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Abstract

Using an experimental mathematics approach, new relations are obtained between Dirichlet-like series for certain periodic coefficients and the moments of certain families of orthogonal polynomials.

In addition to the classical hypergeometric orthogonal polynomials, of Racah type and continuous dual Hahn type, a new similar family of orthogonal polynomials intervenes.

***Keywords*—** Jacobi continued fractions, hypergeometric functions, Dirichlet series, orthogonal polynomials, Racah and Hahn polynomials, moments, experimental mathematics, guess-and-prove

Contents

Introduction

The values at negative integers of the zeta function ζ, and of Dirichlet L-functions, form sequences of numbers of great interest, especially in number theory; they contain in particular Bernoulli numbers, Euler numbers and Springer numbers. A general description of these sequences is given by their *exponential generating series*, for which we know a simple expression in terms of the values of the Dirichlet character.

For the function ζ, and for some characters with small periods, it is known from results of Stieltjes, Touchard, Carlitz and Al-Salam that the *ordinary generating series* of their values at negative integers (possibly omitting the constant term) admits a remarkable expansion as a Jacobi continued fraction[1], see [2, 10], [8, Ch VI §8] and the references therein. By the very strong general connection between Jacobi continued fractions and orthogonal polynomials, this gives a close relation between certain L-functions and certain families of hypergeometric orthogonal polynomials in Askey's hierarchy.

The present article provides a similar relation between families of hypergeometric orthogonal polynomials and sequences of rational numbers related to certain series of Dirichlet type with periodic coefficients in $\{-1, 0, 1\}$.

[1] Because of the cancellation of one value out of two, these Jacobi continued fractions in t often take the form of a simple continued fraction in t^2.

We will content ourselves here with a quick sketch of the context which provides the motivation. By denoting χ the fixed periodic function, the sequences of numbers that we consider are the values at negative integers of functions of the form

$$L_{P,\chi}(s) = \sum_{m\geq 1} \chi(m) \frac{P'(m)}{P(m)^s}, \tag{1}$$

for P a polynomial that does not vanish at positive integers. They are therefore not Dirichlet series in the usual sense.

The justification of the analytic continuation of these functions, and of the fact that the construction given in section 2 indeed provides a description of their values at negative integers, is not made here. A statement of this type, but only for the trivial character, appears in [6]. We therefore content ourselves in this article with using the construction of the section 2 as a starting point which defines a sequence of rational numbers for each choice of χ and P.

We will only consider polynomials P of degree 2 and periodic coefficients χ whose average over a period is zero. The case of the trivial Dirichlet character, for the Riemann function ζ, does not fit into these hypotheses. The study in this case of the similar relation between the Ramanujan-Bernoulli numbers (or median Bernoulli numbers) and the orthogonal polynomials of Racah was started in [5].

Most of the obtained formulas seem new, probably because these sequences of numbers have never been considered before. However, when $P = x^2$, formula (1) simplifies (up to a factor) to the usual Dirichlet series L_χ evaluated at $2s-1$. The negative integer values of $L_{P,\chi}$ are therefore (up to a factor) the odd negative integer values of L_χ. There are several Jacobi continued fractions or Hankel determinants for such sequences in the literature. For instance, the tangent numbers are (up to a factor) the values at odd negative integers of a Dirichlet L-function with coefficients of period 4 and their Hankel determinants are known, see in particular [19, (4.56), (4.57)]. Similarly, the Springer numbers of odd index are the odd negative integer values of the Dirichlet series having as coefficients those of $(z-z^3)/(1+z^4)$ and their Jacobi continued fraction is known, see [21, (5.11)].

Our results are limited to a few simple families of periodic functions having a generating series of the form $(z^a - z^b)/(1-z^c)$, in the same way as previous results concerning the values of Dirichlet's L-functions were only known for a few characters. It is quite possible that a broad search in this class of periodic functions could provide other examples. It would be interesting to arrive at exhaustive results, whether for linear or quadratic polynomials. Another direction would be to look at more general rational functions of the form $(z^a - z^b)(1-z^d)/(1-z^c)$, where new phenomena occur.

On the other hand, one can hardly hope for results of the same type for polynomials P of degree at least 3. Indeed, we are using here orthogonal polynomials which are expressed, by a quadratic change of variables, in terms of a hypergeometric function. There is no such analogue for a change of variables of degree 3 or more in Askey's hierarchy.

Another interesting question concerns the possible existence of q-analogs to our results. There are indeed q-analogs for continued fractions of Bernoulli-Carlitz numbers [7]. For the moment, no q-analogue is known in the case of a quadratic polynomial P, even for the function ζ. The fact that there are q-analogs for hypergeometric orthogonal polynomials leaves nevertheless the possibility open, if not really a positive indication.

From an arithmetic viewpoint, the real Dirichlet characters form the most natural class of periodic functions with values in $\{-1, 0, 1\}$ that can be used as coefficients in a Dirichlet series. This article started with a study in this framework, but then it appeared that this restriction was unnecessary, and that the results extended without any assumption on the multiplicativity of the coefficients.

This work is based on experimental research with several computer algebra software (Fricas [11], SageMath [22] and Maple [17]), and in particular on an exploration of periodic functions that can give rise to continued fractions of the sought type. Some proofs also use these software.

The article is organized as follows. Section 1 contains brief reminders concerning Jacobi fractions in relation to moments of orthogonal polynomials, Dirichlet series with periodic coefficients and hypergeometric functions. Section 2 describes the construction of the sequences of rational numbers that we are going to consider, as well as a description of the proof strategy to establish their interpretation as moments. Then sections 3 and 4 establish the desired results in a series of particular cases, respectively including or not the first term of the sequences of numbers. Appendix A contains reminders of known families of orthogonal polynomials, and the description of a new family. Appendix B demonstrates an auxiliary formula.

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1 Known facts

1.1 Jacobi continued fractions

For the close and classical relationship between Jacobi continued fractions, Hankel determinants and orthogonal polynomials, we refer to [15, §2.7], [16, §5.4] and [23].

For f a power series in t with constant term 1, a Jacobi continued fraction expansion is an expression of the form

$$f(t) = 1/(1 + \mathtt{A}_0 t + \mathtt{B}_0 t^2/(1 + \mathtt{A}_1 t + \mathtt{B}_1 t^2/(1 + \cdots))), \tag{2}$$

where the coefficients form a list of pairs $(\mathtt{A}_n, \mathtt{B}_n)$ for $n \geq 0$. All $\mathtt{B}_n$ must be non-zero.

When the constant term of f is not zero but different from 1, we will normalize implicitly by dividing f by its constant term.

To such a Jacobi continued fraction, we associate a family of orthogonal polynomials. With the convention (2), the associated recurrence for the monic orthogonal polynomials of this family is given by

$$p_{n+1}(X) = (X + \mathtt{A}_n)p_n(X) + \mathtt{B}_{n-1}p_{n-1}(X) \tag{3}$$

with the initial conditions $p_{-1} = 0$ and $p_0 = 1$.

In this situation, the coefficients of the (normalized) power series f form the sequence of moments of the associated family of orthogonal polynomials.

The above conventions for the Jacobi continued fractions differ slightly from the standard (combinatorially natural) convention used, for example, in [23].

1.2 Dirichlet series with periodic coefficients

We refer to the book [9, §9 & §10] for the theory of Dirichlet's L-functions and the description of their values at negative integers. In particular, section 10.2.1 of this book contains general statements about Dirichlet series with periodic coefficients.

Let χ be a function from $\mathbb{N}_{>0}$ to $\{-1, 0, 1\}$, periodic of period N and of zero sum over a period. Note that then $-\chi$ is also such a function. We can associate to the function χ its generating series

$$\sum_{m \geq 1} \chi(m) z^m = Q(z)/(1 - z^N),$$

where $Q(z) = \sum_{m=1}^{N} \chi(m) z^m$ is a polynomial with coefficients in $\{-1, 0, 1\}$, vanishing at $z = 1$, without constant term and of degree at most N. We will often use this description of χ in the form of a rational fraction.

The Dirichlet series for the periodic function χ is defined for s in a complex right half-plane by the series

$$L_\chi(s) = \sum_{m \geq 1} \chi(m) \frac{1}{m^s}$$

and extends into a meromorphic function in the entire complex plane. The one-period zero-sum assumption implies that the resulting function is holomorphic.

The values taken at negative integers by the function L_χ are rational numbers. Let us recall their description, reformulating slightly some well-known statements.

