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Djoko Kamdem, Jonas Koko, Mohamed Mbehou, Toni Sayah

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J.K. Djoko[1], J. Koko[2], M. Mbehou[3], and Toni Sayah[4]

[1]African peer review mechanism, Johannesburg, South Africa, jules.djokokamdem@gmail.com
[2]LIMOS, Université Clermont-Auvergne – CNRS UMR 6158, B.P 10448, 63000 Clermont-Ferrand, France, jonas.koko@uca.fr
[3]Departement de Mathematiques, Faculté des Sciences, Université de Yaounde 1, Yaounde, Cameroun mbehoumoh@gmail.com
[4]Laboratoire de Mathematiques et Applications, Unite de recherche "Mathematiques et Modelisation", Faculté des Sciences, Université Saint-Joseph, B.P 11-514 Riad El Solh, Beirut 1107 2050, Lebanon toni.sayah@usj.edu.lb

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Abstract

In this work, we study theoretically and numerically the equations of Stokes and Navier-Stokes under power law slip boundary condition. We establish existence of a unique solution by using the monotone operators theory for the Stokes equations whereas for the Navier-Stokes equations, we construct the solution by means of Galerkin's approximation combined with some compactness results. Next, we formulate and analyze the finite element approximations associated to these problems. We derive optimal and sub-optimal a priori error estimate for both problems depending how the monotonicity is used. Iterative schemes for solving the non-linear problems are formulated and convergence is studied. Numerical experiments presented confirm the theoretical findings.

Keywords : power law slip boundary condition, Stokes equations, Navier-Stokes equations, finite element method, monotonicity, error estimates, rate of convergence.

1 Introduction

Let $\Omega \subset \mathbb{R}^d$ with $d = 2, 3$ a bounded domain occupied by a fluid with a boundary $\partial\Omega$. We would like to study the incompressible flows governed by the Stokes equations

$$-2\nu \operatorname{div} D\boldsymbol{u} + \nabla p = \boldsymbol{f} \qquad \text{in } \Omega, \tag{1.1}$$

$$\operatorname{div} \boldsymbol{u} = 0 \qquad \text{in } \Omega\ , \tag{1.2}$$

where; ν is the positive viscosity of the fluid velocity, $\boldsymbol{u}(\boldsymbol{x})$, the velocity, the pressure is $p(\boldsymbol{x})$ and $\boldsymbol{f}$ is the external force acting on the fluid, $D\boldsymbol{u}$ is the symmetric part of the velocity gradient defined through $2D\boldsymbol{u} = \nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T$. The equations (1.1) and (1.2) are supplemented by boundary conditions that will be described next. We are interested in (1.1), (1.2) when the position and the direction of the slip boundary condition are taken into account (see [1, 2]). We then assume that

the boundary $\partial\Omega$ of Ω is made of two components S and Γ, such that $\overline{\partial\Omega} = \overline{S \cup \Gamma}$, with $S \cap \Gamma = \emptyset$. We assume the homogeneous Dirichlet condition on S, that is

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } S\ . \tag{1.3}$$

Thus S is the porous or artificial boundary where the fluid is prescribed. On Γ, we assume the impermeability condition

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0 \ \text{ on } \ \Gamma\,, \tag{1.4}$$

where $\boldsymbol{n} : \Gamma \longrightarrow \mathbb{R}^d$ is the normal outward unit vector to Γ. Γ is an impermeable solid surface along which the fluid may slip. The force within the fluid is the Cauchy stress tensor $\boldsymbol{T}$ given through

$$\boldsymbol{T} = 2\nu D\boldsymbol{u} - p\boldsymbol{I} \text{ on } \Omega\,,$$

with $\boldsymbol{I}$ the d-dimensional identity matrix. Taking the scalar product of $\boldsymbol{u}$ and (1.1), one has

$$\int_\Omega \boldsymbol{T} : D\boldsymbol{u}dx + \int_\Gamma (-\boldsymbol{T}\boldsymbol{n})_{\boldsymbol{\tau}} \cdot \boldsymbol{u}_{\boldsymbol{\tau}} d\sigma = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{u}dx \tag{1.5}$$

with $d\sigma$ being the surface measure associated to Γ. We recall that for any vector $\boldsymbol{w}$ define on $\partial\Omega$, we set $\boldsymbol{w}_{\boldsymbol{\tau}} = \boldsymbol{w} - (\boldsymbol{w} \cdot \boldsymbol{n})\boldsymbol{n}$. Thus $(\boldsymbol{T}\boldsymbol{n})_{\boldsymbol{\tau}}$ denotes the projection of the normal stress into the corresponding tangent plane. Now, from a simple interpretation, (1.5) expresses the balance of energy in the system. $\displaystyle\int_\Omega \boldsymbol{T} : D\boldsymbol{u}dx = 2\nu \int_\Omega |D\boldsymbol{u}|^2\, dx$ being the dissipation mechanisms in bulk, while $\displaystyle\int_\Gamma (-\boldsymbol{T}\boldsymbol{n})_{\boldsymbol{\tau}} \cdot \boldsymbol{u}_{\boldsymbol{\tau}} d\sigma$ stands for dissipative processes on the boundary. We need a functional relation between $(\boldsymbol{T}\boldsymbol{n})_{\boldsymbol{\tau}}$ and $\boldsymbol{u}_{\boldsymbol{\tau}}$ that can render the dissipative energy on Γ non negative. For that purpose, the most general relation between $\boldsymbol{u}_{\boldsymbol{\tau}}$ and $(\boldsymbol{T}\boldsymbol{n})_{\boldsymbol{\tau}}$ is the implicit constitutive relation [3]

$$\psi(\boldsymbol{u}_{\boldsymbol{\tau}}, (\boldsymbol{T}\boldsymbol{n})_{\boldsymbol{\tau}}) = 0 \tag{1.6}$$

where ψ is function. The simplest form of (1.6) that ensure the non-negativity of $\displaystyle\int_\Gamma (-\boldsymbol{T}\boldsymbol{n})_{\boldsymbol{\tau}} \cdot \boldsymbol{u}_{\boldsymbol{\tau}} d\sigma$ is the choice

$$(-\boldsymbol{T}\boldsymbol{n})_{\boldsymbol{\tau}} = \alpha \boldsymbol{u}_{\boldsymbol{\tau}} d\sigma \quad , \quad \alpha > 0\ .$$

This is the Navier's slip boundary conditions. If $(\boldsymbol{T}\boldsymbol{n})_{\boldsymbol{\tau}} = \boldsymbol{0}$, then one gets a perfect slip boundary condition, while if $\boldsymbol{u}_{\boldsymbol{\tau}} = \boldsymbol{0}$, then there is no slip. But in this work we are interested in power law slip boundary condition given as follows

$$(\boldsymbol{T}\boldsymbol{n})_{\boldsymbol{\tau}} + |K\boldsymbol{u}_{\boldsymbol{\tau}}|^{s-2} K^2 \boldsymbol{u}_{\boldsymbol{\tau}} = 0 \quad \text{on } \Gamma\,, \tag{1.7}$$

where $|\boldsymbol{v}|^2 = \boldsymbol{v} \cdot \boldsymbol{v}$ is the Euclidean norm. K is an anisotropic tensor, assumed to be uniformly positive definite, symmetric, and bounded. s is a real, strictly positive number representing the flow behaviour index. The tangential shear is a power law function of the tangential velocity. Such a boundary condition arises when the contact surface is lubricated with a thin layer of a non-Newtonian fluid. It is manifest that for $s = 2$ and $K = \boldsymbol{I}$, (1.7) is reduced to the classical Navier's slip condition. In this work we will assume that $s \neq 2$.

We are also interested in the Navier-Stokes equations, where (1.1) is replaced by

$$-2\nu \operatorname{div} \boldsymbol{D}(\boldsymbol{u}) + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla p = \boldsymbol{f} \quad \text{in } \Omega, \tag{1.8}$$

and (1.2)–(1.4), (1.7) are unchanged, and $(\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = \displaystyle\sum_{i=1}^{d} v_i \frac{\partial \boldsymbol{v}}{\partial x_i}$ is the convection term. The solvability of the Stokes and Navier-Stokes equations when (1.3), (1.4) and (1.7) are replaced by

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0 \text{ and } \quad (\boldsymbol{T}\boldsymbol{n})_{\boldsymbol{\tau}} + H(\boldsymbol{u}_{\boldsymbol{\tau}})\boldsymbol{u}_{\boldsymbol{\tau}} = g \quad \text{on } \partial\Omega\,,$$

with g a tangential surface traction and H a bounded, uniformly Lipschitz continuous function are examined by C. Le Roux in [1, 2]. The properties on the function H had helped Le Roux to control the H^1-norm of the velocity with only the slip boundary condition $\boldsymbol{u}\cdot\boldsymbol{n}=0$ on the entire boundary because the Korn's inequality in that context reads (see [4])

$$\int_\Omega |D\boldsymbol{v}|^2 dx + \int_{\partial\Omega} |\boldsymbol{v}_{\boldsymbol{\tau}}|^2 \, d\sigma \geq c \int_\Omega |\nabla \boldsymbol{v}|^2 dx \, .$$

The situation in our hands is different. Indeed, for $s \neq 2$, the function $\boldsymbol{u} \to \int_\Gamma |K\boldsymbol{u}_{\boldsymbol{\tau}}|^{s-2} d\sigma$ does not have the properties enumerated for the function H. Hence the results obtained in [1, 2] are not applicable here. It is therefore clear that the problems described need to be analysed. First, one wishes to find a way to describe/construct the solution(s) of the resulting boundary value problems, and secondly, how can we compute the solutions of the resulting boundary value problems. Our interest to study this type of boundary conditions is mainly motivated by the desire of extending some results known when classical boundary conditions are used. Thus this work also participates in a better mathematical comprehension of fluids with non classical boundary conditions. From the modelling perspective, flows of incompressible Newtonian/Non Newtonian fluids are most of the times analysed with Dirichlet type (or periodic) boundary conditions on the velocity field. But, in some important situations such as oil ducts, water supply, microfluids channels or blood in arteries, some formulations with boundary conditions involving components of the velocity field, stress or pressure are of interest. A recent review of some of the formulations presented in the literature, and their associated boundary conditions, with a focus on applications to air and blood flows can be found in [5]. It is important to observe that most papers dealing with the Stokes or Navier-Stokes equations are written by means of conservation of the momentum in terms of the Laplacian of the velocity, whereas in this text we prefer the equivalent formulation given in terms of the divergence of the symmetric gradient which is useful in the fluid-structure interaction problems. This formulation has many important implications. At the continuous level, the well-posedness of the problems are proved in the same functional spaces as for the standard boundary conditions. Moreover, the discrete analysis allows us to obtain convergence (and even optimal one) with standard inf-sup stable finite element spaces without requiring stabilized formulations. In this work, one also observes that the mathematical and computational models share this important feature: the fluid model is described in terms of mechanical stress tensor, which is more appropriate from the modelling viewpoint, and it is able to properly take into account the fact that the flow continues beyond the boundaries, thanks to the specific form of the boundary conditions.

If one considers that $\partial\Omega$ is not partitioned, with (1.7) and $\boldsymbol{u}\cdot\boldsymbol{n}=0$ on $\partial\Omega$, then one deduces that

$$\int_\Omega |D\boldsymbol{u}|^2 \, dx + \int_{\partial\Omega} |K\boldsymbol{u}_{\boldsymbol{\tau}}|^s \, d\sigma = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{u} dx \, .$$

Hence we will have some difficulties to control the velocity with H^1-norm. To overcome this deficiency, we have assumed that the boundary $\partial\Omega$ is divided into a porous or artificial boundary where the fluid is prescribed, and the other part of the boundary is impermeable which may allow the fluid to slip. Ultimately, we would like to address the problem when the distance between Γ and S is zero, but this would introduce complications that go beyond the scope of this study. The goal of this work is to analyse from the mathematical and numerical point of view both (1.1)–(1.4), (1.7) and (1.2)–(1.4), (1.7) and (1.8). Firstly, for both problems, the existence of solution is constructed without restriction on the data by using monotone theory for the problem related to the Stokes equations, while for the problem associated with the Navier-Stokes equations, we use Galerkin's approximation, Brouwer's fixed point and some compactness results. Secondly,

both variational problems are discretized by finite element schemes in a polygonal or polyhedral domain. We derive existence, uniqueness (conditional for the Navier-Stokes problem), and a priori error estimates for the respective problems. Thirdly, for the implementation of the nonlinear finite element problems, we formulate a solution technique based on the linearization of nonlinear terms. Next, we studied its feasibility, and the convergence of the iterative scheme associated with the problem with the Navier-Stokes equations. Finally, we present some numerical experiments that confirm the theoretical findings. We shall mentioned at at this juncture that, we decide to focus on the implementation of the nonlinear problem related to the Navier-Stokes equations because it has more difficulties/nonlinearities than the problem related to the Stokes equations.

We note that partial differential equations with the nonlinear boundary condition (1.7) (without (1.4)) have been used in the past by [6, 7, 8, 9] when studying bonded elastic materials in solid mechanics. A nonlinear boundary condition similar to power law slip boundary condition (1.7) were studied by [10, 11] for two coupled Stokes fluids with the nonlinear boundary condition representing the transmission condition. The interface condition in the later representing many situations; adhesion, friction, coupling of free-surface and exchange with ground water flows.

The study of flows whose motion is described by the Stokes or Navier-Stokes equations driven by anisotropic slip boundary condition have been the subject of some publications recently (see [1, 2, 12, 13, 14, 15, 16, 17]). But it is important to mention that the numerical analysis (solvability and a priori error estimates) of these models has never been analysed in the literature. Although this work takes into account the vast knowledge on the Stokes/Navier -Stokes analysis, it should be mentioned that the nonlinear slip boundary condition (1.7) adds some difficulties in the treatment of the resulting variational problems. The new challenge in the finite element analysis being the derivation of the uniform a priori error estimate with respect to the parameter in the power law constitutive relation. It should be noted that the power law boundary condition (1.7) we analyse in the work is not the same as the power law fluid analysed in [18, 19, 20, 21]. The later being defined on the flow domain Ω, and the constitutive relation bringing together the Cauchy stress tensor and the symmetric part of the velocity gradient. Finally, we observe that (1.7) is a generalisation of the Navier's slip boundary condition, which is an active research direction (see [22, 23, 24, 25, 26, 27]).

The rest of the paper is organised in the following way. In Section 2, we present the variational problems and study their solvability. Section 3 is devoted to the description and construction of the finite element solution. Section 4 is devoted to the convergence of the finite element solution. Here we establish the convergence of the finite element solution by showing that difference in appropriate norm of the continuous and discrete solution tends to zero with the discretization parameter. In Section 5, we formulate the iterative schemes associated to the finite element approximations. The focus is mainly on the problem associated to the Navier-Stokes equations. We establish some qualitative properties and perform numerical experiments with FreeFem code [37]. We complete this study with some concluding remarks.

2 Analysis of the continuous problems

In this section, we formulate the weak problems associated respectively with (1.1)–(1.4), (1.7), and (1.2)–(1.4), (1.7)–(1.8), and study their well posedness.

2.1 Variational analysis of the Stokes system

We first introduce some notations that will be used throughout this work. Next, we propose and discuss the weak solution of (1.1)–(1.4), and (1.7).

2.1.1 Notations

We adopt the standard definitions of Lebesgue and Sobolev spaces (see [28]). For $1 \leq p < \infty$

$$L^p(\Omega) = \{v : \Omega \to \mathbb{R}; \quad v \text{ is measurable and } \int_\Omega |v(\boldsymbol{x})|^p dx < \infty\},$$

with associated norm

$$\|v\|^p_{L^p(\Omega)} = \int_\Omega |v(\boldsymbol{x})|^p dx$$

for which it is a Banach space, and for the special case $p = 2$, it is a Hilbert space with the norm that will be denoted by $\|\cdot\|$. Finally for $p = \infty$, one has

$$L^\infty(\Omega) = \left\{ v : \Omega \to \mathbb{R}, \quad \left| \begin{array}{r} v \text{ is measurable and there is a constant } c \\ \text{such that } |v(\boldsymbol{x})| \leq c \text{ a.e on } \Omega \end{array} \right. \right\},$$

with the associated norm

$$\|v\|_{L^\infty(\Omega)} = \|v\|_\infty = \inf\{c, \quad |v(\boldsymbol{x})| \leq c \text{ a.e on } \Omega\} \ .$$

Now, for any non-negative integer m and a real number $p \geq 1$, we defined the Sobolev space

$$W^{m,p}(\Omega) = \{v \in L^p(\Omega), \quad \partial^\alpha v(\boldsymbol{x}) \in L^p(\Omega) \ \text{ for all } |\alpha| \leq m\},$$

with $\alpha = (\alpha_1, ..., \alpha_d)$ the multi-index with $|\alpha| = \alpha_1 + \cdots + \alpha_d$, and $\partial^\alpha v(\boldsymbol{x})$ the distributional derivative of v given by

$$\partial^\alpha v = \frac{\partial^{|\alpha|} v}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}} \, .$$

The space $W^{m,p}(\Omega)$ is equipped with the semi-norm and norm

$$|v|^p_{W^{m,p}(\Omega)} = \sum_{|\alpha|=m} \int_\Omega |\partial^\alpha v(\boldsymbol{x})|^p \, dx \ , \quad \|v\|^p_{m,p} = \sum_{0 \leq |\alpha| \leq m} \int_\Omega |\partial^\alpha v(\boldsymbol{x})|^p \, dx \, .$$

The duality between, say, E and its dual E' is denoted as $\langle \ \cdot \ \rangle$. The bold characters denote vector quantities and $\boldsymbol{L}^p(\Omega) = L^p(\Omega)^d, \boldsymbol{W}^{m,p}(\Omega) = W^{m,p}(\Omega)^d$, etc....
We note that $W^{m,2}(\Omega) = H^m(\Omega)$ and $H^0(\Omega)$ coincides with $L^2(\Omega)$, for which the norm and inner product are denoted as $\|\cdot\|$ and $(\cdot, \cdot)$, respectively. In order to introduce functions spaces related to the equations (1.1)–(1.4), (1.7), we recall that (1.5) is re-written as follows

$$2\nu \int_\Omega |D\boldsymbol{u}|^2 dx + \int_\Gamma |K\boldsymbol{u}_{\boldsymbol{\tau}}|^s d\boldsymbol{\sigma} - \langle p, \operatorname{div} \boldsymbol{u} \rangle = \langle \boldsymbol{f}, \boldsymbol{u} \rangle \, . \tag{2.1}$$

Hence, it is manifest from (2.1) that for the velocity one needs $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$, and $\boldsymbol{u}_{\boldsymbol{\tau}} \in \boldsymbol{L}^s(\Gamma)$. Then the pressure must belong to $L^2(\Omega)$ because $\operatorname{div} \boldsymbol{u} \in L^2(\Omega)$. It should be noted that for $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$, $\boldsymbol{v}_{\boldsymbol{\tau}}$ is an element of $\boldsymbol{H}^{1/2}(S) \subset \boldsymbol{L}^2(\Gamma)$ which is not necessarily a subspace of $\boldsymbol{L}^s(\Gamma)$. Hence we have two possibilities.

