Imitation and contrarian behavior: hyperbolic bubbles, crashes and chaos

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Abstract

Imitative and contrarian behaviors are the two typical opposite attitudes of investors in stock markets. We introduce a simple model to investigate their interplay in a stock market where agents can take only two states, bullish or bearish. Each bullish (bearish) agent polls m "friends" and changes her opinion to bearish (bullish) if (1) at least $m\rho_{hb}$ ($m\rho_{bh}$) among the m agents inspected are bearish (bullish) or (2) at least $m\rho_{hh} > m\rho_{hb}$ ($m\rho_{bb} > m\rho_{bh}$) among the m agents inspected are bullish (bearish). The condition (1) (resp. (2)) corresponds to imitative (resp. antagonistic) behavior. In the limit where the number N of agents is infinite, the dynamics of the fraction of bullish agents is deterministic and exhibits chaotic behavior in a significant domain of the parameter space $\{\rho_{hb}, \rho_{bh}, \rho_{hh}, \rho_{bb}, m\}$. A typical chaotic trajectory is characterized by intermittent phases of chaos, quasi-periodic behavior and super-exponentially growing bubbles followed by crashes. A typical bubble starts initially by growing at an exponential rate and then crosses over to a nonlinear power law growth rate leading to a finite-time singularity. The reinjection mechanism provided by the contrarian behavior introduces a finite-size effect, rounding off these singularities and leads to chaos. We document the main stylized facts of this model in the symmetric and asymmetric cases. This model is one of the rare agent-based models that give rise to interesting non-periodic complex dynamics in the "thermodynamic" limit (of an infinite number N of agents). We also discuss the case of a finite number of agents, which introduces an endogenous source of noise superimposed on the chaotic dynamics.

"Human behavior is a main factor in how markets act. Indeed, sometimes markets act quickly, violently with little warning. [...] Ultimately, history tells us that there will be a correction of some significant dimension. I have no doubt that, human nature being what it is, that it is going to happen again and again." Alan Greenspan, Chairman of the Federal Reserve of the USA, before the Committee on Banking and Financial Services, U.S. House of Representatives, July 24, 1998.

1 Introduction

In recent economic and finance research, there is a growing interest in incorporating ideas from social sciences to account for the fact that markets reflect the thoughts, emotions, and actions of real people as opposed to the idealized economic investor whose behavior underlies the efficient market and random walk hypothesis. This was captured by the now famous pronouncement of Keynes (1936) that most investors' decisions "*can only be taken as a result of animal spirits – of a spontaneous urge to action rather than inaction, and not the outcome of a weighed average of benefits multiplied by the quantitative probabilities*". A real investor may intend to be rational and may try to optimize his actions, but that rationality tends to be hampered by cognitive biases, emotional quirks, and social influences. "Behavioral finance" is a growing research field (Thaler (1993), De Bondt and Thaler (1995), Shefrin (2000), Shleifer (2000), Goldberg and von Nitzsch (2001)), which uses psychology, sociology, and other behavioral theories to attempt to explain the behavior of investors and money managers. The behavior of financial markets is thought to result from varying attitudes toward risk, the heterogeneity in the framing of information, from cognitive errors, self-control and lack thereof, from regret in financial decision-making, and from the influence of mass psychology. Assumptions about the frailty of human rationality and the acceptance of such drives as fear and greed are underlying the recipes developed over decades by so-called technical analysts.

There is growing empirical evidence for the existence of herd or "crowd" behavior in speculative markets (Arthur (1987), Bikhchandani *et al.* (1992), Johansen *et al.*(1999, 2000), Orléan (1986, 1990, 1992), Shiller (1984, 2000), Topol (1991), West (1988)). Herd behavior is often said to occur when many people take the same action, because some mimic the actions of others. Herding has been linked to many economic activities, such as investment recommendations (Graham and Dodd (1934), Scharfstein and Stein (1990)), price behavior of IPO's (Initial Public Offering) (Welch (1992)) fads and customs (Bikhchandani *et al.* (1992)), earnings forecasts (Trueman (1994)), corporate conservatism (Zwiebel (1995)) and delegated portfolio management (Maug and Naik (1995)).

Here, we introduce arguably the simplest model capturing the interplay between mimetic and contrarian behavior in a population of N agents taking only two possible states, "bullish" or "bearish" (buying or selling). In the limit of an infinite number $N \to \infty$ of agents, the key variable which is the fraction p of bullish agents follows a chaotic deterministic dynamics on a subspace of positive measure in the parameter space. Before explaining and analyzing the model in subsequent sections, we compare it in three respects to standard theories of economic behavior.

1. Since in the limit $N \to \infty$, the model operates on a purely deterministic basis, it actually challenges the purely external and unpredictable origin of market prices. Our model exploits the continuous mimicry of financial markets to show that the disordered and random aspect of the time series of prices can be in part explained not only by the advent of "random" news and events, but can also be generated by the behavior of the agents fixing the prices.

In the limit $N \to \infty$, the dynamics of prices in our model is deterministic and derives from the theory of chaotic dynamical systems, which have the feature of exhibiting endogenously perturbed motion. After the first papers on the theory of chaotic systems, such as Lorenz (1963), May (1976), (see, *e.g.*, Collet-Eckmann (1980) for an early exposition), a series of economic papers dealt with models mostly of growth—Benhabib-Day (1981), Day (1982, 1983), Stutzer (1980). Later, a vast and varied number of fields of economics were analyzed in the light of the theory of chaos—Grandmont (1985, 1987), Grandmont-Malgrange (1986). They extend from macro-economics—business cycles, models of class struggles, political economy—to micro-economics—models with overlapping generations, optimizing behavior—and touch subjects such as game theory and the theory of finance. The applicability of these theories has been thoroughly tested on the stock market prices—Brock *et al.*(1987), Brock (1988), Brock-Dechert (1988), LeBaron (1988), Brock *et al.*(1991), Hsieh (1989), Scheinkman-LeBaron (1989a,

1989b)—in studies which tried to detect signs of non-linear effects and to nail down the deterministic nature of these prices. While the theoretical models—Van Der Ploeg (1986), De Grauwe-Vansanten (1990), De Grauwe *et al.*(1993)—seem to agree on the relevance of chaotic deterministic dynamics, the empirical studies—Eckmann *et al.*(1988), Hsieh-LeBaron (1988), Hsieh (1989, 1991, 1992), LeBaron (1988), Scheinkman-LeBaron (1989a,b)—are less clear-cut, mostly because of lack of sufficiently long time series (Eckmann-Ruelle (1992)), or, because the deterministic component of market behavior is necessarily overshadowed by the inevitable external effects. An additional source of "noise" is found to result from the finiteness of the number N of agents. For finite N, the deterministically chaotic dynamics of the price is replaced by a stochastic dynamics shadowing the corresponding trajectories obtained for $N \to \infty$.

The model presented here shows a mechanism of price fixing—decisions to buy or sell dictated by comparison with other agents—which is at the origin of an instability of prices. From one period to the next, and in the absence of information other than the anticipations of other agents, prices can continuously exhibit erratic behavior and never stabilize, without diverging. Thus, the model questions the fundamental hypothesis that equilibrium prices have to converge to the intrinsic value of an asset.

2. We can also consider our model in the context of the increasing market volatility of financial markets. The volatility of prices generated by our chaotic model could give a beginning of an explanation of the excess volatility observed on financial markets—Grossman-Shiller (1981), Fama (1965), Flavin (1983), Shiller (1981), West (1988)—which traditional models, such as ARCH, try to incorporate (Engle (1982), Bollerslev *et al.*(1991), Bollerslev (1987)).

3. Finally, we can see speculative bubbles in our model as a natural consequence of mimetism. We can compare this to the two basic trends in explaining the problem of bubbles. The first makes reference to rational anticipations—Muth (1961)—and rests on the hypothesis of efficient markets. With fixed information, and knowing the dynamics of prices, the recurrence relation for the price is seen to depend on the fundamental value and a self-referential component, which tends to cause a deviation from the fundamental value: this is a speculative bubble—Blanchard-Watson (1982). This theory of rational speculative bubbles fails to explain the birth of such events, and even less their collapse, which it does not predict either. Recent developments improve on these traditional approached by combining the rational agents in the economy with irrational "noise" traders (Johansen *et al.* (1999, 2000), Sornette and A. Johansen (2001)). These noise traders are imitative investors who reside on an interaction network. Neighbors of an agent on this network can be viewed as the agent's friends or contacts, and an agent will incorporate his neighbors' views regarding the stock into his own view. These noise traders are responsible for triggering crashes. Sornette and Andersen (2001) develop a similar model in which the noise traders induce a nonlinear positive feedback in the stock price dynamics with an interplay between nonlinearity and multiplicative noise. The derived hyperbolic stochastic finite-time singularity formula transforms a Gaussian white noise into a rich time series possessing all the stylized facts of empirical prices, as well as accelerated speculative bubbles preceding crashes.

The second trend purports to explain speculative bubbles by a limitation of rationality—Shiller (1984, 2000), West (1988), Topol (1991). It allows to incorporate notions which the neo-classical analysis does not take into account: asymmetry of information, inefficiency of prices, heterogeneity of anticipations—Grossman (1977), Grossman-Stiglitz (1980), Grossman (1981), Radner (1972, 1979). In our approach, which follows the second trend, the agents act without knowing the actual effect of their behavior: this contrasts the position of a model-builder—Orléan (1986, 1989, 1990, 1992). This, in turn, can lead to prices which disconnect from the fundamental indicators of economics.

In the present paper we show that self-referred behavior in financial markets can generate chaos and speculative bubbles. They will be seen to be caused by mimetic behavior: bubbles will form due to imitative behavior and collapse when certain agents believe in the advent of a turn of trend, while they observe the behavior of their peers.

Section 2 defines the model. Section 3 provides a qualitative understanding and analysis of its dynamical properties. Section 4 extends it with a quantitative analysis of the phases of speculative bubbles in the symmetric case. Section 5 describes the statistical properties of the price returns derived from its dynamics in the symmetric case. Section 6 discusses the asymmetric case. Section 7 explores some effects introduced by the finiteness $N < \infty$ of the number of agents. In Section 8 we summarize our conclusions.

2 The model

We consider an economy in which the population makes choices between two possible states when tomorrow's price is uncertain. The choice depends on expected capital gains. The portfolio choice then becomes a price expectation problem. Each agent has a different set of informations, obtained by observing other agents. Agents do not operate with reference to fundamental value, but rather with respect to *expected market price*. They are able to make profits if their expectations are judiciously chosen. It is rational for the agent—Keynes (1936), Orléan (1986, 1989), Sornette (2001) (see Chap. 4)—to take into account collective judgments in order to make portfolio profits.

That is why, for constituting expectations at time $t+1$, the information used by an agent is the price expectation, at time t, of a certain sample of *other* agents randomly chosen among the population. This takes into account collective opinion and its expected correctness, that is, their confidence (or absence thereof) in the continuation of a deviation from the fundamental value. Their opinion refers to two kinds of price, market price and fundamental value, as exhibited by Keynes (1936):

1. Speculation relying on short term action and especially market opinion and market price. The most important aspect is the market price expectation, that is, the collective opinion about future market prices.

2. Firm behavior: long term behavior relying on economic reality and fundamental value. This leads agents to detect excessive increase or decrease of market price and thus leads to anticipatory adaptation of the market price. This causes the collapse of the bubble.

The importance of the interplay of these two classes of investing (which can be used by a same agent alternatively), corresponding to fundamental value investors and technical analysts (or trend followers), has been stressed by several recent works (Lux and Marchesi (1999), Farmer and Joshi, 2001) to be essential in order to retrieve the important stylized facts of stock market price statistics. This has recently been incorporated within a macroscopic model of the stock market with a competition between nonlinear trend-followers and nonlinear value investors (Ide and Sornette (2001), Sornette and Ide (2001)). We build on this insight and construct a very simple model of price dynamics, which puts emphasis on the fundamental *nonlinear* behavior of both classes of agents.

These well-known principles generate different kinds of risks between which agents choose by arbitrage. The former is a *competing risk*—Keynes (1936), Orléan (1989)—which leads agents to imitate the collective point of view since the market price includes it. Thus, it is assumed that Keynes' animal spirits may exist. More simply, there is the risk of mistaken expectation: agents believe in a price different from the market price. Keynes uses his famous beauty contest as a parable for stock markets. In order to predict the winner of beauty contest, objective beauty is not very important, but knowledge or prediction of others' prediction of beauty is. In Keynes' view, the optimal strategy is not to pick those faces the player thinks the prettiest, but those the other players are likely to think the average opinion will be, or those the other players will think the others will think the average opinion will be, or even further along this iterative loop.

On the other hand, in the latter case, the emerging price is not necessarily in harmony with economic reality and fundamental value. Self-referred decisions and self-validation phenomena can then indeed lead to speculative bubbles or sunspots—Azariadis (1981), Azariadis-Guesnerie (1982), Blanchard-Watson (1982), Jevons (1871), Kreps (1977). Thus, the latter risk is the result of *precaution*. It addresses the fitting of market price to fundamental value, and by extension, collapse of the speculative bubble.

Both attitudes are likely to be important and are integrated in decision rules. Agents realize an arbitrage between the two kinds of risk we have described. That is why they have both a mimetic behavior and an antagonistic one: they either follow the collective point of view or they have reversed expectations.

We are now going to put these assumptions into the simplest possible mathematical form. We assume that, at any given time t, the population is divided into two parts. Agents are explicitly differentiated as being bullish or bearish in proportion p_t, and $q_t = 1 - p_t$, respectively. The first ones expect an increase of the price, while the bearish ones expect a decrease. The agents then form their opinion for time $t+1$ by sampling the expectations of m other agents at time t, and modifying their own expectations accordingly. The number m of agents polled by a given agent to form her opinion at time $t+1$ is the first important parameter in our model.

We then introduce threshold densities ρ_{hb} and ρ_{hh}. We assume $0 \leq \rho_{hb} \leq \rho_{hh} \leq 1$. A bullish agent will change opinion if at least one of the following propositions is true:

1.At least $m \cdot \rho_{hb}$ among the m agents inspected are bearish.

2. At least $m \cdot \rho_{hh}$ among the m agents inspected are bullish.

The first case corresponds to "following the crowd," while the second case corresponds to the "antagonistic behavior." The quantity ρ_{hb} is thus the threshold for a bullish agent ("haussier") to become bearish ("baissier") for mimetic reasons, and similarly, ρ_{hh} is the threshold for a bullish agent to become bearish because there are "too many" bullish agents. One reason for this behavior is, as we said, that the deviation of the market price from fundamental value is felt to be unsustainable. Another reason is that if many managers tell you that they are bullish, it is probable that they have large "long" positions in the market: they therefore tell you to buy, hoping to be able to unfold in part their position in favorable conditions with a good profit.

The deviation of the threshold ρ_{hb} above the symmetric value $1/2$ is a measure of the "stubbornness" (or "buy-and-hold" tendency) of the agent to keep her position. For $\rho_{hb} = 1/2$, the agent strictly endorses without delay the opinion of the majority and believes in any weak trend. This corresponds to a reversible dynamics. A value $\rho_{hb} > 1/2$ expresses a tendency towards conservatism: a large ρ_{hb} means that the agent will rarely change opinion. She is risk-adverse and would like to see an almost unanimity appearing before changing her mind. Her future behavior has thus a strong memory of her past position. $\rho_{hb} - 1/2$ can be called the bullish "buy-and-hold" index.

The deviation of the threshold ρ_{hh} below 1 quantifies the strength of disbelief of the agent in the sustainability of a speculative trend. For $\rho_{hh} = 1$, she always follows the crowd and is never contrarian. For ρ_{hh} close to $1/2$, she has little faith in trend-following strategies and is closer to a fundamentalist, expecting the price to revert rapidly to its fundamental value. $1 - \rho_{hh}$ can be called the bullish reversal index.

Putting the above rules into mathematical equations we see that the probability P for an agent who is bullish at time t to change his opinion at time $t + 1$ is:

$$P \;=\; \mathrm{Prob}\left(\{x < m \cdot (1 - \rho_{hb})\} \cup \{x > m \cdot \rho_{hh}\}\right) \;, \tag{1}$$

where x is the number of bullish agents found in the sample of m agents.

