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Regular article

Finite-Size Scaling of the majority-voter model above the upper critical dimension

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The majority-voter model is studied by Monte Carlo simulations on hypercubic lattices of dimension $d = 2$ to 7 with periodic boundary conditions. The critical exponents associated to the Finite-Size Scaling of the magnetic susceptibility are shown to be compatible with those of the Ising model. At dimension $d = 4$, the numerical data are compatible with the presence of multiplicative logarithmic corrections. For $d \geq 5$, the estimates of the exponents are close to the prediction $d/2$ when taking into account the dangerous irrelevant variable at the Gaussian fixed point. Moreover, the universal values of the Binder cumulant are also compatible with those of the Ising model. This indicates that the upper critical dimension of the majority-voter model is not $d_c = 6$ as claimed in the literature, but $d_c = 4$ like the equilibrium Ising model.

1. Introduction

The Renormalization Group Theory, pioneered by Wilson and Fisher among others, has provided a deep understanding of the critical behavior of statistical models, such as the Ising model, below their upper critical dimension d_c [1]. Above d_c, the situation is much simpler and the critical exponents take the values predicted by mean-field theory. However, the correct Finite-Size Scaling of thermodynamic averages above the upper critical dimension has been clarified only recently [2].

The critical behavior of the d-dimensional Ising model is described in the continuum limit by the Landau-Ginzburg action [3–5]

$$S[\phi] = \int \left[|\nabla\phi|^2 + r\phi^2 + u\phi^4\right] d^d x. \tag{1.1}$$

Since the action is dimensionless, one can determine the scaling dimensions of the field and of the couplings by power-counting. The scaling dimension of ϕ is $x_\phi = (d-2)/2$ and therefore, the dimension of the coupling u is $y_u = d - 4x_\phi = 4 - d$. It follows that the quartic term is relevant for $d < 4$, marginal at $d = 4$ and irrelevant at $d > 4$. Below the upper critical dimension $d_c = 4$, the critical exponents can only be estimated using the full machinery of the Renormalization Group (RG). In contrast, above the upper critical dimension $d_c = 4$, one expects the critical behavior to be governed by the Gaussian fixed point corresponding to $u = 0$. It turns out that the coupling u is a dangerously irrelevant variable and should be taken into account [6]. From the RG flow equations, it can be shown that under a rescaling $x \to x/b$ the singular part of the free energy density behaves as

$$f(r,u) = b^{-d} f\left(b^2(r+\alpha u) - b^{4-d}\alpha u, b^{4-d}u\right). \tag{1.2}$$

It follows that the critical exponents are not those of the Gaussian fixed point but the mean-field exponents. The standard hyperscaling relation holds only at $d = d_c$ and is violated above. Finite-Size Scaling is also affected by the dangerous irrelevant coupling u. With periodic boundary conditions, the quartic term of

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the action Eq. 1.1 involves the volume V of the system in Fourier space:

$$S = \sum_{\vec{k}} (k^2 + r)|\phi_{\vec{k}}|^2 + \frac{u}{V^2} \sum_{\vec{k}_1,\vec{k}_2,\vec{k}_3} \phi_{\vec{k}_1}\phi_{\vec{k}_2}\phi_{\vec{k}_3}\phi_{-\vec{k}_1-\vec{k}_2-\vec{k}_3}. \tag{1.3}$$

At the pseudo-critical point $r = 0$, the Finite-Size Scaling of the magnetic susceptibility is dominated by the contribution of the $k = 0$ mode [8]:

$$\langle \phi^2 \rangle \sim \frac{\int \phi_0^2 e^{-\frac{u}{V}\phi_0^4} d\phi_0}{\int e^{-\frac{u}{V}\phi_0^4} d\phi_0} \sim V^{1/2} = L^{d/2}. \tag{1.4}$$

It is therefore anomalous with a divergence $L^{d/2}$ with the lattice size L and not $L^{\gamma/\nu}$ with $\gamma/\nu = 2$ as expected at the Gaussian Fixed Point. Similarly, the correlation length scales with the lattice size as $\xi \sim L^{d/d_c}$ [8, 9]. A coherent Finite-Size Scaling theory has been presented based on the new exponent $\wp = d/d_c$ for $d > d_c$ [10]. The later predicts that the magnetic susceptibility diverges with the lattice size with an exponent $\wp\gamma/\nu = d/2$. This theory has recently been extended to quantum phase transitions [11]. Despite some indications that the same Finite-Size Scaling holds with free boundary conditions, the problem is not completely settled [2, 12].