If $1 \leq s < 2$, we have $\|\boldsymbol{u}_{\boldsymbol{\tau}}\|_{L^1(\Gamma)} \leq c\|\boldsymbol{u}_{\boldsymbol{\tau}}\|_{L^s(\Gamma)} \leq c\|\boldsymbol{u}_{\boldsymbol{\tau}}\|_{L^2(\Gamma)}$. Hence the velocity is described in

$$\mathbb{V} = \{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega) : \quad \text{with } \ \boldsymbol{v}|_S = 0 \text{ and } \boldsymbol{v} \cdot \boldsymbol{n}|_\Gamma = 0\} \ ,$$

equipped with the regular $\boldsymbol{H}^1$-norm, while the pressure is in the space

$$M = \{p \in L^2(\Omega) : \quad (p, 1) = 0\} = L^2_0(\Omega) \, .$$

We mentioned that the zero mean value is added on the pressure to avoid the pressure given in (1.1), (1.2) to be determined up to a constant.
For $2 < s$, we have $\|\boldsymbol{u_\tau}\|_{L^2(\Gamma)} \leq c\|\boldsymbol{u_\tau}\|_{L^s(\Gamma)}$. Then the velocity is described in

$$\mathbb{V} = \{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)\ ,\ \boldsymbol{v_\tau} \in \boldsymbol{L}^s(\Gamma)\ :\ \text{with}\ \ \boldsymbol{v}|_S = 0 \text{ and } \boldsymbol{v}\cdot\boldsymbol{n}|_\Gamma = 0\}$$

equipped with the norm

$$|||\boldsymbol{v}|||^2 = \|\boldsymbol{v}\|_1^2 + \|\boldsymbol{v_\tau}\|^2_{L^s(\Gamma)}\ .$$

The pressure is described in the same function space $L^2_0(\Omega)$, because we only need $\langle p, \operatorname{div}\boldsymbol{u}\rangle$ to make sense with $\operatorname{div}\boldsymbol{u} \in L^2(\Omega)$. We shall often use the following Sobolev imbedding (see [28]): for any real number $p \geq 1$ when $d = 2$, or $1 \leq p \leq \dfrac{2d}{d-2}$ when $d \geq 3$, there exist constants c_p and c_p^0 such that

$$\begin{aligned} &\text{for all } \boldsymbol{v} \in \boldsymbol{H}^1(\Omega),\quad \|\boldsymbol{v}\|_{L^p(\Omega)} \leq c_p\|\boldsymbol{v}\|_1, \\ &\text{for all } \boldsymbol{v} \in \mathbb{V}\ ,\quad \|\boldsymbol{v}\|_{L^p(\Omega)} \leq c_p^0\|\nabla\boldsymbol{v}\|\,. \end{aligned} \tag{2.2}$$

We note that when $p = 2$, the second inequality in (2.2) is the Poincaré's inequality. Also of importance is the Korn's inequality which reads (see [4]); there exists a constant c such that

$$\text{for}\ \ \boldsymbol{u} \in \mathbb{V}\ \ ,\qquad \int_\Omega |D\boldsymbol{u}|^2 dx \geq c\int_\Omega |\nabla\boldsymbol{u}|^2 dx\,. \tag{2.3}$$

2.1.2 Variational formulation

In this part, we formulate the weak problem associated with (1.1)–(1.4), (1.7) and construct the weak solution.
We assume once and for all that for all that $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$.

Let $q \in M$, we multiply (1.2) by q, integrate the resulting equation over Ω. Next, we take the dot product between (1.1) and $\boldsymbol{v} \in \mathbb{V}$ and integrate the resulting equation over Ω. After utilization of the Green's formula, (1.3), (1.4) and (1.7), one obtains:

$$\begin{cases} \text{Find } (\boldsymbol{u}, p) \in \mathbb{V}\times M \text{ such that for all } (\boldsymbol{v}, q) \in \mathbb{V}\times M, \\ a(\boldsymbol{u},\boldsymbol{v}) + \displaystyle\int_\Gamma |K\boldsymbol{u_\tau}|^{s-2}K\boldsymbol{u_\tau}\cdot K\boldsymbol{v_\tau}d\sigma - b(\boldsymbol{v}, p) = \ell(\boldsymbol{v})\,, \\ b(\boldsymbol{u}, q) = 0\,, \end{cases} \tag{2.4}$$

with

$$a(\boldsymbol{u},\boldsymbol{v}) \ \ = \ \ 2\nu\int_\Omega D\boldsymbol{u} : D\boldsymbol{v}dx\ ,\ \ b(\boldsymbol{v}, q) = \int_\Omega q\operatorname{div}\boldsymbol{v}dx\ ,\ \ \ell(\boldsymbol{v}) = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}dx,$$

where

$$\boldsymbol{A} : \boldsymbol{B} = \sum_{1\leq i,j\leq d} A_{ij}B_{ij}\,.$$

In the derivation of (2.4), we have also used the identity $\displaystyle\sum_{1\leq i,j\leq d} D_{ij}\boldsymbol{u}\frac{\partial v_i}{\partial x_j} = \sum_{1\leq i,j\leq d} D_{ij}\boldsymbol{u}D_{ij}\boldsymbol{v}$ which is the consequence of the symmetry of $D\boldsymbol{u}$.

Remark 2.1 *In some contributions, the conservation of the momentum is expressed in terms of the Laplacian of the velocity which gives rise to the bilinear form* $\widetilde{a}(\boldsymbol{u},\boldsymbol{v}) = 2\nu\,(\nabla\boldsymbol{v}, \nabla\boldsymbol{u})$*, instead of* $a(\cdot,\cdot)$ *defined before. Although at a continuous level*

$$\operatorname{div}\boldsymbol{u} = 0 \quad \textit{implies that}\ \ 2\operatorname{div} D\boldsymbol{u} = \Delta\boldsymbol{u},$$

from a modelling viewpoint it may be important to work with symmetric tensor. For instance, the problem (1.1)–(1.4) and (1.7) gives directly the natural boundary condition in term of the force (traction force) exerted by the fluid on its boundary.

Now with standard arguments, we can show the following equivalence result.

Lemma 2.1 *The variational formulation (2.4) is equivalent in the sense of distribution to (1.1)–(1.4), (1.7).*

Hence we have the definition.

Definition 2.1 *A couple* $(\boldsymbol{u}, p)$ *is called weak solution of (1.1)–(1.4), (1.7) if* $(\boldsymbol{u}, p) \in \mathbb{V} \times M$ *and satisfy (2.4).*

One of the key step in the study of mixed variational problem like (2.4) is the definition of the kernel of the bilinear form $b(\cdot, \cdot)$ which permits the elimination of the pressure in the problem. Thus we consider the space

$$\mathbb{V}_{\rm div} = \{\boldsymbol{v} \in \mathbb{V}, \quad \text{for all } q \in M, \quad b(\boldsymbol{v}, q) = 0\} = \{\boldsymbol{v} \in \mathbb{V}, \quad \operatorname{div} \boldsymbol{v} = 0\}\,.$$

With the space $\mathbb{V}_{\rm div}$, the variational problem (2.4) is equivalent to

$$\begin{cases} \text{Find } \boldsymbol{u} \in \mathbb{V}_{\rm div} \text{ such that for all } \boldsymbol{v} \in \mathbb{V}_{\rm div}, \\ a(\boldsymbol{u}, \boldsymbol{v}) + \displaystyle\int_\Gamma |K\boldsymbol{u}_{\boldsymbol{\tau}}|^{s-2} K\boldsymbol{u}_{\boldsymbol{\tau}} \cdot K\boldsymbol{v}_{\boldsymbol{\tau}} d\sigma = \ell(\boldsymbol{v})\,. \end{cases} \tag{2.5}$$

In the study of mixed problem like (2.4), the compatibility or inf-sup condition on $b(\cdot, \cdot)$ is very important. It reads; there exists $\beta > 0$ such that

$$\inf_{q \in M} \sup_{0 \neq \boldsymbol{v} \in \mathbb{V}} \frac{b(\boldsymbol{v}, q)}{\|\boldsymbol{v}\|_{\mathbb{V}}\ \|q\|} \geq \beta\,. \tag{2.6}$$

Actually for $1 \leq s < 2$, one has $\mathbb{V} = \{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega) : \quad \text{with} \quad \boldsymbol{v}|_S = 0 \text{ and } \boldsymbol{v} \cdot \boldsymbol{n}|_\Gamma = 0\}$ and $M = L^2_0(\Omega)$. Eq. (2.6) is obtained by observing that $\boldsymbol{H}^1_0(\Omega) \subset \mathbb{V}$ and the pair $(\boldsymbol{H}^1_0(\Omega), L^2_0(\Omega))$ is inf-sup stable (see [29, 30]), hence (2.6) holds.
For $2 < s$, $\mathbb{V} = \{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)\ ,\ \boldsymbol{v}_{\boldsymbol{\tau}} \in L^s(S) : \text{ with } \boldsymbol{v}|_S = 0 \text{ and } \boldsymbol{v} \cdot \boldsymbol{n}|_\Gamma = 0\}$ and $M = L^2_0(\Omega)$. We were not able to obtain (2.6) which is a key point in obtaining the pressure when studying (2.4).

In the rest of this analysis, we consider $1 < s < 2$. This corresponds to the tangential shear thinning with same physical interpretation as the power law fluid (see [18, 19, 20]).

To study (2.5) or (2.4) it is convenient to recall the following monotonicity and continuity properties (see [19, 20]): there exists a constant c independent of $\boldsymbol{x}, \boldsymbol{y}$ elements of $\mathbb{R}^n$ such that for $1 \leq s < 2$;

$$(|\boldsymbol{y}| + |\boldsymbol{x}|)^{2-s} \left(|\boldsymbol{y}|^{s-2}\boldsymbol{y} - |\boldsymbol{x}|^{s-2}\boldsymbol{x}, \boldsymbol{y} - \boldsymbol{x}\right) \geq c|\boldsymbol{x} - \boldsymbol{y}|^2\ , \tag{2.7}$$

and

$$\left|\, |\boldsymbol{x}|^{s-2}\boldsymbol{x} - |\boldsymbol{y}|^{s-2}\boldsymbol{y}\right| \leq c|\boldsymbol{x} - \boldsymbol{y}|^{s-1}\ . \tag{2.8}$$

Now we introduce the mapping $\boldsymbol{v} \longrightarrow \mathcal{A}(\boldsymbol{v})$ defined as follows

$$\text{for all } \boldsymbol{w} \in \mathbb{V}_{\rm div}\ ,\ \mathcal{A}(\boldsymbol{v})\boldsymbol{w} = a(\boldsymbol{v}, \boldsymbol{w}) + \int_\Gamma |K\boldsymbol{v}_{\boldsymbol{\tau}}|^{s-2} K\boldsymbol{v}_{\boldsymbol{\tau}} \cdot K\boldsymbol{w}_{\boldsymbol{\tau}} d\sigma\,. \tag{2.9}$$

We have the following properties for $\mathcal{A}$.

Lemma 2.2 *Let* $1 < s < 2$*, then*

(a) $\mathcal{A}$ *maps* $\mathbb{V}_{\rm div}$ *into its dual* $\mathbb{V}_{\rm div}^{'}$*, and is bounded on all bounded subsets of* $\mathbb{V}_{\rm div}$ *.*

(b) *For all* $\boldsymbol{v}, \boldsymbol{u}$ *elements of* $\mathbb{V}_{\rm div}$

$$\|\mathcal{A}(\boldsymbol{v}) - \mathcal{A}(\boldsymbol{u})\|_{\mathbb{V}'} \leq 2\nu\|\boldsymbol{v} - \boldsymbol{u}\|_1 + c\|K\|^s_{L^\infty(\Gamma)} \|\boldsymbol{u} - \boldsymbol{v}\|_1^{s-1} \ .$$

Proof. Let $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ elements of $\mathbb{V}_{\rm div}$. From Hölder's and Cauchy-Schwarz's inequalities, we have

$$\begin{aligned}\mathcal{A}(\boldsymbol{v})\boldsymbol{w} \leq & 2\nu\|\boldsymbol{v}\|_1\|\boldsymbol{w}\|_1 + \|K\|^s_{L^\infty(\Gamma)} \int_\Gamma |\boldsymbol{v}_{\boldsymbol{\tau}}|^{s-1} |\boldsymbol{w}_{\boldsymbol{\tau}}| \, d\sigma \\ \leq & 2\nu\|\boldsymbol{v}\|_1\|\boldsymbol{w}\|_1 + c\|K\|^s_{L^\infty(\Gamma)}\|\boldsymbol{v}\|_1^{s-1}\|\boldsymbol{w}\|_1 \, ,\end{aligned}$$

so $\|\mathcal{A}(\boldsymbol{v})\|_{\mathbb{V}'} \leq 2\nu\|\boldsymbol{v}\|_1 + c\|K\|^s_{L^\infty(\Gamma)}\|\boldsymbol{v}\|_1^{s-1}$.

Next, from (2.8), Hölder's and Cauchy-Schwarz's inequalities and since K is bounded, one has

$$\begin{aligned}(\mathcal{A}(\boldsymbol{v}) - \mathcal{A}(\boldsymbol{u}), \boldsymbol{w}) \leq & 2\nu\|\boldsymbol{v} - \boldsymbol{u}\|_1\|\boldsymbol{w}\|_1 + \int_\Gamma \Big| \ |K\boldsymbol{u}_{\boldsymbol{\tau}}|^{s-2} K\boldsymbol{u}_{\boldsymbol{\tau}} - |K\boldsymbol{v}_{\boldsymbol{\tau}}|^{s-2} K\boldsymbol{v}_{\boldsymbol{\tau}} \ \Big| \ |K\boldsymbol{w}_{\boldsymbol{\tau}}| \, d\sigma \\ \leq & 2\nu\|\boldsymbol{v} - \boldsymbol{u}\|_1\|\boldsymbol{w}\|_1 + c\|K\|_{L^\infty(\Gamma)} \int_\Gamma |K\boldsymbol{u}_{\boldsymbol{\tau}} - K\boldsymbol{v}_{\boldsymbol{\tau}}|^{s-1} \ \ |\boldsymbol{w}_{\boldsymbol{\tau}}| \, d\sigma \\ \leq & 2\nu\|\boldsymbol{v} - \boldsymbol{u}\|_1\|\boldsymbol{w}\|_1 + c\|K\|^s_{L^\infty(\Gamma)} \|\boldsymbol{u} - \boldsymbol{v}\|_1^{s-1} \ \ \|\boldsymbol{w}\|_1 \ ,\end{aligned}$$

hence $\|\mathcal{A}(\boldsymbol{v}) - \mathcal{A}(\boldsymbol{u})\|_{\mathbb{V}'} \leq 2\nu\|\boldsymbol{v} - \boldsymbol{u}\|_1 + c\|K\|^s_{L^\infty(\Gamma)} \|\boldsymbol{u} - \boldsymbol{v}\|_1^{s-1}$. □

2.1.3 Existence of solutions

Here we construct a solution of (2.4) by using the theory of monotone operator. We first start by indicating the following result

Proposition 2.1 *The problem (2.5) is equivalent to the problem (2.4) .*

This is a standard result for the Stokes problem and we refer the reader to [29, 31] where similar proofs are given. □

Owing to the equivalence in Proposition 2.1, we can restrict the analysis to problem (2.5). We shall prove that $\mathcal{A}$ is strictly monotone, coercive and hemi-continuous in $\mathbb{V}$. Then the existence and uniqueness of (2.5) will follow by standard arguments, see [31, Chapter 2]. We first start with the monotonicity property of $\mathcal{A}$.

Lemma 2.3 *The mapping* $\boldsymbol{v} \to \mathcal{A}\boldsymbol{v}$ *defined by (2.9) is strictly monotone from* $\mathbb{V}_{\rm div}$ *into* $\mathbb{V}_{\rm div}^{'}$*:*

$$\textit{For all } \boldsymbol{v}, \boldsymbol{w} \in \mathbb{V}_{\rm div}, \quad (\mathcal{A}\boldsymbol{v} - \mathcal{A}\boldsymbol{w})(\boldsymbol{v} - \boldsymbol{w}) > 0 \ .$$

Proof. Consider the functional $J : \mathbb{V}_{\rm div} \to \mathbb{R}$ defined by

$$J(\boldsymbol{v}) = \frac{1}{2} a(\boldsymbol{v}, \boldsymbol{v}) + \frac{1}{s} \int_\Gamma |K\boldsymbol{v}_{\boldsymbol{\tau}}|^s \, d\sigma \, .$$

The first derivative of J is

$$J'(\boldsymbol{u}) \cdot \boldsymbol{v} = a(\boldsymbol{u}, \boldsymbol{v}) + \int_\Gamma |K\boldsymbol{u}_{\boldsymbol{\tau}}|^{s-2} K\boldsymbol{u}_{\boldsymbol{\tau}} \cdot K\boldsymbol{v}_{\boldsymbol{\tau}} d\sigma \, , \tag{2.10}$$

and the second derivative is

$$\begin{aligned} J^{(2)}(\boldsymbol{u})\cdot(\boldsymbol{v},\boldsymbol{w}) =& a(\boldsymbol{v},\boldsymbol{w}) + (s-2)\int_\Gamma |K\boldsymbol{u}_{\boldsymbol{\tau}}|^{s-4}\,(K\boldsymbol{u}_{\boldsymbol{\tau}}\cdot K\boldsymbol{w}_{\boldsymbol{\tau}})\,(K\boldsymbol{u}_{\boldsymbol{\tau}}\cdot K\boldsymbol{v}_{\boldsymbol{\tau}})\,d\sigma \\ &+ \int_\Gamma |K\boldsymbol{u}_{\boldsymbol{\tau}}|^{s-2}\,K\boldsymbol{w}_{\boldsymbol{\tau}}\cdot K\boldsymbol{v}_{\boldsymbol{\tau}} d\sigma\,. \end{aligned} \tag{2.11}$$

Note that $J^{(2)}$ is symmetric and positive definite. Indeed $J^{(2)}(\boldsymbol{u})\cdot(\boldsymbol{v},\boldsymbol{w}) = J^{(2)}(\boldsymbol{u})\cdot(\boldsymbol{w},\boldsymbol{v})$ and

$$\begin{aligned} J^{(2)}(\boldsymbol{u})\cdot(\boldsymbol{v},\boldsymbol{v}) =& a(\boldsymbol{v},\boldsymbol{v}) + (s-2)\int_\Gamma |K\boldsymbol{u}_{\boldsymbol{\tau}}|^{s-4}\,(K\boldsymbol{u}_{\boldsymbol{\tau}}\cdot K\boldsymbol{v}_{\boldsymbol{\tau}})\,(K\boldsymbol{u}_{\boldsymbol{\tau}}\cdot K\boldsymbol{v}_{\boldsymbol{\tau}})\,d\sigma \\ &+ \int_\Gamma |K\boldsymbol{u}_{\boldsymbol{\tau}}|^{s-2}\,K\boldsymbol{v}_{\boldsymbol{\tau}}\cdot K\boldsymbol{v}_{\boldsymbol{\tau}} d\sigma\,. \end{aligned}$$

From the assumption on K, one deduces that K^2 is symmetric and uniformly positive definite. Let $\alpha_{\min} > 0$ be the smallest eigenvalue of K^2 over Γ, that is

$$\forall \boldsymbol{x}\in\Gamma,\quad \forall \boldsymbol{w}\in\mathbb{R}^d,\qquad K^2(\boldsymbol{x})\boldsymbol{w}\cdot\boldsymbol{w} = |K(\boldsymbol{x})\boldsymbol{w}|^2 \geq \alpha_{\min}|\boldsymbol{w}|^2\ .$$

Then

$$\forall \boldsymbol{x}\in\Gamma,\quad \forall \boldsymbol{w}\in\mathbb{R}^d,\qquad (K^2(\boldsymbol{x})\boldsymbol{w}\cdot\boldsymbol{w})^{s/2} = |K(\boldsymbol{x})\boldsymbol{w}|^s \geq \alpha_{\min}^{s/2}|\boldsymbol{w}|^s\quad . \tag{2.12}$$

Since $1 \leq s < 2$, one has

$$J^{(2)}(\boldsymbol{u})\cdot(\boldsymbol{v},\boldsymbol{v}) \geq a(\boldsymbol{v},\boldsymbol{v}) - \int_\Gamma |K\boldsymbol{u}_{\boldsymbol{\tau}}|^{s-2}\,|K\boldsymbol{v}_{\boldsymbol{\tau}}|^2\,d\sigma + \int_\Gamma |K\boldsymbol{u}_{\boldsymbol{\tau}}|^{s-2}\,|K\boldsymbol{v}_{\boldsymbol{\tau}}|^2\,d\sigma = a(\boldsymbol{v},\boldsymbol{v}) \geq 0\,.$$

One notes from (2.10) that

$$\begin{aligned} (\mathcal{A}(\boldsymbol{v}) - \mathcal{A}(\boldsymbol{w}))(\boldsymbol{v}-\boldsymbol{w}) =& J'(\boldsymbol{v})\cdot(\boldsymbol{v}-\boldsymbol{w}) - J'(\boldsymbol{w})\cdot(\boldsymbol{v}-\boldsymbol{w}) \\ =& \int_0^1 J^{(2)}(\boldsymbol{w}+\theta(\boldsymbol{v}-\boldsymbol{w}))\cdot(\boldsymbol{v}-\boldsymbol{w},\boldsymbol{v}-\boldsymbol{w})d\theta \geq 0 \end{aligned}$$

because the second derivative is positive definite. We then conclude that $\mathcal{A}$ is monotone (in fact strictly monotone). □

Secondly, we have the following hemi-continuity result for $\mathcal{A}$.

