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Tight Bounds on the Size of 2-Monopolies

(Extended abstract)

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Abstract. This paper deals with the question of the influence of a *monopoly* of vertices, seeking to gain the majority in local neighborhoods in a graph. Say that a vertex v is *r-controlled* by a set of vertices M if the majority of its neighbors at distance r are from M. We ask how large must M be in order to *r-monopolize* the graph, namely, r-control *every vertex*. Tight upper and lower bounds are provided for this problem. For any even $r \geq 2$, an r-monopoly M in an n-vertex graph must be of size $\Omega(n^{3/5})$, and there exist n-vertex graphs with r-monopolies of size $O(n^{3/5})$. As for odd r values, we show that for every odd $k \geq 1$, a $3k$-monopoly M in an n-vertex graph must be of size $\Omega(n^{6/11})$ and that for any odd r there exist n-vertex graphs with r-monopolies of size $O(n^{4/7})$. This settles a problem left open in [L+93, BP95].

1 Introduction

1.1 The problem

Majority voting, in one form or another, is used as a component of fault-tolerant algorithms in a wide variety of contexts, including agreement and consensus problems (cf. [LSP82, Br87, D+88]), quorum system applications (cf. [G79, GMB85, SB94, JRT91]), self-stabilization and local mending [KP95a, KP95b], and more.

In the context of distributed network algorithms, the majority ruling method must be used in a restricted and local fashion, by replicating the data of a processor v only among processors in v's *local vicinity*. The chief reason for this focus on locality is that in the distributed network model, computations are local in nature, namely, the accessibility of data to a processor is greatly affected by its distance from it.

On the other hand, the potential risk in limiting ourselves to local neighborhoods is that it makes it possible for a sufficiently large set of faults to *corrupt*

the majority in some of these neighborhoods. Generally speaking, the ability of failed vertices to influence the voting process is not only a function of their *number* but also a function of their *location* in the network, namely, strategically positioned vertices may have greater influence.

This observation naturally leads to the fundamental problem of characterizing the potential power of a set of failures in a network of processors. In particular, previous work has focused on the notion of *monopolies* in graphs. This notion is defined as follows.

Definition 1. A vertex v in a network $G(V, E)$ is said to be *controlled* by the vertex set M if at least half of its neighbors are in M.

The set M is called a *monopoly* if it controls every vertex in the graph.

The question of bounds for the size of monopolies in the graph was addressed in [L+93], and the following tight answers were provided.

Proposition 2. [L+93]

1. *In every n-vertex graph, a monopoly must be of size* $\Omega(\sqrt{n})$.
2. *There exist (infinitely many) n-vertex graphs with monopolies of size* $O(\sqrt{n})$.

A graph G with a monopoly of size $O(\sqrt{n})$ as in Prop. 2(2) is depicted in Fig. 1. The graph consists of a coalition M of $\sqrt{n}+1$ vertices, connected by a clique, plus a set of $\sqrt{n}$ leaves attached to each vertex of M.

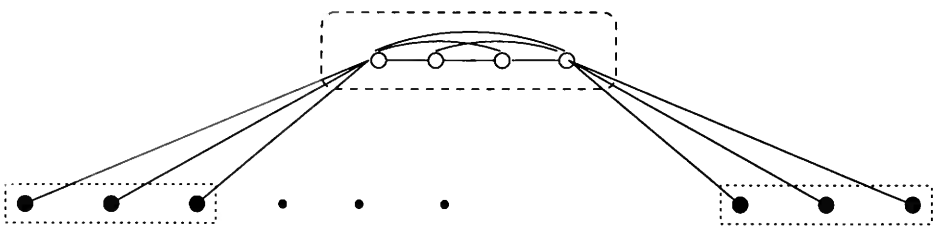

Fig. 1. A graph G with a monopoly M of size $O(\sqrt{n})$.

The above results may be interpreted as the result of the limited scope of our majority voting. Indeed, one may hope to strengthen the quality of the voting by querying vertices to larger distances. Let $\Gamma_r(v)$ denote the r-neighborhood of v in G, i.e., the set of vertices at distance r or less from v. A variant of the above question, in which neighborhoods are replaced by r-neighborhoods for some fixed r, was also discussed in [L+93].

