(t_0 + h)\|)^{\frac{2}{1+\alpha}} \right], \quad (100)$$

where  $M = 2\|C \exp(A \cdot)\|_{L^2}^2 + \|\exp(Ah)\|^2$  and  $F$  is given for any  $s \geq 0$  in (66) as follows:

$$F(t - t_0 - h, s) = \frac{1}{P_{min}} \left[ -\delta \left( 1 - \frac{\nu''}{\nu'} \right) (t - t_0 - h) + (P_{max})^{1 - \frac{\nu''}{\nu'}} (s^{\alpha+1} + s^2)^{1 - \frac{\nu''}{\nu'}} \right]^{\frac{\nu'}{\nu' - \nu''}}. \quad (101)$$

Employing Lemma 3 and the fact that  $F$  is increasing with respect to the second variable, we obtain

$$\|u(t, \cdot) - \hat{u}(t, \cdot)\|_{L^2}^2 + \|Z(t) - \hat{Z}(t)\|^2 \leq M \left[ F(t - t_0 - h, G(\|Z_0 - \hat{Z}_0\|)) + F(t - t_0 - h, G(\|Z_0 - \hat{Z}_0\|))^{\frac{2}{1+\alpha}} \right], \quad (102)$$

where  $G = G_1$  and  $G_1$  is the function defined by (46). In addition, note that using Lemma 4 instead of Lemma 3 we find the same inequality as (102) with  $G = G_2$  and  $G_2$  is defined by (51).

Thus, we finally conclude that the norm  $\|u(t, \cdot) - \hat{u}(t, \cdot)\|_{L^2}^2 + \|Z(t) - \hat{Z}(t)\|^2 \rightarrow 0$ , as  $t \rightarrow \bar{T}_{max}(\|Z_0 - \hat{Z}_0\|) + t_0 + h$ , where  $\bar{T}_{max}$  is given for any  $s \geq 0$  by

$$\bar{T}_{max}(s) = T_{max}(G(s)) = \frac{\nu'}{\delta(\nu' - \nu'')} (P_{max})^{1 - \frac{\nu''}{\nu'}} (G(s)^{\alpha+1} + G(s)^2)^{1 - \frac{\nu''}{\nu'}}. \quad (103)$$

□

## 5 | SIMULATION

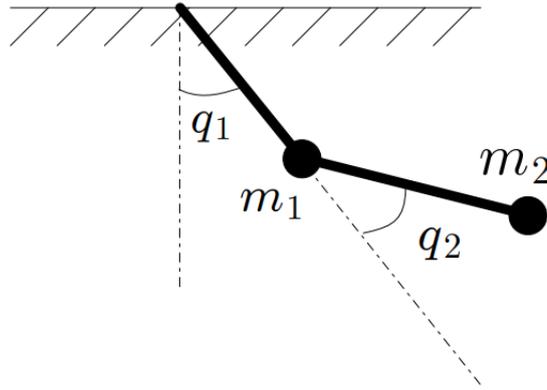
In this section, we focus on a simplified robot manipulator described in Figure 1, where for  $i = 1, 2$ ,  $q_i$  and  $m_i$  are respectively the angle and the mass of the  $i^{\text{th}}$  joint of the robot.

Let us assume that the manipulator is fully actuated, i.e., the number of actuators is equal to 2 (degree of freedom), and that the masses of the links are neglected with respect to the masses of the motors and payload ( $m_2$ ). Then, the model is (1) (i.e.,  $M(q)\ddot{q} + C(q, \dot{q}) + G(q) = \tau$ ), with

$$\begin{aligned} C(q, \dot{q}) &= m_2 l_2 l_1 \sin(q_2) \begin{pmatrix} -\dot{q}_2^2 - 2\dot{q}_1 \dot{q}_2 \\ \dot{q}_1^2 \end{pmatrix}, \\ G(q) &= g \begin{pmatrix} m_2 l_2 \sin(q_1 + q_2) + (m_1 + m_2) l_1 \sin(q_1) \\ m_2 l_2 \sin(q_1 + q_2) \end{pmatrix}, \\ M(q) &= \begin{pmatrix} (m_1 + m_2) l_1^2 + 2m_2 l_1 l_2 \cos(q_2) + m_2 l_2^2 & m_2 l_2^2 + m_2 l_1 l_2 \cos(q_2) \\ m_2 l_2^2 + m_2 l_1 l_2 \cos(q_2) & m_2 l_2^2 \end{pmatrix}, \end{aligned}$$

where for any  $i = 1, 2$ ,  $l_i$  is the  $i^{\text{th}}$  length of the link. Next, (1) with feedback law