Lemma 2.4 *$\mathcal{A}$ is hemi-continuous in $\mathbb{V}_{\rm div}$, that is for $\boldsymbol{u},\boldsymbol{v}$ in $\mathbb{V}_{\rm div}$, the mapping $t \longrightarrow \mathcal{A}(\boldsymbol{u}+t\boldsymbol{v})\boldsymbol{v}$ is continuous from $\mathbb{R}$ into $\mathbb{R}$.*

Proof. Indeed for any t_1, t_2 in $\mathbb{R}$, we can write from (2.10) that

$$\begin{aligned} (\mathcal{A}(\boldsymbol{u}+t_1\boldsymbol{v}) - \mathcal{A}(\boldsymbol{u}+t_2\boldsymbol{v}))\cdot\boldsymbol{v} =& (J'(\boldsymbol{u}+t_1\boldsymbol{v}) - J'(\boldsymbol{u}+t_2\boldsymbol{v}))\cdot\boldsymbol{v} \\ =& (t_1-t_2)\int_0^1 J^{(2)}(\boldsymbol{u}+t_2\boldsymbol{v}+\theta(t_1-t_2)\boldsymbol{v})\cdot(\boldsymbol{v},\boldsymbol{v})d\theta\,. \end{aligned}$$

Since $\boldsymbol{u}$ and $\boldsymbol{v}$ are fixed, it follows from (2.11) that the right hand side term on the second equality tends to zero with $t_1 - t_2$. □

Finally we show that $\mathcal{A}$ is coercive in the following way

Lemma 2.5 *$\mathcal{A}$ is coercive, that is* $\displaystyle\lim_{\|\boldsymbol{v}\|_1\to\infty}\left(\frac{1}{\|\boldsymbol{v}\|_1}\mathcal{A}(\boldsymbol{v})\boldsymbol{v}\right) = \infty$

Proof.

From (2.12), (2.3) and (2.2) one has

$$\frac{1}{\|\boldsymbol{v}\|_1}\mathcal{A}(\boldsymbol{v})\boldsymbol{v} \geq \min(2\nu, \alpha_{\min}^{s/2})\frac{\|D\boldsymbol{v}\|^2 + \|\boldsymbol{v}_{\boldsymbol{\tau}}\|_{L^s(\Gamma)}^s}{\|\boldsymbol{v}\|_1} \geq \min(2\nu, \alpha_{\min}^{s/2})\frac{\|\nabla\boldsymbol{v}\|^2 + \|\boldsymbol{v}_{\boldsymbol{\tau}}\|_{L^s(\Gamma)}^s}{\|\nabla\boldsymbol{v}\|}\,.$$

From which we deduce that

$$\lim_{\|\nabla\boldsymbol{v}\|\to\infty}\frac{\mathcal{A}(\boldsymbol{v})\boldsymbol{v}}{\|\nabla\boldsymbol{v}\|} = \infty\,.$$

□

From Lemma 2.3, Lemma 2.5, and Lemma 2.4, we deduce the following existence result.

Theorem 2.1 *If $1 < s < 2$, then the variational problem (2.4) has a unique solution $(\boldsymbol{u}, p) \in \mathbb{V} \times M$, which moreover satisfy the bounds*

$$\begin{aligned}
&\nu c\|\nabla\boldsymbol{u}\|^2 + \int_S |K\boldsymbol{u}_{\boldsymbol{\tau}}|^s\, d\sigma \leq c\|\boldsymbol{f}\|^2\,,\\
&\|p\| \leq 2\nu\|\nabla\boldsymbol{u}\| + \|K\|_{L^\infty(\Gamma)}^s\,\|\boldsymbol{u}_{\boldsymbol{\tau}}\|_{L^s(\Gamma)}^{s-1} + c\|\boldsymbol{f}\|\,.
\end{aligned}$$

Remark 2.2 *The contribution of the nonstandard boundary condition (1.7) to the a priori estimate in Theorem 2.1 is manifest by the presence of the boundary term $K\boldsymbol{u}_{\boldsymbol{\tau}}$.*

2.2 Variational analysis of the Navier-Stokes system

This part is devoted to the analysis of the problem related to (1.2)–(1.4), (1.7) and (1.8). The weak solution is constructed using a combination of approaches such as; Faedo-Galerkin's approximation, Brouwer's fixed point and compactness properties. The reader will find more details for example in [31, Chap 2].

2.2.1 Existence and uniqueness of solution

Turning to the Navier-Stokes system, the weak formulation associated with (1.2), (1.3), (1.4), (1.7) and (1.8) reads

$$\begin{cases}
\text{Find } (\boldsymbol{u}, p) \in \mathbb{V} \times M \text{ such that for all } (\boldsymbol{v}, q) \in \mathbb{V} \times M,\\
a(\boldsymbol{u}, \boldsymbol{v}) + \displaystyle\int_\Gamma |K\boldsymbol{u}_{\boldsymbol{\tau}}|^{s-2}K\boldsymbol{u}_{\boldsymbol{\tau}} \cdot K\boldsymbol{v}_{\boldsymbol{\tau}} d\sigma + d(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) - b(\boldsymbol{v}, p) = \ell(\boldsymbol{v}),\\
b(\boldsymbol{u}, q) = 0\,,
\end{cases} \tag{2.13}$$

with $d(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = \displaystyle\int_\Omega (\boldsymbol{u}\cdot\nabla)\boldsymbol{v}\cdot\boldsymbol{w}dx$. We recall that

$$\begin{aligned}
&d(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = -d(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{v}) \text{ for all } \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \mathbb{V}_{\text{div}} \times \boldsymbol{H}^1(\Omega) \times \boldsymbol{H}^1(\Omega)\\
&d(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{v}) = 0 \text{ for all } \boldsymbol{u}, \boldsymbol{v} \in \mathbb{V}_{\text{div}} \times \boldsymbol{H}^1(\Omega)\,.
\end{aligned} \tag{2.14}$$

We discuss next, the existence and uniqueness of solution of (2.13). We state the following result.

Proposition 2.2 *For any data $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$, the variational problem (2.13) has at least one solution $(\boldsymbol{u}, p) \in \mathbb{V} \times M$ if $1 < s < 2$, moreover there is a constant c such that*

$$\begin{aligned}
&\nu c\|\boldsymbol{u}\|_1^2 + \int_\Gamma |K\boldsymbol{u}_{\boldsymbol{\tau}}|^s d\sigma \leq \frac{c}{\nu}\|\boldsymbol{f}\|^2\,,\\
&\|p\| \leq \frac{1}{\beta}\left(2\nu\|\boldsymbol{u}\|_1 + \|K\|_{L^\infty(\Gamma)}^s\|\boldsymbol{u}_{\boldsymbol{\tau}}\|_{L^{s-1}(\Gamma)}^{s-1} + c\|\boldsymbol{u}\|_1^2 + \|\boldsymbol{f}\|\right)\,.
\end{aligned}$$

If moreover

$$\nu^2 > c\|\boldsymbol{f}\|\,, \tag{2.15}$$

then the solution is unique.

Proof. It is performed in several steps.
We recall (2.13) is equivalent to

$$\begin{cases} \text{Find } \boldsymbol{u} \in \mathbb{V}_{\rm div} \text{ such that for all } \boldsymbol{v} \in \mathbb{V}_{\rm div}, \\ a(\boldsymbol{u},\boldsymbol{v}) + \displaystyle\int_\Gamma |K\boldsymbol{u}_{\boldsymbol{\tau}}|^{s-2} K\boldsymbol{u}_{\boldsymbol{\tau}} \cdot K\boldsymbol{v}_{\boldsymbol{\tau}} d\sigma + d(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}) = \ell(\boldsymbol{v})\,. \end{cases} \tag{2.16}$$

We first construct the solution $\boldsymbol{u} \in \mathbb{V}_{\rm div}$ solution of (2.16).

Step 1: Existence of approximate solutions.
Since $\mathbb{V}_{\rm div}$ is separable, then there exist an increasing sequence a countable basis $\{\boldsymbol{\psi}_i\}_{i\geq 1}$. Let $\mathbb{K}_m$ be the space spanned by the first m basis functions, $\{\boldsymbol{\psi}_i\}_{1\leq i\leq m}$. The problem (2.16) is discretized in $\mathbb{K}_m$ by the square system of nonlinear equations:

$$\begin{aligned} &\text{Find } \boldsymbol{u}_m \in \mathbb{K}_m \text{ such that for all } \boldsymbol{v} \in \mathbb{K}_m, \\ &a(\boldsymbol{u}_m,\boldsymbol{v}) + \int_\Gamma |K\boldsymbol{u}_{\boldsymbol{\tau},m}|^{s-2} K\boldsymbol{u}_{\boldsymbol{\tau},m} \cdot K\boldsymbol{v}_{\boldsymbol{\tau}} d\sigma + d(\boldsymbol{u}_m,\boldsymbol{u}_m,\boldsymbol{v}) = \ell(\boldsymbol{v})\,. \end{aligned} \tag{2.17}$$

to prove the existence of $\boldsymbol{u}_m$, we use fixed point by considering the mapping $F : \mathbb{K}_m \longrightarrow \mathbb{K}_m$ defined as follows

$$F(\boldsymbol{u})\boldsymbol{v} = a(\boldsymbol{u},\boldsymbol{v}) + \int_\Gamma |K\boldsymbol{u}_{\boldsymbol{\tau}}|^{s-2} K\boldsymbol{u}_{\boldsymbol{\tau}} \cdot K\boldsymbol{v}_{\boldsymbol{\tau}} d\sigma + d(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}) - \ell(\boldsymbol{v})\,.$$

We first show that F is continuous. Let $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}$ in $\mathbb{K}_m$, we have

$$\begin{aligned} |(F(\boldsymbol{u}) - F(\boldsymbol{v}))\boldsymbol{w}| \leq & 2\nu\|\boldsymbol{u}-\boldsymbol{v}\|_1\|\boldsymbol{w}\|_1 + |d(\boldsymbol{u}-\boldsymbol{v},\boldsymbol{u},\boldsymbol{w})| + |d(\boldsymbol{v},\boldsymbol{u}-\boldsymbol{v},\boldsymbol{w})| \\ & + \int_\Gamma \left||K\boldsymbol{u}_{\boldsymbol{\tau}}|^{s-2}K\boldsymbol{u}_{\boldsymbol{\tau}} - |K\boldsymbol{v}_{\boldsymbol{\tau}}|^{s-2}K\boldsymbol{v}_{\boldsymbol{\tau}}\right| |K\boldsymbol{w}_{\boldsymbol{\tau}}|\, d\sigma \\ \leq & 2\nu\|\boldsymbol{u}-\boldsymbol{v}\|_1\|\boldsymbol{w}\|_1 + \|\boldsymbol{u}-\boldsymbol{v}\|_{L^4(\Omega)}\|\nabla\boldsymbol{u}\|\|\boldsymbol{w}\|_{L^4(\Omega)} + \|\boldsymbol{v}\|_{L^4(\Omega)}\|\nabla(\boldsymbol{u}-\boldsymbol{v})\|\|\boldsymbol{w}\|_{L^4(\Omega)} \\ & + \int_\Gamma \left||K\boldsymbol{u}_{\boldsymbol{\tau}}|^{s-2}K\boldsymbol{u}_{\boldsymbol{\tau}} - |K\boldsymbol{v}_{\boldsymbol{\tau}}|^{s-2}K\boldsymbol{v}_{\boldsymbol{\tau}}\right| |K\boldsymbol{w}_{\boldsymbol{\tau}}|\, d\sigma\,, \end{aligned}$$

together with the inequalities (2.8), implies that F is continuous. On the other hand, for $\boldsymbol{v} \in \mathbb{K}_m$, one has

$$\begin{aligned} F(\boldsymbol{v})\boldsymbol{v} = & a(\boldsymbol{v},\boldsymbol{v}) + \int_\Gamma |K\boldsymbol{v}_{\boldsymbol{\tau}}|^s d\sigma - \ell(\boldsymbol{v}) \\ \geq & 2\nu c\|\boldsymbol{v}\|_1^2 + \int_\Gamma |K\boldsymbol{v}_{\boldsymbol{\tau}}|^s\, d\sigma - \|\boldsymbol{f}\|\|\boldsymbol{v}\|_1 \\ \geq & \left(2\nu c\|\boldsymbol{v}\|_1 - \|\boldsymbol{f}\|\right)\|\boldsymbol{v}\|_1 + \int_\Gamma |K\boldsymbol{v}_{\boldsymbol{\tau}}|^s\, d\sigma\,. \end{aligned}$$

Choosing r such that $r > \frac{c}{\nu}\|\boldsymbol{f}\|$, it follows that for any $\boldsymbol{v} \in \mathbb{K}_m$, with $\|\boldsymbol{v}\|_1 = r$, we have $F(\boldsymbol{v})\boldsymbol{v} > 0$. We deduce from a corollary of the Brouwer's fixed point (see [29]), that there exists $\boldsymbol{u}_m \in \mathbb{K}_m$ solution of (2.17).
Step 2: Estimates and passage to the limit.
We take $\boldsymbol{v} = \boldsymbol{u}_m$ in (2.17), using standard inequalities, this gives

$$\nu c\|\boldsymbol{u}_m\|_1^2 + \int_\Gamma |K\boldsymbol{u}_{\boldsymbol{\tau},m}|^s d\sigma \leq \frac{c}{\nu}\|\boldsymbol{f}\|^2\,. \tag{2.18}$$

Hence there exists a subsequence, still denoted by $\boldsymbol{u}_m$ such that

$$\begin{aligned} &\boldsymbol{u}_m \to \boldsymbol{u} \ \text{ weakly in } \ \boldsymbol{H}^1(\Omega), \\ &\mathcal{B}\boldsymbol{u}_m \to \mathcal{W} \quad \text{weakly in } \ \mathbb{V}', \end{aligned} \tag{2.19}$$

because $\|\mathcal{B}\boldsymbol{v}\| \leq c\|\boldsymbol{v}\|_1^{s-1}$, with

$$\mathcal{B}(\boldsymbol{u})\boldsymbol{v} = \int_\Gamma |K\boldsymbol{u}_{\boldsymbol{\tau}}|^{s-2} K\boldsymbol{u}_{\boldsymbol{\tau}} \cdot K\boldsymbol{v}_{\boldsymbol{\tau}} d\sigma.$$

Owing to the compactness of the imbedding of $H^1(\Omega)$ into $L^4(\Omega)$, there exists a subsequence, still denoted by $\boldsymbol{u}_m$, which converges weakly to $\boldsymbol{u} \in H^1(\Omega)$ and strongly in $L^4(\Omega)^d$. We easily check that $\operatorname{div} \boldsymbol{u} = 0$ in Ω, $\boldsymbol{u} \cdot \boldsymbol{n}|_\Gamma = 0$ and $\boldsymbol{u}|_S = 0$. Passing to the limit on m is direct for the linear term because one only needs weak convergence, but for the term $(\boldsymbol{u}_m \cdot \nabla)\boldsymbol{u}_m$, one needs the strong convergence of $\boldsymbol{u}_m$ on $L^4(\Omega)^d$ (see [29, Chap 5, Thm 1.4]). Indeed

$$\begin{aligned} |d(\boldsymbol{u}_m, \boldsymbol{u}_m, \boldsymbol{v}) - d(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v})| &= |d(\boldsymbol{u}_m - \boldsymbol{u}, \boldsymbol{u}_m, \boldsymbol{v}) + d(\boldsymbol{u}, \boldsymbol{u}_m - \boldsymbol{u}, \boldsymbol{v})| \\ &\leq \|\boldsymbol{u}_m - \boldsymbol{u}\|_{L^4(\Omega)} \|\nabla \boldsymbol{u}_m\| \|\boldsymbol{v}\|_{L^4(\Omega)} + |d(\boldsymbol{u}, \boldsymbol{u}_m - \boldsymbol{u}, \boldsymbol{v})| \ , \end{aligned} \tag{2.20}$$

and the right hand side of (2.20) tends to zero with m. Hence $d(\boldsymbol{u}_m, \boldsymbol{u}_m, \boldsymbol{v})$ tends to $d(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v})$. Finally for the term on Γ, we need to show that $\mathcal{W} = \mathcal{B}\boldsymbol{u}$. First, we deduce from the weak convergence on $\boldsymbol{H}^1(\Omega)$, that

$$\boldsymbol{u}_m \to \boldsymbol{u} \ \text{ strongly in } \ \boldsymbol{L}^2(\Omega)) \ .$$

The mapping $\boldsymbol{u} \longmapsto \boldsymbol{u}_{\boldsymbol{\tau}}$ is continuous from $\boldsymbol{H}^1(\Omega)$ to $\boldsymbol{L}^2(\Gamma)$. Then

$$\boldsymbol{u}_m|_\Gamma \to \boldsymbol{u}_{\boldsymbol{\tau}}|_\Gamma \ \text{ weakly in } \boldsymbol{L}^2(\Gamma).$$

Now one can use the fact that the mapping $\boldsymbol{v} \to \mathcal{B}\boldsymbol{v}$ is monotone, and hemi-continuous to show that (see [31, Chap 2]) $\mathcal{W} = \mathcal{B}\boldsymbol{u}$.
Therefore, with the help of the density of $\mathbb{K}_m$ in $\mathbb{V}_{\text{div}}$, one deduces that the limit $\boldsymbol{u}$ satisfies the equation

$$\begin{cases} \text{for all } \boldsymbol{v} \in \mathbb{V}_{\text{div}}, \\ a(\boldsymbol{u}, \boldsymbol{v}) + \displaystyle\int_\Gamma |K\boldsymbol{u}_{\boldsymbol{\tau}}|^{s-2} K\boldsymbol{u}_{\boldsymbol{\tau}} \cdot K\boldsymbol{v}_{\boldsymbol{\tau}} d\sigma + d(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) = \ell(\boldsymbol{v}) \, . \end{cases} \tag{2.21}$$

Taking $\boldsymbol{v} = \boldsymbol{u}$ in (2.21), and using standard inequalities, one obtains the a priori estimate on the velocity announced.
Step 3: Construction of the pressure.
For $1 < s < 2$, the inf-sup condition (2.6) holds and owing to (2.21), there exists a p in M see [29, Chap I, Lemma 4.1] such that

$$\begin{cases} \text{for all } (\boldsymbol{v}, q) \in \mathbb{V} \times M, \\ a(\boldsymbol{u}, \boldsymbol{v}) + \left(|K\boldsymbol{u}_{\boldsymbol{\tau}}|^{s-2} K\boldsymbol{u}_{\boldsymbol{\tau}}, K\boldsymbol{v}_{\boldsymbol{\tau}}\right)_\Gamma + d(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) - \ell(\boldsymbol{v}) = \displaystyle\int_\Omega \operatorname{div} \boldsymbol{v} p dx \\ \displaystyle\int_\Omega q \operatorname{div} \boldsymbol{u} dx = 0 \, . \end{cases} \tag{2.22}$$

The a priori estimate on the pressure is obtained by using the inf-sup condition and the velocity equation.
Step 4: Uniqueness.

