T-norms in Many-Valued Logics: a Representation Theorem in the ABOP Framework

Marie-Jeanne Lesot, Adrien Revault d'Allonnes

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Marie-Jeanne Lesot
Sorbonne Université, LIP6,
F-75005, Paris, France
Email: Marie-Jeanne.Lesot@lip6.fr

Adrien Revault d'Allonnes
Université Paris 8, LIASD, CUBI,
FR-93526, Saint-Denis, France
Email: ara@up8.edu

***Abstract*—This paper proposes a study of a configurable aggregation operator in many-valued logic (MVL), called ABOP, standing for ABating OPerator. It focuses on its capacity of implementing any t-norm: the paper's main theorem offers a constructive proof of the existence of a configuration for any given MVL t-norm, showing it is a generalisation of t-norm operators. Furthermore, it specifies the corresponding parameter values. The paper then considers several examples of classical fuzzy logic t-norms adapted to MVL, and examines the corresponding parameters, as instantiation of the general representation theorem.**

***Index Terms*—many-valued logics, aggregation operator, t-norms**

I. INTRODUCTION

Aggregation operators combine multiple input values into a single output value, where the semantics of the combination can implement various principles, e.g. conjunctive, disjunctive or compromise behaviours, possibly offering additional properties, e.g. reinforcement to name one (see e.g. [1], [4]). This paper focuses on the case of conjunctive aggregation, as performed by the t-norm operators (see e.g. [5]).

It considers this question of conjunctive aggregation in the framework of many-valued logic (MVL) [3], [6] which uses a discrete scale of truth values, intermediate between the classical binary scale $\{0, 1\}$ and the fuzzy real scale $[0, 1]$. In the MVL framework, aggregation is mainly performed using logical operators [3], corresponding to t-norms, t-conorms or implications, and arithmetic operators [9], beyond a purely logical interpretation.

The ABOP operator was introduced in [7] in this MVL theoretical framework, in the applicative context of information scoring: in a nutshell, this scoring aims at measuring the quality of a piece of information and most often relies on the aggregation of evaluations obtained on several dimensions used to decompose the definition of information quality (see e.g. [2]). This, for instance, may include dimensions characterising the source of the considered piece of information, such as its reliability and its competence, in addition to dimensions characterising the content of the piece of information (see e.g. [7]). The ABOP operator was introduced as a general parametric operator implementing a so-called abating influence on its first argument, so that $\forall x, y\ \text{ABOP}(x, y) \leq x$, hence its name ABating OPerator. In the information scoring framework, this property is for instance used to aggregate the effect of source competence with that of source reliability: the idea being that the score of a piece of information provided by a source with low reliability cannot increase simply because the source is competent, i.e. the aggregation of reliability and competence must be lower than reliability alone. In order to enhance the expressiveness of the ABOP operator, no other constraint than its being increasing in both its arguments are imposed to this abating behaviour.

As a consequence, it has been shown that the ABOP parametric family is more general than t-norms [8]: the latter satisfy the previous conditions of monotonicity and abating property, but also impose others (commutativity, associativity and the existence of a neutral element). The ABOP family allows to represent other aggregation behaviours, e.g. with various degrees of severity or the absence of neutral element [8], illustrating its expressive versatility.

This paper proposes to study the reciprocal question, examining whether the set of t-norms is more general than the ABOP family: it studies whether all t-norms are included in the ABOP parametric family, i.e. whether ABOP can represent any t-norm, or whether there exist t-norms that take different forms. The paper provides a negative answer to the latter question, showing that any t-norm can be implemented as a specific ABOP operator: it establishes a representation theorem for any t-norm in the ABOP framework. In addition, it provides a constructive proof of the existence of this configuration for any t-norm and exhibits the corresponding parameter values.

The paper then considers four examples of classical fuzzy logic t-norms, adapted to the MVL framework and specifies their associated configurations, as illustrative cases.

