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► To cite this version:

Sana Ben Abdallah, Dhafer Saidane, Mihaly Petreczky. Application of robust control for CSR formalization and stakeholders interest. Computational Economics, 2023, 64, pp.891-934. 10.1007/s10614-022-10286-y . hal-03832613

HAL Id: hal-03832613

<https://hal.science/hal-03832613v1>

Submitted on 27 Oct 2022

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Computational Economics

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--Manuscript Draft--

Manuscript Number:	CSEM-D-21-00077R2
Full Title:	Application of robust control for CSR formalization and stakeholders interest
Article Type:	Original Research
Keywords:	Sustainability, corporate social responsibility, stakeholder theory, control theory, linear matrix inequalities, Pareto-optimality
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Funding Information:	
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Response to Reviewers:	For a detailed response to reviewer's comments please see the pdf file containing the revised manuscript and our response to reviewers attached to the revised version of our manuscript

Application of robust control for CSR formalization and stakeholders interest

May 13, 2022

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Abstract

In this paper, we propose a new definition of sustainability that includes dynamics and equity. We propose a theoretical framework that allows finding a fair and sustainable strategy for all stakeholders. More precisely, the framework allows calculating a strategy which ensures that in the long run the interests of all the stakeholders are reconciled. In order to calculate a such a strategy, we model stakeholders and actors as dynamical systems in state-space form. Furthermore, we use robust control and linear matrix inequalities to calculate the desired strategy. We use several simulation scenarios to show the effectiveness of our proposed framework.

Keywords: Sustainability, corporate social responsibility, stakeholder theory, control theory, linear matrix inequalities

Declarations of interest: none

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1 Introduction

It is difficult for any company to satisfy the needs of all its stakeholders at the same time. In general, only the interests of the shareholders are protected by the managers. Those of other stakeholders are almost ignored in particular because of their conflicting nature. The sustainable development and corporate social responsibility (CSR) approach tries to harmonize the opposing utilities of the different stakeholders. It defends a compromise between the expectations of these latter, whatever their power. CSR is defined by the European Commission (2002) as "a concept whereby companies integrate social and environmental concerns in their business operations and in their interaction with their stakeholders on a voluntary basis". It is therefore intended for all stakeholders. In general terms, organizations' engagement with stakeholders can be defined as the process of taking into account, the participation or involvement of individuals and groups who influence or are influenced by the company's activities. Although opinions differ on how to integrate them, it seems that the integration of stakeholders is one of the essential ingredients for corporate sustainable excellence. All these approaches pose the question of the distribution of interests, resources, and responsibilities of each actor and highlight the strengths and limitations of stakeholder theory. They all involve not only identifying, classifying and consulting stakeholders but also integrating their interests to advance the business and to be sustainable.

The paper contributes to the literature on CSR. The objective of this paper is to set up a new mathematical framework which allows companies to find strategies for meeting CSR obligations. This framework involves:

- a mathematical formalization of what it means to reconcile stakeholders' interests,
- an algorithm which decides if it is possible to reconcile stakeholders' interests and which computes a strategy for reconciling stakeholders' interest whenever such a strategy exists.

The proposed algorithm requires as input a mathematical model of the company and its stakeholders. This model should be in the form of a dynamical system describing the temporal interaction between the company and its stakeholders. The mathematical definition of reconciling stakeholders' interests involves temporal aspects too. Namely, it relies on the notion of sustainability, which means that stakeholders' interests should be taken into account in the long run too.

This work is the first step of a long program in which we try to develop mathematical tools to help companies to meet CSR requirements. The first step is to propose a formalization of CSR and a strategy achieving it, under the assumption that we have a suitable econometric models. The next step would be to create an appropriate econometric models which can be used in this methodology.

The suggested approach We model the interests of stakeholders by utility functions which depend on attributes. These attributes represent various economic quantities, and they change in time. The change of attributes is modeled by a dynamical system in state-space form. Space-state models Moura et al. (2016); Choi and Park (2013); De Souza et al. (2013) have been widely used in economics. A state-space model describes how attributes are generated from their lag and how certain exogenous actions can influence those attributes. These exogenous actions represent the action of certain actors/stakeholders. The goal is to propose a strategy for choosing these exogenous actions, such that in the long run the attributes get close to an equilibrium point in which the values of the utility functions of the stakeholders are above a certain threshold. These thresholds represent a situation where the interests of each stakeholder is sufficiently taken into account. This means that in the long run, the strategy will lead to a *fair* outcome. This equilibrium point can be chosen to be Pareto-optimal for the stakeholders. By Pareto optimality we mean that in any other point the utility function of at least one stakeholder takes a smaller value. Note that choosing a Pareto optimal equilibrium point is optional.

Moreover, the strategy should be such that the vital interests of the stakeholders are not violated, i.e., at no time the attributes take a value for which the utility of one of the stakeholders descends below a certain level. The situation where the utility level descends below a certain level represents a bad scenario, whereby the corresponding stakeholder is forced to abandon the economic process. It could correspond to resource depletion, bankruptcy or basic needs not being made (if the stakeholder is a human being). We call collections of attribute values *sustainable*, if the utility functions of all stakeholders are above this critical threshold, when evaluated at these attribute values.

This property will be used to define sustainability. Sustainability means that we avoid situations where one of the stakeholder's vital interests are not respected. By avoiding such situations we can ensure that stakeholders will continue to cooperate as their vital interests are respected. This definition of sustainability captures the intuitive meaning of the concept: namely, that the economic process can be continued for a long period without a major crisis. That is, the strategy is *fair, acceptable and sustainable*.

Moreover, the proposed strategy is robustly sustainable i.e., even with the presence of disturbances or modeling errors, our strategy will still be sustainable.

Remarks are in order concerning the origins of the state-space models and the implementation of the strategy.

State-space models may arise in various ways, for example, they can be derived empirically, using data. However, under suitable assumptions they can be viewed as arising from the behavior of rational agents acting using local information Mazumdar et al. (2020); Ratliff et al. (2016). More precisely, assume that each attribute corresponds to an agent who has the right to change them. Note that agents need not coincide with stakeholders, they are two different concepts. Each agent has an utility function which potentially depends on all the attributes. Note that the utility functions of agents are different from those of the stakeholders. We assume that each agent tries to maximize its utility

based on local information: at each time step each agent tries to choose its attribute in such a manner that the utility function of the agent is maximized, assuming that the attributes of other agents remain constant. Moreover, the optimization is done among those attribute values which are close to the current one. That is, each agent tries to apply a locally optimal strategy in the absence of information on the actions of other agents. In such a manner a dynamical system in state-space form arises. The equations of the latter system are directly related to utility functions of agents. In fact, for a large class of systems we can find suitable utility functions such that the game described above results in the given system.

It turns out that the equilibrium point, which might be Pareto-optimal from the point of view of stakeholders, is a local Nash-equilibrium from the point of view of agents. [A more complete treatment of this interpretation can be found Mazumdar et al. \(2020\); Ratliff et al. \(2016\)](#). We stress that the interpretation of state-space models as a result of a game is optional for our framework. We mention it in order to indicate the link between our approach and mathematical tools used in the classical literature on economical modeling.

Concerning the implementation of the strategy, we do not suggest any particular way to implement it. However, the strategy could, in principle, be implemented by imposing a suitable tax on the actors which are responsible for the exogenous actions. Note that imposing a tax in order to achieve CSR does not contradict to the definition of CSR, as CSR does not exclude hard law (taxes). In France, for example, to face new social and environmental challenges, the legislator has intervened on several occasions to outline the main lines in terms of CSR (for example the NRE law, Grenelle Law 1 and 2, ...). However, in other countries (for example, US) CSR relies more on soft law. We would like to stress that implementing the calculated strategy by taxes is only one of the many options, we mention it in order to make a link with classical methods of economic regulation. This paper focuses on finding a strategy the actors could use to meet CSR requirements, and not on the various ways to enforce this strategy. The latter problem is a relevant one, but it requires more research.

We use methods from robust control theory, namely linear matrix inequalities (LMI) to calculate the described strategy Boyd et al. (1994). As we have mentioned it before, state-space models have been extensively used in the economics literature, especially for optimal control of economic models Blueschke et al. (2013); Blueschke and Savin (2017). In contrast to Blueschke et al. (2013) and Blueschke and Savin (2017) the mathematical framework of this paper leads to a robust control problem, not an optimal one. More precisely, we aim at finding a stabilizing feedback which respects a number of constraints and which is robust to perturbations. In optimal control theory we aim at finding a feedback control law which minimizes a certain cost function. While on a technical level the two problems are related in the sense that stabilizing feedbacks may arise as optimal control strategies for a suitably chosen cost function, the two problems are quite different and they can be solved using different algorithms. [Moreover, integrating state and input constraints into optimal control problem in a theoretically sound manner is challenging. In fact, to the best of our knowledge](#)

there is no systematic procedure for doing so.

The need for respecting state and input constrains and the need for robust control strategies was our main the motivation for using LMIs. LMIs are standard tools in control theory. Using LMIs allows us computing a control strategy which is provenly stabilizing, robust, and it guarantees that **state and input constraints are respected and that** certain cost functions satisfy certain inequalities. This is precisely what we need in order to a robust strategy which ensures that all stakeholders are at least minimally satisfied all the time, and are reasonably satisfied in the long run. Moreover, there is an efficient numerical algorithm for solving LMIs and computing the corresponding control strategy.

Note that to the best our knowledge stakeholders' satisfaction has not be addressed in the literature using control theory in general, or by optimal control theory in particular. Based on the discussion above we believe that LMIs are particularly well-suited for addressing sustainable satisfaction of stakeholders' interest in the long run, but other methods could also turn out to be useful.

The structure of the remainder of the paper is as follows. Section 2 will be devoted to the literature review on the stakeholders' theory and CSR. There we will also position our paper with respect to the existing literature on CSR. Section 3 we will present the mathematical framework. In Section 4, a numerical case study will be presented to illustrate the proposed approach. Finally, an overall conclusion will be presented.

2 Related literature: Stakeholder theory

In this paper we define sustainability as stakeholder's satisfaction, and we view various ESG criteria as variables which have impact on the satisfaction of various stakeholders. This is consistent with the various other definitions used ESG literature, since meeting various ESG criteria amounts to satisfying the interests of various stakeholders. In other words, we approach ESG via stakeholders' theory. In particular, the framework proposed in the paper can be used to deal with ESG criteria.

Our view is supported by numerous publications Freeman (2017), which all propose definitions of sustainability which is based on the satisfaction of different stakeholders. In particular, in Ben Abdallah et al. (2020) is defined as "a company's overall ESG performance as it relates to various stakeholder groups: customers, employees, suppliers, executives, managers, investors, and so forth". ». Therefore, ESG criteria are a set of extra-financial criteria used to analyze stakeholder satisfaction and therefore the sustainability of a company.

Indeed, although used throughout the world, there is no exact definition for sustainability. However, this concept is generally understood respect for a company's stakeholders and ESG principles. Dahlsrud (2006) carried out an interesting study on the various definitions listed in the literature between 1980 and 2003, the most cited by 27 authors. The latter are 37 in number, highlighting the main dimensions of sustainability: the relational dimension with stakeholders, the social, economic, and the environmental dimensions (ESG),

and the voluntarist or philanthropic dimension. Several studies have shown the close relationship between ESG criteria and stakeholder. Barnett and Salomon (2012) postulate that the application of strategies related to the implementation of ESG practices requires the participation of stakeholders so that they can adhere to the company's project. According to Koh and Qian (2014), the benefits that companies can acquire from ESG practices depend on their "moral and pragmatic" legitimacy to implement them. They must therefore obtain the support of the stakeholders. They conclude by considering that high-risk companies would be pushed to turn to ESG practices from the moment they can meet the expectations of stakeholders. Attig et al. (2013) demonstrate that within the components inherent to ESG criteria, only those that are positively evaluated and sought after by stakeholders have a positive effect on the rating of companies. We understand here that sustainability is becoming a means of achieving concrete results targeting specific stakeholders.

In fact, the concept of sustainable development encourages companies to involve stakeholders in their governance. The issue of sustainable development is linked to the integration of the expectations and interests of stakeholders in corporate strategy and management Sharma (2001). As Capron and Quairel Lanoizelee (2004) point out, "The concept of stakeholders is ubiquitous in all the literature on Corporate Social Responsibility".

The idea of only appealing to shareholders is now considered obsolete by several experts. The company, as part of a network of actors, must take into account the interests of its stakeholders. Indeed, according to Persais (2005) : "One of the main challenges of the current leader is therefore to integrate the (non-economic) interests of a set of stakeholders and make them compatible with the interests of shareholders". Furthermore, Donaldson and Preston (1995) argue that "all persons or groups with legitimate interests participating in an enterprise do so to obtain benefits and there is no *prima facie* priority of one set of interests and benefits over another",

The stakeholder approach has been the subject of both empirical and theoretical studies. Today, stakeholders are at the heart of the social responsibility mechanisms implemented in companies. According to a broad consensus, stakeholder theory represents a relatively solid foundation, at least well established and recognized, for research on CSR, Business and society relations or business ethics. It is also used in debates on corporate governance and on the relationship between corporate strategy and sustainable development.

The concept of stakeholders is given by Freeman (1984). It is defined as "any group or individual who can affect or is affected by the achievement of the organization objectives." It is a concept that opens towards "a pluralist vision of the organization, an entity open to its environment" Gond and Igalens (2014). The stakeholder theory presents itself as an attempt to found a new theory of the firm integrating its environment to go beyond the traditional profit-making vision of the firm Mercier (2012) . Therefore, this theory seeks to integrate the interests of individuals and groups of people concerning the company and taking into account the social performance of this latter Padioleau (1989).Sternberg (2001) summarizes the concept of stakeholders by "Any person may have an

interest in an organization". The definition of this concept is still the subject of many discussions Pedersen (2006). Furthermore, the stakeholders' approach includes the views of stakeholders and makes them compatible with the views of shareholders. This is one of the most important challenges facing companies.

Some authors have tried to classify the stakeholders in two visions. "Normative" or "Instrumental" Ballet and Bazin (2004). The normative vision is a purely ethical vision, where the company seeks to satisfy all stakeholders, by defining moral guidelines and use these guidelines as the basis for decision making. In contrast, the instrumental view is the consequence of taking into account stakeholders opinions as an essential element that leads to value creation. Managing relationships with stakeholders is a way for the company (directors and shareholders) to achieve its goals. Sharma (2001) returned stakeholders into two groups: economic and non-economic. Economic stakeholders include all stakeholders involved in economic life and in productive activities of the company such as shareholders, suppliers, customers, etc. While the non-economic stakeholders associated with the environmental and the social actors. Also, they are linked to ethical dimensions. In short, stakeholders are defined as suppliers, customers, shareholders, employees, managers, regulators, and civil society ... etc Avkiran and Morita (010a).

Some studies have also tried to prove the positive relationship between financial performance and the inclusion of stakeholder's points of view Luffman et al. (1982); Jones et al. (1999); Hillman and Keim (2001). Some other studies on stakeholder's management also indicated a positive relationship between the plural form in management, the including of all stakeholders opinions, and the financial performance. For instance, Post et al. (2002) showed that among 89 studies, 48 of them showed this positive relationship. Tiras et al. (1998) argue also that companies that hold a good relationship with stakeholders exhibit higher performance. Thus, the integration of stakeholders can reduce risk; enhance the confidence of civil society, and improve the transparency of the regulatory framework Holliday et al. (2002). According to Sharma (2001), "In the short term, the integration of stakeholders can reduce costs and provide opportunities for differentiation. In the long term, it allows the dynamic construction of valuable competitive resources". In most cases, we notice that effective stakeholder management enables banks to design policies for more efficient and stable banking systems Avkiran and Morita (010b).

The novelty of the paper with respect to the related literature We propose a mathematical formalization to calculate a strategy which could be applied to satisfy stakeholders and respond to CSR practices. To our knowledge, there are no other publications which propose a mathematical formalization of the CSR and which take into account temporal aspects. To the best of our knowledge, this paper is the first study which attempts to find a fair and sustainable strategy, for all stakeholders, and which takes into account temporal aspects, and does so in a mathematically rigorous fashion.

3 Methodology

The basic idea is as follows. The degree of satisfaction of each stakeholder at a time instance is a function of the *attributes* at that time instance. This function is called the *utility function* of the stakeholder. Intuitively, attributes correspond to economic variables of interest. The values of some of these attributes can be arbitrarily assigned, they represent actions by some actors. We refer to the latter attributes as *input actions* or *inputs* for short. Which entities are the actors has to be decided for each application separately. Choosing the actors is part of building a suitable mathematical model for the application at hand. Some of the stakeholders may be actors, but not all stakeholders have to be actors. In fact, actors can also be entities which are not stakeholders. That is, stakeholders are entities whose degree of satisfaction interests us, but stakeholders need not be able to undertake actions to change their own satisfaction.

Mathematically, the attributes of stakeholders which cannot be freely chosen, i.e. which are not input actions, will be modeled as states of a controlled dynamical system Weber (2011). The input are the control inputs of that dynamical system. A strategy is then a sequence of input actions. We would like to find a strategy which drives the system to a point in which the value of the utility function of each stakeholder is above a certain threshold. These thresholds correspond to all stakeholders being reasonably satisfied. Moreover, the strategy should be such that when it is applied, the value of the attributes should always remain sustainable, that is, at any time instance, the attributes should take a value for which the value of the utility function of each stakeholder is above a certain critical level. These critical levels represent the minimal levels of satisfaction which we have to guarantee to all stakeholders during the entire evolution of the system. Moreover, we would like the strategy to be in the form of a feedback, i.e., the current input action prescribed by the strategy should be a function of current attribute values. Finally, we would like the strategy to be robust, i.e., to work even in the presence of disturbances or modeling errors.