Let $(\boldsymbol{u}_1, p_1)$ and $(\boldsymbol{u}_2, p_2)$ be two solutions of (2.22). Setting for a while $\boldsymbol{u} = \boldsymbol{u}_1 - \boldsymbol{u}_2$ and $p = p_1 - p_2$, we deduce that

$$a(\boldsymbol{u}, \boldsymbol{v}) + \int_\Gamma \left(|K\boldsymbol{u}_{1,\boldsymbol{\tau}}|^{s-2} K\boldsymbol{u}_{1,\boldsymbol{\tau}} - |K\boldsymbol{u}_{2,\boldsymbol{\tau}}|^{s-2} K\boldsymbol{u}_{2,\boldsymbol{\tau}}\right) \cdot K\boldsymbol{v}_{\boldsymbol{\tau}} d\sigma + d(\boldsymbol{u}_1, \boldsymbol{u}_1, \boldsymbol{v}) - d(\boldsymbol{u}_2, \boldsymbol{u}_2, \boldsymbol{v})$$
$$= \int_\Omega \operatorname{div} \boldsymbol{v} p dx \,.$$

We take successively $\boldsymbol{v} = \boldsymbol{u}_1$, and $\boldsymbol{v} = \boldsymbol{u}_2$, and take the difference of the resulting equations. This gives

$$a(\boldsymbol{u}, \boldsymbol{u}) + \int_\Gamma \underbrace{\left(|K\boldsymbol{u}_{1,\boldsymbol{\tau}}|^{s-2} K\boldsymbol{u}_{1,\boldsymbol{\tau}} - |K\boldsymbol{u}_{2,\boldsymbol{\tau}}|^{s-2} K\boldsymbol{u}_{2,\boldsymbol{\tau}}\right) \cdot (K\boldsymbol{u}_{1,\boldsymbol{\tau}} - K\boldsymbol{u}_{2,\boldsymbol{\tau}})}_{\geq 0} d\sigma = -d(\boldsymbol{u}, \boldsymbol{u}_2, \boldsymbol{u}) \,, \tag{2.23}$$

which implies that

$$2\nu c \|\boldsymbol{u}\|_1^2 \leq \|\boldsymbol{u}\|_{L^4(\Omega)}^2 \|\nabla \boldsymbol{u}_2\| \leq c \|\boldsymbol{u}\|_1^2 \|\boldsymbol{u}_2\|_1 \leq c \frac{\|\boldsymbol{f}\|}{\nu} \|\boldsymbol{u}\|_1^2 \,,$$

which gives us the condition for the unique solvability. Next, the function $p = p_1 - p_2$ satisfies

$$\forall \boldsymbol{v} \in \mathbb{V} \,, \quad \int_\Omega \operatorname{div} \boldsymbol{v} p dx = 0$$

which together with (2.6), implies that $p_1 = p_2$. □

Remark 2.3 *The smallness condition for uniqueness is not surprising given that we have a nonlinear problem. On the other hand, that condition is similar to the uniqueness condition for the Navier-Stokes equations with Dirichlet boundary condition. The similarity in our view is due to the fact that the "extra" nonlinear term is monotone and can be removed from the equation when analysing (see (2.23)).*

The next section is about the construction of finite element solutions of (2.4) and (2.13).

3 Finite element approximation

This section is twofold. First, we introduce the basics tools for the formulation of the finite element approximation of (2.4) and (2.13). Secondly, we discuss the solvability of the nonlinear finite elements problems.

3.1 Preliminaries

From now on, we assume that Ω is a polygon in two dimensions or polyhedron in three dimensions, so it can be entirely triangulated by triangles or tetrahedron, according to the dimensions d. A regular family of triangulations $(\mathcal{T}_h)_h$ of Ω, is a set of closed non degenerate triangles or tetrahedra, called elements, satisfying (see [32])

(i) For each h, $\overline{\Omega}$ is the union of all elements of $\mathcal{T}_h$, that is

$$\overline{\Omega} = \bigcup_{K \in \mathcal{T}_h} K \,.$$

(ii) The ratio of the diameter of an element $K \in \mathcal{T}_h$ to the diameter of its inscribed circle or ball is bounded by a constant independent of h. This is to say that there exists a constant σ, independent of h and K such that

$$\forall\ K \in \mathcal{T}_h\ , \quad \frac{h_K}{\rho_K} = \sigma_K \leq \sigma\ ,$$

where h_K is the diameter of K, and ρ_K is the diameter of the sphere (circle) inscribed in K.

(iii) The intersection of two distinct elements of $\mathcal{T}_h$ is either empty, a common vertex, or an entire common edge or face.

As usual, h is the maximal diameter of all elements of $\mathcal{T}_h$. For every $K \in \mathcal{T}_h$, we denote by $\mathbb{P}_k$ the space of restrictions to K of polynomials of degree k in d variables and total degree at most k. In what follows, c stand for a generic constant which may vary from line to line but are always independent of h.

Remark 3.1 *In this work, we do not address the situation with curved boundary, but we refer the interested readers to the work in [25, 26].*

3.2 Discretization of the Stokes system

Finite element approximation.We discretize the velocity and the pressure respectively in the spaces $\mathbb{V}_h$ and M_h defined as follows

$$\begin{aligned}
\mathbb{V}_h &= \left\{ \boldsymbol{v}_h \in \mathbb{V} \cap \mathcal{C}^0(\overline{\Omega})^d,\ \text{for all } T \in \mathcal{T}_h,\quad \boldsymbol{v}_h|_K \in (\mathbb{P}_1(T) + \text{bubble})^d \right\}\ ,\\
L^2_h &= \left\{ q_h | q_h \in \mathcal{C}^0(\overline{\Omega}),\ q_h|_T \in \mathbb{P}_1,\quad \text{for all } T \in \mathcal{T}_h \right\}\ ,\\
M_h &= \left\{ q_h | q_h \in L^2_h,\quad \int_\Omega q_h dx = 0 \right\}\ ,
\end{aligned} \tag{3.1}$$

where $\mathbb{P}_1(T)$+bubble is the sum of a polynomial of $\mathbb{P}_1(T)$ and a bubble function $b_T(\boldsymbol{x}) = \alpha_1(\boldsymbol{x})\alpha_2(\boldsymbol{x})\alpha_3(\boldsymbol{x})$, for any $T \in \mathcal{T}_h$, and denoting the vertices of $T \in \mathcal{T}_h$ by a_i, $1 \leq i \leq 3$, and its corresponding barycentric coordinates by α_i. Note that $b_T(\boldsymbol{x}) = 0$ on T and that $b_T(\boldsymbol{x}) > 0$ on T.

With the above spaces it is quite natural to approximate (2.4) (with obvious notation) by:

$$\begin{cases}
\text{Find } (\boldsymbol{u}_h, p_h) \in \mathbb{V}_h \times M_h \ \text{ such that for all } (\boldsymbol{v}_h, q_h) \in \mathbb{V}_h \times M_h \\
a(\boldsymbol{u}_h, \boldsymbol{v}_h) + \displaystyle\int_\Gamma |K\boldsymbol{u}_{\boldsymbol{\tau},h}|^{s-2} K\boldsymbol{u}_{\boldsymbol{\tau},h} \cdot K\boldsymbol{v}_{\boldsymbol{\tau},h} d\sigma - b(\boldsymbol{v}_h, p_h) = \ell(\boldsymbol{v}_h)\,, \\
b(\boldsymbol{u}_h, q_h) = 0\,.
\end{cases} \tag{3.2}$$

For $1 < s < 2$, the discrete inf-sup holds, that is, there exists $\widetilde{\beta} > 0$ such that

$$\inf_{q_h \in M_h} \sup_{0 \neq \boldsymbol{v}_h \in \mathbb{V}_h} \frac{b(\boldsymbol{v}_h, q_h)}{\|\boldsymbol{v}_h\|_1\ \|q_h\|} \geq \widetilde{\beta}\,. \tag{3.3}$$

We recall that on the space

$$\mathbb{V}_{\text{div},h} = \left\{ \boldsymbol{v}_h \in \mathbb{V}_h :\quad \int_\Omega q_h \,\text{div}\, \boldsymbol{v}_h dx = 0 \ \text{ for all } q_h \in M_h \right\}\ ,$$

(3.2) is equivalent to:

$$\begin{cases}
\text{Find } \boldsymbol{u}_h \in \mathbb{V}_{\text{div},h} \text{ such that for all } \boldsymbol{v}_h \in \mathbb{V}_{\text{div},h} \\
a(\boldsymbol{u}_h, \boldsymbol{v}_h) + \displaystyle\int_\Gamma |K\boldsymbol{u}_{\boldsymbol{\tau},h}|^{s-2} K\boldsymbol{u}_{\boldsymbol{\tau},h} \cdot K\boldsymbol{v}_{\boldsymbol{\tau},h} d\sigma = \ell(\boldsymbol{v}_h)\,.
\end{cases} \tag{3.4}$$

Remark 3.2 *It is noted that a function in $\mathbb{V}_h$ is reduced to polynomial of degree one on each side e of the element K and also on each boundary side. So, the condition $\boldsymbol{u}_h \cdot \boldsymbol{n}|_\Gamma = 0$ implies that $\boldsymbol{u}_h$ vanishes at each corner of Γ.*

The velocity and pressure are interpolated respectively by the operators $\Pi_h \in \mathcal{L}(\mathbb{V}; \mathbb{V}_h)$ and $r_h \in \mathcal{L}(M; M_h)$ constructed in Girault and Hecht [33, Chap 5]. These operators satisfy, for each real number $\alpha \in [0, 1]$ and for each number $m \geq 2$, the following properties. (1) There exists a constant c, independent of h such that

$$\text{for all } \boldsymbol{v} \in W^{\alpha+1,m}(\Omega)^2 \cap \mathbb{V}\ , \quad |\Pi_h \boldsymbol{v} - \boldsymbol{v}|_{W^{1,m}(\Omega)} \leq c h^\alpha\, |\boldsymbol{v}|_{W^{\alpha+1,m}(\Omega)}\ . \tag{3.5}$$

(2) There exists a constant c, independent of h such that

$$\text{for all } q \in W^{\alpha,m}(\Omega)^2 \cap M\ , \quad \|r_h q - q\|_{L^m(\Omega)} \leq c h^\alpha\, |\boldsymbol{v}|_{W^{\alpha,m}(\Omega)}\ . \tag{3.6}$$

A key issue when studying the mixed formulation (3.2) with the spaces $\mathbb{V}_h$ and M_h is the discrete inf-sup condition (3.3). Its proof consists in using the trick introduce by Fortin (see [29]) based on the operator

$$H(\boldsymbol{v}) = \Pi_h(\boldsymbol{v}) + \sum_{K \in \mathcal{T}_h} a_K(\boldsymbol{v}) b_K$$

where

$$a_K(\boldsymbol{v}) = \frac{1}{\int_K b_K dx} \int_K (\boldsymbol{v} - \Pi_h(\boldsymbol{v})) dx\,.$$

Existence and uniqueness. The Existence of a solution of (3.4) (and (3.2)) can be obtained by repeating the steps of paragraph 2.2. One observes that the variational problem (3.4) is a finite dimensional, square system of nonlinear equations. Thus just like the continuous analysis, we introduce the following mapping; for all $(\boldsymbol{v}_h, \boldsymbol{w}_h) \in \mathbb{V}_{\mathrm{div},h} \times \mathbb{V}_{\mathrm{div},h}$

$$\mathcal{A}_h(\boldsymbol{v}_h)\boldsymbol{w}_h = a(\boldsymbol{v}_h, \boldsymbol{w}_h) + \int_\Gamma |K\boldsymbol{v}_{\boldsymbol{\tau},h}|^{s-2} K\boldsymbol{v}_{\boldsymbol{\tau},h} \cdot K\boldsymbol{w}_{\boldsymbol{\tau},h} d\sigma\,.$$

We can show that for all $1 < s < 2$, $\mathcal{A}_h$ is strictly monotone, coercive and hemi-continuous. Thus, the analog of theorem 2.1 reads as follows.

Theorem 3.1 *For $1 < s < 2$, the problem (3.2) has exactly one solution $(\boldsymbol{u}_h, p_h)$ in $\mathbb{V}_h \times M_h$, and there exists c independent of h such that the following a priori estimates hold*

$$\begin{aligned}
&\|\nabla \boldsymbol{u}_h\|^2 + \int_\Gamma |K\boldsymbol{u}_{\boldsymbol{\tau},h}|^s\, d\sigma \leq c\|\boldsymbol{f}\|^2\,,\\
&\|p_h\| \leq 2\nu\|\nabla \boldsymbol{u}_h\| + \|K\|^s_{L^\infty(\Gamma)}\, \|\boldsymbol{u}_{\boldsymbol{\tau},h}\|^{s-1}_{L^s(\Gamma)} + c\|\boldsymbol{f}\|\,.
\end{aligned}$$

3.3 Analysis of the Navier-Stokes system

Finite element approximation. The velocity and pressure are taken as before. We consider the following finite element scheme

$$\begin{cases}
\text{Find } (\boldsymbol{u}_h, p_h) \in \mathbb{V}_h \times M_h \text{ such that for all } (\boldsymbol{v}_h, q_h) \in \mathbb{V}_h \times M_h,\\
a(\boldsymbol{u}_h, \boldsymbol{v}_h) + \displaystyle\int_\Gamma |K\boldsymbol{u}_{\boldsymbol{\tau},h}|^{s-2} K\boldsymbol{u}_{\boldsymbol{\tau},h} \cdot K\boldsymbol{v}_{\boldsymbol{\tau},h} d\sigma + \widetilde{d}(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{v}_h) - b(\boldsymbol{v}_h, p_h) = \ell(\boldsymbol{v}_h),\\
b(\boldsymbol{u}_h, q_h) = 0\,,
\end{cases} \tag{3.7}$$

with the trilinear form $\widetilde{d}(\cdot,\cdot,\cdot)$ given by R.Temam [35]

$$\widetilde{d}(\boldsymbol{v}_h,\boldsymbol{u}_h,\boldsymbol{w}_h)=d(\boldsymbol{v}_h,\boldsymbol{u}_h,\boldsymbol{w}_h)+\frac{1}{2}\left((\operatorname{div}\boldsymbol{v}_h)\boldsymbol{u}_h,\boldsymbol{w}_h\right)=\frac{1}{2}\left(d(\boldsymbol{v}_h,\boldsymbol{u}_h,\boldsymbol{w}_h)-d(\boldsymbol{v}_h,\boldsymbol{w}_h,\boldsymbol{u}_h)\right)\,.$$

It is noted that $\widetilde{d}(\cdot,\cdot,\cdot)$ is consistent with $d(\cdot,\cdot,\cdot)$ in the sense that

$$\text{for all } (\boldsymbol{v},\boldsymbol{u},\boldsymbol{w})\in\mathbb{V}_{\rm div}\times\boldsymbol{H}^1(\Omega)\times\boldsymbol{H}^1(\Omega)\ \ ,\ \widetilde{d}(\boldsymbol{v},\boldsymbol{u},\boldsymbol{w})=d(\boldsymbol{v},\boldsymbol{u},\boldsymbol{w})\,.$$

Furthermore $\widetilde{d}(\cdot,\cdot,\cdot)$ is anti-symmetry meaning that

$$\text{for all } (\boldsymbol{v}_h,\boldsymbol{u}_h,\boldsymbol{w}_h)\in\boldsymbol{H}^1(\Omega)\times\mathbb{V}_h\times\mathbb{V}_h\ ,\quad \widetilde{d}(\boldsymbol{v}_h,\boldsymbol{u}_h,\boldsymbol{w}_h)=-\widetilde{d}(\boldsymbol{v}_h,\boldsymbol{w}_h,\boldsymbol{u}_h)\ .$$

Existence and uniqueness. The kernel of divergence in $\mathbb{V}_h$ is $\mathbb{V}_{{\rm div},h}$, and for $1\le s<2$, (3.7) is equivalent to

$$\begin{cases}\text{Find } \boldsymbol{u}_h\in\mathbb{V}_{{\rm div},h} \text{ such that for all } \boldsymbol{v}_h\in\mathbb{V}_{{\rm div},h},\\ a(\boldsymbol{u}_h,\boldsymbol{v}_h)+\displaystyle\int_\Gamma|K\boldsymbol{u}_{\boldsymbol{\tau},h}|^{s-2}K\boldsymbol{u}_{\boldsymbol{\tau},h}\cdot K\boldsymbol{v}_{\boldsymbol{\tau},h}d\sigma+\widetilde{d}(\boldsymbol{u}_h,\boldsymbol{u}_h,\boldsymbol{v}_h)=\ell(\boldsymbol{v}_h)\,.\end{cases}\tag{3.8}$$

(3.8) is a finite dimensional, square system of nonlinear equations. We address next the solvability of (3.8) by Brouwer's Fixed point argument. To this end, for fixed $\boldsymbol{u}_h\in\mathbb{V}_{{\rm div},h}$, one introduces $F(\boldsymbol{u}_h)$ in $\mathbb{V}_{{\rm div},h}$ by

$$(F(\boldsymbol{u}_h),\boldsymbol{v}_h)=a(\boldsymbol{u}_h,\boldsymbol{v}_h)+\widetilde{d}(\boldsymbol{u}_h,\boldsymbol{u}_h,\boldsymbol{v}_h)+\int_\Gamma|K\boldsymbol{u}_{\boldsymbol{\tau},h}|^{s-2}K\boldsymbol{u}_{\boldsymbol{\tau},h}\cdot K\boldsymbol{v}_{\boldsymbol{\tau},h}d\sigma-\int_\Omega\boldsymbol{f}\cdot\boldsymbol{v}_h\,dx\,.$$

From the analysis in the continuous formulation, we deduce that the mapping $F:\mathbb{V}_{{\rm div},h}\longrightarrow\mathbb{V}_{{\rm div},h}$ is continuous. Next, using the anti-symmetry property of $\widetilde{d}(\cdot,\cdot,\cdot)$,

$$\begin{aligned}(F(\boldsymbol{v}_h),\boldsymbol{v}_h)=&a(\boldsymbol{v}_h,\boldsymbol{v}_h)+\int_\Gamma|K\boldsymbol{v}_{\boldsymbol{\tau},h}|^s d\sigma-\int_\Omega\boldsymbol{f}\cdot\boldsymbol{v}_h\,dx\\ \ge&\|\boldsymbol{v}_h\|_1\left(2\nu c\|\boldsymbol{v}_h\|_1-\|\boldsymbol{f}\|\right)+\int_\Gamma|K\boldsymbol{v}_{\boldsymbol{\tau},h}|^s d\sigma\,.\end{aligned}$$

Thus $(F(\boldsymbol{v}_h),\boldsymbol{v}_h)$ is non-negative for all $\boldsymbol{v}_h\in\mathbb{V}_h$, with $\|\boldsymbol{v}_h\|_1=r>\dfrac{c}{\nu}\|\boldsymbol{f}\|$. So, this proves existence of at least one solution of (3.8). Using the equivalence between (3.7) and (3.8), we construct the pressure and we have the following result.