$$\tau = M(q)W + C(q, \dot{q}) + G(q), \quad (104)$$



**FIGURE 1** Two-links manipulator.

where  $W = (w_1, w_2)^\top \in \mathbb{R}^2$  is the new input, gives the following closed-loop system:

$$\ddot{q}_i(t) = w_i(t), \quad t \geq 0, \quad i = 1, 2, \quad (105)$$

(where we have used the fact that  $M(q)$  is a symmetric definite positive matrix). However, since  $q$  and  $\dot{q}$  are not directly available by measurements we can not use feedback (104) but instead we can use feedback

$$\hat{\tau} = M(\hat{q})W + C(\hat{q}, \dot{\hat{q}}) + G(\hat{q}). \quad (106)$$

Thus, the system (105) should be replaced by a similar perturbed model,

$$\ddot{q}_i(t) = w_i(t) + \delta_i(\cdot), \quad t \geq 0, \quad i = 1, 2, \quad (107)$$

where we have used the following notation  $(\cdot)$  denotes arguments  $(W(t), q(t), \dot{q}(t), \hat{q}, \dot{\hat{q}}(t))$  (notation used throughout this section) and for  $i = 1, 2$ ,  $\delta_i(\cdot)$  is a disturbance reflecting mismatched models given by

$$(\delta_1(\cdot), \delta_2(\cdot))^\top = M(q)^{-1}[\hat{\tau}(\hat{q}, \dot{\hat{q}}) - M(q)W - C(q, \dot{q}) - G(q)]. \quad (108)$$

This fact explains why such a finite-time robust state estimation has to be performed as seen in the simulations below. Using the following change of coordinates:

$$z_1 = q_1, \quad z_2 = \dot{q}_1, \quad z_3 = q_2, \quad z_4 = \dot{q}_2, \quad (109)$$

the system (107) is transformed into the following two perturbed chains of double integrators:

$$\begin{aligned} \dot{z}_1(t) &= z_2(t), \\ \dot{z}_2(t) &= w_1(t) + \delta_1(\cdot), \\ \dot{z}_3(t) &= z_4(t), \\ \dot{z}_4(t) &= w_2(t) + \delta_2(\cdot), \end{aligned} \quad (110)$$

where  $Z = (z_1, z_2)^\top$ ,  $\bar{Z} = (z_3, z_4)^\top$ , and for  $i = 1, 2$ ,  $\delta_i(t)$  is now given by

$$(\delta_1(\cdot), \delta_2(\cdot))^\top = M(z_3)^{-1}[\hat{\tau}(\hat{z}_1, \hat{z}_2, \hat{z}_3, \hat{z}_4) - M(z_3)W - C(z_2, z_3, z_4) - G(z_1, z_3)]. \quad (111)$$

Then, we take the initialization time  $t_0 = 0s$ , the initial positions  $(z_1(\theta), z_3(\theta))^\top = (0, 0)^\top rad$  for all  $\theta \in [t_0 - h, t_0]$  and  $(z_1(t_0), z_3(t_0))^\top = (1, 2)^\top rad$ , and the initial velocities  $(z_2(\theta), z_4(\theta))^\top = (0, 0)^\top rad/s$  for all  $\theta \in [t_0 - h, t_0]$ . In addition, we assume that  $z_1$  and  $z_3$  are available by delayed measurements, i.e.,

$$\begin{aligned} y_1(t) &= z_1(t - h), \\ y_2(t) &= z_3(t - h), \end{aligned} \quad (112)$$

with a delay of  $h$  units of time.

Next, by following the same steps as in Section 3, we rewrite the two subsystems into a ODE-PDE cascade systems,

$$\begin{aligned}\dot{Z}(t) &= AZ(t) + B [w_1(t) + \delta_1(\cdot)], \\ u_{1,t}(t, x) &= u_{1,x}(t, x), \\ u_1(t, h) &= CZ(t), \\ y_1(t) &= u_1(t, 0),\end{aligned}\tag{113}$$

and

$$\begin{aligned}\dot{\bar{Z}}(t) &= A\bar{Z}(t) + B [w_2(t) + \delta_2(\cdot)], \\ u_{2,t}(t, x) &= u_{2,x}(t, x), \\ u_2(t, h) &= C\bar{Z}(t), \\ y_2(t) &= u_2(t, 0),\end{aligned}\tag{114}$$

