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ABSTRACT. In this paper, we study the cyclicity problem with respect to the forward shift operator S_b acting on the de Branges–Rovnyak space $\mathscr{H}(b)$ associated to a function b in the closed unit ball of H^∞ and satisfying $\log(1-|b|) \in L^1(\mathbb{T})$. We present a characterisation of cyclic vectors for S_b when b is a rational function which is not a finite Blaschke product. This characterisation can be derived from the description, given in [24], of invariant subspaces of S_b in this case, but we provide here an elementary proof. We also study the situation where b has the form $b = (1+I)/2$, where I is a non-constant inner function such that the associated model space $K_I = \mathscr{H}(I)$ has an orthonormal basis of reproducing kernels.

To the memory of Mohamed Zarrabi (1964 – 2021)

1. INTRODUCTION

In this paper, we study the cyclicity problem with respect to the forward shift operator S_b acting on the de Branges–Rovnyak space $\mathscr{H}(b)$, associated to a function b belonging to the closed unit ball of H^∞ and satisfying $\log(1-|b|) \in L^1(\mathbb{T})$. This problem of cyclicity has a long and outstanding history and many efforts have been dedicated to solving it in various reproducing kernel Hilbert spaces. It finds its roots in the pioneering work of Beurling who showed that cyclicity of a function f in the Hardy space H^2 is equivalent to f being outer. Brown and Shields studied the cyclicity problem in the Dirichlet spaces $\mathcal{D}_\alpha$ for polynomials that do not have zeros inside the disc, but that may have some zeros on its boundary. Such functions are cyclic in $\mathcal{D}_\alpha$ if and only if $\alpha \leqslant 1$. They also proved that the set of zeros on the unit circle (in the radial sense) of cyclic functions in the Dirichlet space $\mathcal{D}_\alpha$ has zero logarithmic capacity, and this led them to ask whether any outer function with this property is cyclic [4]. This problem is still open although there has been relevant contributions to the topic by a number of authors; e.g. see [2, 3, 11, 12, 22, 26]. We also mention the paper [10] where the authors prove the Brown–Shields conjecture in the context of some particular Dirichlet type spaces $\mathcal{D}(\mu)$, which happen to be related to our context of de Branges–Rovnyak spaces [6].

The de Branges–Rovnyak spaces $\mathscr{H}(b)$ (see the precise definition in Section 2) have been introduced by L. de Branges and J. Rovnyak in the context of model theory (see [7]). A whole class of Hilbert space contractions is unitarily equivalent to $S^*|\mathscr{H}(b)$, for an appropriate function b belonging to the closed unit ball of H^∞. Here S is the forward shift operator on H^2 and S^*, its adjoint, is the backward shift operator on H^2. The space $\mathscr{H}(b)$ is invariant with respect to S^* for every b in the closed unit ball of H^∞, and S^* defines a

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bounded operator on $\mathscr{H}(b)$, endowed with its own Hilbert space topology. On the contrary, $\mathscr{H}(b)$ is invariant with respect to S if and only if $\log(1-|b|) \in L^1(\mathbb{T})$, i.e. if and only if b is a non-extreme point of the closed unit ball of H^∞. See [18, Corollary 25.2]. In [19], it is proved that if $\log(1-|b|) \in L^1(\mathbb{T})$, the cyclic vectors of $S^*|\mathscr{H}(b)$ are precisely the cyclic vectors of S^* which live in $\mathscr{H}(b)$. Note that cyclic vectors of S^* have been characterised by Douglas–Shapiro–Shields [8]. The result of [19] is based on a nice description, due to Sarason, of closed invariant subspaces of $S^*|\mathscr{H}(b)$. See also [18, Corollary 24.32]. Unfortunately an analogous description of closed invariant subspaces of $S_b = S|\mathscr{H}(b)$ remains an unsolved and difficult problem. In [24], Gu–Luo–Richter give an answer in the case where b is a rational function which is not inner, generalising a result of Sarason [27].

The purpose of this paper is to study the cyclic vectors of S_b when the function b is such that $\log(1-|b|) \in L^1(\mathbb{T})$. In Section 2, we present a quick overview of some useful properties of de Branges–Rovnyak spaces. Then, in Section 3, we give some general facts on cyclic vectors and completely characterise holomorphic functions in a neighborhood of the closed unit disc which are cyclic for S_b. In Section 4, we give a characterisation of cyclic vectors for S_b when b is rational (and not inner). Of course, this characterisation can be derived from the description, given in [24], of invariant subspaces of S_b when b is a non-inner rational function. Nevertheless, we will give a more direct and easier proof of this characterisation. Finally, Section 5 will be devoted to the situation where $b = (1+I)/2$, where I is a non-constant inner function such that the associated model space $K_I = \mathscr{H}(I)$ has an orthonormal basis of reproducing kernels.

2. Preliminaries on $\mathscr{H}(b)$ spaces

2.1. **Definition of de Branges-Rovnyak spaces.** Let

$$\mathrm{ball}(H^\infty) := \Big\{ b \in H^\infty : \|b\|_\infty = \sup_{z\in\mathbb{D}} |b(z)| \leqslant 1 \Big\}$$

be the closed unit ball of H^∞, the space of bounded analytic functions on the open unit disk $\mathbb{D}$, endowed with the sup norm. For $b \in \mathrm{ball}(H^\infty)$, the *de Branges–Rovnyak space* $\mathscr{H}(b)$ is the reproducing kernel Hilbert space on $\mathbb{D}$ associated with the positive definite kernel k_λ^b, $\lambda \in \mathbb{D}$, defined as

$$k_\lambda^b(z) = \frac{1 - \overline{b(\lambda)}b(z)}{1-\overline{\lambda}z}, \quad z \in \mathbb{D}. \tag{2.1}$$

It is known that $\mathscr{H}(b)$ is contractively contained in the well-studied Hardy space H^2 of analytic functions f on $\mathbb{D}$ for which

$$\|f\|_{H^2} := \Big(\sup_{0<r<1} \int_{\mathbb{T}} |f(r\xi)|^2 dm(\xi) \Big)^{\frac{1}{2}} < \infty,$$

where m is the normalised Lebesgue measure on the unit circle $\mathbb{T} = \{\xi \in \mathbb{C} : |\xi| = 1\}$ [9, 21]. For every $f \in H^2$, the radial limit $\lim_{r\to 1^-} f(r\xi) =: f(\xi)$ (even the non-tangential limit $f(\xi) := \lim_{z \underset{\triangleleft}{\to} \xi} f(z)$) exists for m-a.e. $\xi \in \mathbb{T}$, and

$$\|f\|_{H^2} = \Big(\int_{\mathbb{T}} |f(\xi)|^2 dm(\xi) \Big)^{\frac{1}{2}}. \tag{2.2}$$

Though $\mathscr{H}(b)$ is contractively contained in H^2, it is generally not closed in the H^2 norm. It is known that $\mathscr{H}(b)$ is closed in H^2 if and only if $b = I$ is an inner function, meaning

that $|I(\zeta)| = 1$ for a.e. $\zeta \in \mathbb{T}$. In this case, $\mathscr{H}(b) = K_I = (IH^2)^\perp$ is the so-called *model space* associated to I. Note that $K_I = H^2 \cap I\overline{zH^2}$ (see [20, Proposition 5.4]), and then $K_I = \operatorname{Ker} T_{\bar{I}}$, where $T_{\bar{I}}$ is the Toeplitz operator with symbol $\bar{I}$ defined on H^2 as $T_{\bar{I}} f = P_+(\bar{I} f)$, where P_+ denotes the orthogonal projection from L^2 onto H^2.

We refer the reader to the book [28] by Sarason and to the monograph [17], [18] by Fricain and Mashreghi for an in-depth study of de Branges-Rovnyak spaces and their connections to numerous other topics in operator theory and complex analysis.

In this paper, we will always assume that b is a non-extreme point of $\operatorname{ball}(H^\infty)$, which is equivalent to requiring that $\log(1-|b|) \in L^1(\mathbb{T})$. Under this assumption, there is a unique outer function a, called *the pythagorean mate* for b, such that $a(0) > 0$ and $|a|^2 + |b|^2 = 1$ a.e. on $\mathbb{T}$. There are two important subspaces of $\mathscr{H}(b)$ which can be defined via this function a. The first one is the space $\mathscr{M}(a) = aH^2$, equipped with the range norm

$$\|af\|_{\mathscr{M}(a)} = \|f\|_2, \qquad f \in H^2.$$

The second one is $\mathscr{M}(\bar{a}) = T_{\bar{a}} H^2$, equipped also with the range norm

$$\|T_{\bar{a}} f\|_{\mathscr{M}(\bar{a})} = \|f\|_2, \qquad f \in H^2.$$

Note that since a is outer, the Toeplitz operator $T_{\bar{a}}$ is one-to-one and so the above norm is well defined. It is known that $\mathscr{M}(a)$ is contractively contained into $\mathscr{M}(\bar{a})$, which itself is contractively contained into $\mathscr{H}(b)$. See [18, Theorem 23.2]. Note that $\mathscr{M}(a)$ is not necessarily closed in $\mathscr{H}(b)$. See [18, Theorem 28.35] for a characterisation of closeness of $\mathscr{M}(a)$ in $\mathscr{H}(b)$-norm. There is also an important relation between $\mathscr{H}(b)$ and $\mathscr{M}(\bar{a})$ which gives a recipe to compute the norm in $\mathscr{H}(b)$. Indeed, if $f \in H^2$, then $f \in \mathscr{H}(b)$ if and only if there is a function $f^+ \in H^2$ satisfying $T_{\bar{b}} f = T_{\bar{a}} f^+$ (and then f^+ is necessarily unique). Moreover, in this case, we have

$$\|f\|_b^2 = \|f\|_2^2 + \|f^+\|_2^2.$$

Also,

$$\langle f, g\rangle_b = \langle f, g\rangle_2 + \langle f^+, g^+\rangle_2 \quad \text{for every } f, g \in \mathscr{H}(b). \tag{2.3}$$

See [18, Theorem 23.8]. Finally, let us recall that $\mathscr{H}(b) = \mathscr{M}(\bar{a})$ if and only if (a, b) forms a corona pair, that is

$$\inf_{\mathbb{D}} (|a| + |b|) > 0. \tag{HCR}$$

See [18, Theorem 28.7].

A crucial fact on de Branges-Rovnyak space is that the space $\mathscr{H}(b)$ is invariant with respect to the shift operator $S : f \mapsto zf$ if and only if the function b is non-extreme. Since we will consider in this paper only the case where b is non-extreme, $\mathscr{H}(b)$ is indeed invariant by S, and S defines a bounded operator on $\mathscr{H}(b)$, endowed with its own Hilbert space topology, which we will denote by S_b. The functions z^n belong to $\mathscr{H}(b)$ for every $n \geqslant 0$. Actually, we have

$$\operatorname{Span}(z^n : n \geqslant 0) = \mathscr{H}(b), \tag{2.4}$$

where $\operatorname{Span}(A)$ denotes the closed linear span generated by vectors from a certain family A. In other words, the polynomials are dense in $\mathscr{H}(b)$. See [18, Theorem 23.13]. Note that (2.4) exactly means that the constant function 1 is cyclic for S_b.