Definition 3. A vertex v in a network $G(V, E)$ is said to be *r-controlled* by the vertex set M if at least half of the vertices in $\Gamma_r(v)$ are in M.

The set M is called an *r-monopoly* if it r-controls every vertex in the graph.

It is shown in [L+93] that for any fixed $r \geq 1$ there exist (infinitely many) n-vertex graphs with r-monopolies of size $O(n^{2/3})$. However, the question of tight bounds for r-monopolies was left open.

In this paper we improve this bound on the size of r-monopolies, and show a bound that is tight for even r.

Proposition 4.

1. *In every n-vertex graph, an r-monopoly for even $r \geq 2$ must be of size $\Omega(n^{3/5})$.*
2. *For any fixed $r \geq 2$ there exist (infinitely many) n-vertex graphs with r-monopolies of size $O(n^{3/5})$.*
3. *In every n-vertex graph, a $3k$-monopoly for odd $k \geq 1$ must be of size $\Omega(n^{6/11})$*
4. *For any fixed odd r there exist (infinitely many) n-vertex graphs with r-monopolies of size $O(n^{4/7})$.*

In what follows we prove the above claims 1 and 2, and illustrate claim 4. The proof of claim 3 is deferred to the full paper.

1.2 Related work

The variant of a *self-ignoring r-monopoly*, namely, a set M that r-controls every vertex in $V \setminus M$ was studied in [BP95]. That paper also studied the influence of an arbitrary coalition M (that is not a monopoly) as a function of its size.

Certain *dynamic* variants of majority voting problems were studied in the literature, in the context of discrete time dynamical systems. These variants dealt with a setting in which the nodes of the system operate in discrete time steps, and at each step, each node computes the majority in its neighborhood, and adapts the resulting value as its own. Typical problems studied in this setting concern the behavior of the resulting sequence of global states (represented as a vector $\bar{x}^t = (x_1^t, \ldots, x_n^t)$, where x_i^t represents the value at node v_i after time step t). For instance, the fact that the period of such sequences is either one or two is proved (in various contexts) in [GO80, PS83, PT86]. The problem was studied further in [M94c, M94b, M94a]. Also, the applicability of majority voting as a tool for fault-local mending was investigated in [KP95a, KP95b].

For a review of recent results in the area, see [P96].

2 Lower Bounds

Given a graph $G = (V, E)$ and a set $S \subset V$, let $\Gamma_r(S, G)$ denote the r-neighborhood of S in G, i.e., the set of vertices at distance r or less from some node of S in G. (We omit the parameter G when it is clear from the context.) Given a graph $G = (V, E)$, a vertex $x \in V$, and a set $S \subset V$, we denote by $\deg_G(x, S)$ the number of neighbors of x in G belonging to S, namely, $|\Gamma_1(x, G) \cap S|$. (We omit the parameter S when it is the entire vertex set of G; we omit the subscript

G when it is clear from the context.) $D_G(x, y)$ denotes the distance in G between x and y. Given a subset S of V, define the distance from x to S in G as $D_G(x, S) = \min_{y \in S}(D_G(x, y))$.

2.1 The case of 2-monopoly

In this section we concentrate on the case of 2-monopolies. For notational convenience, we will refer to the pair (G, M) as a "2-monopoly" whenever G is a graph (V, E) and $M \subseteq V$ is 2-monopoly for G.

For $i = 1, 2$, let S_i denote the set of vertices at distance i from M, that is, $S_i = \{v \in V \mid D_G(v, M) = i\}$, and let $s_i = |S_i|$. Also let $m = |M|$. Note that if (G, M) is a 2-monopoly then $V = M \cup S_1 \cup S_2$ and $|V| = m + s_1 + s_2$.

The *influence* of a subset $S \subset V$ on a vertex v in the graph G is defined by

$$\mathcal{I}_G(S, v) \;=\; |\{w \in S \mid D_G(v, w) \leq 2\}|.$$

(We omit the subscript G whenever clear from the context.) More generally, the influence of a set S on a set S' is $\mathcal{I}_G(S, S') = \sum_{x \in S'} \mathcal{I}_G(S, x)$.