The above discussion concerns only the Ising model but it is believed to be more general. In particular, the Finite-Size Scaling of percolation above its upper critical dimension $d_c = 6$ has been analyzed in the same way [13]. In this paper, we are interested in the majority voter model which has the peculiarity, as percolation, of not being described by a Hamiltonian. This model is nevertheless believed to belong to the universality class of the Ising model. However, it has been claimed that the upper critical dimension is not $d_c = 4$ but $d_c = 6$ [25]. We performed extensive Monte Carlo simulations and compared the Finite-Size Scaling of the magnetic susceptibility of the majority-voter and Ising models. In the first section, the two models are more precisely defined and details on the Monte Carlo simulations are given. The numerical results are analyzed in the second section. Conclusions follow.

2. Models and simulation details

In this study, we consider hypercubic lattices of dimension d ranging from $d = 2$ to 7. Each node i of this lattice is occupied by a classical Ising spin that can take two possible values $\sigma_i = \pm 1$. The Ising model is defined by the Hamiltonian

$$H(\{\sigma\}) = -\sum_{(i,j)} \sigma_i \sigma_j \tag{2.1}$$

where the sum extends over all pairs of neighboring sites of the lattice. At equilibrium with a thermal bath at the temperature $T = 1/k_B\beta$, the probability of a given spin configuration $\{\sigma_i\}_i$ is given by the celebrated Boltzmann distribution

$$\wp_{\text{Eq.}}(\{\sigma\}) = \frac{1}{\mathcal{Z}} e^{-\beta H(\{\sigma\})}. \tag{2.2}$$

One can find different Markovian dynamics whose stationary distribution is the Boltzmann distribution $\wp_{\text{Eq.}}$. It is convenient to impose the detailed balance condition

$$\wp_{\text{Eq.}}(\{\sigma\}) W(\{\sigma\} \to \{\sigma'\}) = \wp_{\text{Eq.}}(\{\sigma'\}) W(\{\sigma'\} \to \{\sigma\}) \tag{2.3}$$

which is satisfied by the Glauber transition rates consisting in single spin flips

$$W(\{\sigma\} \to \{\sigma'\}) = \frac{1}{N} \sum_i \omega(\sigma_i) \delta_{\sigma'_i,-\sigma_i} \prod_{j \neq i} \delta_{\sigma'_j,\sigma_j} \tag{2.4}$$

with [14]

$$\omega(\sigma_i) = \frac{1}{2}\big[1 - \sigma_i \tanh\big(\beta \sum_j \sigma_j\big)\big] \tag{2.5}$$

where the sum extends over all neighbors j of site i. The dynamics that is generated by these transition rates can be seen as a local equilibration of the spin σ_i in the effective magnetic field created by its neighbors. Note that there are several solutions to the detailed balance condition Eq. 2.3. The Metropolis transition rate is another solution that is commonly used in Monte Carlo simulations [15]. These two dynamics are slow: the dynamical exponent is $z = 2$ away from criticality and slightly larger than 2 at the critical point. To compute equilibrium properties by means of Monte Carlo simulations, it is therefore much more efficient to use cluster algorithms based on non-local spin updates [16, 17].