In an entirely similar way, we introduce thresholds ρ_{bh}, and ρ_{bb}. The thresholds ρ_{bh} and ρ_{bb} have completely symmetric roles when the agent is initially bearish. $\rho_{bh} - 1/2$ can be called the bearish "buy-and-hold" index. $1 - \rho_{bb}$ can be called the bearish reversal index. The probability Q for a bearish agent at time t to become bullish at time $t + 1$ is:

$$Q \;=\; \mathrm{Prob}\left(\{x < m \cdot (1 - \rho_{bh})\} \cup \{x > m \cdot \rho_{bb}\}\right) \;.$$

We can combine these two rules into a *dynamical law* governing the time evolution of the populations. Denoting p_t the proportion of bullish agents in the population at time t, we can find the new proportion, p_{t+1}, at time $t + 1$, by taking into account those agents which have changed opinion according to the *deterministic law* given above. To simplify notation, we let $p_{t+1} = p'$ and $p_t = p$. Then, the above statements are easily used to express p' in terms of p, by using the probability of finding j bullish people among m (Corcos (1993)):

$$\begin{aligned} p' \;&=\; p - p \cdot \sum_{\substack{j \geq m \cdot \rho_{hb} \\ \text{or } j < m \cdot (1 - \rho_{hh})}} \binom{m}{j} p^{m-j} (1-p)^j \\ &+\; (1-p) \cdot \sum_{\substack{j \geq m \cdot \rho_{bh} \\ \text{or } j < m \cdot (1 - \rho_{bb})}} \binom{m}{j} (1-p)^{m-j} p^j \qquad (2) \\ &\equiv\; F_{\underline{\rho},m}(p) \;, \end{aligned}$$

where $\underline{\rho} = \{\rho_{hb},\ \rho_{bh},\ \rho_{hh},\ \rho_{bb}\}$. Thus, the function $F_{\underline{\rho},m}(p)$ completely characterizes the dynamics of the proportion of bullish and bearish populations.

3 Qualitative analysis of the dynamical properties

3.1 The limit $m \to \infty$

The law given by Eq.2 is not easy to analyze, and we give in Fig. 1 a few sample curves $F_{\underline{\rho},m}$. We see that as m gets larger, the curves seem to tend to a limiting curve. Using this observation, our conceptual understanding of the dynamics can be drastically simplified if we consider the problem for a large number m of polled partners. Indeed, it is most convenient to first study the unrealistic problem $m = \infty$ and to view the large m case as a perturbation of this limiting case. The main ingredient in the study of the case $m = \infty$ is the Law of Large Numbers, which we use in a form given in Feller (1966):
Lemma. Let g be a continuous function on $[0, 1]$. Then, for $s \in [0, 1]$,

$$\lim_{m\to\infty} \sum_{j=0}^{m} \binom{m}{j} s^j (1-s)^{m-j} \cdot g(j/m) \;=\; g(s)\;. \tag{3}$$

We apply this lemma to the (piecewise continuous) function $g = f_h$, where f_h is the indicator function of the set defining P:

$$f_h \;=\; \begin{cases} 1\;, & \text{if } x \geq \rho_{hb} \text{ or } x < 1-\rho_{hh}, \\ 0\;, & \text{otherwise}\;. \end{cases} \tag{4}$$

Similarly, we define

$$f_b \;=\; \begin{cases} 1\;, & \text{if } x \geq \rho_{bh} \text{ or } x < 1-\rho_{bb}, \\ 0\;, & \text{otherwise}\;. \end{cases} \tag{5}$$

It is now easy to check that the lemma implies

$$\lim_{m\to\infty} F_{\underline{\rho},m}(p) \;=\; p - p\cdot f_h(1-p) + (1-p)\cdot f_b(p) \;\equiv\; G_{\underline{\rho}}(p)\;. \tag{6}$$

Note again that we do not consider $G_{\underline{\rho}}(p)$ itself as an evolution law for the population of bullish agents, but $G_{\underline{\rho}}$ can serve very well as an approximation for the true laws $F_{\underline{\rho},m}$ for large m. In Fig. 1 we show how the functions $F_{\underline{\rho},m}$ converge to $G_{\underline{\rho}}$.

3.2 Classification of the different regimes

In the preceding section, we have shown how to gain a qualitative understanding of the maps $F_{\underline{\rho},m}$, when m is large. We can now apply in a rather straightforward way the general theory of 1-dimensional discrete time dynamical systems (see *e.g.*, Collet-Eckmann (1980)) to the functions $F_{\underline{\rho},m}$. The recurrence $p_t \to p_{t+1}$ can exhibit several typical behaviors which, for large m depend essentially only on the set of parameters $\underline{\rho}$. We enumerate a few of them and refer the reader to Figs. 2 and 3. In this section, we restrict our attention to the symmetric case $\rho_{hb} = \rho_{bh}$ and $\rho_{hh} = \rho_{bb}$.

1.The most trivial case is the appearance of a stable fixed point. This will occur when the buy-and-hold index $\rho_{hb} - 1/2$ is not too large and the reversal index $1 - \rho_{hh}$ is not too small. For example, this occurs for $\rho_{hh} = \rho_{bb} = 0.75$, $\rho_{hb} = \rho_{bh} = 0.72$, and $m = 60$. Then, the population will equilibrate, and converge to $p = 1 - q \approx 0.68$, or to $p \approx 0.32$ (see upper panel of figure 3).

2. The next more interesting case is the appearance of a limit cycle (of period 2): at successive times, the population of bullish and bearish agents will oscillate between two different values. This happens, *e.g.*, for $\rho_{hh} = \rho_{bb} = 0.76$, with the other parameters as before (see second panel of figure 3).

3. But for certain values of the parameters, *e.g.*, $\rho_{hh} = \rho_{bb} = 0.85$, the sequence of values of p_t is a *chaotic* sequence, with positive Liapunov exponent (cf. Eckmann-Ruelle (1985)). The mechanism for this is really a combination of sufficiently strong buy-and-hold index $\rho_{hb} - 1/2$ and of sufficiently weak reversal index $1 - \rho_{hh}$. This regime thus occurs when the opinion of a trader has a strong memory of her past positions and changes it only when a strong majority appears. This regime also requires a weak belief of the agent in fundamental valuation, as she will believe until very late that a strong bullish or bearish speculative trend is sustainable. Fundamentally, it is this *self-referential* behavior of the anticipations *alone* which is responsible for a deterministic, but seemingly erratic evolution of the

population of bullish and bearish agents. No external noise is needed to make this happen, and in general, we view external stimuli as acting on top of the intrinsic mechanism which we exhibit here (Eckmann (1981)). Note that the set of parameter values $\underline{\rho}$ for which chaos is expected (say, near the values used at the bottom of Fig. 3) has positive Lebesgue measure.

We next consider in more detail the time evolution of p_t for the parameter values of the last frame of Fig. 3, which are typical for the abundant set of "chaotic" parameter values, and we will show how the time evolution exhibits "speculative bubbles." This phenomenon is akin to the notion of intermittency (of "Type I") as known to physicists, see *e.g.*, Manneville (1991) for an exposition. Indeed, we can distinguish two distinct behaviors in the last frame of Fig. 3, which occur repeatedly with more or less pronounced separation. The first process is the "laminar phase," which is seen to occur when the population p_t is near 0.5. Then, the evolution of the population is slow, and the population grows slowly away from 0.5, either monotonically or through an oscillation of period 2, depending on $\underline{\rho}$. This motion is slower when the inspected sample size (m) is larger, reflecting a more stable evolution for less independent agents. When the distance from 0.5 is large, erratic behavior sets in, which persists until the population reaches again a value of about 0.5, at which point the whole scenario repeats. The determinism of the model is reflected by "equal causes lead to equal effects," while its chaotic nature is reflected by the erratic length of the laminar periods, as well as of the bubbles of wild behavior.

Having analyzed qualitatively the evolution of the number of bullish agents, we next describe how the price π_{t+1} of an asset at time $t+1$ is related to the proportion p_t of bullish agents. One can argue (Corcos (1993), Bouchaud and Cont (1998), Farmer (1998)) that the price change $\pi_{t+1} - \pi_t$ from one period to the next is a *monotone* function of p_t (and, perhaps, of π_t). This function is positive when $p_t > 1/2$ and negative when $p_t < 1/2$. If the reaction to a change in p_t is reflected in the prices in the next period, then a bubble in p_t will lead to a speculative bubble in the prices in the next period. Thus, our model predicts the occurrence of bubbles from the behavior of the agents alone. Furthermore, for quite general laws of the form

$$\pi_{t+1} \;=\; H(\pi_t, p_t)\;, \tag{7}$$

a simple application of the chain rule of differentiation leads to the observation that the variable π_t has the same Liapunov exponent as p_t. In fact, this will be the case if $0 < \partial_\pi H < \lambda$ and $\partial_p H > c > 0$, where λ is the Liapunov exponent for p_t, as follows from $\delta\pi_{t+1} = \partial_\pi H \cdot \delta\pi_t + \partial_p H \cdot \delta p_t$. This condition is, in particular, satisfied for a law of the form $\pi_{t+1} = \pi_t + G(p_t)$, where G is strictly monotone. Thus, chaotic behavior of bullish agents leads to chaotic behavior of prices.

In the sequel, we shall take the simplest form of a log-difference of the price linearly proportional to the order unbalance (Farmer (1998)), leading to

$$\ln \pi_{t+1} - \ln \pi_{t+1} \equiv r_{t+1} = \gamma(p_t - \tfrac{1}{2})\;, \tag{8}$$

showing that the return r_t calculated over one period is proportional to the imbalance $p_t - \frac{1}{2}$. Thus, the properties of the return time series can be derived directly from those of p_t as we document below.

To summarize this qualitative analysis of the case of an infinite number N of agents, we observe a time evolution which, while satisfying certain criteria of randomness (such as possessing an absolutely continuous invariant measure and exhibiting a positive Liapunov exponent—cf. Eckmann-Ruelle (1985)) at the same time exhibits some regularities on short time scales, since it is deterministic. Our model thus establishes that straightforward fundamental conditions may suffice to generate chaotic stock market behavior, depending on the parameter values. If the market adjusts present market price on the basis of expectations and mimicry—self-referred behavior—then chaotic evolution of the population will also imply chaotic evolution of prices.

4 Quantitative analysis of the speculative bubbles within the chaotic regime in the symmetric case

For an infinite number N of agents and in the symmetric case $\rho_{hb} = \rho_{bh} \equiv \rho_1$ and $\rho_{hh} = \rho_{bb} \equiv \rho_2$, let us rewrite the dynamical evolution (2) of the system as

$$p' = p - p\sum_{j=0}^{m}\binom{m}{j}p^{m-j}(1-p)^j f\left(\frac{j}{m}\right) + (1-p)\sum_{j=0}^{m}\binom{m}{j}(1-p)^{m-j}p^j f\left(\frac{j}{m}\right), \qquad (9)$$

where

$$f(x) = \begin{cases} 1, \text{ if } x \geq \rho_1 \text{ or } x < 1-\rho_2 \\ 0, \text{ otherwise.} \end{cases} \qquad (10)$$

Let us define

$$g_m(p) = \sum_{j=0}^{m}\binom{m}{j}p^{m-j}(1-p)^j f\left(\frac{j}{m}\right) , \qquad (11)$$

which yields

$$p' = F_m(p) = p - p\cdot g_m(p) + (1-p)\cdot g_m(1-p) . \qquad (12)$$

This expression (12) generalizes (6) to arbitrary m.

As was described in the previous section, this system can exhibit chaotic behavior for certain values of the parameters. An example is given in figure 4 which shows a long time series, showing many positive bubbles and negative bubbles interrupted by chaotic oscillatory phases. For the time being, we do not worry about the existence of the negative bubbles, which are rarely if ever observed in real markets: this is an artifact of the symmetry $\rho_{hb} = \rho_{bh} \equiv \rho_1$ and $\rho_{hh} = \rho_{bb} \equiv \rho_2$, that we shall relax later. Keeping the symmetry assumption simplified the theoretical analysis without changing the key results obtained below.

Let us consider the first bubble developing in the time interval from $t = 35$ to $t = 546$ as seen in figure 5-a). Figure 5-b plots the logarithm of $p - 1/2$ as a function of linear time: the linear trend from $t = 35$ to $t \approx 480$ seen in the lower panel qualifies an exponential growth $p - 1/2 \propto e^{\kappa t}$ (with $\kappa > 0$) followed by a super-exponential growth accelerating so much as to give the impression of reaching a singularity in finite-time.

To understand this phenomenon, we plot the logarithm of $F_m(p) - p$ versus the logarithm of $p - 1/2$ in figure 6 for three different values of $m = 30, 60$ and 100. Two regimes can be observed.

1. For small $p - 1/2$, the slope of $\log_{10}(F_m(p) - p)$ versus $\log_{10}(p - 1/2)$ is 1, *i.e*,

$$p' - p \equiv F_m(p) - p \simeq \alpha(m)\left(p - \frac{1}{2}\right) . \qquad (13)$$

   This expression (13) explains the exponential growth observed at early times in figure 5.

2. For larger $p - 1/2$, the slope of $\log_{10}(F_m(p) - p)$ versus $\log_{10}(p - 1/2)$ increases above 1 and stabilizes to a value $\mu(m)$ before decreasing again due to the reinjection produced by the contrarian mechanism. The interval in $p - 1/2$ in which the slope is approximately stabilized at the value $\mu(m)$ enables us to write

$$F_m(p) - p \simeq \beta(m)\left(p - \frac{1}{2}\right)^{\mu(m)} , \quad \text{with } \mu > 1 . \qquad (14)$$

These two regimes can be summarized in the following phenomenological expression for $F_m(p)$:

$$F_m(p) = \frac{1}{2} + (1 - 2g_m(1/2) - g'_m(1/2))\left(p - \frac{1}{2}\right) + \beta(m)\left(p - \frac{1}{2}\right)^{\mu(m)} , \qquad (15)$$

$$= \frac{1}{2} + \left(p - \frac{1}{2}\right) + \alpha(m)\left(p - \frac{1}{2}\right) + \beta(m)\left(p - \frac{1}{2}\right)^{\mu(m)} \quad \text{with } \mu > 1 , \qquad (16)$$

and

$$\alpha(m) = -2g_m(1/2) - g'_m(1/2)\ . \tag{17}$$

This expression can be obtained as an approximation of the exact expansion derived in the Appendix.

In order to check the hypothesis (16), we numerically solve the following problem

$$\min_{\{\alpha,\beta,\mu\}} \left|\left| F_m(p) - \frac{1}{2} - [1+\alpha]\left(p-\frac{1}{2}\right) - \beta\left(p-\frac{1}{2}\right)^{\mu} \right|\right|^2\ , \tag{18}$$

which amounts to constructing the best approximation of the exact map $F_m(p)$ in terms of an effective power law acceleration (see (20) below). The results obtained for $m = 60$ interacting agents and $\rho_{hb} = \rho_{bh} = 0.72$ and $\rho_{hh} = \rho_{bb} = 0.85$ are given in table 1 and shown in figure 7. The numerical values of α are in good agreement with the theoretical prediction : $\alpha(m) = F'_m(1/2) - 1$ which yields $\alpha(m) \simeq 0.011$ in the present case ($m = 60$, $\rho_{hb} = \rho_{bh} = 0.72$ and $\rho_{hh} = \rho_{bb} = 0.85$). As a first approximation, we can consider that the exponent μ is fixed over the interval of interest, which is reasonable according to the very good quality of the fits shown in figure 7. We can conclude from this numerical investigation that $\mu(m) \in [3, 4]$. A finer analysis shows however that the exponent μ is in fact not perfectly constant but shifts slowly from about 3 to 4 as p increases. This should be expected as the function $F_m(p)$ contains many higher-order terms. We can also note that the parameter $p_c = (\beta/\alpha)^{-1/\mu}$, which defines the typical scale of the crossover remains constant and equal to $p_c \simeq 0.70$ for all the fits (except for the largest interval $p - 1/2 < 0.2$, for which p_c = 0.8). In sum, the procedure (18) and its results show that the effective power law representation (16) is a cross-over phenomenon: it is not dominated by the "critical" value $\rho_{hb} = \rho_{bh}$ of the jump of the map obtained in the limit of large m.