Another tool which will turn out to be useful when studying the cyclicity for the shift operator is the notion of *multiplier*. Recall that the set $\mathfrak{M}(\mathscr{H}(b))$ of multipliers of $\mathscr{H}(b)$ is defined as

$$\mathfrak{M}(\mathscr{H}(b)) = \{\varphi \in \mathrm{Hol}(\mathbb{D}) : \varphi f \in \mathscr{H}(b), \forall f \in \mathscr{H}(b)\}.$$

Using the closed graph theorem, it is easy to see that when $\varphi \in \mathfrak{M}(\mathscr{H}(b))$, then M_φ, the multiplication operator by φ, is bounded on $\mathscr{H}(b)$. The algebra of multipliers is a Banach algebra when equipped with the norm $\|\varphi\|_{\mathfrak{M}(\mathscr{H}(b)} = \|M_\varphi\|_{\mathcal{L}(\mathscr{H}(b))}$. Using standard arguments, we see that $\mathfrak{M}(\mathscr{H}(b)) \subseteq H^\infty \cap \mathscr{H}(b)$. In general, this inclusion is strict. See [18, Example 28.24]. However, we will encounter below a situation (when b is a rational function which is not a finite Blaschke product) where we have the equality $\mathfrak{M}(\mathscr{H}(b)) = H^\infty \cap \mathscr{H}(b)$.

2.2. **Some properties of the reproducing kernels of H^2 in de Branges-Rovnyak spaces.** Recall that we are supposing that b is non-extreme. If we denote by $k_\lambda(z) = (1-\overline{\lambda}z)^{-1}$ the reproducing kernel of H^2 at the point $\lambda \in \mathbb{D}$, then k_λ belongs to $\mathscr{H}(b)$ and

$$\mathrm{Span}(k_\lambda : \lambda \in \mathbb{D}) = \mathscr{H}(b). \tag{2.5}$$

See [18, Corollary 23.26] or [23, Lemma 7]. We also know (see [18, Theorem 23.23]) that $bk_\lambda \in \mathscr{H}(b)$ for every $\lambda \in \mathbb{D}$, and that for every $f \in \mathscr{H}(b)$ we have

$$\langle f, k_\lambda\rangle_b = f(\lambda) + \frac{b(\lambda)}{a(\lambda)} f^+(\lambda) \quad \text{and} \quad \langle f, bk_\lambda\rangle_b = \frac{f^+(\lambda)}{a(\lambda)}. \tag{2.6}$$

Using these two equations, we can produce an interesting complete family in $\mathscr{H}(b)$ which will be of use to us.

Lemma 2.1. *Let b be a non-extreme point of the closed unit ball of H^∞, and let c be a complex number with $|c| < 1$. Then*

$$\mathit{Span}(k_\mu - cbk_\mu : \mu \in \mathbb{D}) = \mathscr{H}(b).$$

Proof. Let $h \in \mathscr{H}(b)$, and assume that for every $\mu \in \mathbb{D}$, h is orthogonal in $\mathscr{H}(b)$ to $k_\mu - cbk_\mu$. According to (2.6), we have

$$0 = h(\mu) + \frac{b(\mu)}{a(\mu)} h^+(\mu) - \overline{c}\frac{h^+(\mu)}{a(\mu)}.$$

This can be rewritten as $ah = -bh^+ + \overline{c}h^+$. Multiplying this equality by $\bar{b}$ and using the fact that $|a|^2 + |b|^2 = 1$ a.e. on $\mathbb{T}$, we obtain

$$a(\bar{b}h - \bar{a}h^+) = -(1-\overline{c}\bar{b})h^+.$$

Note that $|1-\overline{c}\bar{b}| \geqslant 1 - |c| > 0$, and so the last identity can be written as

$$\frac{\bar{b}h - \bar{a}h^+}{1-\overline{c}\bar{b}} = -\frac{h^+}{a}.$$

On the one hand, this equality says that $\frac{h^+}{a}$ belongs to L^2 and since a is outer, we have $\frac{h^+}{a} \in H^2$. See [25, page 43]. On the other hand, by definition of h^+, the function $\bar{b}h - \bar{a}h^+$ belongs to $\overline{H^2_0}$ and since $(1-\overline{c}\bar{b})^{-1}$ is in $\overline{H^\infty}$, we also have $\frac{h^+}{a} \in \overline{H^2_0}$. Then $\frac{h^+}{a}$ belongs to $H^2 \cap \overline{H^2_0} = \{0\}$. Finally we get that $h^+ = 0$ and thus that $h = 0$. □

2.3. **Boundary evaluation points on $\mathscr{H}(b)$.** An important tool in the cyclicity problem will be the boundary evaluation points for $\mathscr{H}(b)$. It is known that the description of these points depends on the inner-outer factorisation of b. Recall that any b in $\mathrm{ball}(H^\infty)$ can be decomposed as

$$b(z) = B(z)S_\sigma(z)O(z), \qquad z \in \mathbb{D}, \tag{2.7}$$

where

$$B(z) = \gamma \prod_{n\geqslant 1} \left(\frac{|a_n|}{a_n} \frac{a_n - z}{1 - \overline{a}_n z} \right) \quad \text{is a Blaschke product,}$$

with $|\gamma| = 1$, $a_n \in \mathbb{D}$ for every $n \geqslant 1$, and $\sum_{n\geqslant 1}(1 - |a_n|) < +\infty$,

$$S_\sigma(z) = \exp\left(- \int_{\mathbb{T}} \frac{\xi + z}{\xi - z}\, d\sigma(\xi) \right) \quad \text{is a singular inner function,}$$

with σ a positive finite Borel measure on $\mathbb{T}$ which is singular with respect to the Lebesgue measure, and

$$O(z) = \exp\left(\int_{\mathbb{T}} \frac{\xi + z}{\xi - z} \log|b(\xi)|\, dm(\xi) \right)$$

is the outer part of b. Now, let $E_0(b)$ be the set of points $\zeta \in \mathbb{T}$ satisfying the following condition:

$$\sum_n \frac{1 - |a_n|^2}{|\zeta - a_n|^2} + \int_{\mathbb{T}} \frac{d\sigma(\xi)}{|\zeta - \xi|^2} + \int_{\mathbb{T}} \frac{\big|\log|b(\xi)|\big|}{|\zeta - \xi|^2}\, dm(\xi) < \infty. \tag{2.8}$$

It is proved in [16] that for every point $\zeta \in \mathbb{T}$, every function $f \in \mathscr{H}(b)$ has a non-tangential limit at ζ if and only if $\zeta \in E_0(b)$. This is also equivalent to the property that b has an *angular derivative (in the sense of Carathéodory) at* ζ, meaning that b and b' both have a non-tangential limit at ζ and $|b(\zeta)| = 1$. Moreover, in this case, the linear map

$$f \longmapsto f(\zeta) := \lim_{\substack{z\to\zeta \\ \sphericalangle}} f(z) \tag{2.9}$$

is bounded on $\mathscr{H}(b)$. The function k_ζ^b defined by

$$k_\zeta^b(z) = \frac{1 - \overline{b(\zeta)}b(z)}{1 - \overline{\zeta}z}, \quad z \in \mathbb{D},$$

belongs to $\mathscr{H}(b)$, and

$$\langle f, k_\zeta^b \rangle_b = f(\zeta) \quad \text{for every } f \in \mathscr{H}(b).$$

We call the function k_ζ^b *the reproducing kernel of* $\mathscr{H}(b)$ *at the point* ζ, and (2.9) means that the reproducing kernels k_z^b tend weakly to k_ζ^b as $z \in \mathbb{D}$ tends non-tangentially to ζ. See [18, Theorem 25.1]. There is also a nice connection between the boundary evaluation points and the point spectrum of S_b^* in the case where b is a non-extreme point in $\mathrm{ball}(H^\infty)$: for $\zeta \in \mathbb{T}$, we have that

$$\bar{\zeta} \text{ is an eigenvalue for } S_b^* \text{ if and only if } b \text{ has an angular derivative at } \zeta. \tag{2.10}$$

The boundary evaluation points play a particular role in the description of certain orthogonal basis of reproducing kernels in model spaces K_I, the so-called *Clark basis*. Given an inner function I and $\alpha \in \mathbb{T}$, recall that by Herglotz theorem, there is a unique

finite positive Borel measure σ_α on $\mathbb{T}$, singular with respect to the Lebesgue measure, such that

$$\frac{1-|I(z)|^2}{|\alpha - I(z)|^2} = \int_{\mathbb{T}} \frac{1-|z|^2}{|\xi - z|^2}\, d\sigma_\alpha(\xi), \qquad z \in \mathbb{D}. \tag{2.11}$$

The collection $(\sigma_\alpha)_{\alpha\in\mathbb{T}}$ is the family of *Clark measures* of I.

Let $E_\alpha = \{\zeta \in E_0(I) : I(\zeta) = \alpha\}$. By [18, Theorem 21.5], the point ζ belongs to E_α if and only if the measure σ_α has an atom at ζ. In this case,

$$\sigma_\alpha(\{\zeta\}) = \frac{1}{|I'(\zeta)|} = \frac{1}{\|k_\zeta^I\|_2^2}. \tag{2.12}$$

See [18, Theorems 21.1 and 21.5]. When σ_α is a discrete measure, its support is exactly the set E_α, which is necessarily countable, and we write it as

$$E_\alpha = \{\zeta_n : n \geqslant 1\} = \{\zeta \in E_0(I) : I(\zeta) = \alpha\}. \tag{2.13}$$

Then, in this case, Clark proved in [5] that the family $\{k_{\zeta_n}^I : n \geqslant 1\}$ forms an orthogonal basis of K_I (and the family $\{k_{\zeta_n}^I/\|k_{\zeta_n}^I\|_2 : n \geqslant 1\}$ forms an orthonormal basis of K_I). It is called *the Clark basis of* K_I associated to point $\alpha \in \mathbb{T}$.