We will use control theory to find such a strategy. Finding strategies for influencing dynamical systems is the core topic of control theory Weber (2011), Franklin et al. (2001).

The proposed methodology thus consists of the following steps:

- Choice of the equilibrium point

The first step is to choose an equilibrium point which is also a sustainable state. By a sustainable state we mean a state where the value of the utility function of each stakeholder is above a certain threshold. By an equilibrium point we mean a state such that if the system is started in this point, then it will always remain there. The concept of an equilibrium point is a standard one in the theory of dynamical systems Hirsch et al. (2013). The strategy we are looking for is one that forces the states of the dynamical system to approach the desired equilibrium point as time progresses. That is, in the language of control theory Franklin et al. (2001), the strategy is *stabilizing*.

The motivation for using the concept of equilibrium point is the following. It is a standard practice in control theory Weber (2011); Franklin et al. (2001) to consider the problem of driving a dynamical system to a desired equilibrium (in the language of control theory, stabilizing the system around the chosen equilibrium point). In turn, the equilibrium is chosen in such a manner that the behavior of the system in the vicinity of the equilibrium point has some desirable properties. This choice of the equilibrium point together with the use of a strategy which drives the system to the chosen equilibrium point ensures that in the long run the behavior of the system has the desired properties, and it keeps these desired properties in the presence of small enough perturbations.

- Calculating a safe set and a feedback strategy

We calculate a set such that it contains the equilibrium point and such that all the elements of this set are sustainable. Recall that by sustainability we mean that the utility function of each stakeholder is above a certain critical value. We will call such set a *safe set*. In parallel to calculating a safe set we also calculate a *stabilizing* strategy such that when the strategy is applied, the safe set is *invariant*. By stabilizing we mean that when applying this strategy, in the absence of disturbances, the state of the system converges to the chosen equilibrium point. By invariance we mean that if the initial collection of attributes in this set, then at any time instance the attributes at that time instance will also be in that set.

As a consequence, if the initial state of the system is in the safe set, then the future states of the system will always remain there and in the absence of disturbances they will converge towards the equilibrium point. Moreover, even in the presence of small enough disturbances on the states, the future states of the system will remain in the safe set and hence the future states will be sustainable. If the initial state is not in the safe set, then the proposed strategy is not guaranteed to achieve convergence to the equilibrium point while ensuring that the states remain sustainable.

- Application of the strategy: feedback.

The calculated strategy will be in the form of feedback. That is, at each time instance, the action prescribed by the strategy is a function of the current state values. The use of feedback and the properties of the safe set guarantee that the strategy is robust. If the actual attribute values differ slightly from the ones prescribed by the model, due to external shocks (disturbances) or modeling error, but they are still in the safe set, then the application of the feedback will ensure sustainability and convergence to the equilibrium point in the absence of further disturbances. This property of feedback strategies is widely used in engineering Franklin et al. (2001).

In Subsection 3.1 we will represent the details of our framework. In Subsection 3.2 we will discuss the relationship between our framework with some mathematical concepts which are classical in economic literature.

3.1 Mathematical framework

In mathematical terms, we model the time evolution of attributes as a discrete-time state-space model Kailath (1980) of the form

$$X(t+1) = F(X(t), U(t)) \quad (1)$$

where $X(t) = (X_1(t), \dots, X_n(t))^T \in \mathbb{R}^n$ is the vector of those attribute values at time $t = 0, 1, \dots$, which are not freely chosen by some external actors, $U(t) = (U_1(t), \dots, U_m(t))^T \in \mathbb{R}^m$ is the vector of input actions by actors. That is, $U(t)$ represents those attribute values which can be freely chosen by some actors. Examples of actions $U(t)$ could be increase in minimal wage, or change in required solvency ratio, etc. The function $F : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ is the *state-transition* function. The function F describes how the current attribute values and the input actions of actors influence the attribute values in the future. That is, at the next time step, the values of those attributes which are not input actions are determined by values of these attributes at the previous time step and by input actions at the previous time step. We call $X(t)$ the *state* of (1) and we call $U(t)$ the *input* of (1). That is, the states of (1) correspond to attributes whose values cannot be freely chosen, and inputs of (1) correspond to attributes whose values can freely be chosen by actors.

In order to define utility functions of stakeholders, we will be interested in the following subsets of state vectors

$$\mathcal{X} = \{(x_1, \dots, x_n) \mid x_i \in [x_{i,min}, x_{i,max}], i = 1, \dots, n\}, \quad (2)$$

and of input vectors

$$\mathcal{U} = \{(u_1, \dots, u_m) \mid u_i \in [u_{i,min}, u_{i,max}], i = 1, \dots, m\}. \quad (3)$$

That is, we are interested in vectors of states and inputs such that the i th attribute is to take values in the interval $[x_{i,min}, x_{i,max}]$ and the j th action is assumed to take values in $[u_{j,min}, u_{j,max}]$. **The reason for introducing the sets \mathcal{X} and \mathcal{U} is that in practice the economic variables which correspond to attributes and input actions tend to take values from a bounded set. Moreover, the utility functions of the stakeholders tend to be monotone in the attributes. Hence, the smallest (or highest) attribute value corresponds to the best or worst outcome from the point of view of one of the stakeholders. The same comment can be made on the input actions. We will call \mathcal{X} the set of *admissible states* and we will call \mathcal{U} the set of *admissible input*.**

Assume that there are N stakeholders and for each stakeholder there is an utility function $f_i : \mathcal{X} \times \mathcal{U} \rightarrow [0, 1]$, $i = 1, \dots, N$. Intuitively, if $f_i(x, u)$ is close to zero, then the state and action pair (x, u) is not favorable for the stakeholder, if the value $f_i(x, u)$ is close to 1, then the stakeholder is satisfied. Note that the utility functions of the stakeholders are defined only for those state and input pairs (x, u) which belong to $\mathcal{X} \times \mathcal{U}$. The reason for this is that it is difficult to define utility functions on an unbounded set. Of course, with a suitable change of variables the case of unbounded domain of definition can be transformed to

the bounded one, but such changes of variables tend to be non-linear and could create technical issues. For this reason we prefer to stick to partially defined utility functions.

We fix a set of values $\{f_{i,min}\}_{i=1}^N$ which represent the desired minima of the utility functions.

We will call a state and input pair $(X(t), U(t)) \in \mathbb{R}^n \times \mathbb{R}^m$ *sustainable*, if $(X(t), U(t)) \in \mathcal{X} \times \mathcal{U}$, and

$$\forall i = 1, 2, \dots, N : f_i(X(t), U(t)) \geq f_{i,min}, \quad (4)$$

i.e., if in this state and input the value of the utility function of each stakeholder is well-defined and it is greater than a certain minimal value.

In the sequel, we will concentrate on the case when $U(t)$ is determined as a function of $X(t)$, i.e., $U(t) = \mathcal{F}(X(t))$ for some function \mathcal{F} . In this case, we say that the state $X(t)$ is sustainable, if $(X(t), U(t))$, $U(t) = \mathcal{F}(X(t))$ is sustainable.

Our goal is to find a strategy, i.e., a function $\mathcal{F} : \mathcal{X} \rightarrow \mathcal{U}$ such that with the choice $U(t) = \mathcal{F}(X(t))$, the state of the system (1) will become sustainable. Moreover, we would like the strategy to be robustly sustainable, i.e., to be sustainable in the presence of disturbances or modeling errors.

In order to make the computation of the desired strategy easier, we will make some simplifying assumptions on the right-hand side of (1). Namely, we will assume that $F(x, u)$ is affine, i.e. it is of the form

$$F(x, u) = Ax + Bu + h$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$ are suitable matrices, $h \in \mathbb{R}^n$ is a suitable vector.

Assuming that F is affine is a standard practice in control engineering Franklin et al. (2001); Sontag (1998). In a nutshell, affine systems arise via linearization of non-linear systems around an equilibrium point, and they represent an approximation of the original non-linear system around this equilibrium point. Moreover, as it is well-known from standard control theory Khalil (2002); Sontag (1998), a feedback law which stabilizes the linearized model also stabilizes the original system, provided that the initial state of the original system is close enough to the equilibrium point around which the linearization was calculated. We will present a more detailed discussion on this topic when discussing the robustness properties of the feedback strategy generated by the proposed method in Subsection 3.1.5 and in Remark 2, Appendix A. In a nutshell, the proposed method can be applied even when F is not affine, if we approximate F by its linearization around a suitably chosen equilibrium point. In this case, the proposed strategy will still work, if the initial state of the true system is close enough to the equilibrium point.

For the reasons described above, linearized models of non-linear systems are widely used for control design, as they allow calculating a control law which locally achieves the control objectives. Moreover, by choosing more robust control laws, the effect of the modeling error, i.e., the error which arises due to approximating non-linear dynamics by an affine model, can often be sufficiently reduced. This approach is not always satisfactory, and sometimes a non-linear

model has to be used. Note that it is possible to use LMIs for non-linear models by embedding them into polytopic models or linear-parameter varying systems Boyd et al. (1994); Mohammadpour and Scherer (2012). This latter approach is a standard practice in control engineering. However, in the present paper we preferred to stick to affine dynamical systems to be able to focus on the main ideas rather than on technicalities. Extending the approach of this paper to linear parameter-varying models would go beyond the intended scope of the paper

We stress that the assumption that F is affine is made only to simplify computation, and it is not related to the control objectives described in (4).

Hence, we assume that the dynamical system (1) takes the form

$$X(t+1) = AX(t) + BU(t) + h. \quad (5)$$

If we consider the equation (5) line by line, then the change in the value of the i th attribute is

$$X_i(t+1) = \sum_{j=1}^n a_{i,j} X_j(t) + \sum_{l=1}^m b_{i,l} U_l(t)$$

$$A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ a_{21} & \dots & a_{2n} \\ \vdots & \dots & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & \dots & b_{1m} \\ b_{21} & \dots & b_{2m} \\ \vdots & \dots & \vdots \\ b_{n1} & \dots & b_{nm} \end{bmatrix}$$

Hence, if $a_{i,j}$ is positive (negative), it means that the increase in the value of the j th attribute leads to an increase (decrease) in the value of the i th attribute in the next time step. For example if $X_1(t)$ is profitability at time t , and $X_{10}(t)$ is the fixed wage at time t , then by increasing $X_{10}(t)$ we expect $X_1(t+1)$ to decrease (increase of wage leads to decrease of profitability), and hence $a_{1,10}$ should be negative

In order to find a suitable strategy we will carry out the following steps.

3.1.1 Choice of an equilibrium point

We find vectors $x_0 \in \mathcal{X}$, $u_0 \in \mathcal{U}$ such that

- $x_0 = Ax_0 + Bu_0 + h = F(x_0, u_0)$ (i.e. (x_0, u_0) is an equilibrium point, that is if the (x, u) is a solution of (1) such that $x(0) = x_0$ and $u(0) = u_0$ for all t , then $x(t) = x_0$).
- (x_0, u_0) is a sustainable pair.
- $f_i(x_0, u_0) = f_{i,t}$, $i = 1, \dots, N$ for some target values $f_{i,t} \geq f_{i,min}$ of the utility functions.

That is, (x_0, u_0) is such that if the system (1) is started in the initial state x_0 and the input $U(t)$ is constant and it equals u_0 , then the solution $X(t)$ will

be equal to x_0 . In other words, an equilibrium point is such that if the system is in that point, then it will never leave it. Moreover, in the equilibrium point the utility functions take the target values $f_{i,t}$.

The idea behind this is to choose $f_{i,t}$ in such a manner that all stakeholders are satisfied, e.g., $f_{i,t}$ is larger than 0.5. In order to find (x_0, u_0) , $x_0 = (x_{0,1}, \dots, x_{0,n})^T$ $u_0 = (u_{0,1}, \dots, u_{0,m})^T$ the following non-linear programming problem should be solved:

$$\begin{aligned} x_0 &= Ax_0 + Bu_0 + h \\ f_i(x_0, u_0) &= f_{i,t}, \quad i = 1, \dots, N \\ x_{i,min} &\leq x_{0,i} \leq x_{i,max}, \quad i = 1, \dots, n \\ u_{j,min} &\leq u_{0,j} \leq u_{j,max}, \quad j = 1, \dots, m. \end{aligned} \tag{6}$$

Once the equilibrium point has been chosen, the next step will be to design a strategy for choosing $U(t)$ which drives (1) to the chosen equilibrium point.

At a first glance the idea of choosing an equilibrium model might appear to be unusual. Note, however, that this is a standard practice in control theory. More precisely, in control theory the desired behavior of the system is usually the behavior which the system exhibits at a certain equilibrium point. One then first finds an equilibrium point in which the system exhibits the desired behavior, and the one designs a strategy to drive the system to that equilibrium point.

To illustrate this point, let us recall the typical textbook example of regulator design, see Franklin et al. (2001). That is, consider a dynamical system of the form (5) and consider a variable $Y(t) = C(X(t))$, where C is a $1 \times n$ matrix. In regulator design, the goal is to find a strategy for choosing $U(t)$ such that

$$\lim_{t \rightarrow \infty} Y(t) = r.$$

In this case, the usual procedure Franklin et al. (2001) is as follows: we compute the equilibrium point x_0 such that

$$r = h(x_0) \text{ and } x_0 = F(x_0, u_0)$$

for a suitable u_0 . If

$$F(x, u) = Ax + Bu + h,$$

, there is a single input, i.e., B has one column, then

$$u_0 = \frac{r - C(I - A)^{-1}h}{C(I - A)^{-1}B}.$$

and $x_0 = (I - A)^{-1}(Bu_0 + h)$. If we compute a matrix K such that all the eigenvalues of $A - BK$ are within the unit disc, then the strategy

$$U(t) = -K(X(t) - x_0) + u_0$$

achieves the objective that

$$\lim_{t \rightarrow \infty} Y(t) = r.$$

Our approach follows the same philosophy: we choose an equilibrium point at which the behavior of the system has the desired properties and then design a strategy for driving the system to that equilibrium point.

3.1.2 Choice of the strategy

Let us choose the strategy \mathcal{F} as a feedback

$$U(t) = -K(X(t) - x_0) + u_0 \quad (7)$$

where K is a $m \times n$ matrix. That is, at every step, the input applied to the system depends on the current state.

We would like to find K and a positive definite $n \times n$ matrix Q , such that the following holds.

Let us associate with Q the following ellipsoidal set \mathcal{P} centered around x_0 :

$$\mathcal{P} = \{x \in \mathbb{R}^n \mid (x - x_0)^T Q^{-1} (x - x_0) < 1\}. \quad (8)$$

We will refer to \mathcal{P} as the safe set.

Then the matrix K and the matrix Q should satisfy the following conditions:

- **Stability** If we use (7), then $X(t)$ converges to x_0 . Notice that if $X(t)$ converges to x_0 , then $U(t)$ defined by (7) converges to u_0 .
- **Invariance** If $X(0)$ belongs to \mathcal{P} and $U(t)$ is chosen as in (7), then $X(t)$ belongs to \mathcal{P} for all t .
- **Safety** If $X(t)$ belongs to \mathcal{P} and $U(t)$ satisfies (7), then for all $i = 1, \dots, N$, $(X(t), U(t))$ is a sustainable pair.
- **Constraint satisfaction** \mathcal{P} should be a subset of \mathcal{X} and for any $x \in \mathcal{P}$, $-K(x - x_0) + u_0 \in \mathcal{U}$.

In other words, the goal of the computational method proposed in this paper is to find a strategy of the form (7) and a set of the form (8) which satisfies the conditions above. The corresponding matrices K and Q are determined using Linear Matrix Inequalities (LMIs), the details are presented in Appendix A

Next, we will discuss the intuition behind the choice of strategies of the form (7) and the requirements on the set (8) formulated above.

The strategy (7) is a feedback strategy: the input action $U(t)$ at time t is determined by the attribute vector $X(t)$ at time t . The intuition behind (7) is that we look at the difference between the current attribute vector $X(t)$ and the desired equilibrium state x_0 . Then the control action $U(t)$ is proportional to that difference, in fact $U(t)$ is an affine function of that difference. The elements of K are the coefficients of this affine function. More precisely,

$$U_i(t) = \sum_{j=1}^n K_{i,j} (X_j(t) - x_{0,j}) + u_{0,i},$$

i.e., $K_{i,j}$ is the coefficient with which we multiply the term $X_j(t) - x_{0,j}$ representing the deviation of the j th component of the attribute vector from the desired value. We choose the matrix K so that the application of the strategy (7) makes the system (1) *asymptotically stable* at the equilibrium point (x_0, u_0) Sontag (1998), i.e. any solution $X(t)$ of (1) for the choice of $U(t)$ as in (7) is such that $X(t)$ converges to x_0 and $U(t)$ converges to u_0 .

In principle, in order to choose the matrix K such that (7) stabilizes (5), it is sufficient to choose K so that $A - BK$ is a Schur matrix, i.e., all of the eigenvalues of $A - BK$ are inside the unit disk. This follows from standard control engineering, see Franklin et al. (2001). There are several classical methods for choosing a matrix K for which $A - BK$ is a Schur matrix, such as pole placement or optimal control Franklin et al. (2001); Sontag (1998). However, stability is not sufficient for our purposes, we would also like to make sure that during the evolution of (5) all states and inputs are admissible and sustainable. This cannot be achieved by the classical techniques of pole placement or optimal control Franklin et al. (2001); Sontag (1998).