Using the $\mathcal{I}$ notation, a vertex v is 2-controlled by M if $\mathcal{I}(M, v) \geq \mathcal{I}(S_1 \cup S_2, v)$, and a subset S of V is 2-controlled if every vertex $x \in S$ is.

In what follows we make frequent use (sometimes without mention) of the following three immediate properties of $\mathcal{I}$.

Proposition 5.

1. *$\mathcal{I}$ is a symmetric function, i.e., $\mathcal{I}(S, S') = \mathcal{I}(S', S)$ for every $S, S' \subseteq V$.*
2. *$\mathcal{I}(S, S) \leq |S|^2$ for every $S \subseteq V$, and*
3. *$\mathcal{I}$ is monotone nonincreasing in the size of E, namely, if $G' = (V, E')$ for $E' \subseteq E$, then $\mathcal{I}_{G'}(S, S') \leq \mathcal{I}_G(S, S')$ for every $S, S' \subseteq V$.*

Lemma 6. If (G, M) is a 2-monopoly, then $\mathcal{I}(S_1 \cup S_2, M) \leq m^2$ and $\mathcal{I}(S_1 \cup S_2, S_1 \cup S_2) \leq m^2$.

Proof As all the nodes are 2-controlled, we have for every $v \in V$,

$$\mathcal{I}(S_1 \cup S_2, v) \;\leq\; \mathcal{I}(M, v). \tag{1}$$

By summing (1) on all $v \in M$ we get

$$\mathcal{I}(S_1 \cup S_2, M) \;\leq\; \mathcal{I}(M, M) \;\leq\; m^2, \tag{2}$$

proving the first claim. On the other hand, by summing (1) on all $v \in S_1 \cup S_2$ we get

$$\mathcal{I}(S_1 \cup S_2, S_1 \cup S_2) \;\leq\; \mathcal{I}(M, S_1 \cup S_2) \;=\; \mathcal{I}(S_1 \cup S_2, M), \tag{3}$$

and combining (2) and (3), the lemma follows. ∎

Our main two lemmas bound the size of S_1 and S_2 with respect to M.

Lemma 7. If (G, M) is a 2-monopoly, then $s_1 \leq m^{3/2}$.

Proof For every vertex $y \in S_1$, we assign a unique neighbor $p(y)$ in M as its *parent*. For every $x \in M$, let $\alpha(x)$ denote the number of children assigned to it. Note that $s_1 = \sum_{x \in M} \alpha(x)$. Also note that for every vertex $y \in S_1$, $\alpha(p(y)) \leq \mathcal{I}(y, S_1) \leq \mathcal{I}(y, M)$. Therefore

$$\sum_{x \in M} \alpha^2(x) = \sum_{y \in S_1} \alpha(p(y)) \leq \sum_{y \in S_1} \mathcal{I}(y, M) = \mathcal{I}(S_1, M) \leq m^2.$$

By convexity, this sum is minimal when all the values of $\alpha(x)$ are equal; that happens when $\alpha(x) = s_1/m$ as $s_1 = \sum_{x \in M} \alpha(x)$. Hence $m(s_1/m)^2 \leq m^2$, yielding the claim. ∎

Consider a graph G and a set of vertices M with S_1 and S_2 defined as above. Let E_1 denote the set of edges connecting M and S_1. The following lemma holds even if M is not a 2-monopoly.

Lemma 8. If S_2 is 2-controlled by M, then $|E_1| \geq s_2$.

Proof Construct a bipartite graph $B = (S_2, E_1, E')$ by defining the edge set E' as follows. For $z \in S_2$, $y \in S_1$ and $x \in M$ such that $(x, y) \in E_1$, we connect z to (x, y) in B if z is adjacent to y in G. (See Fig. 2(a) and 2(b)).