The paper is structured as follows: Section II offers a short reminder about aggregation in Many-Valued Logics, focusing on the ABOP parametric operator family. Section III provides the main result of the paper, establishing the representation theorem for any t-norm in the ABOP framework. It also comments the general form of the induced parameter configuration and the minimal structure of any t-norm in this framework. Section IV illustrates the application of the representation theorem to several reference t-norm cases. Section V concludes the paper.

II. Aggregation in Many-Valued Logics

The classical many-valued framework [3], [6] uses M truth degrees of a discrete and totally ordered set $\mathcal{L}_M = \{\tau_0, \ldots, \tau_{M-1}\}$, where $\tau_\alpha \leq \tau_\beta \Leftrightarrow \alpha \leq \beta$. These span, at a granularity varying with M, the different levels of veracity, from τ_0, meaning 'false', to τ_{M-1}, for 'true'.

This section provides a brief reminder on the topic of aggregation in this framework, focusing on the two types of operators considered in the paper: it first gives the general definition of MVL t-norms, transposing the definition given in fuzzy logic, and then gives the definition of the AbOp parametric family.

A. T-norms in Many-Valued Logics

The general definition of an aggregation operator with two arguments in MVL is a mapping $F : \mathcal{L}_M \times \mathcal{L}_M \longrightarrow \mathcal{L}_M$. Considering only the case with two arguments is not a limitation in this paper, as the t-norms it studies are associative by definition (see below) and their general definition can thus be derived from the two argument case.

Transposing the properties stated in the fuzzy logic framework, a t-norm $\top$ on $\mathcal{L}_M \times \mathcal{L}_M$ is an aggregation operator that satisfies the following contraints, for any $\tau_\alpha, \tau_\beta, \tau_\gamma \in \mathcal{L}_M$

- commutativity: $\top(\tau_\alpha, \tau_\beta) = \top(\tau_\beta, \tau_\alpha)$
- associativity: $\top(\top(\tau_\alpha, \tau_\beta), \tau_\gamma) = \top(\tau_\alpha, \top(\tau_\beta, \tau_\gamma))$
- monotonicity: if $\tau_\alpha \leq \tau_\gamma$, $\top(\tau_\alpha, \tau_\beta) \leq \top(\tau_\gamma, \tau_\beta)$
- neutral element: $\top(\tau_{M-1}, \tau_\alpha) = \tau_\alpha$

Several examples are given and studied in Section IV. They for instance include the Zadeh t-norm $F_Z(\tau_\alpha, \tau_\beta) = \min(\tau_\alpha, \tau_\beta)$.

In MVL, as the domain is finite, with cardinal M, the definition of any t-norm, and more generally of any aggregation operator, is equivalent to specifying the M^2 coefficient of the matrix that gives the result of $F(\tau_\alpha, \tau_\beta)$ for any couple $(\tau_\alpha, \tau_\beta) \in \mathcal{L}_M^2$. However, such an explicit and exhaustive definition makes the interpretation of the considered operator difficult, as it prevents from having an intuition of the operator behaviour. Functional definitions, e.g. setting $F = \min$, help understanding the underlying principle. As will be discussed below, one of the advantages of the AbOp operator is to provide a graphical representation that improves the legibility, as well as a sparse matrix with fewer than M^2 parameters.

B. AbOp Definition

The AbOp operator, introduced in [7] and characterised in [8], is a parametric family of MVL aggregation operators. It depends on a sparse $M \times M$ matrix of parameters κ:

$$\kappa \subseteq \{\kappa_\alpha^\gamma \in \mathcal{L}_M, \alpha, \gamma \in [\![0, M-1]\!]\} \tag{1}$$

that must satisfy the constraints discussed in the next subsection. The operator is defined as:

$$F_\kappa(\tau_\alpha, \tau_\beta) = \min\{\tau_\gamma \in \mathcal{L}_M | \tau_\beta \leq \kappa_\alpha^\gamma\} \tag{2}$$

As detailed in [8], Figure 1 illustrates such a function in the specific case of a 5-valued logic, i.e. $\mathcal{L}_5$: the horizontally aligned discs represent the degrees of $\mathcal{L}_5$, from τ_0 to τ_4, corresponding either to possible values of τ_α or possible outputs.