Introducing the notation $\epsilon = p - 1/2$, the dynamics associated with the effective map (16) can be rewritten

$$\epsilon' - \epsilon = \alpha(m)\epsilon + \beta(m)\epsilon^{\mu(m)}, \tag{19}$$

which, in the continuous time limit, yields

$$\frac{d\epsilon}{dt} = \alpha(m)\epsilon + \beta(m)\epsilon^{\mu(m)}\ . \tag{20}$$

Thus, for small ϵ, we obtain an exponential growth rate

$$\epsilon_t \sim e^{\alpha(m)t}\ , \tag{21}$$

while for large enough ϵ

$$\epsilon_t \sim (t_c - t)^{-\frac{1}{\mu(m)-1}}\ . \tag{22}$$

For example, for $m = 60$ with $\rho_{hb} = \rho_{bh} = 0.72$ and $\rho_{hh} = \rho_{bb} = 0.85$, we can check on figure 6 that $\mu(m) = 3$, which yields for large ϵ:

$$p_t - \frac{1}{2} \sim \frac{1}{\sqrt{t_c - t}}\ . \tag{23}$$

The prediction (23) implies that plotting $(p_t - 1/2)^{-2}$ as a function of t should be a straight line in this regime. This non-parametric test is checked in figure 8 on five successive bubbles. This provides a confirmation of the effective power law representation (16) of the map. The fact that it is the lowest

Optimization Domain	α	β	μ
$0 \le p - \frac{1}{2} \le 0.05$	0.011	11.67	3.27
$0 \le p - \frac{1}{2} \le 0.10$	0.013	43.66	3.77
$0 \le p - \frac{1}{2} \le 0.15$	0.014	60.32	3.91
$0 \le p - \frac{1}{2} \le 0.20$	0.004	30.64	3.54

Table 1: Optimized parameters α, β and μ for several optimization interval with m=60 interacting agents and $\rho_{hb} = \rho_{bh} = 0.72$ and $\rho_{hh} = \rho_{bb} = 0.85$

estimate $\mu \approx 3$ shown in table 1 which dominates in figure 8 results simply from the fact that it is the longest transient corresponding to the regime where p is closest to the unstable fixed point $1/2$. This is visualized in figure 8 by the horizontal dashed lines indicating the levels $p - 1/2 = 0.05, 0.01$ and 0.2. This demonstrates that most of the visited values are close to the unstable fixed point, which determines the effective value of the nonlinear exponent $\mu \approx 3$.

With the price dynamics (8), the prediction (22) implies that the returns r_t should increase in an accelerating super-exponential fashion at the end of a bubble, leading to a price trajectory

$$\pi_t = \pi_c - C(t_c - t)^{\frac{\mu(m)-2}{\mu(m)-1}} , \tag{24}$$

where π_c is the culminating price of the bubble reached at $t = t_c$ when $\mu(m) > 2$, such the finite-time singularity in r_t gives rise only to an infinite slope of the price trajectory. The behavior (24) with an exponent $0 < \frac{\mu(m)-2}{\mu(m)-1} < 1$ has been documented in many bubbles (Sornette *et al.* (1996), Johansen *et al.* (1999, 2000), Johansen and Sornette (1999, 2000), Sornette and Johansen (2001), Sornette and Andersen (2001), Sornette (2001)). The case $m = 60$ with $\rho_{hb} = \rho_{bh} = 0.72$ and $\rho_{hh} = \rho_{bb} = 0.85$ shown in figure 6 leads to $\frac{\mu(m)-2}{\mu(m)-1} = 1/2$, which is in reasonable agreement with previously reported values.

Interpreted within the present model, the exponent $\frac{\mu(m)-2}{\mu(m)-1}$ of the price singularity gives an estimation of the "connectivity" number m through the dependence of μ on m documented in figure 6. Such a relationship has already been argued by Johansen *et al.*, (2000) at a phenomenological level using a mean-field equation in which the exponent is directly related to the number of connections to a given agent.

5 Statistical properties of price returns in the symmetric case

Using the price dynamics (8), the distribution of $p - 1/2$ is the same as the distribution of returns, which is the first statistical property analyzed in econometric work (Campbell *et al.* (1997), Lo and MacKinlay (1999), Lux (1996), Pagan (1996), Plerou et al (1999), Laherrère and Sornette (1998)). Note that the distribution of $p - 1/2$ is nothing but the invariant measure of the chaotic map $p'(p)$ which can be shown to be continuous with respect to the Lebesgue measure (Eckmann and Ruelle (1985)). Figure 9 shows the cumulative distribution of $r_t \propto p_t - 1/2$. Notice the two breaks at $|p - 1/2| = 0.28$, which are due to the existence of weakly unstable periodic orbits corresponding to a transient oscillation between bullish and bearish states.

Figure 10 plots in double logarithmic scales the survival distribution of $r_t \propto p_t - 1/2$ for $m = 30, 60$ and 100. For $m = 60$, we can observe an approximate power law tail but the exponent is smaller than 1 in contradiction with the empirical evidence which suggests a tail of the survival probability with exponents $3 - 5$. In the other cases, we cannot conclude on the existence of a power law regime, but it is obvious that the tail behavior of the distribution function depends on the number m of polled agents.

Figure 11 shows the behavior of the autocorrelation function for $m = 60$ and $m = 100$, with the same values of the other parameters $\rho_{hb} = \rho_{bh} = 0.72$ and $\rho_{hh} = \rho_{bb} = 0.85$. For $m = 100$, the presence of the weakly unstable orbits is felt much stronger, which is reflected in 1) a very strong periodic component of the correlation function and 2) its slow decay. Even for $m = 60$, the correlation function does not decay fast enough compared to the typical duration of speculative bubbles to be in quantitative agreement with empirical data. This anomalously large correlation of the returns is obviously related to the deterministic dynamics of the returns. We thus expect that including stochastic noise due to a finite number N of agents (see below) and adding external noise due to "news" will whiten r_t significantly.

Figure 12 compares the correlation function for the returns time series $r_t \propto p_t - 1/2$ and the volatility time series defined as $|r_t|$. The volatility is an important measure of risks and thus plays an important role in portfolio managements and option pricing and hedging. Note that taking the absolute value of the return removes the one source of irregularity stemming from the change of sign of $r_t \propto p_t - 1/2$ to focus on the local amplitudes. We observe in figure 12 a significantly longer correlation time for the volatility. Moreover, the correlation function of the volatility first decays exponentially and then as a power law. This behavior has previously been documented in many econometric works (Ding *et al.* (1993), Ding and Granger (1996), Müller *et al.* (1997), Dacorogna *et al.* (1998), Arneodo *et al.* (1998), Ballocchi *et al.* (1999), Muzy *et al.* (2001)).

6 Asymmetric cases

We have seen that the symmetric case $\rho_{hb} = \rho_{bh}$ and $\rho_{hh} = \rho_{bb}$ is plagued by the weakly unstable periodic orbits which put a strong and unrealistic imprint on the statistical properties of the return time series. It is natural to argue that breaking the symmetry will destroy the strength of these periodic orbits.

From a behavioral point of view, it is also quite clear that the attitude of an investor is not symmetric. One can expect a priori a stronger bullish buy-and-hold index $\rho_{hb} - 1/2$ than bearish buy-and-hold index $\rho_{bh} - 1/2$: one is a priori more prone to hold a position in a bullish market than in a bearish one. Similarly, we expect a smaller bullish reversal index $1 - \rho_{hh}$ than bearish reversal index $1 - \rho_{bb}$: speculative bubbles are rarely seen on downward trends as it is much more common that increasing prices are favorably perceived and can be sustained much longer without reference to the fundamental price.

Such an asymmetry has been clearly demonstrated empirically in the difference between the rate of occurrence and size of extreme drawdowns compared to drawups in stock market time series (Johansen and Sornette (2001)). Drawdowns (drawups) are defined as the cumulative losses (gains) from the last local maximum (minimum) to the next local minimum (maximum). Drawdowns and drawups are very interesting because they offer a more natural measure of real market risks than the variance, the value-at-risk or other measures based on fixed time scale distributions of returns. For the major stock market indices, there are very large drawdowns which are "outliers" while drawups do not exhibit such drastic change of regime. For major companies, drawups of amplitude larger than 15% occur at a rate about twice as large as the rate of drawdowns, but with lower absolute amplitude.

Figure 13 compares the dynamics for the symmetric system (upper panel (a)) and for the asymmetric system (lower panel (b)). It is clear that, as expected, the number of periodic orbits decreases significantly in the asymmetric system. However, there are still an unrealistic number of negative bubbles. It is not possible to increase the asymmetry sufficiently strongly without exiting from the chaotic regime. This unrealistic feature is thus an intrinsic property and limitation of the present model. We shall indicate in the conclusion possible extensions and remedies.

Figure 14 compares the cumulative distributions of $p - 1/2$ for $m = 60$ for the symmetric and asymmetric cases. The strong effect of the weakly unstable periodic orbits observed in the periodic case has disappeared. In addition, the tail of the distribution decays faster in the asymmetric case, in better (but still not very good) agreement with empirical data.

Figure 15 shows the correlation function of the returns for a symmetric and an asymmetric case. In the asymmetric case, there is no trace of oscillations but the decay is slightly slower.

7 Finite size effects

Until now, our analysis has focused on the limit of an infinite number $N \to \infty$ of agents, in which each agent polls randomly m agents among N. In this limit, we have shown that, for a large domain in the parameter space, the dynamics of the returns is chaotic with interesting and qualitatively realistic properties.

7.1 Finite-size effects in other models

We now investigate finite-size effects resulting from a finite number N of interacting agents trading on the stock market. This issue of the role of the number of agents has recently been investigated vigorously with surprising results. First, Egenter *et al.*, (1999) studied the N-dependence of the dynamical properties of price time series of the Kim-Markowitz (1989) and of the Lux-Marchesi (1999) models. They found that, if this number N goes to infinity, nearly periodic oscillations occur and the statistical properties of the price time series become completely unrealistic. Stauffer (1999) reviewed this work and others such as the Levy-Levy-Solomon (1995, 2000) model: realistically looking price fluctuations are obtained for $N \sim 10^2$, but for $N \sim 10^6$ the prices vary smoothly in a nearly periodic and thus unrealistic way. The model proposed by Farmer (1998) suffers from the same problem: with a few hundred investors, most investors are fundamentalists during calm times, but bursts of high volatility coincide with large fractions of noise traders. When N becomes much larger, the fraction of noise traders goes to

zero in contradiction to reality. On a somewhat different issue, Huang and Solomon (2001) have studied finite-size effects in dynamical systems of price evolution with multiplicative noise. They find that the exponent of the Pareto law obtained in stochastic multiplicative market models is crucially affected by a finite N and may cause in the absence of an appropriate social policy extreme wealth inequality and market instability. Another model (apart from ours) where the market may stay realistic even for $N \to \infty$ seems to be the Cont-Bouchaud percolation model (2001). However, this only occurs for an unrealistic tuning of the percolation concentration to its critical value. Thus, in most cases, the limit $N \to \infty$ leads to a behavior of the simulated markets which becomes quite smooth or periodic and thus predictable, in contrast to real markets. Our model which remains (deterministically) chaotic is thus a significant improvement upon this behavior. We trace this improvement on the highly nonlinear behavior resulting from the interplay between the imitative and contrarian behavior. It has thus been argued (Stauffer (1999)) that, if these previous models are good descriptions of markets, then real markets with their strong random fluctuations are dominated by a rather limited number of large players: this amounts to assume that the hundred most important investors or investment companies have much more influence than the millions of less wealthy private investors.

There is another class of models, the minority games (Challet and Zhang (1997)), in which the dynamics remains complex even in the limit $N \to \infty$. It has been established that the fluctuations of the sum of the aggregate demand have an approximate scaling with similar sized fluctuations (volatility/standard deviation) for any N and m for the scale scaled variable $2^m/N$, where m is the memory length (Challet *et al.* (2000)). In a generalization, the so-called Grand Canonical version of the Minority Game (Jefferies *et al.* (2001)), where the agents have a confidence threshold that prevents them from playing if their strategies have not been successful over the last T turns, the dynamics can depend more sensitively on N: as N becomes small, the dynamics can become quite different. For large N, the complexity remains.

The difference between the limit $N \to \infty$ considered up to now in this paper and the case of finite N is that p_t is no more the fraction of bullish agents. For finite N, p_t must be interpreted as the probability for an agent to be bullish. Of course, in the limit of large N, the law of large numbers ensures that the fraction of bullish agents becomes equal to the probability for an agent to be bullish. There are several ways to implement a finite-size effect. We here discuss only the two simplest ones.

7.2 Finite external sampling of an infinite system

Consider a system with an infinite number of agents for which the fraction p_t of bullish agents is governed by the deterministic dynamics (2). At each time step t, let us sample a finite number N of them to determine the fraction of bullish agents. We get a number n, which is in general close but not exactly equal to Np_t due to statistical fluctuations. The probability to find n bullish agents among N agents is indeed given by the binomial law

$$\Pr(n) = \binom{N}{n} p^n (1-p)^{N-n} \, . \tag{25}$$

This shows that the observed proportion $\tilde{p} = n/N$ of bullish agents is asymptotically normal with mean p and standard deviation $1/\sqrt{p(1-p)N}$: $\Pr(\tilde{p}) \sim \mathcal{N}(p, 1/\sqrt{p(1-p)N})$. Iterating the sampling among N agents at each time step gives a noisy dynamics $\tilde{p}_t$ shadowing the true deterministic one.

Figure 16 compares the dynamics of the deterministic p_t corresponding to $N \to \infty$ (panel (a)) with $\tilde{p}_t$ for a number $N = m + 1 = 61$ of sampled agents among the infinite ensemble of them (panel (b)). Panel (c) is the "noise" time series defined as $\tilde{p}_t - p_t$, *i.e*, by subtracting the time series of panel (a) from the time series of panel (b). The noise time series of panel (c) thus represents the statistical fluctuations due to the finite sampling of agents'opinions. Figure 16-b shows the characteristic volatility clusters which is one of the most important stylized properties of empirical time series.

For large N, we can write

$$\tilde{p}_t = p_t + \frac{1}{\sqrt{p_t(1-p_t)N}} W_t \tag{26}$$

where $\{W_t\}$ are iid gaussian variables with zero mean and unit variance. Therefore, the correlation

function $\text{Corr}_N(\tau)$ at lag $\tau \neq 0$ is obtained from that for $N \to \infty$ by multiplication by a constant factor:

$$\text{Corr}_N(\tau) = \frac{N\text{Var}(p)}{\text{E}[1/\{p(1-p)\}] + N\text{Var}(p)} \times \text{Corr}_\infty(\tau) \quad \text{and } \tau \neq 0, \tag{27}$$

$$\simeq \text{Corr}_\infty(\tau) \quad \text{for large } N\ , \tag{28}$$

where $\text{E}[x]$ denotes the expectation of x with respect to the continuous invariant measure of the dynamical system (2). Note that $\text{E}[1/\{p(1-p)\}]$ always exists for $m < \infty$ since the support of the continuous measure of (2) with respect to Lebesgue measure is bounded from below by a value strictly larger than 0 and from above by a value strictly less than 1. Figure 17 shows that the correlation function of $\tilde{p}_t$ is very close to that of the deterministic trajectory p_t.

To quantify further the impact of the statistical noise stemming from the finite size of the market, figures 18 and 19 show the return maps of $\tilde{p}_t$, *i.e*, $\tilde{p}_{t+1}$ as a function of $\tilde{p}_t$, for $m = 60$ polled agents among a total number $N = 61$ of agents (fig 18) and $N = 600$ (fig 19).

Figure 20 shows the price trajectory obtained by $\pi_t = \pi_{t-1}\ \exp[\tilde{r}_t]$ in linear and logarithmic scale. The super-exponential acceleration of the price giving rise to sharp peaks in the semi-logarithmic representation (Roehner and Sornette (1998)) is clearly visible.

7.3 Finite number N of agents

We now introduce a genuine finite stock market with N agents. We assume that the agents do not know the exact number N of agents in the market (this is realistic) and they are in contact with only m other agents that they poll at each time period. Not knowing the true value of N but assuming it to be large, it is rational for them to develop the best predictor of the dynamics by assuming the ideal case of an infinite number of agents with m polled agents and thus use the deterministic dynamics (2) as their best predictor.

At each time period t, each agent thus chooses randomly m agents that she polls. She then counts the number of bullish and bearish agents among her polled sample of m agents. This number divided by m gives her an estimation $\hat{p}_t$ of the probability p_t be to bullish at time t. Introducing this estimation in the deterministic equation (2), the agent obtains a forecast $\hat{p}'$ of the true probability p' to be bullish at the next time step.