2.4. **A description of $\mathscr{H}(b)$ when b is a rational function.** Although the contents of the space $\mathscr{H}(b)$ may seem mysterious for a general non-extreme $b \in \mathrm{ball}(H^\infty)$, it turns out that when b is a rational function (and not a finite Blaschke product – in which case b is an inner function, and thus extreme), the description of $\mathscr{H}(b)$ is quite explicit. Since our b is a non-extreme point of $\mathrm{ball}(H^\infty)$, it admits a pythagorean mate a, which is also a rational function. In fact, the function a can be obtained from the Fejér–Riesz theorem (see [13]). Let $\zeta_1, \ldots, \zeta_n$ denote the *distinct* roots of a on $\mathbb{T}$, with corresponding multiplicities $m_1, \ldots, m_n$, and define the polynomial a_1 by

$$a_1(z) := \prod_{k=1}^{n} (z - \zeta_k)^{m_k}. \tag{2.14}$$

Results from [6, 13] show that $\mathscr{H}(b)$ has an explicit description as

$$\mathscr{H}(b) = a_1 H^2 \oplus \mathscr{P}_{N-1} = \mathcal{M}(a_1) \oplus \mathscr{P}_{N-1}, \tag{2.15}$$

where $N = m_1 + \cdots + m_n$, and $\mathscr{P}_{N-1}$ denotes the set of polynomials of degree at most $N-1$. Since a/a_1 is invertible in $H^\infty = \mathfrak{M}(H^2)$, note that $\mathcal{M}(a) = \mathcal{M}(a_1)$. The notation $\oplus$ above denotes a topological direct sum in $\mathscr{H}(b)$. But this sum may not be an orthogonal one. See [13]. In particular, $\mathcal{M}(a_1) \cap \mathscr{P}_{N-1} = \{0\}$. Moreover, if $f \in \mathscr{H}(b)$ is decomposed with respect to (2.15) as

$$f = a_1 \widetilde{f} + p, \quad \text{where } \widetilde{f} \in H^2 \text{ and } p \in \mathscr{P}_{N-1}, \tag{2.16}$$

an equivalent norm on $\mathscr{H}(b)$ (to the natural one induced by the positive definite kernel k_λ^b, $\lambda \in \mathbb{D}$, above) is

$$|||a_1 \widetilde{f} + p|||_b^2 := \|\widetilde{f}\|_{H^2}^2 + \|p\|_{H^2}^2. \tag{2.17}$$

Note that the functions $\widetilde{f} \in H^2$ and $p \in \mathscr{P}_{N-1}$ appearing in the decomposition (2.16) are unique. It is important to note that $|||\cdot|||_b$ is only equivalent to the original norm $\|\cdot\|_b$ associated to the kernel in (2.1), and its scalar product as well as the reproducing kernels and the adjoints of operators defined on $\mathscr{H}(b)$ will be different. However, the cyclicity

problem for S_b does not depend on the equivalent norm we consider. So, in the rational case, there is no problem to work with the norm given by (2.17).

Note also that when the zeros $\zeta_1, \dots, \zeta_n$ of the polynomial a_1 are simple (i.e. when $m_k = 1$, $1 \leqslant k \leqslant n$), then the space $\mathscr{H}(b)$ coincides with a Dirichlet type space $\mathcal{D}(\mu)$, where μ is a finite sum of Dirac masses at the points ζ_k, $1 \leqslant k \leqslant n$. See [6]. So our results are also connected to the works [11] and [12] on the cyclicity problem for Dirichlet spaces.

Using (2.16) and the standard estimate that any $g \in H^2$ satisfies

$$|g(z)| \leqslant \frac{\|g\|_2}{\sqrt{1-|z|^2}} \quad \text{for all } z \in \mathbb{D}, \tag{2.18}$$

we see that for fixed $1 \leqslant k \leqslant n$ and for each $f \in \mathscr{H}(b)$ we have

$$f(\zeta_k) = \lim_{\substack{z \to \zeta_k \\ \sphericalangle}} f(z) = p(\zeta_k), \tag{2.19}$$

where $f = a_1 \widetilde{f} + p$ with $\widetilde{f} \in H^2$ and $p \in \mathscr{P}_{N-1}$. In particular,

$$E_0(b) = \{\zeta_k : 1 \leqslant k \leqslant n\}. \tag{2.20}$$

Finally, let us mention that when $b \in \operatorname{ball}(H^\infty)$ is a rational function and not a finite Blaschke product, then $\mathfrak{M}(\mathscr{H}(b)) = H^\infty \cap \mathscr{H}(b)$. See [15].

2.5. **A description of $\mathscr{H}(b)$ when $b = (1+I)/2$, with I an inner function.** There is another situation where we have an explicit description of the space $\mathscr{H}(b)$: this is when $b = \frac{1+I}{2}$ and I is an inner function with $I \not\equiv 1$. In this case, b is a non-extreme point of $\operatorname{ball}(H^\infty)$, and its Pythagorean mate (up to a unimodular constant) is $a = \frac{1-I}{2}$. Moreover, (a, b) satisfies (HCR), since $|a|^2 + |b|^2 \geqslant \frac{1}{2}$ on $\mathbb{D}$. In particular $\mathscr{H}(b) = \mathcal{M}(\bar{a})$, with equivalent norms.

Under the assumption that $I(0) \neq 0$, it is proved in [14] that

$$\mathscr{H}(b) = \mathcal{M}(a) \overset{\perp}{\oplus}_b K_I, \tag{2.21}$$

where the direct sum $\overset{\perp}{\oplus}_b$ is orthogonal with respect to the $\mathscr{H}(b)$ norm. In particular, every $f \in \mathscr{H}(b)$ can be written in a unique way as

$$f = (1-I)g_1 + g_2, \qquad \text{with } g_1 \in H^2 \text{ and } g_2 \in K_I. \tag{2.22}$$

It turns out that the same proof holds without any assumption on the value of $I(0)$. For completeness's sake, we present it in Lemma 2.2 below. We also give an equivalent norm on $\mathscr{H}(b)$ analogue to (2.17).

Lemma 2.2. *Let I be an inner function with $I \not\equiv 1$, and let $b = (1+I)/2$. Then the following assertions hold:*

(i) $\mathscr{H}(b) = (1-I)H^2 \overset{\perp}{\oplus}_b K_I$, where $\overset{\perp}{\oplus}_b$ denotes an orthogonal direct sum in $\mathscr{H}(b)$;
(ii) if for $f = (1-I)g_1 + g_2 \in \mathscr{H}(b)$, $g_1 \in H^2$, $g_2 \in K_I$, we define

$$|||f|||_b^2 = ||g_1||_2^2 + ||g_2||_2^2,$$

then $|||\,.\,|||_b$ is a norm on $\mathscr{H}(b)$ which is equivalent to $||\,.\,||_b$.

Proof. (i) We have $\mathscr{H}(b) = \mathcal{M}(\bar{a})$ with equivalent norms, where $a = \frac{1-I}{2}$ is the Pythagorean mate of b. Also, $\frac{\bar{a}}{a} = \frac{1-\bar{I}}{1-I} = -\bar{I}$ a.e. on $\mathbb{T}$, and thus $T_{\bar{a}/a} = -T_{\bar{I}}$. Hence $\operatorname{Ker} T_{\bar{a}/a} = \operatorname{Ker} T_{\bar{I}} = K_I$. Moreover, $T_{a/\bar{a}} = -T_I$ has closed range, and thus

$$H^2 = \operatorname{Ran}(T_I) \overset{\perp}{\oplus} \operatorname{Ker}(T_I^*) = \operatorname{Ran}(T_I) \overset{\perp}{\oplus} \operatorname{Ker}(T_{\bar{I}}) = T_{a/\bar{a}} H^2 \overset{\perp}{\oplus} K_I \tag{2.23}$$

(the sign $\overset{\perp}{\oplus}$ denotes here an orthogonal direct sum in H^2). Using now the fact that $T_{\bar{a}}$ is an isometry from H^2 onto $\mathcal{M}(\bar{a}) = T_{\bar{a}} H^2$ (equipped with the range norm), applying $T_{\bar{a}}$ to the equation (2.23), and using the identity $T_{\bar{a}}\, T_{a/\bar{a}} = T_a$, we obtain

$$\mathscr{H}(b) = \mathcal{M}(\bar{a}) = \mathcal{M}(a) \overset{\perp}{\oplus}_{\bar{a}} T_{\bar{a}}\, K_I,$$

where the notation $\overset{\perp}{\oplus}_{\bar{a}}$ represents an orthogonal direct sum with respect to the range norm on $\mathcal{M}(\bar{a})$. Since $T_{\bar{I}}\, K_I = \{0\}$ and $\bar{a} = (1-\bar{I})/2$, we have $T_{\bar{a}}\, K_I = (Id - T_{\bar{I}})\, K_I = K_I$, and so

$$\mathscr{H}(b) = \mathcal{M}(\bar{a}) = \mathcal{M}(a) \overset{\perp}{\oplus}_{\bar{a}} K_I = (1-I)H^2 \overset{\perp}{\oplus}_{\bar{a}} K_I. \tag{2.24}$$

It now remains to prove that the direct sum in this decomposition of $\mathscr{H}(b)$ is in fact orthogonal with respect to the $\mathscr{H}(b)$ norm.

Let $f \in H^2$ and $g \in K_I$. Our aim is to show that $\langle (1-I)f, g\rangle_b = 0$. Note that

$$T_{\bar{b}}\, g = T_{(1+\bar{I})/2}\, g = \frac{1}{2}\, g = T_{\bar{a}}\, g$$

from which it follows that

$$g^+ = g. \tag{2.25}$$

Moreover, since $\bar{b}\, a = -\bar{a}\, b$ a.e. on $\mathbb{T}$, we have

$$T_{\bar{b}}\big((1-I)f\big) = P_+\,(2\bar{b}\, af) = -P_+(2\bar{a}\, bf) = T_{\bar{a}}\,(-2bf),$$

whence we get

$$\big((1-I)f\big)^+ = -2bf = -(1+I)f. \tag{2.26}$$

By (2.3), (2.25) and (2.26), it follows that

$$\langle (1-I)f, g\rangle_b = \langle (1-I)f, g\rangle_2 - \langle 2bf, g\rangle_2 = \langle (1-I-2b)f, g\rangle_2 = -2\,\langle If, g\rangle_b = 0$$

because g belongs to K_I.

(ii) Since $\mathscr{H}(b) = (1-I)H^2 \overset{\perp}{\oplus}_b K_I$, we have

$$||(1-I)g_1 + g_2||_b^2 = ||(1-I)g_1||_b^2 + ||g_2||_b^2, \qquad g_1 \in H^2,\ g_2 \in K_I.$$

But observe that by (2.3) and (2.26) we have

$$||(1-I)g_1||_b^2 = ||(1-I)g_1||_2^2 + ||(1+I)g_1||_2^2 = 4||g_1||_2^2,$$

while we get from (2.3) and (2.25) that $||g_2||_b^2 = 2||g_2||_2^2$. Thus

$$||(1-I)g_1 + g_2||_b^2 = 4||g_1||_2^2 + 2||g_2||_2^2,$$

and from this the norm $|||\,.\,|||_b$ is easily seen to be equivalent to $||\,.\,||_b$. □

The next result is an analogue of (2.19) for the case where $b = (1+I)/2$ with respect to decomposition (2.22).