The majority-voter model is another Markovian dynamics for Ising spins on a lattice. As the Glauber dynamics, it consists in single-spin flips, i.e. the transition rates are of the form Eq. (2.4) but with

$$\omega(\sigma_i) = \frac{1}{2}\big[1 - \lambda\sigma_i S\big(\sum_j \sigma_j\big)\big] \tag{2.6}$$

where $S(x)$ is the sign function defined by $S(x) = 1$ for $x > 0$, -1 for $x < 0$, and 0 for $x = 0$. Note that in Ref. [25], the parameter λ is denoted $\tanh\beta_T$. In contrast to Glauber dynamics, the majority-voter model does not satisfy the detailed balance condition Eq. 2.3. No Hamiltonian can be associated to this model and the parameter λ is not related to any temperature. The majority-voter dynamics can be used to study the spreading of opinion in a population [18]. Each node i of the lattice is associated to a voter and the spin σ_i to his answer to a binary question. At each time, each voter adapts his choice according to the majority opinion of his neighbors. Note that the majority-voter model should not be confused with the voter model [1]. The Ising-Glauber and majority-voter models are special cases of the more general transition rates

$$\omega(\sigma_i) = \frac{1}{2}\big[1 - \lambda\sigma_i \tanh\big(\beta \sum_j \sigma_j\big)\big]. \tag{2.7}$$

The Ising-Glauber model is recovered with the choice $\lambda = 1$ and the majority-voter model with $\beta \to +\infty$. As a consequence, the majority-voter model appears as an Ising-Glauber model at zero temperature with an additional noise.

Even though the majority-voter model cannot be associated with any Hamiltonian, it is believed that the averages computed in the stationary distribution display the same critical behavior as the equilibrium Ising model [19]. Measurements of the static critical exponents of the 2D majority-voter model by Monte Carlo simulations indeed support this idea [20, 23]. The dynamical exponent z and the initial critical slip exponent θ were also shown to be compatible with those of the Ising model [21]. A more recent Monte Carlo simulation of the 3D majority-voter model also found critical exponents in the Ising universality class [22, 24]. However, an extensive Monte Carlo simulations in dimensions $d = 2$ to 7 reached the conclusion that the upper critical dimension of the majority-voter model is not $d_c = 4$, like the Ising model, but $d_c = 6$ [25].

In this work, the critical behavior of the Ising-Glauber and majority-voter models is studied by Monte Carlo simulations. Much more accurate estimates of the critical exponents of the Ising model could have been computed with cluster algorithms. However, our goal is here to study both models with local dynamics and with the same number of Monte Carlo iterations in order to compare exponents with similar error bars. The code was parallelized with the Cuda language and run on GPUs Nvidia Tesla P100 and GTX 1080. Hypercubic lattices of dimension $d = 2$ to 7 were considered with periodic boundary conditions. The largest lattice sizes that could be reached are 1024 in 2D, 120 in 3D, 36 in 4D, 18 in 5D, 12 in 6D and 8 in 7D. 10^5 iterations were performed to thermalize the system and 10^6 iterations were used to compute the averages. Several independent simulations were performed and the error was estimated as $\sigma/\sqrt{N}$ where σ is the standard deviation among the N independent simulations.

[1] In the voter model, each voter chooses one of its neighbours and adopts his choice.

Figure 1. Susceptibility χ of the 5D Majority-Voter model (left) and of the 5D Ising model (right) versus the parameters λ and β. The different curves correspond to different lattice sizes.

