



HAL
open science

On Exotic Linear Materials: 2D Elasticity And Beyond

Guangjin Mou, Boris Desmorat, Robin Turlin, Nicolas Auffray

► **To cite this version:**

Guangjin Mou, Boris Desmorat, Robin Turlin, Nicolas Auffray. On Exotic Linear Materials: 2D Elasticity And Beyond. International Journal of Solids and Structures, 2023, 264, pp.112103. 10.1016/j.ijsolstr.2022.112103 . hal-03834502

HAL Id: hal-03834502

<https://hal.science/hal-03834502>

Submitted on 30 Oct 2022

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

ON EXOTIC LINEAR MATERIALS: 2D ELASTICITY AND BEYOND

G. MOU, B. DESMORAT, R. TURLIN, AND N. AUFFRAY

ABSTRACT. The symmetry classes of a linear constitutive law define the different types of anisotropy that can be modelled by it. However, the spaces of linear materials are very rich and a whole range of intermediate possibilities exists beyond symmetry classes. Geometric methods developed to characterise spaces of linear materials in a very fine way allow these intermediate possibilities to be detected. Materials with non-standard anisotropic properties associated with these intermediate possibilities are called exotic materials. In this paper, we provide a mathematical and mechanical definition of what an exotic material is. Using these definitions, we conclude that, for 2D linear elasticity, there is only one possible exotic material that meets our criteria. An example of a unit cell producing such exotic material is determined. Finally, the enumeration result obtained for elasticity is generalised to other bi-dimensional linear constitutive laws.

1. INTRODUCTION

Anisotropy characterises the way a physical property varies with respect to material directions. Linear properties, such as elasticity or conductivity, are encoded using constitutive tensors. Depending on their order, these tensors can model different types of anisotropies ranging from complete anisotropy to isotropy. These different possibilities are called *symmetry classes*, and have attracted much interest in recent years (Forte and Vianello, 1996; Geymonat and Weller, 2002; Le Quang and He, 2011; Olive, 2019).

The geometrical tools developed to determine the symmetry classes of a tensor space have also revealed the richness of these spaces, as well as the existence of a whole range of intermediate possibilities beyond the symmetry classes. Indeed, these tools allow to describe the linear material space in a very fine way and to detect materials with non-standard anisotropic properties. These intermediate possibilities are referred to here as *exotic*.

This possibility was identified very early on by authors working in this field (Rychlewski, 2001; Vannucci, 2002; He, 2004). For example, in the case of 2D linear elasticity, P. Vannucci has identified a particular situation which he calls R_0 -orthotropy. He says about it:

"The existence of a particular type of planar orthotropic material, [...] but, [...], not linked to a particular type of elastic symmetry condition. For this reason, the existence of this type of material cannot be revealed only by the use of certain symmetry conditions on the Cartesian components of \mathbb{C}^1 ." (Vannucci, 2002)

At the same moment, J. Rychlewski made some related observations and noticed that

"The fact is that materials of appreciably different anisotropy can behave, in certain situations and in certain aspects, in a completely similar or simply identical way. In particular, some essentially anisotropic materials may retain some important characteristics of isotropic materials." (Rychlewski, 2001)

But these early works seem to be curiosities and have not aroused much interest. This is owing to two major lacks that limited the applicability of these discoveries to the community:

- (1) lack of an *operative* mathematical definition of what an exotic elasticity is. In the contributions cited, only particular cases of exotic elastic materials have been studied, and no systematic classification has been undertaken. This is certainly due to the lack of a good definition for such a study;
- (2) to the best of our knowledge, mesostructures generating these exotic properties were not determined at this time². It is understandable because the topology optimisation was not as mature a technique as it is now (Ferrer, Cante, Hernández and Oliver, 2018; Wu, Sigmund and Groen, 2021) and additive manufacturing was also less democratised than now. In fact, there was also something missing with regard to practical applications.

¹ \mathbb{C} being the elasticity tensor.

²It should be noted, however, that P. Vannucci has determined a stacking sequence of elementary orthotropic layers producing a global elastic behaviour R_0 -orthotropic for a laminated plate (Vannucci, 2002).

Today, these practical locks have been broken. Very efficient multi-scale topology optimisation codes have been developed by the applied mathematics and mechanics communities (Amstutz, Giusti, Novotny and De Souza Neto, 2010; Amstutz, 2021; Laurain, 2018). And further, 3D printers have almost become standard laboratory tools. These tools have already been applied to the design of architected materials. However, all these studies are concerned with the optimisation of specific coefficients of the stiffness (or compliance) tensor in a given base. While this intuition-based method is effective in some cases, it is still rather home-made and does not exploit the progresses that have been made in describing the geometry of the elastic material space.

It is hence important to revisit this topic because the understanding of exotic linear behaviours opens up multiple possibilities for smart optimal design of architected materials that can accommodate seemingly incompatible design constraints. This is particularly true in 3D, and concerns both static and dynamic applications (Durand, Lebée, Seppecher and Sab, 2022; Camar-Eddine and Seppecher, 2003; Bückmann, Schittny, Thiel, Kadic, Milton and Wegener, 2014).

To go further in the design of custom elastic materials and to exploit exotic symmetry sets, it is necessary to understand the geometry of the elastic tensor space. In particular, how to characterise the strata of this space, i.e. the subsets of tensors of the same anisotropy type. It is from this description that we will be able to bring out the mathematical concept of exotic anisotropy.

The questions we ask ourselves and which this article proposes to answer are the following:

- (1) what is the mathematical definition for exotic materials ?
- (2) what are the properties of these materials ?
- (3) can we, *a priori*, list the number of exotic situations ?

In order to provide a mathematical definition while keeping a reasonable level of complexity, we will work in this article in a 2D framework, while keeping in mind the will to generalise it to 3D.

Following the observations made by Vanucci (Vanucci, 2002) and Rychlewski (Rychlewski, 2001), we propose in this paper to adopt the following *mechanical* definition of what an *exotic* linear material is:

- (1) **Specific design:** they satisfy constraints independent of those imposed by symmetry arguments;
- (2) **Hypersymmetric:** they produce more symmetrical behaviour than that imposed by material symmetries.

It should be noted that some specifically designed materials may have interesting non-standard properties while not meeting the hypersymmetry requirement. These materials will be referred to as *semi-exotic*. As will be seen, this mechanical definition will lead to a mathematical one and is operative since allowing to enumerate the exotic sets of a constitutive tensor space. As we shall see, in this context, general results can be obtained.

To avoid being too abstract, our main result will be inferred from the study of bidimensional elasticity. As such the first five sections will mainly be devoted to linear elasticity. In section 2 the main geometrical concepts are introduced and illustrated. The section 3 will introduce the concept of symmetry classes and the geometry of elastic strata. Having these tools at hands, the number of exotic elastic classes is determined in section 4. It will be shown that, from the definition retained, there is only one type of exotic symmetry in 2D, which is the one identified by P. Vanucci (Vanucci, 2002). This is due to the extremely simple harmonic structure of the elasticity tensor. In section 5, a multiscale topology optimisation approach is used to determine a local geometry producing such exotic effect in 2D linear elasticity. The reasoning used in section 4 is then generalised to treat more general constitutive tensor spaces. This generalisation is the object of the section 6. Applications to the piezoelectric law and to Cosserat elasticity are considered. As the harmonic structures of these constitutive laws are more complex, their number of exotic sets increases considerably. These results reveal that these coupled constitutive laws are highly non-trivial and opened to numerous exotic materials with possible mechanical applications.

Two appendix sections are also present in the document. Appendix A is dedicated to the main notations used in the article, and Appendix B to the symmetry properties (classes, invariants,...) of the compliance tensor. These sections have been moved to the appendix to improve the readability of the main part of the manuscript.

2. LINEAR ELASTICITY IN A NUTSHELL

2.1. The space of elasticity tensors. In the field of linear elasticity, the constitutive law is a local linear relation between the second-order symmetric Cauchy stress tensor $\underline{\underline{\sigma}}$ and the second-order symmetric infinitesimal strain tensor $\underline{\underline{\varepsilon}}$:

$$\underline{\underline{\sigma}} = \underline{\underline{\mathbb{C}}} : \underline{\underline{\varepsilon}},$$

in which $\underline{\underline{\sigma}}$ and $\underline{\underline{\varepsilon}}$ belong to $S^2(\mathbb{R}^2)$, the space of bi-dimensional symmetric second-order tensors, and $\underline{\underline{\mathbb{C}}}$ is the elasticity tensor, element of the following vector space,

$$\mathbb{E}la := \{ \underline{\underline{\mathbb{C}}} \in \otimes^4 \mathbb{R}^2 | C_{(ij)(kl)} \}, \quad \dim \mathbb{E}la = 6,$$

in which the notation $(ij)(kl)$ indicates the minor index symmetries and $\underline{\underline{ij}} \underline{\underline{kl}}$ the major one. Let see now how elasticity tensors are transformed when subjected to an isometry, i.e. to a transformation belonging to the orthogonal group

$$O(2) := \{ \mathbf{g} \in GL(2), \mathbf{g}^T = \mathbf{g}^{-1} \}.$$

$O(2)$ -action on an element $\underline{\underline{\mathbb{C}}}$ of $\mathbb{E}la$ gives a new element $\overline{\underline{\underline{\mathbb{C}}}}$ of $\mathbb{E}la$,

$$\overline{\underline{\underline{\mathbb{C}}}} = \mathbf{g} \star \underline{\underline{\mathbb{C}}}, \quad \mathbf{g} \in O(2),$$

in which the star product \star stands for the standard tensorial action. In components, in relation to a rectangular basis of \mathbb{R}^2 , this action is as follows:

$$\overline{C}_{ijkl} = g_{ip} g_{jq} g_{kr} g_{ls} C_{pqrs} \quad i = \{1, 2\}.$$

$\mathbb{E}la$ can also be viewed as

$$\mathbb{E}la = \mathbb{V} \otimes^s \mathbb{V}, \quad \text{with } \mathbb{V} = S^2(\mathbb{R}^2).$$

i.e. as a symmetric second-order tensor on $\mathbb{V} \simeq \mathbb{R}^3$. This picture is important for physical applications since energetic aspects, such as definite positiveness of the strain energy, are attached to this representation.

Consider $O(3)$ the set of rigid transformations acting on $\mathbb{V} \simeq \mathbb{R}^3$.

$$O(3) := \{ \mathbf{g} \in GL(3), \mathbf{g}^T = \mathbf{g}^{-1} \}.$$

Its action on an element $\underline{\underline{\mathbb{C}}}$ of $\mathbb{E}la$ gives a new element $\widehat{\underline{\underline{\mathbb{C}}}}$ of $\mathbb{E}la$,

$$\widehat{\underline{\underline{\mathbb{C}}}} = \widehat{\mathbf{g}} \otimes \underline{\underline{\mathbb{C}}}, \quad \widehat{\mathbf{g}} \in O(3),$$

in which the product \otimes stands for the standard tensorial action on \mathbb{R}^3 . In components, in relation to a rectangular basis of \mathbb{R}^3 , this action is as follows:

$$\widehat{C}_{IJK} = \widehat{g}_{IP} \widehat{g}_{JQ} \widehat{g}_{KR} C_{PQR} \quad I = \{1, 2, 3\}.$$

For being physically admissible, an elasticity tensor, considered as a quadratic form on $\mathbb{V} = S^2(\mathbb{R}^2)$, should be positive definite, meaning that its eigenvalues λ_i should verify

$$\exists M \in \mathbb{R}^{*+}, \quad 0 < \lambda_i \leq M.$$

Let us denote by $\mathbb{E}la^+$ the set of elasticity tensors that satisfy this requirement. We have the following property

$$\forall \underline{\underline{\mathbb{C}}} \in \mathbb{E}la^+, \exists! \underline{\underline{\mathbb{S}}} \in \mathbb{E}la^+, \underline{\underline{\mathbb{S}}} : \underline{\underline{\mathbb{C}}} = \underline{\underline{\mathbb{C}}} : \underline{\underline{\mathbb{S}}} = \underline{\underline{\mathbb{1}}},$$

with $\underline{\underline{\mathbb{1}}} = \underline{\underline{\mathbb{I}}} \otimes \underline{\underline{\mathbb{I}}}$ the identity of $S^2(\mathbb{R}^2)$, and $\underline{\underline{\mathbb{I}}}$ the identity of \mathbb{R}^2 . Such an element is known as the compliance tensor and allows to invert the constitutive law

$$\underline{\underline{\sigma}} = \underline{\underline{\mathbb{C}}} : \underline{\underline{\varepsilon}} \Leftrightarrow \underline{\underline{\varepsilon}} = \underline{\underline{\mathbb{S}}} : \underline{\underline{\sigma}}.$$

In the following the notation $\underline{\underline{\mathbb{S}}} = \underline{\underline{\mathbb{C}}}^{-1}$ and $\underline{\underline{\mathbb{C}}} = \underline{\underline{\mathbb{S}}}^{-1}$ will be used. The inversion is an $O(3)$ -equivariant automorphism of $\mathbb{E}la^+$, meaning that

$$\forall \mathbf{G} \in O(3), \quad \forall \underline{\underline{\mathbb{C}}} \in \mathbb{E}la^+, \quad (\mathbf{G} \otimes \underline{\underline{\mathbb{C}}})^{-1} = \mathbf{G} \otimes (\underline{\underline{\mathbb{C}}})^{-1}.$$

In other words, the following diagram commutes

$$\begin{array}{ccc} \underset{\cong}{\mathbb{C}} & \xrightarrow{(\cdot)^{-1}} & \underset{\cong}{\mathbb{S}} \\ \downarrow \mathbf{G} & & \downarrow \mathbf{G} \\ \underset{\cong}{\widehat{\mathbb{C}}} & \xrightarrow{(\cdot)^{-1}} & \underset{\cong}{\widehat{\mathbb{S}}} \end{array}$$

2.2. From elasticity tensors to elastic materials. When subjected to an isometry, the nature of an elastic material does not change. On the contrary, its elasticity tensor will vary. It results that, generically, multiple elasticity tensors are associated to the same elastic material. To speak intrinsically of an elastic material, it is necessary to remove information attached to a particular elasticity tensor.