Lemma 2.3. *Let I be an inner function, $I \not\equiv 1$, and let $b = (1+I)/2$. Let $\zeta \in E_0(I)$ be such that $I(\zeta) = 1$. Then $\zeta \in E_0(b)$. Moreover, if $f = (1-I)g_1 + g_2$, $g_1 \in H^2$ and $g_2 \in K_I$, then $f(\zeta) = g_2(\zeta)$.*

Proof. As mentioned above, since $\zeta \in E_0(I)$ the function g_2 has a non-tangential limit at the point ζ. Thus it remains to prove that $(1-I)g_1$ has a zero non-tangential limit at ζ. To this purpose, write for $z \in \mathbb{D}$

$$\begin{aligned}(1-I(z))g_1(z) =& \frac{1-\overline{I(\zeta)}I(z)}{1-\overline{\zeta}z}(1-\overline{\zeta}z)g_1(z)\\ =& k_\zeta^I(z)\,\overline{\zeta}\,(\zeta - z)g_1(z)\\ =& \langle k_\zeta^I, k_z^I\rangle_2\,\overline{\zeta}\,(\zeta-z)g_1(z).\end{aligned}$$

Now, since $\zeta \in E_0(I)$, k_z^I tends weakly to k_ζ^I as z tends to ζ non-tangentially. Hence

$$\lim_{\substack{z\to\zeta\\ \sphericalangle}} \langle k_\zeta^I, k_z^I\rangle_2 = \|k_\zeta^I\|_2^2 < \infty.$$

Moreover, using the estimate (2.18), we obtain that

$$\lim_{\substack{z\to\zeta\\ \sphericalangle}} (\zeta - z)g_1(z) = 0,$$

from which it follows that

$$\lim_{\substack{z\to\zeta\\ \sphericalangle}} (1-I(z))g_1(z) = 0.$$

□

In the case where $b = (1+I)/2$ and I is an inner function with $I \not\equiv 1$, there is no complete characterisation of multipliers for $\mathscr{H}(b)$. Nevertheless, we have at our disposal a sufficient condition which will be useful for our study of cyclicity. Before stating this result (Lemma 2.6) on multipliers, we recall a well-known property of model spaces, of which we provide a proof for completeness's sake.

Lemma 2.4. *Let I be an inner function and let $f \in K_I$ and $g \in K_I \cap H^\infty$. Then $fg \in K_{I^2}$.*

Proof. Using that $K_I = H^2 \cap I\overline{zH^2}$, write $f = I\overline{z\widetilde{f}}$ and $g = I\overline{z\widetilde{g}}$, with $\widetilde{f}, \widetilde{g} \in H^2$. Since $g \in H^\infty$, we indeed have $|\widetilde{g}| = |g| \in L^\infty(\mathbb{T})$, and thus $\widetilde{g} \in H^\infty$. Moreover, $fg \in H^2$, and

$$fg = I^2\overline{z^2\widetilde{f}\widetilde{g}},$$

whence $fg \in H^2 \cap I^2\overline{zH^2} = K_{I^2}$. □

In the case where $b = (1+I)/2$, the de Branges-Rovnyak space $\mathcal{H}(b)$ contains a sequence of model spaces.

Lemma 2.5. *Let I be an inner function, $I \not\equiv 1$, and let $b = (1+I)/2$. Then the following assertions hold:*

(a) *the function I is a multiplier of $\mathscr{H}(b)$;*
(b) *for every $n \geqslant 1$, $K_{I^n} \subseteq \mathscr{H}(b)$.*

Proof. (a): Let $f \in \mathscr{H}(b)$. According to (2.22), we can decompose f as $f = (1-I)g_1 + g_2$ with $g_1 \in H^2$ and $g_2 \in K_I$. Then

$$If = (1-I)(Ig_1) + Ig_2 = (1-I)(Ig_1 - g_2) + g_2$$

and $Ig_1 - g_2 \in H^2$ and $g_2 \in K_I$. Thus, using one more time (2.22), it follows that $If \in \mathscr{H}(b)$.

(b): We argue by induction. For $n = 1$, the property follows from Lemma 2.2. Assume that for some $n \geqslant 1$, $K_{I^n} \subseteq \mathscr{H}(b)$. It is known that $K_{I^{n+1}} = K_I \oplus IK_{I^n}$. See [20, Lemma 5.10]. The conclusion now follows from the induction assumption and (a). □

Here is now our sufficient condition for $f \in \mathscr{H}(b)$ to be a multiplier of $\mathscr{H}(b)$.

Lemma 2.6. *Let I be an inner function, $I \not\equiv 1$, and let $b = (1+I)/2$. Assume that f decomposes as $f = (1-I)g_1 + g_2$, with $g_1 \in H^\infty$ and $g_2 \in H^\infty \cap K_I$. Then $f \in \mathfrak{M}(\mathscr{H}(b))$.*

Proof. We need to show that for every $\varphi \in \mathscr{H}(b)$, we have $\varphi f \in \mathscr{H}(b)$. According to (2.22), write $\varphi = (1-I)\varphi_1 + \varphi_2$, with $\varphi_1 \in H^2$ and $\varphi_2 \in K_I$. Then

$$\varphi f = (1-I)\varphi_1 f + \varphi_2 f.$$

Since $f \in H^\infty$, $\varphi_1 f \in H^2$, and so the first term $(1-I)\varphi_1 f$ belongs to $(1-I)H^2$ which is contained in $\mathscr{H}(b)$. Thus it remains to prove that $\varphi_2 f \in \mathscr{H}(b)$. In order to deal with this term, write

$$\varphi_2 f = (1-I)\varphi_2 g_1 + g_2\varphi_2,$$

and as before, since $g_1 \in H^\infty$, the term $(1-I)\varphi_2 g_1$ belongs to $(1-I)H^2$, and so to $\mathscr{H}(b)$. It remains to prove that $g_2\varphi_2 \in \mathscr{H}(b)$. Lemma 2.4 implies that $g_2\varphi_2 \in K_{I^2}$, and the conclusion follows now directly from Lemma 2.5.

□

3. Some basic facts on cyclic vectors for the shift operator

Recall that if T is a bounded operator on a Hilbert space $\mathscr{H}$, then a vector $f \in \mathscr{H}$ is said to be cyclic for T if the linear span of the orbit of f under the action of T is dense in $\mathscr{H}$, i.e. if

$$\mathrm{Span}(T^n f : n \geqslant 0) = \overline{\{p(T)f : p \in \mathbb{C}[X]\}} = \mathscr{H}.$$

When $T = S_b$ is the shift operator on $\mathscr{H}(b)$, we have $p(S_b)f = pf$ for every $f \in \mathscr{H}(b)$ and every polynomial $p \in \mathbb{C}[X]$. Thus a function $f \in \mathscr{H}(b)$ is cyclic for S_b if and only if

$$\overline{\{pf : p \in \mathbb{C}[X]\}} = \mathscr{H}(b).$$

In fact, it is sufficient to approximate the constant function 1 by elements of the form pf, $p \in \mathbb{C}[X]$, to get that f is cyclic for S_b.

Lemma 3.1. *Let b a non-extreme point in $\mathrm{ball}(H^\infty)$ and $f \in \mathscr{H}(b)$. Then the following assertions are equivalent:*

(a) *f is cyclic for S_b.*
(b) *There exists a sequence of polynomials $(p_n)_n$ such that*

$$\|p_n f - 1\|_b \to 0, \text{ as } n \to \infty.$$

Proof. Follows immediately from the density of polynomials in $\mathscr{H}(b)$ and the boundedness of S_b. □

The general meaning of our next result is that the set of zeros of a cyclic vector $f \in \mathscr{H}(b)$ for S_b cannot be too large.

Lemma 3.2. *Let b a non-extreme point in $\mathrm{ball}(H^\infty)$ and $f \in \mathscr{H}(b)$. Assume that f is cyclic for S_b. Then we have the following properties:*

(a) *f is outer;*

(b) *for every* $\zeta \in E_0(b)$, $f(\zeta) \neq 0$.

Proof. (a) Since f is cyclic for S_b, there exists a sequence of polynomials $(p_n)_n$ such that

$$\|p_n f - 1\|_b \to 0, \text{ as } n \to \infty. \tag{3.1}$$

Now, using the fact that $\mathscr{H}(b)$ is contractively contained into H^2, we get that

$$\|p_n f - 1\|_2 \to 0, \text{ as } n \to \infty.$$

That proves that f is cyclic for S in H^2, and so f is outer by Beurling's theorem.

(b) Since the functional $f \longmapsto f(\zeta)$ is bounded on $\mathscr{H}(b)$ for every $\zeta \in E_0(b)$, we deduce from (3.1) that

$$|p_n(\zeta) f(\zeta) - 1| \to 0, \text{ as } n \to \infty$$

for every $\zeta \in E_0(b)$. This property implies directly that $f(\zeta) \neq 0$. □

We will encounter in the sequel of the paper some situations where the converse of Lemma 3.2 is also true, i.e. where conditions (a) and (b) of Lemma 3.2 give a necessary and sufficient condition for a function $f \in \mathscr{H}(b)$ to be cyclic.

We now provide some elementary results concerning cyclic functions for S_b.

Lemma 3.3. *Let b a non-extreme point in* $\operatorname{ball}(H^\infty)$. *Suppose that $f \in \mathfrak{M}(\mathscr{H}(b))$ and that $1/f \in \mathscr{H}(b)$. Then f is cyclic for S_b.*

Proof. Using (2.4), we see that there exists a sequence of polynomials $(p_n)_n$ such that

$$\|p_n - f^{-1}\|_b \to 0, \text{ as } n \to \infty.$$

Now, since $f \in \mathfrak{M}(\mathscr{H}(b))$, the multiplication operator by f is bounded on $\mathscr{H}(b)$, and thus we get that

$$\|p_n f - 1\|_b \to 0, \text{ as } n \to \infty,$$

which by Lemma 3.1 implies that f is cyclic for S_b. □

In the following result, the set $\operatorname{Hol}(\overline{\mathbb{D}})$ denotes the space of analytic functions in a neighborhood of the closed unit disc $\overline{\mathbb{D}}$.