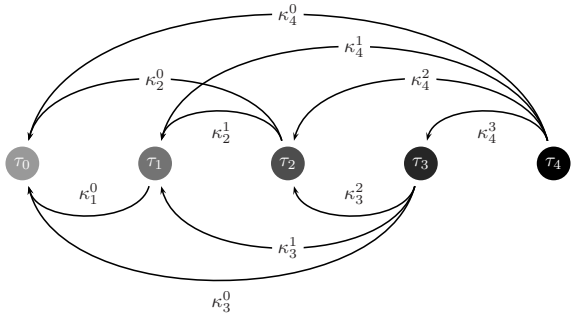

Fig. 1. An example of $F(\tau_\alpha, \tau_\beta)$, in $\mathcal{L}_5$, reproduced from [7].

The arrow labels represent the parameters κ_α^γ, constraining the transitions from τ_α to τ_γ. The computation of $F_\kappa(\tau_\alpha, \tau_\beta)$ then consists in starting from the disc labelled τ_α and following the arrows whose labels κ_α^γ are greater than τ_β. The output is the lowest accessible such τ_γ, that is the leftmost pointed to disc, with $\kappa_\alpha^\gamma \geq \tau_\beta$.

This graphical representation can be seen as increasing the interpretability and legibility of the operator behaviour. For instance, the fact that all arrows in Figure 1 point to the left is an illustration of the abating property: the output of the aggregation is less than the starting point, the first argument.

In the information scoring case mentioned in the introduction, the discs correspond to the score whose value is initialised with the source reliability. Let us for instance consider it takes the middle value τ_3. Then the aggregation with the source competence corresponds to a transition to a disc pointed to by an arrow starting from the disc τ_3. The competence value, consider value τ_4, is compared to the transition costs κ_3^γ, to determine the accepted transitions, defined as the ones for which this cost is greater than τ_4. The aggregation result is then the leftmost disc pointed to by an accepted transition. Due to the considered arrows, that are all pointing to the left, it is necessarily lower than the starting τ_3 value.

C. AbOp Parameters

The parameters κ given in (1) must satisfy several constraints to ensure that the function defined in (2) is *well*-defined and that it possesses the desired properties [8]: as mentioned in the introduction, the AbOp operator is required to have an abating influence on its first argument and to be increasing in both its argument (see [7]). This section summarises these constraints, with the κ parameters presented in their matrix form.

Note that not all monotonic functions are an AbOp instanciation, since the main constraint, the abating behaviour, is not a requirement for monotonicity. For instance, the basic case $F(x, y) = y$ is monotonic in both its arguments but does not possess the desired abating influence on x.

Formally, four conditions are imposed to the parameter matrix κ (see [8] for a discussion and justification): it must hold that

(P1) each row has at least one occurrence of the value τ_{M-1}, on the diagonal or in a column located to the left the diagonal,
(P2) the values on each row are sorted in increasing order,
(P3) given any reference position a_{ij} of the matrix, all values located in the lower left submatrix (i.e. a_{kl} such that $k \geq i$ and $l \leq j$) are less than the value of a_{ij}.
(P4) some cells in the matrix may be empty

The matrix sparsity is crucial since, otherwise, it would be more relevant and no more expensive to define, explicitly and directly, the value of $F(\tau_\alpha, \tau_\beta)$ for any couple $(\tau_\alpha, \tau_\beta) \in \mathcal{L}_M^2$. Sparsity has two causes, namely the existence of redundant cells and that of undefined cells.

Indeed, some cells are redundant insofar as on each row α, a unique occurrence of each observed value among the κ_α^γ can be kept. Indeed, as the ABOP operator relies on the $\min$ operator, repetitions of this value to the right would not modify the operator result.