In order to achieve the latter objective, we introduced the set (8). The set (8) is parameterized by the positive definite matrix Q , and it represents an ellipsoid around the desired equilibrium point x_0 . The matrix Q is related to K as follows. If the strategy (7) is applied, then the matrix Q plays the role of a Lyapunov function Sontag (1998), i.e. it can be thought of as a distance measure from any attribute vector to the equilibrium point which decreases during the evolution of the underlying dynamical system eq. (1). More precisely, $V(X(t)) := (X(t) - x_0)^T Q^{-1} (X(t) - x_0)$ can be viewed as the square of the distance between $X(t)$ and x_0 , and if (7) is used, then $0 \leq V(X(t+1)) < V(X(t))$ unless $X(t) = x_0$. In fact, we choose Q so that the matrix $(A - KB)^T Q^{-1} - Q^{-1}$ is negative definite. Again, using standard arguments Sontag (1998), the fact that $0 \leq V(X(t+1)) < V(X(t))$ ensures that (8) is invariant under (5) if eq. (7) is used: if the strategy (7) is applied, and $X(0)$ is in the set \mathcal{P} , then all the subsequent states $X(1), X(2), \dots$ of (1) will be in the set \mathcal{P} .

In addition, when choosing Q we make sure that the elements of \mathcal{P} are admissible states and if (7) is used to calculate the input action for an element of \mathcal{P} , the corresponding input action will also be admissible. That is, if $X(t)$ is an element of \mathcal{P} and $U(t)$ satisfies (7), then the constraints $X_k(t) \in [x_{k,min}, x_{k,max}]$ and $U_i(t) \in [u_{i,min}, u_{i,max}]$, $i = 1, \dots, m$, $k = 1, \dots, n$ are satisfied. This requirement is not surprising, since all states and inputs of interest live in \mathcal{X} and in \mathcal{U} respectively.

Moreover, we choose Q in such a manner that if an attribute vector is in the set defined (8), the utility functions of each stakeholder exceed the minimal thresholds $f_{i,min}$ from eq. (4), i.e., if $X(t) \in \mathcal{P}$, then $f_i(X(t), U(t)) \geq f_{i,min}$, if $U(t)$ is chosen as prescribed in (7).

These properties imply that if the system (5) is started in the set \mathcal{P} defined in (8), and (7) is applied, then the solution of (5) will remain in this set and will in fact converge to the equilibrium point. Moreover, the solution and the corresponding input defined by (7) will be admissible and sustainable: for any t , $(X(t), U(t))$ will satisfy (4) and $X(t), U(t)$ will belong to \mathcal{X} and \mathcal{U} respectively.

Finally, the use of the set (8) also allows us to achieve our objectives in a robust manner if the initial state is in this set, as describe below, in Subsection 3.1.5.

3.1.3 Summary

That is, we choose an equilibrium point (x_0, u_0) , and a strategy (7) and a set \mathcal{P} containing x_0 , such that if the initial state $X(0)$ belongs to the set \mathcal{P} , it is also true that for all t , $(X(t), U(t))$ is sustainable. holds for all t . That is, if \mathcal{P} is the set of sustainable initial states, such that if the system is started in such a sustainable initial state, then its state will always be sustainable. This remains true even in the presence of small perturbation or modeling error.

Moreover, in the absence of perturbation, attribute vector $X(t)$ will converge to x_0 , so not only $(X(t), U(t))$ is sustainable, but eventually the value of the utility functions $f_i(X(t), U(t))$ will be close to $f_i(x_0, u_0)$.

3.1.4 Algorithm for calculating the desired strategy

In order to calculate the desired strategy, we propose to use tools from robust control, namely linear matrix inequalities (LMIs) Boyd et al. (1994). The details and the technical assumptions are described in Appendix A.

3.1.5 Robustness of the strategy with respect to perturbation and modeling error

Assume that the true system is not (5) but

$$\tilde{X}(t+1) = A\tilde{X}(t) + BU(t) + h + d(t)$$

for some disturbance $d(t)$ such that $\|d(t)\|$ is sufficiently small, then with the strategy $U(t) = -K(\tilde{X}(t) - x_0) + u_0$ the system with perturbation $d(t)$ will still be sustainable, although $\tilde{X}(t)$ will no longer converge to x_0 . Indeed, if $\tilde{X}(t) \in \mathcal{P}$, then $F(\tilde{X}, U(t)) = A\tilde{X}(t) + BU(t) + h_0 \in \mathcal{P}$ and $U(t) = -K(\tilde{X}(t) - x_0) + u_0$ ⁴, by the invariance property, and hence for small enough $d(t)$, $F(\tilde{X}(t), U(t)) + d(t)$ will be in \mathcal{P} . In Remark 2, Appendix A we make the argument above more precise.

In particular, assume that the true system (1) is not affine. Furthermore, assume that (x_0, u_0) is chosen so that $x_0 = F(x_0, u_0)$, i.e., by solving (6) with $x_0 = Ax_0 + Bu_0 + h$ being replaced by $x_0 = F(x_0, u_0)$. Assume that A, B, h are the linearization of F around x_0, u_0 . The latter means that $A = \frac{dF}{dx}(x, u)|_{x=\tilde{x}_0, u=\tilde{u}_0}$, $B = \frac{dF}{du}(x, u)|_{x=\tilde{x}_0, u=\tilde{u}_0}$, $h = \tilde{x}_0 - A\tilde{x}_0 - B\tilde{u}_0$. It then follows that (x_0, u_0) is an equilibrium point of (5). Moreover, the solution of the true system $\tilde{X}(t+1) = F(\tilde{X}(t), U(t))$ can be written as

$$\tilde{X}(t+1) = A\tilde{X}(t) + BU(t) + h + d(t)$$

⁴If $x \in \mathcal{P}$, the i th component of $u = -K(x - x_0) + u_0$ belongs to $[u_{i,min}, u_{i,max}]$

with $d(t) = F(\tilde{X}(t), U(t)) - A\tilde{X}(t) + BU(t)$. Let us apply (7), computed using the matrices A, B, h , to the true non-affine system $\tilde{X}(t+1) = F(\tilde{X}(t), U(t))$, i.e., assume that $U(t) = -K(\tilde{X}(t) - x_0) + u_0$. It turns out that if $\tilde{X}(0)$ is close enough to x_0 , then $\tilde{X}(t)$ will remain in the safe set \mathcal{P} and in fact $\tilde{X}(t)$ will converge to the equilibrium point as $t \rightarrow \infty$. In particular, in this case, the true solution $\tilde{X}(t)$ and the corresponding input $U(t)$ will always be both admissible and sustainable. That is, even if F is not affine, if we use an affine approximation of F , the resulting strategy (7) will still achieve the control objectives, if the initial state of the true system is close enough to the equilibrium point. Intuitively, this is due to the fact that in this case the disturbance $d(t)$ is small at the beginning, and as we approach the equilibrium point, it will get smaller. A more precise argument is presented in Remark 2, Appendix A

3.2 Relationship with the classical approach

Interpretation of the result of feedback policy in terms of Pareto optimality The proposed approach can be viewed as an attempt to achieve a Pareto-optimal (hence socially acceptable) outcome for all stakeholders. More precisely, we can choose the equilibrium point (x_0, u_0) as follows:

$$(x_0, u_0) = \operatorname{argmax}_{(x,u) \in \mathcal{X} \times \mathcal{U} : F(x,u)=x, f_i(x,u) \geq f_{i,min}, i=1,\dots,N} \sum_{i=1}^N f_i(x, u) \quad (9)$$

This choice then guarantees that (x_0, u_0) is a Pareto-optimal point for the utility functions $f_i(x, u)$, $i = 1, \dots, N$, i.e., it represents a socially desirable outcome. If the set of solutions of the constraints

$$\mathcal{C} = \{(x, u) \in \mathcal{X} \times \mathcal{U} \mid F(x, u) = x, f_i(x, u) \geq f_{i,min}, i = 1, \dots, N\} \quad (10)$$

is not empty, then there is always a solution to (9), if the utility functions f_i are continuous, as \mathcal{C} is a compact set.

Note that our framework does not require the equilibrium point to be Pareto optimal. This is just a possibility, which we mention in order to relate our approach to more traditional concepts used in economics. In fact, it may happen that our framework is able to find a strategy for an equilibrium point which is not Pareto optimal, while it fails to do so for a Pareto optimal equilibrium point. That is, requiring Pareto optimality may lead to loss of generality.

Implementation of the feedback policy via taxation The strategy $U(t)$ then can be thought of as a policy to enforce the Pareto-optimal outcome. The strategy can be enforced by imposing a tax on the actors. Assume for the sake of simplicity that there is one actor, i.e., $m = 1$, and the actor is the N th stakeholder. Note that the attributes change according to the dynamic equation $X(t+1) = F(X(t), U(t))$, so the various stakeholders are not able to change their attributes and hence the values of their utility functions, except the N th

stakeholder. In this case, if we assume that the N th stakeholder chooses the value $U(t)$ in such a manner that

$$U(t) = \operatorname{argmax}_{u_{1,min} \leq u \leq u_{2,min}} (f_N(X(t), u) + I(X(t), u)), \quad (11)$$

where $I(X(t), u)$ is the tax to pay, then by assuming that $f_N(X(t), u)$ is smooth in u and $\frac{d^2 f_N(x, u)}{du^2} < 0$ for all $x \in \mathcal{X}$, $u \in [u_{1,min}, u_{1,max}]$, and by choosing

$$I(x, u) = - \left(\frac{df_N(x, v)}{dv} \Big|_{v=-K(x-x_0)+u_0} \right) u, \quad (12)$$

it follows that $U(t) = -K(X(t) - x_0) + u_0$. That is, with a suitable choice of I , the optimal strategy of the agent will be to follow the control law $U(t)$.

That is, the calculated control law can be used by regulatory body to implement a policy via taxing, such that under this policy the attributes of the stakeholders converge to a Pareto-optimum.

We mention the possibility of imposing the tax just to illustrate that is possible to use classical method taxation for implement the proposed strategy. However, we do not suggest that such a tax should be implemented. The strategy can be implemented using different mechanism not only through taxes. Note that CSR does not exclude applying hard law (taxes), but does not impose it either. Whether the strategy should be implemented by taxes or any other method is subject of future research.

Justification of the state-space representation It is possible to view the state-space representation (1) as a system which arises as a result of interaction of rational agents, following the ideas of Mazumdar et al. (2020); Ratliff et al. (2016). More precisely, let us assume that the function $F(X, U)$ describing the dynamics arises as follows:

$$F_i(X, U) = X_i + \mu \frac{\mathbf{u}_i(X, U)}{dx_i} \quad (13)$$

where \mathbf{u}_i is a smooth function

$$\mathbf{u}_i : \mathbb{R}^{n+m} \rightarrow [0, 1]$$

The interpretation of (13) is as follows. We assume that there are n agent and the i th agent would like to maximize it's utility function \mathbf{u}_i . The utility function \mathbf{u}_i depends on attributes $X = (X_1, \dots, X_n)$, $X_i \in \mathbb{R}$, $i = 1, \dots, n$ and on the input action $U \in \mathbb{R}^m$. Let us assume that the i th agent controls the i th attribute X_i , and it tries to choose X_i in such a manner that the value of \mathbf{u}_i is maximized. The control action U is determined externally, it is a parameter of the game, but it cannot be changed by any of the actors.

Following Mazumdar et al. (2020); Ratliff et al. (2016) the agent can be assumed to follow the following gradient descend scheme:

$$X_i(t+1) = X_i(t) + \mu \frac{\mathbf{u}_i(X_1(t), \dots, X_n(t), U(t))}{dx_i} \quad (14)$$

Note that (14) coincides with the dynamical system (1), if F satisfies (13).

The equation (14) allows for the following interpretation: at step t the i th agent tries to apply a gradient descend algorithm with rate $\mu > 0$ in order to maximize \mathbf{u}_i with respect to X_i . We assume that the agent has access to the current value of the attributes controlled by the other agents. Moreover, (14) accounts for the possibility that the input action U can be changed at each step by an external actor too.

More precisely, the i th agent wants to choose the next value $X_i(t+1)$ in such a manner that

$$u_i(X_1(t), \dots, X_{i-1}(t), X_i(t+1), X_{i+1}(t), \dots, X_n(t), U(t))$$

is as large as possible. That is, each agent assumes that the attributes of the other agents will not change and it tries to adjust its own attribute value in such a manner that it optimizes its own utility function. However, each agent has only limited information about its own utility function, and it can optimize it only locally, i.e., around $X_i(t)$. It is then reasonable for the i th agent to opt for changing $X_i(t)$ in the direction which increases the utility the most. It is well-known that for some $\lambda > 0$,

$$\begin{aligned} \mathbf{u}_i(X_1(t), \dots, X_n(t), U(t)) &\leq \\ \mathbf{u}_i(X_1(t), \dots, X_{i-1}(t), X_i(t) + \lambda \frac{\mathbf{u}_i(X(t), U(t))}{dx_i}, X_{i+1}(t), \dots, X_n(t), U(t)), \end{aligned}$$

where $X(t) = (X_1(t), \dots, X_n(t))^T$, so $\frac{\mathbf{u}_i(X(t), U(t))}{dx_i}$ is the direction into which $X_i(t)$ should be changed in order to increase the utility function \mathbf{u}_i . That is, intuitively, locally minimizing $\mathbf{u}_i(X(t), U(t))$ with respect to $X_i(t)$ amounts choosing the next value for $X_i(t)$ as $X_i(t) + \mu \frac{\mathbf{u}_i(X(t), U(t))}{dx_i}$ for a suitable $\mu > 0$.

That is, if all agents update their attribute following this update rule, and the external actors which control U also keep updating $U(t)$ at each step, we arrive at (14). This idea was formalized in Mazumdar et al. (2020)⁵

Note that (14) can be viewed as the discretization of the following differential equation

$$\frac{d}{d\tau} X_i(\tau) = \frac{\mathbf{u}_i(X(\tau), U(\tau))}{dx_i}. \quad (15)$$

Assume now (x_0, u_0) is an equilibrium point of that differential equation, i.e.

$$\frac{\mathbf{u}_i(x_0, u_0)}{dx_i} = 0,$$

and the second derivative

$$\frac{d^2 \mathbf{u}_i(x_0, u_0)}{dx_i^2}$$

⁵Note that in Mazumdar et al. (2020) the agents were assumed to minimize their costs \mathbf{c}_i , instead in our setting we talk about maximizing utility \mathbf{u}_i : by setting say $\mathbf{c}_i = 1 - \mathbf{u}_i$ the two frameworks are equivalent.

is strictly negative. It was shown in (Ratliff et al., 2016, Theorem 1) that then x_0 is the local Nash equilibrium of the game with n players, where the utility function of the i th player is $X \mapsto \mathbf{u}_i(X, u_0)$. The notion of a local Nash equilibrium was defined in (Ratliff et al., 2016, Definition 1). Intuitively, one assumes that the set of actions of the players has a topology, and one define a local Nash equilibrium of the game as a Nash equilibrium of a restriction of the game to an open set of actions.

It is well-known that the equilibrium points of the differential equation (15) and (14) coincide. Indeed, (x_0, u_0) is an equilibrium point of (14) if and only if

$$\dot{x}_0 = x_0 + \mu \frac{\mathbf{u}_i(x_0, u_0)}{dx_i} = 0,$$

which for $\mu > 0$ is equivalent to $\frac{\mathbf{u}_i(x_0, u_0)}{dx_i} = 0$. The latter is precisely the definition of an equilibrium point of the differential equation (15).

The discussion above leads to the following remark. Consider an equilibrium point (x_0, u_0) of (1), under the assumption that F satisfies (13). In this case (1) coincides with (14). Assume that $\frac{d^2 \mathbf{u}_i(x_0, u_0)}{dx_i^2} < 0$. Then by (Ratliff et al., 2016, Theorem 1) x_0 is a local Nash equilibrium of a game with n players. The latter game is such that the action set of each player is a subset of \mathbb{R} and the utility of the i th player is the function $\mathbb{R}^n \ni X \mapsto \mathbf{u}_i(X, u_0)$.

That is, an equilibrium point of (1) is a local Nash equilibrium w.r.t. the game above, under some mild assumptions. As it was explained in Subsection 3.2, the method of this paper proposes to choose the equilibrium points (1) as Pareto optimal points w.r.t. utility functions f_i , $i = 1, \dots, N$. That is, we have two types of utility functions:

- \mathbf{u}_i , $i = 1, \dots, n$, are the utility functions which determine the interaction among various attributes and which define the mechanism of the time evolution of the attribute vectors,
- f_i , $i = 1, \dots, N$ are the utility functions of the stakeholders.

and equilibrium points of (1) can be local Nash equilibriums w.r.t. the first set of utility functions, and independently, they may also be Pareto-optimal points w.r.t. the second set of utility functions.

Note that the dynamics of the form (5) arise via the following choice of \mathbf{u}_i :

$$\mathbf{u}_i(X, U) = \frac{1}{\mu} \left(\frac{1}{2}(a_{ii} - 1)X_i^2 + \sum_{j=1, i \neq j}^n a_{i,j} X_j X_i + \sum_{j=1}^m b_{i,j} U_j X_i + h_i X_i \right). \quad (16)$$

where $X = (X_1, \dots, X_n)^T$ and $U = (U_1, \dots, U_m)^T$. Moreover, $\frac{d^2 \mathbf{u}_i}{dx_i^2}(x_0, u_0) < 0$, if and only if $a_{i,i} < 1$. That is, if $\mathbf{u}_i(x, u)$ is of the form (16) and $a_{ii} < 1$, then (x_0, u_0) is a local Nash equilibrium of the game above.

Note that for applying our framework it is not necessary to assume that the state-space representation arises as the result of the game described above. We have presented this interpretation in order to relate our assumption to traditional approaches used in economics.

4 Numerical case study

In this example we will consider the behavior of stakeholders in the financial institutions (banks). However, our model can be applied not only to financial institution but in fact to any scenario of several rational stakeholders. The choice of this example was motivated by previous research in this field Ben Abdallah et al. (2018), Ben Abdallah et al. (2020).