The first step is to define the set of all elasticity tensors describing the same elastic material. Two stiffness tensors $\underset{\cong}{\mathbb{C}}, \underset{\cong}{\overline{\mathbb{C}}} \in \mathbb{Ela}$ are said to be equivalent, and denoted $\underset{\cong}{\mathbb{C}} \sim \underset{\cong}{\overline{\mathbb{C}}}$, when they are related by an orthogonal transformation, namely

$$\underset{\cong}{\mathbb{C}} \sim \underset{\cong}{\overline{\mathbb{C}}} \iff \exists \mathbf{g} \in O(2) \mid \underset{\cong}{\mathbb{C}} = \mathbf{g} \star \underset{\cong}{\overline{\mathbb{C}}}.$$

In such case, $\underset{\cong}{\mathbb{C}}$ and $\underset{\cong}{\overline{\mathbb{C}}}$ describe the same elastic material. The collection of all elasticity tensors describing an elastic material is a geometric object called the orbit of $\underset{\cong}{\mathbb{C}}$ and defined as

$$\text{Orb}(\underset{\cong}{\mathbb{C}}) = \{ \underset{\cong}{\overline{\mathbb{C}}} \in \mathbb{Ela}, \exists \mathbf{g} \in O(2) \mid \underset{\cong}{\overline{\mathbb{C}}} = \mathbf{g} \star \underset{\cong}{\mathbb{C}} \}.$$

It results that an elastic material corresponds to the orbit $\text{Orb}(\underset{\cong}{\mathbb{C}})$ of a tensor $\underset{\cong}{\mathbb{C}} \in \mathbb{Ela}$. Instead of considering geometric sets in \mathbb{Ela} , we can also construct a space $\mathbb{Ela}/O(2)$, called the *orbit space*, in which each point corresponds to an orbit in \mathbb{Ela} , i.e. to an elastic material (Abud and Sartori, 1983; Olive, Kolev and Auffray, 2017). A natural question is therefore how to designate points in $\mathbb{Ela}/O(2)$.

To speak of an elastic material independently of an elasticity tensor we need to define functions on \mathbb{Ela} that are constant over each orbit and take different values on different orbits. In other words, we need to determine a *separating set* Desmorat, Auffray, Desmorat, Olive and Kolev (2021), i.e.

Definition 2.1 (Separating set). *A finite set $\mathcal{S} := \{\kappa_1, \dots, \kappa_r\}$ of $O(2)$ -invariant functions is a separating set of $\mathbb{Ela}/O(2)$ if for any $\underset{\cong}{\mathbb{C}}, \underset{\cong}{\overline{\mathbb{C}}}$ in \mathbb{Ela}*

$$\text{Orb}(\underset{\cong}{\mathbb{C}}) = \text{Orb}(\underset{\cong}{\overline{\mathbb{C}}}) \iff \kappa_i(\underset{\cong}{\mathbb{C}}) = \kappa_i(\underset{\cong}{\overline{\mathbb{C}}}), \quad i = 1, \dots, r.$$

A separating set \mathcal{S} is said to be minimal if any strict subset \mathcal{S}' of \mathcal{S} is no longer a separating set.

2.3. Invariant algebra. Let consider the *algebra* $\mathbb{R}[\mathbb{Ela}]$ of polynomial functions on \mathbb{Ela}

$$\begin{array}{ccc} p : \mathbb{Ela} & \rightarrow & \mathbb{R} \\ \underset{\cong}{\mathbb{C}} & \mapsto & p(\underset{\cong}{\mathbb{C}}). \end{array}$$

A polynomial function in $\underset{\cong}{\mathbb{C}}$ is a polynomial in the components of $\underset{\cong}{\mathbb{C}}$ expressed with respect to a given basis. We can further consider the algebra $\mathbb{R}[\mathbb{Ela}]^{O(2)}$ of $O(2)$ -invariant polynomials:

$$\mathbb{R}[\mathbb{Ela}]^{O(2)} := \left\{ p \in \mathbb{R}[\mathbb{Ela}], \quad p(\mathbf{g} \star \underset{\cong}{\mathbb{C}}) = p(\underset{\cong}{\mathbb{C}}), \quad \forall \mathbf{g} \in O(2), \quad \underset{\cong}{\mathbb{C}} \in \mathbb{Ela} \right\}.$$

As a consequence of Hilbert's finiteness theorem (Sturmfels, 2008), such an algebra is finitely generated and any finite generating set $\{I_1, \dots, I_N\}$ of $\mathbb{R}[\mathbb{Ela}]^{O(2)}$ is called an *integrity basis* (Weyl, 1946; Vianello, 1997). The generating property means that any $O(2)$ -invariant polynomial $J \in \mathbb{R}[\mathbb{Ela}]^{O(2)}$ is a polynomial function in I_1, \dots, I_N :

$$J(\underset{\cong}{\mathbb{C}}) = p(I_1(\underset{\cong}{\mathbb{C}}), \dots, I_N(\underset{\cong}{\mathbb{C}})), \quad \underset{\cong}{\mathbb{C}} \in \mathbb{Ela},$$

where p is a polynomial in N variables. An integrity basis is *minimal* if no proper subset of it is an integrity basis. Knowing an *integrity basis* is interesting for applications since their elements:

- **Generate the algebra of $O(2)$ -invariant polynomials:** any $O(2)$ -polynomial function can be written as a polynomial in the elements of the integrity basis;

- **Separate the O(2)-orbits:** the invariants of the integrity basis take the same value if evaluated on two sets of constitutive tensors that just differ up to an isometry, and take different values if not.

The ability of expressing O(2)-invariant polynomials is important in practice since it allows to specify sets of materials in the orbit space. Said differently, the family of elastic materials can be defined by relations of the following forms

$$V := \{X \in \text{Ela}/\text{O}(2) \mid p_i(X) = 0, \text{ and, } p_q(X) \leq 0, p \in \mathbb{R}[\text{Ela}]^{\text{O}(2)}\}.$$

The question is now to determine such an integrity basis. For the case of $\mathbb{E}la$ the answer is known and will be detailed in subsection 2.5. But to reach this point, the first step is to decompose the space $\mathbb{E}la$ into a collection of O(2)-irreducible spaces. This decomposition is known as the harmonic decomposition and, in the next subsection, its explicit decomposition will be provided.

2.4. Harmonic decomposition of elasticity tensors. The harmonic decomposition is an O(2)-equivariant isomorphism between the $\text{E}la$ and a direct sum of harmonic tensor spaces

$$\mathbb{E}la \simeq 2\mathbb{K}^0 \oplus \mathbb{K}^2 \oplus \mathbb{K}^4.$$

in which \mathbb{K}^n denotes the space of n -th order completely symmetric and traceless tensors on \mathbb{R}^2 , called *harmonic tensors*³. Due to the multiplicity of the space \mathbb{K}^0 , the explicit harmonic decomposition is not defined uniquely. This results in multiple possibilities for choosing a basis for the isotropic components.

Let us denote by f an explicit harmonic decomposition, we have

$$\underset{\approx}{\mathbb{C}} = f(\alpha, \beta, \underset{\approx}{\mathbf{h}}, \underset{\approx}{\mathbf{H}}),$$

and the O(2)-equivariance property

$$\forall \mathbf{g} \in \text{O}(2), \mathbf{g} \star \underset{\approx}{\mathbb{C}} = f(\alpha, \beta, \mathbf{g} \star \underset{\approx}{\mathbf{h}}, \mathbf{g} \star \underset{\approx}{\mathbf{H}}).$$

The harmonic decomposition splits any elasticity tensor $\underset{\approx}{\mathbb{C}}$ into an isotropic part defined by two scalars α and β completed by an anisotropic part comprising $\underset{\approx}{\mathbf{h}} \in \mathbb{K}^2$ and $\underset{\approx}{\mathbf{H}} \in \mathbb{K}^4$. The set of $\{\underset{\approx}{\mathbf{h}}, \underset{\approx}{\mathbf{H}}\}$ will be referred to as the *harmonic bouquet* of $\underset{\approx}{\mathbb{C}}$, and denoted by $\text{HB}(\underset{\approx}{\mathbb{C}})$.

Among the different possibilities⁴, the Clebsch-Gordan harmonic decomposition (Auffray, Abdoul-Anziz and Desmorat, 2021a) will be considered here:

Proposition 2.2. *The tensor $\underset{\approx}{\mathbb{C}} \in \mathbb{E}la$ admits the uniquely defined Clebsch-Gordan Harmonic Decomposition associated to the family of projectors $\{\underset{\approx}{\mathbb{J}}, \underset{\approx}{\mathbb{K}}\}$ ⁵:*

$$(2.1) \quad \underset{\approx}{\mathbb{C}} = \alpha \underset{\approx}{\mathbb{J}} + \beta \underset{\approx}{\mathbb{K}} + \frac{1}{2}(\underset{\approx}{1} \otimes \underset{\approx}{\mathbf{h}} + \underset{\approx}{\mathbf{h}} \otimes \underset{\approx}{1}) + \underset{\approx}{\mathbf{H}},$$

in which $\{\alpha, \beta, \underset{\approx}{\mathbf{h}}, \underset{\approx}{\mathbf{H}}\}$ are elements of $\mathbb{K}^0 \times \mathbb{K}^0 \times \mathbb{K}^2 \times \mathbb{K}^4$ defined from $\underset{\approx}{\mathbb{C}}$ as follows:

\mathbb{K}^0	\mathbb{K}^2	\mathbb{K}^4
$\beta = \underset{\approx}{\mathbb{K}} :: \underset{\approx}{\mathbb{C}}$	$\underset{\approx}{\mathbf{h}} = \underset{\approx}{\mathbb{J}} : \underset{\approx}{\mathbb{C}} : \underset{\approx}{1}$	
$\alpha = \underset{\approx}{\mathbb{D}} :: \underset{\approx}{\mathbb{J}}$		$\underset{\approx}{\mathbf{H}} = \underset{\approx}{\mathbb{D}} - \frac{\alpha}{2} \underset{\approx}{\mathbb{J}}$

where $\underset{\approx}{\mathbb{D}} = \underset{\approx}{\mathbb{J}} : \underset{\approx}{\mathbb{C}} : \underset{\approx}{\mathbb{J}}$ denotes the deviatoric part of $\underset{\approx}{\mathbb{C}}$.

From a mechanical standpoint, this decomposition can be interpreted as follows

$$\begin{pmatrix} \underset{\approx}{\sigma}^d \\ \underset{\approx}{\sigma}^s \end{pmatrix} = \begin{pmatrix} \underset{\approx}{\mathbf{H}} + \alpha \underset{\approx}{\mathbb{J}} & \frac{1}{2} \underset{\approx}{\mathbf{h}} \otimes \underset{\approx}{1} \\ \frac{1}{2} \underset{\approx}{1} \otimes \underset{\approx}{\mathbf{h}} & \beta \underset{\approx}{\mathbb{K}} \end{pmatrix} \begin{pmatrix} \underset{\approx}{\varepsilon}^d \\ \underset{\approx}{\varepsilon}^s \end{pmatrix},$$

³In \mathbb{R}^2 , we have $\dim \mathbb{K}^n = 2$ for $n > 0$ and 1 for $n = \{-1, 0\}$.