As a consequence, (P1), together with (P2), implies that κ is a lower triangular matrix: only the leftmost occurrence of each value need be specified and, according to (P2), each row contains one occurrence of the value τ_{M-1}, whose rightmost allowed position is the diagonal. Therefore the matrix need not contain any value in its upper right part and its definition contains at most $M(M-1)/2$ coefficients. Note that redundancy can also occur in the lower triangular part, reducing further the number of coefficients.

In addition, the matrix sparsity is also due to the fact that some cells must be considered as undefined since they have no allowed value. This is for instance the case of cells located, on any row, to the left of τ_0. Indeed they would point to excluded outputs, i.e. values which can never be reached.

As a consequence, the instantiation of ABOP consists in choosing, in each row, the position of τ_{M-1}, on the diagonal or in a column located to the left of the diagonal and possibly the position of τ_0 and of their intermediate values, satisfying the inter-row constraints and possibly leaving some cells empty.

The rest of the paper uses this representation.

III. ENCODING MVL T-NORMS IN ABOP

This section contains the main result of the paper, the representation theorem stating that any MVL t-norm can be represented as an ABOP instantiation, and providing the associated parameter values. It first establishes a lemma used in the proof, showing that a specific matrix of size $M \times M$ with values in $\mathcal{L}_M$ satisfies the conditions specifying an ABOP configuration. The theorem then shows that this configuration induces the considered t-norm.

A. Preliminary Result

This section studies a specific matrix, defined from a considered t-norm, and shows that it constitutes an allowed parametrisation of an ABOP operator.

Lemma 1.
Given a t-norm $\top$ on $\mathcal{L}_M \times \mathcal{L}_M$, the parameters defined as

$$\kappa_\alpha^\beta = \max\{\tau_\gamma | \top(\tau_\alpha, \tau_\gamma) \leq \tau_\beta\} \tag{3}$$

adequately define an ABOP *function F_κ.*

Note that this definition of κ_α^β is nothing more than the definition of the residual implication based on t-norm $\top$. The following section, in particular Theorem 1, will use said r-implication defined criteria to define the original t-norm, as an ABOP configuration.

Proof. The proof consists in showing that all conditions stated in II-C are satisfied.

(P1) holds because, for any α,

$$\kappa_\alpha^\alpha = \max\{\tau_\gamma | \top(\tau_\alpha, \tau_\gamma) \leq \tau_\alpha\} = \tau_{M-1}$$

Indeed, $M-1$ belongs to the set of γ candidates as $\top(\tau_\alpha, \tau_{M-1}) = \tau_\alpha$ as τ_{M-1} is $\top$'s neutral element. Computing the maximum over the set of γ candidates then leads to $\kappa_\alpha^\alpha = \tau_{M-1}$ as τ_{M-1} is $\mathcal{L}_M$'s maximal element.

(P2) holds because for any α, β, δ, if $\beta \leq \delta$, $\kappa_\alpha^\beta \leq \kappa_\alpha^\delta$. Indeed, if $\beta \leq \delta$, for any γ, $\top(\tau_\alpha, \tau_\gamma) \leq \tau_\beta$ implies that $\top(\tau_\alpha, \tau_\gamma) \leq \tau_\delta$. Therefore,

$$\{\tau_\gamma | \top(\tau_\alpha, \tau_\gamma) \leq \tau_\beta\} \subseteq \{\tau_\gamma | \top(\tau_\alpha, \tau_\gamma) \leq \tau_\delta\}$$

Computing the maxima over these two sets implies the expected inequality $\kappa_\alpha^\beta \leq \kappa_\alpha^\delta$

(P3) holds because of $\top$'s monotonicity: for any α, β, η, δ such that $\eta \geq \alpha$ and $\delta \leq \beta$ (i.e. κ_η^δ is in the lower left submatrix of κ_α^β), (P2) implies that $\kappa_\eta^\delta \leq \kappa_\eta^\beta$, and it holds that $\kappa_\eta^\beta \leq \kappa_\alpha^\beta$. Indeed for any γ, $\top$'s monotonicity implies $\top(\tau_\alpha, \tau_\gamma) \leq \top(\tau_\eta, \tau_\gamma)$, therefore

$$\{\tau_\gamma | \top(\tau_\eta, \tau_\gamma) \leq \tau_\beta\} \subseteq \{\tau_\gamma | \top(\tau_\alpha, \tau_\gamma) \leq \tau_\beta\}$$

Computing the maxima over these two sets implies $\kappa_\eta^\beta \leq \kappa_\alpha^\beta$.