We associate to χ the linear form Ψ_χ on the space of polynomials in x defined by the formula

$$t\frac{\sum_{m=1}^{N}\chi(m)e^{mt}}{1-e^{Nt}} = -\sum_{n\geq 0}\Psi_\chi(x^n)\frac{t^n}{n!} = -\Psi_\chi(e^{xt}), \tag{4}$$

as an equality between power series in the variable t, where N is the period of χ. Then, for all $n \geq 1$, we have the relation

$$L_\chi(1-n) = -\frac{\Psi_\chi(x^n)}{n}. \tag{5}$$

The values $\Psi_\chi(x^n)$ are the analogues for the function χ of the Bernoulli numbers (which correspond in the same way to the constant function equal to 1). The assumption that χ has zero-sum over a period implies that $\Psi_\chi(1) = 0$.

Let φ_χ be the linear form defined by

$$\varphi_\chi(x^n) = \Psi_\chi\left(\int_0 x^n dx\right) = -L_\chi(-n), \tag{6}$$

for $n \geq 0$. By linearity, we then have, for any polynomial E,

$$\varphi_\chi(E(x)) = \Psi_\chi\left(\int_0 E(x)dx\right).$$

We deduce the exponential generating series

$$\sum_{n\geq 0}\varphi_\chi(x^n)\frac{t^n}{n!} = \frac{\sum_{m=1}^{N}\chi(m)e^{mt}}{e^{Nt}-1} \tag{7}$$

for values of φ_χ on monomials.

We have the identity

$$\varphi_\chi(E(x+N)) - \varphi_\chi(E(x)) = \sum_{m=1}^{N}\chi(m)E(m), \tag{8}$$

for any polynomial E. It suffices to prove it for the monomials x^n for all $n \geq 0$. This is obtained by multiplying (7) by $e^{Nt}-1$ then comparing the coefficients of t^n.

We also have a variant of the previous identity when the fraction

$$F_\chi = \frac{\sum_{m=1}^{N}\chi(m)z^m}{z^N-1}$$

is not reduced. We can then multiply (7) by the denominator of the fraction evaluated in $z = e^t$, and compare the coefficients of t^n again. Any partially reduced expression of F_χ can also be used in this way.

1.2.1 Action by dilation by an integer factor

If $\ell \geq 1$ is an integer, we can consider the action of ℓ by dilation on the function χ, i.e. the function $\bar{\chi}$ defined by

$$\bar{\chi}(\ell m) = \chi(m)$$

for all $m \geq 1$ and $\bar{\chi}(m) = 0$ if ℓ does not divide m. At the level of the associated fraction, this corresponds to the substitution $F_{\bar{\chi}}(z) = F_{\chi}(z^{\ell})$.

For the Dirichlet series, we find the relation

$$L_{\bar{\chi}}(s) = \ell^{-s} L_{\chi}(s).$$

We also deduce the following formulas, for any polynomial P:

$$\Psi_{\bar{\chi}}(P(x)) = \ell^{-1}\Psi_{\chi}(P(\ell x)) \quad \text{and} \quad \varphi_{\bar{\chi}}(P(x)) = \varphi_{\chi}(P(\ell x)).$$

These relations are well compatible with the formula (5).

1.2.2 Dirichlet characters

Any real Dirichlet character provides an example of a periodic function with values in $\{-1, 0, 1\}$. The cancellation condition of the sum over a period is verified exactly when the character is not trivial, by orthogonality.

Here is a table of such characters that appear next. We give each time the period N, the list of values over a period $\{0, \ldots, N-1\}$ and the generating series in the form of a fraction.

$$\left| \begin{array}{c|c|c|c} \chi_3 & 3 & 0, 1, -1 & \frac{z-z^2}{1-z^3} \\ \chi_4 & 4 & 0, 1, 0, -1 & \frac{z-z^3}{1-z^4} \\ \chi_6 & 6 & 0, 1, 0, 0, 0, -1 & \frac{z-z^5}{1-z^6} \end{array} \right| \tag{9}$$

This finite list of characters is in fact delimited by a simple condition on the associated fraction, which must be of the form

$$\frac{z - z^a}{1 - z^d},$$

which results in finding the integers d whose Euler's totient function $\varphi(d)$ equals 2.

1.3 Hypergeometric functions

We will use throughout the article the standard notations for the hypergeometric functions [1, 12], and in particular, for $k \geq 0$ integer, the notation $(\alpha)_k$ for the ascending factorial, defined by

$$(\alpha)_k = \prod_{i=0}^{k-1} (\alpha + i)\,.$$

For $k \geq 0$, we introduce the notation

$$M_k(\alpha, \beta) = (\beta - \alpha)(\alpha)_k(\beta)_k.$$

Proposition 1.1. *For all $k \geq 0$, we have the expansion*

$$M_k(\alpha, \beta) = \sum_{\ell=0}^{k+1} c_{k,\ell} \cdot (\alpha)_{k+\ell}, \tag{10}$$

with

$$c_{k,\ell} = \frac{(-k-1)_\ell (k+2)_\ell}{(1)_\ell (k+1)_\ell} (\alpha + \beta + k + \ell)_{k+1-\ell}.$$

Proof. In terms of hypergeometric functions, the right summation in (10) is the product of the initial term

$$(\alpha + \beta + k)_{k+1} \cdot (\alpha)_k$$

by the finite sum

$${}_3F_2\left(\begin{matrix} -k-1, k+2, \alpha+k \\ k+1, \alpha+\beta+k \end{matrix}; 1\right).$$

By a hypergeometric identity (deflation followed by Chu-Vandermonde), the latter is equal to

$$(\beta - \alpha)\frac{(\beta)_k}{(\alpha + \beta + k)_{k+1}},$$

which concludes the proof. □

Lemma 1.2. *For $k \geq 0$ and α, β two parameters, we have*

$$\sum_{\ell=0}^{k+1} \frac{(-k-1)_\ell}{(1)_\ell} (\alpha + k + \ell)_{k+1-\ell} (\beta)_{k+\ell+1} = (\beta)_{k+1}(\alpha - \beta - 1)_{k+1}.$$

Proof. In terms of hypergeometric functions, this sum is written

$$(\alpha + k)_{k+1}(\beta)_{k+1}\, {}_2F_1\left(\begin{matrix} -k-1, \beta+k+1 \\ \alpha+k \end{matrix}; 1\right).$$

But the hypergeometric function is equal, by Chu-Vandermonde, to

$$(\alpha - \beta - 1)_{k+1}/(\alpha + k)_{k+1},$$

which concludes the proof. □

Lemma 1.3. *We have the following formula:*

$${}_2F_1\left(\begin{matrix} -m, m+1 \\ x \end{matrix}; 1/2\right) = \frac{2^m((x-m)/2)_m}{(x)_m},$$

for any integer $m \geq 0$.

Proof. This is a direct consequence of the classical formula [3, §2.4, (3)]

$$ {}_2F_1\left(\begin{matrix}1-a, a\\ c\end{matrix};1/2\right) = \frac{\Gamma(c/2)\Gamma((c+1)/2)}{\Gamma((a+c)/2)\Gamma((c+1-a)/2)}, $$

by taking $c = x$ and $a = -m$. □

2 General setting and shifted setting

2.1 General setting

We fix a periodic function χ from $\mathbb{N}_{>0}$ to $\{-1, 0, 1\}$, assumed to have zero sum over a period. We are interested in the quadratic polynomial $x(x+u)$, where u will be either a variable or a small integer or a half-integer. Following the discussion in the introduction, we consider the power series

$$ f_\chi = \sum_{n\geq 1} \Psi_\chi\big(\,(x(x+u))^n\,\big)t^{n-1}/n. \tag{11} $$

The parameter u is often omitted from the notation. Note that the power series associated with $-\chi$ is $-f_\chi$, because $\Psi_{-\chi} = -\Psi_\chi$.

We could have considered more generally

$$ f_{\chi,P} = \sum_{n\geq 1} \Psi_\chi\left(P^n\right) t^{n-1}/n, $$

for a polynomial P of the form $\lambda x^2 + ux + v$ with $\lambda \neq 0$. In the future interpretation of the coefficients of the power series $f_{\chi,P}$, normalized as the moments of a family of orthogonal polynomials, we can reduce by a dilation to the case $\lambda = 1$, because

$$ f_{\chi,\lambda P}(t) = \lambda f_{\chi,P}(\lambda t). $$

Moreover, the fact that $\Psi_\chi(1)$ is zero implies that a translation by v of the variable of the orthogonal polynomials corresponds to an addition of v to the polynomial P. We can therefore reduce to the case $v = 0$ without loss of generality.