⁴In 2D, the different decompositions are almost identical and their differences only concern the isotropic part. In 3D, on the other hand, the choice of a decomposition also involves the anisotropic components and is therefore more important.

⁵The deviatoric projector $\underset{\approx}{\mathbb{J}}$ and spheric projector $\underset{\approx}{\mathbb{K}}$ are defined in Appendix A.

in which $\underset{\sim}{t}^s, \underset{\sim}{t}^d$ denote, respectively, the spheric and deviatoric part of $\underset{\sim}{t}$. Hence,

$$\begin{cases} \underset{\sim}{\sigma}^d = \underset{\sim}{\mathbb{H}} : \underset{\sim}{\varepsilon}^d + \alpha \underset{\sim}{\varepsilon}^d + \frac{1}{2} \text{tr}(\underset{\sim}{\varepsilon}^s) \underset{\sim}{\mathbb{h}} \\ \underset{\sim}{\sigma}^s = \frac{1}{2} \left(\underset{\sim}{\mathbb{h}} : \underset{\sim}{\varepsilon}^d \right) \underset{\sim}{1} + \beta \underset{\sim}{\varepsilon}^s \end{cases}.$$

where $\text{tr}(\underset{\sim}{\varepsilon}^s)$ is the trace of $\underset{\sim}{\varepsilon}^s$.

2.5. Integrity basis for elasticity tensors. Integrity bases for the $O(2)$ -action are known since the second-half of the 90' (Blinowski, Ostrowska-Maciejewska and Rychlewski, 1996; Vianello, 1997). Consider the following quantities:

$$I_1 = \alpha, \quad J_1 = \beta, \quad I_2 = \underset{\sim}{\mathbb{h}} : \underset{\sim}{\mathbb{h}}, \quad J_2 = \underset{\sim}{\mathbb{H}} :: \underset{\sim}{\mathbb{H}}, \quad I_3 = \underset{\sim}{\mathbb{h}} : \underset{\sim}{\mathbb{H}} : \underset{\sim}{\mathbb{h}}.$$

Those elements are polynomial functions of $\underset{\sim}{\mathbb{C}}$ that are $O(2)$ -invariant. In the notation, the subscript indicates the degree of the polynomial in the elasticity tensor. We have the following result (Vianello, 1997):

Theorem 2.3. *A minimal integrity basis for $O(2)$ -action on $\mathbb{E}la$ is given by*

$$\mathcal{IB} = (I_1, J_1, I_2, J_2, I_3).$$

These elements are free, meaning that they are not related by any polynomial relation. They satisfy, however, the following inequalities

$$(2.2) \quad I_2 \geq 0, \quad J_2 \geq 0, \quad I_2^2 J_2 - 2I_3^2 \geq 0.$$

As such they define a closed domain of \mathbb{R}^5 which will be denoted by \mathcal{V} . Note that the last inequality implies that $I_3 = 0$ as soon as I_2 or J_2 is zero.

We define the following application from $\mathbb{E}la$ to $\mathbb{E}la/O(2)$ which associates to a tensor its (uniquely defined) elastic material:

$$\mathcal{IB}(\underset{\sim}{\mathbb{C}}) := \left(I_1(\underset{\sim}{\mathbb{C}}), J_1(\underset{\sim}{\mathbb{C}}), I_2(\underset{\sim}{\mathbb{C}}), J_2(\underset{\sim}{\mathbb{C}}), I_3(\underset{\sim}{\mathbb{C}}) \right).$$

3. SYMMETRY CLASSES

In the previous section, no mention was made of the spatial invariance properties that an elasticity tensor $\underset{\sim}{\mathbb{C}}$ may possess. The purpose of this section is to introduce this notion and to see how these properties translate into the orbit space. Let us start by defining a totally generic elasticity tensor with the lowest level of spatial invariance.

3.1. Genericity. Elasticity tensors $\underset{\sim}{\mathbb{C}}$, i.e. satisfying $I_2^2 J_2 - 2I_3^2 > 0$ are said to be generic. Almost all elasticity tensors are generic, meaning that the probability is 1 of randomly picking elasticity tensors satisfying these relations.

From a geometric point of view, the anisotropic harmonic bouquet $\mathbb{HB} = \{\underset{\sim}{\mathbb{h}}, \underset{\sim}{\mathbb{H}}\}$ is non degenerated, i.e.

- neither $\underset{\sim}{\mathbb{h}}$ nor $\underset{\sim}{\mathbb{H}}$ is nil;
- they are not *aligned*, meaning that $(\underset{\sim}{\mathbb{h}} * \underset{\sim}{\mathbb{h}}) \times \underset{\sim}{\mathbb{H}} \neq 0$;

in which $*$ and \times stand, respectively, for the *harmonic* and *generalised cross product* as defined in Appendix A. This last condition motivates the introduction of the concept of *Homogeneous Harmonic Bouquet* $\mathbb{HHB} = \{\underset{\sim}{\mathbb{h}} * \underset{\sim}{\mathbb{h}}, \underset{\sim}{\mathbb{H}}\}$, in which all elements belong to the same harmonic space and therefore transform in the same way. This notion will allow us to introduce the *harmonic normal form* of an elasticity tensor. To this end, let us introduce $\mathcal{K}^n = (\mathbf{K}_1^n, \mathbf{K}_2^n)$ the orthonormal basis of \mathbb{K}^n constructed from the canonical basis \mathcal{B} of \mathbb{R}^2 (Desmorat, Olive, Auffray, Desmorat and Kolev, 2020). Since elasticity tensors sharing the same orbit have conjugate \mathbb{HHB} , the following result is natural

$$\forall \underset{\sim}{\mathbb{C}} \in \mathbb{E}la, \exists \mathbf{g} \in O(2) \text{ s.t. } (\mathbf{g} \star \underset{\sim}{\mathbb{H}})_1 = 0, (\mathbf{g} \star \underset{\sim}{\mathbb{H}})_2 > 0$$

with $(\underset{\sim}{\mathbb{H}})_i = \underset{\sim}{\mathbb{H}} :: \mathbf{K}_i^4$. A generic tensor $\underset{\sim}{\mathbb{C}}$ is in its harmonic normal form, if $(\underset{\sim}{\mathbb{H}})_1 = 0$, and $(\underset{\sim}{\mathbb{H}})_2 > 0$. Geometrically, this corresponds to the configuration depicted on Figure 1.

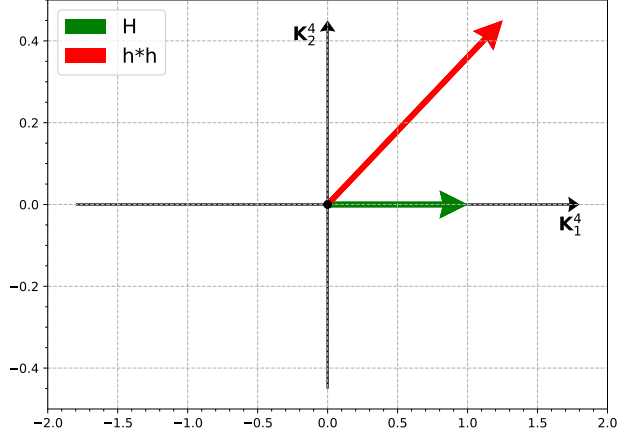


FIGURE 1. Harmonic normal form of a generic elasticity tensor.

Harmonic bouquets not satisfying the genericity conditions are said *degenerated*. Tensors that are invariant w.r.t. to some spatial transformations usually posses degenerated bouquet⁶. But, as will be seen, this mechanism is not surjective, meaning that some degenerated systems may not imply a specific spatial invariance. This is particularly true for tensors with a large harmonic bouquet.

3.2. From symmetry groups to symmetry classes. Let consider the image $\bar{\mathbb{C}}$ of \mathbb{C} by an isometric transformation:

$$\bar{\mathbb{C}} = \mathbf{g} \star \mathbb{C}, \quad \mathbf{g} \in O(2).$$

Depending on \mathbb{C} , for some transformations, the resulting tensor may be identical to the original one. The set of such transformations constitutes its symmetry group

$$G_{\mathbb{C}} := \{\mathbf{g} \in O(2) \mid \mathbb{C} = \mathbf{g} \star \mathbb{C}\}.$$

Tensors on the same orbit have conjugate symmetry groups. A weaker equivalence relation than being on the same orbit can be defined and consists only in having a conjugate symmetry group. This weaker equivalence relation among elements of $\mathbb{E}la$ is defined as follows

$$\mathbb{C} \approx_{\cong} \bar{\mathbb{C}} \quad \Leftrightarrow \quad \{\exists \mathbf{g} \in O(2) \mid G_{\bar{\mathbb{C}}} = \mathbf{g} G_{\mathbb{C}} \mathbf{g}^{-1}\}.$$

This relation indicates that two tensors are equivalent if their symmetry groups are conjugate. The equivalence classes for this relation are called strata. More specifically, in what follows $\Sigma_{[H]}$ will denote the equivalence class of elasticity tensors having their symmetry group conjugate to H . In other words, $[H]$ is the symmetry class of the elements of the (open) stratum $\Sigma_{[H]}$ (Auffray, Kolev and Petitot, 2014; Auffray, Kolev and Olive, 2017). The space of 2D elasticity tensors is divided into 4 strata (He and Zheng, 1996; Vianello, 1997; Auffray et al., 2017):

$$\mathbb{E}la = \Sigma_{[Z_2]} \cup \Sigma_{[D_2]} \cup \Sigma_{[D_4]} \cup \Sigma_{[O(2)]}.$$

In mechanical terms, $\Sigma_{[Z_2]}$ corresponds to the set of biclinic materials, $\Sigma_{[D_2]}$ to the set of orthotropic materials. $\Sigma_{[D_4]}$ to the set of tetragonal materials and $\Sigma_{[O(2)]}$ to isotropic materials⁷. These four strata are organised as follows⁸:

$$(3.1) \quad \Sigma_{[Z_2]} \xrightarrow{(\tilde{h} * \tilde{h}) \times \tilde{H} = 0} \Sigma_{[D_2]} \xrightarrow{\tilde{h} : \tilde{h} = 0} \Sigma_{[D_4]} \xrightarrow{\tilde{H} : \tilde{H} = 0} \Sigma_{[O(2)]}.$$

Geometrically, the harmonic normal forms of elasticity tensors in the three remaining classes are⁹:

⁶The case of even-order tensors is a bit special since generic even-order tensors are Z_2 -invariant.

⁷The group notation Z_k and D_k are defined in Appendix A.

⁸It should be noted that such an in-line structure is exceptional and is very specific to 2D linear elasticity.

⁹A tetragonal tensor \mathbb{C} is in its harmonic normal form, if $(\tilde{h} * \tilde{h})_1 = 0$, and $(\tilde{h} * \tilde{h})_2 > 0$.

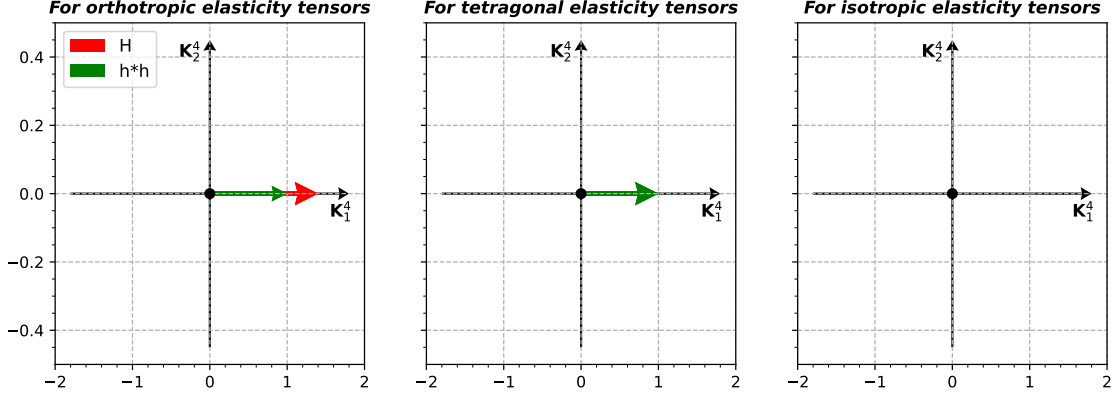


FIGURE 2. Harmonic normal form of non generic elasticity tensors.