where  $A$ ,  $B$ , and  $C$  are given in (6). The observers are then given by,

$$\begin{aligned}\dot{\hat{Z}}(t) &= A\hat{Z}(t) + Bw_1(t) - e^{Ah}\mathcal{L}(y_1(t) - \hat{u}_1(t, 0)), \\ \hat{u}_{1,t}(t, x) &= \hat{u}_{1,x}(t, x) - Ce^{Ax}\mathcal{L}(y_1(t) - \hat{u}_1(t, 0)), \\ \hat{u}_1(t, h) &= C\hat{Z}(t),\end{aligned}\tag{115}$$

and

$$\begin{aligned}\dot{\hat{\bar{Z}}}(t) &= A\hat{\bar{Z}}(t) + Bw_2(t) - e^{Ah}\mathcal{L}(y_2(t) - \hat{u}_2(t, 0)), \\ \hat{u}_{2,t}(t, x) &= \hat{u}_{2,x}(t, x) - Ce^{Ax}\mathcal{L}(y_2(t) - \hat{u}_2(t, 0)), \\ \hat{u}_2(t, h) &= C\hat{\bar{Z}}(t),\end{aligned}\tag{116}$$

where  $\mathcal{L}$  is given in (15) and (16).

In this case, since we do not design a predictor feedback, one can only ensure robustness of the controller to small delays (although the finite-time convergence of the observer is guaranteed for larger delays).

To simulate the systems (113), (114), (115), and (116), we first discretize them using the two-step variant of the Lax-Friedrichs numerical method introduced in<sup>39</sup>, then we use its corresponding solver in Matlab. For simplicity, we chose the parameters of  $\mathcal{L}$  as follows:  $k_1 = k_2 = 4$ ,  $\alpha_2 = 0.8$  and  $\alpha_1 = 0.9$  for both observers, and the delay  $h = 0.04s$ . The rest of the parameters are estimated numerically as follows: the eigenvalues of the matrix  $P$  in (72),  $P_{min} = 0.2150$ ,  $P_{max} = 2.2162$ , the coefficient  $\delta = 5.9525$  in (81) which is calculated in Matlab using the function *fmincon*, the state  $z - \hat{z}$  at time  $t_0 + h$ :  $z(t_0 + h) - \hat{z}(t_0 + h) = (0.9933, -0.3206, 2.0048, 0.2360)^\top$ . Therefore, the settling-time of each subsystem is numerically estimated using (67):  $T_{1,max}(\|z(t_0 + h) - \hat{z}(t_0 + h)\|) = 3.2996s$  and  $T_{2,max}(\|z(t_0 + h) - \hat{z}(t_0 + h)\|) = 3.5380s$ .

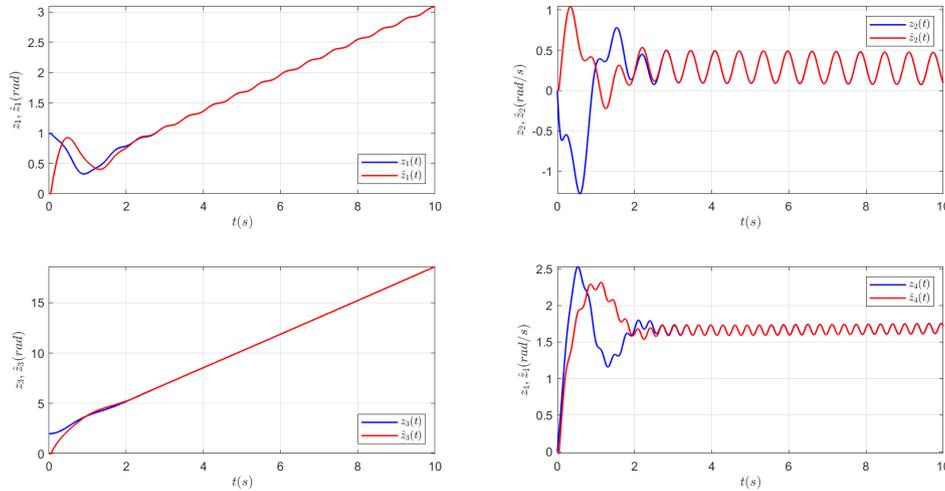
Next, in Figure 2 we give the evolution of the states  $z_1$ ,  $z_2$ ,  $z_3$ , and  $z_4$  of the system (110) in solid blue lines and the estimated states  $\hat{z}_1$ ,  $\hat{z}_2$ ,  $\hat{z}_3$ , and  $\hat{z}_4$  in dashed red lines with the delay  $h = 0.04s$ , the feedback  $\hat{\tau}$  in (106) instead of  $\tau$  and the new inputs  $w_1(t) = 2 \sin(10t)$  and  $w_2(t) = 1.4 \sin(20t)$ . Then, in Figure 3 we show the evolution of the error states  $(\tilde{z}_1, \tilde{z}_2)^\top$  of the ODE part of the error system linked to the subsystem (113) where we have used the feedback  $\hat{\tau}$  (104), multiplied by  $\exp(Ah)$ . In addition, we can observe that the solutions converge to the origin in a finite-time less than  $T_{1,max} + h + t_0 = 3.3396s$  (numerical estimation using (67)): the log plot (not shown) confirmed that  $T_{1,max} + h + t_0 \approx 3.31s$ . On the right hand side, we show the evolution of the error state of the PDE part of the error system. Similarly, Figure 4 shows the evolution of the error states  $(\tilde{z}_3, \tilde{z}_4)^\top$  of the ODE part of the error system linked to the subsystem (114) with the feedback  $\hat{\tau}$  (104), multiplied by  $\exp(Ah)$ . In addition, we can observe that the solutions converge to the origin in a finite-time less than  $T_{2,max} + h + t_0 = 3.5780s$  (numerical estimation using (67)): the log plot (not shown) confirmed that  $T_{2,max} + h + t_0 \approx 3.5s$ . On the right hand, we show the evolution of the error state of the PDE part of the error system. Finally, Figure 5 present on the left, the evolution of the norm