Corollary 3.4. *Let b a non-extreme point in* $\operatorname{ball}(H^\infty)$. *Let $f \in Hol(\overline{\mathbb{D}})$ and assume that $\inf_{\overline{\mathbb{D}}} |f| > 0$. Then f is cyclic for S_b.*

Proof. When b is a non-extreme point in $\operatorname{ball}(H^\infty)$, we have $\operatorname{Hol}(\overline{\mathbb{D}}) \subseteq \mathfrak{M}(\mathscr{H}(b))$. See [18, Theorem 24.6]. Hence $f \in \mathfrak{M}(\mathscr{H}(b))$. Moreover, the conditions on f also imply that $1/f \in \operatorname{Hol}(\overline{\mathbb{D}})$. In particular, $1/f \in \mathscr{H}(b)$. It remains to apply Lemma 3.3 in order to get that f is cyclic. □

Lemma 3.5. *Let $f_1, f_2 \in \mathfrak{M}(\mathscr{H}(b))$. Then the following assertions are equivalent:*

(a) *the product function $f_1 f_2$ is cyclic for S_b;*
(b) *each of the functions f_1 and f_2 is cyclic for S_b.*

Proof. $(a) \implies (b)$: Assume that $f_1 f_2$ is cyclic. By symmetry, it suffices to prove that f_1 is cyclic. Let $\varepsilon > 0$. There exists a polynomial q such that $\|q f_1 f_2 - 1\|_b \leqslant \epsilon$. Now since the polynomials are dense in $\mathscr{H}(b)$, we can also find a polynomial p such that

$$\|f_2 q - p\|_b \leqslant \frac{\varepsilon}{\|f_1\|_{\mathfrak{M}(\mathscr{H}(b))}}.$$

Thus

$$\|p f_1 - 1\|_b \leqslant \|p f_1 - f_1 f_2 q\|_b + \|f_1 f_2 q - 1\|_b \leqslant \|f_1\|_{\mathfrak{M}(\mathscr{H}(b))} \|p - f_2 q\|_b + \varepsilon \leqslant 2\varepsilon,$$

which proves that f_1 is cyclic.

$(b) \implies (a)$: Assume that f_1 and f_2 are cyclic for $\mathscr{H}(b)$. Let $\varepsilon > 0$. There exists a polynomial p such that $\|pf_1 - 1\|_b \leqslant \epsilon$. On the other hand, there is also a polynomial q such that

$$\|qf_2 - 1\|_b \leqslant \frac{\varepsilon}{\|pf_1\|_{\mathfrak{M}(\mathscr{H}(b))}}.$$

Now we have

$$\|pqf_1f_2 - 1\|_b \leqslant \|pqf_1f_2 - pf_1\|_b + \|pf_1 - 1\|_b \leqslant \|pf_1\|_{\mathfrak{M}(\mathscr{H}(b))}\|qf_2 - 1\|_b + \varepsilon \leqslant 2\varepsilon.$$

Hence the function f_1f_2 is cyclic. □

Our next result is motivated by the Brown–Shields conjecture and the work [10] for Dirichlet type spaces $\mathcal{D}(\mu)$. Indeed, let μ be a positive finite measure on $\mathbb{T}$, and let $\mathcal{D}(\mu)$ be the associated Dirichlet space (i.e. the space of holomorphic functions on $\mathbb{D}$ whose derivatives are square-integrable when weighted against the Poisson integral of the measure μ). It is shown in [6], [23] that in some cases, Dirichlet spaces and de Branges-Rovnyak spaces are connected. More precisely, let $b \in \text{ball}(H^\infty)$ be a rational function (which is not a finite Blaschke product), and let a be its pythagorean mate. Let also μ be a positive finite measure on $\mathbb{T}$. Then $\mathcal{D}(\mu) = \mathscr{H}(b)$ with equivalent norms if and only if the zeros of a on $\mathbb{T}$ are all simple, and the support of μ is exactly the set of these zeros [6]. In the context of Dirichlet spaces, the authors of [10] prove the Brown–Shields conjecture when the measure μ has countable support, using two notions of capacity (which they denote $c_\mu(F)$ and $c^a_\mu(F)$ respectively) and showing that they are comparable: $c_\mu(F) \leqslant c^a_\mu(F) \leqslant 4\, c_\mu(F)$ for every $F \subseteq \mathbb{T}$ ([10, Lemma 3.1]). In the same spirit, we introduce the following notions of capacity in $\mathscr{H}(b)$-spaces. For a set $F \subseteq \mathbb{T}$, we define $c_1(F)$ and $c_2(F)$ as

$$c_1(F) = \inf\{\|f\|_b : f \in \mathscr{H}(b),\ |f| \geqslant 1 \text{ a.e. on a neighborhood of } F\},$$

and

$$c_2(F) = \inf\{\|f\|_b : f \in \mathscr{H}(b),\ |f| = 1 \text{ a.e. on a neighborhood of } F\}.$$

Observe that $c_1(F) \leqslant c_2(F)$. We do not know if $c_1(F)$ and $c_2(F)$ are comparable in general in our context of de Branges-Rovnyak spaces.

Our next result should be compared to [10, Lemma 3.2].

Theorem 3.6. *Let b a non-extreme point in $\text{ball}(H^\infty)$ and $\zeta \in \mathbb{T}$. Consider the following assertions:*

(a) *$z - \zeta$ is not cyclic for S_b;*
(b) *$\zeta \in E_0(b)$;*
(c) *$c_1(\{\zeta\}) > 0$;*
(d) *$c_2(\{\zeta\}) > 0$.*

Then $(a) \iff (b)$, $(b) \implies (d)$ and $(c) \implies (a)$.

Proof. $(b) \implies (a)$: follows immediately from Lemma 3.2.

$(a) \implies (b)$: our assumption (a) exactly means that

$$[z - \zeta] := \text{Span}((z - \zeta)z^n : n \geqslant 0) \subsetneq \mathscr{H}(b).$$

Denote by π the orthogonal projection from $\mathscr{H}(b)$ onto $[z - \zeta]^\perp$. First note that $\pi(1) \neq 0$, otherwise we would have $1 \in [z - \zeta]$ and then the function $z - \zeta$ would be cyclic for S_b, which is a contradiction.

Let us now prove that $[z-\zeta]^\perp = \mathbb{C}\pi(1)$. For every $g \in [z-\zeta]^\perp$ and every $n \geqslant 0$, we have

$$0 = \langle g, (z-\zeta)z^n\rangle_b = \langle g, z^{n+1}\rangle_b - \overline{\zeta}\langle g, z^n\rangle_b.$$

From this, we immediately get that

$$\langle g, z^n\rangle_b = \overline{\zeta}^n \langle g, 1\rangle_b, \quad n \geqslant 0. \tag{3.2}$$

This implies that $\langle \pi(1), 1\rangle_b \neq 0$ (otherwise, by (3.2) we would have that $\pi(1)$ is orthogonal to z^n for every $n \geqslant 0$, which implies that $\pi(1) = 0$). Secondly, if we define $c := \frac{\langle g,1\rangle_b}{\langle \pi(1),1\rangle_b}$, then we have $\langle g - c\pi(1), z^n\rangle_b = 0$ for every $n \geqslant 0$. By the density of polynomials in $\mathscr{H}(b)$, we deduce that $g = c\pi(1)$, which proves that $[z-\zeta]^\perp$ is of dimension 1, generated by $\pi(1)$.

Now consider the continuous linear functional $\rho : \mathbb{C}\pi(1) \longrightarrow \mathbb{C}$ defined by $\rho(\alpha\pi(1)) = \alpha$ for every $\alpha \in \mathbb{C}$. Let us check that for every $n \geqslant 0$,

$$(\rho \circ \pi)(z^n) = \zeta^n. \tag{3.3}$$

For $n = 0$, this is true by definition. Assume that (3.3) is satisfied for some integer $n \geqslant 0$. Then,

$$(\rho \circ \pi)(z^{n+1}) = (\rho \circ \pi)(z^n(z-\zeta)) + \zeta(\rho \circ \pi)(z^n) = \zeta^{n+1}.$$

By induction, we deduce (3.3) and by linearity, for any polynomial p, we have $(\rho \circ \pi)(p) = p(\zeta)$. Now, using the continuity of ρ and π, we obtain that there exists a constant $C > 0$ such that

$$|p(\zeta)| \leqslant C\|p\|_b, \qquad \text{for any polynomial } p \in \mathbb{C}[X].$$

Denote by L_ζ the linear functional defined on $\mathbb{C}[X]$ by $L_\zeta(p) = p(\zeta)$, $p \in \mathbb{C}[X]$. Then L_ζ is continuous on $\mathbb{C}[X]$ endowed with the topology of $\mathscr{H}(b)$. Hence it extends to a continuous linear map on $\mathscr{H}(b)$. By the Riesz representation theorem, there exists a unique vector $h_\zeta \in \mathscr{H}(b)$, $h_\zeta \neq 0$, such that

$$p(\zeta) = L_\zeta(p) = \langle p, h_\zeta\rangle_b, \qquad \text{for any polynomial } p \in \mathbb{C}[X].$$

Now, note that for any polynomial p, we have

$$\langle p, S_b^* h_\zeta\rangle_b = \langle zp, h_\zeta\rangle_b = \zeta p(\zeta) = \langle p, \overline{\zeta} h_\zeta\rangle_b,$$

whence, using (2.4), $S_b^* h_\zeta = \overline{\zeta} h_\zeta$. In particular, $\overline{\zeta}$ belongs to the point spectrum of S_b^*. But by (2.10), this implies that b has an angular derivative at ζ, which is equivalent to the property that $\zeta \in E_0(b)$. Note that the function h_ζ is in fact the reproducing kernel k_ζ^b of $\mathscr{H}(b)$ at the point ζ.

$(b) \implies (d)$: assume now that $\zeta \in E_0(b)$. Let $f \in \mathscr{H}(b)$ be such that $|f| = 1$ a.e. on a neighborhood $\mathcal{O}$ of ζ. Let us consider the inner-outer factorisation of $f = f_i f_o$, where f_i is the inner part and f_o the outer part of f. Since by definition $|f_i| = 1$ a.e. on $\mathbb{T}$, we have $|f_o| = 1$ a.e. on $\mathcal{O}$. Moreover, $f_o \in \mathscr{H}(b)$ and $\|f_o\|_b \leqslant \|f\|_b$. Indeed, $f_o = T_{\bar{f}_i} f$, where $T_{\bar{f}_i}$ is the Toeplitz operator with symbol $\bar{f}_i$ and $\mathscr{H}(b)$ is invariant with respect to co-analytic Toeplitz operators. Furthermore,

$$\|f_o\|_b = \|T_{\bar{f}_i} f\|_b \leqslant \|f_i\|_\infty \|f\|_b = \|f\|_b.$$

See [18, Theorem 18.13]. Since f_o is outer and $\log|f_o| = 0$ a.e. on $\mathcal{O}$, we have

$$f_o(z) = \lambda \exp\left(\int_{\mathbb{T}\setminus\mathcal{O}} \frac{\xi + z}{\xi - z} \log|f_o(\xi)|\, dm(\xi)\right),$$

for some constant $\lambda \in \mathbb{T}$. Hence f_o is analytic in a neighborhood of ζ and in particular, we deduce that $|f_o(\zeta)| = 1$. Using now the fact that $\zeta \in E_0(b)$, we know that there exists a constant $C > 0$ such that $|g(\zeta)| \leqslant C\|g\|_b$ for every $g \in \mathscr{H}(b)$. Hence

$$1 = |f_o(\zeta)| \leqslant C\|f_o\|_b \leqslant C\|f\|_b$$

for every function $f \in \mathscr{H}(b)$ such that $|f| = 1$ a.e. on a neighborhood $\mathcal{O}$ of ζ. We deduce that $c_2(\{\zeta\}) \geqslant C^{-1} > 0$.