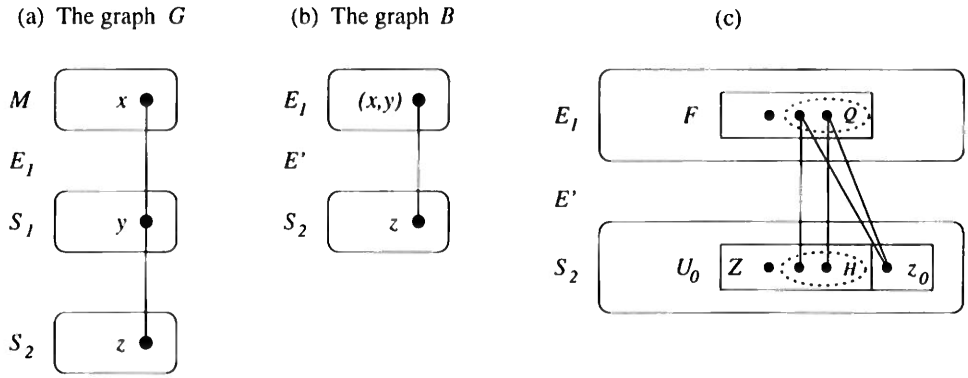

Fig. 2. Proof of Lemma 8.

We prove the lemma by showing that B admits a matching saturating S_2. This is proved by relying on Hall's Lemma (cf. [Be76]). By Hall's Lemma, B admits a matching touching every vertex of S_2 if and only if $|\Gamma_1(U, B)| \geq |U|$ for every subset $U \subseteq S_2$.

This property is proved by contradiction. Suppose that this property does not hold, namely, there exists a "deficit" set U, such that $|\Gamma_1(U, B)| < |U|$. Let U_0 be a minimal size deficit set. Note that U_0 is not a singleton (since no singletons in S_2 are in deficit). Let $F = \Gamma_1(U_0, B)$. Pick an arbitrary node $z_0 \in U_0$, and let $Z = U_0 \setminus \{z_0\}$. The bipartite graph B' induced by Z and $\Gamma_1(Z, B)$ obeys the

condition of Hall's Lemma, hence it admits a matching saturating the vertices of Z. Moreover, note that

$$|Z| + 1 = |U_0| > |\Gamma_1(U_0, B)| \geq |\Gamma_1(Z, B)| \geq |Z|,$$

so $|F| = |Z|$, and hence B' admits a perfect matching $\mathcal{M}$ between Z and F. Now let $Q = \Gamma_1(z_0, B) \subseteq F$, and let H be the set of nodes in Z matched by $\mathcal{M}$ with the edges of Q (see Fig. 2(c)). Note that

$$|\Gamma_2(z_0, G) \cap M| \;\leq\; |Q| = |H| \;\leq\; |\Gamma_2(z_0, G) \cap S_2| - 1$$

(counting z_0 itself, and possibly more S_2 vertices currently not in U_0). This implies that z_0 is not 2-controlled, leading to contradiction.

Consequently, B admits a matching saturating S_2, and hence $s_2 \leq |E_1|$. ∎

Lemma 9. If (G, M) is a 2-monopoly, then $s_2 \leq O(m^{5/3})$.

Proof Consider a 2-monopoly (G, M). Let $\delta(y) = \deg_G(y, M)$ for each node $y \in S_1$. Let d_0 be an integer to be fixed later. Let A_1 (resp. B_1) be the set of vertices $y \in S_1$ with $\delta(y) < d_0$ (resp. $\delta(y) \geq d_0$). Let B_2 be the set of nodes in S_2 adjacent to some node in B_1, and $A_2 = S_2 \setminus B_2$. Let $a_2 = |A_2|$ and $b_2 = |B_2|$.

First, we note that $m^2 \geq \mathcal{I}(B_2, M) \geq b_2 \cdot d_0$, leading to

$$b_2 \;\leq\; \frac{m^2}{d_0} \tag{4}$$

Secondly, let t be the number of edges between M and A_1. Note that a vertex $z \in A_2$ has all its S_1 neighbors in A_1, and consequently, A_2 is 2-controlled by M in the subgraph of G induced by $M \cup A_1 \cup A_2$. Therefore $t \geq a_2$ by Lemma 8. These edges form paths of length 2 from A_1 to A_1 via M. The number of such paths is $K = \sum_{x \in M} \deg(x, A_1)^2$. By convexity, this number is minimal when the degrees $\deg(x, A_1)$ are equal, in which case $\deg(x, A_1)) = \frac{t}{m} \geq \frac{a_2}{m}$ for any $x \in M$. Hence