In our case study we consider the following three stakeholders : managers, regulators, and customers.

The first stakeholder is the management. The behavior of managers directly influences the bank's results since they are primarily responsible for its profitability. They always seek to meet the needs of different stakeholders. So their satisfaction is seen as a guarantee for the future of the bank. In our case study the satisfaction of managers will depend on two attributes: evolution of annual remuneration and return on assets (ROA) of the bank. The motivation for this choice is as follows. The satisfaction of the management can be measured by financial incentives such as the evolution of annual remuneration. It is obvious that the higher the evolution of remuneration, the greater the satisfaction of managers . As executives, managers must comply with all relevant regulations and banking standards. To capture this, we propose to use return on assets to measure the manager's ability to generate income for the bank.

The second stakeholder is the regulator. The main objective of regulators is to supervise banking institutions, to ensure the stability of the financial system and to protect the interests of depositors. To achieve these objectives, the supervisory authorities impose prudential standards in order to minimize the various risks. This is why risk control is considered to be the main criterion of performance from the point of view of regulators. For this reason, in our case study the satisfaction of the regulator depends on the following two attributes: the ratio of non-performing loans to total loans, and the ratio of liquid assets to total assets.

The third stakeholder is the customer. Customer loyalty is becoming a strategic objective for the various banks. The survival of the bank is generally linked to the satisfaction of customers, since it finds its main resources in the operations carried out by them. For this reason, the customer and his satisfaction are at the heart of the concerns of banking institutions. Banks are expected to attract customers through a number of competitive advantages. We assess customer satisfaction through the following two attributes: interest receivable to loans and bank fees on deposit.

The attribute "Bank Fees to deposit" will play the role of the input action, as it is not influenced by the other attributes and can freely be set by the

corresponding stakeholder ("Manager").

4.1 Mathematical model

The use case involves several economic quantities, and in order to apply our method to it, we need to a mathematical model which describes the interaction between these quantities. From the existing literature one can extract a qualitative information on these interactions. For instance, it seems clear that increase in non-performing loans is likely to have a negative impact on ROA. However, it is more difficult to find quantitative models of all these relationships. More precisely, while there is a significant body of literature on quantitative models involving some of these variables (for example, Bourke (1989); Nguyen et al. (2018); Murphy (1985) ; Arshadi and Edward (1987), etc.), to the best of our knowledge there is no work proposing a quantitative dynamical model which integrates all the variables used in our example. In fact, even the models which describe the relationship between a few variables require separate papers.

Moreover, existing literature tends to assume that the actors involved maximize their profit, and this assumption is used in deriving econometric models. That is, the actors are already assumed to apply a strategy which is optimal for them, and the available models describe the relationships between the variables under such a strategy. In contrast, in our paper the assumption of profit maximization is no longer used, as we aim at capturing stakeholder's theory in our framework. As a result, the models we need should describe the relationship between the variables for any strategy, including the ones which are not necessarily optimal for some of the actors involved.

To sum up, to the best of our knowledge, there is no off-the-shelf econometric model for this use case. Moreover, the existing literature (for example, Bourke (1989); Nguyen et al. (2018); Murphy (1985) ; Arshadi and Edward (1987), etc.) indicates that constructing such an econometric model would be challenging, requiring several separate papers, and hence would go beyond the scope of the present paper. For this reason we illustrate our approach with a case study which captures the basic problems considered in stakeholder's theory, but which lacks an off-the-shelf econometric model.

For this reason, we decided to illustrate our approach with a case study which captures the basic problems considered in stakeholder's theory, but which lacks an off-the-shelf econometric model. For this case study we construct a model which captures qualitative knowledge of the relationship between the economic variables involved. More precisely, the dynamical model which describes the interaction between attributes is based on qualitative knowledge of these interactions, and on our beliefs regarding the relative strengths of these interactions. However, the choice of utility functions for the stakeholders is based on real data Ben Abdallah et al. (2018, 2020). Below we will describe the corresponding models in detail.

Our model is of the form (1), more precisely, of the form (5). That is, to define model, it is necessary to define the state space, the input space, the set of admissible states and inputs, the utility functions f_i , the corresponding

target values of $f_{i,t}$ for $i = 1, \dots, N$, the corresponding minimal values $f_{i,min}$, $i = 1, 2, \dots, N$, the equilibrium point x_0, u_0 , and the matrices A and B and the vector h from in (5). Below we will only indicate the main steps, the precise numerical values will be presented in Appendix B.

Choice of the state space and the input space For this example, \mathbb{R}^5 is the state space and \mathbb{R} is the input space. The set of admissible states \mathcal{X} is of the form (2) with $n = 5$ and the set of admissible inputs \mathcal{U} is of the form (3) with $m = 1$. That is, each attribute and input are assumed to take values in the interval $[x_{i,min}, x_{i,max}]$, $i = 1, \dots, 5$ and $\mathcal{U} = [u_{1,min}, u_{1,max}]$. The values of $x_{i,min}, x_{i,max}$, $i = 1, 2, 3, 4, 5$ and of $u_{1,min}, u_{1,max}$ are shown in Table 1 in Appendix B.

Determination of the utility function

The utility function $f_i(x, u)$ is determined using multi-attribute utility approach due to Keeney and Raiffa (1976). This approach is based on the analyzing of multiple variables simultaneously and assembling them in one single value. In our case, the performance of each stakeholder is considered as a multi-attribute utility function (MAUF) ($f_i(x, u)$) and each attribute is considered as a single attribute utility functions (SAUF) ($g_k(x, u)$) i.e., we have 3 MAUF and 6 SAUF (2 SAUF for each stakeholder).

The first step consists in determining the attribute vectors (i.e. x_1, x_2, x_3, x_4, x_5 and u_6). In general, these latter are assumed to be linear or exponential. According to Kim and Song (2009), *"when SAUFs are assumed to be linear or exponential, they are sufficient for most cases and their forms are solid"*. The SAUFs are determined using the ASSESS ⁶(to determine the three intermediate values) and LAB Fit (to fit the utility functions) software ⁷.

Equation (17) represents the forms of the single utility functions of the various attributes corresponding to risk-averse, risk-seeking and risk-neutral, respectively.

$$g_k(x, u) = \begin{cases} a - be^{(-cx)} \\ a + be^{(cx)} \\ a + b(cx) \end{cases} \quad (17)$$

According to the expert's answers, the utility functions for each attribute are of the form:

$$g_k(x, u) = \begin{cases} a_k + b_k c_k \frac{x_k}{100} & k \in \{1, 2, 4\} \\ a_k + b_k e^{c_k \frac{x_k}{100}} & k = 3 \\ a_k + b_k e^{c_k \frac{u}{100}} & k = 5 \\ a_k + b_k e^{c_k \frac{x_k - 1}{100}} & k = 6 \end{cases} \quad (18)$$

⁶<http://faculty.insead.edu/delquie/ASSESS.htm>

⁷<http://www.labfit.net/>

The scaling constants, the constants a_k, b_k, c_k , $k = 1, \dots, 6$ are shown in Table 5, Appendix B.

To assess the utility function, a scaling constant (k_{ij}, k_i) is determined by the expert for each attribute and stakeholder, using ASSESS software, to establish the relevance of some with regard to other.

After determining the weights of the different attributes, we there by deduced the SAUF of the different attributes and consequently the MAUF of the different stakeholders. The expert's opinion and our utility functions were obtained by using the same procedure and data set as the ones used in Ben Abdallah et al. (2018). More details can be found in previous studies (see Ben Abdallah et al. (2020); Ben Abdallah et al. (2018) and Rebai et al. (2015)) The resulting MAUF of different stakeholders is of the following form:

$$f_i(x, u) = \frac{((K_i k_{2i-1} g_{2i-1}(x, u) + 1)(K_i k_{2i} g_{2i}(x, u) + 1) - 1)}{K_i}, \quad i = 1, 2, 3 \quad (19)$$

where $f_i(x, u)$ is the utility function for each stakeholder. k_i is the scaling constant for each attribute. K is the overall scaling constant. The values of the constants K_i, k_i , $i = 1, 2, 3$ are presented in Table 5, Appendix B.

Choice of the matrices A and B of the model (5) We construct the matrices A and B of (5) according to the the known interaction between the attributes. More precisely, the attributes of our model correspond to well-known economic quantities, e.g., ROA, bank deposit fees, non-performing loans, etc.

More precisely, we relied on assumptions from the literature review (such as, Bourke (1989); Nguyen et al. (2018); Murphy (1985) ; Arshadi and Edward (1987), etc. to determine the signs of entries of the matrices A and B . This was done in two steps:

Step 1. First, we constructed matrices \hat{A} and \hat{B} , entries of which live in $[-1, 1]$. Matrix \hat{A} is composed of the attributes that can be influenced by the other attributes. Matrix \hat{B} is composed of attributes that are not influenced by any other attribute. We consider these attributes as inputs of the model. A negative entry indicates a negative feedback, a positive one indicates a positive feedback. That is, if the entry (i, j) of $[\hat{A} \quad \hat{B}]$ is positive, then value of the i th attribute increases if value the j th one increases. If this entry is negative, then the value of the i attribute decreases, if the value of the j th attribute increases. Finally, the columns of \hat{B} correspond to the attributes which influence others, but are not influenced by other attributes, in the absence of any assumptions on the strategy employed by the actors. That is, the columns of \hat{B} correspond to actions the various actors could in principle undertake.

The signs of the entries of \hat{A} and \hat{B} can be derived from the qualitative knowledge on the interaction between the economic variables which are represented by attributes. However, we still need to quantify these influences. This was done by using our own belief on the relative strengths of these interactions. In order to do so, these entries were chosen to be are between -1 and 1 ⁸,

⁸ -1 if the attribute negatively affects the other attribute and 1 if it positively affects it

and the absolute value of a given entry reflects our belief in the strength of the relationship which corresponds to it. That is, if the absolute value of the entry is close to 1, then the relationship is strong, i.e., if the (i, j) th entry of $[\hat{A} \quad \hat{B}]$ has an absolute value close to 1, then the i th attribute depends strongly on the j th one). Conversely, if the absolute value is close 0, then we believe that the interaction is weak.

The entries of the matrices \hat{A} and \hat{B} in Table 2 in Appendix B, and the precise economic considerations for choosing the signs of the entries of \hat{A} , \hat{B} can be found in the discussion in Appendix B, before Table 2.

Step 2. Second, we applied a re-scaling to the entries of \hat{A} and \hat{B} . More precisely, from the discussion above it follows that the entries of \hat{A} and \hat{B} are numbers in the interval $[-1, 1]$. However, the values of the attributes belong to different intervals, hence the matrices \hat{A} and \hat{B} could be viewed as adequate models only for re-normalized attribute vectors, which take their values in the interval $[-1, 1]$. Alternatively, the matrices \hat{A} and \hat{B} have to be rescaled in order to describe the dynamics of the true attribute vectors, hence the rescaling and shift by b . We therefore consider: $A = 0.5T^{-1}\hat{A}T$, $T^{-1}\hat{B} = B$ for a suitable diagonal matrix T . The values of A, B, T as indicated in Table 3 in Appendix B.

Choice of the equilibrium point (x_0, u_0) and the vector h of (5) In order to find an equilibrium point (x_0, u_0) we solve the following nonlinear programming problem:

$$\begin{aligned} f_i(x_0, u_0) &= f_{i,t}, \quad i = 1, \dots, N \\ x_{i,min} &\leq x_{0,i} \leq x_{i,max}, \quad i = 1, \dots, n \\ u_{j,min} &\leq u_{0,j} \leq u_{j,max}, \quad j = 1, \dots, m. \end{aligned} \tag{20}$$

using `fmincon` function of Matlab for $f_{1,t} = 0.7$, $f_{2,t} = 0.7$, $f_{3,t} = 0.65$. Note that (20) differs from (6), as the first constraint of (6), namely $x_0 = Ax_0 + Bu_0 + h$ is absent from (20). This is due to the difficulty assigning a value to h in the absence of any data, as expert's opinion does not tell much about the vector h . In fact, in some sense, the vector h determines the equilibrium of the system: for any pair (x_0, u_0) , if we choose $h = x_0 - Ax_0 - Bu_0$, then (x_0, u_0) will be an equilibrium point for (5). Prompted by this observation, and by the lack of any other method to assign h in the absence of measurement data, we propose to find first a candidate equilibrium point (x_0, u_0) by solving (20), and then to choose $h = x_0 - Ax_0 - Bu_0$. With this choice of h , (x_0, u_0) will then be a true equilibrium point, in particular, it will be a solution of (6). The values of x_0, u_0 and h are indicated in Table 3 in Appendix B.

Interpretation of the model using utility functions As it was pointed out in Subsection 3.2, the dynamical system (1) can be interpreted as a result of rational behavior of agents trying to optimize their own utility functions. More precisely, if we assume that there are as many agents as state components, the i th agent can influence the i th state component and at every

time instance it does so by choosing $X_i(t+1)$ in the vicinity of $X_i(t)$ so that $\mathbf{u}_i(X_1(t), \dots, X_{i-1}(t), X_i(t+1), X_{i+1}(t), \dots, X_n(t), U(t))$ is maximal. Here \mathbf{u}_i is the utility function associated with the agent which manages the i th attribute. Then the right-hand side of (1) is of the form (13). If the dynamical system is of the form (5), then this corresponds to the choice of \mathbf{u}_i of the form (16). For the numerical example at hand the corresponding choices of \mathbf{u}_i , $i = 1, \dots, 5$ are presented in Table 4.

Calculating the strategy We calculate the strategy using the method described in Section 3.1 and in Appendix A. In order to apply the methods, the minimal thresholds for utility functions are chosen to be $f_{i,min} = 0.4$, $i = 1, 2, 3$.

The matrix K defining the strategy of the form (7) and the matrix Q defining the ellipsoid (8) can be found in Table 7 in Appendix B.2. The details of the application of the method can be found in Appendix B.2. As it was explained in Subsection 3.2, the strategy could be implemented using an appropriately formulated taxation policy.

4.2 Interpretation of the results

We simulated result of the system with the strategy (7), if the system is started from an initial state inside \mathcal{P} , and if the system is started from an initial state outside of \mathcal{P} . The values of these two initial states are presented in Table 8, Appendix B.3.

When the initial state is in \mathcal{P} the system we simulated was of the form

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t) + h + d(t) \\ u(t) &= -K(x - x_0) + u_0 \end{aligned}$$

The disturbance $d(t)$ represents the modeling error or external disturbances. While the calculation of the strategy was done for the case $d(t) = 0$, it is robust with respect to small enough disturbances, so it makes sense to study its behavior under perturbations. We performed simulation with $d(t) = 0$ and with the choice of $d(t)$ as in (32)-(33), see Appendix B.3 for a detailed explanation.

The results are shown in Figures 1, 2, 3 and 4. For Figures 1 and 3, the blue line represents the stakeholder's utility function and the red line represent the utility at equilibrium. For Figures 2 and 4, the blue line represents the state component and the red line represent the equilibrium value.

We start with the case where the system is started from an initial state inside \mathcal{P} , and it is run in the absence of perturbations (i.e. $d(t) = 0$). Figures 1 and 2 below show the variation of the different stakeholders and attributes respectively.

Figure 1: Utility functions when the initial state is in \mathcal{P} (see Table 8), no perturbations ($d(t) = 0$)

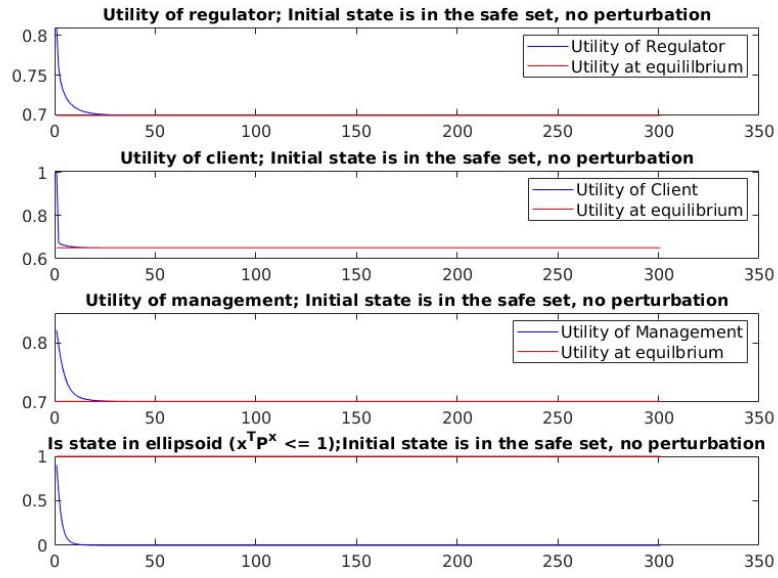
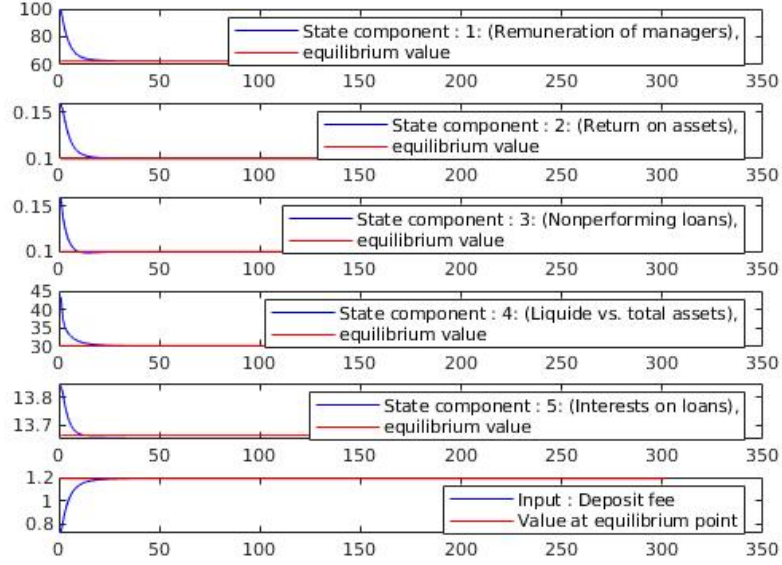


Figure 2: State components and the input when the initial state is in \mathcal{P} (see Table 8), no perturbations ($d(t) = 0$)



As shown in Figures 1 and 2, we started with a sustainable utilities and states respectively. We see that the utility functions converge to the utility at equilibrium (i.e., 0.7 and 0.65) and the state components converge to the equilibrium values. Obviously, this situation guarantees us that the strategy will still be sustainable.