Proposition 3.1 *The finite element problem (3.7) admits at least one solution $(\boldsymbol{u}_h,p_h)\in\mathbb{V}_h\times M_h$, and there exist positive constant c such that*

$$\begin{aligned}&\nu c\|\boldsymbol{u}_h\|_1^2+\int_\Gamma|K\boldsymbol{u}_{\boldsymbol{\tau},h}|^s d\sigma\le\frac{c}{\nu}\|\boldsymbol{f}\|^2\,,\\ &\widetilde{\beta}\|p_h\|\le 2\nu\|\boldsymbol{u}_h\|_1+\|K\|^s_{L^\infty(\Gamma)}\|\boldsymbol{u}_{\boldsymbol{\tau},h}\|^{s-1}_{L^{s-1}(\Gamma)}+\widetilde{c}\|\boldsymbol{u}_h\|_1^2+\|\boldsymbol{f}\|\,.\end{aligned}$$

The estimate above is obtained in the standard way (see proposition 2.2). One can observe the small difference between the estimate on the pressure in proposition 2.2 and proposition 3.1. This difference being the fact that in the discrete setting $\operatorname{div}\boldsymbol{u}_h$ is non zero.
We end this analysis with the result below.

Proposition 3.2 *Let $(\boldsymbol{u}_h,p_h)\in\mathbb{V}_h\times M_h$ be the solution of (3.7). There exists a positive constant $\widetilde{c}$ depending only on Ω such that if for ν, $\boldsymbol{f}$, the relation*

$$\nu^2\ge\widetilde{c}\|\boldsymbol{f}\|\,,$$

is satisfied, then the solution of (3.7) is unique.

Proof. Let $(\boldsymbol{u}_1, p_1)$ and $(\boldsymbol{u}_2, p_2)$ be two solutions of (3.7). Setting for a while $\boldsymbol{u} = \boldsymbol{u}_1 - \boldsymbol{u}_2$ and $p = p_1 - p_2$. the equation (2.23) is

$$a(\boldsymbol{u}, \boldsymbol{u}) + \int_\Gamma \underbrace{\left(|K\boldsymbol{u}_{1,\boldsymbol{\tau}}|^{s-2} K\boldsymbol{u}_{1,\boldsymbol{\tau}} - |K\boldsymbol{u}_{2,\boldsymbol{\tau}}|^{s-2} K\boldsymbol{u}_{2,\boldsymbol{\tau}}\right) \cdot (K\boldsymbol{u}_{1,\boldsymbol{\tau}} - K\boldsymbol{u}_{2,\boldsymbol{\tau}})}_{\geq 0} d\sigma = -\widetilde{d}(\boldsymbol{u}, \boldsymbol{u}_2, \boldsymbol{u}) \,, \tag{3.9}$$

this leads to

$$2\nu c\|\boldsymbol{u}\|_1^2 \leq \|\boldsymbol{u}\|_{L^4(\Omega)}^2 \|\nabla \boldsymbol{u}_2\| \leq \widetilde{c}\|\boldsymbol{u}\|_1^2 \|\boldsymbol{u}_2\|_1 \leq \widetilde{c}\frac{\|\boldsymbol{f}\|}{\nu}\|\boldsymbol{u}\|_1^2 \,,$$

which gives us the condition for the unique solvability. Secondly, the function p satisfies

$$\forall \boldsymbol{v} \in \mathbb{V}_h \ , \quad \int_\Omega \operatorname{div} \boldsymbol{v} p dx = 0$$

which together with the discrete inf-sup condition (3.3), implies that $p_1 = p_2$. □

Remark 3.3 *The condition obtained for the unique solvability of (3.7) in Proposition 3.2, is not the same condition obtained in (2.15) for the unique solvability of (2.13). This difference being that in the discrete setting* $\operatorname{div} \boldsymbol{u}_h$ *is non zero. Hence the constants are not the same.*

At this juncture, knowing that solution of (3.7) and (3.2) are well defined and computable, we next study the convergence of the finite element solution $(\boldsymbol{u}_h, p_h)$. This exercise is very important in numerical analysis because by doing it we check the "reliability" of the approximation.

4 A priori error analysis

We now study the convergence of the finite element solution $(\boldsymbol{u}_h, p_h)$ by computing an upper bound of the quantity $\|\boldsymbol{u} - \boldsymbol{u}_h\|_1 + \|p - p_h\|$. The problems (3.7) and (3.2) are well defined, hence study the convergence of the finite element solution make perfect sense. It is important to recall that without uniqueness no convergence result can be formulated.

4.1 A priori error analysis: the Stokes equations

We start with the convergence of the solution $(\boldsymbol{u}_h, p_h)$ of (3.2). It holds that

Theorem 4.1 *For $1 < s < 2$, let $(\boldsymbol{u}, p)$ be the solution of (2.4) and $(\boldsymbol{u}_h, p_h)$ the solution of (3.2). Then there is a positive constant c independent of h such that*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_1 + \|p - p_h\| + \|\boldsymbol{u}_{\boldsymbol{\tau}} - \boldsymbol{u}_{\boldsymbol{\tau},h}\|_\Gamma \leq c \inf_{\boldsymbol{v}_h \in \mathbb{V}_h} \|\boldsymbol{u}_{\boldsymbol{\tau}} - \boldsymbol{v}_{\boldsymbol{\tau},h}\|_\Gamma^{1/2} + c \inf_{\boldsymbol{v}_h \in \mathbb{V}_h} \|\nabla(\boldsymbol{u} - \boldsymbol{v}_h)\| + c \inf_{q_h \in M_h} \|q_h - p\| \,.$$

Proof. **Step 1**: Inf-sup condition.

Let $(\boldsymbol{u}_h, p_h)$ be the unique solution of (3.2). Let $q_h \in M_h$, then $p_h - q_h \in M_h$ and in particular, there exists $\widetilde{\beta}$ independent of h such that

$$\widetilde{\beta}\|p_h - q_h\| \leq \sup_{0 \neq \boldsymbol{v}_h \in \mathbb{V}_h \cap \{\boldsymbol{v}_h|_\Gamma = 0\}} \frac{b(\boldsymbol{v}_h, p_h - q_h)}{\|\boldsymbol{v}_h\|_1} \,. \tag{4.1}$$

We take the difference between (2.4) and (3.2) and obtain (after observing that $\mathbb{V}_h \subset \mathbb{V}$ and $M_h \subset M$)

$$\begin{cases} \text{for all } (\boldsymbol{v}_h, q_h) \in \mathbb{V}_h \times M_h \ , \\ a(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{v}_h) + \left(|K\boldsymbol{u}_{\boldsymbol{\tau}}|^{s-2} K\boldsymbol{u}_{\boldsymbol{\tau}} - |K\boldsymbol{u}_{\boldsymbol{\tau},h}|^{s-2} K\boldsymbol{u}_{\boldsymbol{\tau},h} \,,\, K\boldsymbol{v}_{\boldsymbol{\tau},h}\right)_\Gamma \\ = b(\boldsymbol{v}_h, p - p_h) \,, \\ b(\boldsymbol{u} - \boldsymbol{u}_h, q_h) = 0 \,. \end{cases} \tag{4.2}$$

We solve for $b(\boldsymbol{v}_h, p_h - q_h)$ in the first equation of (4.2), report the result to (4.1), and using Cauchy-Schwarz's, Hölder's and trace's inequality gives

$$\begin{aligned}\widetilde{\beta}\|p_h - q_h\| &\leq \sup_{0\neq \boldsymbol{v}_h\in \mathbb{V}_h\cap\{\boldsymbol{v}_h|_\Gamma=0\}} \frac{|a(\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{v}_h) - b(\boldsymbol{v}_h, q_h - p)|}{\|\boldsymbol{v}_h\|_1} \\ &\leq 2\nu\|\nabla(\boldsymbol{u}-\boldsymbol{u}_h)\| + \|q_h - p\|\,. \end{aligned} \tag{4.3}$$

Step 2: Monotonicity.
Set $\mathcal{B}\boldsymbol{u}_{\boldsymbol{\tau}} = |K\boldsymbol{u}_{\boldsymbol{\tau}}|^{s-2}K\boldsymbol{u}_{\boldsymbol{\tau}}$. Let $\boldsymbol{w}_h \in \mathbb{V}_h$, take $\boldsymbol{v}_h = \boldsymbol{w}_h - \boldsymbol{u}_h$, (4.2) leads to

$$\begin{cases} \text{for all } (\boldsymbol{w}_h, q_h) \in \mathbb{V}_h \times M_h\ , \\ a(\boldsymbol{u}-\boldsymbol{u}_h, \boldsymbol{w}_h - \boldsymbol{u}_h) + \displaystyle\int_\Gamma (\mathcal{B}\boldsymbol{u}_{\boldsymbol{\tau}} - \mathcal{B}\boldsymbol{u}_{\boldsymbol{\tau},h})\cdot(K\boldsymbol{w}_{\boldsymbol{\tau},h} - K\boldsymbol{u}_{\boldsymbol{\tau},h})d\sigma = b(\boldsymbol{w}_h - \boldsymbol{u}_h, p - p_h)\,, \\ b(\boldsymbol{u}-\boldsymbol{u}_h, q_h - p_h) = 0\,. \end{cases}$$

which is re -written as follows

$$\begin{cases} a(\boldsymbol{w}_h - \boldsymbol{u}_h, \boldsymbol{w}_h - \boldsymbol{u}_h) + \displaystyle\int_\Gamma (\mathcal{B}\boldsymbol{u}_{\boldsymbol{\tau}} - \mathcal{B}\boldsymbol{u}_{\boldsymbol{\tau},h})\cdot(K\boldsymbol{w}_{\boldsymbol{\tau},h} - K\boldsymbol{u}_{\boldsymbol{\tau},h})d\sigma \\ = a(\boldsymbol{w}_h - \boldsymbol{u}, \boldsymbol{w}_h - \boldsymbol{u}_h) + b(\boldsymbol{w}_h - \boldsymbol{u}_h, p - p_h) \\ \\ b(\boldsymbol{w}_h - \boldsymbol{u}_h, p - p_h) = b(\boldsymbol{w}_h - \boldsymbol{u}, q_h - p_h) + b(\boldsymbol{u}_h - \boldsymbol{w}_h, q_h - p)\,. \end{cases} \tag{4.4}$$

We use the second equation of (4.4) in the first one and obtain that

$$\begin{aligned} &a(\boldsymbol{w}_h - \boldsymbol{u}_h, \boldsymbol{w}_h - \boldsymbol{u}_h) + \int_\Gamma (\mathcal{B}\boldsymbol{u}_{\boldsymbol{\tau}} - \mathcal{B}\boldsymbol{u}_{\boldsymbol{\tau},h})\cdot(K\boldsymbol{w}_{\boldsymbol{\tau},h} - K\boldsymbol{u}_{\boldsymbol{\tau},h})d\sigma \\ =&a(\boldsymbol{w}_h - \boldsymbol{u}, \boldsymbol{w}_h - \boldsymbol{u}_h) + b(\boldsymbol{w}_h - \boldsymbol{u}, q_h - p_h) + b(\boldsymbol{u}_h - \boldsymbol{w}_h, q_h - p)\,. \end{aligned} \tag{4.5}$$

From linearity we have

$$\begin{aligned} &\int_\Gamma (\mathcal{B}\boldsymbol{u}_{\boldsymbol{\tau}} - \mathcal{B}\boldsymbol{u}_{\boldsymbol{\tau},h})\cdot(K\boldsymbol{w}_{\boldsymbol{\tau},h} - K\boldsymbol{u}_{\boldsymbol{\tau},h})d\sigma \\ =&\int_\Gamma (\mathcal{B}\boldsymbol{u}_{\boldsymbol{\tau}} - \mathcal{B}\boldsymbol{u}_{\boldsymbol{\tau},h})\cdot(K\boldsymbol{w}_{\boldsymbol{\tau},h} - K\boldsymbol{u}_{\boldsymbol{\tau}})d\sigma + \int_\Gamma (\mathcal{B}\boldsymbol{u}_{\boldsymbol{\tau}} - \mathcal{B}\boldsymbol{u}_{\boldsymbol{\tau},h})\cdot(K\boldsymbol{u}_{\boldsymbol{\tau}} - K\boldsymbol{u}_{\boldsymbol{\tau},h})d\sigma\,. \end{aligned} \tag{4.6}$$

Thus (4.5) becomes

$$\begin{aligned} &a(\boldsymbol{w}_h - \boldsymbol{u}_h, \boldsymbol{w}_h - \boldsymbol{u}_h) + \int_\Gamma (\mathcal{B}\boldsymbol{u}_{\boldsymbol{\tau}} - \mathcal{B}\boldsymbol{u}_{\boldsymbol{\tau},h})\cdot(K\boldsymbol{u}_{\boldsymbol{\tau}} - K\boldsymbol{u}_{\boldsymbol{\tau},h})d\sigma \\ =&a(\boldsymbol{w}_h - \boldsymbol{u}, \boldsymbol{w}_h - \boldsymbol{u}_h) + b(\boldsymbol{w}_h - \boldsymbol{u}, q_h - p_h) + b(\boldsymbol{u}_h - \boldsymbol{w}_h, q_h - p) \\ &+ \int_\Gamma (\mathcal{B}\boldsymbol{u}_{\boldsymbol{\tau}} - \mathcal{B}\boldsymbol{u}_{\boldsymbol{\tau},h})\cdot(K\boldsymbol{u}_{\boldsymbol{\tau}} - K\boldsymbol{w}_{\boldsymbol{\tau},h})d\sigma \end{aligned} \tag{4.7}$$

From (2.7), it was shown that (see [20]): there exist α_1, α_2 two positive constants such that

$$\alpha_1 \frac{\|\boldsymbol{u}_{\boldsymbol{\tau}} - \boldsymbol{u}_{\boldsymbol{\tau},h}\|_\Gamma^2}{\alpha_2 + \|\boldsymbol{u}_{\boldsymbol{\tau}}\|_\Gamma^{2-s} + \|\boldsymbol{u}_{\boldsymbol{\tau},h}\|_\Gamma^{2-s}} \leq \int_\Gamma (\mathcal{B}\boldsymbol{u}_{\boldsymbol{\tau}} - \mathcal{B}\boldsymbol{u}_{\boldsymbol{\tau},h})\cdot(K\boldsymbol{u}_{\boldsymbol{\tau}} - K\boldsymbol{u}_{\boldsymbol{\tau},h})\,d\sigma\,. \tag{4.8}$$

Now since $1 < s < 2$, we use (2.8), (4.8), triangle's inequality and (2.3) in (4.7) and get

$$\begin{aligned} &2\nu c\|\nabla(\boldsymbol{w}_h - \boldsymbol{u}_h)\|^2 + \alpha_1 \frac{\|\boldsymbol{u}_{\boldsymbol{\tau}} - \boldsymbol{u}_{\boldsymbol{\tau},h}\|_\Gamma^2}{\alpha_2 + \|\boldsymbol{u}_{\boldsymbol{\tau}}\|_\Gamma^{2-s} + \|\boldsymbol{u}_{\boldsymbol{\tau},h}\|_\Gamma^{2-s}} \\ \leq&2\nu\|\nabla(\boldsymbol{w}_h - \boldsymbol{u})\|\|\nabla(\boldsymbol{w}_h - \boldsymbol{u}_h)\| + \|\nabla(\boldsymbol{w}_h - \boldsymbol{u})\|\|q_h - p_h\| + \|\nabla(\boldsymbol{u}_h - \boldsymbol{w}_h)\|q_h - p\| \\ &+ c\|\boldsymbol{u}_{\boldsymbol{\tau}} - \boldsymbol{u}_{\boldsymbol{\tau},h}\|_\Gamma^{s-1}\|\boldsymbol{u}_{\boldsymbol{\tau}} - \boldsymbol{w}_{\boldsymbol{\tau},h}\|_\Gamma \end{aligned}$$

from which after utilization of (4.3), the estimates in Theorem 2.1, Theorem 3.1, and young's inequality leads to

$$\begin{aligned}\|\nabla(\boldsymbol{w}_h-\boldsymbol{u}_h)\|^2+\|\boldsymbol{u}_{\boldsymbol{\tau}}-\boldsymbol{u}_{\boldsymbol{\tau},h}\|_\Gamma^2 \leq & c\,\|\boldsymbol{u}_{\boldsymbol{\tau}}-\boldsymbol{w}_{\boldsymbol{\tau},h}\|_\Gamma+c\|\nabla(\boldsymbol{u}-\boldsymbol{w}_h)\|^2\\ &+c\|q_h-p\|^2\,.\end{aligned}\tag{4.9}$$

Hence the result announced is obtained after utilization of triangle's inequality and (4.9). □

From Theorem 4.1, one concludes that the a priori error is independent of s, dominated by the interpolation error on the slip zone Γ, and depends on the regularity of the weak solution on the slip zone Γ. In the next result, we demonstrate that the a priori error may depend on s if one uses the monotonicity of $\mathcal{B}$ differently.

Theorem 4.2 *For $1<s<2$, let $(\boldsymbol{u},p)$ be the solution of (2.4) and $(\boldsymbol{u}_h,p_h)$ the solution of (3.2). Then there is a positive constant c independent of h such that*

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_1+\|p-p_h\|\leq c\inf_{\boldsymbol{v}_h\in\mathbb{V}_h}\|\boldsymbol{u}_{\boldsymbol{\tau}}-\boldsymbol{v}_{\boldsymbol{\tau},h}\|_\Gamma^{s-1}+c\inf_{\boldsymbol{v}_h\in\mathbb{V}_h}\|\nabla(\boldsymbol{u}-\boldsymbol{v}_h)\|+c\inf_{q_h\in M_h}\|q_h-p\|\,.$$

Proof. We follow the proof of Theorem 4.1.