$(c) \implies (a)$: by contradiction, assume that $z - \zeta$ is cyclic for S_b. Then, for every $\varepsilon > 0$, we can find a polynomial q such that $\|q(z-\zeta) - 1\|_b \leqslant \varepsilon$. Note that the value of the polynomial $q(z-\zeta)-1$ at ζ is -1. So by continuity, we can find a neighborhood $\mathcal{O}$ of ζ on $\mathbb{T}$ such that $|q(z-\zeta) - 1| \geqslant 1/2$ on $\mathcal{O}$. Hence $|2(q(z-\zeta) - 1)| \geqslant 1$ on $\mathcal{O}$ and by definition of $c_1(\{\zeta\})$, we obtain that

$$c_1(\{\zeta\}) \leqslant 2\|q(z-\zeta) - 1\|_b \leqslant 2\varepsilon.$$

Since this is true for every $\varepsilon > 0$, we deduce that $c_1(\{\zeta\}) = 0$, which contradicts (c). □

If we knew that $c_1(F)$ and $c_2(F)$ were comparable, assertions (a) to (d) in Theorem 3.6 would be equivalent. This motivates the following question:

Question 3.4. (i) Does there exist $\kappa > 0$ such that $c_2(F) \leqslant \kappa\, c_1(F)$ for every $F \subseteq \mathbb{T}$?
(ii) Is it true that $c_1(F) > 0$ if and only if $c_2(F) > 0$?

Remark 3.7. It can be easily seen from Theorem 3.6 that the condition $\inf_{\overline{\mathbb{D}}} |f| > 0$ in Corollary 3.4 is not necessary for f to be cyclic in $\mathscr{H}(b)$. Indeed, let $b(z) = \frac{1+z}{2} S_{\delta_1}(z)$, where S_{δ_1} is the singular inner function associated to δ_1, the Dirac measure at point 1. See (2.7). It is clear that

$$\int_{\mathbb{T}} \frac{d\delta_1(\xi)}{|\zeta_0 - \xi|^2} = \frac{1}{|\zeta_0 - 1|^2} = \infty$$

when $\zeta_0 = 1$. Hence $1 \notin E_0(b)$. Therefore, by Theorem 3.6, the function $z - 1$ is cyclic for S_b while $\inf_{\overline{\mathbb{D}}} |z - 1| = 0$.

Corollary 3.8. *Let b a non-extreme point in $\mathrm{ball}(H^\infty)$. Let p be a polynomial. The following assertions are equivalent:*

(a) *p is cyclic for S_b.*
(b) *$p(z) \neq 0$ for every $z \in \mathbb{D} \cup E_0(b)$.*

Proof. $(a) \implies (b)$: follows immediately from Lemma 3.2.

$(b) \implies (a)$: factorise the polynomial p as $p(z) = c\prod_{j=1}^n (z - \zeta_j)$, where by definition the roots ζ_j belong to $\mathbb{C}\backslash(\mathbb{D} \cup E_0(b))$. On the one hand, if $|\zeta_j| > 1$, then, according to Corollary 3.4, the function $z - \zeta_j$ is cyclic for S_b. On the other hand, if $|\zeta_j| = 1$, then $\zeta_j \notin E_0(b)$ and Theorem 3.6 implies that the function $z - \zeta_j$ is also cyclic for S_b. Thus, for every $1 \leqslant j \leqslant n$, the function $z - \zeta_j$ is cyclic and it follows from Lemma 3.5 that p itself is cyclic for S_b. □

This result can be slightly generalised:

Corollary 3.9. *Let b a non-extreme point in $\mathrm{ball}(H^\infty)$. Let $f \in Hol(\overline{\mathbb{D}})$. The following assertions are equivalent:*

(a) *f is cyclic for S_b.*
(b) *f is outer and $f(\zeta) \neq 0$ for every $\zeta \in E_0(b)$.*

Proof. $(a) \implies (b)$: follows from Lemma 3.2.

$(b) \implies (a)$: since f is outer and $f \in \mathrm{Hol}(\overline{\mathbb{D}})$, f does not vanish on the unit disc and has at most a finite number of zeros on $\mathbb{T}$ (otherwise by compactness and the uniqueness principle for holomorphic functions, f would vanish identically). Let $\zeta_1, \zeta_2, \dots, \zeta_n$ be the (possible) zeros of f on $\mathbb{T}$. Then there exists a function $g \in \mathrm{Hol}(\overline{\mathbb{D}})$ with $\inf_{\overline{\mathbb{D}}} |g| > 0$ such that

$$f(z) = \prod_{j=1}^{n} (z - \zeta_j) g(z), \qquad z \in \overline{\mathbb{D}}.$$

Our assumption implies that for every $1 \leqslant j \leqslant n$, $\zeta_j \notin E_0(b)$, and thus by Theorem 3.6, the function $z - \zeta_j$ is cyclic for S_b. Moreover, by Corollary 3.4, the function g is also cyclic. Now it follows from Lemma 3.5 that f itself is cyclic for S_b. □

Example 3.10. Let b a non-extreme point in $\mathrm{ball}(H^\infty)$. For every $\lambda \in \mathbb{D}$, k_λ is a cyclic vector for S_b. Indeed, it is clear that $k_\lambda(z) = (1 - \overline{\lambda} z)^{-1}$ satisfies the conditions of Corollary 3.9. Hence k_λ is cyclic.

In particular, by (2.5), the set of cyclic vectors for S_b spans a dense subspace in $\mathscr{H}(b)$.

4. The rational case

The main result of this section is a characterisation of cyclic functions for S_b when b is a rational function which is not a finite Blaschke product. As mentioned already in the Introduction, this result can be derived from the work [24] by Luo – Gu – Richter, but we provide here an elementary proof, the ideas of which will turn out to be also relevant to the case where $b = (1 + I)/2$ (see Section 5 below). Note that Theorem 4.1 extends a result proved in [19] in the particular case where $b(z) = (1 + z)/2$.

Theorem 4.1. *Let $b \in \mathrm{ball}(H^\infty)$ and assume that b is rational (but not a finite Blaschke product). Let a_1 be the associated polynomial given by* (2.14)*, and let $f \in \mathscr{H}(b)$. Then the following assertions are equivalent:*

(a) *f is cyclic for S_b.*
(b) *f is an outer function and for every $1 \leqslant k \leqslant n$, $f(\zeta_k) \neq 0$.*

Proof. $(a) \implies (b)$: according to (2.20), we know that $E_0(b) = \{\zeta_k : 1 \leqslant k \leqslant n\}$. Hence this implication follows from Lemma 3.2.

$(b) \implies (a)$: according to (2.16), write $f = a_1 \widetilde{f} + p$, where $\widetilde{f} \in H^2$ and $p \in \mathscr{P}_{N-1}$. By (2.19), $p(\zeta_k) \neq 0$, $1 \leqslant k \leqslant n$. Let now $r \in \mathscr{P}_{N-1}$ be the unique polynomial satisfying the following interpolation properties: for every $1 \leqslant k \leqslant n$,

$$r^{(j)}(\zeta_k) = \begin{cases} \frac{1}{p(\zeta_k)} & \text{if } j = 0 \\ -\frac{1}{p(\zeta_k)} \sum_{\ell=0}^{j-1} \binom{j}{\ell} r^{(\ell)}(\zeta_k) p^{(j-\ell)}(\zeta_k) & \text{if } 1 \leqslant j \leqslant m_k - 1. \end{cases}$$

This polynomial r can be constructed using Hermite polynomial interpolation, see for instance [1, Chapter 1, E. 7]. By Leibniz's rule, we easily see that for every $1 \leqslant k \leqslant n$ and $0 \leqslant j \leqslant m_k - 1$, we have $(rp - 1)^{(j)}(\zeta_k) = 0$. Hence a_1 divides the polynomial $rp - 1$. In other words, there exists a polynomial q such that $rp - 1 = a_1 q$. Using that f is outer, we can find a sequence of polynomials $(q_n)_n$ such that $\|q_n f + r\widetilde{f} + q\|_2 \to 0$ as $n \to \infty$. Define

now a sequence of polynomials (p_n) by $p_n = a_1 q_n + r$, $n \geqslant 1$. Observe that

$$\begin{aligned} p_n f - 1 =& (a_1 q_n + r) f - 1 = a_1 q_n f + r(a_1 \widetilde{f} + p) - 1 \\ =& a_1(q_n f + r\widetilde{f}) + rp - 1 = a_1(q_n f + r\widetilde{f} + q). \end{aligned}$$

Then it follows from (2.17) that

$$\|\!|p_n f - 1|\!\|_b = \|\!|a_1(q_n f + r\widetilde{f} + q)|\!\|_b = \|q_n f + r\widetilde{f} + q\|_2 \to 0 \text{ as } n \to \infty.$$

Therefore f is cyclic for S_b. □

Example 4.2. Let $b(z) = \frac{1}{2}(1 - z^2)$. Then it is proved in [13] that $a(z) = c(z - i)(z + i)$, for some constant c. Thus, according to Theorem 4.1, a function $f \in \mathscr{H}(b)$ is cyclic for S_b if and only if f is outer, $f(i) \neq 0$ and $f(-i) \neq 0$.

5. The case where $b = (1 + I)/2$

Our main result in this section is the following:

Theorem 5.1. *Let I be an inner function, $I \not\equiv 1$, and assume that its Clark measure σ_1 associated to point 1 (defined in (2.11)) is a discrete measure. Let $\{\zeta_n : n \geqslant 1\} = \{\zeta \in E_0(I) : I(\zeta) = 1\}$. Let $b = (1 + I)/2$, and $f \in \mathscr{H}(b)$ which we decompose according to (2.22) as $f = (1 - I)g_1 + g_2$, with $g_1 \in H^2$, $g_2 \in K_I$. Assume that:*

(a) *$g_1, g_2 \in H^\infty$;*
(b) *f is outer;*
(c) *we have*

$$\sum_{n \geqslant 1} \frac{1}{|f(\zeta_n)|^2 |I'(\zeta_n)|} < \infty.$$

Then f is cyclic for S_b.

Proof. The proof proceeds along the same lines as in the rational case. According to Lemma 2.6, $f \in \mathfrak{M}(\mathscr{H}(b))$ and by Lemma 2.3 we have $f(\zeta_n) = g_2(\zeta_n)$, $n \geqslant 1$.

First step: We claim that there exists a sequence of functions $(\psi_n)_n$ in $\mathscr{H}(b)$ such that $\|\psi_n f - 1\|_b \to 0$ as $n \to \infty$.