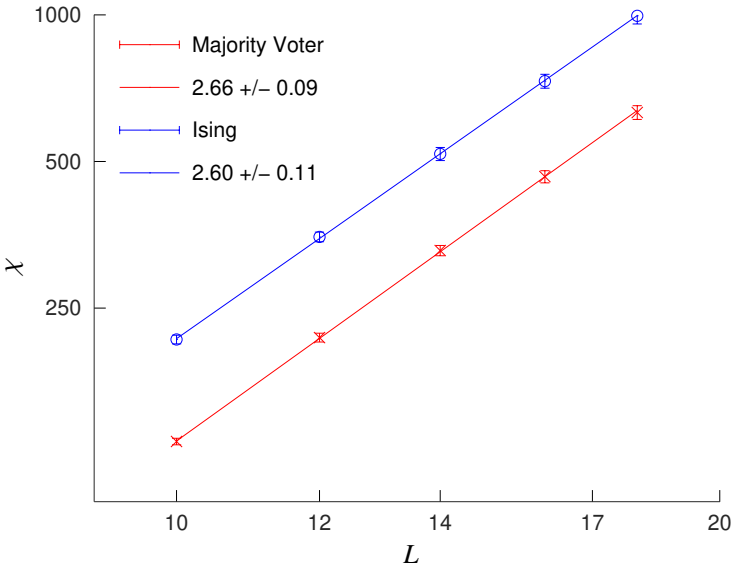

Figure 2. Maximum of the susceptibility of the 5D Majority-Voter model (cross) and of the 5D Ising model (circle) versus the lattice size L. The solid lines are power-law fits of the data. The estimated critical exponents γ/ν are indicated in the legend.

3. Numerical results

On Fig. 1, the magnetic susceptibility

$$\chi = L^d \left[\langle m^2 \rangle - \langle m^2 \rangle \right] \tag{3.1}$$

is plotted versus the parameter λ of the majority-voter model and β of the Ising model in dimension $d = 5$. For each lattice size, the pseudo-critical parameters $\lambda_c(L)$ and $\beta_c(L)$ were estimated as the location of the maximum of the susceptibility. To improve the accuracy, a quadratic fit of the data is first performed over the points for which $\chi \geq 0.7 \max \chi$ and the maximum is computed from the parameters of the fit. On Fig. 2, the magnetic susceptibility at the pseudo-critical point is plotted versus the lattice size. The critical exponent $\wp\gamma/\nu$ is estimated from a simple power-law fit over all data. The estimates are collected in Table 1. The critical exponents $\wp\gamma/\nu$ of the majority-voter and Ising models are compatible within error bars, except at $d = 4$ where the two error bars do not overlap but are very close to each other (the distance between them is 0.03). Note that $d = 4$ is the upper critical dimension of the Ising model and, as will be shown in this paper, of the majority-voter model. RG predicts the presence of logarithmic corrections in the scaling behavior of the ϕ^4 model with temperature [3–5]. These corrections were shown to be present in Finite-Size Scaling too [26]. In dimension $d > 4$, the critical exponents $\wp\gamma/\nu$ of both the majority-voter and Ising model are close to the prediction $d/2$ at the Gaussian fixed point when taking into account the dangerous irrelevant variable, although not compatible within error bars.

As already discussed above, the critical behavior of the magnetic susceptibility of the Ising model is expected to involve multiplicative logarithmic corrections at the upper critical dimension $d_c = 4$. It can be inferred that the Finite-Size Scaling of the magnetic susceptibility at its maximum is of the form

$$\chi \sim L^{d/2} (\ln L)^{\hat{\gamma}/\hat{\nu}} \tag{3.2}$$

where $\hat{\gamma}/\hat{\nu}$ is known to be $1/2$ for the Ising model at the critical temperature β_c [26]. To test this scaling for the majority-voter model, the quantity χ/L^2 has been plotted versus $\ln L$ with a logarithmic scale on Fig. 4. A very nice power-law behavior can be observed for both the majority-voter model and the Ising