$$K \;\geq\; \left(\frac{a_2}{m}\right)^2 \cdot m \;=\; \frac{a_2^2}{m}.$$

As we want to compute the influence $\mathcal{I}(A_1, A_1)$, we have to take into account the fact that a vertex in A_1 may influence another vertex in A_1 via more than one 2-path. But as for any $y \in A_1$ we have $\deg(y, M) \leq d_0$, the number of different 2-paths via M that can contribute the (same) influence of some $y' \in A_1$ on y is at most d_0. Consequently

$$m^2 \;\geq\; \mathcal{I}(S_1, S_1) \;\geq\; \mathcal{I}(A_1, A_1) \;\geq\; \frac{K}{d_0} \;\geq\; \frac{a_2^2}{m d_0}.$$

and we get

$$a_2 \;\leq\; \sqrt{d_0} \cdot m^{3/2}. \tag{5}$$

The bounds specified by inequalities (4) and (5) are optimized by fixing $d_0 = \Theta(m^{\frac{1}{3}})$ (more exactly $d_0 = (4m)^{\frac{1}{3}}$), where we get

$$a_2 = O(m^{\frac{5}{3}}), \qquad b_2 = O(m^{\frac{5}{3}}).$$

So $s_2 = a_2 + b_2 = O(m^{5/3})$. ∎

Since $|V| = s_1 + s_2 + m$, by Lemmas 7 and 9 we have the following.

Proposition 10. Every 2-monopoly (G, M) satisfies $m = \Omega(|V|^{3/5})$.

2.2 Lower Bound for $2k$-Monopolies

The generalization to $2k$-monopolies is straightforward.

Proposition 11. If (G, M) is a $2k$-monopoly then $m = \Omega(|V|^{3/5})$.

Proof Define G^k, as the graph on vertex set V with an edge between x and y if and only if $D_G(x, y) \leq k$. Then clearly (G, M) is a $2k$-monopoly only if (G^k, M) is a 2-monopoly. ∎

The lower bound proof for the case of $3k$-monopolies for odd k is deferred to the full paper.

3 Upper Bounds

Proposition 12. There exist r monopolies (G, M) with $m = O(|V|^{3/5})$.

Proof To prove this we construct a 2-monopoly associated to a parameter t with $m = \Theta(t^3)$, $s_1 = \Theta(t^4)$, and $|V| = \Theta(t^5)$, that is, $|V| = \Theta(m^{\frac{3}{5}})$. The nodes $x \in S_1$ also satisfy $\delta(x) = O(t) = O(m^{1/3})$. Clearly these parameters ensure the lower bound. An outline of the construction is given in figure 3. The construction technique can be extended to the case of r-monopolies for $r > 2$; this extension is deferred to the full version of the paper.

- $V = M_2 \cup M_1 \cup S_1 \cup S_2$.
- M_2 is a clique of size t^3, composed of t sets of t^2 vertices.
- M_1 is a independent set of size t^3 composed of t sets of t^2 vertices. The ith set in M_1 is connected to the ith set in M_2 by a complete bipartite graph. Each set of size t^2 is decomposed into subsets of size t.
- S_1 is a independent set of size t^4, composed of t^2 sets of size t^2. The ith set in S_1 (of size t^2) is connected to the ith subset in M_1 (of size t) by a complete bipartite graph.
- S_2 is a independent set of size $t^5 - t^4$, composed of t^4 sets of size $t - 1$. The nodes of the ith set in S_2 are connected to the ith node of S_1.

In order to show that the construction gives a 2-monopoly achieving the lower bound it is enough to count the influences for the four types of vertices (in M_2, M_1, S_1, S_2). The following table summarizes the counts.