The symmetry class of the elements belonging to the *open* stratum $\Sigma_{[H]}$ is *exactly* $[H]$. The closed strata $\bar{\Sigma}_{[H]}$ on its side contains elements which symmetry classes are at least $[H]$. Since the lattice of symmetry classes is linear (3.1):

$$\begin{aligned}\bar{\Sigma}_{[Z_2]} &= \Sigma_{[Z_2]} \cup \Sigma_{[D_2]} \cup \Sigma_{[D_4]} \cup \Sigma_{[O(2)]} (= \mathbb{E}la), \\ \bar{\Sigma}_{[D_2]} &= \Sigma_{[D_2]} \cup \Sigma_{[D_4]} \cup \Sigma_{[O(2)]}, \\ \bar{\Sigma}_{[D_4]} &= \Sigma_{[D_4]} \cup \Sigma_{[O(2)]}, \\ \bar{\Sigma}_{[O(2)]} &= \Sigma_{[O(2)]}.\end{aligned}$$

At the exception of $\Sigma_{[O(2)]}$, open strata are not vector spaces. The simplest reason for this is that the identity element does not belong to strata other than the isotropic one. For the closed strata, the situation is different. Since elements within a stratum have conjugate symmetry groups, the stability with regard to linear combination is not automatic. It can be proved that for 2D linear elasticity, all closed strata, except the orthotropy one, are vector spaces (Vianello, 1997; Antonelli, Desmorat, Kolev and Desmorat, 2022).

3.3. Polynomial invariant conditions characterising symmetry classes. For the least symmetric class, that is for biclinic elastic materials, the polynomial invariants of \mathcal{IB} are algebraically independent. A biclinic material is described by five independent quantities, that is by a point in $\mathcal{V} \subset \mathbb{R}^5$. The location of this point is not any, since constrained by the relations (2.2). For elastic materials with higher symmetries, polynomial relations (also called syzygies) between elements of \mathcal{IB} appear. For example, for (at least) orthotropic materials the following polynomial relation is satisfied:

$$I_2^2 J_2 - 2I_3^2 = 0,$$

hence orthotropic materials belong to $\partial\mathcal{V}$ which is a 4D surface in \mathbb{R}^5 . These relations for all strata are provided in the following table.

TABLE 1. Polynomial conditions for membership of an open stratum

stratum	Tensor representations	Polynomial conditions
$\Sigma_{[Z_2]}$	$(\alpha, \beta, \underset{\sim}{h}, \underset{\approx}{H})$	$I_2^2 J_2 - 2I_3^2 > 0$
$\Sigma_{[D_2]}$	$(\alpha, \beta, \underset{\sim}{h}, \underset{\approx}{H})$	$I_2^2 J_2 - 2I_3^2 = 0$ and $I_2 \neq 0$
$\Sigma_{[D_4]}$	$(\alpha, \beta, 0, \underset{\approx}{H})$	$I_2 = 0$ and $J_2 \neq 0$
$\Sigma_{[O(2)]}$	$(\alpha, \beta, 0, 0)$	$I_2 + J_2 = 0$

Polynomial transitions are summed-up on the following lattice:

$$\Sigma_{[Z_2]} \xrightarrow{I_2^2 J_2 - 2I_3^2 = 0} \Sigma_{[D_2]} \xrightarrow{I_2 = 0} \Sigma_{[D_4]} \xrightarrow{J_2 = 0} \Sigma_{[O(2)]}.$$

In the case of 2D elasticity, the geometry of the elastic material space can be visualised. Figure 3 depicts the

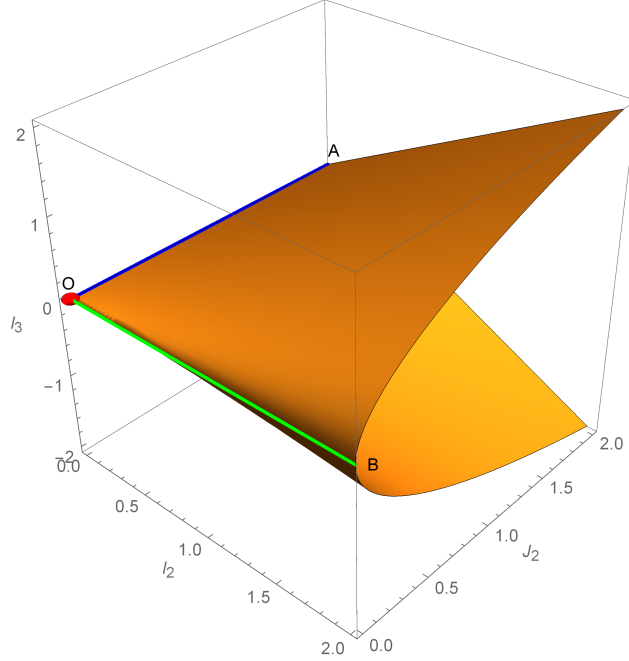


FIGURE 3. Semi-algebraic variety of elastic materials with respect to (I_2, J_2, I_3) ¹⁰.

elastic material space with respect to (I_2, J_2, I_3) . The surface, which corresponds to the polynomial equation: $I_2^2 J_2 - 2I_3^2 = 0$ contains all the *at-least-orthotropic* materials (stratum $\overline{\Sigma}_{[D_2]} = \Sigma_{[D_2]} \cup \Sigma_{[D_4]} \cup \Sigma_{[O(2)]}$). The condition $I_2^2 J_2 - 2I_3^2 > 0$ indicates on which side of the orthotropic surface are the biclinic materials located (stratum $\Sigma_{[Z_2]}$). Finally, we get that, independently of the values of the isotropic invariants I_1 and J_1 :

- point O corresponds to isotropic materials (stratum $\Sigma_{[O(2)]}$);
- open ray $]OA)$ corresponds to tetragonal materials (stratum $\Sigma_{[D_4]}$);
- surface without $\{O\} \cup]OA) \cup]OB)$ corresponds to ordinary orthotropic materials (stratum $\Sigma_{[D_2]}$);
- biclinic materials (stratum $\Sigma_{[Z_2]}$) are strictly inside the volume defined by the surface.

It can be observed for isotropic and tetragonal materials that spatial symmetries imply the vanishing of invariant polynomials. But this mechanism does not exhaust all possibilities since, for instance, no spatial symmetry results from the conditions $J_2 = 0$ or $I_3 = 0$. It follows that if one wants to design a mesostructure such that the effective elasticity tensor verifies these relationships, this cannot be done by imposing symmetry restrictions alone. These relations must be imposed by a specific design of the mesostructure. This aspect satisfies the first point of the properties defining *exotic* materials, as stated in the introduction.

It may be tempting to define as *exotic* any material defined by the vanishing of polynomial quantities not associated to a symmetry invariance. However, it is not sufficient because it does not guarantee the second requirement, which is to produce a paradoxical behaviour that is more symmetrical in appearance than expected. Elastic materials that do not fulfil this second requirement will be referred to as *semi-exotic*.

In order to select from the many possibilities those that really define exotic materials, we will return to geometrical considerations in the next section. At the core of this approach is the clip product, a tool for deducing the symmetry classes of a tensor space from its harmonic decomposition. It is important to note that this approach extends almost directly to 3D situations (Olive, 2019). This is the great strength of this approach.

4. TOWARDS EXOTIC ELASTIC MATERIALS

As previously said, an elasticity material will be said to be *exotic*, provided

- (1) it satisfy constraints independent of those that may be imposed by symmetry arguments;
- (2) its behaviour appears to be more symmetrical than that imposed by the material symmetries.

¹⁰Without taking the positive definiteness condition into account.

It should be noted that the second point excludes isotropic materials from the family of exotic materials. Indeed, since the material is already fully isotropic, a specific design, even if possible and potentially interesting, cannot produce a paradoxical increase of symmetry. Therefore, our definition here is adapted to anisotropic exotic behaviour.

4.1. Symmetry classes and clips operation. To establish the symmetry classes of a tensor space, symmetry classes of harmonic tensor spaces \mathbb{K}^n will first be considered. Using the *clips operator* defined in (Auffray et al., 2017), these results can be combined.

Let $\mathfrak{J}(\mathbb{K}^n)$ denotes the set of all isotropy classes associated to \mathbb{K}^n . These classes are given by the following theorem (cf. Appendix A for notations):

Theorem 4.1. *The symmetry classes of \mathbb{K}^n are:*

$$\mathfrak{J}(\mathbb{K}^n) = \begin{cases} n \geq 1, & \{[D_n], [O(2)]\} \\ n = -1, & \{[SO(2)], [O(2)]\} \\ n = 0, & \{[O(2)]\} \end{cases} .$$

with the convention that $D_1 = Z_2^{\sigma_x}$.

From this result, the symmetry classes of $\mathbb{K}^p \oplus \mathbb{K}^q$ are obtained as

$$\mathfrak{J}(\mathbb{K}^p \oplus \mathbb{K}^q) = \mathfrak{J}(\mathbb{K}^p) \odot \mathfrak{J}(\mathbb{K}^q).$$

in which \odot is the clips product. Clips operations on conjugacy classes $[H_i]$ of closed $O(2)$ -subgroups are defined as follows:

$$[H_1] \odot [H_2] = \bigcup_{g \in O(2)} H_1 \cap (gH_2g^{-1}).$$

The result clips operations are given in the following table (Auffray et al., 2017):

\odot	[Id]	$[Z_2^{\sigma_x}]$	$[Z_n]$	$[D_n]$	$[SO(2)]$	$[O(2)]$
[Id]	[Id]					
$[Z_2^{\sigma_x}]$	[Id]	[Id], $[Z_2^{\sigma_x}]$				
$[Z_m]$	[Id]	[Id]	$[Z_{d(n,m)}]$			
$[D_m]$	[Id]	[Id], $[Z_2^{\sigma_x}]$	$[Z_{d(n,m)}]$	$[Z_{d(n,m)}]$, $[D_{d(n,m)}]$		
$[SO(2)]$	[Id]	[Id]	$[Z_n]$	$[Z_n]$	$[SO(2)]$	
$[O(2)]$	[Id]	$[Z_2^{\sigma_x}]$	$[Z_n]$	$[D_n]$	$[SO(2)]$	$[O(2)]$

Notations: $Z_1 := \text{Id}$, $D_1 := Z_2^{\sigma_x}$, $d := \text{gcd}(n, m)$.

4.2. Space of elasticity tensors. The symmetry classes of $\mathbb{E}la$ can be obtained using clips products. This determination start with the harmonic structure of $\mathbb{E}la$:

$$\mathbb{E}la \simeq 2\mathbb{K}^0 \oplus \mathbb{K}^2 \oplus \mathbb{K}^4.$$

Since $\mathfrak{J}(\mathbb{K}^0) = [O(2)]$ the symmetry classes of $\mathbb{E}la$ are given by

$$\mathfrak{J}(\mathbb{E}la) = \mathfrak{J}(\mathbb{K}^2) \odot \mathfrak{J}(\mathbb{K}^4).$$

The symmetry classes of the harmonic components are

$$\mathfrak{J}(\mathbb{K}^2) = \{[D_2], [O(2)]\}, \quad \mathfrak{J}(\mathbb{K}^4) = \{[D_4], [O(2)]\}.$$

i.e. either the component associated with \mathbb{K}^n is not nil and its symmetry class is $[D_n]$ or it is nil and then it is isotropic.

The symmetry classes of the harmonic bouquet $\{\underset{\sim}{h}, \underset{\approx}{H}\}$ are obtained by the different possible clips of these elementary symmetry classes:

$\mathfrak{J}(\mathbb{K}^2) \odot \mathfrak{J}(\mathbb{K}^4)$	$[D_4]$	$[O(2)]$
$[D_2]$	$[Z_2], [D_2]$	$[D_2]$
$[O(2)]$	$[D_4]$	$[O(2)]$

Several things can be observed:

- (1) the stratification of $\mathbb{E}1a$ is retrieved;
- (2) non-nil covariants $\underset{\sim}{h}$ and $\underset{\sim}{H}$ generates classes:
 - $[Z_2]$, which corresponds to a generic orientation of the pair $(\underset{\sim}{h}, \underset{\sim}{H})$, i.e.

$$(\underset{\sim}{h} * \underset{\sim}{h}) \times \underset{\sim}{H} \neq 0,$$

- $[D_2]$, which corresponds to the *alignment* of the pair, i.e.

$$(\underset{\sim}{h} * \underset{\sim}{h}) \times \underset{\sim}{H} = 0.$$

- (3) the symmetry class $[D_2]$ is obtained in two different manners¹¹

$$[D_2]_{(\underset{\sim}{h}, \underset{\sim}{H})} = \{[D_2]_{\underset{\sim}{h}} \odot [D_4]_{\underset{\sim}{H}}, [D_2]_{\underset{\sim}{h}} \odot [O(2)]_{\underset{\sim}{H}}\},$$

It results that the stratum $\Sigma_{[D_2]}$ can be divided into two subsets:

$$\Sigma_{[D_2]} = \Sigma_{[D_2]}^g \cup \Sigma_{[D_2]}^e.$$

with the harmonic bouquet being of type $\left([D_2]_{\underset{\sim}{h}}, [D_4]_{\underset{\sim}{H}}\right)$ for $\Sigma_{[D_2]}^g$ while being $\left([D_2]_{\underset{\sim}{h}}, [O(2)]_{\underset{\sim}{H}}\right)$ for $\Sigma_{[D_2]}^e$. The first subset will be said *generic*¹², while the second will be called *exotic*. Elements in $\Sigma_{[D_2]}^e$, which corresponds to the polynomial condition $J_2 = 0$, satisfy the following points:

- (1) this restriction does not only come from a symmetry requirement but also should satisfy an extra constraint;
- (2) the cancellation of $\underset{\sim}{H}$ results in an orthotropic material for which the deviatoric elasticity is isotropic, and thus more symmetrical than it should be. This will be clearer in the forthcoming Equation 4.2.