In order to construct the sequence $(\psi_n)_n$, let us first consider the function r given by

$$r = \sum_{n=1}^{\infty} \frac{1}{f(\zeta_n)} \frac{k_{\zeta_n}^I}{\|k_{\zeta_n}^I\|_2^2}.$$

Recall that by (2.12), $\|k_{\zeta_n}^I\|_2^2 = |I'(\zeta_n)|$. Combining this with condition (c) and the fact that the family $(k_{\zeta_n}^I / \|k_{\zeta_n}^I\|_2)_n$ forms an orthonormal basis of K_I (since σ_1 is discrete, Clark's theorem holds true), we see that the series defining the function r is convergent in K_I. In others words, $r \in K_I$ and $r(\zeta_n) = 1/f(\zeta_n) = 1/g_2(\zeta_n)$ for every $n \geqslant 1$.

Let us now prove that $rg_2 - 1 \in (1 - I)H^2$. Observe that Lemma 2.4 implies that $rg_2 \in K_{I^2} = K_I \oplus IK_I$. Hence there exist $\varphi_1, \varphi_2 \in K_I$ such that $rg_2 - 1 = \varphi_1 + I\varphi_2 - 1$. Since $r(\zeta_n)g_2(\zeta_n) - 1 = 0$, we have $\varphi_1(\zeta_n) + \varphi_2(\zeta_n) - 1 = 0$ for every $n \geqslant 1$. Note that $(1 - \overline{I(0)})^{-1}(1 - \overline{I(0)}I) = ((1 - \overline{I(0)})^{-1} k_0^I \in K_I$ and

$$(1 - \overline{I(0)})^{-1}(1 - \overline{I(0)} I(\zeta_n)) = 1 \qquad \text{for every } n \geqslant 1.$$

Since the family $\left(k_{\zeta_n}^I\right)_n$ is complete in K_I, and since $\varphi_1(\zeta_n) + \varphi_2(\zeta_n) = 1$ for every $n \geqslant 1$, we deduce that

$$\varphi_1 + \varphi_2 = (1 - \overline{I(0)})^{-1}(1 - \overline{I(0)}I).$$

Hence

$$\begin{aligned} rg_2 - 1 = \varphi_1 + I\varphi_2 - 1 &= -\varphi_2 + (1 - \overline{I(0)})^{-1}(1 - \overline{I(0)}I) + I\varphi_2 - 1 \\ &= (1 - I)(-\varphi_2) + (1 - \overline{I(0)})^{-1}(1 - \overline{I(0)}I) - 1. \end{aligned}$$

Observe that

$$(1 - \overline{I(0)})^{-1}(1 - \overline{I(0)}I) - 1 = (1 - \overline{I(0)})^{-1}\overline{I(0)}(1 - I),$$

from which it follows that

$$rg_2 - 1 = (1 - I)(-\varphi_2 + \overline{I(0)}(1 - \overline{I(0)})^{-1}).$$

This proves that $rg_2 - 1$ belongs to $(1 - I)H^2$. Write $rg_2 - 1$ as $rg_2 - 1 = (1 - I)g_3$, with $g_3 \in H^2$.

Using now that f is outer, and that $rg_1 \in H^2$ (as $g_1 \in H^\infty$) we can find a sequence of polynomials $(q_n)_n$ such that $\|q_n f + rg_1 + g_3\|_2 \to 0$, as $n \to \infty$. We then define for each $n \geqslant 1$ a function ψ_n as

$$\psi_n := (1 - I)q_n + r,$$

where q_n and r are defined above. Note that $\psi_n \in (1 - I)H^2 + K_I = \mathscr{H}(b)$ and

$$\begin{aligned} \psi_n f - 1 =& (1 - I)q_n f + rf - 1 \\ =& (1 - I)q_n f + (1 - I)rg_1 + rg_2 - 1 \\ =& (1 - I)(q_n f + rg_1 + g_3). \end{aligned}$$

It follows from Lemma 2.2 that there exists a positive constant C such that

$$\begin{aligned} \|\psi_n f - 1\|_b =& \|(1 - I)(q_n f + rg_1 + g_3)\|_b \\ \leqslant& C\, \|\!|(1 - I)(q_n f + rg_1 + g_3)|\!\|_b \\ =& C\, \|q_n f + rg_1 + g_3\|_2, \end{aligned}$$

from which it follows that $\|\psi_n f - 1\|_b \to 0$ as $n \to \infty$.

Second step: Let us now prove that there exists a sequence of polynomials $(p_n)_n$ such that $\|p_n f - 1\|_b \to 0$ as $n \to \infty$.

By the density of polynomials in $\mathscr{H}(b)$, we can find a sequence of polynomials $(p_n)_n$ such that $\|p_n - \psi_n\|_b \to 0$ as $n \to \infty$. Now write

$$\begin{aligned} \|p_n f - 1\|_b \leqslant& \|p_n f - \psi_n f\|_b + \|\psi_n f - 1\|_b \\ \leqslant& \|f\|_{\mathfrak{M}(\mathscr{H}(b))} \|p_n - \psi_n\|_b + \|\psi_n f - 1\|_b, \end{aligned}$$

and by the choice of the sequence $(p_n)_n$ and the first step, we get the conclusion of the second step.

We finally conclude that f is cyclic for S_b. □

Remark 5.2. If I is an inner function such that, for some $\alpha \in \mathbb{T}$, its Clark measure σ_α is a discrete measure and $I \not\equiv \alpha$, then we may apply Theorem 5.1 replacing I by $\bar{\alpha}I$ and $b = (1 + I)/2$ by $b = (1 + \bar{\alpha}I)/2$.

Example 5.3. Let $I = S_{\delta_1}$ be the inner function associated to the measure δ_1:

$$I(z) = \exp\Big(-\frac{1 + z}{1 - z}\Big).$$

In this case we can compute explicitly the Clark basis of K_I associated to point 1. We have $E_1 = \{\zeta \in E_0(I) : I(\zeta) = 1\} = \{\zeta_n \ ; \ n \in \mathbb{Z}\}$ with

$$\zeta_n = \frac{2i\pi n - 1}{2i\pi n + 1} \quad \text{and} \quad I'(\zeta_n) = -\frac{1}{2}(2i\pi n + 1)^2, \ n \in \mathbb{Z}.$$

Therefore, if $f \in \mathcal{H}(b)$ is outer, with $f = (1 - I)g_1 + g_2$, $g_1 \in H^\infty$, $g_2 \in K_I \cap H^\infty$, and if

$$\sum_{n\in\mathbb{Z}} \frac{1}{|f(\zeta_n)|^2} \cdot \frac{1}{4n^2\pi^2 + 1} < +\infty,$$

then f is cyclic for S_b.

Remark 5.4. There exists a recipe to construct an inner function satisfying the hypothesis of Theorem 5.1. Let σ be a positive discrete measure on $\mathbb{T}$ and let $H\sigma$ be its Herglotz transform,

$$H\sigma(z) = \int_{\mathbb{T}} \frac{\xi + z}{\xi - z}\, d\sigma(\xi), \qquad z \in \mathbb{D}.$$

We easily see that $H\sigma$ defines an analytic function on $\mathbb{D}$ and satisfies $\Re e(H\sigma(z)) \geqslant 0$ for every $z \in \mathbb{D}$. Now define a function I on $\mathbb{D}$ as

$$I = \frac{H\sigma - 1}{H\sigma + 1}.$$

Since $\Re e H\sigma \geqslant 0$, it is easy to check that $I \in H^\infty$ and $|I| \leqslant 1$. Moreover, for every $0 < r < 1$ and $\zeta \in \mathbb{T}$, we have

$$|I(r\zeta)| = \frac{(\Re e(H\sigma(r\zeta)) - 1)^2 + (\Im m(H\sigma(r\zeta))^2}{(\Re e(H\sigma(r\zeta)) + 1)^2 + (\Im m(H\sigma(r\zeta))^2}. \tag{5.1}$$

Since σ is a singular measure, it is well-known that for almost all $\zeta \in \mathbb{T}$, we have

$$\Re e(H\sigma(r\zeta)) = \int_{\mathbb{T}} \frac{1 - r^2}{|\xi - r\zeta|^2}\, d\sigma(\xi) \to 0 \quad \text{as } r \to 1^-.$$

See [17, Corollary 3.4]. Moreover, the radial limit of $\Im m(H\sigma)$ also exists and is finite for almost all $\zeta \in \mathbb{T}$. See [17, page 113]. Thus, it follows from (5.1) that $|I(\zeta)| = 1$ for almost all $\zeta \in \mathbb{T}$, meaning that I is an inner function. Of course, we have $I \not\equiv 1$. Now, we easily check that

$$\frac{1 - |I(z)|^2}{|1 - I(z)|^2} = \Re e(H\sigma(z)) = \int_{\mathbb{T}} \frac{1 - |z|^2}{|\xi - z|^2}\, d\sigma(\xi),$$

which implies by unicity of the Clark measure that $\sigma_1 = \sigma$. Therefore I satisfies the assumptions of Theorem 5.1.

Corollary 5.5. *Let I be an inner function, $I \not\equiv 1$, and assume that σ_1 is a discrete measure. Let $b = (1 + I)/2$. Then k_λ^b is a cyclic vector for S_b for every $\lambda \in \mathbb{D}$.*

Proof. Let us prove that k_λ^b satisfies the assumptions $(a), (b)$ and (c) of Theorem 5.1. First, using that $b = (1 + I)/2$, straightforward computations show that

$$\frac{1 - \overline{b(\lambda)}b(z)}{1 - \overline{\lambda}z} = (1 - I(z))\frac{1}{4} \cdot \frac{(1 - \overline{I(\lambda)})}{1 - \overline{\lambda}z} + \frac{1}{2} \cdot \frac{1 - \overline{I(\lambda)}I(z)}{1 - \overline{\lambda}z}.$$

In other words, k_λ^b can be written as $k_\lambda^b = (1 - I)g_1 + g_2$, with $g_1 = \frac{1}{4}(1 - \overline{I(\lambda)})k_\lambda$ and $g_2 = \frac{1}{2}k_\lambda^I$. In particular, $g_1, g_2 \in H^\infty$ and k_λ^b satisfies the assumption (a).