The results of the simulation for $d(t) \neq 0$ can be seen on Figures 3 and 4. These figures show the utility functions and state components for the case when the system was started with the initial state from \mathcal{P} (see Table 8), in the presence of the perturbation $d(t)$ defined in (32)-(33).

Figure 3: Utility functions when the initial state is in \mathcal{P} (see Table 8) and perturbations are present ($d(t)$ as in (33))

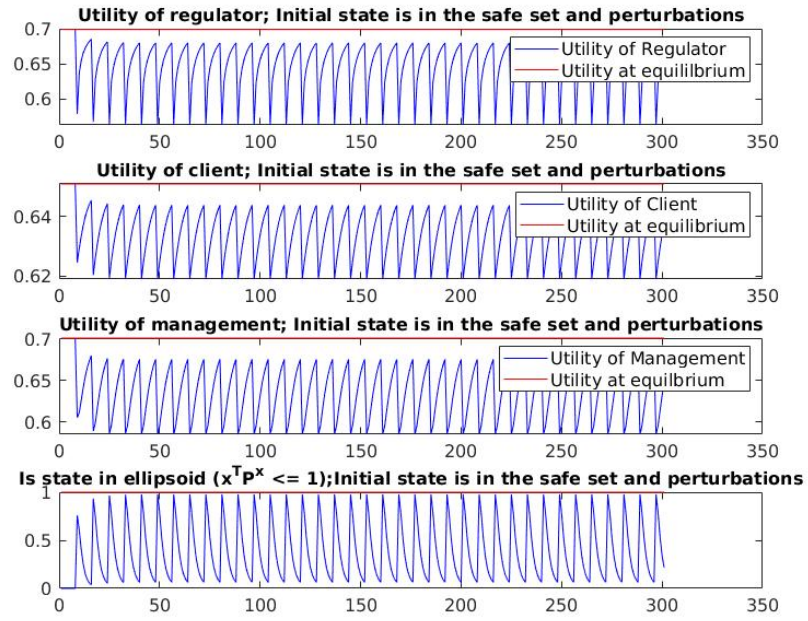
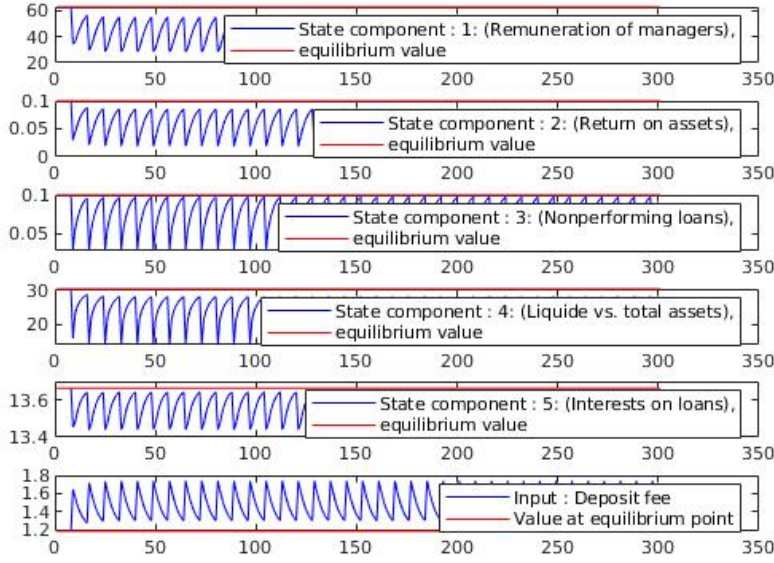


Figure 4: State components and the input when the initial state is in \mathcal{P} (see Table 8) and perturbations are present ($d(t)$ as in (33))



Looking at Figures 4 and 3, we see a variation of the utility functions and the state components. We see that the proposed strategy is sustainable, and it remains so even in the presence of disturbance as long as the disturbance is small enough to keep the state in the safe set. Note that the state components will no longer converge towards the equilibrium point.

Besides the simulations in the safe set (with and without disturbance), we also performed another simulation, but this time with an initial state that does belong to the safe set and without disturbance. In this case study, we obtained similar results as the previous ones, i.e., the proposed strategy reaches a sustainable level. But it is not guaranteed to remain in this sustainable behavior.

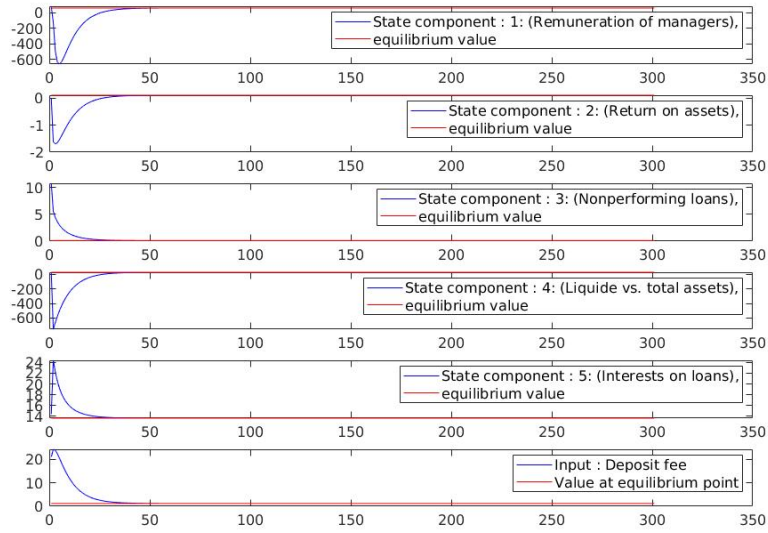
More precisely, when the initial state is not in the set \mathcal{P} , then even in the absence of disturbances, i.e., when $d(t) = 0$, the system

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t) + h + d(t) \\ u(t) &= -K(x - x_0) + u_0 \end{aligned}$$

did not respect the state constraints of \mathcal{X} , more precisely, the first state component $x_1(t)$ evolved beyond the interval $[x_{1,min}, x_{2,min}]$ before finally converging to the equilibrium point. This is not surprising, as the proposed method guarantees sustainability only if the initial state is in the safe set \mathcal{P} . In particular,

when the initial state is not in the safe set, the state trajectory is allowed to become unsustainable before getting close to the equilibrium and becoming sustainable. In particular, for this simulation the utility functions will not always take values in $[0, 1]$, as the states are not always in the admissible set \mathcal{X} . This is illustrated in Figure 5.

Figure 5: Utility functions when the initial state is not in \mathcal{P} (see Table 8), no perturbation ($d(t) = 0$), no enforcement of the state constraint



Even if we assume that the state constraints are enforced, the state is not guaranteed to be sustainable if the initial state is not in the safe set \mathcal{P} . To show this, we simulated the following system

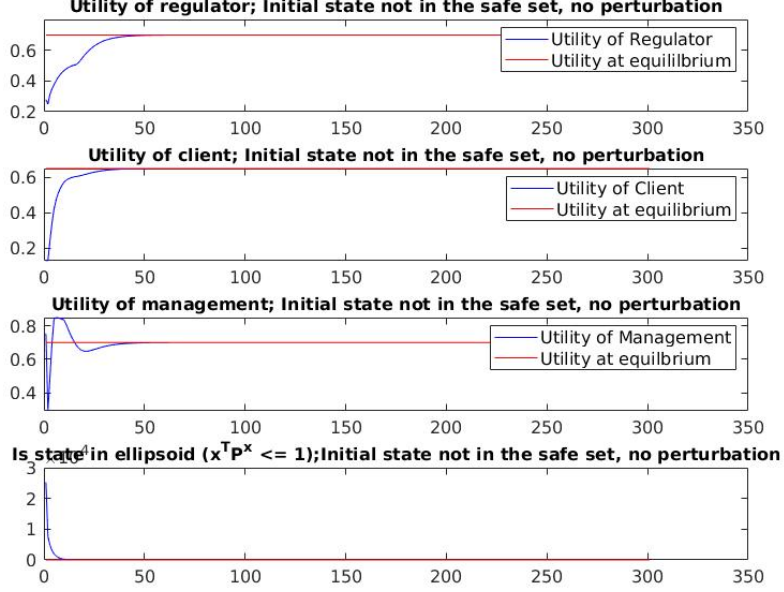
$$\begin{aligned} x(t+1) &= \sigma_x(Ax(t) + Bu(t) + h) + d(t) \\ u(t) &= \sigma_u(-K(x - x_0) + u_0) \end{aligned}$$

with

$$\begin{aligned}
\sigma_x((x_1, \dots, x_n)^T) &= (\sigma_{1,x}(x_1), \dots, \sigma_{n,x}(x_n))^T \\
\sigma_{i,x}(x_i) &= \begin{cases} x_i & x_i \in [x_{i,min}, x_{i,max}] \\ x_{min} & x_i < x_{i,min} \\ x_{max} & x_i > x_{i,max} \end{cases} \\
\sigma_u((u_1, \dots, u_m)^T) &= (\sigma_{1,u}(u_1), \dots, \sigma_{m,u}(u_m))^T \\
\sigma_{i,u}(u_i) &= \begin{cases} u_i & u_i \in [u_{i,min}, u_{i,max}] \\ u_{min} & u_i < u_{i,min} \\ u_{max} & u_i > u_{i,max} \end{cases} \quad rl
\end{aligned}$$

The saturation functions σ_x and σ_u were used in order to make sure that the states and inputs stay in the sets \mathcal{X} and \mathcal{U} respectively. The latter was necessary because for states and inputs outside these sets, the utility function are not valid (their values no longer belong to the interval $[0, 1]$). The initial state was chosen not from \mathcal{P} , as in Table 8, Appendix B.3, and we simulated for $d(t) = 0$. The result is shown in Fig 6 below. We can see that the utility functions will converge to the desired value in this case, however the utility function may descend below the minimal value of 0.4.

Figure 6: Utility functions when the initial state is not in \mathcal{P} (see Table 8), no perturbation ($d(t) = 0$), state and input constraints are enforced by saturation functions



Finally, we applied greedy input, i.e. we applied a strategy which is the best possible or the worst possible for one of the stakeholders. For our case study, we chose to increase the deposit fees for customers. That is, we chose the worst possible value for the customer. When applying this strategy to the original system $x(t+1) = Ax(t) + Bu(t)$, the state did not satisfy the constraint $x(t) \in \mathcal{X}$ and hence the values of the utility functions were no longer in the interval $[0, 1]$. We also tried to simulate this strategy for the system where the state constraints were enforced by a saturation function, just like in the case of initial states outside the safe set, that is, we simulated the system

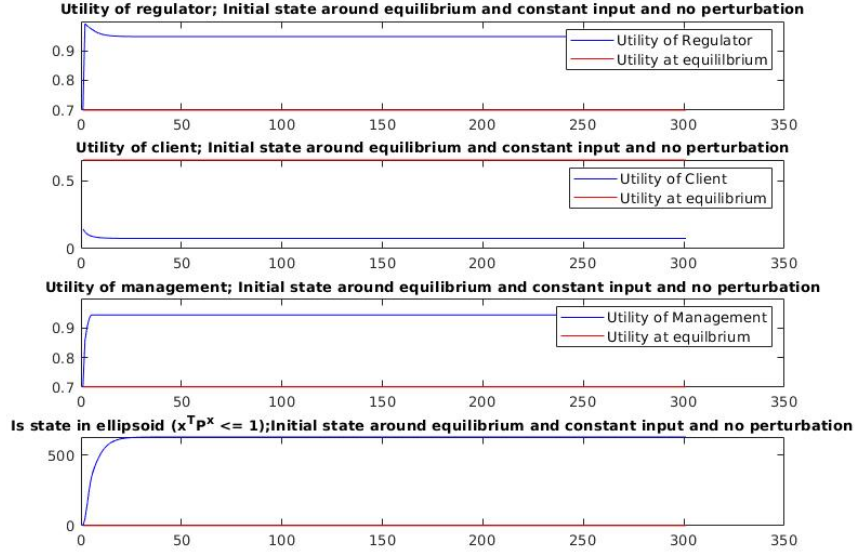
$$\begin{aligned} x(t+1) &= \sigma_x(Ax(t) + Bu(t) + h) \\ u(t) &= u_{max} \end{aligned}$$

with

$$\begin{aligned} \sigma_x((x_1, \dots, x_n)^T) &= (\sigma_{1,x}(x_1), \dots, \sigma_{n,x}(x_n))^T \\ \sigma_{i,x}(x_i) &= \begin{cases} x_i & x_i \in [x_{i,min}, x_{i,max}] \\ x_{min} & x_i < x_{i,min} \\ x_{max} & x_i > x_{i,max} \end{cases} \end{aligned}$$

The initial state was chosen to be the equilibrium point x_0 . The results of the latter simulation are presented in Figure 7. Looking at Figure 7, we notice that this action caused a remarkable decrease in the utility of the client (almost 0). (i.e that the client is not at all satisfied). In contrast, we note a remarkable increase of the utilities of the manager and regulator (almost 1) (i.e that the manager and regulator are totally satisfied). This action allows us to show the advantage of our model and the importance of the strategy found, i.e., of a strategy that ensures the satisfaction of all stakeholders.

Figure 7: Utility functions when the initial state is the equilibrium (see Table 8), there is constant greedy input and no perturbation ($d(t) = 0$), state constraints are enforced by saturation functions



5 Conclusion

The objective of this paper was to propose a theoretical framework that allows finding an acceptable strategy for all stakeholders by applying control theory. To illustrate our approach, we used an academic example. Note that our method requires a mathematical model in a state-space form that describes the interaction of various stakeholders. For the example at hand, we used a model which is not estimated from econometric data. The reason for this was the lack of off-the-shelf models suitable for our purposes.

Despite these shortcomings, we believe that the example of this paper demonstrated the feasibility of our method.

Our approach helps us to correct some dangerous bad management practices, or deadly sins, that explain the current instability of companies. Our framework contributes to transforming the conflictual system, formed of a group of individuals with disparate objectives, to a group of individuals acting rationally in the name of a common objective. One of the main reasons for the unsustainability of companies is the excessive search for profit. Our approach consists in finding a strategy that allows everyone to reach an acceptable situation without looking for the ‘best’ situation. Furthermore, in general, stakeholders seek to maximize their profit against the other party. Our approach is to increase the claims of each stakeholder without exceeding a certain threshold. Finally, we propose a long-term vision instead of a short-term vision. Indeed, our strategy ensures that the system will reach a certain sustainable state and remain in this sustainable state.

Practical implications The work that we are doing is the evaluation of strategies that allow the connection between stakeholders in the most beneficial way for all. This evaluation work can be performed by extra-financial rating agencies. Indeed, they can propose for companies a set of solutions (strategies) to ensure that all stakeholders are satisfied. This can improve their scoring process by giving additional notes, for example, on the application of good strategies. These scores could be viewed as a form (reputational) tax. Alternatively, tax policy could be adjusted to push stakeholders to apply good strategies. The construction of these strategies can then constitute the opportunity to stimulate a new form of negotiation that allows moving to a consensual and cooperative model, open to a truly sustainable economic environment.

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A Calculating the strategy and a safe invariant set using linear matrix inequalities

In order to calculate the matrix K and the set \mathcal{P} described in Section 3.1, we use so called linear matrix inequalities (LMIs) Boyd et al. (1994). More precisely, we assume that the utility functions are piecewise quadratic and are of the form

$$f_i(X, U) = \begin{bmatrix} X - x_0 \\ U - u_0 \end{bmatrix}_i^T Q_{i,k} \begin{bmatrix} X - x_0 \\ U - u_0 \end{bmatrix} + H_{i,k} \begin{bmatrix} X - x_0 \\ U - u_0 \end{bmatrix} + g_{i,k}, \text{ if } (X - x_0, U - u_0) \in \mathcal{P}_{i,k} \quad (21)$$

for suitably sized matrices $Q_{i,k}, H_{i,k}$ and scalar $g_{i,k}$, $i = 1, \dots, N$, $k = 1, \dots, D_i$, where the sets $\mathcal{P}_{i,k}$ are polyhedral sets of the form

$$\mathcal{P}_{i,k} = \{(x, u) \in \mathbb{R}^n \times \mathbb{R}^m \mid C_{i,k} \begin{bmatrix} x \\ u \end{bmatrix} + c_{i,k} \leq 0\} \quad (22)$$

for suitable matrices $C_{i,k} \in \mathbb{R}^{r \times (n+m)}$ and vectors $c_{i,k} \in \mathbb{R}^r$, such that the union of all the sets $\mathcal{P}_{i,k} \cap \mathcal{X}$ covers the whole space \mathcal{X} .

Remark 1. If the utility functions are not of the form (21), then they can be approximated with arbitrary accuracy by piecewise-quadratic functions of the form (21). This follows from the universal approximation property of piecewise-quadratic functions, which is a consequence of the universal approximation property of piecewise-constant functions (which is a subclass of piecewise quadratic functions).