2.1.1 Action by dilation by an integer factor

If we replace χ by its image $\bar{\chi}$ by the dilation by a factor $\ell \geq 1$, and simultaneously u by ℓu, we find that

$$ f_{\bar{\chi},\ell u}(t) = \ell f_{\chi,u}(\ell^2 t). $$

In the future interpretation of the coefficients of this power series as moments, this amounts to a dilation by a factor ℓ^2 in the variable of the orthogonal polynomials. We thus consider it to be an equivalent situation.

2.1.2 Experimental approach

Our procedure follows a classic approach in experimental mathematics, and is based on the *guess-and-prove* paradigm. From this point of view, our methodology is close to that of the article [18], whose objective is nevertheless different.

For each choice of χ and u, we can first easily calculate the first coefficients $\mathtt{A}_n$ and $\mathtt{B}_n$ of the expansion of f_χ in Jacobi continued fraction under the form (2).

In favorable cases, one can *guess* an expression, in the form of a rational function in n, for the coefficients $\mathtt{A}_n$ and $\mathtt{B}_n$ thus obtained. We can then introduce a family p_n^χ of orthogonal polynomials defined by the recurrence (3) for these coefficients $\mathtt{A}_n$ and $\mathtt{B}_n$.

The next step is to identify these polynomials as part of a family of orthogonal polynomials. Most of the cases studied involve hypergeometric orthogonal polynomials in the Askey hierarchy: either *Racah polynomials* or *continuous dual Hahn type polynomials*. This requires an affine change of variables, both in the variable Y of the orthogonal polynomials, but also in an auxiliary variable y. We end up with orthogonal polynomials p_n^χ in one variable X, which are expressed using a hypergeometric function whose parameters depend on x, with the relation $X = x(x+u)$.

It then remains to check that the moments of this family of orthogonal polynomials p_n^χ are indeed the coefficients of the power series f_χ, once normalized after division by its constant term. For this, we introduce the linear form Λ_χ (which plays the role of a measure) on the polynomials in X such that

$$\Lambda_\chi(nX^{n-1}) = \frac{1}{f_\chi(0)}\Psi_\chi\big(x^n(x+u)^n\big)$$

for $n \geq 1$.

By Favard's theorem [8, Thm. 4.4], it then suffices to show on the one hand that $\Lambda_\chi(1) = 1$ (which is immediate by the definitions of Λ_χ above and of f_χ in (11)), and on the other hand that Λ_χ vanishes on the polynomials $p_n^\chi(X)$ for $n \geq 1$. We then observe, using the definition of Λ_χ and the equality $X = x(x+u)$, that

$$\Lambda_\chi(p_n^\chi(X)) = \Lambda_\chi\left(\partial_X \int_0 p_n^\chi(X)dX\right) =$$
$$\frac{1}{f_\chi(0)}\Psi_\chi\left(\int_0 p_n^\chi(x(x+u))(2x+u)dx\right) = \frac{1}{f_\chi(0)}\varphi_\chi((2x+u)p_n^\chi(x(x+u))).$$

We are therefore reduced to proving in each case the following statement:

(N) For $n \geq 1$, the expression $\varphi_\chi((2x+u)p_n^\chi(x(x+u)))$ vanishes.

Once this statement has been proved, we will have obtained the following statement in each particular case.

Theorem 2.1. *The family of orthogonal polynomials p_n^χ has for moments the coefficients of the function $f_\chi/f_\chi(0)$.*

By the general theory connecting Jacobi continued fractions, Hankel determinants and orthogonal polynomials, one can then deduce not only the existence of the expansion in Jacobi continued fractions which was used as the experimental starting point, but also statements on the factorization of Hankel determinants for the list of moments.

2.1.3 Proof strategy

The general proof strategy for the statement **(N)** is as follows. We first obtain evaluations of φ_χ on expressions of the form

$$\left(\frac{cx+d}{e}\right)_k$$

for some integers (c, d, e). From this we deduce evaluations of φ_χ on polynomials of the form

$$\left(\frac{(c-c')x+(d-d')}{e}\right)\left(\frac{cx+d}{e}\right)_k\left(\frac{c'x+d'}{e}\right)_k,$$

using the identity of Proposition 1.1. We finally use these second evaluations to calculate the value of φ_χ on the series expansion of the hypergeometric functions and obtain the desired cancellation.

2.2 Shifted setting

In parallel, we are also interested in power series of the form

$$f_\chi^+ = \sum_{n\geq 2} \Psi_\chi\big(\,(x(x+u))^n\,\big)t^{n-2}/n,$$

obtained by forgetting the constant term of f_χ and dividing by t. Again, u is either a variable or a small integer, often omitted in notation. Note that $f_{-\chi}^+ = -f_\chi^+$.

We can then, in the same way as previously, first guess in certain cases an expression for the coefficients $\mathtt{A}_n$ and $\mathtt{B}_n$ of the expansion of f_χ^+ in Jacobi continued fraction of the form (2), then identify the orthogonal polynomials associated with these coefficients. We denote by $p_n^{\chi,+}$ these orthogonal polynomials.

We introduce the associated linear form (measure):

$$\Lambda_\chi^+(nX^{n-2}) = \frac{1}{f_\chi^+(0)}\Psi_\chi\left(x^n(x+u)^n\right)$$

for $n \geq 2$. Then,

$$\Lambda_\chi^+(nX^{n-2}) = \Lambda_\chi^+\left(\frac{1}{X}\partial_X X^n\right).$$

We then observe, using the definition of Λ_χ^+ and the relation $X = x(x+u)$, that

$$\Lambda_\chi^+(p_n^{\chi,+}(X)) = \Lambda_\chi^+\left(\frac{1}{X}\partial_X \int_0 p_n^{\chi,+}(X)XdX\right) =$$
$$\frac{1}{f_\chi^+(0)}\Psi_\chi\left(\int_0 p_n^{\chi,+}(x(x+u))x(x+u)(2x+u)dx\right) =$$
$$\frac{1}{f_\chi^+(0)}\varphi_\chi((2x+u)x(x+u)p_n^{\chi,+}(x(x+u))).$$

We are therefore reduced to proving in each case the following statement:

(N+) For $n \geq 1$, the expression $\varphi_\chi((2x+u)x(x+u)p_n^{\chi,+}(x(x+u)))$ vanishes.

Once this statement has been proved, we will have obtained the following statement in each particular case.

Theorem 2.2. *The family of orthogonal polynomials $p_n^{\chi,+}$ has for moments the coefficients of the function $f_\chi^+/f_\chi^+(0)$.*

3 Fractions of the form $(z^a - z^b)/(1 - z^c)$ without shift

Throughout this section, we place ourselves in the situation described in section 2.1. We will consider in this section, and in the next one, periodic coefficients which correspond to fractions of the form

$$\frac{z^a - z^b}{1 - z^c}, \tag{12}$$

for a, b, c positive integers such that c does not divide $b - a$. (According to the table (9), this includes the case of the Dirichlet characters χ_3, χ_4 and χ_6.)

We denote by $\Psi_{a,b,c}$ and $\varphi_{a,b,c}$ the linear forms associated with the fraction (12) by formulas (4) and (6) in Section 1.2. We denote by $f_{a,b,c}$ the power series of type (11) with coefficients defined by $\Psi_{a,b,c}$. Throughout the section, u will denote an indeterminate.

Up to exchanging a and b, which replaces $f_{a,b,c}$ by its opposite and leaves the quotient $f_{a,b,c}/f_{a,b,c}(0)$ unchanged, we may assume $a < b$.

We note here for later use the identity

$$\varphi_{a,b,c}(E(x+c)) - \varphi_{a,b,c}(E(x)) = E(a) - E(b), \tag{13}$$

for any polynomial E; this is a special case of equality (8).

3.1 Preliminaries

We will now evaluate $\varphi_{a,b,c}$ on hypergeometric expressions containing ascending factorials.