Results of the simulations of this model are shown in figure 21. We observe a significantly stronger "noise" compared to the previous section, which is expected since the noise is itself injected in the dynamical equation at each time step. As a consequence, the correlation function of the returns and of the volatility decay faster than their deterministic counterpart. The correlation of the volatility still decays about ten times slower than the correlation of the returns, but this clustering of volatility is not sufficiently strong compared to empirical facts.

Other more realistic models of a finite number of agents can be introduced. For instance, at time t, consider an agent among the N. She chooses m other agents randomly and polls them. Each of them is either bullish or bearish as a result of decisions taken during the previous time period. She then counts the number of bullish agents among the m, and then determines her new attitude using the rules (1). If she is polled at time $t+1$ by another agent, her attitude will be the one determined from t to $t+1$. In this way, we never refer to the deterministic dynamics p_t but only to its underlying rules. As a consequence, this deterministic dynamics does not exert an attraction that minimizes the effect of statistical fluctuations due to finite sizes. This approach is similar to going from a Fokker-Planck equation (equation (2)) to a Langevin equation with finite-size effects. This class of models will be investigated elsewhere.

8 Conclusions

The traditional concept of stock market dynamics envisions a stream of stochastic "news" that may move prices in random directions. This paper, in contrast, demonstrates that certain types of deterministic behavior—mimicry and contradictory behavior alone—can already lead to chaotic prices.

If the market prices are assumed to follow the p_t behavior, our description refers to the well-known evolution of the speculative bubbles. Such apparent regularities often occur in the stock market and form the basis of the so-called "technical analysis" whereby traders attempt to predict future price movements

by extrapolating certain patterns from recent historical prices. Our model provides an explanation of birth, life and death of the speculative bubbles in this context.

While the traditional theory of rational anticipations exhibits and emphasizes self-reinforcing mechanisms, without either predicting their inception nor their collapse, the strength of our model is to justify the occurrence of speculative bubbles. It allows for their collapse by taking into account the combination of mimetic and antagonistic behavior in the formation of expectations about prices.

The specific feature of the model is to combine these two Keynesian aspects of speculation and enterprise and to derive from them behavioral rules based on collective opinion: the agents can adopt an imitative and gregarious behavior, or, on the contrary, anticipate a reversal of tendency, thereby detaching themselves from the current trend. It is this duality, the continuous coexistence of these two elements, which is at the origin of the properties of our model: chaotic behavior and the generation of bubbles.

It is a common wisdom that deterministic chaos leads to fundamental limits of predictability because the tiny inevitable fluctuations in those chaotic systems quickly snowball in unpredictable ways. This has been investigated in relation with for instance long-term weather patterns. However, in the context of our models, we have shown that the chaotic dynamics of the returns alone cannot be the limiting factor for predictability, as it contains too much residual correlations. Endogenous fluctuations due to finite-size effects and external news (noise) seem to be needed as important factors leading to the observed randomness of stock market prices. The relation between these extrinsic factors and the intrinsic ones studied in this paper will be explored elsewhere.

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Appendix

We expand $F_m(p)$ around the fixed point $p = 1/2$, so that, using the symmetry of $F_m(p)$

$$F_m(p) = \frac{1}{2} + F'_m(1/2) \cdot \left(p - \frac{1}{2}\right) + F^{(3)}_m(1/2) \cdot \left(p - \frac{1}{2}\right)^3 + \cdots \tag{29}$$

First of all, it is obvious to show by recursion that

$$\begin{aligned} F'_m(1/2) &= 1 - 2g_m(1/2) - g'_m(1/2) & (30)\\ F^{(2k+1)}_m(1/2) &= -2(2k+1)g^{(2k)}_m(1/2) - g^{(2k+1)}_m(1/2) \text{ if } k > 0\,. & (31) \end{aligned}$$

The problem thus amounts to calculating the derivatives of g_m.

Some simple algebraic manipulations allow to obtain

$$\begin{aligned} g'_m(p) &= m \sum_{j=0}^{m-1} \binom{m-1}{j} p^{m1-j}(1-p)^j \left[f\left(\frac{j}{m}\right) - f\left(\frac{j+1}{m}\right)\right] & (32)\\ &= -m \sum_{j=0}^{m-1} \binom{m-1}{j} p^{m1-j}(1-p)^j \Delta_1 f_m(j), & (33) \end{aligned}$$

where $\Delta_1 f_m(\cdot)$ is the first order discrete derivative of $f\left(\frac{\cdot}{m}\right)$, which yields

$$g'_m\left(\frac{1}{2}\right) = -\frac{m}{2^{m-1}} \sum_{j=0}^{m-1} \binom{m-1}{j} \Delta_1 f_m(j)\,. \tag{34}$$

By recursion, it is easy to prove that

$$g^{(k)}_m\left(\frac{1}{2}\right) = \frac{(-1)^k\, m!}{2^{m-k}\, k!} \sum_{j=0}^{m-k} \binom{m-k}{j} \Delta_k f_m(j) \tag{35}$$

and $\Delta_k f_m(\cdot)$ is the k^{th} order discrete derivative of $f\left(\frac{\cdot}{m}\right)$:

$$\Delta_k f_m(j) = \sum_{i=0}^{k} \binom{k}{i} (-1)^i f\left(\frac{j+i}{m}\right). \tag{36}$$

Finally,

$$\begin{aligned} F^{(2k+1)}_m(1/2) &= \frac{m!}{2^{m-2k-1}\,(2k)!} \left[\frac{1}{2k+1} \sum_{j=0}^{m-2k-1} \binom{m-2k-1}{j} \Delta_{2k+1} f_m(j) \right.\\ &\quad \left. -(2k+1) \sum_{j=0}^{m-2k} \binom{m-2k}{j} \Delta_{2k} f_m(j) \right]. & (37) \end{aligned}$$

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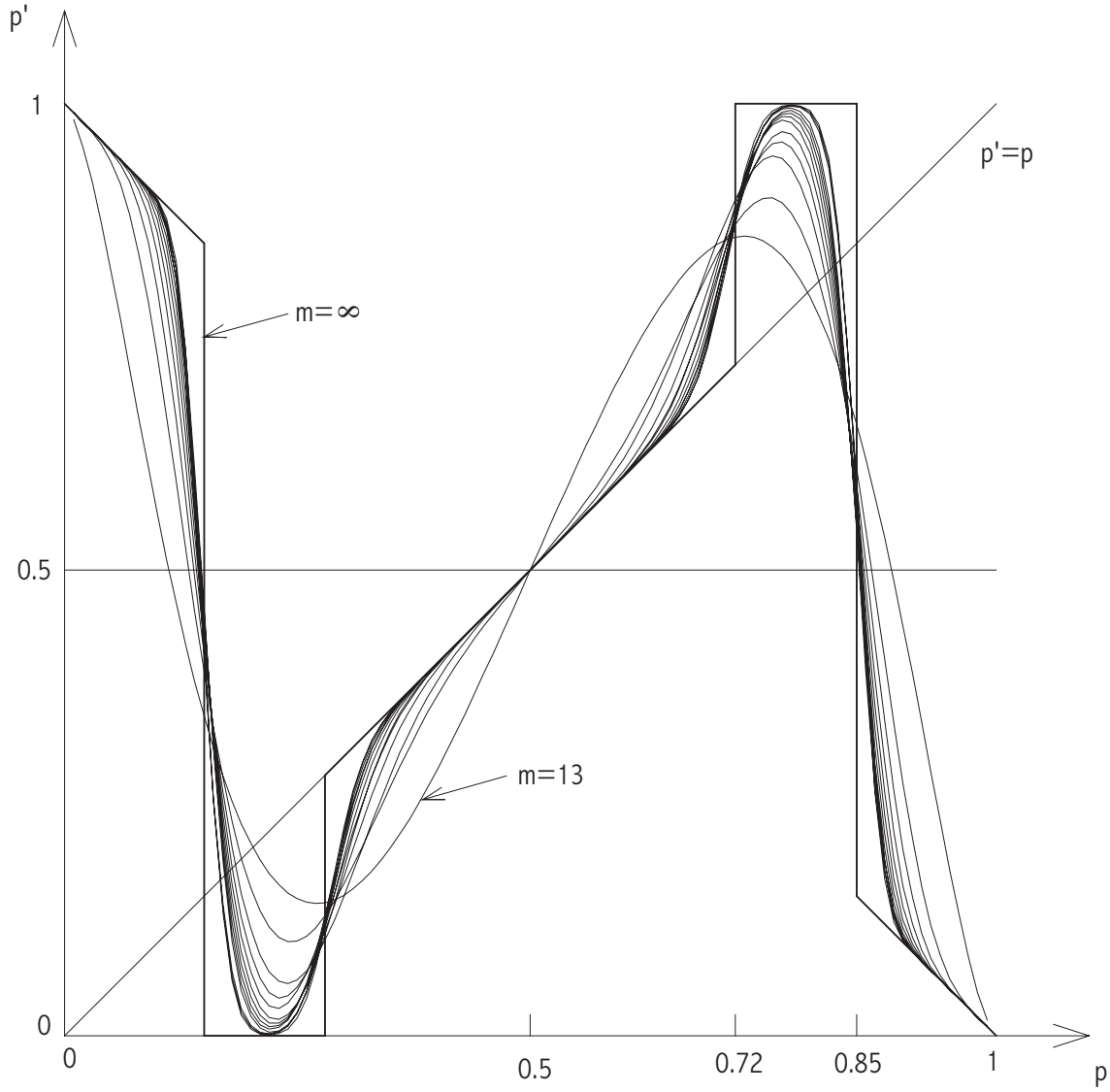

Figure 1: The family of functions $F_{\underline{\rho},m}(p)$ for $\rho_{hb} = \rho_{bh} = 0.72$ and $\rho_{hh} = \rho_{bb} = 0.85$. The curves are for $m = 13 + j \cdot 26, j = 0, \ldots, 13$. Note the convergence to the function $G_{\underline{\rho}}$, (indicated by $m = \infty$).

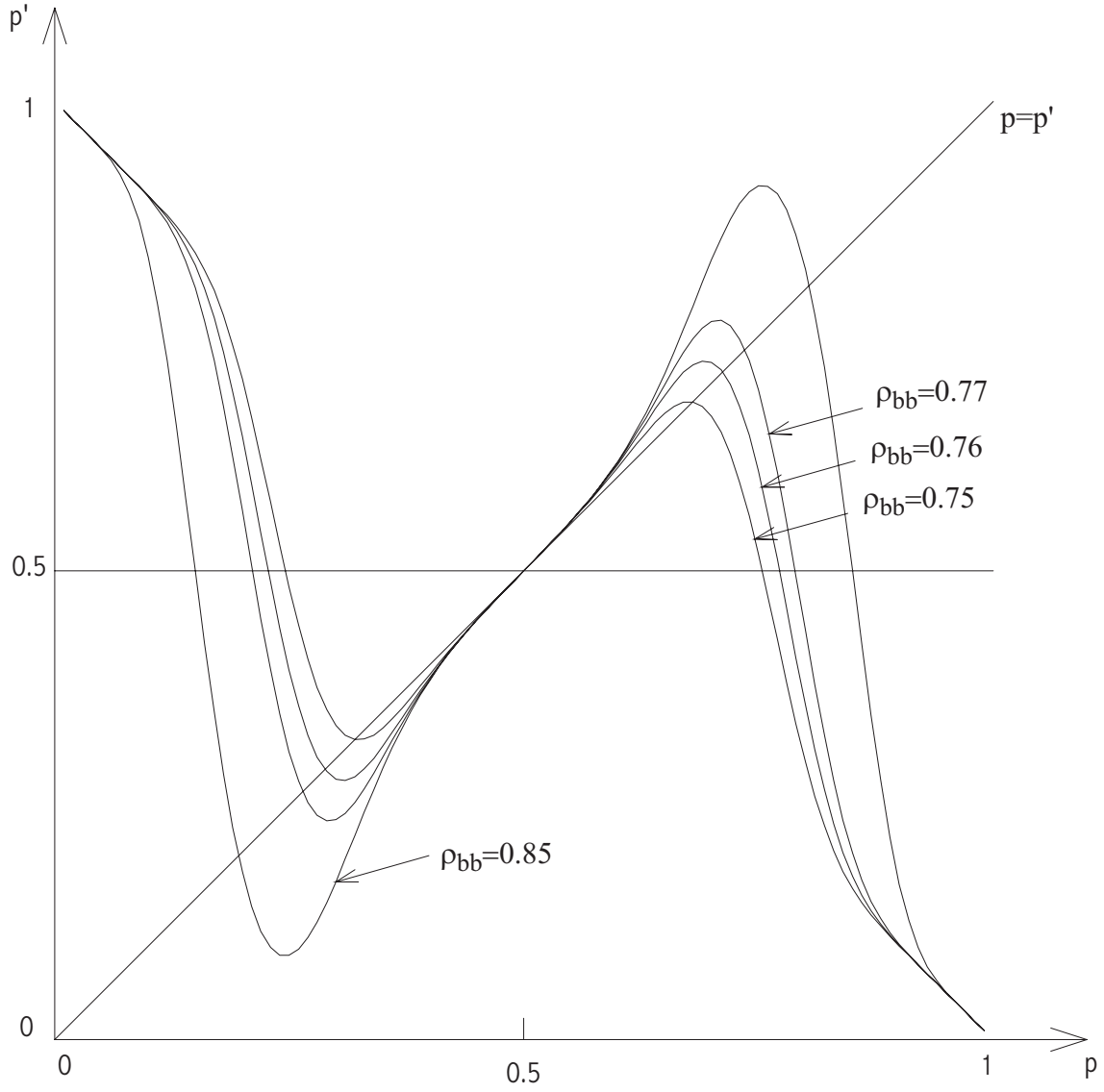

Figure 2: Four curves $F_{\underline{\rho},m}$, for $m = 60$ and $\rho_{hb} = \rho_{bh} = 0.72$, with $\rho_{hh} = \rho_{bb} = 0.75, 0.76, 0.77, 0.85$.

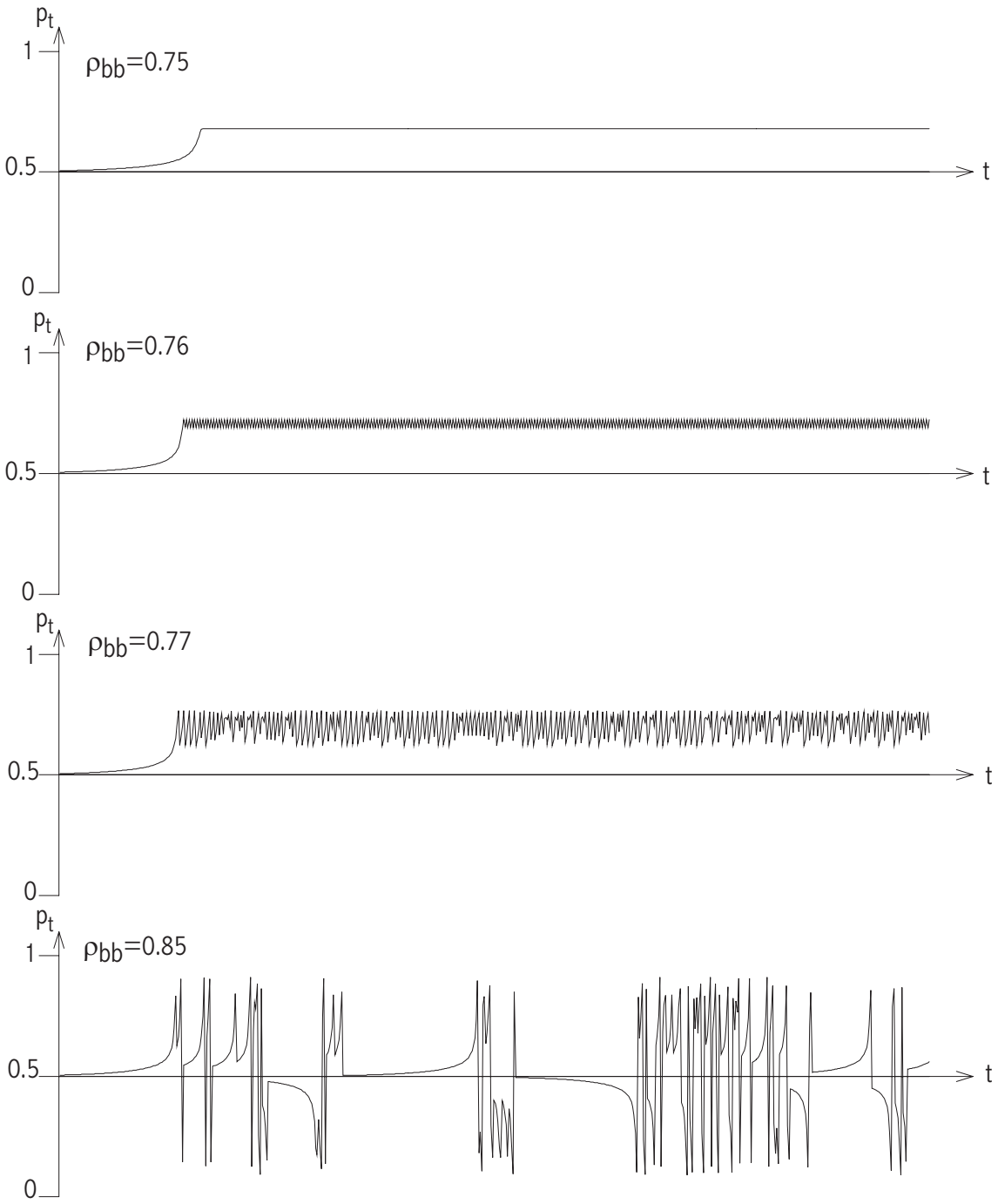

Figure 3: The time series for the same parameter values as in Fig. 2. Note that, for $\rho_{hh} = \rho_{bb} = 0.75$, one has convergence to a bullish equilibrium, for 0.76 a bullish period 2, for 0.77 a bullish, but chaotic behavior. The most interesting case is $\rho_{hh} = \rho_{bb} = 0.85$, where calm periods alternate in a seemingly random fashion with speculative bubbles.