(P4) holds because the definition in (3) specifies all values, not leading to any undefined value.

As a consequence, the matrix sparsity only results from possibly redundant values. □

B. Representation Theorem

This section proves the main theorem of the paper, offering a constructive proof on an ABOP configuration for any given t-norm. Specifically, this configuration is the one stated in Lemma 1:

Theorem 1.
Any t-norm $\top$ on $\mathcal{L}_M \times \mathcal{L}_M$ is encoded in the ABOP *configuration defined by the parameters*

$$\kappa_\alpha^\beta = \max\{\tau_\gamma | \top(\tau_\alpha, \tau_\gamma) \leq \tau_\beta\}$$

Proof. Let $\top$ be any t-norm on $\mathcal{L}_M \times \mathcal{L}_M$ and κ as stated in (3). Expanding on Lemma 1, these κ define a proper ABOP F_κ. To show the theorem holds one must show that, for any τ_α, τ_β, it holds that $F_\kappa(\tau_\alpha, \tau_\beta) = \top(\tau_\alpha, \tau_\beta)$. We show this equality by showing two inequalities.

First, let us show $F_\kappa(\tau_\alpha, \tau_\beta) \leq \top(\tau_\alpha, \tau_\beta)$. By definition, $F_\kappa(\tau_\alpha, \tau_\beta) = \min\{\tau_\gamma | \tau_\beta \leq \kappa_\alpha^\gamma\}$, let us show that $\top(\tau_\alpha, \tau_\beta)$

belongs to the set of γ candidates, i.e. $\tau_\beta \leq \kappa_\alpha^{\top(\tau_\alpha,\tau_\beta)}$. By definition (3), $\kappa_\alpha^{\top(\tau_\alpha,\tau_\beta)} = \max\{\tau_\delta | \top(\tau_\alpha,\tau_\delta) \leq \top(\tau_\alpha,\tau_\beta)\}$, and τ_β obviously belongs to the set of τ_δ candidates.

Computing the minimum over the set of γ candidates then leads to the expected $F_\kappa(\tau_\alpha,\tau_\beta) \leq \top(\tau_\alpha,\tau_\beta)$.

Second, let us show $F_\kappa(\tau_\alpha,\tau_\beta) \geq \top(\tau_\alpha,\tau_\beta)$. From (2), $F_\kappa(\tau_\alpha,\tau_\beta)$ is defined as a minimum over a set of γ candidates, let us show that $\top(\tau_\alpha,\tau_\beta)$ is lower than any of these candidates: choose τ_γ such that $\tau_\beta \leq \kappa_\alpha^\gamma$ (its existence is guaranteed by the fact that κ define an adequate F_κ). Let us show that $\top(\tau_\alpha,\tau_\beta) \leq \tau_\gamma$.

As $\top$ is increasing in its first argument, $\tau_\beta \leq \kappa_\alpha^\gamma$ implies $\top(\tau_\alpha,\tau_\beta) \leq \top(\tau_\alpha,\kappa_\alpha^\gamma)$. Now by definition of κ_α^γ as a maximum over τ_δ candidates, as stated in (3), κ_α^γ is one of these candidates and satisfies $\top(\tau_\alpha,\kappa_\alpha^\gamma) \leq \tau_\gamma$.