We have

$$\begin{aligned}\int_\Gamma(\mathcal{B}\boldsymbol{u}_{\boldsymbol{\tau}}-\mathcal{B}\boldsymbol{u}_{\boldsymbol{\tau},h})\cdot(K\boldsymbol{w}_{\boldsymbol{\tau},h}-K\boldsymbol{u}_{\boldsymbol{\tau},h})d\sigma=&\int_\Gamma(\mathcal{B}\boldsymbol{u}_{\boldsymbol{\tau}}-\mathcal{B}\boldsymbol{w}_{\boldsymbol{\tau},h})\cdot(K\boldsymbol{w}_{\boldsymbol{\tau},h}-K\boldsymbol{u}_{\boldsymbol{\tau},h})d\sigma\\&+\int_\Gamma(\mathcal{B}\boldsymbol{w}_{\boldsymbol{\tau},h}-\mathcal{B}\boldsymbol{u}_{\boldsymbol{\tau},h})\cdot(K\boldsymbol{w}_{\boldsymbol{\tau},h}-K\boldsymbol{u}_{\boldsymbol{\tau},h})d\sigma\,.\end{aligned}\tag{4.10}$$

Thus from (4.10), inserting the second equation of (4.4) into the first one yields

$$\begin{aligned}&2\nu\|D(\boldsymbol{w}_h-\boldsymbol{u}_h)\|^2+\int_\Gamma(\mathcal{B}\boldsymbol{w}_{\boldsymbol{\tau},h}-\mathcal{B}\boldsymbol{u}_{\boldsymbol{\tau},h})\cdot(K\boldsymbol{w}_{\boldsymbol{\tau},h}-K\boldsymbol{u}_{\boldsymbol{\tau},h})\,d\sigma\\=&-(\mathcal{B}\boldsymbol{u}_{\boldsymbol{\tau}}-\mathcal{B}\boldsymbol{w}_{\boldsymbol{\tau},h},K\boldsymbol{w}_{\boldsymbol{\tau},h}-K\boldsymbol{u}_{\boldsymbol{\tau},h})_\Gamma+a(\boldsymbol{w}_h-\boldsymbol{u},\boldsymbol{w}_h-\boldsymbol{u}_h)\\&+b(\boldsymbol{w}_h-\boldsymbol{u},q_h-p_h)+b(\boldsymbol{u}_h-\boldsymbol{w}_h,q_h-p)\,.\end{aligned}\tag{4.11}$$

From (2.7), it was shown that (see [20]): there exist α_1,α_2 two positive constants such that

$$\alpha_1\frac{\|\boldsymbol{w}_{\boldsymbol{\tau},h}-\boldsymbol{u}_{\boldsymbol{\tau},h}\|_\Gamma^2}{\alpha_2+\|\boldsymbol{w}_{\boldsymbol{\tau},h}\|_\Gamma^{2-s}+\|\boldsymbol{u}_{\boldsymbol{\tau},h}\|_\Gamma^{2-s}}\leq\int_\Gamma(\mathcal{B}\boldsymbol{w}_{\boldsymbol{\tau},h}-\mathcal{B}\boldsymbol{u}_{\boldsymbol{\tau},h})\cdot(K\boldsymbol{w}_{\boldsymbol{\tau},h}-K\boldsymbol{u}_{\boldsymbol{\tau},h})\,d\sigma\,.\tag{4.12}$$

Now since $1<s<2$, we use (2.8), (4.12), triangle's inequality and (2.3) in (4.11) and get

$$\begin{aligned}&2\nu c\|\nabla(\boldsymbol{w}_h-\boldsymbol{u}_h)\|^2+\alpha_1\frac{\|\boldsymbol{w}_{\boldsymbol{\tau},h}-\boldsymbol{u}_{\boldsymbol{\tau},h}\|_\Gamma^2}{\alpha_2+c\|\boldsymbol{w}_{\boldsymbol{\tau},h}-\boldsymbol{u}_{\boldsymbol{\tau}}\|_\Gamma^{2-s}+c\|\boldsymbol{u}_{\boldsymbol{\tau}}\|_\Gamma^{2-s}+\|\boldsymbol{u}_{\boldsymbol{\tau},h}\|_\Gamma^{2-s}}\\\leq&c\int_\Gamma|K\boldsymbol{u}_{\boldsymbol{\tau}}-K\boldsymbol{w}_{\boldsymbol{\tau},h}|^{s-1}\,|K\boldsymbol{w}_{\boldsymbol{\tau},h}-K\boldsymbol{u}_{\boldsymbol{\tau},h}|\,d\sigma+2\nu\|\nabla(\boldsymbol{u}-\boldsymbol{w}_h)\|\|\nabla(\boldsymbol{w}_h-\boldsymbol{u}_h)\|\\&+\|\nabla(\boldsymbol{w}_h-\boldsymbol{u})\|\|q_h-p_h\|+\|\nabla(\boldsymbol{u}_h-\boldsymbol{w}_h)\|\|q_h-p\|\\\leq&c\,\|\boldsymbol{u}_{\boldsymbol{\tau}}-\boldsymbol{w}_{\boldsymbol{\tau},h}\|_\Gamma^{s-1}\,\|\boldsymbol{w}_{\boldsymbol{\tau},h}-\boldsymbol{u}_{\boldsymbol{\tau},h}\|_\Gamma+2\nu\|\nabla(\boldsymbol{u}-\boldsymbol{w}_h)\|\|\nabla(\boldsymbol{w}_h-\boldsymbol{u}_h)\|\\&+\|\nabla(\boldsymbol{w}_h-\boldsymbol{u})\|\|q_h-p_h\|+\|\nabla(\boldsymbol{u}_h-\boldsymbol{w}_h)\|\|q_h-p\|\,,\end{aligned}$$

from which after utilization of (4.3), the trace's inequality, and young's inequality leads to

$$\begin{aligned}&\|\nabla(\boldsymbol{w}_h-\boldsymbol{u}_h)\|^2+c_1\frac{\|\boldsymbol{w}_{\boldsymbol{\tau},h}-\boldsymbol{u}_{\boldsymbol{\tau},h}\|_\Gamma^2}{c+c\|\boldsymbol{w}_{\boldsymbol{\tau},h}-\boldsymbol{u}_{\boldsymbol{\tau}}\|_\Gamma^{2-s}+c\|\boldsymbol{u}_{\boldsymbol{\tau}}\|_\Gamma^{2-s}+c\|\boldsymbol{u}_{\boldsymbol{\tau},h}\|_\Gamma^{2-s}}\\\leq&c\,\|\boldsymbol{u}_{\boldsymbol{\tau}}-\boldsymbol{w}_{\boldsymbol{\tau},h}\|_\Gamma^{2(s-1)}+c\|\nabla(\boldsymbol{u}-\boldsymbol{w}_h)\|^2+c\|q_h-p\|^2\,,\end{aligned}\tag{4.13}$$

which together with the triangle's inequality gives the asserted result. □

Remark 4.1 *It is manifest that the rate of convergence in Theorem 4.2 depends on the parameter s and the regularity of the weak solution on the slip zone Γ. This is different to the conclusion obtained in Theorem 4.1. Hence the relations (4.6), and (4.10) play a big role in the analysis of the convergence of the finite element solution $(\boldsymbol{u}_h, p_h)$ of (3.2).*

Remark 4.2 *From (4.13), we deduce that*

$$\begin{aligned}\|\boldsymbol{w}_{\boldsymbol{\tau},h} - \boldsymbol{u}_{\boldsymbol{\tau},h}\|_\Gamma \leq & c\left(c + \|\boldsymbol{w}_{\boldsymbol{\tau},h} - \boldsymbol{u}_{\boldsymbol{\tau}}\|_\Gamma^{(2-s)/2} + \|\boldsymbol{u}_{\boldsymbol{\tau}}\|_\Gamma^{(2-s)/2} + \|\boldsymbol{u}_{\boldsymbol{\tau},h}\|_\Gamma^{(2-s)/2}\right) \\ & \times \left(\|\boldsymbol{u}_{\boldsymbol{\tau}} - \boldsymbol{w}_{\boldsymbol{\tau},h}\|_\Gamma^{s-1} + c\|\nabla(\boldsymbol{u} - \boldsymbol{w}_h)\| + c\|q_h - p\|\right).\end{aligned}$$

Remark 4.3 *(a) Taking $(\boldsymbol{w}_h, q_h) = (\Pi_h \boldsymbol{u}, r_h p)$, the right hand sides of the error estimates obtained in theorem 4.1 or theorem 4.2 tend to zero as h tends to zero.*

(b) Let $(\boldsymbol{u}, p)$ in $H^2(\Omega)^2 \times H^1(\Omega)$, we take $(\boldsymbol{w}_h, q_h) = (\Pi_h \boldsymbol{u}, r_h p)$ and we recall that (see [34, p.39]), there exists c such that

$$\|\boldsymbol{u}_{\boldsymbol{\tau}} - \boldsymbol{w}_{\boldsymbol{\tau},h}\|_\Gamma \leq c\,\|\boldsymbol{u} - \boldsymbol{w}_h\|^{1/2}\,\|\boldsymbol{u} - \boldsymbol{w}_h\|_1^{1/2}\,.$$

Then

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_1 + \|p - p_h\| \leq c \begin{cases} h^{\min(3(s-1)/2,1)} & \text{from theorem 4.2} \\ h^{3/4} & \text{from theorem 4.1}. \end{cases}$$

(c) If $\boldsymbol{u}_{\boldsymbol{\tau}} \in \boldsymbol{H}^2(\Gamma)$, then $\|\boldsymbol{w}_{\boldsymbol{\tau},h} - \boldsymbol{u}_{\boldsymbol{\tau}}\|_\Gamma \leq ch^2$. So assuming that the weak solution $(\boldsymbol{u}, p)$ is in $H^2(\Omega)^2 \times H^1(\Omega)$, we take $(\boldsymbol{w}_h, q_h) = (\Pi_h \boldsymbol{u}, r_h p)$, and we deduce that

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_1 + \|p - p_h\| \leq c \begin{cases} h^{\min(2(s-1),1)} & \text{from theorem 4.2} \\ h & \text{from theorem 4.1}. \end{cases}$$

4.2 A priori error analysis: the Navier-Stokes system

We address the convergence of the finite element approximation $(\boldsymbol{u}_h, p_h)$, solution of (3.7) by showing that the upper bound of the term $\|\boldsymbol{u} - \boldsymbol{u}_h\|_1 + \|p - p_h\|$ tend to zero when h tends to zero. This a priori error estimate is obtained with the assumption that the problems (3.7) and (2.13) are uniquely solvable. In this sub-section, we continue in the footsteps of Theorem 4.1, and Theorem 4.2. It holds that

Theorem 4.3 *Let $(\boldsymbol{u}, p)$ be the solution given by (2.13) and $(\boldsymbol{u}_h, p_h)$ the solution of (3.7). Assume that*

$$c\|\boldsymbol{f}\| \leq \nu^2\,.$$

Then there is a positive constant c independent of h such that for $1 < s < 2$

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_1 + \|\boldsymbol{u}_{\boldsymbol{\tau}} - \boldsymbol{u}_{\boldsymbol{\tau},h}\|_\Gamma + \|p - p_h\| \leq c \inf_{\boldsymbol{w}_h \in \mathbb{V}_h} \|\boldsymbol{u}_{\boldsymbol{\tau}} - \boldsymbol{w}_{\boldsymbol{\tau},h}\|_\Gamma^{1/2} + c \inf_{\boldsymbol{w}_h \in \mathbb{V}_h} \|\nabla(\boldsymbol{u} - \boldsymbol{w}_h)\| + c \inf_{q_h \in M_h} \|q_h - p\|\,.$$

Proof. **Step 1**: Inf-sup condition. We subtract (2.13) to (3.7), and taking into account the fact that $\mathbb{V}_h$ is a subset of $\mathbb{V}$, we arrived at

$$\begin{cases} \forall \boldsymbol{v}_h \in \mathbb{V}_h\,, \\ b(\boldsymbol{v}_h, p_h - q_h) = b(\boldsymbol{v}_h, p - q_h) + a(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{v}_h) + \displaystyle\int_\Gamma \left(|K\boldsymbol{u}_{\boldsymbol{\tau},h}|^{s-2} K\boldsymbol{u}_{\boldsymbol{\tau},h} - |K\boldsymbol{u}_{\boldsymbol{\tau}}|^{s-2} K\boldsymbol{u}_{\boldsymbol{\tau}}\right) \cdot K\boldsymbol{v}_{\boldsymbol{\tau},h} d\sigma \\ + \widetilde{d}(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{u}, \boldsymbol{v}_h) + \widetilde{d}(\boldsymbol{u}_h, \boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{v}_h)\,. \end{cases}$$

Using Cauchy-Schwarz's, Hölder's and trace's inequality, one gets

$$\begin{aligned} b(\boldsymbol{v}_h, p_h - q_h) \leq & 2\nu\|\nabla(\boldsymbol{u}-\boldsymbol{u}_h)\|\|\boldsymbol{v}_h\|_1 + \|q_h - p\|\|\boldsymbol{v}_h\|_1 + c\|\boldsymbol{u}-\boldsymbol{u}_h\|_1 \left(\|\boldsymbol{u}\|_1 + \|\boldsymbol{u}_h\|_1\right)\|\boldsymbol{v}_h\|_1 \\ & + \int_\Gamma \left(|K\boldsymbol{u}_{\boldsymbol{\tau},h}|^{s-2} K\boldsymbol{u}_{\boldsymbol{\tau},h} - |K\boldsymbol{u}_{\boldsymbol{\tau}}|^{s-2} K\boldsymbol{u}_{\boldsymbol{\tau}} \right) \cdot K\boldsymbol{v}_{\boldsymbol{\tau},h} d\sigma \end{aligned}$$

which after inserting it in (4.1) and making use of (2.8), and the estimates in proposition 2.2 and proposition 3.1, one has

$$\begin{aligned} \widetilde{\beta}\|p_h - q_h\| &\leq 2\nu\|\nabla(\boldsymbol{u}-\boldsymbol{u}_h)\| + \|q_h - p\| + c\|\boldsymbol{u}-\boldsymbol{u}_h\|_1 \left(\|\boldsymbol{u}\|_1 + \|\boldsymbol{u}_h\|_1\right) \\ &\leq 2\nu\|\nabla(\boldsymbol{u}-\boldsymbol{u}_h)\| + \|q_h - p\| + \frac{c}{\nu}\|\boldsymbol{f}\|\|\boldsymbol{u}-\boldsymbol{u}_h\|_1 . \end{aligned} \tag{4.14}$$

Step 2: Monotonicity. We let $\mathcal{B}\boldsymbol{u}_{\boldsymbol{\tau}} = |K\boldsymbol{u}_{\boldsymbol{\tau}}|^{s-2} K\boldsymbol{u}_{\boldsymbol{\tau}}$. We subtract (2.13) to (3.7) and obtain

$$\begin{cases} a(\boldsymbol{u}-\boldsymbol{u}_h, \boldsymbol{v}_h) + \widetilde{d}(\boldsymbol{u}-\boldsymbol{u}_h, \boldsymbol{u}, \boldsymbol{v}_h) + \widetilde{d}(\boldsymbol{u}_h, \boldsymbol{u}-\boldsymbol{u}_h, \boldsymbol{v}_h) \\ + \left(\mathcal{B}\boldsymbol{u}_{\boldsymbol{\tau}} - \mathcal{B}\boldsymbol{u}_{\boldsymbol{\tau},h}, K\boldsymbol{v}_{\boldsymbol{\tau},h}\right)_\Gamma = b(\boldsymbol{v}_h, p - p_h) , \\ \\ b(\boldsymbol{u}-\boldsymbol{u}_h, q_h) = 0 . \end{cases} \tag{4.15}$$

Let $\boldsymbol{w}_h \in \mathbb{V}_h$, we take $\boldsymbol{v}_h = \boldsymbol{w}_h - \boldsymbol{u}_h$ in (4.15), and after using linearity of operators involved, one gets

$$\begin{cases} a(\boldsymbol{u}-\boldsymbol{w}_h, \boldsymbol{w}_h-\boldsymbol{u}_h) + a(\boldsymbol{w}_h-\boldsymbol{u}_h, \boldsymbol{w}_h-\boldsymbol{u}_h) + \displaystyle\int_\Gamma \left(\mathcal{B}\boldsymbol{u}_{\boldsymbol{\tau}} - \mathcal{B}\boldsymbol{u}_{\boldsymbol{\tau},h}\right)\cdot\left(K\boldsymbol{u}_{\boldsymbol{\tau}} - K\boldsymbol{u}_{\boldsymbol{\tau},h}\right) d\sigma \\ + \widetilde{d}(\boldsymbol{u}-\boldsymbol{w}_h, \boldsymbol{u}, \boldsymbol{w}_h-\boldsymbol{u}_h) + \widetilde{d}(\boldsymbol{w}_h-\boldsymbol{u}_h, \boldsymbol{u}, \boldsymbol{w}_h-\boldsymbol{u}_h) \\ + \widetilde{d}(\boldsymbol{u}_h, \boldsymbol{u}-\boldsymbol{w}_h, \boldsymbol{w}_h-\boldsymbol{u}_h) + \widetilde{d}(\boldsymbol{u}_h, \boldsymbol{w}_h-\boldsymbol{u}_h, \boldsymbol{w}_h-\boldsymbol{u}_h) \\ = -\displaystyle\int_\Gamma \left(\mathcal{B}\boldsymbol{u}_{\boldsymbol{\tau}} - \mathcal{B}\boldsymbol{u}_{\boldsymbol{\tau},h}\right)\cdot\left(K\boldsymbol{w}_{\boldsymbol{\tau},h} - K\boldsymbol{u}_{\boldsymbol{\tau}}\right) d\sigma + b(\boldsymbol{w}_h-\boldsymbol{u}_h, p-p_h) \\ \\ b(\boldsymbol{w}_h-\boldsymbol{u}_h, p-p_h) = b(\boldsymbol{w}_h-\boldsymbol{u}, q_h-p_h) + b(\boldsymbol{u}_h-\boldsymbol{w}_h, q_h-p) . \end{cases} \tag{4.16}$$

Thus inserting the second equation into the first one gives

$$\begin{aligned} & 2\nu\|D(\boldsymbol{w}_h-\boldsymbol{u}_h)\|^2 + \int_\Gamma \left(\mathcal{B}\boldsymbol{u}_{\boldsymbol{\tau}} - \mathcal{B}\boldsymbol{u}_{\boldsymbol{\tau},h}\right)\cdot\left(K\boldsymbol{u}_{\boldsymbol{\tau}} - K\boldsymbol{u}_{\boldsymbol{\tau},h}\right) d\sigma \\ & = -\widetilde{d}(\boldsymbol{u}-\boldsymbol{w}_h, \boldsymbol{u}, \boldsymbol{w}_h-\boldsymbol{u}_h) - \widetilde{d}(\boldsymbol{w}_h-\boldsymbol{u}_h, \boldsymbol{u}, \boldsymbol{w}_h-\boldsymbol{u}_h) - \widetilde{d}(\boldsymbol{u}_h, \boldsymbol{u}-\boldsymbol{w}_h, \boldsymbol{w}_h-\boldsymbol{u}_h) \\ & - \int_\Gamma \left(\mathcal{B}\boldsymbol{u}_{\boldsymbol{\tau}} - \mathcal{B}\boldsymbol{u}_{\boldsymbol{\tau},h}\right)\cdot\left(K\boldsymbol{w}_{\boldsymbol{\tau},h} - K\boldsymbol{u}_{\boldsymbol{\tau}}\right) d\sigma - a(\boldsymbol{u}-\boldsymbol{w}_h, \boldsymbol{w}_h-\boldsymbol{u}_h) \\ & + b(\boldsymbol{w}_h-\boldsymbol{u}, q_h-p_h) + b(\boldsymbol{u}_h-\boldsymbol{w}_h, q_h-p) . \end{aligned} \tag{4.17}$$

Now since $1 < s < 2$, then using (2.8) and (2.3) in (4.17), one obtains

$$\begin{aligned} & 2\nu c\|\nabla(\boldsymbol{w}_h-\boldsymbol{u}_h)\|^2 + \int_\Gamma \left(\mathcal{B}\boldsymbol{u}_{\boldsymbol{\tau}} - \mathcal{B}\boldsymbol{u}_{\boldsymbol{\tau},h}\right)\cdot\left(K\boldsymbol{u}_{\boldsymbol{\tau}} - K\boldsymbol{u}_{\boldsymbol{\tau},h}\right) d\sigma \\ \leq & \frac{c}{\nu}\|\boldsymbol{f}\|\|\nabla(\boldsymbol{u}-\boldsymbol{w}_h)\|\|\nabla(\boldsymbol{w}_h-\boldsymbol{u}_h)\| + \frac{c}{\nu}\|\boldsymbol{f}\|\|\nabla(\boldsymbol{w}_h-\boldsymbol{u}_h)\|^2 + \frac{c}{\nu}\|\boldsymbol{f}\|\|\nabla(\boldsymbol{u}-\boldsymbol{w}_h)\|\|\nabla(\boldsymbol{w}_h-\boldsymbol{u}_h)\| \\ & + c\|K\|^s_{L^\infty(\Gamma)} \|\boldsymbol{u}_{\boldsymbol{\tau}} - \boldsymbol{u}_{\boldsymbol{\tau},h}\|^{s-1}_{L^2(\Gamma)} \|\boldsymbol{w}_{\boldsymbol{\tau},h} - \boldsymbol{u}_{\boldsymbol{\tau}}\|_{L^2(\Gamma)} + 2\nu\|\nabla(\boldsymbol{u}-\boldsymbol{w}_h)\|\|\nabla(\boldsymbol{w}_h-\boldsymbol{u}_h)\| \\ & + \|\nabla(\boldsymbol{w}_h-\boldsymbol{u})\|\|q_h-p_h\| + \|\nabla(\boldsymbol{u}_h-\boldsymbol{w}_h)\|\|q_h-p\| , \end{aligned}$$

from which after utilization of (4.14), (4.12), the trace's inequality, and young's inequality leads to

$$\begin{aligned} & \left(\nu - \frac{c}{\nu}\|\boldsymbol{f}\|\right)\|\nabla(\boldsymbol{w}_h-\boldsymbol{u}_h)\|^2 + \alpha_1 \frac{\|\boldsymbol{u}_{\boldsymbol{\tau},h} - \boldsymbol{u}_{\boldsymbol{\tau}}\|^2_\Gamma}{c + c\|\boldsymbol{u}_{\boldsymbol{\tau}}\|^{2-s}_\Gamma + \|\boldsymbol{u}_{\boldsymbol{\tau},h}\|^{2-s}_\Gamma} \\ \leq & c\left\|\boldsymbol{u}_{\boldsymbol{\tau}} - \boldsymbol{w}_{\boldsymbol{\tau},h}\right\|_\Gamma + c\|\nabla(\boldsymbol{u}-\boldsymbol{w}_h)\|^2 + c\|q_h-p\|^2 , \end{aligned}$$

which together with the triangle's inequality gives the result announced.