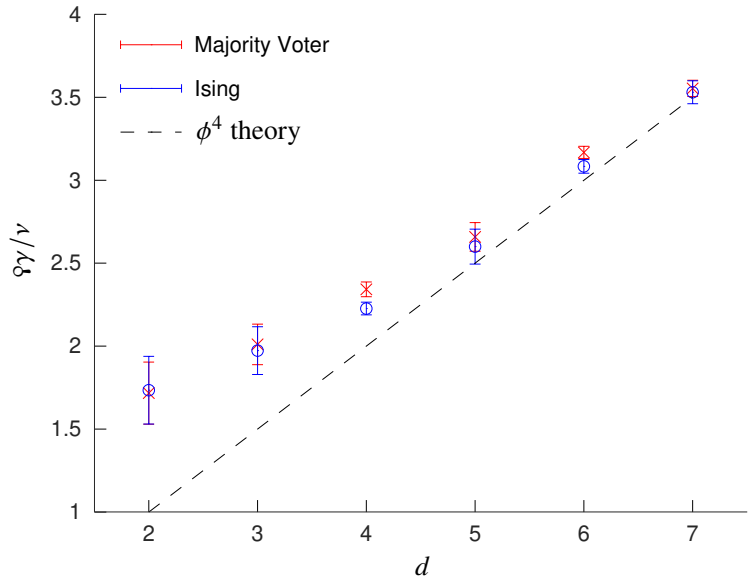

Figure 3. Critical exponent $\wp\gamma/\nu$ obtained from the Finite-Size Scaling of the susceptibility of the Majority-Voter model and the Ising model versus the dimension d of the lattice. The dashed line is the prediction $d/2$ at the Gaussian fixed point with dangerous irrelevant variables that is expected to hold above the upper critical dimension.

Table 1. Estimates of the critical exponent $\wp\gamma/\nu$ for the Majority Voter model (left) and the Ising model (center) on an hypercubic lattice of dimension d. Known values for the Ising model are given in the right column. At $d = 3$, the value is a recent estimate by Functional RG [27]. Other values are exact. At $d = d_c = 4$, multiplicative logarithmic corrections are present.

d	Majority Voter	Ising	Known
2	1.72(19)	1.73(20)	1.75
3	2.01(12)	1.97(14)	1.96370(20)
4	2.34(4)	2.23(4)	2 (+log)
5	2.66(9)	2.60(11)	2.5
6	3.17(4)	3.08(4)	3
7	3.55(5)	3.53(7)	3.5

model. The exponent $\hat{\gamma}/\hat{\nu}$ is estimated to be 0.94(10) for the majority-voter model and 0.62(9) for the Ising model. For the Ising model, the known value $\hat{\gamma}/\hat{\nu} = 1/2$ is slightly outside of the error bar of our estimate. Note also that, for both the majority-voter and the Ising models, the magnetic susceptibility can also be fitted with logarithmic corrections in dimensions $d > 4$ but with an exponent $\hat{\gamma}/\hat{\nu}$ decreasing with d (0.41(22) at $d = 5$ and 0.30(6) at $d = 6$ for the majority-voter model).

On figure 5, the Binder cumulant

$$U = 1 - \frac{\langle m^4 \rangle}{3\langle m^2 \rangle^2} \tag{3.3}$$

is plotted versus the parameters λ and β in dimension $d = 5$. The curves for different lattice sizes are expected to cross at the critical parameters λ_c and β_c in the thermodynamic limit and the value of the Binder cumulant at the crossing points is expected to be universal. For all dimensions, very similar values are observed for the majority-voter and Ising models. In dimensions $d = 5$, we have estimated the universal values to be $U_*(\infty) \simeq 0.27(7)$ for the majority-voter model and $U_*(\infty) \simeq 0.30(6)$ for the Ising model. Estimates at dimensions $d = 2$ to 7 are given in Table 2. They were obtained by a linear interpolation of the Binder cumulant to estimate more accurately the crossing points. The value $U_*(L)$ at the crossing is then fitted with the law $U(L) = U_*(\infty) + b/L$. The absence of error bars for the largest