By transitivity, it follows that $\top(\tau_\alpha,\tau_\beta) \leq \tau_\gamma$. □

C. Minimal Structure

This section proposes to take a look at the effect t-norm properties have on the ABOP operator and the general form of the associated parameter matrix: for any t-norm, the non-redundant representation of the matrix induced by (3) has the following two properties, where (T1) specialises (P1) and (T2) is an additional characterisation with respect to the general properties stated in Section II-C:

(T1) its diagonal terms equal τ_{M-1},

(T2) its bottom row contains all values from τ_0 to τ_{M-1}.

Indeed, (T1) has been established in the proof of Lemma 1, when proving (P1) holds, and (T2) is a direct consequence of the general κ_α^β definition (3): for $\alpha = M-1$, by definition of $\top$'s neutral element, for any τ_γ, $\top(\tau_{M-1},\tau_\gamma) = \tau_\gamma$, and the greatest τ_γ such that $\top(\tau_{M-1},\tau_\gamma) = \tau_\gamma \leq \tau_\beta$ equals τ_β. This means that $\kappa_{M-1}^\beta = \tau_\beta$.

The global form of the matrix is thus as follows, where the upper part of the matrix is empty and its lower part is to be filled according to (3) and filtered with respect to the redundancy principle:

$$\begin{bmatrix} \tau_{M-1} & & & & & & \\ & \tau_{M-1} & & & & & \\ & & \tau_{M-1} & & & & \\ & & & \tau_{M-1} & & & \\ & & & & \ddots & & \\ \tau_0 & \tau_1 & & \cdots & & \tau_{M-2} & \tau_{M-1} \end{bmatrix}$$

Equivalently, the graphical representations of the the ABOP operator for all t-norms have common labelled arrows: from one disc to itself, labelled with τ_{M-1}, and from disc τ_{M-1} to each disc. In order to improve the legibility of these representations and to ease their comparison across t-norms, these common arrows are displayed in grey on Figures 2 to 5.

IV. SOME EXAMPLES OF MVL T-NORM ENCODINGS

This section proposes to study the general theorem established in the previous section and to examine the ABOP configuration it induces for several reference cases, namely the Zadeh, Łukasiewicz, nilpotent minimum and drastic t-norms successively.

For each of them, the general equation (3) is specialised for the considered t-norm, the resulting matrix is shown as well as the associated graphical representation.

A. Zadeh t-norm

The instantiation of Theorem 1 in the case of the Zadeh t-norm leads to the following expression:

Corollary 1.
The Zadeh t-norm $\top(\tau_\alpha,\tau_\beta) = \min(\tau_\alpha,\tau_\beta)$ is encoded in the ABOP *configuration defined by*

$$\kappa_\alpha^\beta = \begin{cases} \tau_{M-1} & \text{if } \tau_\alpha \leq \tau_\beta \\ \tau_\beta & \text{otherwise} \end{cases}$$

Proof. According to Theorem 1 applied to Zadeh t-norm, $\kappa_\alpha^\beta = \max\{\tau_\gamma | \min(\tau_\alpha,\tau_\gamma) \leq \tau_\beta\}$.

If $\tau_\alpha \leq \tau_\beta$,

$$\begin{aligned} \kappa_\alpha^\beta &= \max(\max\{\tau_\gamma \leq \tau_\alpha | \min(\tau_\alpha,\tau_\gamma) \leq \tau_\beta\}, \\ &\qquad \max\{\tau_\gamma > \tau_\alpha | \min(\tau_\alpha,\tau_\gamma) \leq \tau_\beta)\} \\ &= \max(\max\{\tau_\gamma \leq \tau_\alpha | \tau_\gamma \leq \tau_\beta\}, \\ &\qquad \max\{\tau_\gamma > \tau_\alpha | \tau_\alpha \leq \tau_\beta\}) \\ &= \max(\tau_\alpha, \tau_{M-1}) = \tau_{M-1} \end{aligned}$$

If $\tau_\alpha > \tau_\beta$, in the above decomposition, the second set is empty and the first set has maximum τ_β, leading to $\kappa_\alpha^\beta = \tau_\beta$. □