□

Following the proof of Theorem 4.2, we can show that

Theorem 4.4 *Let $(\boldsymbol{u}, p)$ be the solution given by (2.13) and $(\boldsymbol{u}_h, p_h)$ the solution of (3.7). Assume that*

$$c\|\boldsymbol{f}\| \leq \nu^2 .$$

Then there is a positive constant c independent of h such that for $1 < s < 2$

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_1 + \|p - p_h\| \leq c \inf_{\boldsymbol{w}_h \in \mathbb{V}_h} \|\boldsymbol{u}_{\boldsymbol{\tau}} - \boldsymbol{w}_{\boldsymbol{\tau},h}\|_\Gamma^{s-1} + c \inf_{\boldsymbol{w}_h \in \mathbb{V}_h} \|\nabla(\boldsymbol{u} - \boldsymbol{w}_h)\| + c \inf_{q_h \in M_h} \|q_h - p\| .$$

The next section is concerned with the solution strategy for (3.2) and (3.7), and their numerical simulations.

5 Iterative schemes and Simulations

The finite element problems (3.2) and (3.7) are nonlinear. Hence iterative/incremental schemes need to be formulated for their effective resolution. It should be noted that since s is in $(1, 2)$, the expression $(|K\boldsymbol{u}_{\boldsymbol{\tau},h}|^{s-2}K\boldsymbol{u}_{\boldsymbol{\tau},h}, K\boldsymbol{v}_{\boldsymbol{\tau},h})_\Gamma$ has a poor differentiability. So a numerical solution of (3.2) or (3.7) with Newton's method or conjugate gradient may not converge at all. A direct strategy to make the above systems less nonlinear and weaken the nonlinearities in the systems is to linearize the nonlinear terms by considering the approximations

$$(|K\boldsymbol{u}_{\boldsymbol{\tau},h}^{n+1}|^{s-2}K\boldsymbol{u}_{\boldsymbol{\tau},h}^{n+1}, K\boldsymbol{v}_{\boldsymbol{\tau},h})_\Gamma \approx (|K\boldsymbol{u}_{\boldsymbol{\tau},h}^{n}|^{s-2}K\boldsymbol{u}_{\boldsymbol{\tau},h}^{n+1}, K\boldsymbol{v}_{\boldsymbol{\tau},h})_\Gamma,$$

$$\widetilde{d}(\boldsymbol{u}_h^{n+1}, \boldsymbol{u}_h^{n+1}, \boldsymbol{v}_h) \approx \widetilde{d}(\boldsymbol{u}_h^{n}, \boldsymbol{u}_h^{n+1}, \boldsymbol{v}_h) .$$

In the lines that follows, we concentrate on the iterative method for solving (3.7) since the nonlinear system of equations (3.2) can be viewed as a special case of (3.7).

5.1 Iterative scheme for (3.7)

We solve (3.7) by adopting the following strategy. Let $\boldsymbol{u}_h^0 \in \mathbb{V}_h$ be a given initial guess, we generate the sequence $(\boldsymbol{u}^n, p_h^n)$ for $n \geq 1$, until an adequate stopping condition is satisfied, by computing $(\boldsymbol{u}_h^{n+1}, p_h^{n+1})$ solution of

$$\begin{cases} \text{for all } (\boldsymbol{v}_h, q_h) \in \mathbb{V}_h \times M_h , \\ a(\boldsymbol{u}_h^{n+1}, \boldsymbol{v}_h) + \left(|K\boldsymbol{u}_{\boldsymbol{\tau},h}^{n}|^{s-2}K\boldsymbol{u}_{\boldsymbol{\tau},h}^{n+1} \ , \ K\boldsymbol{v}_{\boldsymbol{\tau},h}\right)_\Gamma + \widetilde{d}(\boldsymbol{u}_h^{n}, \boldsymbol{u}_h^{n+1}, \boldsymbol{v}_h) - b(\boldsymbol{v}_h, p_h^{n+1}) = \ell(\boldsymbol{v}_h) , \\ b(\boldsymbol{u}_h^{n+1}, q_h) = 0 . \end{cases} \tag{5.1}$$

After the convergence of the system (5.1), we get the numerical solution $\boldsymbol{u}_h$ of the discrete system (3.7). One of the principal interests of the scheme (5.1) (which by the way is a fixed point algorithm) is that it converges to the unique fixed point for any starting point, but that convergence is slow.

Remark 5.1 *Similar to the iterative scheme (5.1), the iterative scheme we may consider for (3.2) is the following: Let $\boldsymbol{u}_h^0$ a given element of $\mathbb{V}_h$. For $n \geq 1$ and having $\boldsymbol{u}_h^n$ in $\mathbb{V}_h$, we compute until*

an adequate stopping condition is satisfied, $(\boldsymbol{u}_h^{n+1}, p_h^{n+1})$ *solution of*

$$\begin{cases} \text{for all } (\boldsymbol{v}_h, q_h) \in \mathbb{V}_h \times M_h\,, \\ a(\boldsymbol{u}_h^{n+1}, \boldsymbol{v}_h) + \displaystyle\int_\Gamma |K\boldsymbol{u}_{\boldsymbol{\tau},h}^n|^{s-2} K\boldsymbol{u}_{\boldsymbol{\tau},h}^{n+1} \cdot K\boldsymbol{v}_{\boldsymbol{\tau},h} d\sigma - b(\boldsymbol{v}_h, p_h^{n+1}) = \ell(\boldsymbol{v}_h)\,, \\ b(\boldsymbol{u}_h^{n+1}, q_h) = 0\,. \end{cases} \tag{5.2}$$

After the convergence of the system (5.2)*, we get the numerical solution* $\boldsymbol{u}_h$ *of the discrete system (3.2).*

The variational problem (5.1) is linear, and from Babuska-Brezzi's theory for mixed problems (see [29, 30]), $(\boldsymbol{u}_h^{n+1}, p_h^{n+1})$ is well defined. Continuing on the qualitative analysis on (5.1), we claim that

Lemma 5.1 *(see [36]) The algorithm (5.1) is consistent with (3.7) in the sense that if* $(\boldsymbol{u}_h^1, p_h^1) = (\boldsymbol{u}_h, p_h)$ *with* $(\boldsymbol{u}_h, p_h)$ *solution of (3.7), then for all* $n \geq 2$, $(\boldsymbol{u}_h^n, p_h^n) = (\boldsymbol{u}_h, p_h)$.

Proof. It is done by indiction on n. So, we assume that $(\boldsymbol{u}_h^n, p_h^n) = (\boldsymbol{u}_h, p_h)$, and we will show that $(\boldsymbol{u}_h^{n+1}, p_h^{n+1}) = (\boldsymbol{u}_h, p_h)$.

We use the induction hypothesis, take the difference between (5.1) and (3.7) and obtain that

$$\begin{cases} \text{for all } (\boldsymbol{v}_h, q_h) \in \mathbb{V}_h \times M_h\,, \\ a(\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h, \boldsymbol{v}_h) + \Big(|K\boldsymbol{u}_{\boldsymbol{\tau},h}|^{s-2} K\boldsymbol{u}_{\boldsymbol{\tau},h}^{n+1} - |K\boldsymbol{u}_{\boldsymbol{\tau},h}|^{s-2} K\boldsymbol{u}_{\boldsymbol{\tau},h}, K\boldsymbol{v}_{\boldsymbol{\tau},h}\Big)_\Gamma \\ + \widetilde{d}(\boldsymbol{u}_h, \boldsymbol{u}_h^{n+1}, \boldsymbol{v}_h) - \widetilde{d}(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{v}_h) - b(\boldsymbol{v}_h, p_h^{n+1} - p_h) = 0\,, \\ b(\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h, q_h) = 0\,. \end{cases} \tag{5.3}$$

We take $\boldsymbol{v}_h = \boldsymbol{u}_h - \boldsymbol{u}_h^{n+1}$ and $q_h = p_h - p_h^{n+1}$ in (5.3), and obtain

$$a(\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h, \boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h) + \int_\Gamma |K\boldsymbol{u}_{\boldsymbol{\tau},h}|^{s-2} \left|K\boldsymbol{u}_{\boldsymbol{\tau},h}^{n+1} - K\boldsymbol{u}_{\boldsymbol{\tau},h}\right|^2 d\sigma = 0$$

from which we deduce that $\boldsymbol{u}_h^{n+1} = \boldsymbol{u}_h$.
Equations (5.3) become

$$\text{for all } \boldsymbol{v}_h \in \mathbb{V}_h\,, \quad b(\boldsymbol{v}_h, p_h^{n+1} - p_h) = 0\,. \tag{5.4}$$

We use the inf-sup condition on $b(\cdot,\cdot)$ and one obtains that $p_h^{n+1} = p_h$. □

In order to justify why we use the iterative scheme (5.1) (implemented in paragraph 5.2) to compute the solution of (3.7), we shall prove a convergence result. First we state that the sequence generated by Algorithm (5.1) is bounded.

Lemma 5.2 *Let* $(\boldsymbol{u}_h^{n+1}, p_h^{n+1})$ *the solution of the algorithm (5.1). Then there exists a constant* c *independent of* h *such that*

$$\|\nabla \boldsymbol{u}_h^{n+1}\| + \|p_h^{n+1}\| \leq c\,.$$

Proof. We take $\boldsymbol{v}_h = \boldsymbol{u}_h^{n+1}$ in (5.1) and obtain

$$a(\boldsymbol{u}_h^{n+1}, \boldsymbol{u}_h^{n+1}) + \int_\Gamma |K\boldsymbol{u}_{\boldsymbol{\tau},h}^n|^{s-2} \left|K\boldsymbol{u}_{\boldsymbol{\tau},h}^{n+1}\right|^2 d\sigma = \ell(\boldsymbol{u}_h^{n+1}) \leq c\|\boldsymbol{f}\|\|D\boldsymbol{u}_h^{n+1}\|$$

which implies that

$$\|\nabla \boldsymbol{u}_h^{n+1}\| \leq \frac{c}{\nu}\|\boldsymbol{f}\|\,.$$

From the inf-sup condition, we obtain that

$$\|p_h^{n+1}\| \leq c\|\boldsymbol{f}\| + \frac{c}{\nu^2}\|\boldsymbol{f}\|^2 .$$

□

We have the following abstract convergence result

Theorem 5.1 *Let $(\boldsymbol{u}_h^{n+1}, p_h^{n+1}) \in \mathbb{V}_h \times M_h$ the solution of (5.1). Let $(\boldsymbol{u}_h, p_h)$ solution of (3.7). Assume that there exists c positive constant independent of h such that $\|\boldsymbol{f}\| \leq \dfrac{\nu^2}{c}$. Then*

$$\lim_{n\to\infty} \boldsymbol{u}_h^n = \boldsymbol{u}_h \text{ weakly in} \mathbb{V} .$$

Proof. By virtue of the bound obtained in Lemma 5.2, there is a subsequence, still denoted by $\{\boldsymbol{u}_h^n\}_n$ such that

$$\lim_{n\to\infty} \boldsymbol{u}_h^n = \widetilde{\boldsymbol{u}}_h \quad \text{weakly in} \quad \boldsymbol{H}^1(\Omega) . \tag{5.5}$$

Clearly, we need to show that $\widetilde{\boldsymbol{u}}_h = \boldsymbol{u}_h$, meaning that

$$\begin{cases} \text{for all } \boldsymbol{v}_h \in \mathbb{V}_{\mathrm{div},h} , \\ a(\widetilde{\boldsymbol{u}}_h, \boldsymbol{v}_h) + \left(|K\widetilde{\boldsymbol{u}}_{\boldsymbol{\tau},h}|^{s-2}K\widetilde{\boldsymbol{u}}_{\boldsymbol{\tau},h} \ , \ K\boldsymbol{v}_{\boldsymbol{\tau},h}\right)_\Gamma + \widetilde{d}(\widetilde{\boldsymbol{u}}_h, \widetilde{\boldsymbol{u}}_h, \boldsymbol{v}_h) = \ell(\boldsymbol{v}_h) . \end{cases} \tag{5.6}$$

Actually, since $\mathbb{V}_{\mathrm{div},h}$ is closed, it follows from (5.5) that $\widetilde{\boldsymbol{u}}_h \in \mathbb{V}_{\mathrm{div},h}$. Owing to the compactness of the imbedding of $H^1(\Omega)$ into $L^4(\Omega)$, there exists a subsequence, still denoted by $\{\boldsymbol{u}_h^n\}_n$, which converges weakly to $\widetilde{\boldsymbol{u}}_h \in \boldsymbol{H}^1(\Omega)$ and strongly in $\boldsymbol{L}^4(\Omega)$. For $\boldsymbol{v}_h \in \mathbb{V}_{\mathrm{div},h}$, passing to the limit on n in (5.1) is direct for the linear term because one only needs weak convergence, but for the term $\widetilde{d}(\cdot,\cdot,\cdot)$, one needs the strong convergence of $\boldsymbol{u}_h^n$ on $\boldsymbol{L}^4(\Omega)$ (see [29, Chap 5, Thm 1.4]). In fact

$$\begin{aligned} \left|\widetilde{d}(\boldsymbol{u}_k^n, \boldsymbol{u}_h^{n+1}, \boldsymbol{v}_h) - \widetilde{d}(\widetilde{\boldsymbol{u}}_h, \widetilde{\boldsymbol{u}}_h, \boldsymbol{v}_h)\right| &= \left|\widetilde{d}(\boldsymbol{u}_k^n - \widetilde{\boldsymbol{u}}_h, \boldsymbol{u}_h^{n+1}, \boldsymbol{v}_h) + \widetilde{d}(\widetilde{\boldsymbol{u}}_h, \boldsymbol{u}_h^{n+1} - \widetilde{\boldsymbol{u}}_h, \boldsymbol{v}_h)\right| \\ &\leq \|\boldsymbol{u}_k^n - \widetilde{\boldsymbol{u}}_h\|_{L^4(\Omega)}\|\nabla \boldsymbol{u}_h^{n+1}\|\|\boldsymbol{v}_h\|_{L^4(\Omega)} + \left|\widetilde{d}(\widetilde{\boldsymbol{u}}_h, \boldsymbol{u}_h^{n+1} - \widetilde{\boldsymbol{u}}_h, \boldsymbol{v}_h)\right| \\ &\longrightarrow 0 . \end{aligned}$$

Next, for the term on Γ, we deduce from the weak convergence of $\boldsymbol{u}_h^n$ on $\boldsymbol{H}^1(\Omega)$, that

$$\boldsymbol{u}_h^n \to \widetilde{\boldsymbol{u}}_h \quad \text{strongly in} \quad \boldsymbol{L}^2(\Omega) \quad ,$$

which together with the continuity of the trace implies that

$$\boldsymbol{u}_h^n|_\Gamma \to \widetilde{\boldsymbol{u}}_{\boldsymbol{\tau},h}|_\Gamma \text{ weakly in } \boldsymbol{L}^2(\Gamma).$$

But from [31, Chap 2], we have that

$$\lim_n \left(|K\boldsymbol{u}_{\boldsymbol{\tau},h}^n|^{s-2}K\boldsymbol{u}_{\boldsymbol{\tau},h}^{n+1} \ , \ K\boldsymbol{v}_{\boldsymbol{\tau},h}\right)_\Gamma = \left(|K\widetilde{\boldsymbol{u}}_{\boldsymbol{\tau},h}|^{s-2}K\widetilde{\boldsymbol{u}}_{\boldsymbol{\tau},h} \ , \ K\boldsymbol{v}_{\boldsymbol{\tau},h}\right)_\Gamma .$$

Hence, $\widetilde{\boldsymbol{u}}_h$ satisfy (5.6), which has a unique $\boldsymbol{u}_h$ if $\|\boldsymbol{f}\| \leq \dfrac{\nu^2}{c}$. Then $\widetilde{\boldsymbol{u}}_h = \boldsymbol{u}_h$

Since the solution of (5.6) is unique (under the condition $\|\boldsymbol{f}\| \leq \dfrac{\nu^2}{c}$), then the entire sequence converges weakly to $\boldsymbol{u}_h$. □

5.2 Simulations

In this section we study the numerical behaviour of the algorithm (5.1) which is more general than the algorithm (5.2). The test problems used are designed to illustrate the numerical behaviour of the algorithm in two dimensions.

All the numerical simulations showed in this section are performed with FreeFem code [37] by taking $\boldsymbol{u}_h^0 = \mathbf{0}$ and by using the following stopping criterion:

$$\frac{||\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h^n||_1}{||\boldsymbol{u}_h^{n+1}||_1} \leq 10^{-5}. \tag{5.7}$$

In all examples, $\mathbb{V}_h$ and M_h are given by (3.1).