Lemma 3.1. *For all integers $k \geq j \geq 0$, we have*

$$\varphi_{a,b,c}\left(\left(\frac{-x+b}{c}-j\right)_k\right) = -\frac{1}{k+1}\left(\frac{b-a}{c}-j\right)_{k+1}.$$

Proof. We write $N_k(x)$ for $\left(\frac{-x+b}{c}-j\right)_k$. We check that

$$N_{k+1}(x+c) - N_{k+1}(x) = -(k+1)N_k(x)$$

for all $k \geq 0$. We apply to this the linear form $\varphi_{a,b,c}$ and we use (13) on the left side to obtain

$$N_{k+1}(a) - N_{k+1}(b) = -(k+1)\varphi_{a,b,c}(N_k(x)).$$

It remains to remark that $N_{k+1}(b) = (-j)_{k+1}$ vanishes, because $k \geq j$. □

We set, for $k \geq j \geq 0$,

$$M_{k,j}^{(b,c)}(x,u) := M_k\left(\frac{-x+b}{c}-j, \frac{x+u+b}{c}-j\right). \tag{14}$$

Proposition 3.2. *For all integers $k \geq j \geq 0$, we have*

$$\varphi_{a,b,c}\left(M_{k,j}^{(b,c)}(x,u)\right) = -\frac{1}{k+1}\left(\frac{b-a}{c}-j\right)_{k+1}\left(\frac{u+b+a-c}{c}-j\right)_{k+1}. \tag{15}$$

Proof. By specializing Proposition 1.1, we have the expansion

$$M_{k,j}^{(b,c)}(x,u) = \sum_{\ell=0}^{k+1} c_{k,\ell}\left(\frac{-x+b}{c}-j\right)_{k+\ell}, \tag{16}$$

with

$$c_{k,\ell} = \frac{(-k-1)_\ell(k+2)_\ell}{(1)_\ell(k+1)_\ell}\left(\frac{u+2b}{c}+k+\ell-2j\right)_{k+1-\ell}, \tag{17}$$

which expresses $M_{k,j}^{(b,c)}(x,u)$ as a linear combination of polynomials $\left(\frac{-x+b}{c}-j\right)_\ell$ with coefficients independent of x.

By applying Lemma 3.1 to the image by $\varphi_{a,b,c}$ of the expression (16) with coefficients (17), we find for $\varphi_{a,b,c}(M_{k,j}^{(b,c)}(x,u))$ a hypergeometric formula:

$$\frac{-1}{k+1}\sum_{\ell=0}^{k+1}\frac{(-k-1)_\ell}{(1)_\ell}\left(\frac{u+2b}{c}+k+\ell-2j\right)_{k+1-\ell}\left(\frac{b-a}{c}-j\right)_{k+\ell+1}.$$

In terms of hypergeometric functions, this gives

$$\frac{-1}{k+1}\left(\frac{u+2b}{c}+k-2j\right)_{k+1}\left(\frac{b-a}{c}-j\right)_{k+1}\times \\ {}_2F_1\left(\begin{matrix}-k-1,(b-a)/c+k+1-j\\(u+2b)/c+k-2j\end{matrix};1\right).$$

Then using the Chu-Vandermonde formula in the form

$${}_2F_1\left(\begin{matrix}-k-1,(b-a)/c+k+1-j\\(u+2b)/c+k-2j\end{matrix};1\right)=\frac{((u+b+a-c)/c-j)_{k+1}}{((u+2b)/c+k-2j)_{k+1}},$$

we find the term on the right of (15). □

3.2 Associated orthogonal polynomials

We recall that, for any triple $(a,b,c)\in(\mathbb{N}\setminus\{0\})^3$ such that $a<b$ and such that c does not divide $b-a$, we have defined the power series $f_{a,b,c}$ of $\mathbb{Q}[u][[t]]$ by

$$f_{a,b,c}:=\sum_{n\geq 1}\Psi_{a,b,c}\left(x^n(x+u)^n\right)\frac{t^{n-1}}{n},$$

where $\Psi_{a,b,c}$ is a linear form which acts on any polynomial E of $\mathbb{Q}[u,x]$ by

$$\Psi_{a,b,c}(E)=\sum_{k\geq 1}k!\cdot[x^k]E\cdot[t^{k-1}]\frac{e^{at}-e^{bt}}{e^{tc}-1}.$$

By calculating the first terms of the series expansion of

$$f_{a,b,c}=\frac{(a-b)\,(u+a-c+b)}{c}\cdot\left(1+\frac{(a+b-c)u+a^2+b^2-ac-bc}{2}t+\cdots\right)$$

we can *guess* for the power series $f_{a,b,c}$ divided by its constant term a Jacobi continued fraction of the form (2), with

$$\mathtt{A}_n=(c^2n^2+c^2n+u(c-b-a)+a(c-a)+b(c-b))/2 \tag{18}$$

and

$$\mathtt{B}_n=-\frac{(cn+c+a-b)(cn+c+b-a)(n+1)^2}{4(2n+1)(2n+3)}(cn-u+2c-a-b)(cn+u+a+b), \tag{19}$$

for $n\geq 0$.

In order to *prove* this Jacobi continued fraction for $f_{a,b,c}$, we introduce the orthogonal polynomials $p_n^{(a,b,c)}(X)$ defined by the recurrence (3) with the above coefficients. We first show that they belong to the family of orthogonal hypergeometric Racah polynomials.

Proposition 3.3. *The orthogonal polynomials $p_n^{(a,b,c)}$ are given, up to a multiplicative constant, by the formula*

$$ {}_4F_3\left(\begin{matrix} -n, n+1, (-x+b)/c, (x+u+b)/c \\ 1, (u+a+b)/c, (b-a+c)/c \end{matrix}; 1\right) \tag{20} $$

as functions of $X(x) = x(x+u)$.

Proof. Formula (31) for the orthogonal Racah polynomials, for the parameters $\alpha = (u+a+b-c)/c$, $\beta = -(u+a+b-c)/c$, $\gamma = (b-a)/c$ and $\delta = (u+a+b-c)/c$ gives

$$ {}_4F_3\left(\begin{matrix} -n, n+1, -y, y+u/c+2b/c \\ 1, (u+a+b)/c, (b-a+c)/c \end{matrix}; 1\right) $$

as a function of $Y(y) = y(y+u/c+2b/c)$. We pass to the formula (20) by the change of variables

$$ \begin{cases} X = c^2 Y + b(b+u), \\ x = cy + b. \end{cases} $$

The same change of variables performed in formula (30) for the recurrence of the Racah polynomials gives the coefficients $\mathtt{A}_n$ of (18) and $\mathtt{B}_n$ of (19). This proves the proposition. □

Note: the expression (19) simplifies when $2(b-a) = c$. In this case, the orthogonal polynomials actually belong to the continuous dual Hahn type.

We now want to prove that the coefficients of the quotient of $f_{a,b,c}$ by its constant term are indeed the moments of the family of orthogonal polynomials (20). As explained in section 2, we are reduced to proving the following lemma, which gives **(N)**.

Lemma 3.4. *The expression*

$$ \varphi_{a,b,c}\left((2x+u)p_n^{(a,b,c)}(x(x+u))\right) $$

vanishes for any $n \geq 1$.

Proof. By expanding the explicit formula (20) for the polynomials $p_n^{(a,b,c)}$, this expression becomes the finite sum

$$ \varphi_{a,b,c}\left(\sum_{k\geq 0}(2x+u)\frac{(-n)_k(n+1)_k((-x+b)/c)_k((x+u+b)/c)_k}{(1)_k(1)_k((u+a+b)/c)_k((b-a+c)/c)_k}\right) $$

in which we recognize a factor $M_k^{(b,c)}(x,u)$ defined by (14). We can therefore use the evaluation (15) of $\varphi_{a,b,c}$ and obtain a hypergeometric sum which simplifies (up to a factor) to

$$ {}_2F_1\left(\begin{matrix} -n, n+1 \\ 2 \end{matrix}; 1\right). $$

This vanishes for $n \geq 1$ by Chu-Vandermonde. □

4 Fractions of the form $(z^a - z^b)/(1 - z^c)$ with shift

We keep the same notations and the same condition: a, b, c are positive integers such that c does not divide $b - a$.

We can assume without loss of generality that $a < b$. This will not always be useful, in particular for not distinguishing cases.

Throughout this section, we place ourselves in the situation described in section 2.2.

4.1 Case $c = b$

We consider in this section fractions of the form $\frac{z^a - z^b}{1 - z^b}$, for a, b positive integers such that b does not divide a. The parameter u is an indeterminate.

We guess the coefficients of the Jacobi continued fraction:

$$\mathtt{A}_n = \frac{(2b^2(n+1)^2 + (b-2a)u - b^2 + 2ab - 2a^2)(n+1)^2}{(2n+1)(2n+3)}$$

and

$$\mathtt{B}_n = -\frac{(bn+u+b+a)(bn-u+2b-a)(bn+b+a)(bn+2b-a)(n+2)(n+1)}{4(2n+3)^2}.$$

We identify these coefficients as those of the recurrence for the Racah polynomials with parameters $\alpha = (b-a)/b$, $\beta = a/b$, $\gamma = (u+a)/b$, $\delta = (b-a)/b)$, modulo the change of variables

$$\begin{cases} X = b^2 Y + bu + b^2, \\ x = b(y+1). \end{cases}$$

The orthogonal polynomials p_n are therefore given, up to a multiplicative constant, by the formula

$${}_4F_3\left(\begin{matrix} -n, n+2, (-x+b)/b, (x+u+b)/b \\ (u+a+b)/b, 2, 1+(b-a)/b \end{matrix}; 1\right) \tag{21}$$

as functions of $X(x) = x(x+u)$.