We find matrices $Q \in \mathbb{R}^{n \times n}$, and $Y \in \mathbb{R}^{n \times m}$, $Q > 0$ ⁹, by solving the following system of linear matrix inequalities (LMI)

$$\begin{bmatrix} -Q & QA^T - Y^T B \\ AQ - BY & -Q \end{bmatrix} < 0, Q > W \quad (23)$$

$$\begin{bmatrix} Q & Qe_{i,n+m} \\ e_{i,n+m}^T Q & \mu_i^2 \end{bmatrix} > 0, \quad i = 1, \dots, n \quad (24)$$

$$\begin{bmatrix} Q & e_j Y \\ (e_j Y)^T & \mu_{n+j}^2 \end{bmatrix} > 0, \quad j = 1, \dots, m \quad (25)$$

where $W \in \mathbb{R}^{n \times n}$ and $\mu_i, i = 1, \dots, n + m$ are design parameters chosen by the user, and $x_{0,i}, u_{0,j}, i = 1, \dots, n, j = 1, \dots, m$ denote the i th and j th component of x_0 and u_0 respectively. The parameters $\mu_i, i = 1, \dots, n + m$ are chosen so that if $(x_i - x_{0,i})^2 \leq \mu_i^2$, then $x_i \in [x_{min,i}, x_{max,i}]$ for all $i = 1, \dots, n$, and if $(u_j - u_{0,j})^2 \leq \mu_{n+j}^2$, then $u_j \in [u_{min,j}, u_{max,j}]$, $j = 1, \dots, m$.

This can be achieved by choosing $\mu_i \leq \min\{x_{max,i} - x_{0,i}, x_{0,i} - x_{min,i}\}$, $i = 1, \dots, n$, and $\mu_{n+j} \leq \min\{u_{max,j} - u_{0,j}, u_{0,j} - u_{min,j}\}$, $j = 1, \dots, m$.

The matrix W should be chosen as a symmetric positive semi-definite matrix and it can be used to control the size of the ellipsoid \mathcal{P} .

⁹ $Z > 0$, $Z < 0$ means that the matrix Z is positive (negative) definite.

We then set $K = YQ^{-1}$ and

$$\mathcal{P} = \{x \in \mathbb{R}^n \mid (x - x_0)^T Q^{-1} (x - x_0) < 1\}. \quad (26)$$

Finally, we verify that the following LMI with the indeterminate $\tau > 0$, $\tau_{i,k,l} > 0$, $i = 1, \dots, N$, $k = 1, \dots, D_i$, $l = 1, \dots, r$ has a solution

$$\begin{aligned} & \begin{bmatrix} \mathcal{S}^T Q_{i,k} \mathcal{S} & 0.5(H_{i,k} \mathcal{S})^T \\ 0.5 H_{i,k} \mathcal{S} & g_{i,k} - f_{i,min} \end{bmatrix} + \tau \begin{bmatrix} Q^{-1} & 0 \\ 0 & -1 \end{bmatrix} + \\ & + \sum_{l=1}^r \tau_{i,k,l} \begin{bmatrix} 0 & (e_{l,r}^T C_{i,k} \mathcal{S})^T 0.5 \\ (e_{l,r}^T C_{i,k} \mathcal{S}) 0.5 & e_l^T c_{i,k} \end{bmatrix} > 0, \end{aligned} \quad (27)$$

where $\mathcal{S} = \begin{bmatrix} I_n \\ -K \end{bmatrix}$, and $e_{i,d}$ is the i th standard unit vector of \mathbb{R}^d , i.e., all the elements of e_i are zeros, except the i th one, which is 1.

The intuition behind the equations (23) – (27) is the following.

1. LMI (23) ensures that the feedback $U(t) = -K(X(t) - x_0) + u_0$ will stabilize the system $X(t+1) = AX(t) + BU(t) + h$, $\lim_{t \rightarrow \infty} X(t) = x_0$.
2. LMI (24) ensures that if $x \in \mathcal{P}$, then the i th component x_i of x satisfies $x_i \in [x_{i,min}, x_{i,max}]$. Likewise, (25) ensures that if $x \in \mathcal{P}$ and $u = -K(x - x_0) + u_0$ then the j th component u_j of u satisfies $u_j \in [u_{j,min}, u_{j,max}]$.
3. LMI (27) ensures that if $x \in \mathcal{P}$ and $u = -K(x - x_0) + u_0$ then $f_i(x, u) \geq f_{i,min}$.

To sum up, the matrix Q is calculated so that all the elements of the ellipsoid \mathcal{P} are sustainable and satisfy the constraints on the attributes and actions.

From classical results of control theory it then follows that the matrix K and the set \mathcal{P} satisfies the conditions described in the previous section. The solution of (23)-(27) is calculated using classical numerical tools YALMIP and its interface with Matlab.

Remark 2 (Robustness of the control strategy). Below we discuss the robustness of control strategy (7) computed above. More precisely, assume that

$$\tilde{X}(t+1) = A\tilde{X}(t) + BU(t) + h + d(t).$$

It can be shown using standard techniques Boyd et al. (1994) that for all $t \geq 0$,

$$(\tilde{X}(t+1) - x_0)^T Q^{-1} (\tilde{X}(t+1) - x_0) \leq (1 - \mu)(\tilde{X}(t) - x_0)^T Q^{-1} (\tilde{X}(t) - x_0) + \gamma^2 \|d(t)\|_2^2.$$

for suitable $\mu, \gamma > 0$.

This remark can be used for analyzing robustness for two scenarios: either when $d(t)$ is a perturbation or modelling error, or when $d(t)$ is the modelling

error due to linearization. In the former case the magnitude of $d(t)$ does not necessarily depend on $X(t)$, while in the latter case the magnitude of $d(t)$ depends on how close $X(t)$ is to the equilibrium point.

Perturbation which is potentially independent of the state Let us consider the scenario where we do not assume that $d(t)$ decreases if $X(t)$ gets closer to the equilibrium point, i.e., $d(t)$ is some generic external disturbance or modelling error. Then for all $t \geq 0$,

$$\begin{aligned} & (\tilde{X}(t+1) - x_0)^T Q^{-1} (\tilde{X}(t+1) - x_0) \leq \\ & (1 - \mu)(\tilde{X}(t) - x_0)^T Q^{-1} (\tilde{X}(t) - x_0) + \gamma^2 \|d(t)\|_2^2 \leq \\ & (\tilde{X}(t) - x_0)^T Q^{-1} (\tilde{X}(t) - x_0) + \gamma^2 \|d(t)\|_2^2 \end{aligned}$$

and by applying the inequality above repeatedly, it follows using standard arguments Boyd et al. (1994) that

$$\begin{aligned} & (\tilde{X}(t) - x_0)^T Q^{-1} (\tilde{X}(t) - x_0) \leq \\ & (\tilde{X}(0) - x_0)^T Q^{-1} (\tilde{X}(0) - x_0) + \gamma^2 \sum_{s=0}^{t-1} \|d(s)\|_2^2 \end{aligned}$$

This then implies that if the energy of the disturbance d is sufficiently small, i.e.,

$$\sum_{s=0}^{t-1} \|d(s)\|_2^2 \leq C,$$

where

$$C \leq \frac{1 - (\tilde{X}(0) - x_0)^T Q^{-1} (\tilde{X}(0) - x_0)}{\gamma^2},$$

then the perturbed state $\tilde{X}(t)$ will remain in \mathcal{P} , as in this case

$$\begin{aligned} & (\tilde{X}(t) - x_0)^T Q^{-1} (\tilde{X}(t) - x_0) \leq \\ & (\tilde{X}(0) - x_0)^T Q^{-1} (\tilde{X}(0) - x_0) + \gamma^2 \sum_{s=0}^{t-1} \|d(s)\|_2^2 \leq 1 \end{aligned}$$

The magnitude of the constant C depends on the constant γ^2 and the state $\tilde{X}(0)$ at time 0: the closer the initial state $\tilde{X}(0)$ is to the equilibrium point, the larger perturbation the system can tolerate without leaving the safe set \mathcal{P} . Note that $\sum_{s=0}^{t-1} \|d(s)\|_2^2 \leq C$ achieved when the magnitude of d is small, and most of the time d is zero, i.e., there are some shocks time-to-time, and they do not take place all too often.

That is, for small enough disturbances, if the state of the system is in \mathcal{P} , it will always remain there. Since the elements x of \mathcal{P} are sustainable (more precisely, $(x, u = -K(x - x_0) + u_0)$ is sustainable) it shows that the proposed strategy is robustly sustainable.

Perturbation is due to linearization error Let us now consider the case when the true F is a non-linear function and the matrices A , B and the vector

h have been obtained by linearizing F around an equilibrium point $(\tilde{x}_0, \tilde{u}_0)$ such that $\tilde{x}_0 = F(\tilde{x}_0, \tilde{u}_0)$. In this case, following the standard linearization procedure Khalil (2002); Sontag (1998); Franklin et al. (2001), $A = \frac{dF}{dx}(x, u)|_{x=\tilde{x}_0, u=\tilde{u}_0}$, $B = \frac{dF}{du}(x, u)|_{x=\tilde{x}_0, u=\tilde{u}_0}$, $h = \tilde{x}_0 - A\tilde{x}_0 - B\tilde{u}_0$. In particular, if $(\tilde{x}_0, \tilde{u}_0)$ is an equilibrium point of (5). For the sake of simplicity, let us assume that the equilibrium point (x_0, u_0) which is chosen by solving (6) coincides with $(\tilde{x}_0, \tilde{u}_0)$. Note that if the true non-linear F is known, then the proposed procedure can be modified in such a manner that (6) is solved with $x_0 = Ax_0 + Bu_0 + h$ being replaced by $x_0 = F(x_0, u_0)$, in which case $(\tilde{x}_0, \tilde{u}_0) = (x_0, u_0)$ will automatically hold.

It then follows that

$$F(x, u) = Ax + Bu + h + d(x, u)$$

such that if

$$\lim_{(x,u) \rightarrow (x_0, u_0)} \frac{\|d(x)\|_2^2}{\|x - x_0\|_2^2 + \|u - u_0\|_2^2} = 0.$$

Assume now that

$$\tilde{X}(t+1) = F(\tilde{X}(t), U(t))$$

and (7) is applied to $\tilde{X}(t)$ instead of $X(t)$. It then follows that

$$\tilde{X}(t+1) = A\tilde{X}(t) + BU(t) + d(t)$$

where $d(t) = d(\tilde{X}(t), U(t)) = d(X(t), -K(\tilde{X}(t) - x_0) + u_0)$. It then follows Khalil (2002); Sontag (1998) that for any $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that $\|d(t)\|_2^2 \leq \epsilon \|\tilde{X}(t) - x_0\|_2^2 (1 + \text{trace} K K^T) \leq \epsilon (\tilde{X}(t) - x_0)^T Q^{-1} (\tilde{X}(t) - x_0) m$, if $\|(\tilde{X}(t) - x_0)\|_2^2 \leq \delta(\epsilon)$, where m is a suitable constant which depends only on Q and K . From

$$(\tilde{X}(t+1) - x_0)^T Q^{-1} (\tilde{X}(t+1) - x_0) \leq (1 - \mu) (\tilde{X}(t) - x_0)^T Q^{-1} (\tilde{X}(t) - x_0) + \gamma^2 \|d(t)\|_2^2$$

it then follows that

$$(\tilde{X}(t+1) - x_0)^T Q^{-1} (\tilde{X}(t+1) - x_0) \leq (1 - \mu + \epsilon m \gamma^2) (\tilde{X}(t) - x_0)^T Q^{-1} (\tilde{X}(t) - x_0),$$

if $\|\tilde{X}(t) - x_0\|_2^2 \leq \delta(\epsilon)$. There exists $1 \geq \delta'(\epsilon) > 0$ such that if $(\tilde{X}(t) - x_0)^T Q^{-1} (\tilde{X}(t) - x_0) \leq \delta'(\epsilon)$, then $\|\tilde{X}(t) - x_0\|_2^2 \leq \delta(\epsilon)$. If ϵ is chosen so that $\mu \geq \epsilon m \gamma^2$, then for $(\tilde{X}(t) - x_0)^T Q^{-1} (\tilde{X}(t) - x_0) \leq \delta'(\epsilon)$, it holds that

$$(\tilde{X}(t+1) - x_0)^T Q^{-1} (\tilde{X}(t+1) - x_0) < (\tilde{X}(t) - x_0)^T Q^{-1} (\tilde{X}(t) - x_0).$$

That is, $V(\tilde{X}(t)) = (\tilde{X}(t) - x_0)^T Q^{-1} (\tilde{X}(t) - x_0)$ is a Lyapunov function for the system $\tilde{X}(t+1) = F(\tilde{X}(t), U(t))$, where $U(t)$ as in (7) with $X(t)$ being replaced by $\tilde{X}(t)$, if the latter system is restricted to the ball $B(x_0) = \{x \in \mathbb{R}^n \mid (x - x_0)^T Q^{-1} (x - x_0) < \delta'(\epsilon)\}$. That is, if $\tilde{X}(0) \in B(x_0)$, i.e., $\tilde{X}(0)$ is close enough to x_0 , then $\lim_{t \rightarrow \infty} \tilde{X}(t) = x_0$, i.e., the true non-linear system is stable under the control law of (7). Moreover, in this case $\tilde{X}(t)$ remains in the set

$B(x_0)$ for all $t \geq 0$. Note that $B(x_0)$ is a subset of the safe set \mathcal{P} . Hence, if $\tilde{X}(t)$ belongs to $B(x_0)$, then $\tilde{X}(t)$ and $U(t)$ are both admissible and sustainable. In other words, (7) will achieve the control objectives even if the assumption that F is affine is not true, as long as A, B, h are chosen to be the linearization of F around the chosen equilibrium point and the initial state is close enough to x_0 .

B Numerical example

B.1 Tables with the parameters of the example

Table 1: Selected Attributes

Stackholders	Attributes	Range
Managers:	Attribute x_1 : Annual remuneration evolution (x_1)	[-0.67 , 103]
	Attribute x_2 : Return on assets (ROA) (x_2)	[-2 , 1]
Regulator:	Attribute x_3 : Non-performing loans to total loans (x_3)	[0, 15]
	Attribute x_4 : Liquid assets to total assets (x_4)	[9.86 , 63.91]
Customers	Attribute x_5 : Interest receivable to loans (x_5)	[1.52,31.05]
	Attribute u : Bank Fees to deposit (u_1)	[0.51, 14]

As explained above, the values of \hat{A} and \hat{B} were filled according to the literature review. For example, the majority of previous empirical studies have shown that managerial remuneration has a positive impact on ROA (Murphy, 1985) as well as on liquidity (Nguyen, 2018). However, assumptions from economics indicate that Non-performing loans negatively affect the remuneration of managers. All other things being equal. Hence our assumptions for the attribute x_1 "Annual remuneration evolution" (first row):

Assumption 1: ROA affects positively the remuneration of managers (first row, second column).

Assumption 2: Non-performing loans affects negatively the remuneration of managers (first row, third column).

Assumption 3: Liquid assets to total assets affects positively the remuneration of managers (first row, fourth column).

Assumption 4: Interest receivable to loans affects positively the remuneration of managers (first row, fifth column).

The same procedure was applied for the other attributes.