It should be noted that this is the only situation satisfying our definition that emerges through the clips symmetry analysis. This exotic situation has been identified yet in the literature and is sometimes known as the R_0 -orthotropy¹³. We just demonstrates that it was in fact, and according to the considered definition, the only possibility for $\mathbb{E}1a$ in 2D. For example, the apparently interesting case corresponding to $I_3 = 0$, does not produce a paradoxical symmetric situation and therefore cannot be considered as an exotic elastic material. This situation is called *semi-exotic*.

4.3. R_0 -orthotropic materials. Exotic materials are represented by the harmonic components:

$$(4.1) \quad \underset{\sim}{C} = f(\alpha, \beta, \underset{\sim}{h}, 0).$$

The set of such elastic materials is located on Figure 3 on the open ray $]OB)$. It corresponds to a subset of the stratum $\Sigma_{[D_2]}$. An important remark is that this property is not stable with respect to inversion. The harmonic components for $\underset{\sim}{S} = \underset{\sim}{C}^{-1}$ are denoted by $\{\alpha^-, \beta^-, \underset{\sim}{h}^-, \underset{\sim}{H}^-\}$, and their expressions can be found in Appendix B. It can be observed that, in the case of $\underset{\sim}{H} = 0$, the expression of $\underset{\sim}{H}^-$ reduces to,

$$\underset{\sim}{H}^- = \frac{2}{\Delta} (2\underset{\sim}{h} * \underset{\sim}{h}) \neq 0,$$

Hence,

$$\underset{\sim}{C} \in \Sigma_{[D_2]}^e \Rightarrow \underset{\sim}{S} \notin \Sigma_{[D_2]}^e.$$

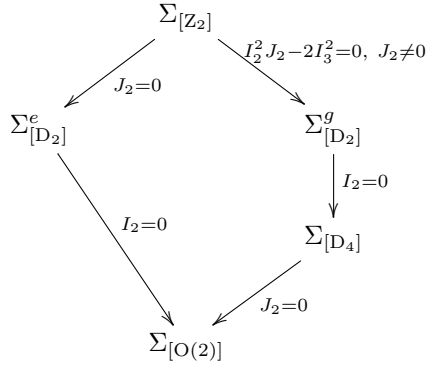
and conversely. A symmetry class is intrinsic to an elastic material, it is a property that does not depend on the choice of its description in terms of stiffness or compliance. Specifically, exotic sets of elastic materials can not be considered as symmetry classes. It results that exotic orthotropic materials can either be defined with respect to stiffness or with respect to compliance. But these materials are distinct since their respective inverse are not exotic (Vannucci, 2002). In the polar literature, stiffness exotic orthotropic materials are referred to as R_0 -orthotropic, while those in compliance are referred to as r_0 -orthotropic.

In the end, this gives the following complete structure of transition between the different strata:

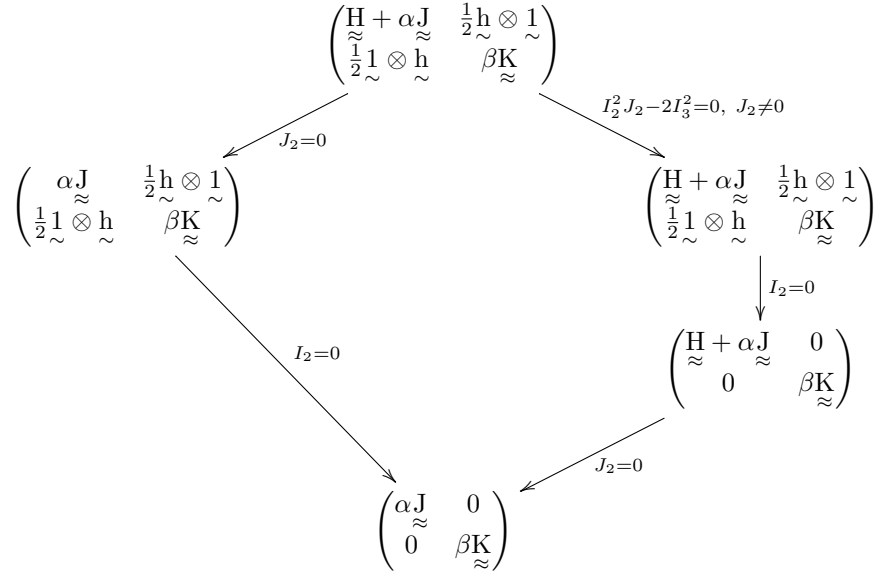
¹¹The notation $[H]_X$ indicates the symmetry class of X

¹²It should be noted that $\Sigma_{[DD_2]}^g$ is divided into two separate connected components, the membership of an elastic material to one or the other component is indicated by the sign of I_3 .

¹³This name comes from the polar parameterisation of bi-dimensional elasticity tensors (Vannucci, 2002)



which can be detailed as a coupled elastic law as follows



This gives the following anisotropic elasticity law for a R_0 -orthotropic material

$$(4.2) \quad \begin{cases} \sigma_{\sim}^d = \alpha \varepsilon^d + \frac{1}{2} \text{tr}(\varepsilon^s) \mathbb{h} \\ \sigma_{\sim}^s = \frac{1}{2} (\mathbb{h} : \varepsilon^d) \mathbb{1} + \beta \varepsilon^s \end{cases} .$$

Finally, it can be shown that the set $\bar{\Sigma}_{[D_2]}^e$ is linear, i.e. a vector space. This means that this set is stable by linear combinations, a property that is not generically fulfilled by orthotropic tensors.

4.4. Cauchy anisotropy. It is a well-known fact that by imposing an extra constrain, the index symmetry of the elasticity tensor can be increased. As a result, it becomes completely symmetric with respect to index permutation. The constraint to be imposed is usually known as the *Cauchy relation* (Poincaré, 1892; Hehl and Itin, 2002). Does Cauchy relation define an exotic set of materials? The answer is given in 2D by the following results:

Lemma 4.2. *A 2D elasticity tensor \mathbb{C} with an explicit expression (2.1) is totally index symmetric if and only if*

$$(4.3) \quad 2\alpha = \beta.$$

Proof. Following the definition of harmonic tensors, the anisotropic part of an elasticity tensor denoted by $\mathbb{C} = f(0, 0, \mathbb{h}, \mathbb{H})$ is totally symmetric¹⁴. As a result, we focus here on the isotropic part. Isotropic elasticity

¹⁴It is a very specific case of 2D that the tensor $(\mathbb{1} \otimes \mathbb{h} + \mathbb{h} \otimes \mathbb{1})$ belongs to $S^4(\mathbb{R}^2)$

tensor reads:

$$\underset{\approx}{\mathbb{C}} = \alpha \underset{\approx}{\mathbb{J}} + \beta \underset{\approx}{\mathbb{K}}.$$

In the case of 2D, some properties are satisfied:

$$\underset{\approx}{\mathbb{J}} = \frac{1}{2} (\underset{\approx}{\mathbb{I}}_2 + \underset{\approx}{\mathbb{I}}_3 - \underset{\approx}{\mathbb{I}}_1), \quad \underset{\approx}{\mathbb{K}} = \frac{1}{2} \underset{\approx}{\mathbb{I}}_1,$$

and a totally symmetric tensor $\underset{\approx}{\mathbb{C}}^{sym}$ is defined by:

$$\underset{\approx}{\mathbb{C}}^{sym} = \lambda (\underset{\approx}{\mathbb{I}}_1 + \underset{\approx}{\mathbb{I}}_2 + \underset{\approx}{\mathbb{I}}_3)$$

with λ any real number, $(\underset{\approx}{\mathbb{I}}_1)_{ijkl} = \delta_{ij}\delta_{kl}$, $(\underset{\approx}{\mathbb{I}}_2)_{ijkl} = \delta_{il}\delta_{jk}$ and $(\underset{\approx}{\mathbb{I}}_3)_{ijkl} = \delta_{ik}\delta_{jl}$.

We get

$$\underset{\approx}{\mathbb{C}} = \frac{\beta}{2} \underset{\approx}{\mathbb{I}}_1 + \frac{\alpha}{2} (\underset{\approx}{\mathbb{I}}_2 + \underset{\approx}{\mathbb{I}}_3 - \underset{\approx}{\mathbb{I}}_1).$$

$2\alpha = \beta$ implies that $\underset{\approx}{\mathbb{C}}$ is totally symmetric because the coefficients for $\underset{\approx}{\mathbb{I}}_1$, $\underset{\approx}{\mathbb{I}}_2$ and $\underset{\approx}{\mathbb{I}}_3$ are identical, and vice versa. \square

This result can also be obtained using the polar formalism (Vannucci and Desmorat, 2016). Obviously, this constraint can not be only enforced by imposing symmetry requirements and hence resort on a specific design. But since relying on a constraint upon isotropic components, the resulting behaviour does not produce a paradoxical increase of symmetry. Hence in 2D, and according to a strict application of our definition, the set of Cauchy anisotropic materials is not an exotic set but a semi-exotic one¹⁵. The Equation B.1 which provides expressions for the covariants of $\underset{\approx}{\mathbb{S}}$ in terms of those of $\underset{\approx}{\mathbb{C}}$ shows that the Cauchy relations cannot be satisfied at the same time by $\underset{\approx}{\mathbb{C}}$ and $\underset{\approx}{\mathbb{S}}$. Hence, as for R_0 -orthotropic materials, this properties is not stable with respect to inversion.

5. OPTIMAL DESIGN OF A R_0 -ORTHOTROPIC UNIT CELL

As topology optimisation algorithms are nowadays very efficient, it is possible to determine the local geometry producing macroscopic exotic effects. This approach will be used here for the determination of the geometry of a periodic unit cell producing a R_0 -orthotropic elastic behaviour¹⁶. It should be noted that, in the case of laminated plates, sequence of laminations producing R_0 -orthotropic materials are known (Vannucci, 2002). However, the same result for a two-phase unit cell is, up to authors' best knowledge, new. This result is interesting for the design of architected materials. Before dealing with the example at hand, let us briefly introduce the used numerical method.

5.1. Multiscale topology optimisation. Let Ω be the domain of the periodic unit cell. Once an origin is chosen, points in Ω are designated by their position vector \underline{x} . This domain Ω is supposed to be constituted of two phases, one rigid Ω_m for the matrix and the other Ω_i soft for the inclusion. These phases satisfy:

$$\Omega = \Omega_m \cup \Omega_i, \quad \emptyset = \Omega_m \cap \Omega_i.$$

In the present approach, the inclusion is made of a very soft material in order to *simulate* a void. This approach is sometimes referred to as the **ersatz material** method, and is known to be convergent in the case of linear elasticity Allaire, Jouve and Toader (2004). Therefore, the elastic tensor field will have two values, depending on whether \underline{x} is in the matrix or in the inclusion.

$$\underset{\approx}{\mathbb{C}}(\underline{x}) = \begin{cases} \underset{\approx}{\mathbb{C}}^m, & \underline{x} \in \Omega_m \\ \gamma \underset{\approx}{\mathbb{C}}^m, & \underline{x} \in \Omega_i \end{cases}.$$

with γ the contrast parameter between the phases. We emphasise the following two points:

- in what follows, $\underset{\approx}{\mathbb{C}}^m$ will be taken as isotropic, but more general situations can as well be considered;

¹⁵The 3D response is different anyway and leads to a truly exotic set of materials.

¹⁶It can be noted that the same calculation could also be carried out with respect to the compliance tensor, hence leading to a r_0 -orthotropic material.