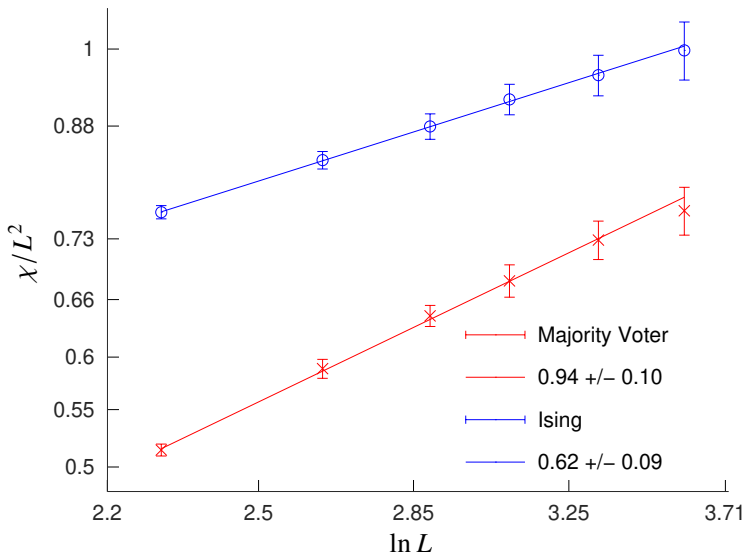

Figure 4. Maximum of the susceptibility of the 4D Majority-Voter model (cross) and of the 4D Ising model (circle) divided by L^2 versus the logarithm $\ln L$ of the lattice size L. The solid lines are power-law fits of the data. The estimated exponents $\hat{\gamma}/\hat{\nu}$ are indicated in the legend.

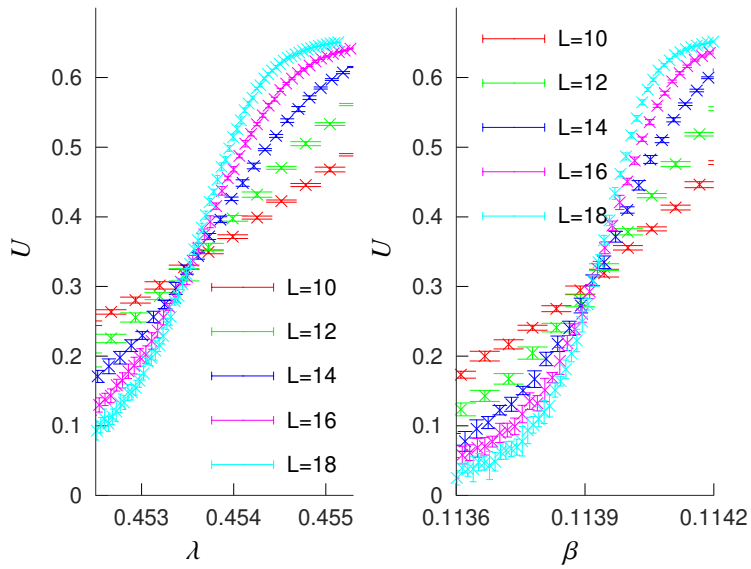

Figure 5. Binder cumulant U of the 5D Majority-Voter model (left) and of the 5D Ising model (right) versus the parameters λ and β. The different curves correspond to different lattice sizes.

Table 2. Estimates of the universal Binder cumulant $U_*(\infty)$ for the Majority Voter model (left) and the Ising model (right) on an hypercubic lattice of dimension d.

d	Majority Voter	Ising
2	0.62(4)	0.62(4)
3	0.47(6)	0.46(6)
4	0.36(2)	0.33(2)
5	0.27(7)	0.30(6)
6	0.31(5)	0.29(3)
7	0.40	0.33

lattice sizes is due to a too small number of points in the fit. The accuracy on the Binder cumulant is unfortunately not sufficient to estimate ν from the Finite-Size Scaling of $\frac{d}{d\lambda}U$ and $\frac{d}{d\beta}U$.

Conclusions

In conclusion, we have provided numerical evidences that the majority-voter model belongs to he universality class of the equilibrium Ising model for any dimension $2 \leq d \leq 7$. The immediate consequence is that the upper critical dimension of the majority-voter model is $d_c = 4$, like the Ising model. Above the upper critical dimension, the Finite-Size Scaling of the magnetic susceptibility is indeed close to the prediction $\chi \sim L^2$ at the Gaussian fixed point with a dangerous irrelevant variable. For both the majority-voter and Ising models, the small deviation to this law may be attributed to the too small lattice sizes that could be reached, and therefore to scaling corrections. At the upper critical dimension, multiplicative logarithmic corrections are present in both the majority-voter and Ising models but with a different exponent. Note that this is not a proof that $d_c = 4$ since the data can also be fitted with logarithmic corrections for $d > 4$, although with smaller exponents.

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