Note that this simplifies to continuous dual Hahn polynomials when $b = 2a$.

It remains to prove **(N+)** for this family of polynomials and the linear form $\varphi_{a,b,b}$.

Lemma 4.1. *The expression*

$$\varphi_{a,b,b}\left((2x+u)(x(x+u))p_n(x(x+u))\right)$$

vanishes for all $n \geq 1$.

Proof. By expanding the explicit formula (21) for the polynomials p_n, this expression becomes (up to a factor) the finite sum

$$\varphi_{a,b,c}\left(\sum_{k\geq 0}(2x+u)\frac{(-n)_k(n+2)_k(-x/b)_{k+1}((x+u)/b)_{k+1}}{(1)_k(2)_k((u+a+b)/b)_k(1+(b-a)/b)_k}\right)$$

in which we recognize a factor $M_{k+1}(-x/b,(x+u)/b)$. We can therefore use the formula

$$\varphi_{a,b,b}\left(M_k(-x/b,(x+u)/b)\right)=\frac{-1}{k+1}(-a/b)_{k+1}((u+a-b)/b)_{k+1}$$

which results from the proposition 3.2 with $j=1$, and obtain a hypergeometric sum which simplifies (up to a factor) to

$${}_2F_1\left(\begin{matrix}-n,n+2\\3\end{matrix};1\right).$$

This vanishes for $n\geq 1$ by the Chu-Vandermonde identity. □

4.2 Case $u=c-a$

We consider in this section fractions of the form $\frac{z^a-z^b}{1-z^c}$, for a,b positive integers such that c does not divide $a-b$. The parameter u is set equal to $c-a$. We further assume that c does not divide b.

We guess the coefficients of the Jacobi continued fraction:

$$\mathtt{A}_n=\frac{(n+1)^2(2c^2n^2+4c^2n+2c^2-ac-2b^2+2ab)}{(2n+1)(2n+3)}$$

and

$$\mathtt{B}_n=-\frac{(cn+c+a-b)(cn+2c-b)(cn+c+b)(cn+2c+b-a)(n+1)(n+2)}{4(2n+3)^2}.$$

We note in these coefficients a symmetry when exchanging b and $a-b$.

We identify these coefficients as those of the recurrence for the Racah polynomials with parameters $\alpha=b/c,\beta=(cb)/c,\gamma=(a-b)/c,\delta=b/c$, modulo the change of variables

$$\begin{cases}X=c^2Y+ca,\\ x=cy+a.\end{cases}$$

The orthogonal polynomials p_n are therefore given, up to a multiplicative constant, by the formula

$${}_4F_3\left(\begin{matrix}-n,n+2,(-x+a)/c,(x+c)/c\\(b+c)/c,2,(a-b+c)/c\end{matrix};1\right) \tag{22}$$

as functions of $X(x) = x(x+c-a)$.

Note that this simplifies to dual Hahn polynomials when $c = 2b$ or $c = 2(a-b)$.

It remains to prove **(N+)** in this case.

Lemma 4.2. *The expression*

$$\varphi_{a,b,c}\left((2x+c-a)x(x+c-a)p_n(x(x+c-a))\right)$$

vanishes for all $n \geq 1$.

Proof. By expanding the explicit formula (22) for the polynomials p_n and by integrating the factors x and $x+c-a$ in the ascending factorials, this expression becomes (up to a factor) the finite sum

$$\varphi_{a,b,c}\left(\sum_{k\geq 0}(2x+c-a)\frac{(-n)_k(n+2)_k((-x+a-c)/c)_{k+1}(x/c)_{k+1}}{(1)_k(2)_k((b+c)/c)_k((a-b+c)/c)_k}\right).$$

We can therefore use the following lemma, and obtain a hypergeometric sum which simplifies (up to a factor) to

$${}_2F_1\left(\begin{matrix}-n, n+2\\ 3\end{matrix};1\right).$$

This vanishes for $n \geq 1$ by the Chu-Vandermonde identity. □

Lemma 4.3. *For* $k \geq 1$,

$$\varphi_{a,b,c}((2x+c-a)((-x+a-c)/c)_k(x/c)_k) = \frac{(b-c)(a-b-c)}{(k+1)c}(b/c)_k((a-b)/c)_k.$$

Proof. The argument of $\varphi_{a,b,c}$ is $cM_k((-x+a-c)/c, x/c)$. The formula is just Proposition 3.2 after swapping a and b, with $j = 1$ and $u = c-a$. □

4.3 Case $c = a+b$ and $u = c/2$

In this section, we consider fractions of the form $\frac{z^a - z^b}{1-z^{a+b}}$, for positive integers a and b. The parameter u is set equal to $(a+b)/2$. To simplify the notations, we set $c = a+b$. The situation studied is then symmetric in a, b.

We guess the coefficients of the Jacobi continued fraction:

$$\mathtt{A}_n = c^2n^2/2 + c^2n + c^2/4 + ab$$

and

$$\mathtt{B}_n = -\frac{c^2}{16}(cn+3a+b)(cn+a+3b)(n+1)^2.$$

We identify these coefficients as those of the continuous dual Hahn type with parameters $\mathtt{a}, \mathtt{b}, \mathtt{c} = (3/2, (3a-b)/(2c), (3b-a)/(2c))$, modulo the change of variables

$$\begin{cases} X = \frac{c^2}{4} Y - c^2/16, \\ x = \frac{c}{2} y - c/4. \end{cases}$$

The orthogonal polynomials p_n are therefore given, up to a multiplicative constant, by the formula

$$p_n(X) = {}_3F_2\left(\begin{matrix} -n, (-2x)/c+1, (2x)/c+2 \\ (3a+b)/c, (3b+a)/c \end{matrix} ; 1 \right) \tag{23}$$

as functions in $X(x) = x(x+c/2)$.

It remains to prove **(N+)** in this case.

Lemma 4.4. *The expression*

$$\varphi_{a,b,a+b}\left((2x+c/2)x(x+c/2)p_n(x(x+c/2))\right)$$

vanishes for all $n \geq 1$.

Proof. We expand the argument of $\varphi_{a,b,a+b}$ by using the hypergeometric expression (23). We then use the lemma below. We get a multiple of

$${}_1F_0\left(\begin{matrix} -n \\ \end{matrix} ; 1 \right) = (1-1)^n,$$

which vanishes for $n \geq 1$. □

Lemma 4.5. *For* $k \geq 0$,

$$\varphi_{a,b,a+b}((2x+c/2)((-2x)/c)_{k+1}((2x)/c+1)_{k+1}) = \frac{c}{2}((a-b)/c)_{k+2}((b-a)/c+1)_{k+1}. \tag{24}$$

Proof. We first notice that the argument of $\varphi_{a,b,a+b}$ is $M_{k+1}(-2x/c, 2x/c+1)$ times $c/2$. We are going to simplify by the factor $c/2$, which is also present in the right-hand side of (24). We expand the argument using Proposition 1.1 to get

$$\sum_{\ell=0}^{k+2} \frac{(-k-2)_\ell (k+\ell+2)(k+2+\ell)_{k+2-\ell}}{(1)_\ell (k+2)} \varphi_{a,b,a+b}\big((-2x/c)_{k+1+\ell}\big).$$

Using the lemma below, we find

$$-(2k+3)! \sum_{\ell=0}^{k+2} \frac{(-k-2)_\ell (k+\ell+2)}{(1)_\ell (k+2) 2^{k+1+\ell}} \sum_{j=0}^{k+1+\ell} \frac{2^{j-1}}{(j+1)!} \big((-2a/c)_{j+1} - (-2b/c)_{j+1}\big).$$

By Proposition B.1 at $N = 1$, this becomes

$$\frac{(-1)^{k+1}}{2k+4}((-2a/c-k-1)_{2k+4} - (-2b/c-k-1)_{2k+4}).$$

It only remains to factor this difference to obtain the desired result. □

Lemma 4.6.

$$\varphi_{a,b,a+b}(((-2x)/c)_k) = \frac{-k!}{2^k}\left(\sum_{j=0}^{k} \frac{2^{j-1}}{(j+1)!}((-2a/c)_{j+1} - (-2b/c)_{j+1})\right).$$

Proof. We set $N_k(x) = (-2x/c)_k$ and write Z_k for $\varphi_{a,b,a+b}(N_k(x))$. Then

$$N_{k+1}(x+c/2) - N_{k+1}(x) = -(k+1)N_k(x),$$

therefore

$$N_{k+1}(x+c) - N_{k+1}(x) = -(k+1)(N_k(x) + N_k(x+c/2)).$$

By applying $\varphi_{a,b,a+b}$ to the two equations above, we deduce

$$2Z_k = -\frac{1}{k+1}(N_{k+1}(a) - N_{k+1}(b)) + kZ_{k-1}.$$

In conclusion,

$$\frac{2^k Z_k}{k!} = -\sum_{j=0}^{k} \frac{2^{j-1}}{(j+1)!}(N_{j+1}(a) - N_{j+1}(b)).$$

□

4.4 Case $c = 2(b-a)$ and $u = b-a$

In this section, we consider fractions of the form

$$\frac{z^a - z^b}{1 - z^{2(b-a)}} = \frac{z^a}{1 + z^{b-a}},$$

for a, b positive integers. The parameter u is set equal to $b-a$. We assume here that $a < b$ and that $b-a$ does not divide a.