$\|Z(t) - \hat{Z}(t)\|^2 + \|\bar{Z}(t) - \hat{\bar{Z}}(t)\|^2 + \sum_{i=1}^2 \|u_i(t, \cdot) - \hat{u}_i(t, \cdot)\|_{L^2}^2$  of the error systems (in a logarithmic scale), where is shown to converge to zero in finite-time less than  $T_{max} + h + t_0 \approx 3.5s$  (compare to 3.578s obtained using (67)). On the right, we observe the evolution of the outputs  $y_1(t) = z_1(t - h) = u_1(t, 0)$  and  $y_2(t) = z_3(t - h) = u_2(t, 0)$  in solid lines and the estimated outputs  $\hat{u}_1(t, 0)$  and  $\hat{u}_2(t, 0)$  in dashed lines.

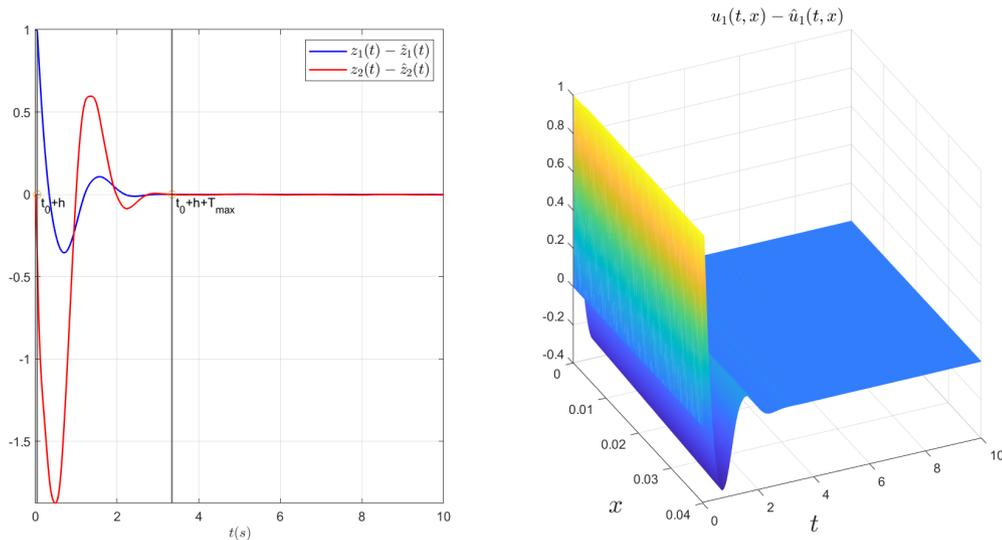
In Figures 6, 7, 8, and 9, we add a noise measurement of power 0.0001 and we give the same simulations as before and verify the impact of noise to the convergence of the closed-loop. As anticipated, the observer is robust with respect to the modeled

uncertainties and the noise measurement. However, we only ensure a finite-time convergence to a neighborhood of the origin characterized by the magnitude of the noise.