5.2.1 Rate of convergence

In this first example, we consider the following geometry $\Omega = (0,1)^2$ where each edge is divided into N segments of equal length. Thus the corresponding mesh contains $2N^2$ elements. We set $\nu = 1$, $K = I$ and $\boldsymbol{f} = (x^2 + y^2, 1)^T$. We take the anisotropic slip boundary condition (1.7), on right and top boundaries of Ω, that is on $\Gamma = \{(x,y)|\ \ x = 1 \text{ or } y = 1\} = \overline{\Gamma_2 \cup \Gamma_3}$, and the homogeneous Dirichlet boundary condition $\boldsymbol{u} = \mathbf{0}$ on the bottom and left boundaries $S = \overline{\Gamma_1 \cup \Gamma_4}$. We have $s \in (1,2)$, and we are interested in computing the rate of convergence of the finite element solution $(\boldsymbol{u}_h, p_h)$. We recall that (see Theorem 4.3)

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_1 + \|p - p_h\| \leq ch \quad \text{if} \quad (\boldsymbol{u}, p) \in \boldsymbol{H}^2(\Omega) \times H^1(\Omega)\,, \boldsymbol{u}_{\boldsymbol{\tau}} \in \boldsymbol{H}^2(\Gamma)\,. \tag{5.8}$$

As we do not have the exact solution $\boldsymbol{u}$ corresponding to this case, we will approximate it by computing the numerical solution of the iterative system (5.1) for $N = 400$ which will be designated as the reference solution say, $(\boldsymbol{u}_{ref}, p_{ref})$ and which depends on $s \in (1,2)$. Next, we compute the numerical solution for $N = 10, 15, 20, \ldots, 50$. Fig. 1 show the graph of the relative error

$$E_{err} = \frac{||\boldsymbol{u}_{ref} - \boldsymbol{u}_h||_1 + ||p_{ref} - p_h||}{||\boldsymbol{u}_{ref}||_1 + ||p_{ref}||}$$

with respect to $h = \dfrac{1}{N}$ in logarithmic scale for $s = 1.2, 1.5, 1.8$. These lines have slopes $1.07, 1.05, 1.04$ respectively. We can deduce for this particular case that the numerical slope is one and is independent of s. In all these cases, the numerical slopes satisfy the condition (5.8)

5.2.2 Flow in rectangle

In this second example, we consider the following geometry $\Omega = (0, l_1) \times (0, l_2)$ with $l_1 = 10, l_2 = 1$. We divide the edges of the boundary $\partial\Omega$ into segments of equal lengths, and we define $N = 40$ to be the number of such mesh segments per unit length. We set $\nu = 1$, $\boldsymbol{f} = \mathbf{0}$, $s = 3/2$ or $s = 4/3$, and the tensor K one of this two cases:

$$K_1 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad K_2 = \begin{bmatrix} 5 & -1 \\ -1 & 4 \end{bmatrix}.$$

We consider the isotropic slip boundary condition (1.4), (1.7), on bottom (Γ_1) and top (Γ_3) boundaries of Ω, that is on $\Gamma = \{(x,y)|\ \ y = 0 \text{ or } y = l_2\} = \overline{\Gamma_1 \cup \Gamma_3}$. A non-homogeneous Dirichlet boundary condition is imposed on the left (Γ_4) and right (Γ_2) boundaries of Ω, that is $S = \{(x,y)|\ \ x = 0 \text{ or } x = l_1\} = \overline{\Gamma_2 \cup \Gamma_4}$:

$$\boldsymbol{u}|_{\Gamma_2} = \boldsymbol{u}|_{\Gamma_4} = \begin{bmatrix} y(1-y) \\ -y(1-y) \end{bmatrix}. \tag{5.9}$$

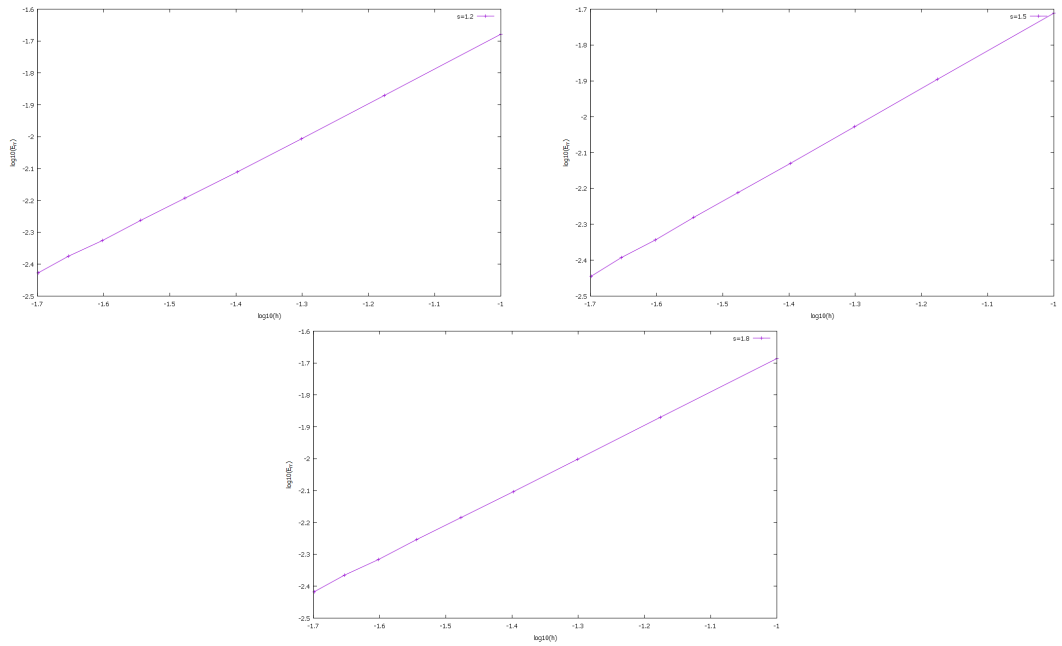

Figure 1: The *a priori error* estimates with respect to the mesh step h in logarithmic scale for $s = 1.2, 1.5, 1.8$.

Thus, the velocity on S has a parabolic profile with a critical point at $y = 1/2$ with $\boldsymbol{u}|_S = (1/4, -1/4)^T$. Thus it is manifest that anisotropic slip is expected on the upper and lower boundaries. In Figure 2 we have shown the velocity profiles for $s = 3/2$ and $K = K_1$ and $K = K_2$. On S, one notes the parabolic profiles of the velocity. The fluid along Γ and far away from S is unidirectional.

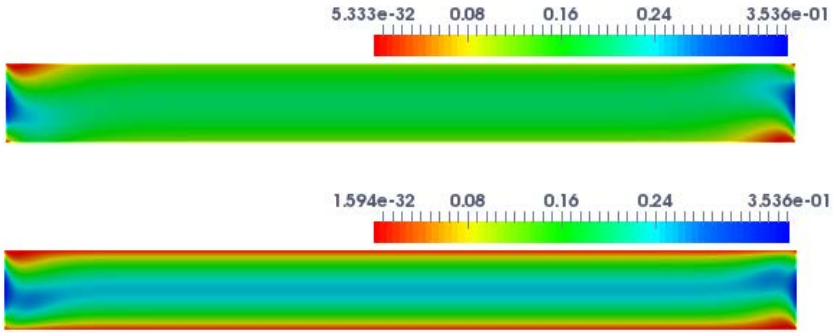

Figure 2: The velocity plots for $s = 3/2$, $K = K_1$ (up) and $K = K_2$ (down).

Figure 3 shows the graphs $x \to u_1(x, 1/2)$ and $x \to u_2(x, 1/2)$ for $s = 3/2$ and $K = K_1$ or K_2. One notes that u_1 (resp. u_2) is equal to $1/4$ (resp. $-1/4$) on $x = 0$ (resp. $x = 10$) which is coherent with the boundary condition (5.9) for $y = 1/2$. Furthermore, Figure 4 plots the tangential velocity on Γ_1 and Γ_3 ($\boldsymbol{u_\tau}(x, 0)$ and $\boldsymbol{u_\tau}(x, 1)$) for $s = 3/2$ and $K = K_1$.
Next, Figure 5 show the velocity plots on the line $y = 1/2$ for $K = K_1$ and $s = 4/3$ or $3/2$. We note the dependence on s of the velocity on the line $y = 1/2$.

5.2.3 Driven cavity flow

In this paragraph, we consider the numerical results for the Lid Driven cavity with the Navier-Stokes flows under the power law slip boundary condition by the numerical scheme (5.1). This

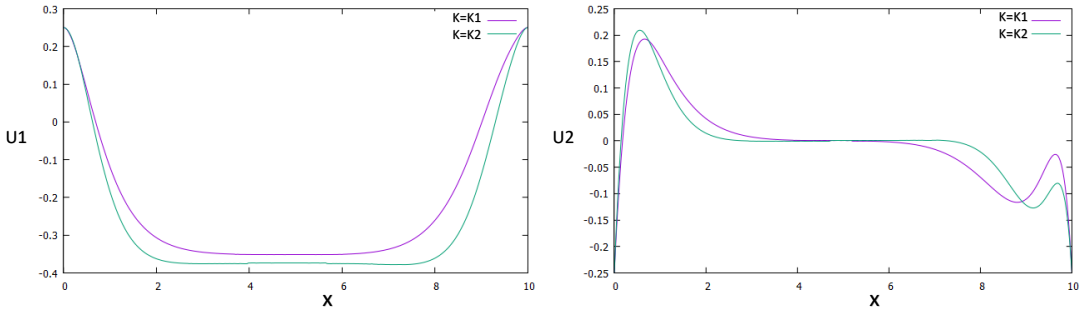

Figure 3: The numerical velocity (u_1 right, u_2 left) on the line $y = 1/2$ for $s = 3/2$, $K = K_1$ and $K = K_2$.

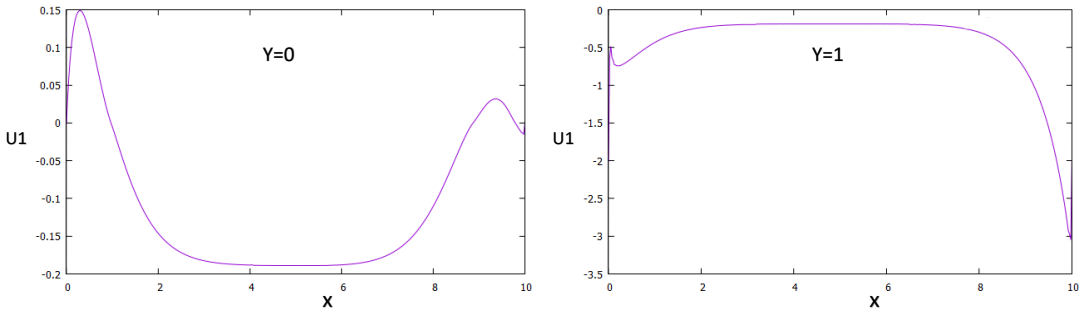

Figure 4: The tangential velocity on the line $y = 0$ (left) and $y = l_2$ (right) for $s = 3/2$ and $K = K_1$.

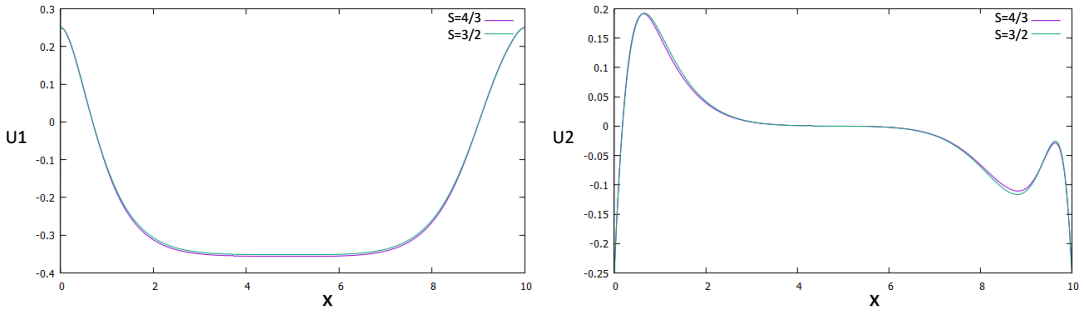

Figure 5: The numerical velocity (u_1 right, u_2 left) on the line $y = 1/2$ for $K = K_1$, $s = 4/3$ or $s = 3/2$.

problem is very popular example that can be seen in [38, 39, 40]. The fluid is confined in the domain $\Omega = (0,1)^2$ and the boundary condition for the velocity field on the top is given by

$$\boldsymbol{u} = (16\,x^2(x-1)^2, 0)^T \quad \text{on} \quad y = 1 \quad 0 < x < 1\,.$$

Thus one has a smooth start of the flow, and the lid velocity attains its maximum, $\boldsymbol{u} = (1,0)^T$, at the centre $x = 1/2$. This boundary condition has the effect of avoiding local singularities at the top-right and top-left corners as it ensures both the velocity and velocity gradient vanish at the corners. On the other parts of the boundary, we consider the isotropic slip boundary condition. In all the numerical results considered in this section, we set $K = I$, $\boldsymbol{f} = \boldsymbol{0}$ and $\nu = 1/Re$ where Re designates the Reynolds number.
Here also, each edge is divided into $N = 100$ segments of equal length. Thus the corresponding mesh contains $2N^2$ elements.
We begin ours numerical results with $Re = 1$. Figures 6 and 7 shows the numerical velocity for $s = 1.1, 1.5$ and 1.9. One notes the formation of a vortex which is typical for this problem. Furthermore, we show in figure 8 the first and second components of the velocity on the line $y = 1/2$ for $s = 1.1, 1.5$ and 1.9. Again, one observes the dependence on s of the velocity on the line $y = 1/2$.

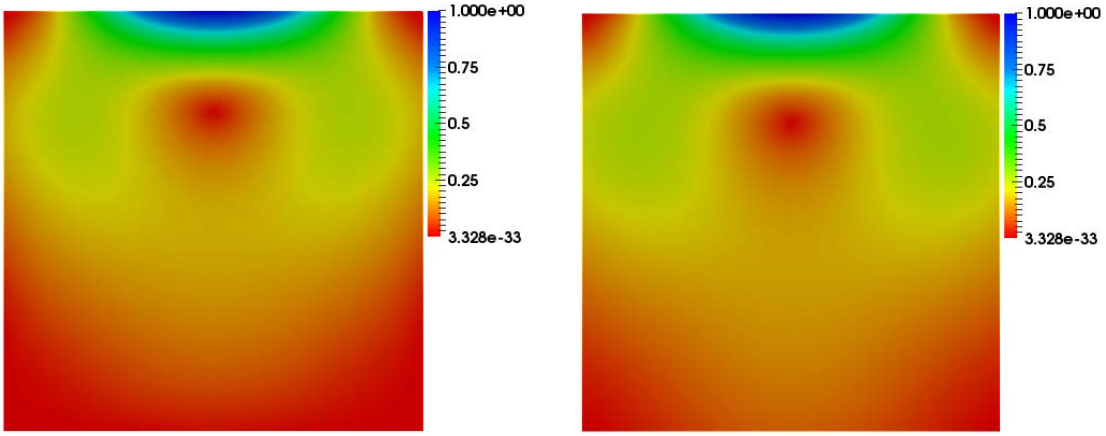

Figure 6: The numerical velocity for $s = 1.1$ (left) and $s = 1.5$ (right).

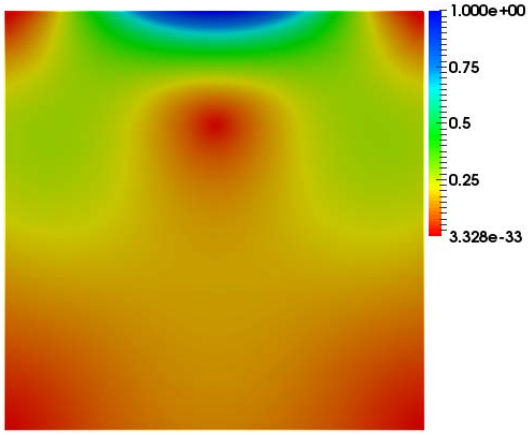

Figure 7: The numerical velocity for $s = 1.9$.

Next, we take $s = 3/2$ and $Re = 1000$ and 2000. Figure 9 shows the repartition of the velocity for the corresponding Reynolds numbers. Thus, our algorithm converges when the Reynolds number

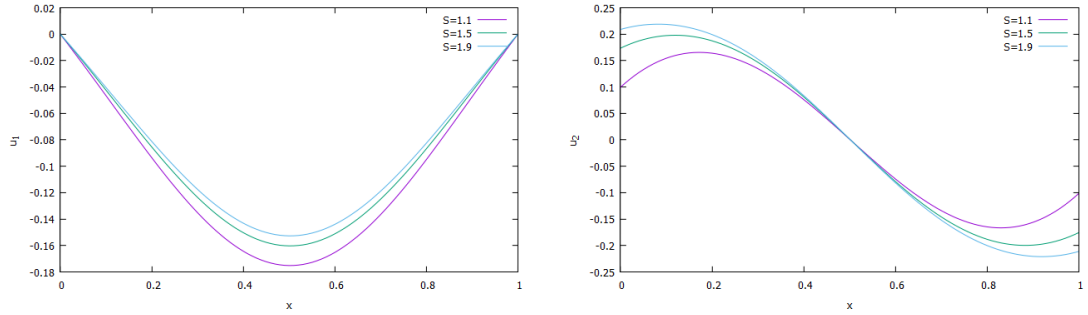

Figure 8: The numerical velocity (u_1, u_2) for $s = 1.1, 1.5$ and 1.9.

is big.

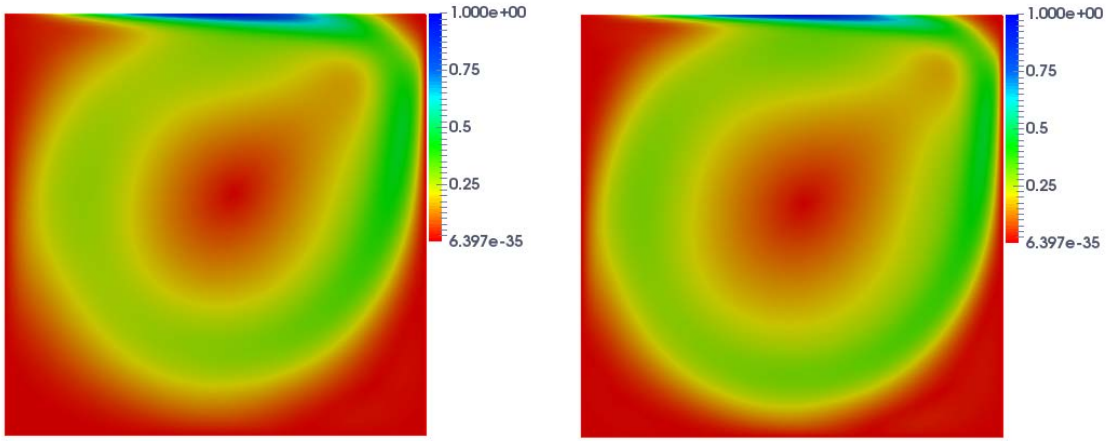

Figure 9: The numerical velocity for $s = 3/2$, $Re = 1000$ (left) and $Re = 2000$ (right).

5.3 Conclusions

The goal of the present work is to propose a novel formulation for the Stokes and Navier-Stokes equations with a direction dependent slip on a part of the boundary. The variational formulations can be regarded are perturbations of classical variational formulations encountered with standard boundary conditions. In this work we have

− Constructed the weak solutions of the Stokes and Navier-Stokes equations with power law slip boundary condition by using the same functional framework as for the standard boundary conditions.

− Formulated and studied the finite element approximations associated to these problems. In particular we have derived optimal convergence when the tangential part of the velocity on the slip zone has enough regularity.

− Constructed and studied an iterative scheme associated to the equations of Navier-Stokes, and derived some qualitative properties.

− The numerical experiments reveal optimal convergence rate, which is uniform with respect to the power law parameter s. As expected, one can observe the dependence of the velocity with respect to the power law parameter s in all examples treated. Finally, the convergence of the iterative scheme is observed for big Reynolds number.

The current methodology should be further extended, in particular by

(i) devising an adaptive mesh strategy based on a posteriori error estimate;

(ii) devising an adapted discretization approach for the case of a curved boundary and application in computational hemodynamics;

(iii) implementation of the current problems by combining alternative directions method of multiplier(s) and augmented Lagrangian formulations;

(iv) extending the present analysis to generalised non-Newtonian models;

(v) extending the present analysis to discontinuous Galerkin methods and virtual element approach;

(vi) extending the present analysis to time dependent problem.

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conflict of interest statement. None

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