We guess the coefficients of the Jacobi continued fraction:

$$\mathtt{A}_n = 2(b-a)^2n^2 + 4(b-a)^2n + 2b^2 - 4ab + a^2$$

and

$$\mathtt{B}_n = -(b-a)^2((b-a)n + 2b - a)((b-a)n + 2b - 3a)(n+1)^2.$$

We identify these coefficients as those of the continuous dual Hahn recurrence with parameters $(3/2, 1-(a+b)/(2(b-a)), (a+b)/(2(b-a)))$, modulo the change of variables

$$\begin{cases} X = (b-a)^2(Y-1/4), \\ x = (b-a)(y-1/2). \end{cases}$$

We therefore obtain polynomials defined, up to a multiplicative constant, by

$$ {}_3F_2\left(\begin{matrix} -n, 1-x/(b-a), 2+x/(b-a) \\ (2b-a)/(b-a), (2b-3a)/(b-a) \end{matrix}; 1\right) \tag{25}$$

for $X = x(x+b-a)$.

We want to prove **(N+)**:

Proposition 4.7. *The expression*

$$\varphi_{a,b,2(b-a)}((2x+b-a)(x(x+b-a))p_n(x(x+b-a))$$

vanishes for all $n \geq 1$.

Proof. We start by replacing $p_n(x(x+b-a))$ by its expression (25) expanded as a sum. We can then incorporate the factors x and $x+b-a$ each in an ascending factorial in the numerator. By using the formula (26) at $k+1$, the summation simplifies to ${}_1F_0\left(\begin{matrix} -n \\ \end{matrix}; 1\right)$, which vanishes for any positive integer. □

We therefore need the following exact value.

Proposition 4.8. *For* $k \geq 0$,

$$\varphi_{a,b,2(b-a)}((2x+b-a)(-x/(b-a))_k(1+x/(b-a))_k) = \\ (b-a)(-a/(b-a))_{k+1}(b/(b-a))_k. \tag{26}$$

Proof. The argument of $\varphi_{a,b,2(b-a)}$ is $b-a$ times $M_k(-x/(b-a), x/(b-a)+1))$. We will simplify by the factor $b-a$, which is also present in the right-hand side of (26). We expand the argument using Proposition 1.1 to get

$$\sum_{\ell=0}^{k+1} \frac{(-k-1)_\ell(k+\ell+1)(k+1+\ell)_{k+1-\ell}}{(1)_\ell(k+1)} \varphi_{a,b,a+b}\big((-x/(b-a))_{k+\ell}\big).$$

Using the lemma below, we find

$$-(2k+1)! \sum_{\ell=0}^{k+1} \frac{(-k-1)_\ell(k+\ell+1)}{(1)_\ell(k+1)2^{k+\ell}} \sum_{j=0}^{k+\ell} \frac{2^{j-1}}{j!}(-a/(b-a))_j.$$

By the Proposition B.1 at $N = 0$, this becomes

$$(-1)^k(-a/(b-a)-k)_{2k+1}.$$

An easy rewriting allows to obtain the desired result. □

Lemma 4.9. *For all $k \geq 0$,*

$$\varphi_{a,b,2(b-a)}((-x/(b-a))_k) = -\frac{k!}{2^k}\sum_{j=0}^{k}\frac{2^{j-1}(-a/(b-a))_j}{j!}.$$

Proof. For any polynomial E, we have

$$\varphi(E(x+b-a)) + \varphi(E(x)) = -E(a).$$

By setting $N_k(x) := (-x/(b-a))_k$, we check that

$$N_{k+1}(x+b-a) - N_{k+1}(x) = -(k+1)N_k(x),$$

from where we derive, by applying φ, that

$$2\varphi(N_{k+1}) = -N_{k+1}(a) + (k+1)\varphi(N_k).$$

We can finally take the sum:

$$\frac{2^k\varphi(N_k)}{k!} = -\sum_{j=0}^{k}\frac{2^{j-1}N_j(a)}{j!}.$$

□

4.5 Case $c = 2(b-a)$ and $u = 3(b-a)$

To simplify the notations we will, in this section only, replace $b-a$ by the letter d. This choice of parameters then corresponds to fractions of the form

$$\frac{z^a}{1+z^d}$$

for $u = 3d$. The case $a = 1, d = 2$ is related to the Dirichlet character χ_4 of (9).

We denote by $\varphi_d = \varphi_{a,d+a,2d}$ the linear form associated with this fraction. We have $\varphi_d(1) = -1/2$ and $\varphi_d(x) = d/4 - a/2$.

By first computing the first terms of the continued fraction, one can determine the first corresponding orthogonal polynomials. These polynomials q_n are not in Askey's hierarchy and are perhaps new. We introduce in the Appendix A.3 some polynomials p_n which depend on a parameter. We then have

$$q_n(X) = q_n(x(x+3d)) = p_n\left(\frac{x}{d}\left(\frac{x}{d}+3\right)\right),$$

where the polynomials p_n are taken for the parameter a/d.

We will now check the criterion **(N+)** for the polynomials q_n and the linear form φ_d.

Proposition 4.10. *For any $n \geq 1$, the expression $\varphi_d\left((2x+3d)(x(x+3d))q_n(x(x+3d)\right)$ vanishes.*

Proof. By using Lemma 4.11 and the definition (33) specialized at the parameter a/d, we find a sum of 4 terms,

$$\begin{aligned} d(d+a)(2d+a)((2n+2)d^2-a^2){}_3F_2\left(\begin{matrix}-n,2,1-a/d\\ 1,2-a/d\end{matrix};1\right) \\ -a(d+a)(3d+a)((2n+2)d^2-a^2){}_2F_1\left(\begin{matrix}-n,1-a/d\\ 2-a/d\end{matrix};1\right) \\ -ad^2(d+a)(2d+a){}_4F_3\left(\begin{matrix}-n,2,-a/d,1-a/(2d)\\ 1,2-a/d,-a/(2d)\end{matrix};1\right) \\ +a^2d(d+a)(3d+a){}_3F_2\left(\begin{matrix}-n,-a/d,1-a/(2d)\\ 2-a/d,-a/(2d)\end{matrix};1\right). \end{aligned}$$

We can check that this sum vanishes for all $n > 0$. □

Lemma 4.11. *For any $k \geq 0$,*

$$\begin{aligned} \varphi_d((2x+3d)x(x+3d)((x+d-a)/d)_k((-x-2d-a)/d)_k = \\ d(d+a)(2d+a)(2)_k(-a/d)_k - a(d+a)(3d+a)(1)_k(-a/d)_k. \end{aligned}$$

Proof. To calculate this, we cut it in two pieces, using

$$x(x+3d) = (x-a)(x+3d+a) + a(3d+a).$$

The result then follows from the Lemma 4.12. □

Lemma 4.12. *We have, for all $k \geq 0$,*

$$\varphi_d((2x+3d)((x-a)/d)_{k+1}((-x-3d-a)/d)_{k+1}) = (2d+a)(k+1)!(-1-a/d)_{k+1},$$

and

$$\varphi_d((2x+3d)((x+d-a)/d)_k((-x-2d-a)/d)_k) = -(d+a)k!(-a/d)_k.$$

Proof. These two formulas can be written as

$$(-d)\varphi_d(M_{k+1}((x-a)/d,(-x-3d-a)/d))$$

and

$$(-d)\varphi_d(M_k((x+d-a)/d,(-x-2d-a)/d)).$$

We will give the proof of the first formula, the second being similar. We use the expansion of M_k given by Proposition 1.1:

$$(-d)\sum_{\ell=0}^{k+2}\frac{(-k-2)_\ell(k+\ell+2)((-3d-2a)/d+k+1+\ell)_{k+2-\ell}}{(1)_\ell(k+2)}\times \\ \varphi_d(((x-a)/d)_{k+1+\ell}).$$

We then use the Lemma 4.13 to obtain

$$(-d)\sum_{\ell=0}^{k+2}\frac{(-k-2)_\ell(k+\ell+2)!(-2a/d+k-2+\ell)_{k+2-\ell}}{(1)_\ell(k+2)2^{k+\ell+2}},$$

which writes

$$(-d)(k+1)!\frac{(-2a/d+k-2)_{k+2}}{2^{k+2}}{}_2F_1\left(\begin{matrix}-k-2,k+3\\ k-2-2a/d\end{matrix};1/2\right).$$

We then use Lemma 1.3 with $m=k+2$ to conclude. □

Lemma 4.13. *We have*

$$\varphi_d((x-a)/d)_k)=k!/2^{k+1}$$

for $k>0$ *and*

$$\varphi_d((x-a)/d+1)_k)=-k!/2^{k+1}$$

for $k\geq 0$.