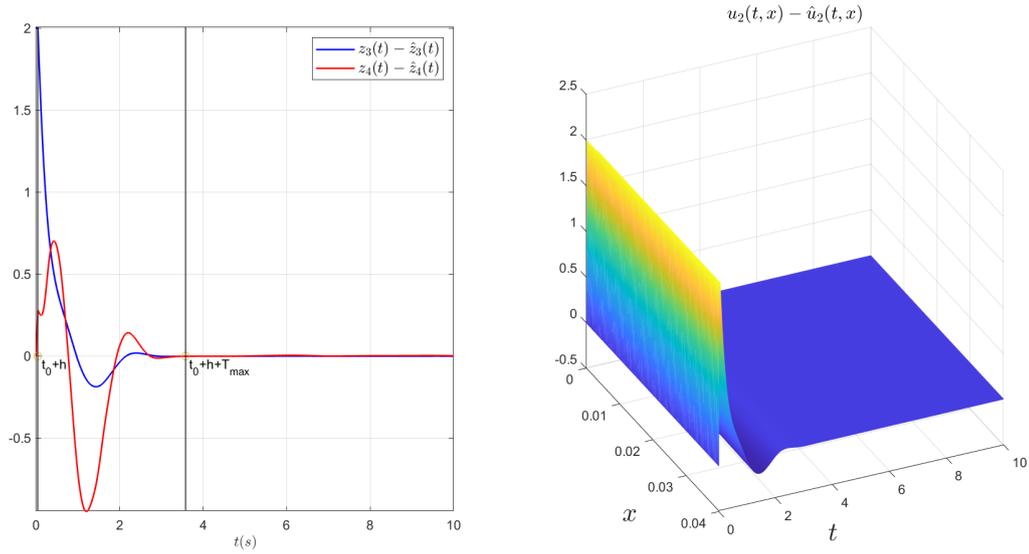
*Remark 3.* Now, following Remark 2 with  $C = 1.7$ ,  $\alpha$ ,  $k_1$ , and  $k_2$  as chosen in the above simulations (i.e.,  $k_1 = k_2 = 4$ ,  $\alpha = \alpha_2 = 0.8$ ), one can numerically estimate (94) as follows:  $T_{1,max} = 16.9744s$  and  $T_{2,max} = 18.4054s$ . Indeed, to get these estimates, we have  $l_1 = 2.5049$ ,  $l_2 = 0.25$ ,  $V_{0,1}(z(t_0+h) - \hat{z}(t_0+h)) = 0.6883$ ,  $V_{0,2}(z(t_0+h) - \hat{z}(t_0+h)) = 3.2034$  and  $\bar{\delta} = 1.075$  which is calculated using the Matlab function *fmincon*. We can clearly see that these two values are far from the settling time observed in the simulations (see Figures 3, 4 and 5) and our estimates using (67) ( $T_{1,max} = 3.3396s$  and  $T_{2,max} = 3.5380s$ ).



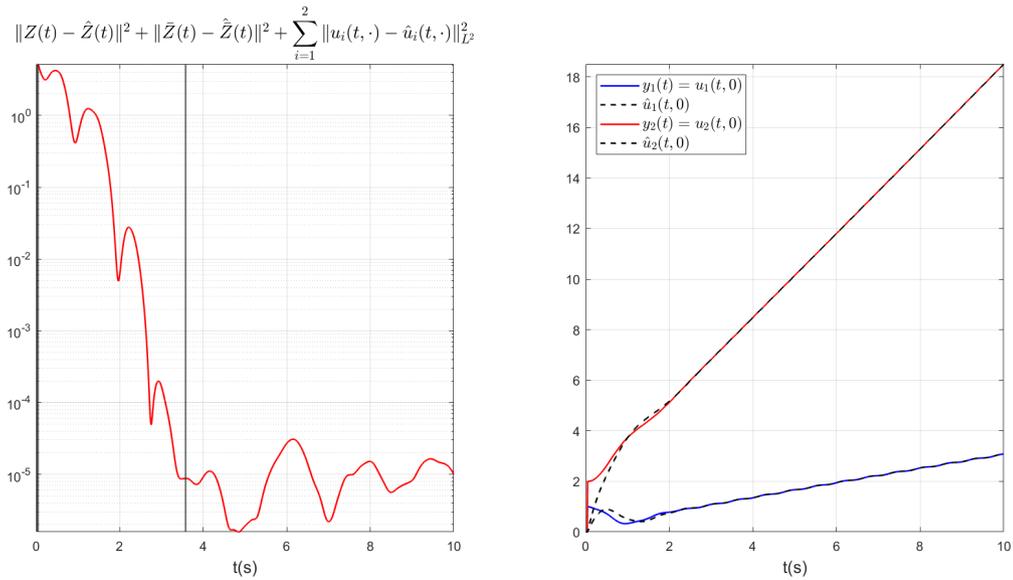
**FIGURE 2** The evolution of the states  $z_1$ ,  $z_2$ ,  $z_3$ , and  $z_4$  of the system (110) in solid blue lines and the estimated states  $\hat{z}_1$ ,  $\hat{z}_2$ ,  $\hat{z}_3$ , and  $\hat{z}_4$  in dashed red lines, with feedback  $\hat{t}$  in (106) and without noise measurement, where the delay  $h = 0.04s$ , the initial positions  $(z_1(0), z_3(0))^T = (1, 2)^T$ , the initial velocities  $(z_2(0), z_4(0))^T = (0, 0)^T m/s$ , and the input  $w_1(t) = 2 \sin(10t)$  and  $w_2(t) = 1.4 \sin(20t)$ .