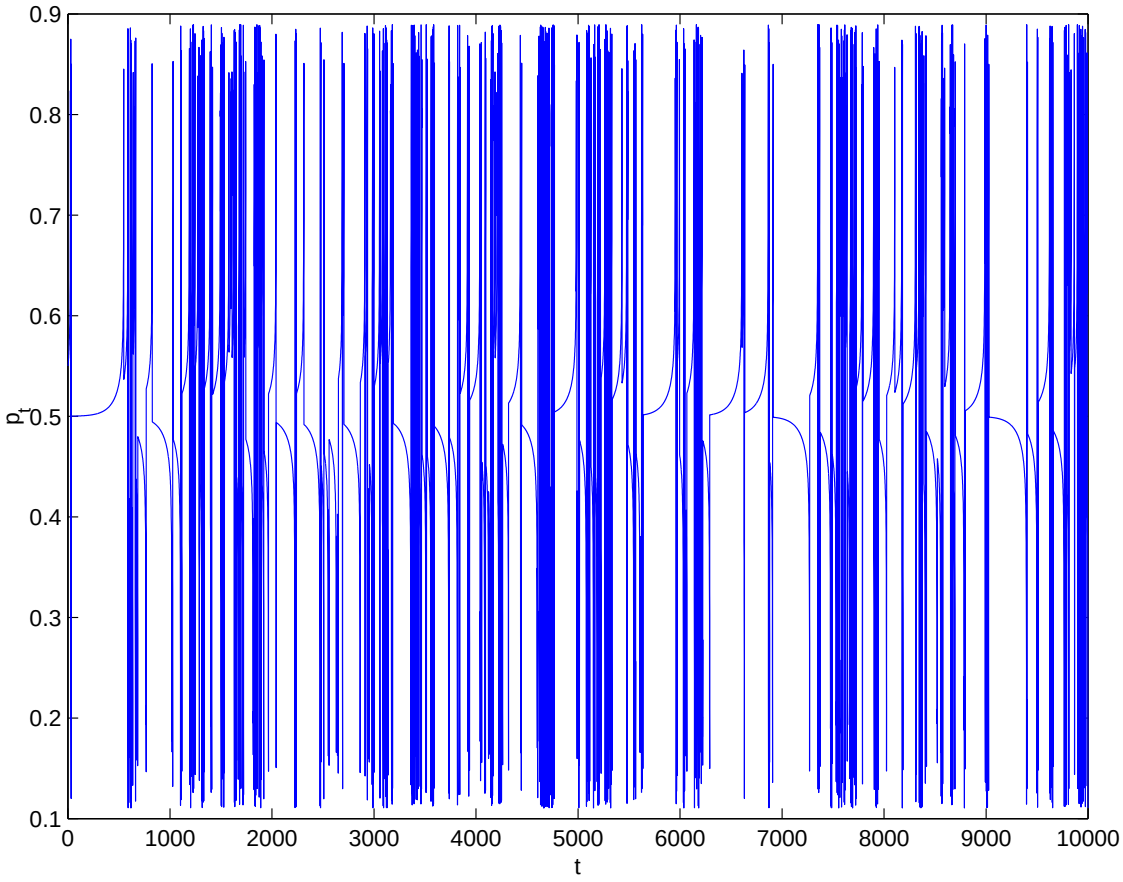

Figure 4: Evolution of the system over 10000 time steps for $N = \infty$, $m = 60$ polled agents and the parameters $\rho_{hb} = \rho_{bh} = 0.72$ and $\rho_{hh} = \rho_{bb} = 0.85$.

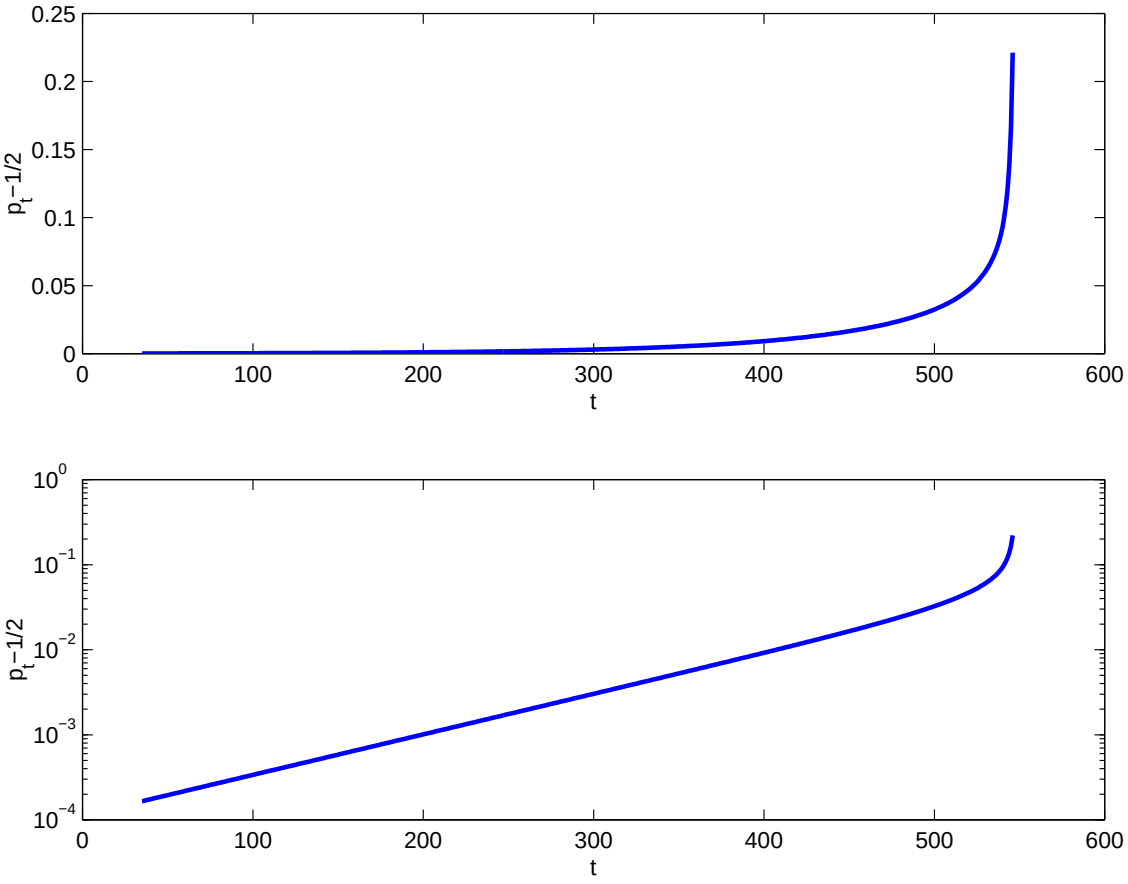

Figure 5: The first bubble of figure 4 for $N = \infty$ agents with $m = 60$ polled agents and parameters $\rho_{hb} = \rho_{bh} = 0.72$ and $\rho_{hh} = \rho_{bb} = 0.85$.

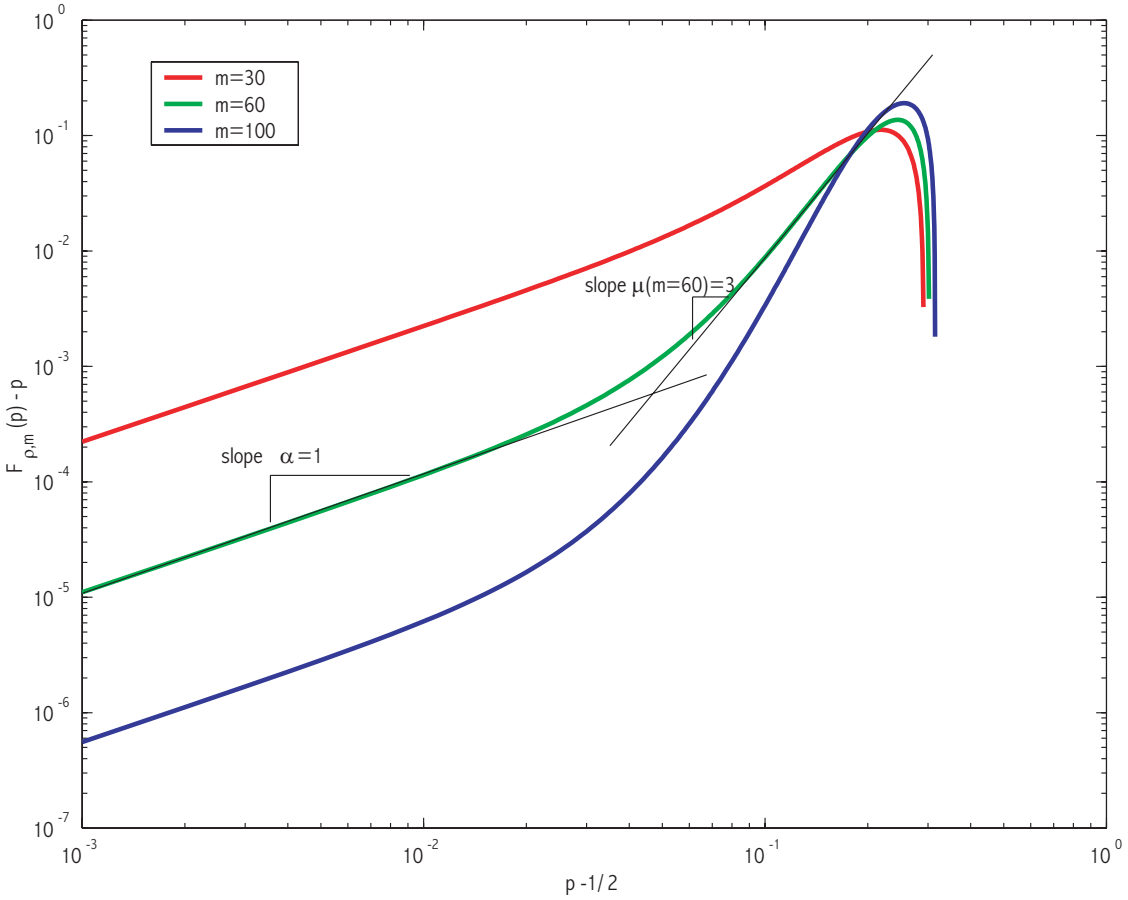

Figure 6: The logarithm of $F_m(p) - p$ versus the logarithm of $p - 1/2$ for three different values of $m = 30, 60$ and 100, with $\rho_{hb} = \rho_{bh} = 0.72$ and $\rho_{hh} = \rho_{bb} = 0.85$. Note the transition from a slope 1 to a large effective slope before the reinjection due to the contrarian mechanism.

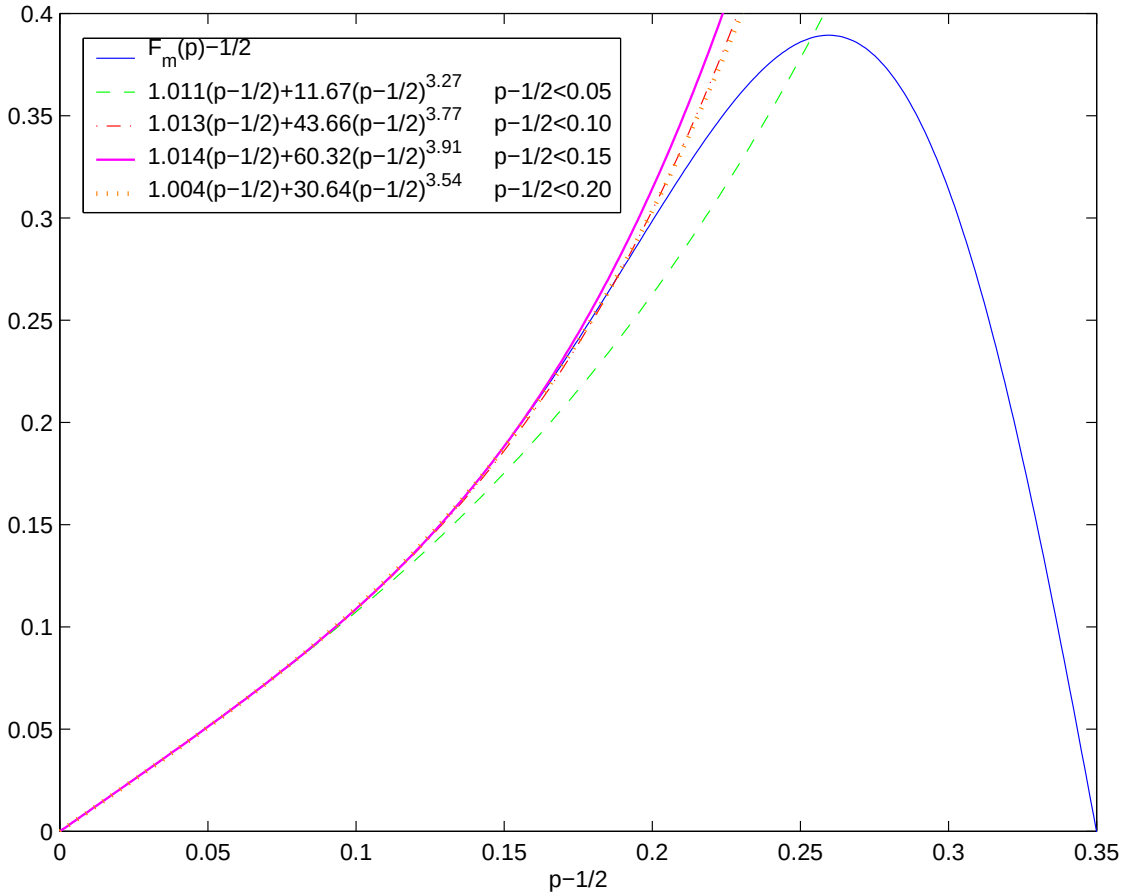

Figure 7: Approximation of the function $F_m(p) - \frac{1}{2}$ by the function $f(p) = [1+\alpha]\left(p+\frac{1}{2}\right) + \beta\left(p+\frac{1}{2}\right)^{\mu}$ over different p-intervals, for $m = 60$ interacting agents and parameters $\rho_{hb} = \rho_{bh} = 0.72$ and $\rho_{hh} = \rho_{bb} = 0.85$.