Proof. By definition of φ_d,

$$\varphi_d(E(x+d))+\varphi_d(E(x))=-E(a)$$

for any polynomial E. Moreover, by denoting $N_k(x):=((x-a)/d)_k$, we check that

$$N_{k+1}(x)-N_{k+1}(x-d)=(k+1)N_k(x)$$

for all $k\geq 0$. Applying φ_d to this equality, we find

$$2\varphi_d(N_{k+1})-(-1)_{k+1}=(k+1)\varphi_d(N_k),$$

which implies that $\varphi_d(N_k)=(k+1)!/2^k\varphi_d(N_1)$. We conclude by using that $\varphi_d(N_1)=1/4$. The second part of the statement is proved in the same way. □

4.6 Two other cases

We first consider the case $c=(a+b)/2$ and $u=c/2=(a+b)/4$, assuming that a and b are positive, $a\neq b$, $b\neq 3a$ and $a\neq 3b$. The situation is symmetric in a and b. Let $D=(b-a)/2$. We will work with the parameters c and D.

We then guess the coefficients of the Jacobi continued fraction:

$$\mathtt{A}_n=-D^2+c^2n+\frac{c^2}{2}n^2+K_n \tag{27}$$

where

$$K_n = \frac{(c+4D)\cdot(c-4D)c^2}{16(2n+1)(2n+3)} \cdot \frac{(c^2n^2 + (2c^2-4D^2)n + c^2/2 - 2D^2)}{(2D^2n + (c/2)^2 - D^2)\cdot(2D^2n + (c/2)^2 + D^2)}. \tag{28}$$

and

$$\mathtt{B}_n = -\frac{(\frac{c}{2})^2(2n+1)(2n+5)(\frac{c}{2}n + c - D)(\frac{c}{2}n + c + D)}{(2n+3)^2(2D^2n + (c/2)^2 + D^2)^2} \cdot (2D^2n + (c/2)^2 - D^2)(2D^2n + (c/2)^2 + 3D^2)(n+1)^2. \tag{29}$$

This simplifies to a continuous dual Hahn type of parameters $(-1, 3/2, 2)$ if $3a = 5b$ or $3b = 5a$, *i.e.* when $c = \pm 4D$.

On the other hand, we consider the case $b = a + 3c/2$ (therefore c even) and $u = c/2$, assuming that a and c are positive and that a is not a multiple of $c/2$.

We have a symmetry under the transformation $(a, c) \mapsto (-a-c, c)$. A fixed point of this transformation is c, an anti-fixed point is $D = a + c/2$. We will work with the parameters c and D.

We then guess the coefficients of the Jacobi continued fraction, and we find exactly the same formulas (27), (28) and (29) as in the previous case. This second case therefore involves the same orthogonal polynomials.

One could hope for a formula for these monic orthogonal polynomials, in the form of a linear combination of hypergeometric terms, similar to the formula (33) obtained in A.3. We have not been able to find such a formula.

A Hypergeometric orthogonal polynomials

A.1 Racah orthogonal polynomials

We recall here the known expressions for the recurrence of Racah's orthogonal polynomials, taken *verbatim* from the book by Koekoek and Swarttouw [13, 14, Section 1.2]. Set

$$A_n = \frac{(n+\alpha+1)(n+\alpha+\beta+1)(n+\beta+\delta+1)(n+\gamma+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}$$

and

$$C_n = \frac{n(n+\alpha+\beta-\gamma)(n+\alpha-\delta)(n+\beta)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)}.$$

Then the *monic* orthogonal Racah polynomials satisfy the recurrence

$$p_{n+1}(Y) = (Y + A_n + C_n)p_n(Y) - A_{n-1}C_n p_{n-1}(Y). \tag{30}$$

The general formula for *non-monic* Racah polynomials is

$$ {}_4F_3\left(\begin{matrix} -n, n+\alpha+\beta+1, -y, y+\gamma+\delta+1 \\ \alpha+1, \beta+\delta+1, \gamma+1 \end{matrix}; 1\right) \tag{31} $$

as functions in $Y(y) = y(y+\gamma+\delta+1)$.

A.2 Orthogonal polynomials of continuous dual Hahn type

We recall here the known expressions for the recurrence of orthogonal polynomials of continuous dual Hahn type, taken from the book by Koekoek and Swarttouw [13, 14], Section 1.3. We will, however, slightly modify the notations of this book.

We must first take care to distinguish between two variables, the variable x which appears in the form ix in the parameters of the hypergeometric function ${}_3F_2$, and the variable X which is the argument of the orthogonal polynomials $p_n(X)$. These variables are linked by the relation $X = x^2$.

We perform the changes of variables $Y = -X$ and $y = ix$ in the notations of Koekoek and Swarttouw. The new variables are therefore linked by $Y = y^2$. It is also necessary to pay attention to the sign change of the dominant term of the orthogonal polynomials induced by this change of variables. The final result is given below, without going into details.

Set

$$ A_n = (n+\mathtt{a}+\mathtt{b})(n+\mathtt{a}+\mathtt{c}) \quad \text{and} \quad C_n = n(n+\mathtt{b}+\mathtt{c}-1). $$

Then the *monic* orthogonal polynomials of continuous dual Hahn type satisfy the recurrence

$$ p_{n+1}(Y) = (Y + A_n + C_n - \mathtt{a}^2)p_n(Y) - A_{n-1}C_n p_{n-1}(Y). \tag{32} $$

The general formula for polynomials of continuous dual Hahn type *non-monic* is

$$ {}_3F_2\left(\begin{matrix} -n, -y+\mathtt{a}, y+\mathtt{a} \\ \mathtt{a}+\mathtt{b}, \mathtt{a}+\mathtt{c} \end{matrix}; 1\right) $$

as functions in $Y(y) = y^2$.

A.3 A new family of orthogonal polynomials

We will use the parameter a, with $a \notin \mathbb{N}$.

We consider the polynomials $p_n(X)$ defined for $n \geq 0$ by the formula

$$ \begin{aligned} (2(n+1)-a^2)\,{}_4F_3\left(\begin{matrix} -n, x+1-a, -x-2-a, 1-a \\ 1, 2-a, -a \end{matrix}; 1\right) \\ - a\,{}_4F_3\left(\begin{matrix} -n, x+1-a, -x-2-a, 1-a/2 \\ 1, 2-a, -a/2 \end{matrix}; 1\right) \end{aligned} \tag{33} $$

and by the relation $X = x(x+3)$. The first two terms are

$$p_0(X) = (1-a)\,(a+2)\,, \quad p_1(X) = \frac{\left(a^2+2a-2\right)}{a}X - \frac{\left(a^2+2a-4\right)\left(a^2+2a-1\right)}{a}.$$

By deflation, we find the simpler formula

$$\frac{n(a-x-1)(a+x+2)(2a+a^2-2(n+1))}{a(a-2)}\,{}_3F_2\left(\begin{matrix}-n+1, x+2-a, -x-1-a\\ 2, 3-a\end{matrix};1\right)$$
$$+(2(n+1)-a^2-a)\,{}_3F_2\left(\begin{matrix}-n, x+1-a, -x-2-a\\ 1, 2-a\end{matrix};1\right).$$

We can rewrite this expression in the form

$$\frac{n(a-x-1)(a+x+2)(2a+a^2-2(n+1))}{a(a-2)n!(-a+3)_{n-1}}u_n(X) + \frac{(2(n+1)-a^2-a)}{n!(-a+2)_n}v_n(X), \tag{34}$$

by introducing the monic polynomials

$$u_n(X) = n!\cdot(3-a)_{n-1}\cdot{}_3F_2\left(\begin{matrix}-n+1, x+2-a, -x-1-a\\ 2, 3-a\end{matrix};1\right),$$

and

$$v_n(X) = n!\cdot(2-a)_n\cdot{}_3F_2\left(\begin{matrix}-n, x+1-a, -x-2-a\\ 1, 2-a\end{matrix};1\right).$$

The first terms of these sequences of polynomials are

$$u_0(X) = 0, \quad u_1(X) = 1, \quad u_2(X) = X - a^2 - a + 8$$

and

$$v_0(X) = 1, \quad v_1(X) = X - b - 1, \quad v_2(X) = X^2 - 2\,(b-3)\,X + b^2 - 5b - 10,$$

where $b = (a-1)(a+3) - 2$.