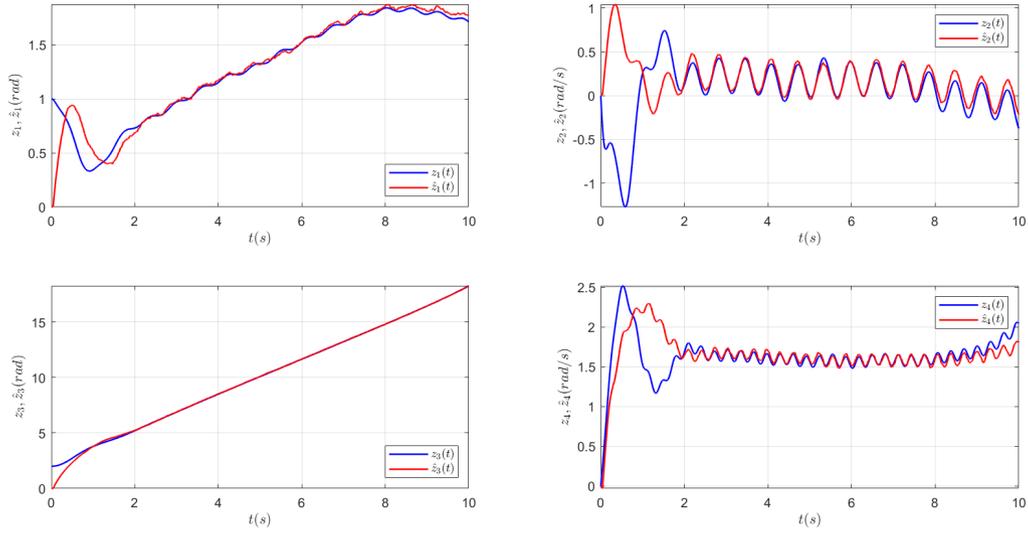
**FIGURE 3** On the left hand, we have the evolution of the error states of the ODE part of the error system linked to the subsystem (113) multiplied by  $\exp(Ah)$  with feedback  $\hat{t}$  in (106) and without noise measurement, where  $h = 0.04s$  and  $T_{1,max} = 3.3396s$ . On the right hand, the evolution of the error state of the PDE part of the error system with feedback  $\hat{t}$  in (106) and without noise measurement.



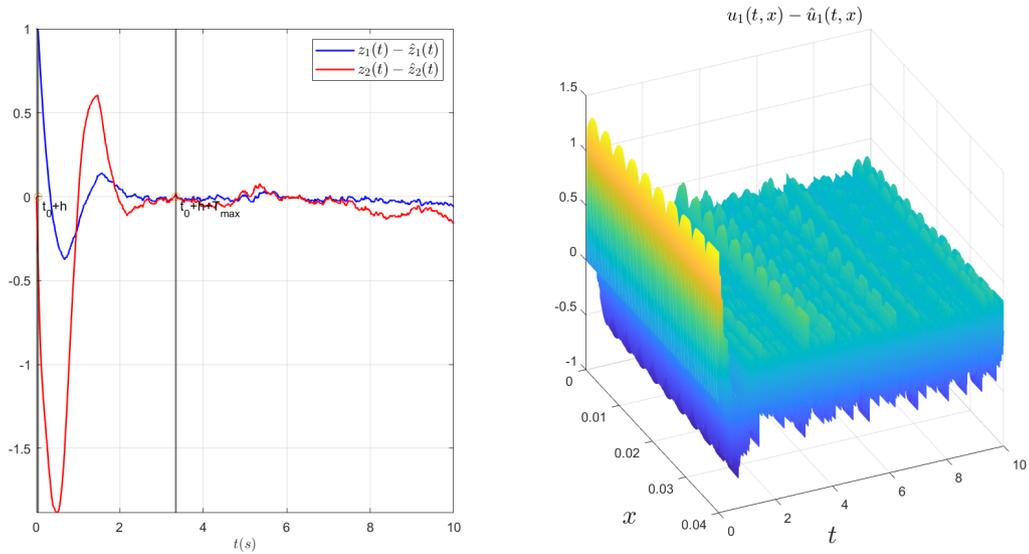
**FIGURE 4** On the left hand, we have the evolution of the error states of the ODE part of the error system linked to the subsystem (114) multiplied by  $\exp(Ah)$  with feedback  $\hat{\tau}$  in (106) and without noise measurement, where  $h = 0.04s$  and  $T_{2,max} = 3.5380s$ . On the right hand, the evolution of the error state of the PDE part of the error system with feedback  $\hat{\tau}$  in (106) and without noise measurement.