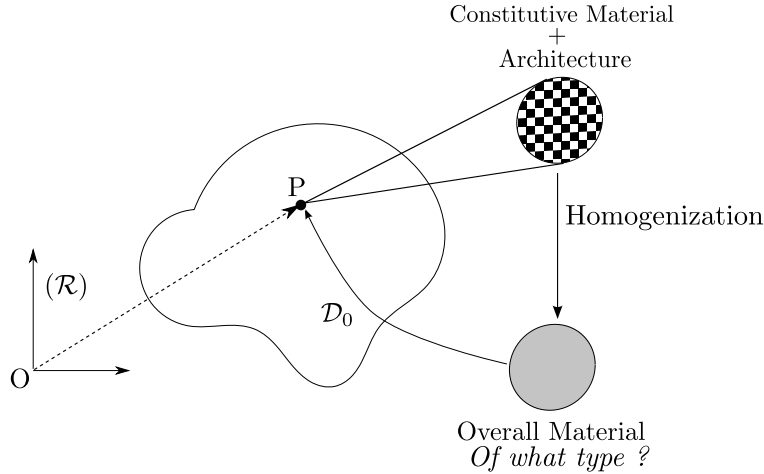


FIGURE 4. Multi-scale based homogenization model

- the periodic homogenisation theory allows defining an effective tensor \mathbb{C}^h that describe the "mean" behaviour of the heterogeneous unit cell.

The objective function of the optimisation process will be a function¹⁷ f of \mathbb{C}^h , the effective elasticity tensor calculated by periodic homogenisation from Ω Bakhvalov and Panasenko (2012). It can directly be the component of the tensor or more involved functions of it. Here, polynomial functions of \mathbb{C}^h will be considered.

The layout of this specific mesostructure is determined by using a multiscale topology optimisation algorithm introduced by S. Amstutz and coworkers. As this is not the purpose of this article, the method will not be detailed, and we refer the interested reader to the following references (Amstutz and Andrä, 2006; Amstutz et al., 2010). In broad terms, this is an optimisation code using a level-set description of the phases (Amstutz et al., 2010) and whose level set evolution is based on the topological derivative technique (Novotny and Sokółowski, 2012; Amstutz, 2021). It should be mentioned that the minima found using this approach are only local, and that the method is therefore very sensitive to the geometry used for the initialization.

5.2. A R_0 -orthotropic unit cell. We aim here at providing a geometry of a unit cell that generates an effective R_0 -orthotropic elastic material. We do not claim that it is the only specific design producing such exotic behaviour, nor the best one.

For our computation, the rigid matrix phase is constituted of an elastic isotropic material with:

$$E_m = 1, \quad \nu_m = 0.3.$$

while the soft is considered with $E_i = \gamma$, with $\gamma = 10^{-4}$ and $\nu_i = 0.3$.

The initial unit cell contains an ellipsoidal soft inclusion (cf. Figure 5), and is intended to initiate the algorithm from an effective orthotropic material. The considered level set is

$$\Psi(x, y) = \cos^2(0, 55\pi(x - 0, 5)) \cos^2(0, 3\pi(y - 0, 5)) - 0, 95.$$

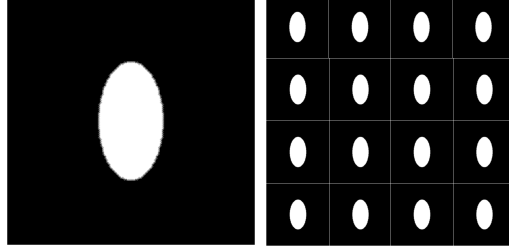
A structured mesh is considered in order to preserve, at best, the symmetries of the unit cell, and hence of the effective tensor. The following calculations were performed using a mesh of 57600 standard linear triangular elements (Tri3).

The matrix form of the initial effective tensor resulting from these choices is¹⁸

$$[\mathbb{C}^h]_{\approx_0} = 10^{-1} \begin{pmatrix} 7.33 & 2.31 & 0 \\ 2.31 & 9.03 & 0 \\ 0 & 0 & 5.32 \end{pmatrix}_{\mathcal{K}}.$$

¹⁷If wanted, the cost function can be supplemented with constraints on the total volume of a phase, or on the perimeter of the interface between phases.

¹⁸The matrix representations are provided here using the Kelvin convention, i.e. $C_{33} = 2C_{1212}$.



(A) Initial unit cell (B) Initial lattice (as-used for computation) sembly of unit cells)

FIGURE 5. Initialisation

with the following set of invariants

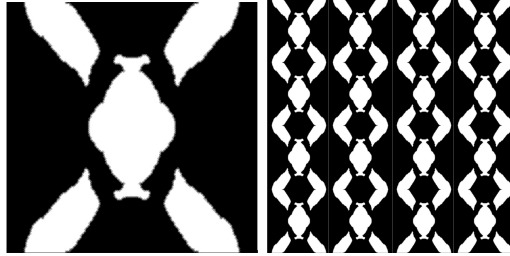
$$(I_1, J_1, I_2, J_2, I_3) = (5.59 \cdot 10^{-1}, 1.05, 1.44 \cdot 10^{-2}, 1.51 \cdot 10^{-3}, 3.97 \cdot 10^{-4}).$$

The ratio $\frac{J_2}{I_2} \sim 10\%$ is not small enough to consider J_2 as being negligible with respect to I_2 . Hence the initial elastic material should be considered as orthotropic, but not R_0 -orthotropic.

The cost function to be minimised by the algorithm was chosen to be

$$h(\underset{\approx}{\mathbb{C}}^h) = aJ_2(\underset{\approx}{\mathbb{C}}^h) + \frac{b}{I_2(\underset{\approx}{\mathbb{C}}^h)}.$$

and we specifically consider $(a, b) = (1000, 1)$. This condition corresponds to the characterisation of the open set $\Sigma_{[D_2]}^e$ (4.1). Without considering any constraint on the total volume of matter, adding the term $\frac{b}{I_2(\underset{\approx}{\mathbb{C}}^h)}$ prevents the trivial solution of a unit domain without inclusion¹⁹. Considering this cost function, the optimal mesostructure depicted on Figure 6 is obtained.



(A) Initial unit cell (B) Initial lattice

FIGURE 6. Optimised design

The matrix form of the effective tensor obtained at the end of the optimisation process is

$$[\underset{\approx}{\mathbb{C}}^h]_{\approx f} = 10^{-1} \cdot \begin{pmatrix} 0.53 & 0.60 & 0 \\ 0.60 & 4.41 & 0 \\ 0 & 0 & 1.86 \end{pmatrix}_{\mathcal{K}}.$$

The invariants of the effective stiffness tensor of the optimal geometry are

$$(I_1, J_1, I_2, J_2, I_3) = (1.86 \cdot 10^{-1}, 3.07 \cdot 10^{-1}, 7.53 \cdot 10^{-2}, 5.00 \cdot 10^{-7}, 3.77 \cdot 10^{-5}).$$

The optimized elastic material is obviously R_0 -orthotropic since J_2 is negligible with respect to I_2 . To the contrary, the invariants of the associated compliance tensor are

$$(I_1, J_1, I_2, J_2, I_3) = (1.05 \cdot 10^1, 9.49, 1.94 \cdot 10^2, 5.23 \cdot 10^1, 9.95 \cdot 10^2).$$

which obviously shows that the resulting material is not r_0 -orthotropic. This numerical result substantiates the statement in subsection 4.3 that R_0 -orthotropic is not inverse stable.

¹⁹Such a trivial case is isotropic and hence verify $J_2 = I_2 = 0$.

6. GENERALIZATION TO OTHER TWO-DIMENSIONAL CONSTITUTIVE LAWS

The aim of this section is twofold. First, it will be a question of deriving a general result concerning linear constitutive laws in \mathbb{R}^2 . We will then illustrate this result on two non-trivial situations: the piezoelectric law, and the Cosserat elasticity.

6.1. A general theorem. The approach used for elasticity in section 4 can be generalised to any linear constitutive law in \mathbb{R}^2 based on the fact that, in \mathbb{R}^2 , harmonic spaces are bi-dimensional. To obtain the number of exotic sets, one must enumerate all the particular geometric configurations that can occur between the components of the harmonic decomposition of a given tensor. In practice, one must enumerate:

- the number of possible cancellations of harmonic components;
- for a given configuration, the number of possible alignments between the non-zero harmonic components.

and subtract from this the number of symmetry classes. The number of exotic sets is given by the following result:

Theorem 6.1. *Consider \mathbb{T} a space of bidimensional constitutive tensors. Let N and M be, respectively, the number of harmonic spaces of order > 0 and -1 in the harmonic structure of \mathbb{T} . The number of exotic anisotropic sets of \mathbb{T} is*

$$\#\mathcal{E} = \left(\sum_{p=0}^N \binom{N}{p} (2^p - p) \right) 2^M - \#\mathcal{C}.$$

with $\#\mathcal{C}$ the number of $O(2)$ symmetry classes of \mathbb{T} .

Proof. As indicated by Theorem 4.1, the symmetry classes of \mathbb{K}^n are:

$$\mathfrak{J}(\mathbb{K}^n) = \begin{cases} n \geq 1, & \{[D_n], [O(2)]\} \\ n = -1, & \{[SO(2)], [O(2)]\} \\ n = 0, & \{[O(2)]\} \end{cases}.$$

Thus, the \mathbb{K}^0 play no role in the counting of classes, the \mathbb{K}^{-1} are on or off but independent of orientation, while the non-zero $\mathbb{K}^{n \geq 1}$ have an orientation. It results that, for $p, q \geq 1$,

$$\mathfrak{J}(\mathbb{K}^p \otimes \mathbb{K}^q) = \{[Z_{d(p,q)}], [D_{d(p,q)}], [D_p], [D_q], [O(2)]\}.$$

with $d(p, q) := \gcd(p, q)$. Since no relative orientation are involved, the situation is simpler when $p = q = -1$ and

$$\mathfrak{J}(\mathbb{K}^{-1} \otimes \mathbb{K}^{-1}) = \{[SO(2)], [O(2)]\}.$$

Therefore, the presence of M harmonic spaces of type \mathbb{K}^{-1} in the harmonic decomposition of \mathbb{T} generate 2^M different combinations.

Let N , the number of harmonic spaces $\mathbb{K}^{n \geq 1}$ in the harmonic decomposition of \mathbb{T} . Let us first assume that none of the associated harmonic components is zero. We then have a collection of N non-zero vectors. Among them, some can be aligned with others. We need to count the different alignments that can occur. There is a configuration with no alignment, $\binom{N}{2}$ configurations with a pair of aligned vectors, $\binom{N}{3}$ with 3 vectors and so on... At the end, we can count $2^N - N$ configurations going from the generic configuration to the complete alignment. Suppose now that among those N harmonic components p of them are null. There are $\binom{N}{p}$ different manner to cancel p components among N , and each of them generated $2^p - p$ alignment configurations. Hence by combining the vanishing and the alignments of harmonic components we obtain

$$\left(\sum_{p=0}^N \binom{N}{p} (2^p - p) \right),$$

different configurations generated by harmonic tensors of order greater than 0. The total number of configurations is then obtained by taking into account the configurations of hemitropic components \mathbb{K}^{-1} , hence

$$\#\mathcal{S} = \left(\sum_{p=0}^N \binom{N}{p} (2^p - p) \right) 2^M.$$

To obtain the number of exotic situations, we need to remove those that generate genuine symmetry classes. Thus, the final formula used to obtain the number of exotic sets showed in Theorem 6.1 is obtained. \square

6.2. Application to coupled constitutive laws. This result is applied here to two classical coupled constitutive laws highlighted in the literature: piezoelectricity and Cosserat elasticity.

6.2.1. *Piezoelectricity.* This constitutive law couple mechanical state with the electric one. The *electrical* state is described by two vector fields: the electric displacement \tilde{d} and the electric field \tilde{e} . As in the mechanical situation, these fields are connected by a constitutive law $\tilde{\mathcal{L}}$ that describes the behaviour of each different material. For linear conductivity, this relation can be written

$$\tilde{d} = \tilde{S} \cdot \tilde{e}$$

in which \tilde{S} is a second-order tensor, known as the *permittivity tensor* Landau and Lifshitz (1984). For non-centro symmetric materials these two phenomena are not independent but coupled Landau and Lifshitz (1984). In this situation the constitutive law reads

$$(6.1) \quad \begin{cases} \tilde{\sigma} = \tilde{C} : \tilde{\varepsilon} - \tilde{e} \cdot \tilde{P} \\ \tilde{d} = \tilde{P} : \tilde{\varepsilon} + \tilde{S} \cdot \tilde{e} \end{cases}$$

in which a third-order tensor \tilde{P} , known as the *piezoelectricity tensor*, responsible for the coupling appears²⁰. The different constitutive tensors are summed up in the following table.