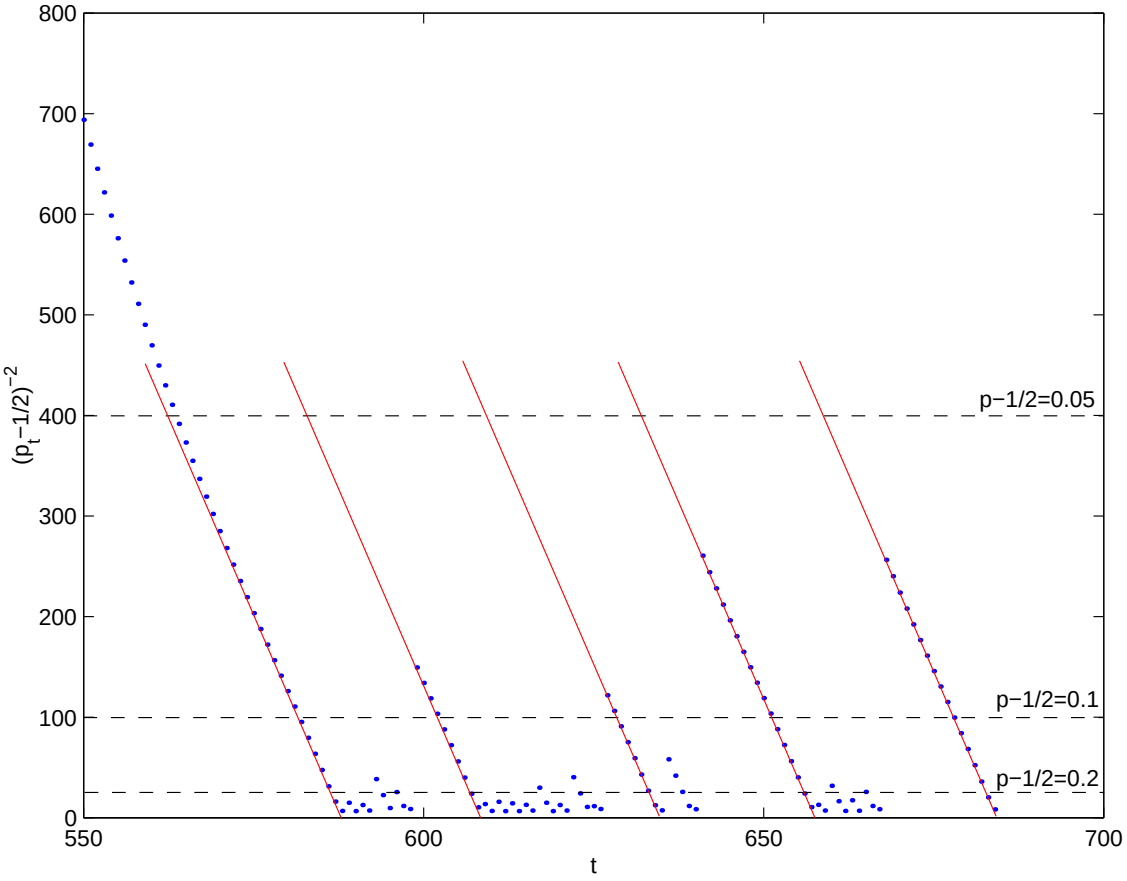

Figure 8: $\frac{1}{(p_t-1/2)^2}$ versus t to qualify the finite time singularity predicted by (23) for $m = 60$ with $\rho_{hb} = \rho_{bh} = 0.72$ and $\rho_{hh} = \rho_{bb} = 0.85$. The points are obtained from the time series p_t and the straight continuous lines are the best linear fits. The horizontal dashed lines indicate the levels $p - 1/2 = 0.05, 0.01$ and 0.2 to demonstrate that most of the visited values are close to the unstable fixed point, which determines the effective value of the nonlinear exponent $\mu \approx 3$.

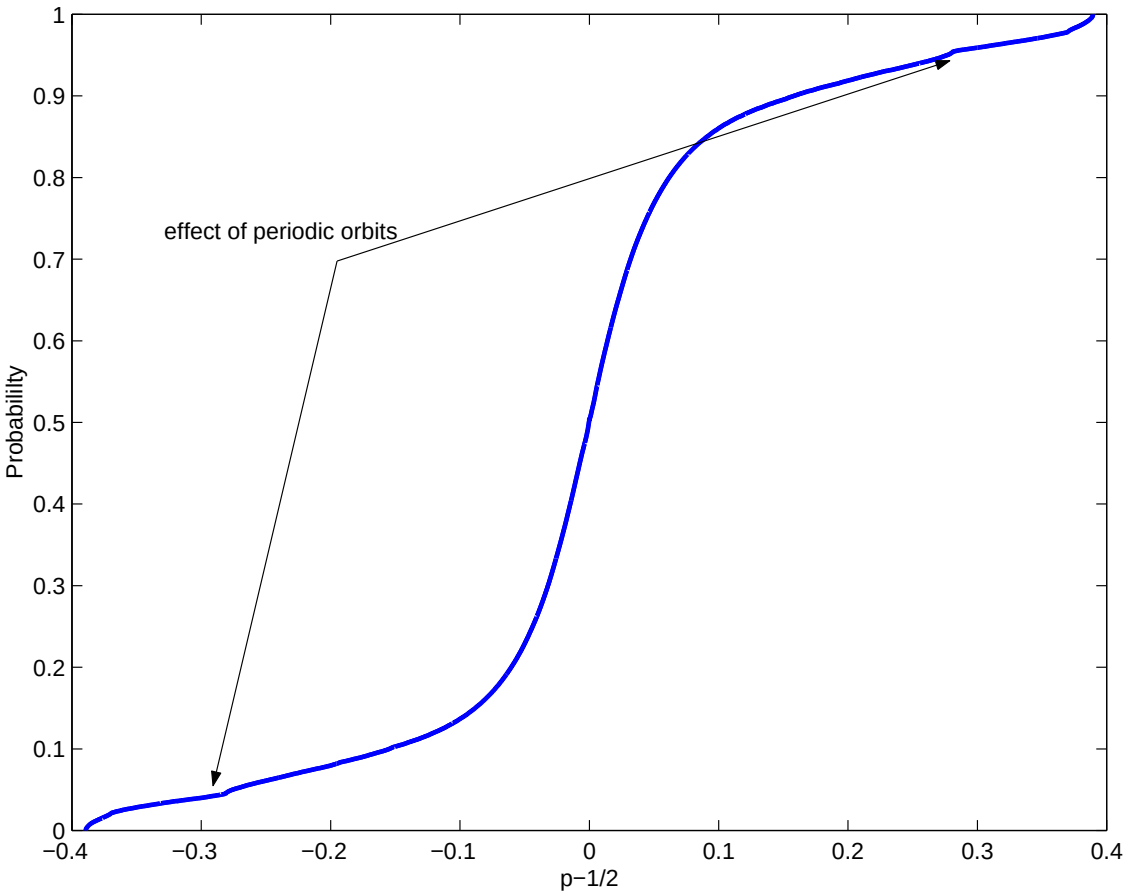

Figure 9: Cumulative distribution for $m = 60$ polled agents and the parameters $\rho_{hb} = \rho_{bh} = 0.72$ and $\rho_{hh} = \rho_{bb} = 0.85$.

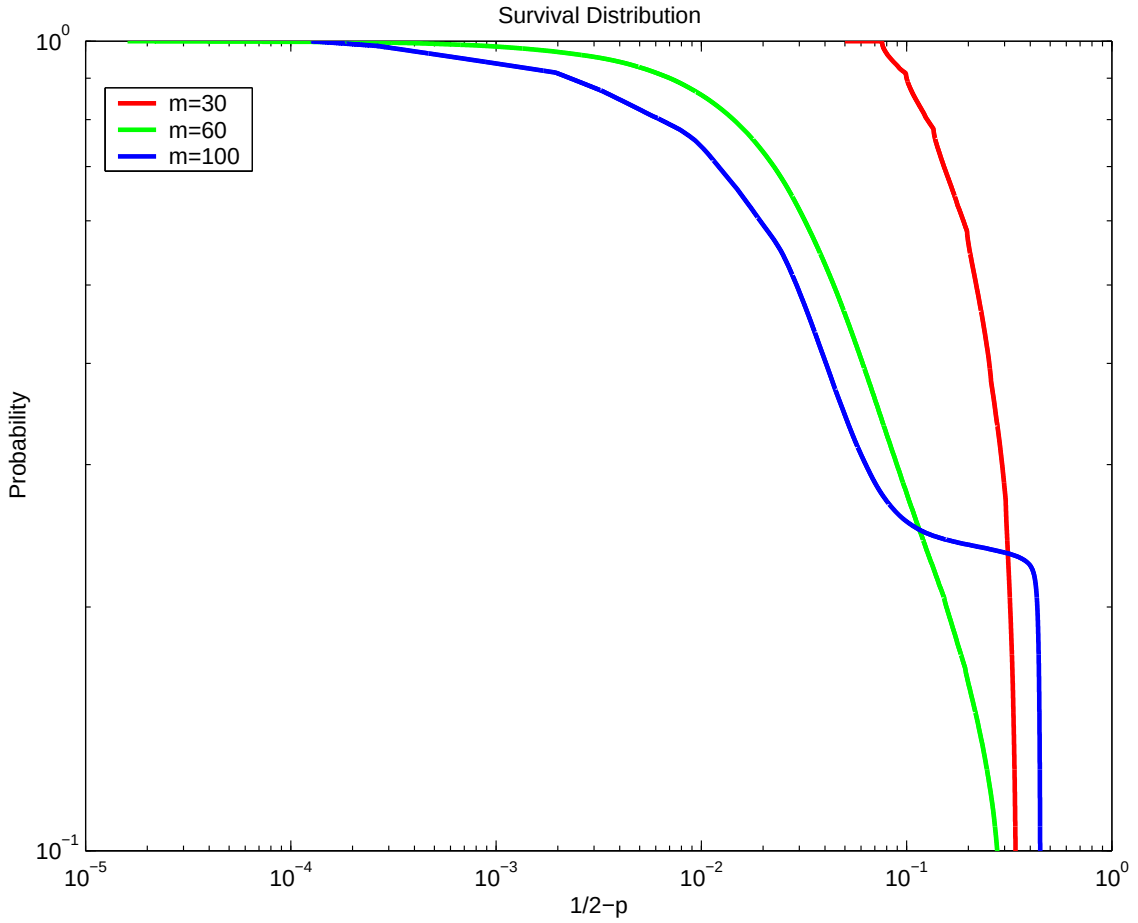

Figure 10: Survival distribution for $m = 30, 60$ and 100 polled agents and parameters $\rho_{hb} = \rho_{bh} = 0.72$ and $\rho_{hh} = \rho_{bb} = 0.85$

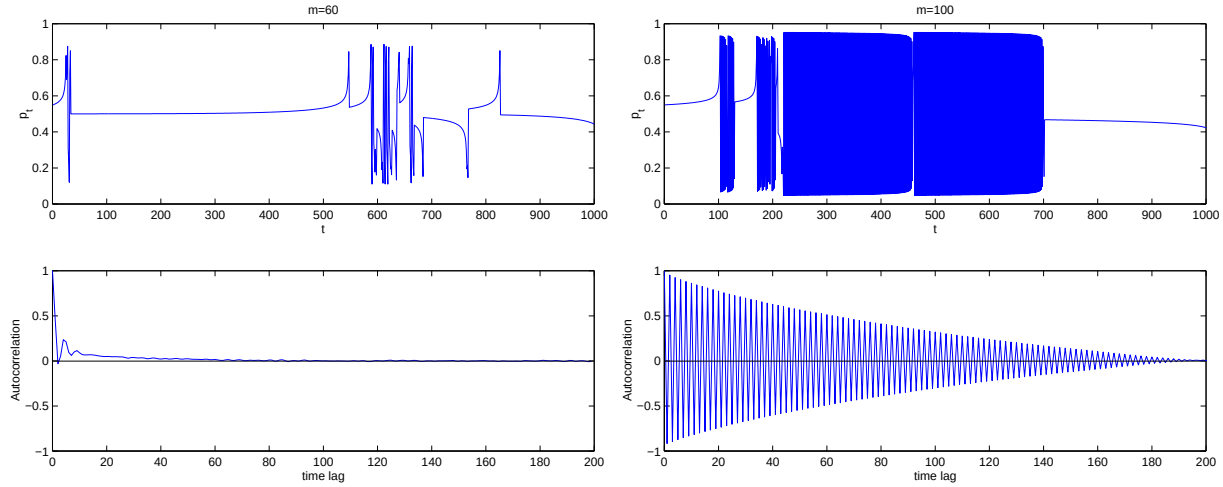

Figure 11: The upper panels represent the time series p_t for $m = 60$ (left) and $m = 100$ (right). The lower panels represents the corresponding autocorrelation function of $r_t \propto p - 1/2$ for $m = 60$ (left) and $m = 100$ (right) with the same parameters $\rho_{hb} = \rho_{bh} = 0.72$ and $\rho_{hh} = \rho_{bb} = 0.85$.

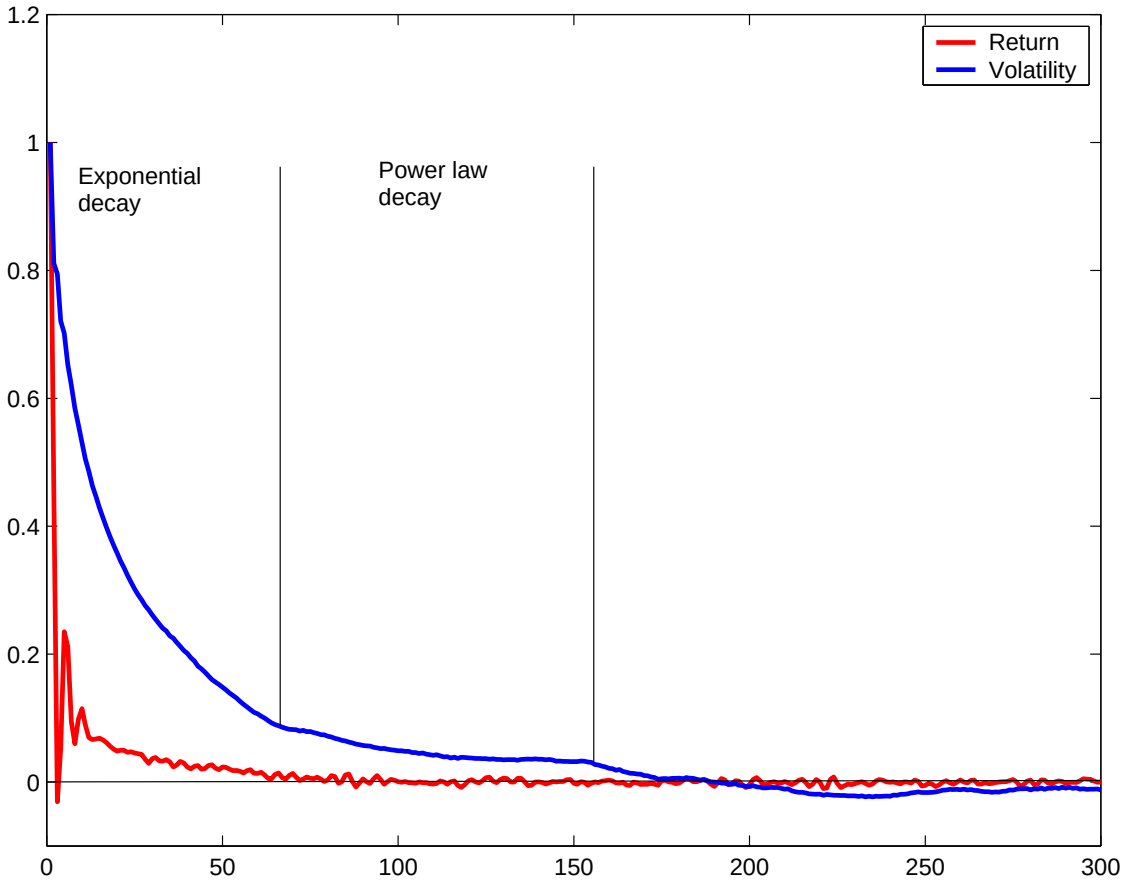

Figure 12: Autocorrelation function of the returns and of the volatility for $m = 60$ polled agents and the parameters $\rho_{hb} = \rho_{bh} = 0.72$ and $\rho_{hh} = \rho_{bb} = 0.85$.