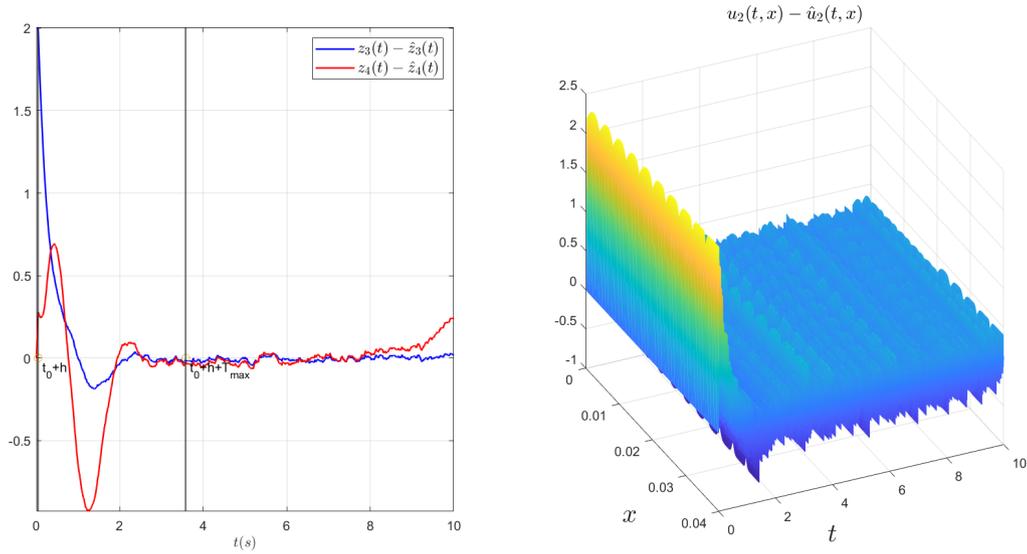
**FIGURE 5** On the left, the evolution of the norm  $\|Z(t) - \hat{Z}(t)\|^2 + \|\bar{Z}(t) - \hat{\bar{Z}}(t)\|^2 + \sum_{i=1}^2 \|u_i(t, \cdot) - \hat{u}_i(t, \cdot)\|_{L^2}^2$  of the two error systems (plotted in logarithmic scale) with feedback  $\hat{\tau}$  in (106) and without noise measurement, with the delay  $h = 0.04$ , and  $T_{max} = 3.5380s$ . On the right, the evolution of the outputs  $y_1(t) = z_1(t - h) = u_1(t, 0)$  and  $y_2(t) = z_3(t - h) = u_2(t, 0)$  in solid lines and the estimated outputs  $\hat{u}_1(t, 0)$  and  $\hat{u}_2(t, 0)$  in dashed with feedback  $\hat{\tau}$  in (106) and lines without noise measurement.