Table 2: Matrices \hat{A} and \hat{B} originating from expert's opinion

\hat{A}					\hat{B}	
1	0.8	-0.2	0.5	0.2	0.3	
0.2	1	-0.4	0.6	0.5	0.2	
0	-0.4	1	0	0.7	0	
0.2	0.5	-0.8	1	0.4	0.2	
0	0.3	0.5	0.5	1	0.4	

Table 3: State-space transformation T and the model parameters $A = 0.5T^{-1}\hat{A}T$, $T^{-1}\hat{B} = B$, h , equilibrium x_0, u_0

T		0.0248	0	0	0	0	0
		0	10	0	0	0	0
		0	0	10	0	0	0
		0	0	0	0.049	0	0
		0	0	0	0	3.3333	
		0	0	0	0	0	0.9487
A		0.5	161.4498	-40.3624	0.4949	13.4541	
		0.0002	0.5	-0.2	0.0015	0.0833	
		0	-0.2	0.5	0	0.1167	
		0.0505	50.9759	-81.5615	0.5	13.5936	
		0	0.45	0.75	0.0037	0.5	
B					12.1087		
					0.02		
					0		
					4.0781		
					0.12		
x_0	$[62.64 \quad 0.1 \quad 0.1 \quad 30.25 \quad 13.67]^T$						
u_0	1.1902						
h	$[-194.0018 \quad -1.1525 \quad -1.5241 \quad -175.5697 \quad 6.4577]^T$						

Table 4: Utility functions for agents managing each attribute

$\mathbf{u}_1(x, u)$	$-0.5 \cdot 0.5x_1^2 + 161.4498x_2x_1 - 40.3624x_3x_1 + 0.4949x_4x_1 + 13.4541x_5x_1 + 12.1087u_1x_1 - 175.43x_1$
$\mathbf{u}_2(x, u)$	$-0.5 \cdot 0.5x_2^2 + 0.0002x_1x_2 - 0.2x_3x_2 + 0.0015x_4x_2 + 0.0833x_5x_2 + 0.0200u_2x_2 - 1.07x_2$
$\mathbf{u}_3(x, u)$	$-0.5 \cdot 0.5x_3^2 - 0.2x_2x_3 + 0.1167x_5x_3 - 1.42x_3$
$\mathbf{u}_4(x, u)$	$-0.5 \cdot 0.5x_4^2 + 0.0505x_1x_4 + 50.9759x_2x_4 - 81.5615x_3x_4 + 13.5936x_5x_4 + 4.0781u_4x_4 - 161.54x_4$
$\mathbf{u}_5(x, u)$	$-0.5 \cdot 0.5x_5^2 + 0.45x_2x_5 + 0.75x_3x_5 + 0.0037x_4x_5 + 0.12u_5x_5 + 6.1x_5$

B.2 Calculating a strategy

We would like to apply the method of Appendix A to the numerical example. However, to this end, we have to solve a small technical issue. Namely, not all

Table 5: Scaling constant and constants

Stakeholders	Attributes	K_i	k_{ij}	a_k	b_k	c_k
Managers	Attribute x_1 : Annual remuneration evolution	-0.35854	0.59	0.3535	0.7625	0.7625
	Attribute x_2 : Return on assets (ROA)		0.52	0.597	5.686	5.686
Regulator	Attribute x_3 : Non-performing loans to total loans	-0.409837	0.52	-0.2919	1.276	-9.923
	Attribute x_4 : Liquid assets to total assets		0.61	0.1441	1.342	1.342
Customers	Attribute x_5 : Interest receivable to loans	-0.333333	0.5	-0.08086	1.172	-8.528
	Input u : Bank Fees to deposit		0.6	-0.428	1.461	-8.653

Table 6: Constants

Stakeholders	Attributes	k	$x_{k,min}$	$s_{k,2}$	$s_{k,3}$	$s_{k,4}$	$x_{k,max}$
Managers	Attribute x_1 : Annual remuneration evolution	1	-67 %	10%	48%	79%	103%
	Attribute x_2 : Return on assets (ROA)	2	-2 %	-0.24%	0.16%	0.64%	1%
Regulator	Attribute x_3 : Non-performing loans to total loans	3	0%	8%	5.5%	1.7%	15%
Customers	Attribute x_4 : Liquid assets to total assets	4	9.86%	20.5%	30.5%	50%	63.91%
	Attribute x_5 : Interest receivable to loans	5	1.52%	14.5%	9%	3%	31.05%
Customers	Inputs	k	$u_{1,min}$	$s_{k,2}$	$s_{k,3}$	$s_{k,4}$	$u_{1,max}$
	Input action u : Bank Fees to deposit	6	0.51%	9%	5.5%	2%	14%

the utility functions are of the form (21). More precisely, for $i = 1$, the utility function $f_1(x, u)$ is of the form (21), with $D_1 = 1$, $\mathcal{P}_{1,1} = \mathbb{R}^{n+m}$ ($C_{1,1} = 0, c_{1,1} = 0$), and with the following choice of $Q_{1,1}, H_{1,1}, g_{1,1}$:

$$Q_{1,1} = 10^{-03} \begin{bmatrix} 0 & -0.1034 & 0 & 0 & 0 & 0 \\ -0.1034 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$H_{1,1} = [0.0030 \quad 0.1555 \quad 0 \quad 0 \quad 0 \quad 0]$$

$$g_{1,1} = 0.4958$$

However, for $i = 1, 2$, the utility functions f_i are not of the form (21). In order to be able to apply Appendix A, for $i = 2, 3$, the utility functions f_i will be approximated by functions $f_{i,pl}$, $i = 2, 3$ of the form (21). This will be done as follows. For $i = 2, 3$, the utility functions f_i can easily be approximated by functions of the form (21) as follows.

For $k = 1, 2, 3, 4, 5$, let $x_{k,max} = s_{k,1} > s_{k,2} > s_{k,3} > s_{k,4} > s_{k,5} = x_{k,min}$ and for $k = 6$ let $u_{1,max} = s_{k,1} > s_{k,2} > s_{k,3} > s_{k,4} > s_{k,5} = u_{1,min}$ be such that $g_k(s_{k,j}) = 0.25j$, $j = 1, 2, 3, 4, 5$. The values of $s_{k,j}$, $j = 1, 2, 3, 4, 5, 6$, $k = 1, 2, 3, 4, 5, 6$ are presented in Table 6. For each $k = 3, 4, 5, 6$, the utility functions g_k are approximated by piecewise-linear functions $g_{k,pl}$

$$g_{k,pl}(x, u) = n_{k,j}^T \begin{bmatrix} x \\ u \end{bmatrix} + b_{k,j} \quad \text{if } R_{k,j} \begin{bmatrix} x \\ u \end{bmatrix} + r_{k,j} \leq 0, \quad j = 1, \dots, 4$$

$$n_{k,j} = \frac{0.25}{s_{k,j+1} - s_{k,j}} E_k$$

$$b_{k,j} = \frac{0.25 s_{k,j}}{s_{k,j+1} - s_{k,j}} \quad (28)$$

$$E_4 = e_{4,n+1}^T, E_3 = e_{3,n+1}^T, E_5 = e_{6,n+1}^T, E_6 = e_{5,n+1}^T$$

$$R_{k,j} = \begin{bmatrix} E_k \\ -E_k \end{bmatrix}, r_{k,j} = \begin{bmatrix} s_{k,j} \\ s_{k,j+1} \end{bmatrix}$$

The vectors $n_{k,j}, b_{k,j}, R_{k,j}, r_{k,j}$ can readily be computed using the values $s_{k,j}, s_{k,j+1}$ in Table 6, $k = 3, 4, 5, 6$, $j = 1, \dots, 4$. We then approximate the functions f_2, f_3 by the following functions

$$f_{i,pl}(x, u) = \frac{((K_i k_{2i-1} g_{2i-1,pl}(x, u) + 1)(K_i k_{2i} g_{2i,pl}(x, u) + 1) - 1)}{K_i}, \quad i = 2, 3 \quad (29)$$

where K_i, k_i , $i = 2, 3$ are the same as in (19). It then follows that $f_{2,pl}$ satisfies

(21) with $D_i = 4$ and for all $j = 1, \dots, 4$,

$$\begin{aligned}
Q_{2,j} &= \frac{N_4 N_{3,j}^T}{K_2} \\
H_{2,j} &= \frac{d_{3,j} N_4 + d_4 N_{3,j}}{K_2} + 2 \begin{bmatrix} x_0 \\ u_0 \end{bmatrix}^T Q_{2,j} \\
g_{2,j} &= \frac{d_{3,j} d_4 - 1}{K_2} - \begin{bmatrix} x_0 \\ u_0 \end{bmatrix}^T Q_{2,j} \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} + H_{2,j} \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} \\
N_{3,j} &= K_2 k_3 n_{3,j}^T, \quad d_{3,j} = K_2 k_3 b_{3,j} + 1 \\
N_4 &= b_4 c_4 K_2 k_4, \quad d_4 = a_4 K_2 k_4 + 1 \\
C_{2,j} &= R_{3,j}, \quad c_{2,j} = r_{3,j} + R_{3,j} \begin{bmatrix} x_0 \\ u_0 \end{bmatrix}.
\end{aligned} \tag{30}$$

Similarly $f_{3,pl}$ satisfies (21) with $D_i = 16$ and for all $j = 1, \dots, 16$, $j = 4(j_1 - 1) + j_2$, $j_1, j_2 = 1, \dots, 4$,

$$\begin{aligned}
Q_{3,j} &= \frac{N_{5,j_1} N_{6,j_2}^T}{K_3} \\
H_{2,j} &= \frac{d_{5,j_1} N_{6,j_1} + d_{6,j_2} N_{5,j_1}}{K_3} + 2 \begin{bmatrix} x_0 \\ u_0 \end{bmatrix}^T Q_{3,j} \\
g_{3,j} &= \frac{d_{5,j_1} d_{6,j_2} - 1}{K_3} - \begin{bmatrix} x_0 \\ u_0 \end{bmatrix}^T Q_{3,j} \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} + H_{3,j} \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} \\
N_{5,j_1} &= K_3 k_5 n_{5,j_1}^T, \quad d_{5,j_1} = K_3 k_5 b_{5,j_1} + 1 \\
N_{6,j_2} &= K_3 k_6 n_{6,j_2}^T, \quad d_{6,j_2} = K_3 k_6 b_{6,j_2} + 1 \\
C_{3,j} &= \begin{bmatrix} R_{5,j_1} \\ R_{6,j_1} \end{bmatrix} \quad c_{3,j} = \begin{bmatrix} r_{5,j_1} \\ r_{6,j_2} \end{bmatrix} + C_{3,j} \begin{bmatrix} x_0 \\ u_0 \end{bmatrix}.
\end{aligned} \tag{31}$$

The numerical values of $Q_{i,j}, H_{i,j}, g_{i,j}, C_{i,j}, c_{i,j}$, $i = 2, 3$, $j = 1, \dots, D_i$ can readily be computed from the values $K_2, K_3, k_3, k_4, k_5, k_6$, $n_{k,j}, b_{k,j}$, $k = 3, 4, 5, 6$, $j = 1, \dots, 4$. and (30)–(31).

We then apply Appendix A to the functions $f_1, f_{2,pl}, f_{3,pl}$ as utility functions, i.e., the utility function of the first stakeholder will be f_1 , and the utility function of the stakeholder i will be $f_{i,pl}$ for $i = 1, 2$. The resulting matrices K and Q are presented in Table 7.

Table 7: Strategy (8) and safe set (8) for the numerical example

K		0.0054	3.5116	−1.8580	0.0104	0.1990	
Q^{-1}		0.0013	−0.1195	−0.1102	−0.0006	−0.0402	
		−0.1195	224.1473	−51.7612	−0.2257	−15.9296	
		−0.1102	−51.7612	222.6623	−0.2288	−16.6081	
		−0.0006	−0.2257	−0.2288	0.0052	−0.0683	
		−0.0402	−15.9296	−16.6081	−0.0683	24.3354	

B.3 Initial states and disturbances used for simulation

In the dynamical model was simulated using different strategies, initial states, and was subjected to disturbances. Three different initial states were used: inside the safe set, outside the safe set, and around the equilibrium point. The numerical values of these initial states are described in Table 8.

Table 8: Initial states used for simulations

Initial state $X(0)$ in \mathcal{P}	$[99.23 \quad 0.16 \quad 0.16 \quad 43.38 \quad 13.84]^T$
Initial state $X(0)$ is not in \mathcal{P}	$[85 \quad 0.00 \quad 10.7828 \quad 27 \quad 14.41]^T$
Initial state $X(0)$ equals the equilibrium point x_0	$[62.64 \quad 0.1 \quad 0.1 \quad 30.25 \quad 13.6635]^T$

For the simulations, when the disturbance $d(t)$ was not zero, it was chosen as follows:

$$d(t) = \begin{cases} -(Ax(t) + Bu(t) - \lambda d) & t = kN \\ 0 & \text{otherwise} \end{cases} \quad (32)$$

i.e., $d(t)$ models a periodic change in the state, which occurs with a period N . For the simulation we have chosen $\lambda = 0.7$, $N = 8$ and

$$d = [40.3624 \quad 0.1000 \quad 0.1000 \quad 20.3904 \quad 0.3000 \quad 0.6802]^T \quad (33)$$

Response letter

We would like to thank the Editor-in-Chief Pr. Hans Amman and the anonymous Reviewers for their contributions to increase the quality of our paper. We very much appreciate the encouraging, critical and constructive remarks, and valuable suggestions on this manuscript. In this revised version of the manuscript, we did our best to address all suggestions and advices raised by the reviewers.

The outline of this report is as follows. In Section 1 we present our response to Reviewer 4, in Section 2 we present our response to Reviewer 2.

In the revised version, for the convenience of the reviewers, all the modifications are displayed in red.

1 Reviewer #4 :

1. *I am happy to read this interesting manuscript but I do not think it is suitable to be published in Computational Economics.*

Response : First of all, we would like to express our heartfelt appreciation for your critical and constructive comments on our manuscript. They are extremely valuable and beneficial in helping us revise the paper.

2. *The cited literature is insufficient to support the author's innovations and discoveries. Sustainability has been generally agreed on a combination of ESG framework and many works have address such a issue.*

Response :

In this paper we define sustainability as stakeholder's satisfaction, and we view various ESG criteria as variables which have impact on the satisfaction of various stakeholders. This is consistent with the various other definitions used ESG literature, since meeting various ESG criteria amounts to satisfying the interests of various stakeholders. In other words, we approach ESG via stakeholders' theory. In particular, the framework proposed in the paper can be used to deal with ESG criteria. We believe that the proposed approach reflects the sustainability literature, as it incorporates explicitly the issue of ESG via stakeholders' satisfaction, which is at the heart of sustainability approach.

We believe the reviewer's observation is important. Sustainability, among other things, means taking ESG criteria into account. That is why we

added a new paragraph in the 'Related literature' section (page 5) that highlights the importance of ESG dimensions to all stakeholders in particular and to company's sustainability in general : Our view is supported by numerous publications [Freeman, 2017], which all propose definitions of sustainability which is based on the satisfaction of different stakeholders. In particular, in [Ben Abdallah et al., 2020] as : 'a company's overall ESG performance as it relates to various stakeholder groups : customers, employees, suppliers, executives, managers, investors, and so forth'. Therefore, ESG criteria are a set of extra-financial criteria used to analyze stakeholder satisfaction and therefore the sustainability of a company. Indeed, although used throughout the world, there is no exact definition for sustainability. However, this concept is generally understood respect for a company's stakeholders and ESG principles. [Dahlsrud, 2006] carried out an interesting study on the various definitions listed in the literature between 1980 and 2003, the most cited by 27 authors. The latter are 37 in number, highlighting the main dimensions of sustainability : the relational dimension with stakeholders, the social, economic, and the environmental dimensions (ESG), and the voluntarist or philanthropic dimension. Several studies have shown the close relationship between ESG criteria and stakeholder. [Barnett and Salomon, 2012] postulate that the application of strategies related to the implementation of ESG practices requires the participation of stakeholders so that they can adhere to the company's project. According to [Koh and Qian, 2014], the benefits that companies.

We can acquire from ESG practices depend on their 'moral and pragmatic' legitimacy to implement them. They must therefore obtain the support of the stakeholders. They conclude by considering that high-risk companies would be pushed to turn to ESG practices from the moment they can meet the expectations of stakeholders. [Attig et al., 2013] demonstrate that within the components inherent to ESG criteria, only those that are positively evaluated and sought after by stakeholders have a positive effect on the rating of companies. We understand here that sustainability is becoming a means of achieving concrete results targeting specific stakeholders. In fact, the concept of sustainable development encourages companies to involve stakeholders in their governance. The issue of sustainable development is linked to the integration of the expectations and interests of stakeholders in corporate strategy and management [Sharma, 2001].

3. *The writing style, for me, has deviated from the mainstream form of expression. It is difficult for me to directly find the exact meaning of what the author is trying to express.*

Response : Thank you for this valuable observation. We have done a thorough review in English and corrected the mistakes in the revised manuscript. We hope that the revised manuscript is more clear.

4. *The attribute/action framework is not well explained theoretically. When*

there is uncertainty, conflicts between stakeholders are difficult to express with maximum or minimum values. In addition, deterministic causality between attributes and actions is not well documented.

Response : The minimal and maximal values of the attributes do not depend on the conflicts between stakeholders. The intuition behind the minimal and maximal values of the attributes is as follows. Since utility functions tend to be monotonically increasing or decreasing attributes, minimal and maximal values of the attributes correspond to extreme levels of satisfaction or dissatisfaction of the various stakeholders. For example, if the utility function is monotonically increasing in a certain attribute, then the minimal value of this attribute corresponds to stakeholder being the least satisfied, and the maximal value corresponds to the stakeholder being the most satisfied, if all the other attributed are fixed, i.e., all the other things being equal.

However, conflicts between stakeholders comes from the action of the attributes over time, i.e., if we increase the value of some attributes (which is favorable to some stakeholders) then it can lead to a reduction in the value of another attribute in the next step (which will lead to a reduction in the utility function of the other stakeholders). This interaction is captured by the dynamic equation eq. (1).

In the revised version we tried to explain this point in more details.

5. *The functional of $F(x,u)$ is vague. The link between Equation (4) and affine function $F(x, u)$ should be well clarified. Although saturation function in Equation (5) can simplify the reasoning/modeling, it still needs to be explained after reviewing related literature.*

Response : In the new version of the paper, we do not use saturation functions in eq. (5). The reason that saturation functions was used in eq. (5) in the very first version of the paper (the version submitted for the previous round of reviews) was to make sure that the attribute vectors are admissible and hence the utility functions are well-defined. Note that for the utility functions to take values in the interval $[0, 1]$, it was necessary for the attribute vectors to be admissible, i.e. take values from a bounded set. In principle, if $F(x, u)$ is affine, it is no guaranteed that the solution of eq. (1) remains admissible for any choice of input. Adding saturation functions solved this problem. However, if the input is chosen according to a stabilizing control law e. (7), and the initial state is in the safe set eq. (8), then the solution of eq. (1) will be admissible, i.e., it will belong to a bounded set and the utility functions will take values in $[0, 1]$. Hence, the use of saturation functions in the definition of F was superfluous and for this reason we removed it.

More precisely, in the first version of the paper we assumed that $F(x, u) = \sigma_x(Ax + Bu + h)$, where σ_x was a saturation function such that for any admissible state $x \in \mathcal{X}$, $\sigma_x(x) = x$. This was necessary because we defined the dynamical system eq. (1) only on the set of admissible states \mathcal{X} . Hence, we had to ensure that $F(x, u)$ belongs to \mathcal{X} for any u and

any x from \mathcal{X} . Note that we are only interested in admissible attribute vectors, as the utility functions are defined only for admissible attribute vectors. The control strategy eq. (7) was constructed in such a way that it ensured that $AX(t) + BU(t) + h$ is admissible, for any t , if $X(0)$ was in the safe set. Hence, under eq. (7), $X(t+1) = AX(t) + BU(t) + h = \sigma_x(AX(t) + BU(t) + h)$. That is, the use of saturation functions was not really necessary.