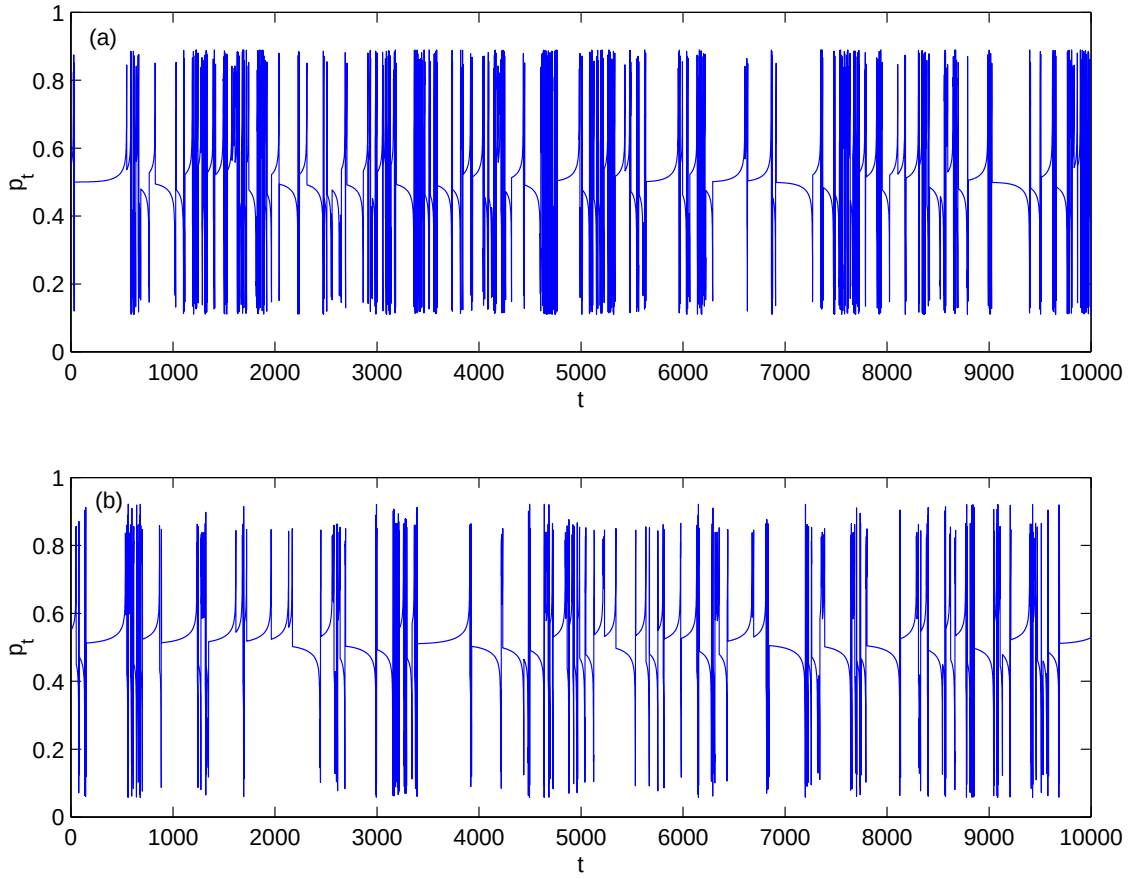

Figure 13: Time evolution of p_t over 10000 time steps for $m = 60$ polled agents in (a) a symmetric case $\rho_{hb} = \rho_{bh} = 0.72$ and $\rho_{hh} = \rho_{bb} = 0.85$ and (b) an asymmetric case $\rho_{hb} = 0.72$, $\rho_{bh} = 0.74$, $\rho_{hh} = 0.85$ and $\rho_{bb} = 0.87$.

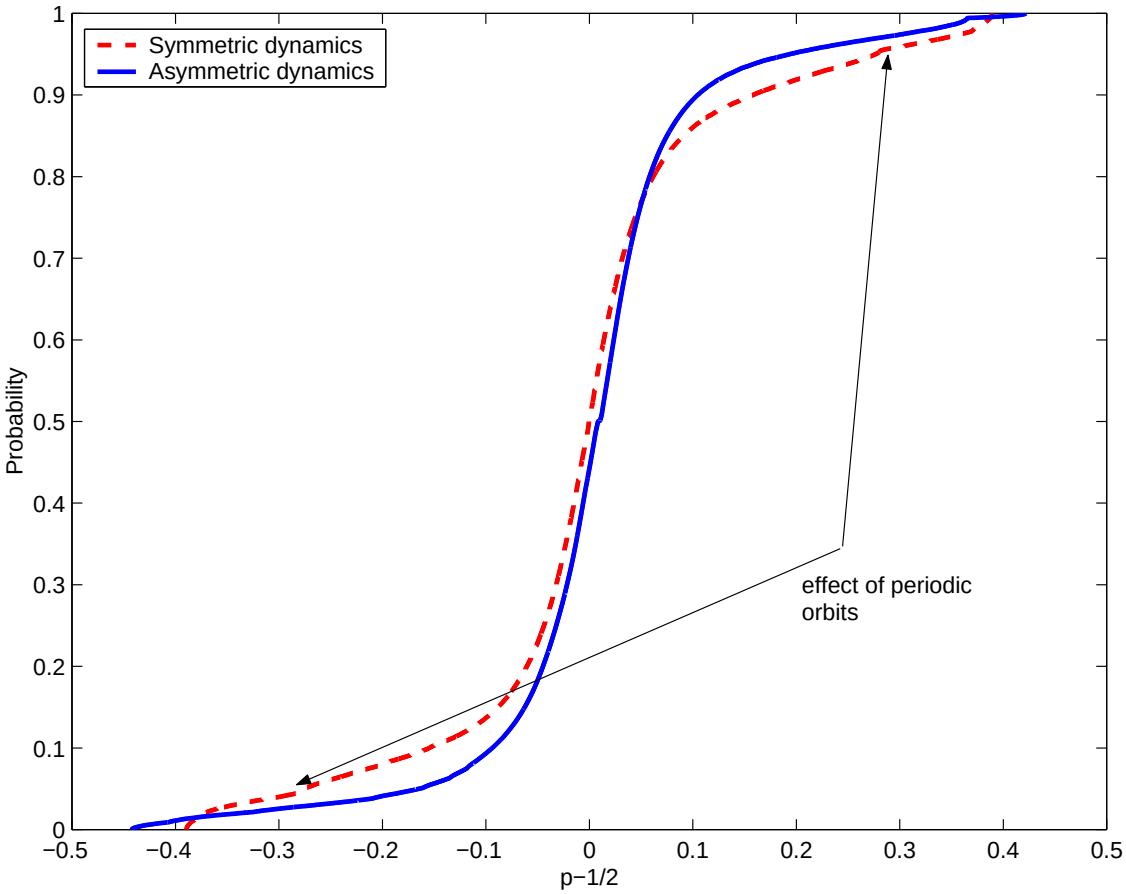

Figure 14: Distribution function of $p - 1/2$ for $m = 60$ polled agents and parameters $\rho_{hb} = \rho_{bh} = 0.72$ and $\rho_{hh} = \rho_{bb} = 0.85$ (dashed line) and $\rho_{hb} = 0.72$, $\rho_{bh} = 0.74$, $\rho_{hh} = 0.85$ and $\rho_{bb} = 0.87$ (continuous line).

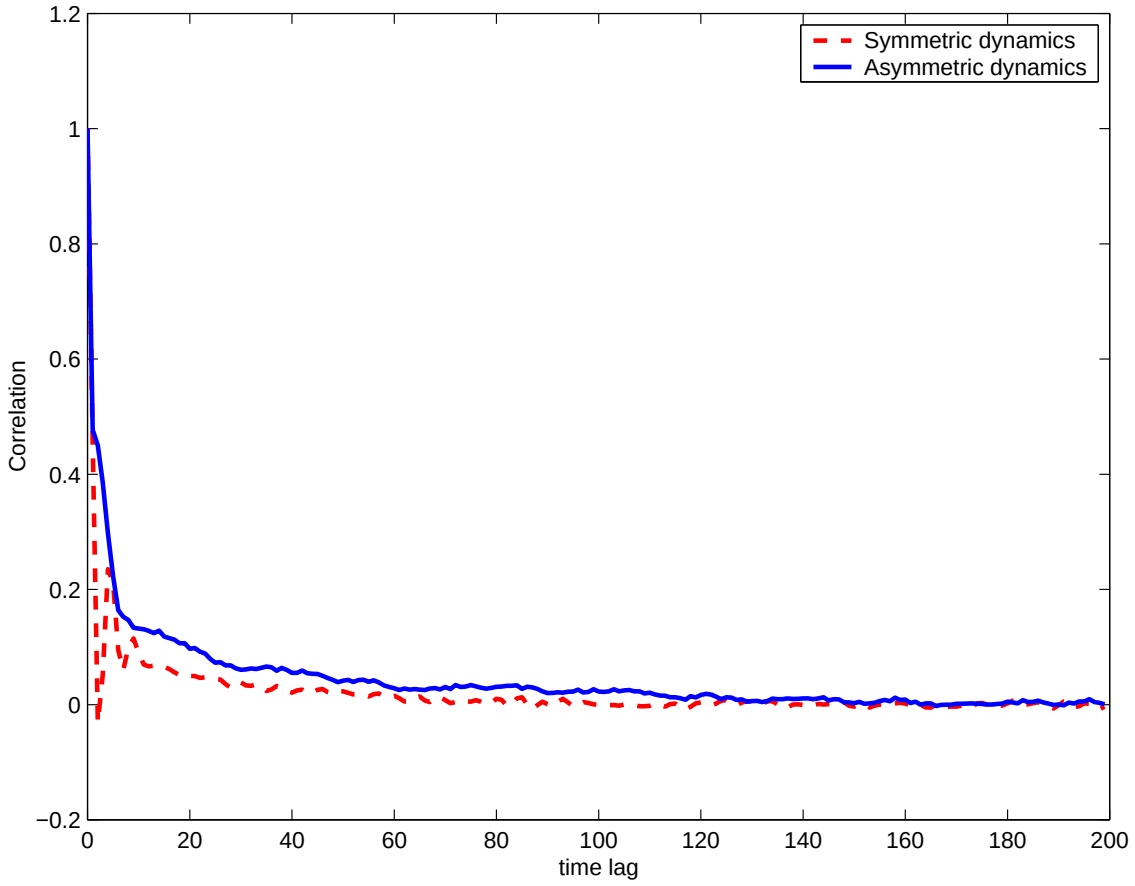

Figure 15: Correlation function for $m = 60$ polled agents and parameters $\rho_{hb} = \rho_{bh} = 0.72$ and $\rho_{hh} = \rho_{bb} = 0.85$ (dashed line) and $\rho_{hb} = 0.72$, $\rho_{bh} = 0.74$, $\rho_{hh} = 0.85$ and $\rho_{bb} = 0.87$ (continuous line).

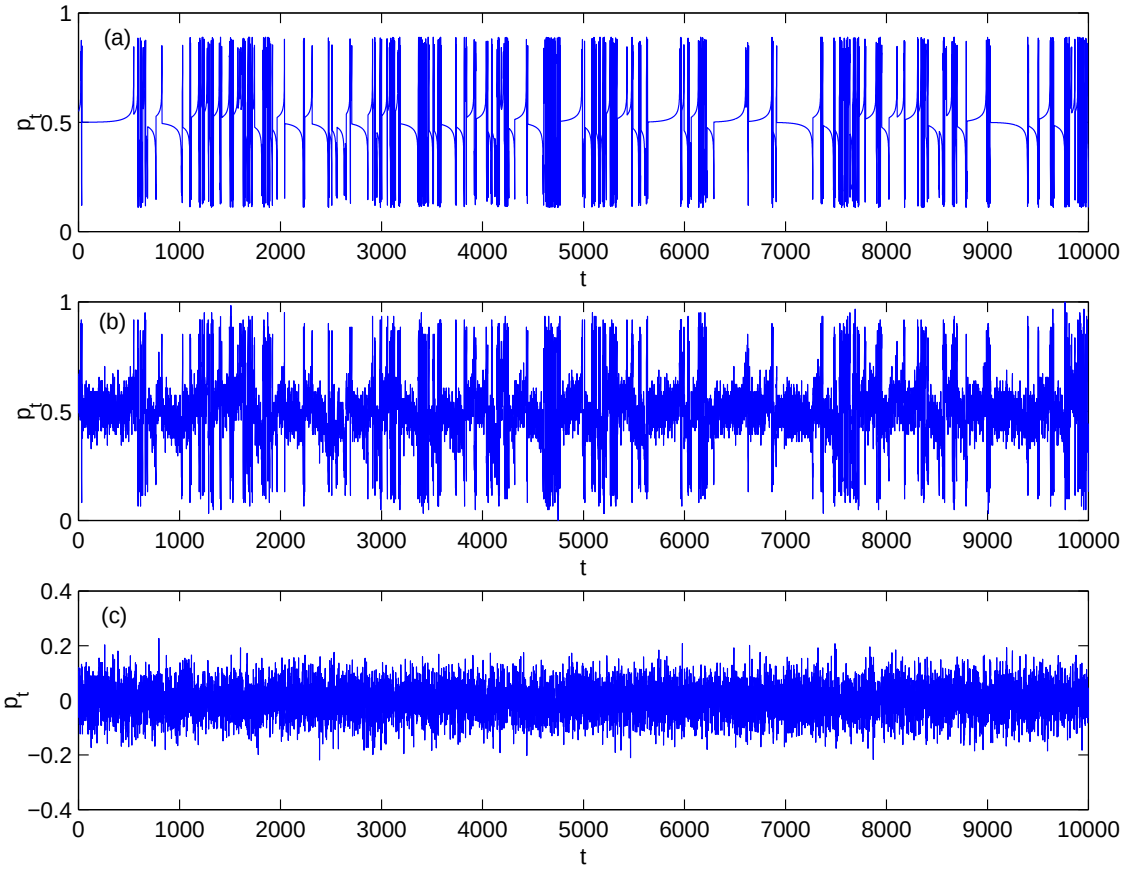

Figure 16: Time evolution of p_t over 10000 time steps for $m = 60$ polled agents with (a) $N = \infty$, (b) $N = m + 1 = 61$ agents and parameters $\rho_{hb} = \rho_{bh} = 0.72$ and $\rho_{hh} = \rho_{bb} = 0.85$. The panel (c) represents the noise due to the finite size of the system and is obtained by subtracting the time series in panel (a) from the time series in panel (b).

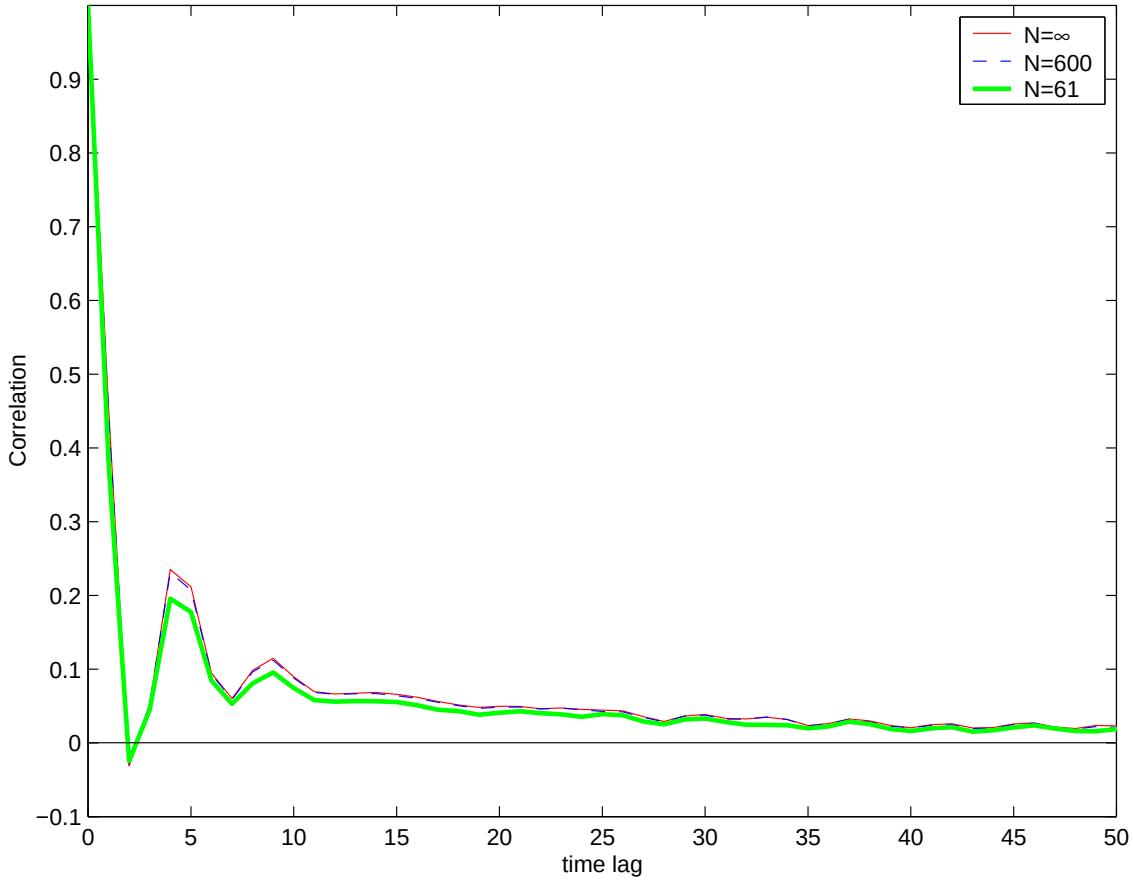

Figure 17: Correlation function for $m = 60$ polled agents with $N = \infty$ (thin line), $N = 600$ (dashed line) and $N = 61$ (continuous line) agents and parameters $\rho_{hb} = \rho_{bh} = 0.72$ and $\rho_{hh} = \rho_{bb} = 0.85$.