Observe now that $\Re e(1 - \overline{b(\lambda)}b(z)) \geqslant 0$ and $\Re e(1 - \overline{\lambda}z) \geqslant 0$ for every $z \in \mathbb{D}$, which implies that the functions $1 - \overline{b(\lambda)}b(z)$ and $1 - \overline{\lambda}z$ are outer. See [25, page 67]. So k_λ^b is outer

as the quotient of two outer functions. It remains to check that k_λ^b satisfies assumption (c). But

$$|k_\lambda^b(\zeta_n)| = \left|\frac{1-b(\lambda)}{1-\overline{\lambda}\zeta_n}\right| \geqslant \frac{|1-b(\lambda)|}{1+|\lambda|},$$

and the property (c) follows from the fact that

$$\sum_{n\geqslant 1} \frac{1}{|I'(\zeta_n)|} = \sum_{n\geqslant 1} \sigma_1(\{\zeta_n\}) \leqslant \sigma_1(\mathbb{T}) < +\infty.$$

Thus k_λ^b satisfies the assumptions $(a), (b), (c)$ of Theorem 5.1, and k_λ^b is cyclic for S_b. □

It should be noted that in Corollary 5.5, the reproducing kernels $f = k_\lambda^b$, $\lambda \in \mathbb{D}$, which are cyclic for S_b, are such that $1/f \in H^\infty$. As we already observed in Lemma 3.3, certain invertibility conditions for f make cyclicity easier. Using Theorem 5.1, we now construct a family of functions f which are cyclic for S_b but are such that $1/f \notin H^2$.

Example 5.6. Let I be a non-constant inner function, and assume that σ_1 is a discrete measure. Let $b = (1+I)/2$ and $f = (1+I)k_\lambda^I$ for some $\lambda \in \mathbb{D}$. Then f is cyclic for S_b and $1/f \notin L^2$.

Proof. First observe that

$$f = (1-I)(-k_\lambda^I) + 2k_\lambda^I,$$

so that $f = (I-I)g_1 + g_2$, with $g_2 = -2g_1 = 2k_\lambda^I \in H^\infty \cap K_I$. In particular, f satisfies condition (a) of Theorem 5.1. Moreover, the function f is outer as the product of two outer functions (use the same arguments as in the proof of Corollary 5.5). Finally, since $|f(\zeta_n)| = 2|k_\lambda^I(\zeta_n)| \geqslant |1-I(\lambda)|$, f satisfies condition (c). Hence by Theorem 5.1, f is cyclic for S_b.

Let us now check that $1/f \notin L^2$. First observe that there exist two positive constants C_1 and C_2 such that

$$C_1 \frac{1}{|1+I(\zeta)|} \leqslant \frac{1}{|f(\zeta)|} \leqslant C_2 \frac{1}{|1+I(\zeta)|} \quad \text{for a.e. } \zeta \in \mathbb{T},$$

because

$$\frac{1-|I(\lambda)|}{2} \leqslant |k_\lambda^I(\zeta)| \leqslant \frac{2}{1-|\lambda|}.$$

Now assume that $1/f$ belongs to L^2. Then $1/(1+I) \in L^2$. But since $1+I$ is outer, we get that $1/(1+I) \in H^2$. See [25, page 43]. As

$$\frac{1}{1+I} = \overline{\frac{I}{1+I}} \quad \text{for a.e. } \zeta \in \mathbb{T},$$

we deduce that $1/(1+I)$ also belongs to $\overline{H^2}$. Thus $1/(1+I)$ is constant, which is a contradiction. □

In the context of Corollary 5.5, it is easy to see that (a, b) forms a corona pair, and we have seen that k_λ^b is cyclic for S_b. In fact, this cyclicity result holds true under the (HCR) condition only.

Proposition 5.7. *Let b be a non-extreme point in $\mathrm{ball}(H^\infty)$, and let a be its pythagorean mate. Assume that (a, b) satisfies (HCR). Then the following. assertions hold:*

(a) *k_λ^b is cyclic for S_b for every $\lambda \in \mathbb{D}$;*

(b) *If b is furthermore assumed to be outer, then bk_λ is also cyclic for S_b for every $\lambda \in \mathbb{D}$. In particular, b is a cyclic vector for S_b.*

Proof. (a) We have $p(S_b)k_\lambda^b = (1 - \overline{b(\lambda)}b)p(S_b)k_\lambda$ for every polynomial $p \in \mathbb{C}[X]$. Since (a, b) satisfies (HCR), we have $\mathscr{H}(b) = \mathscr{M}(\bar{a})$, and b is a multiplier of $\mathscr{H}(b)$. See [18, Theorems 28.7 and 28.3]. In particular, the multiplication operator $T = M_{1-\overline{b(\lambda)}b}$ is bounded on $\mathscr{H}(b)$ and we have

$$p(S_b)k_\lambda^b = Tp(S_b)k_\lambda \qquad \text{for every polynomial } p \in \mathbb{C}[X].$$

Since k_λ is cyclic for S_b (see Example 3.10), in order to check that k_λ^b is also cyclic for S_b it is sufficient to check that T has dense range. Let $h \in \mathscr{H}(b)$ be such that $h \perp \text{Range}(T)$. Then $h \perp Tk_\mu = k_\mu - \overline{b(\lambda)}bk_\mu$ for every $\mu \in \mathbb{D}$. Lemma 2.1 now implies that $h = 0$, proving that T has dense range. It follows that k_λ^b is cyclic for S_b.

(b) The proof of (b) proceeds along the same lines of (a). We have

$$p(S_b)bk_\lambda = Vp(S_b)k_\lambda \qquad \text{for every polynomial } p \in \mathbb{C}[X],$$

where $V = M_b$ is the multiplication operator by b. As previously, in order to show that bk_λ is cyclic for S_b, it is sufficient to check that V has a dense range. Let $h \in \mathscr{H}(b)$ be such that $h \perp \text{Range}(V)$. Then $h \perp Vk_\mu = bk_\mu$ for every $\mu \in \mathbb{D}$. By (2.6), it then follows that $h^+(\mu) = 0$ for every $\mu \in \mathbb{D}$. Then $h^+ = 0$ and $T_{\overline{b}}h = T_{\bar{a}}h^+ = 0$. But, since b is outer, $T_{\overline{b}}$ is one-to-one, which implies that $h = 0$. It then follows that bk_λ is cyclic for S_b. □

We finish the paper with the following question:

Question 5.2. Does Proposition 5.7 hold true without the assumption that (a, b) satisfies (HCR)?

REFERENCES

[1] Peter Borwein and Tamás Erdélyi. *Polynomials and polynomial inequalities*, volume 161 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.

[2] Leon Brown. Invertible elements in the Dirichlet space. *Canad. Math. Bull.*, 33(4):419–422, 1990.

[3] Leon Brown and William Cohn. Some examples of cyclic vectors in the Dirichlet space. *Proc. Amer. Math. Soc.*, 95(1):42–46, 1985.

[4] Leon Brown and Allen L. Shields. Cyclic vectors in the Dirichlet space. *Trans. Amer. Math. Soc.*, 285(1):269–303, 1984.

[5] Douglas N. Clark. One dimensional perturbations of restricted shifts. *J. Analyse Math.*, 25:169–191, 1972.

[6] Constantin Costara and Thomas Ransford. Which de Branges-Rovnyak spaces are Dirichlet spaces (and vice versa)? *J. Funct. Anal.*, 265(12):3204–3218, 2013.

[7] Louis de Branges and James Rovnyak. *Square summable power series*. Holt, Rinehart and Winston, New York-Toronto, Ont.-London, 1966.

[8] Ronald G. Douglas, Harold S. Shapiro, and Allen L. Shields. On cyclic vectors of the backward shift. *Bull. Amer. Math. Soc.*, 73:156–159, 1967.

[9] Peter L. Duren. *Theory of H^p spaces*. Academic Press, New York, 1970.

[10] Omar El-Fallah, Youssef Elmadani, and Karim Kellay. Cyclicity and invariant subspaces in Dirichlet spaces. *J. Funct. Anal.*, 270(9):3262–3279, 2016.

[11] Omar El-Fallah, Karim Kellay, and Thomas Ransford. Cyclicity in the Dirichlet space. *Ark. Mat.*, 44(1):61–86, 2006.

[12] Omar El-Fallah, Karim Kellay, and Thomas Ransford. On the Brown-Shields conjecture for cyclicity in the Dirichlet space. *Adv. Math.*, 222(6):2196–2214, 2009.

[13] Emmanuel Fricain, Andreas Hartmann, and William T. Ross. Concrete examples of $\mathscr{H}(b)$ spaces. *Comput. Methods Funct. Theory*, 16(2):287–306, 2016.

[14] Emmanuel Fricain, Andreas Hartmann, and William T. Ross. Range spaces of co-analytic Toeplitz operators. *Canad. J. Math.*, 70(6):1261–1283, 2018.

[15] Emmanuel Fricain, Andreas Hartmann, and William T. Ross. Multipliers between range spaces of co-analytic Toeplitz operators. *Acta Sci. Math. (Szeged)*, 85(1-2):215–230, 2019.
[16] Emmanuel Fricain and Javad Mashreghi. Boundary behavior of functions in the de Branges-Rovnyak spaces. *Complex Anal. Oper. Theory*, 2(1):87–97, 2008.
[17] Emmanuel Fricain and Javad Mashreghi. *The theory of $\mathcal{H}(b)$ spaces. Vol. 1*, volume 20 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2016.
[18] Emmanuel Fricain and Javad Mashreghi. *The theory of $\mathcal{H}(b)$ spaces. Vol. 2*, volume 21 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2016.
[19] Emmanuel Fricain, Javad Mashreghi, and Daniel Seco. Cyclicity in non-extreme de Branges–Rovnyak spaces. In *Invariant subspaces of the shift operator*, volume 638 of *Contemp. Math.*, pages 131–136. Amer. Math. Soc., Providence, RI, 2015.
[20] Stephan Ramon Garcia, Javad Mashreghi, and William T. Ross. *Introduction to model spaces and their operators*, volume 148 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2016.
[21] John B. Garnett. *Bounded analytic functions*, volume 96 of *Pure and Applied Mathematics*. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1981.
[22] Håkan Hedenmalm and Allen Shields. Invariant subspaces in Banach spaces of analytic functions. *Michigan Math. J.*, 37(1):91–104, 1990.
[23] Karim Kellay and Mohamed Zarrabi. Two-isometries and de Branges–Rovnyak spaces. *Complex Anal. Oper. Theory*, 9(6):1325–1335, 2015.
[24] Shuaibing Luo, Caixing Gu, and Stefan Richter. Higher order local Dirichlet integrals and de Branges–Rovnyak spaces. *Adv. Math.*, 385:Paper No. 107748, 47, 2021.
[25] Nikolai K. Nikolski. *Operators, functions, and systems: an easy reading. Vol. 1*, volume 92 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2002. Hardy, Hankel, and Toeplitz, Translated from the French by Andreas Hartmann.
[26] Stefan Richter and Carl Sundberg. Multipliers and invariant subspaces in the Dirichlet space. *J. Operator Theory*, 28(1):167–186, 1992.
[27] Donald Sarason. Doubly shift-invariant spaces in H^2. *J. Operator Theory*, 16(1):75–97, 1986.
[28] Donald Sarason. *Sub-Hardy Hilbert spaces in the unit disk*, volume 10 of *University of Arkansas Lecture Notes in the Mathematical Sciences*. John Wiley & Sons, Inc., New York, 1994. A Wiley-Interscience Publication.

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