Motivated by the observation above, in the two last versions, we dropped the use of saturation functions when stating our assumptions on F . Instead we defined eq. (1) on \mathbb{R}^n and we included the requirement that the solutions of eq. (1) under the control law eq. (7) should be admissible into control objectives. This formulation seemed more natural and realistic. In the revised version, we use saturation functions only when simulating the numerical examples for strategies different from eq. (7), or for initial states which are not in the safe set defined eq. (8). In those cases one may obtain non-admissible states for which it is impossible to evaluate the utility functions. However, for the sake of completeness we preferred to present numerical results even in that case, by assuming that there is a mechanism for enforcing admissibility. This does not create a bias in favor of our method, as it increases the number of alternatives with which we compare our method.

We agree with the reviewer, in the first version of the manuscript the link between eq. (4) and affine function $F(x, u)$ were not sufficiently explained. We corrected them in the revised version of the manuscript. In addition, after introducing the assumption that F is affine, we present a discussion motivating the choice of F and reviewing the related literature, see the text in red, immediately after the introduction of the assumption that $F(x, u)$ is affine.

In a nutshell, assuming that F is affine is a standard practice in control engineering [Franklin et al., 2001, Sontag, 1998]. The intuition behind this assumption is that we approximate the true non-linear system by its linearization around a suitable equilibrium point. The resulting linearized system is affine. The true system can then be thought of as linearized system perturbed by an additive noise term. If we stabilize the linearized system by a suitable feedback law, then the same feedback law will stabilize the true system, if the true system is started sufficiently close to the equilibrium point (i.e., the difference between the true dynamics and the linearized one is small), [Khalil, 2002, Sontag, 1998]. This approach often works sufficiently well in practice, but, of course, it is not always satisfactory.

In particular, in Subsection 3.1.5 of the revised version we argue that under mild conditions, the feedback law proposed by the paper will achieve its objectives for the true system, even if the latter is not affine. The main idea is to apply the methodology to the linearization of the true system around an equilibrium point. The resulting control law will achieve the

control objectives if the initial state is close enough to the equilibrium point.

The assumption that $F(x, u)$ is affine has nothing to do with equation eq. (4) : eq. (4) defines what we mean by sustainability, i.e., it defines the smallest acceptable values for the utility functions of the stakeholders. However, the assumption that $F(x, u)$ is affine makes it easier to find a control strategy for achieving sustainability, i.e. for achieving eq. (4). In fact, the assumption that $F(x, u)$ is affine helps solving most of control problems which involve stabilization around an equilibrium point. That is, the usefulness of this assumption is not specific for the control objective eq. (4).

In the revised version of the paper we tried to explain these points more clearly, please see the text in red after stating the assumption that $F(x, u)$ is affine. In addition, we added an explanation on how to apply the proposed methods to affine approximation of non-linear systems in Subsection 3.1.5 and Remark 2 of the revised version.

6. *What is the reasoning of Equation (8) ? It is unclear.*

Response :

We thank the reviewer for this comment. In the revised version we added additional text explaining eq. (8) and the related equation eq. (7). Below we will briefly recall the motivation and derivation of eq. (8).

In the manuscript eq. (7) describes a strategy for choosing actions and eq. (8) describes the corresponding safe invariant set.

The very goal of the computational method proposed in this paper is to find a strategy of the form eq. (7) and a set of the form eq. (8) which satisfies a number of conditions. That is, we to compute the matrices K and Q simultaneously.

The conditions which K and Q should satisfy are as follows :

- the strategy eq. (7), which is determined by the matrix K , drives the system to the chosen equilibrium state,
- if the strategy eq. (7) is applied, then the set eq. (8) is invariant with respect to the dynamical system eq. (1) and it is safe. By invariance we mean that if $U(t) = (U_1(t), \dots, U_m(t))^T$ is chosen according to eq. (7), then if the initial attribute vector $X(0)$ is in the set \mathcal{P} defined by eq. (8), then for any time instance t , the attribute vector $X(t)$ at time t will also be in \mathcal{P} . By safety we mean the following : when the attribute vector (e.g., the state of eq. (1)) is in the set eq. (8), then the values of the utility functions for that attribute vector are above the thresholds from eq. (4).

The strategy eq. (7) is a feedback strategy : the input action $U(t) = (U_1(t), \dots, U_m(t))^T$ at time t is determined by the attribute vector $X(t)$ at time t . The intuition behind eq. (7) is that we look at the difference between the current attribute vector $X(t)$ and the desired equilibrium state x_0 . Then the control action $U(t)$ is proportional to that difference, in fact $U(t)$ is an affine function of that difference. The elements of K

are the coefficients of this affine function. More precisely,

$$U_i(t) = \sum_{j=1}^n K_{i,j}(X_j(t) - x_{0,j}) + u_{0,i},$$

i.e., $K_{i,j}$ is the coefficient with which we multiply the term $X_j(t) - x_{0,j}$ representing the deviation of the j th component of the attribute vector from the desired value. The matrix K is chosen in such a manner that if eq. (7) is used, then the dynamical system eq. (1) will be globally asymptotically Lyapunov stable [Sontag, 1998], in particular, its solution will converge to the equilibrium state.

The set eq. (8) is parameterized by the positive definite matrix Q , and it represents an ellipsoid around the desired equilibrium point x_0 . The matrix Q is related to K as follows. If the strategy eq. (7) is applied, then the matrix Q plays the role of a Lyapunov function for the system eq. (1) under strategy eq. (7), i.e. it can be thought of as a distance measure from any attribute vector to the equilibrium point which decreases during the evolution of the underlying dynamical system eq. (1). More precisely, $V(X(t)) := (X(t) - x_0)^T Q^{-1} (X(t) - x_0)$ can be viewed as the square of the distance between $X(t)$ and x_0 , and if eq. (7) is used, then $0 \leq V(X(t+1)) < V(X(t))$ unless $X(t) = x_0$. In fact, Q is chosen so that the matrix $(A - KB)^T Q^{-1} - Q^{-1}$ is negative definite. This ensures that eq. (8) is invariant under eq. (1) if eq. (7) is used.

In addition, when choosing Q we make sure that if an attribute vector is in the set defined in eq. (8), then the utility function of each stakeholder exceeds the minimal thresholds $f_{i,min}$ from eq. (4), i.e., if $X(t) \in \mathcal{P}$, then $f_i(X(t), U(t)) \geq f_{i,min}$, if $U(t)$ is chosen as prescribed in eq. (7). Moreover, the matrix Q is calculated in such a manner, that any element $X(t)$ of \mathcal{P} satisfies the constraints $X_k(t) \in [x_{k,min}, x_{k,max}]$ and if $U_i(t) = \sum_{j=1}^n K_{i,j}(X_j(t) - x_{0,j}) + u_{0,i}$, then $U_i(t) \in [u_{i,min}, u_{i,max}]$, $i = 1, \dots, m$, $k = 1, \dots, n$. That is, the elements of \mathcal{P} are admissible states and if eq. (7) is used to calculate the input action for an element of \mathcal{P} , the corresponding input action will also be admissible.

The properties of eq. (7) and eq. (8) imply that if the system eq. (1) is started in the set \mathcal{P} defined in eq. (8), then the solution of eq. (1) will remain in this set and will in fact converge to the equilibrium point. In particular, the state of eq. (1) and the input produced by eq. (7) will be admissible. Moreover, the value of utility functions of each stakeholder along the solutions of eq. (1) will be above the thresholds defined in eq. (4). Moreover, if during the evolution of eq. (1) the attribute vector is perturbed, but the perturbation is small enough so that the perturbed vector still lies in \mathcal{P} , then

- the perturbed trajectory of eq. (1) will still be admissible,
- the input actions generated by eq. (7) using the perturbed attribute vectors are still admissible, and

— for the perturbed action vector $X(t)$ eq. (4) will still hold, i.e., the value of utility functions will exceed the designated thresholds.

That is, the proposed strategy eq. (7) is robust, i.e., it achieves sustainable states even under perturbations. Moreover, note that if there is no perturbation after certain time instance $t_1 > 0$, i.e., $d(t) = 0$ for all $t > t_1$, then $\tilde{X}(t)$ will still converge to the equilibrium state x_0 as $t \rightarrow +\infty$.

In the revised version, we added Subsection 3.1.5 and Remark 2 to explain in more detail the robustness properties of the proposed control strategy.

7. *I do not think choosing the equilibrium point is proper, it there anyway to derive it ? If yes, how ? If not, why ? The authors should clearly address these issues.*

Response : We thank the reviewer for this remark. We agree with the reviewer that for an autonomous dynamical system it is problematic to choose a globally stable equilibrium point. However, eq. (1) defines a dynamical system with input. Such systems may have several equilibrium points, in fact, for any choice u_0 of the input, the system

$$X(t+1) = F(X(t), U(t))$$

may have an equilibrium point x_0 which satisfies

$$x_0 = F(x_0, u_0).$$

For example, if

$$F(x, u) = Ax + Bu + h$$

and A is an invertible matrix, then x_0 is uniquely determined by

$$x_0 = (I - A)^{-1}(Bu_0 + h).$$

Note that if the system

$$X(t+1) = F(X(t), U(t))$$

arises by discretizing a differential equation, then A tends to be invertible.

In control theory, choosing equilibrium points is a standard practice. More precisely, in control theory the desired behavior of the system tend to correspond to the behavior which the system exhibits at a certain equilibrium point. Designing controllers then boils down to the following two steps :

- find an equilibrium point at which the system exhibits properties which are required by the control objectives,
- find a control law which drives the system to that equilibrium point.

To illustrate this point, let us recall the typical textbook example of regulator design [Franklin et al., 2001]. That is, consider a dynamical system of the form eq. (1), i.e.

$$X(t+1) = F(X(t), U(t))$$

and let us assume that

$$F(X(t), U(t)) = AX(t) + BU(t) + h.$$

The goal of regulator design is to make sure that a scalar valued linear function of the state approaches a certain constant value r . More precisely, assume that

$$y(t) = CX(t),$$

where C is a $1 \times n$ matrix, and the goal is to find a strategy for choosing $U(t)$ such that

$$\lim_{t \rightarrow \infty} CX(t) = r.$$

In this case, based on r , we compute the equilibrium point x_0 such that

$$r = Cx_0 \text{ and } x_0 = F(x_0, u_0)$$

for a suitable u_0 . If

$$F(x, u) = Ax + Bu + h,$$

and there is a single input action, i.e., $m = 1$, then

$$u_0 = \frac{r - C(I - A)^{-1}h}{C(I - A)^{-1}B}.$$

If we compute a matrix K such that all the eigenvalues of $A - BK$ are within the unit disc, then the strategy

$$U(t) = -K(X(t) - x_0) + u_0$$

achieves the objective that

$$\lim_{t \rightarrow \infty} CX(t) = r.$$

Note that one option for computing K is to solve a suitably designed optimal control problem [Franklin et al., 2001]. Another method is to use pole placement [Franklin et al., 2001] or to solve LMIs. The disadvantage of both optimal control and pole placement is that they cannot take into account explicitly constraints on magnitude of the state and input variables. That is, if in addition to converging to x_0 the components of the state and the input should be bounded, then this requirement can be achieved only by trial and error when using pole placement or optimal control. In contrast, this constraints can directly be integrated into LMIs [Boyd et al., 1994].

The approach of this paper follows the same philosophy. We try to find an equilibrium point x_0, u_0 for which the utility functions exceed a certain thresholds. As soon as the equilibrium point are chosen, then it is sufficient to find a strategy which drives the system eq. (1) to the chosen equilibrium point. By applying this strategy, we can ensure that in the

long run the utility function of each stakeholder will exceed the target threshold. That is, we achieve a favorable outcome for all participants. The challenge which remains is to ensure that while approaching the equilibrium points the states and the inputs remain admissible and sustainable (i.e., the states and inputs are bounded and the values of the utility functions are above the corresponding minimal thresholds). This latter requirement motivates us to use LMIs.

In the revised version we added a short version of the explanation above in the paragraph where the choice of the equilibrium point is introduced. The additional text is displayed in red in the revised version.

8. *There is an error in the second line below Equation (12).*

Response : We thank the reviewer for this observation. We double checked eq. (12), we hope that in the revised version eq. 12 and the text below it are correct. In the first version, there was indeed an error after eq. (12).

9. *The section "Justification of the state-space representation" should be re-written by clarifying the related literature and authors' own contribution. I am confused by this section and did not find any evidence of how the Nash equilibrium has been proven.*

Response : We thank the reviewer for the valuable comment. In the revised version we rewrote the mentioned section in order to make it more clear. Note that the discussion of that section was intended as a heuristic explanation linking state-space representations with games. In particular, the discussion was not rigorous, the original text did not constitute a formal mathematical proof. We did not claim that the presented link was new either. In fact, the same idea was formalized in [Mazumdar et al., 2020, Ratliff et al., 2016]. In the revised version we made the discussion more precise and we cited the relevant literature.

In a nutshell, state-space representations can be viewed as implementations of a gradient descend algorithm for finding a Nash equilibrium of a certain game. The gradient descend algorithm corresponds to the behavior of rational but myopic agents, i.e., agents which act on local information. The equilibrium points of the state-space representations correspond to so called differential Nash equilibria [Ratliff et al., 2016, Definition 3], and by [Ratliff et al., 2016, Theorem 1] they are local Nash equilibria as defined in [Ratliff et al., 2016, Definition 1].

In the revised version we cite [Mazumdar et al., 2020, Ratliff et al., 2016] and we clearly state that the interpretation presented in the paragraph "Justification of the state-space representation" is based on [Mazumdar et al., 2020, Ratliff et al., 2016].

10. *Section 4.1., i.e., numerical table B.1, I do not think it is acceptable for assigning these attributes to the underlying stakeholders, which is too simple to effectively validate the proposed model. In addition, there is no financial literature support. It is too simple to valid the proposed model.*

Response : The purpose of the example is to illustrate the possible use of the mathematical framework proposed in the paper. We agree

that a better model would be desirable, but, developing a more detailed model remains a topic of future research. The reason for this is that our framework requires dynamical models which describe several aspects of banking and its interplay with the stakeholder's interest, and there is no such model in the literature. Several authors have attempted to model certain aspects of banking (for example, profits, manager's remuneration, etc. [4], [15], [14] , [1], etc.), but there is no model integrating all of them. Moreover, the literature indicates that modeling each of these aspects separately is a paper in itself [17].

Presenting both the mathematical framework and an integrated model of banks would not fit the scope of one single paper. We would like to point out that the example we used is partially based on the economic literature : the utility functions were taken from the literature [3], and the choice of the parameters of the dynamic model is motivated by economic relationships known from the literature.

In addition, when building the model for the numerical example, we tried to incorporate the results of the literature when choosing the parameters of the model. In the revised version we added a paragraph that explains the economic basis behind the value of the parameters of the numerical example, please see pages 15-20 and page 34-37 of the revised manuscript. More precisely, the signs of the entries of the matrices A and B of the example are based on hypotheses from the literature.

We agree that it would be preferable to use better models for the numerical example. However, in the light of the discussion above, we are compelled to leave building such models as a topic for future research.

11. *The author should clarify how the proposed framework has been applied with optimal control framework in a dynamics setting, particularly, when finding feasible strategy for all stakeholders ?*

Response : We thank the reviewer for this comment. We would like to clarify that the approach of the present manuscript does not use optimal control. Instead, we aim at finding a strategy which keeps the utility functions at a certain level. That is, the mathematical framework which is solved in the present paper is not an optimal control framework. To the best of our knowledge, there are no results on using optimal control for stakeholder satisfaction. In fact, the control objectives considered in this paper are not formally compatible with the one of optimal control. Namely, while optimal control could be used to find a stabilizing feedback control. However, integrating state and input constraints into optimal control problem in a theoretically sound manner is challenging. In fact, there is no systematic procedure for doing so. In any case, to the best of our knowledge, there has been no prior work using classical optimal control for stakeholders' satisfaction. In the revised version, these issues are discussed on page 4-5, we added additional text (in red) to further clarify the relationship with optimal control framework.

2 Reviewer 2 General Comments :

1. *This revised version of the paper aims to use robust modeling to capture CSR aspects with a Pareto optimal solution for stakeholder groups with distinct objectives. This version of the paper can be accepted. The authors have expanded the text and have clarified many of the previous problems with the paper. There were a number of typos that have been corrected. The additions have improved the paper. The strongest part of the paper is that it attempts to model and quantify the CSR objectives of the stakeholders. CSR is current topic faced by firms, and firms' performance in the area of CSR has been difficult to quantify or measure. Thus, there is a basis for research contributions in this area.*

Response : We are very happy to hear that you are satisfied with our revision. We are grateful to you for suggesting those changes which increased the quality of our paper.

2. *The weakest part of the paper is that model is not applied to any firm using statistical data. Further, it is not able to demonstrate why a robust control approach to CSR is superior to other approaches, or how it could be integrated with other approaches.*

Response : We thank the reviewer for the constructive remark. We agree with the reviewer. In the paper we did not use models based on statistical data. This choice is motivated by the novelty of the proposed approach. More precisely, as we explained it in the revised version, the reason for not using models based on statistical data is that we have not found such models in the literature which are suitable for the proposed methodology. Please note that most of papers using control theory do not address the problem of estimating models from statistical data. Instead, they use models which were already published in the literature, see for example [Blueschke et al., 2013, Blueschke and Savin, 2017]. The reason for this is that estimating dynamical models from data is a non-trivial research problem, and the results are often published as separate papers. In our case the challenge is even greater, as we need models which model the interaction of several stakeholders. However, most of the literature deals with models which describe one stakeholder separately.

To sum up, developing models based on statistical data is a research problem on its own which we plan to address this issue in subsequent papers.

Concerning the advantages of robust control with respect to other approaches, we are not aware of other papers using dynamical systems and control theory for CSR. That is, there are no obvious alternatives with which we could make a useful comparison.

In the revised version, we tried to better explain these points.

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