Tensor	Symmetries	T	Physical meaning
\tilde{C}	$T_{(ij) (kl)}$	Ela	Fourth-order elasticity tensor
\tilde{P}	$T_{i(jk)}$	Piez	Piezoelectric tensor
\tilde{S}	$T_{(ij)}$	Con	Dielectric susceptibility tensor

The determination of the number of exotic sets requires the knowledge of the harmonic structure of the constitutive tensors. These structure together with the number of exotic sets are provided in the table below:

T	\mathcal{H}	N	M	$\#\mathcal{S}$	$\#\mathcal{C}$	$\#\mathcal{E}$
Ela	$2\mathbb{K}^0 \oplus \mathbb{K}^2 \oplus \mathbb{K}^4$	2	0	5	4	1
Piez	$2\mathbb{K}^1 \oplus \mathbb{K}^3$	3	0	15	4	11
Con	$\mathbb{K}^0 \oplus \mathbb{K}^2$	1	0	2	2	0

We denote the space of the piezoelectric law by $\mathcal{P}iez$, its harmonic structure is obtained from those of the constitutive tensors which compose it

$$\mathcal{P}iez = 3\mathbb{K}^0 \oplus 2\mathbb{K}^1 \oplus 2\mathbb{K}^2 \oplus \mathbb{K}^3 \oplus \mathbb{K}^4.$$

Knowing that this law has 7 regular classes, the number of exotic sets is then determined

$$N = 6; \quad M = 0; \quad \#\mathcal{C} = 7 \quad \Rightarrow \quad \#\mathcal{E} = 530.$$

We observe that the number of exotic set of the coupled law is not the sum of the exotic set of its constituents.

²⁰Depending on the considered set of primary variables, four different conventions can be used to express the law of piezoelectricity Meitzler, Tiersten, Warner, Berlincourt, Couquin and Welsh III (1988). The one chosen here is regarded as the most general according to the *IEEE Standard on Piezoelectricity* Meitzler et al. (1988). In any case, the results are independent of the chosen convention

6.3. Cosserat elasticity. We consider here the classical formulation of Cosserat elasticity in small-strain as introduced for instance in (Eremeyev, Lebedev and Altenbach, 2012; Forest, 2005), using the linear stretch strain tensor \underline{e} and the linear wryness tensor $\underline{\kappa}$. By duality we define the stress tensors: \underline{s} the asymmetric stress tensor, and \underline{m} the couple-stress tensor. It should be emphasised that, in contrast to the standard elasticity, the strain and stress tensors \underline{e} and \underline{s} are not symmetric. Consequently, the constitutive law of linear cosserat elasticity is expressed as:

$$\begin{cases} \underline{s} = \underline{A} : \underline{e} - \underline{B} \cdot \underline{\kappa} \\ \underline{m} = \underline{B}^T : \underline{e} + \underline{d} \cdot \underline{\kappa} \end{cases}$$

The harmonic structure of this coupled elastic law has been derived in (Auffray, El Ouafa, Rosi and Desmorat, 2021b), associated results are provided in the following table:

TABLE 2. Number of exotic sets for tensor spaces in linear cosserat elasticity

Tensor	Symmetries	T	\mathcal{H}	N	M	$\#\mathcal{S}$	$\#\mathcal{C}$	$\#\mathcal{E}$
\underline{A}	$\underline{T}_{ij\ kl}$	Cos	$\mathbb{K}^{-1} \oplus 3\mathbb{K}^0 \oplus 2\mathbb{K}^2 \oplus \mathbb{K}^4$	3	1	30	6	24
\underline{B}	\underline{T}_{ijk}	Cou	$3\mathbb{K}^1 \oplus \mathbb{K}^3$	4	0	49	4	45
\underline{d}	$\underline{T}_{(ij)}$	Rot	$\mathbb{K}^0 \oplus \mathbb{K}^2$	1	0	2	2	0

We denote by \mathcal{C} os the space of the Cosserat elasticity law, its harmonic structure is obtained from those of the constitutive tensors which compose it

$$\mathcal{C}os \simeq \mathbb{K}^{-1} \oplus 4\mathbb{K}^0 \oplus 3\mathbb{K}^1 \oplus 3\mathbb{K}^2 \oplus \mathbb{K}^3 \oplus \mathbb{K}^4.$$

Knowing that this law has 10 regular classes (Auffray et al., 2021b), the number of exotic sets is then determined:

$$N = 1; \quad M = 1; \quad \#\mathcal{C} = 10 \quad \Rightarrow \quad \#\mathcal{E} = 11064.$$

Following these two examples, we can observe that the number of exotic sets for a given constitutive tensor space is significantly greater than the sum of the numbers of exotic sets for its constituent subspaces.

7. CONCLUSION

The main findings of this paper are as follows:

- a mathematical definition of what is an exotic set of materials has been obtained;
- this definition allowed us to verify that the space of 2D elasticity tensors has only one exotic class. This set had been well identified in the literature and we showed that, with respect to our definition, there are no others;
- however, our definition allowed us to obtain a general result concerning all 2D constitutive tensor spaces;
- in the case of R_0 – *orthotropy*, an elementary architected cell generating such behaviour has been identified for the first time.

The important question now is what happens for 3D elasticity. This will be the subject of future work, but the great strength of the approach proposed here is that, via the use of adapted clips product Olive (2019), it extends to \mathbb{R}^3 . It should be noted, however, that the situation will be extremely more complex.

ACKNOWLEDGEMENT

The authors acknowledge the support of the French Agence Nationale de la Recherche (ANR), under grant ANR-19-CE08-0005 (project Max-Oasis).

APPENDIX A. NOTATIONS

Throughout this paper, the physical space is modeled on the Euclidean space \mathcal{E}^2 with \mathbb{E}^2 its associated vector space. Once an arbitrary reference point chosen, those spaces can be associated and $\mathcal{P} = \{\mathbf{e}_1, \mathbf{e}_2\}$ will denote an orthonormal basis of \mathbb{E}^2 . For forthcoming needs, let also define $\mathcal{K} = \{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3\}$ the orthonormal canonical basis of \mathbb{R}^3 , \mathcal{K} will be referred to as the Kelvin basis.

- $\mathbb{T}^n(\mathbb{R}^2)$: $= \otimes^n \mathbb{R}^2$, the vector space of 2D tensors of order n ;
- $\mathbb{S}^n(\mathbb{R}^2)$: the subspace of $\mathbb{T}^n(\mathbb{R}^2)$ of totally symmetric tensor space with respect to the permutation of its indices;
- \mathbb{I} : fourth-order identity tensor of $\mathbb{S}^2(\mathbb{R}^2)$, $I_{ijkl} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$;
- \mathbb{K} : spherical projector for \mathbb{R}^2 , $K_{ijkl} = (\frac{1}{2}\mathbb{1} \otimes \mathbb{1})_{ijkl} = \frac{1}{2}\delta_{ij}\delta_{kl}$;
- \mathbb{J} : deviatoric projector for \mathbb{R}^2 , $J_{ijkl} = I_{ijkl} - K_{ijkl} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \delta_{ij}\delta_{kl})$;
- \mathbb{D}_k : the dihedral group with $2k$ elements generated by $\mathbb{R}(2\pi/k)$ and σ_x , in which $\mathbb{R}(\theta)$ is a rotation by an angle θ and σ_x is the reflection across the line normal to \underline{x} ;
- \mathbb{Z}_k : the cyclic group with k elements generated by $\mathbb{R}(2\pi/k)$;
- $\mathbb{Z}_2^{\sigma_x}$: the cyclic group with 2 elements generated by σ_x ;
- \simeq : the isomorphism relation;
- \otimes : standard tensor product and \otimes^n indicates its n power;
- \oplus : the direct sum of vector spaces;
- $\overset{(r)}{\cdot}$: the r -contraction of two tensors $\mathbb{T}_1 \in \mathbb{T}^{n_1}(\mathbb{R}^2)$ and $\mathbb{T}_2 \in \mathbb{T}^{n_2}(\mathbb{R}^2)$ in any orthonormal basis, $(\mathbb{T}_1 \overset{(r)}{\cdot} \mathbb{T}_2)_{i_1 \dots i_{n_1-r} j_{r+1} \dots j_{n_2}} := T_{i_1 \dots i_{n_1-r} k_1 \dots k_r}^1 T_{k_1 \dots k_r j_{r+1} \dots j_{n_2}}^2$, which is a tensor of order $n_1 + n_2 - 2r$;
- $(\cdot)^s$: complete symmetrisation of a tensor;
- \odot : symmetric tensor product between two tensors $\mathbb{S}_1 \in \mathbb{S}^{n_1}(\mathbb{R}^2)$ and $\mathbb{S}_2 \in \mathbb{S}^{n_2}(\mathbb{R}^2)$, $\mathbb{S}_1 \odot \mathbb{S}_2 := (\mathbb{S}_1 \otimes \mathbb{S}_2)^s \in \mathbb{S}^{n_1+n_2}(\mathbb{R}^2)$;
- $*$: the harmonic product between two harmonic tensors $\mathbb{K}_1 \in \mathbb{K}^{n_1}$ and $\mathbb{K}_2 \in \mathbb{K}^{n_2}$, it is defined as $\mathbb{K}_1 * \mathbb{K}_2$ the projection of the classical tensor product on $\mathbb{K}^{n_1+n_2}$. This product is computed as follows

$$(\mathbb{K}_1 * \mathbb{K}_2) = \mathbb{K}_1 \odot \mathbb{K}_2 - \frac{1}{2}(\mathbb{1} \otimes (\mathbb{K}_1 \cdot \mathbb{K}_2))^s$$

- \times : the skew-symmetric contraction between two totally symmetric tensors $\mathbb{S}_1 \in \mathbb{S}^{n_1}(\mathbb{R}^2)$ and $\mathbb{S}_2 \in \mathbb{S}^{n_2}(\mathbb{R}^2)$, it is defined as:

$$(\mathbb{S}_1 \times \mathbb{S}_2) := -(\mathbb{S}_1 \cdot \boldsymbol{\epsilon} \cdot \mathbb{S}_2)^s \in \mathbb{S}^{n_1+n_2-2}(\mathbb{R}^2),$$

where $\boldsymbol{\epsilon}$ is the 2D Levi-Civita tensor. In any orthonormal basis $(\mathbf{e}_1, \mathbf{e}_2)$, we get $\epsilon_{ij} = \det(\mathbf{e}_i, \mathbf{e}_j)$ and

$$(\mathbb{S}_1 \times \mathbb{S}_2)_{i_1 \dots i_{n_1+n_2-2}} = -\left(\epsilon_{jk} S_{j i_1 \dots i_{n_1}}^1 S_{k i_{n_1+1} \dots i_{n_1+n_2-2}}^2\right)^s.$$

APPENDIX B. SYMMETRY CLASSES OF THE COMPLIANCE TENSOR

From a physical point of view, and for a given material, the stiffness tensor and the compliance tensor are two equivalent ways of describing the macroscopic behaviour resulting from an identical micro-structure. It therefore seems natural that their symmetry groups, and thus their symmetry classes, coincide. This seems so natural that, up to authors best knowledge, no mathematical proof of this point has been given. A simple proof of that point is provided here.

Theorem B.1. *If \mathbb{C} and \mathbb{S} are two elasticity tensors satisfying $\mathbb{S} : \mathbb{C} = \mathbb{C} : \mathbb{S} = \mathbb{I}$ then $\mathbb{G}_{\mathbb{C}} = \mathbb{G}_{\mathbb{S}}$.*

Proof. Let consider \mathbb{C} as a symmetric second-order tensor on $S^2(\mathbb{R}^3)$, and denote by $[\mathbb{C}]$ its matrix representation with respect to a basis of \mathbb{R}^3 . Let $p_{\mathbb{C}}(X)$ be the characteristic polynomial of $[\mathbb{C}]$,

$$p(X) = X^3 - \sigma_1 X^2 + \sigma_2 X - \sigma_3$$

in which σ_k are elementary symmetric polynomials:

$$\begin{cases} \sigma_1 = \lambda_1 + \lambda_2 + \lambda_3 = \text{tr}(\mathbb{C}), \\ \sigma_2 = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3, \\ \sigma_3 = \lambda_1\lambda_2\lambda_3 = \det(\mathbb{C}), \end{cases}$$

It should be noted that symmetric polynomial are $O(3)$ -invariant polynomial of \mathbb{C} , i.e.

$$\forall \mathbf{g} \in O(3), \sigma_k(\mathbf{g} \star \mathbb{C}) = \sigma_k(\mathbb{C})$$

Hence since σ_k are $O(3)$ -invariant, they are also $O(2)$ -invariant for any $O(2)$ subgroups of $O(3)$.