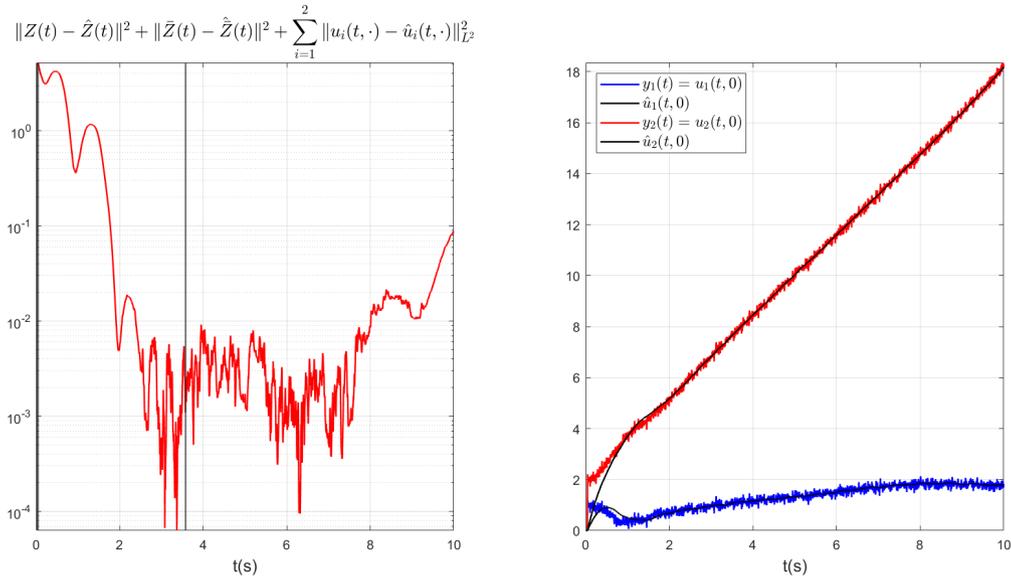
**FIGURE 6** The evolution of the states  $z_1$ ,  $z_2$ ,  $z_3$ , and  $z_4$  of the system (110) in blue lines and the estimated states  $\hat{z}_1$ ,  $\hat{z}_2$ ,  $\hat{z}_3$ , and  $\hat{z}_4$  in red lines, with feedback  $\hat{\tau}$  in (106) and a noise of power 0.0001, where the delay  $h = 0.04s$ , the initial positions  $(z_1(0), z_3(0))^T = (1, 2)^T$ , the initial velocities  $(z_2(0), z_4(0))^T = (0, 0)^T m/s$ , and the input  $w_1(t) = 2 \sin(10t)$  and  $w_2(t) = 1.4 \sin(20t)$ .



**FIGURE 7** On the left hand, we have the evolution of the error states of the ODE part of the error system linked to the subsystem (113) multiplied by  $\exp(Ah)$  with feedback  $\hat{\tau}$  in (106) and a noise of power 0.0001, where  $h = 0.04s$  and  $T_{1,max} = 3.3024s$ . On the right hand, the evolution of the error state of the PDE part of the error system with feedback  $\hat{\tau}$  in (106) and a noise of power 0.0001.



**FIGURE 8** On the left hand, we have the evolution of the error states of the ODE part of the error system linked to the subsystem (114) multiplied by  $\exp(Ah)$  with feedback  $\hat{\tau}$  in (106) and a noise of power 0.0001, where  $h = 0.04$ s and  $T_{2,max} = 3.5381$ s. On the right hand, the evolution of the error state of the PDE part of the error system with feedback  $\hat{\tau}$  in (106) and a noise of power 0.0001.



**FIGURE 9** On the left, the evolution of the norm  $\|Z(t) - \hat{Z}(t)\|^2 + \|\bar{Z}(t) - \hat{\bar{Z}}(t)\|^2 + \sum_{i=1}^2 \|u_i(t, \cdot) - \hat{u}_i(t, \cdot)\|_{L^2}^2$  of the two error systems (plotted in logarithmic scale) with feedback  $\hat{\tau}$  in (106) and a noise of power 0.0001, and the delay  $h = 0.04$ , and  $T_{max} = 3.5381$ s. On the right, the evolution of the outputs  $y_1(t) = z_1(t - h) = u_1(t, 0)$  and  $y_2(t) = z_3(t - h) = u_2(t, 0)$  in solid lines and the estimated outputs  $\hat{u}_1(t, 0)$  and  $\hat{u}_2(t, 0)$  in dashed lines with feedback  $\hat{\tau}$  in (106) and a noise of power 0.0001.

## 6 | CONCLUSION

In this paper, we dealt with the problem of finite-time estimation of second-order linear time-invariant (LTI) systems with a delayed output. The main ideas rely on rewriting the system into an ODE-PDE cascade setting. The nonlinear gains are designed such that the error observer system is finite-time stable. To do so, we choose a suitable nonlinear target system satisfying a chosen finite-time convergence property proven using Lyapunov-based analysis and homogeneity tools. Finally, we use the invertibility of the backstepping transformation to pass this property to the error system.

Future work will generalize these results to  $n$ th-dimensional LTI systems. Moreover, one can also similarly design a robust prescribed-time estimation of LTI systems in the presence of measurement delays integrating the ideas of<sup>32</sup> and<sup>40</sup>. Finally, one can also address the dual problem of finite-time stabilization for LTI systems with delayed input using a nonlinear backstepping approach and homogeneity tools.

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**How to cite this article:** Zekraoui S, Espitia N, Perruquetti W. Finite-time estimation of second-order linear time-invariant systems in the presence of delayed measurement. *Int J Robust Nonlinear Control*. 2022;1-26. doi:10.1002/rnc.6224