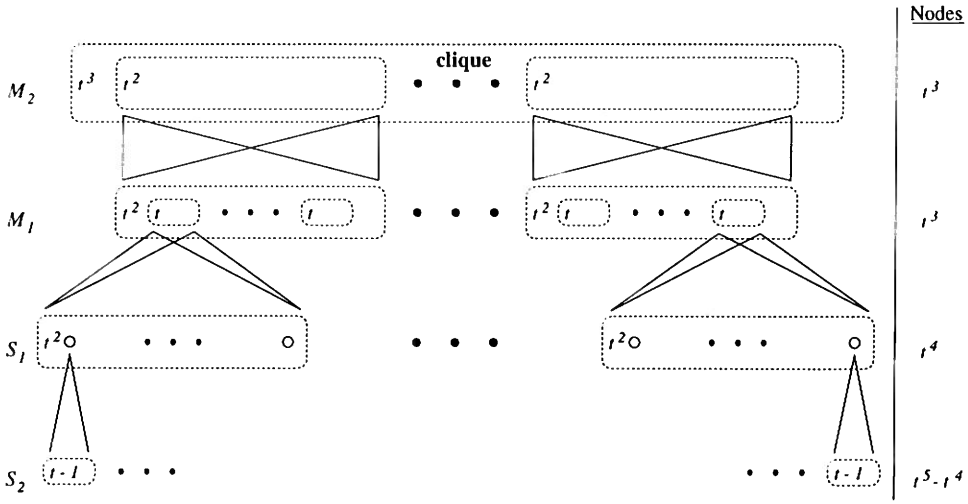

Fig. 3. A 2-monopoly achieving the upper bound.

	$x \in M_2$	$x \in M_1$	$x \in S_1$	$x \in S_2$
$\mathcal{I}(x, M)$	$2t^3$	$t^3 + t^2$	$t^2 + t$	t
$\mathcal{I}(x, S)$	t^3	t^3	$t^2 + t - 1$	t

∎

Proposition 13. There exist 3-monopolies (G, M) with $m = O(|V|^{4/7})$.

Proof To prove this we construct a 3-monopoly associated to a parameter t with $m = \Theta(t^4)$, $s_1 = \Theta(t^5)$, and $|V| = \Theta(t^7)$, that is, $|V| = \Theta(m^{\frac{4}{7}})$. Clearly these parameters ensure the lower bound. An outline of the construction is given in figure 4), and the details are deferred to the full version of the paper. ∎

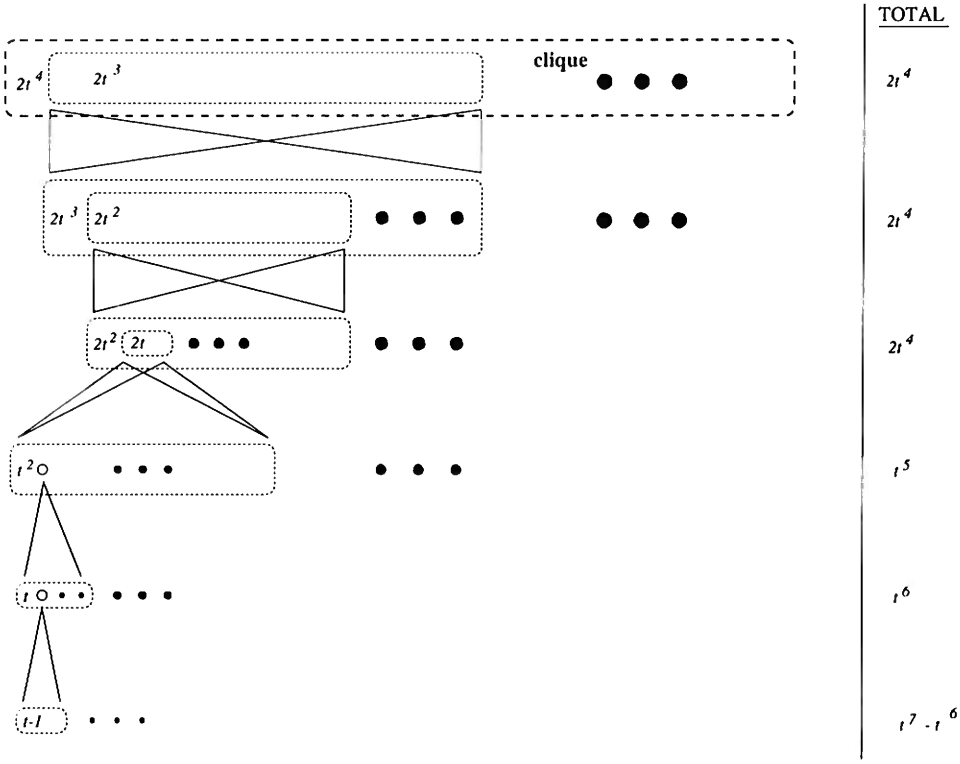

Fig. 4. A 3-monopoly achieving the upper bound.

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