These polynomials $u_n(X)$ and $v_n(X)$ are of *continuous dual Hahn* type, and therefore satisfy the recurrences (32), which can be written

$$u_{n+1} = (X + 9/4 + (n+1)(n-a+2) + (n-1)(n+2+a) - (1/2-a)^2)u_n$$
$$- n(n+1-a)(n-1)(n+2+a)u_{n-1}$$

and

$$v_{n+1} = (X + 9/4 + (n+1)(n-a+2) + n(n+a+3) - (1/2+a)^2)v_n$$
$$- n^2(n-a+1)(n+3+a)v_{n-1}.$$

From these recurrences and from identity (34), it follows that the sequence $(p_n(X))_{n\geq 0}$ satisfies a linear recurrence with polynomial coefficients

(in n, in x and in a) of order 4. This recurrence relation can be computed explicitly using closure algorithms for P-recursive sequences, such as those described in [20, §2.1].

It appears that this recurrence is not the one of minimal order verified by $(p_n(X))_{n\geq 0}$, and that this sequence satisfies the recurrence of order 2:

$$\begin{aligned}(a+n+3)\,(n+1)\,(b+1-2n)^2\,p_n(X) - ((b+1-2n)\,(b+3-2n)\,X + 8n^4\\ -8\,(b-2)\,n^3 + 2\left(b^2-14b-23\right)n^2 + 4\,(3b+7)\,bn - b^3 + 19b + 34\right)p_{n+1}(X)\\ + (n+2)\,(2-a+n)\,(b+3-2n)^2\,p_{n+2}(X) = 0.\end{aligned} \tag{35}$$

There are several ways to prove recurrence (35). A first approach consists in first conjecturing this recurrence from the first 20 terms $p_0(X), \ldots, p_{19}(X)$. This can be done using *guessing* algorithms, see [20, §1]. Then, denoting by S_n the shift operator which associates to the sequence $(u_n)_{n\geq 0}$ the sequence $(u_{n+1})_{n\geq 0}$, by $R_4(n, S_n)$ the (proven) recurrence operator of order 4 and by $C_2(n, S_n)$ the (conjectural) recurrence operator of order 2, we can verify by a Euclidean right division in the ring $\mathbb{Q}(a, x)\langle n, S_n\rangle$ that there exists an operator $Q_2(n, S_n)$ such that $R_4 = Q_2 C_2$. In particular, the sequence $(Q_2 \cdot p_n(X))_{n\geq 0}$ of the left-hand side of (35) satisfies a recurrence of order 2 whose leading term does not vanish at any natural number. As the first two terms of this sequence are zero, we conclude that the sequence $(Q_2 \cdot p_n(X))_{n\geq 0}$ is zero, so that the conjectured recurrence (35) is proved.

Another approach to prove (35) consists in directly using a *recurrence minimization* algorithm, which takes as input the recurrence of order 4 satisfied by the sequence $(p_n(X))_{n\geq 0}$ and the first terms $p_0(X), \ldots, p_3(X)$, and which returns the minimal-order recurrence verified by $(p_n(X))_{n\geq 0}$. Such an algorithm is implemented in Maple, in the command LREtools[MinimalRecurrence], based on results in [24]. Applied to our situation, this algorithm proves that the recurrence (35) is the minimal-order recurrence satisfied by $(p_n(X))_{n\geq 0}$.

From (35), it follows that $p_n(X)$ has degree n with the leading coefficient

$$\frac{(n+1-a)\cdot\left(a^2+2a-2n\right)}{n!\cdot a\cdot(2-a)_n}. \tag{36}$$

To show that the polynomials $p_n(X)$ are orthogonal polynomials, it is now sufficient to observe that the equations (35) and (36) entail that the monic version of these polynomials satisfies the recurrence (3) with coefficients

$$\mathtt{A}_n = 2(n+1)^2 - a(a+2)\frac{(2n-3a-a^2)(2(n+1)-a-a^2)}{(2n-a(a+2))(2(n+1)-a(a+2))}$$

and

$$\mathtt{B}_n = -\frac{(2(n+2)-a(a+2))(2n-a(a+2))}{(2(n+1)-a(a+2))^2}\times (n+3+a)(n+1-a)(n+1)^2,$$

for $n \geq 0$.

We end with two remarks. First, we observe a symmetry under the involution $a \mapsto -2-a$ in $\mathtt{A}_n$ and $\mathtt{B}_n$.

Then, the same simplification occurs formally in the coefficients $\mathtt{A}_n$ and $\mathtt{B}_n$ when $a=0$ or $a=-2$:

$$\mathtt{A}_n = 2(n+1)^2 \quad \text{and} \quad \mathtt{B}_n = -n(n+1)(n+2)(n+3).$$

However the coefficient $\mathtt{B}_0$ is then zero, so this particular case defines orthogonal polynomials q_n but does not give rise to a Jacobi continued fraction. The family of orthogonal polynomials $q_n(X-2)$ is essentially identical to the case $\nu = 1$ of a family considered in [4], see also the sequence **A130182**.

B An auxiliary formula

We need the following formula.

Proposition B.1. *For all integers $0 \leq k$ and $0 \leq N$, we have*

$$\sum_{\ell=0}^{k+1} \frac{(-k-1)_\ell (k+\ell+1)}{(1)_\ell} \sum_{j=0}^{k+\ell} 2^{j-k-\ell} \frac{(x)_{j+N}}{(j+N)!} = (-1)^{k+1}(2k+2)\frac{(x-k)_{2k+1+N}}{(2k+1+N)!}.$$

Proof. By exchanging summations, we find

$$\sum_{0 \leq j \leq 2k+1} \frac{2^j (x)_{j+N}}{(j+N)!} \sum_{0 \leq \ell \leq k+1} \frac{(-k-1)_\ell (k+l+1)}{2^{k+\ell}\ell!},$$

where the summation variable ℓ also satisfies $j-k \leq \ell$. Using Lemma B.2 below to simplify the internal sum, we get

$$2 \sum_{k+1 \leq j \leq 2k+1} \frac{(x)_{j+N}}{(j+N)!} \frac{(-1)^{j-k}(k+1)!}{(j-k-1)!(2k+1-j)!}.$$

We can write this as

$$(-1)\frac{(x)_{k+1+N}(2k+2)}{(k+1+N)!} {}_2F_1\left(\begin{matrix} x+k+1+N, -k \\ k+2+N \end{matrix}; 1\right).$$

By Gauss's summation theorem, the hypergeometric evaluation is equal to

$$\frac{(1-x)_k}{(k+2+N)_k},$$

which allows us to conclude. □

Lemma B.2. *For $0 \leq k$ and $0 \leq M \leq k$,*

$$\sum_{0 \leq \ell \leq M} \frac{(-k-1)_\ell (k+\ell+1)}{2^{k+\ell} \ell!} = (-1)^M \frac{(k+1)!}{2^{M+k} M!(k-M)!} \tag{37}$$

and

$$\sum_{0 \leq \ell \leq k+1} \frac{(-k-1)_\ell (k+\ell+1)}{2^{k+\ell} \ell!} = 0. \tag{38}$$

Proof. To prove formula (37), we start from the observation that for all $0 \leq \ell \leq k$, we have the *telescopic* identity

$$\frac{(-k-1)_\ell (k+\ell+1)}{2^{k+\ell} \ell!} = \frac{(-1)^\ell \, \ell \binom{k+1}{\ell}}{2^{k+\ell-1}} - \frac{(-1)^{\ell+1} \, (\ell+1) \binom{k+1}{\ell+1}}{2^{k+\ell}}. \tag{39}$$

Then, for all $0 \leq M \leq k$, formula (37) is deduced by summing the identity (39) over ℓ between 0 and M.

On the other hand, identity (39) holds true for $\ell = k+1$, and a new summation of (39) for ℓ between 0 and $k+1$ proves equality (38). □

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