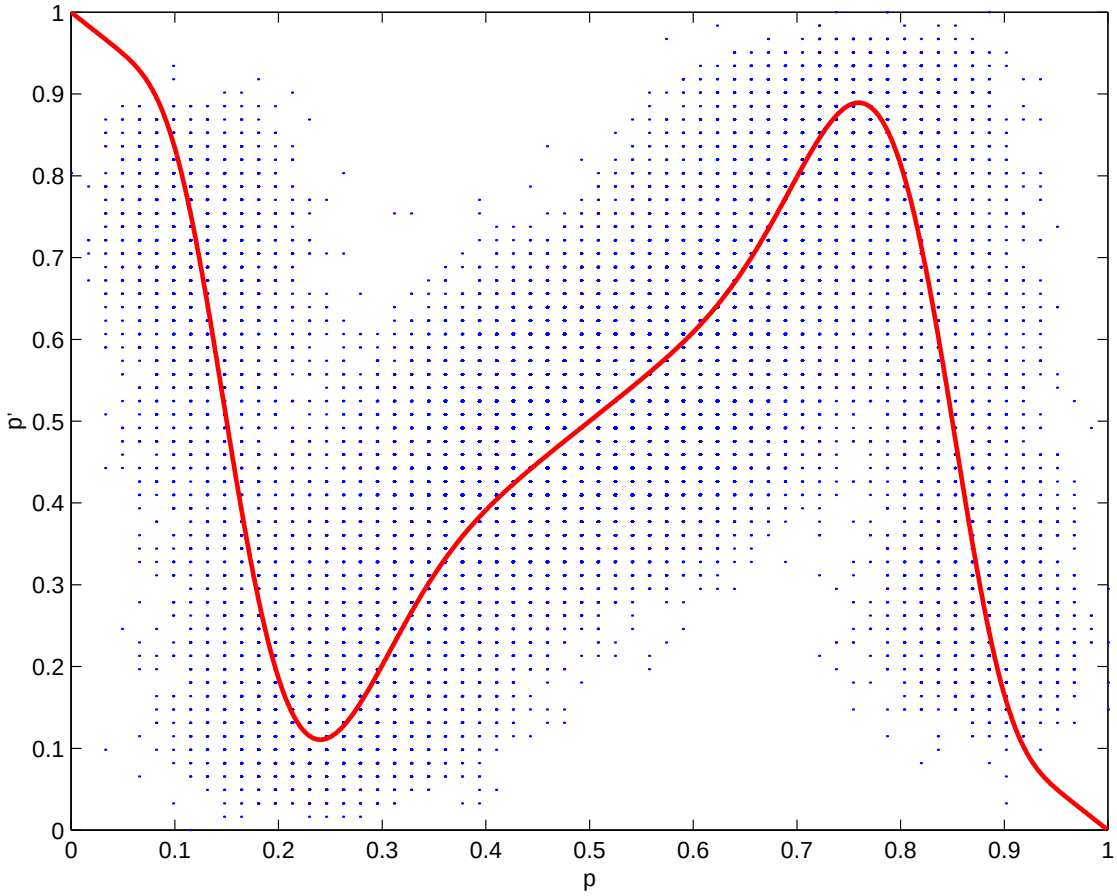

Figure 18: Return map of the fraction of bullish agents for $m = 60$ polled agents among $N = 61$ agents (points) and the deterministic trajectory (continuous line) corresponding to $N = \infty$ agents. The parameters are $\rho_{hb} = \rho_{bh} = 0.72$ and $\rho_{hh} = \rho_{bb} = 0.85$.

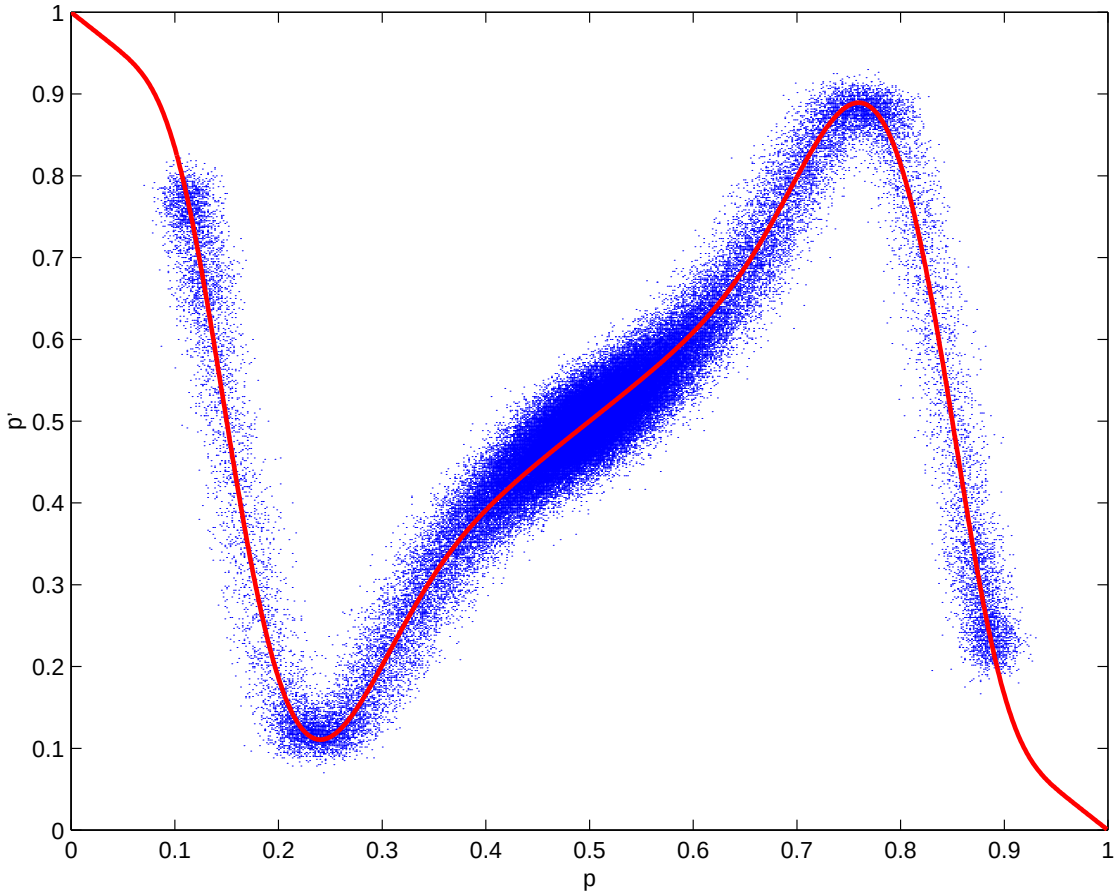

Figure 19: Return map of the fraction of bullish agents for $m = 60$ polled agents among $N = 600$ agents (points) and the deterministic trajectory (continuous line) corresponding to $N = \infty$ agents. The parameters are $\rho_{hb} = \rho_{bh} = 0.72$ and $\rho_{hh} = \rho_{bb} = 0.85$.

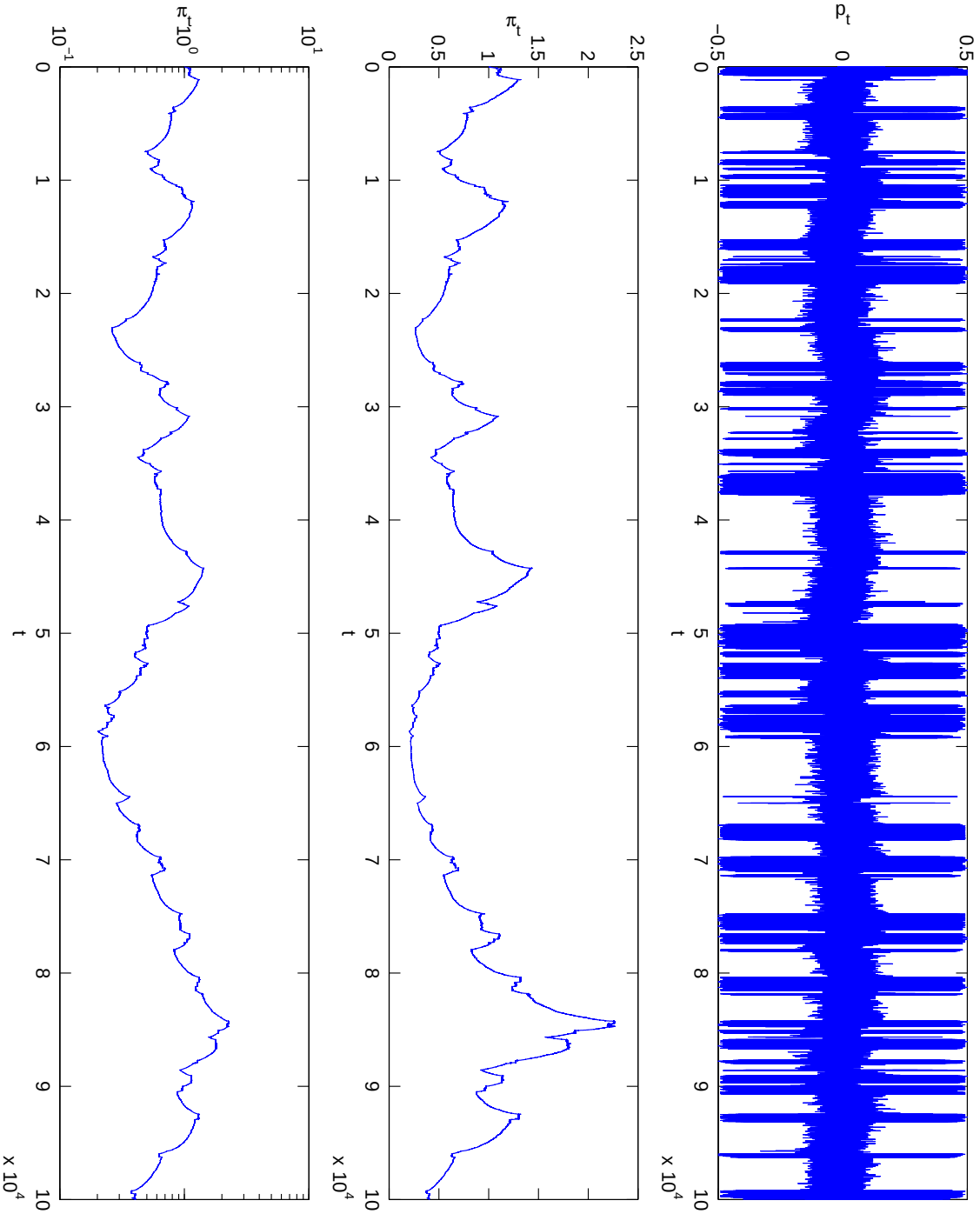

Figure 20: Upper panel: return trajectory $\tilde{r}_t = \gamma \tilde{p}_t - 1/2$ for $m = 100$, $N = 100$, $\rho_{hb} = \rho_{bh} = 0.72$ and $\rho_{hh} = \rho_{bb} = 0.85$ and $\gamma = 0.01$. Middle panel: price trajectory obtained by $\pi_t = \pi_{t-1} \exp[\tilde{r}_t]$. Lower panel: same as the middle panel with π_t shown in logarithmic scale. Note the "flat trough-sharp peak" structure of the log-price trajectory (Roehner Sornette (1998)).

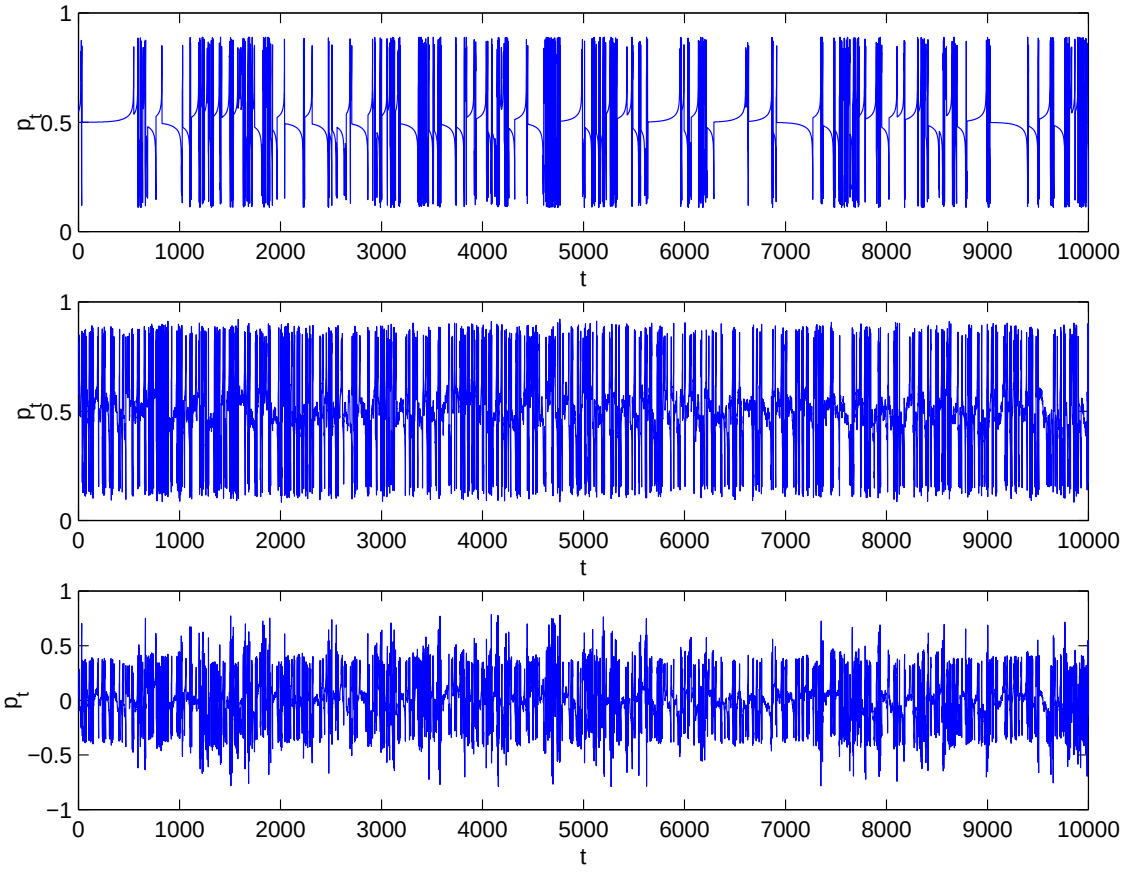

Figure 21: Evolution of the system over 10000 time steps for $m = 60$ polled agents with (upper panel) $N = \infty$, (second panel) $N = m + 1 = 61$ and parameters $\rho_{hb} = \rho_{bh} = 0.72$ and $\rho_{hh} = \rho_{bb} = 0.85$. The lower panel represents the "noise" introduced by the finite size of the system and is obtained by subtracting the upper panel from the second panel.