From Cayley-Hamilton theorem, it is known that $p(\mathbb{C}) = 0$, i.e.

$$p(\mathbb{C}) = \mathbb{C}^3 - \sigma_1 \mathbb{C}^2 + \sigma_2 \mathbb{C} - \sigma_3 \mathbb{I} = \mathbf{0}$$

Multiplying this relation on the left by \mathbb{S} , and since $\sigma_3 \neq 0$, we obtain that

$$\mathbb{S} = \frac{1}{\sigma_3} (\sigma_2 \mathbb{I} - \sigma_1 \mathbb{C} + \mathbb{C}^2)$$

Hence \mathbb{S} is polynomial in \mathbb{C} , $\mathbb{S} = P(\mathbb{C})$. Let $\mathbf{g} \in G(\mathbb{C})$,

$$\mathbf{g} \star \mathbb{S} = \mathbf{g} \star P(\mathbb{C}) = P(\mathbf{g} \star \mathbb{C}) = P(\mathbb{C}) = \mathbb{S}$$

hence, $G(\mathbb{C}) \subset G(\mathbb{S})$. Since \mathbb{C} can be expressed in the same way as a polynomial function in \mathbb{S} , the same reasoning leads to the reverse inclusion and to the final conclusion that

$$G(\mathbb{C}) = G(\mathbb{S})$$

□

The explicit decomposition of $\mathbb{S} = \mathbb{C}^{-1}$ has the same structure as that of \mathbb{C}

$$\mathbb{C}^{-1} = \mathbb{S} = f(\alpha^-, \beta^-, \mathfrak{h}^-, \mathbb{H}^-),$$

in which $\{\alpha^-, \beta^-, \mathfrak{h}^-, \mathbb{H}^-\}$ denotes the harmonic components of \mathbb{C}^{-1} . These components can be expressed in terms of those of \mathbb{C}

$$(B.1) \quad \alpha^- = \frac{1}{4\Delta} (4\alpha\beta - \mathfrak{h} : \mathfrak{h}), \quad \beta^- = \frac{1}{2\Delta} (2\alpha^2 - \mathbb{H} :: \mathbb{H}), \quad \mathfrak{h}^- = \frac{1}{\Delta} (\mathbb{H} : \mathfrak{h} - \alpha\mathfrak{h}), \quad \mathbb{H}^- = \frac{1}{2\Delta} (\mathfrak{h} * \mathfrak{h} - 2\beta\mathbb{H}),$$

with Δ defined as:

$$\Delta = \alpha^2\beta - \frac{1}{2}\beta\mathbb{H} :: \mathbb{H} - \frac{1}{2}\alpha\mathfrak{h} : \mathfrak{h} - \frac{1}{2}\mathfrak{h} : \mathbb{H} : \mathfrak{h},$$

The invariants of $\mathbb{S} = \mathbb{C}^{-1}$ are denoted $(I_1^-, J_1^-, I_2^-, J_2^-, I_3^-)$. They are rational functions of those of \mathbb{C} :

$$\begin{aligned} I_1^- &= \frac{1}{4\Delta} (4I_1J_1 - I_2), & J_1^- &= \frac{1}{2\Delta} (2I_1^2 - J_2), \\ I_2^- &= \frac{1}{\Delta^2} \left(\frac{1}{2}J_2I_2 - 2I_1I_3 + I_1^2I_2 \right), & J_2^- &= \frac{1}{4\Delta^2} \left(\frac{1}{2}I_2^2 - 4J_1I_3 + 4J_1^2J_2 \right), \end{aligned}$$

and

$$I_3^- = \frac{1}{2\Delta^3} \left(I_3^2 - \frac{1}{4}I_2^2J_2 - J_1J_2I_3 - I_1I_2I_3 + 2I_1J_1I_2J_2 + \frac{1}{2}I_1^2I_2^2 - 2I_1^2J_1I_3 \right),$$

with Δ the determinant of \mathbb{C} :

$$\Delta = \frac{1}{2}(I_3 - I_1I_2 - J_1J_2 + 2I_1^2J_1).$$

It is obvious that for \mathbb{C} to be invertible $\Delta \neq 0$.

ACKNOWLEDGEMENT

This document exposes results from the research project ANR-19-CE08-0005 (project Max-Oasis) funded by the French National Research Agency (ANR).

REFERENCES

- Abud, M., Sartori, G., 1983. The geometry of spontaneous symmetry breaking. *Annals of Physics* 150, 307–372. <https://www.sciencedirect.com/science/article/pii/0003491683900179>.
- Allaire, G., Jouve, F., Toader, A.M., 2004. Structural optimization using sensitivity analysis and a level-set method. *Journal of computational physics* 194, 363–393.
- Amstutz, S., 2021. An introduction to the topological derivative. *Engineering Computations* <https://doi.org/10.1108/EC-07-2021-0433>.
- Amstutz, S., Andrä, H., 2006. A new algorithm for topology optimization using a level-set method. *Journal of computational physics* 216, 573–588. <https://doi.org/10.1016/j.jcp.2005.12.015>.
- Amstutz, S., Giusti, S.M., Novotny, A.A., De Souza Neto, E.A., 2010. Topological derivative for multi-scale linear elasticity models applied to the synthesis of microstructures. *International Journal for Numerical Methods in Engineering* 84, 733–756. <https://doi.org/10.1002/nme.2922>.
- Antonelli, A., Desmorat, B., Kolev, B., Desmorat, R., 2022. Distance to plane elasticity orthotropy by Euler–Lagrange method. *Comptes Rendus. Mécanique* 350, 413–430. <https://doi.org/10.5802/crmeca.122>.
- Auffray, N., Abdoul-Anziz, H., Desmorat, B., 2021a. Explicit harmonic structure of bidimensional linear strain-gradient elasticity. *European Journal of Mechanics-A/Solids* 87, 104202.
- Auffray, N., El Ouafa, S., Rosi, G., Desmorat, B., 2021b. Anisotropic structure of two-dimensional linear cosserat elasticity. *Mathematics and Mechanics of Complex Systems* <https://hal-upec-upem.archives-ouvertes.fr/hal-03287608>.
- Auffray, N., Kolev, B., Olive, M., 2017. Handbook of bi-dimensional tensors: Part i: Harmonic decomposition and symmetry classes. *Mathematics and Mechanics of Solids* 22, 1847–1865. <https://doi.org/10.1177/1081286516649017>.
- Auffray, N., Kolev, B., Petitot, M., 2014. On anisotropic polynomial relations for the elasticity tensor. *Journal of Elasticity* 115, 77–103. <https://doi.org/10.1007/s10659-013-9448-z>.
- Bakhvalov, N.S., Panasenko, G., 2012. Homogenisation: averaging processes in periodic media: mathematical problems in the mechanics of composite materials. volume 36. Springer Science & Business Media.
- Blinowski, A., Ostrowska-Maciejewska, J., Rychlewski, J., 1996. Two-dimensional hooke’s tensors-isotropic decomposition, effective symmetry criteria. *Archives of Mechanics* 48, 325–345. <http://am.ippt.pan.pl/am/article/view/v48p325/463>.
- Bückmann, T., Schittny, R., Thiel, M., Kadic, M., Milton, G.W., Wegener, M., 2014. On three-dimensional dilational elastic metamaterials. *New Journal of Physics* 16. <https://doi.org/10.1088/1367-2630/16/3/033032>, [arXiv:1310.3719](https://arxiv.org/abs/1310.3719).
- Camar-Eddine, M., Seppecher, P., 2003. Determination of the Closure of the Set of Elasticity Functionals. *Archive for Rational Mechanics and Analysis* 170, 211–245. <https://doi.org/10.1007/s00205-003-0272-7>.
- Desmorat, B., Olive, M., Auffray, N., Desmorat, R., Kolev, B., 2020. Computation of minimal covariants bases for 2d coupled constitutive laws. *arXiv preprint arXiv:2007.01576* <https://doi.org/10.48550/arXiv.2007.01576>.
- Desmorat, R., Auffray, N., Desmorat, B., Olive, M., Kolev, B., 2021. Minimal functional bases for elasticity tensor symmetry classes. *Journal of Elasticity* 147, 201–228.
- Durand, B., Lebée, A., Seppecher, P., Sab, K., 2022. Predictive strain-gradient homogenization of a pantographic material with compliant junctions. *Journal of the Mechanics and Physics of Solids* 160, 104773. <https://doi.org/10.1016/j.jmps.2021.104773>, doi:10.1016/j.jmps.2021.104773.
- Eremeyev, V., Lebedev, L., Altenbach, H., 2012. Foundations of micropolar mechanics. Springer Science & Business Media.
- Ferrer, A., Cante, J.C., Hernández, J.A., Oliver, J., 2018. Two-scale topology optimization in computational material design: An integrated approach. *International Journal for Numerical Methods in Engineering* 114, 232–254. <https://doi.org/10.1002/nme.5742>.

- Forest, S., 2005. Mechanics of cosserat media—an introduction. Ecole des Mines de Paris, Paris , 1–20.
- Forte, S., Vianello, M., 1996. Symmetry classes for elasticity tensors. *Journal of Elasticity* 43, 81–108. <https://doi.org/10.1007/BF00042505>.
- Geymonat, G., Weller, T., 2002. Classes de symétrie des solides piézoélectriques. *Comptes Rendus Mathématique* 335, 847–852. [https://doi.org/10.1016/S1631-073X\(02\)02573-6](https://doi.org/10.1016/S1631-073X(02)02573-6).
- He, Q.C., 2004. Characterization of the anisotropic materials capable of exhibiting an isotropic Young or shear or area modulus. *International Journal of Engineering Science* 42, 2107–2118. <https://doi.org/10.1016/j.ijengsci.2004.04.009>.
- He, Q.C., Zheng, Q.S., 1996. On the symmetries of 2d elastic and hyperelastic tensors. *Journal of elasticity* 43, 203–225. <https://doi.org/10.1007/BF00042501>.
- Hehl, F.W., Itin, Y., 2002. The cauchy relations in linear elasticity theory. *Journal of Elasticity* 66, 185–192. <https://doi.org/10.1023/A:1021225230036>.
- Landau, L.D., Lifshitz, E.M., 1984. *Electrodynamics of continuous media*. volume 8. elsevier.
- Laurain, A., 2018. A level set-based structural optimization code using FEniCS. *Structural and Multidisciplinary Optimization* 58, 1311–1334. <https://doi.org/10.1007/s00158-018-1950-2>, [arXiv:1705.01442](https://arxiv.org/abs/1705.01442).
- Le Quang, H., He, Q.C., 2011. The number and types of all possible rotational symmetries for flexoelectric tensors. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences* 467, 2369–2386. <https://doi.org/10.1098/rspa.2010.0521>.
- Meitzler, A., Tiersten, H., Warner, A., Berlincourt, D., Couquin, G., Welsh III, F., 1988. Ieee standard on piezoelectricity.
- Novotny, A., Sokółowski, J., 2012. *Topological derivatives in shape optimization*. Springer Science & Business Media.
- Olive, M., 2019. Effective computation of $so(3)$ and $oo(3)$ linear representation symmetry classes. *Mathematics and Mechanics of Complex Systems* 7, 203–237.
- Olive, M., Kolev, B., Auffray, N., 2017. A minimal integrity basis for the elasticity tensor. *Archive for Rational Mechanics and Analysis* 226, 1–31. <https://doi.org/10.1007/s00205-017-1127-y>.
- Poincaré, H., 1892. *Leçons sur la théorie de l'élasticité*. G. Carré.
- Rychlewski, J., 2001. Elastic waves under unusual anisotropy. *Journal of the Mechanics and Physics of Solids* 49, 2651–2666. [https://doi.org/10.1016/S0022-5096\(01\)00083-7](https://doi.org/10.1016/S0022-5096(01)00083-7).
- Sturmfels, B., 2008. *Algorithms in Invariant Theory*. <https://doi.org/10.1007/978-3-211-77417-5>.
- Vannucci, P., 2002. A Special Planar Orthotropic Material. *Journal of Elasticity* 67, 81–96. <https://doi.org/10.1023/A:1023949729395>.
- Vannucci, P., Desmorat, B., 2016. Plane anisotropic rari-constant materials. *Mathematical Methods in the Applied Sciences* 39, 3271–3281. <https://doi.org/10.1002/mma.3770>.
- Vianello, M., 1997. An integrity basis for plane elasticity tensors. *Archives of Mechanics* 49, 197–208. <https://am.ippt.pan.pl/am/article/viewFile/v49p197/519>.
- Weyl, H., 1946. *The Classical Groups: their invariants and representations*. Princeton University Press. <http://www.jstor.org/stable/j.ctv3hh48t>.
- Wu, J., Sigmund, O., Groen, J.p., 2021. Topology optimization of multi-scale structures: a review. *Structural and Multidisciplinary Optimization* 63, 1455–1480. doi:10.1007/s00158-021-02881-8.

(IDA) INSTITUT D'ALEMBERT, CNRS UMR 7190, SORBONNE UNIVERSITÉ, FRANCE

(MSME) LABORATOIRE MODÉLISATION ET SIMULATION MULTI-ECHELLE (MSME) - UMR 8208 CNRS, FRANCE