In the matrix representation, the subdiagonal values are thus constant columnwise:

$$\kappa_Z = \begin{bmatrix} \tau_{M-1} & & & & & & \\ \tau_0 & \tau_{M-1} & & & & & \\ \tau_0 & \tau_1 & \tau_{M-1} & & & & \\ \tau_0 & \tau_1 & \tau_2 & \tau_{M-1} & & & \\ \vdots & \vdots & \vdots & & \ddots & & \\ \tau_0 & \tau_1 & \tau_2 & \cdots & \cdots & \tau_{M-1} \end{bmatrix}$$

They lead to a non-sparse lower triangular matrix, with the maximal number of parameters $M(M-1)/2$.

This matrix is, as expected, consistent with the proposition in [8]. The difference comes from the way the result is obtained: in [8], this configuration is arbitrarily considered and shown to lead to the Zadeh t-norm. Using Theorem 1 allows to establish this matrix with justification. The associated graphical representation is shown in Figure 2.

B. Łukasiewicz t-norm

The instantiation of Theorem 1 in the case of the Łukasiewicz t-norm leads to the following expression:

Corollary 2.
The Łukasiewicz t-norm $\top(\tau_\alpha,\tau_\beta) = \tau_{\max(\alpha+\beta-(M-1),0)}$ is encoded in the ABOP *configuration defined by*

$$\kappa_\alpha^\beta = \begin{cases} \tau_{M-1-\alpha+\beta} & \text{if } \alpha \geq \beta \\ \tau_{M-1} & \text{otherwise} \end{cases}$$

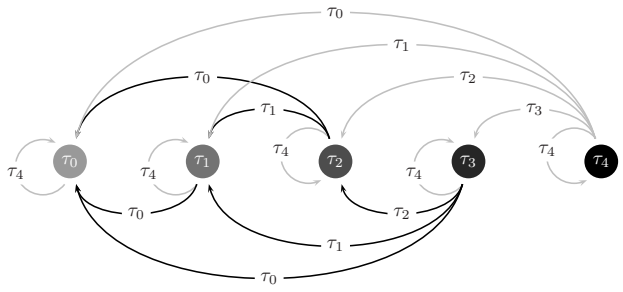

Fig. 2. Zadeh t-norm, F_Z, in $\mathcal{L}_5$

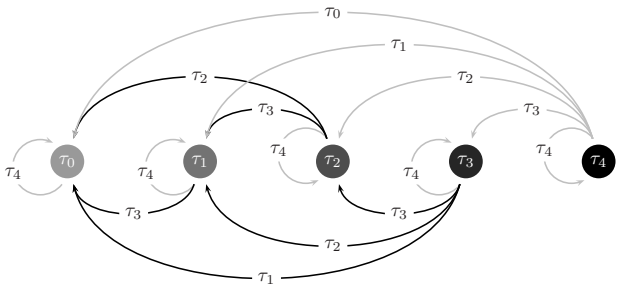

Fig. 3. Łukasiewicz t-norm, $F_{\text{Ł}}$, in $\mathcal{L}_5$

Proof. Applying Theorem 1 to Łukasiewicz t-norm leads to

$$\begin{aligned}
\kappa_\alpha^\beta &= \max\{\tau_\gamma \in \mathcal{L}_M | \tau_{\max(\alpha+\gamma-(M-1),0)} \leq \tau_\beta\} \\
&= \max\{\tau_\gamma \in \mathcal{L}_M | \max(\alpha+\gamma-(M-1),0) \leq \beta\} \\
&= \max(\max\{\tau_\gamma \in \mathcal{L}_M | \alpha+\gamma-(M-1) \leq 0\}, \\
&\qquad \max\{\tau_\gamma \in \mathcal{L}_M | 0 < \alpha+\gamma-(M-1) \leq \beta\}) \\
&= \max(\max\{\tau_\gamma \in \mathcal{L}_M | \gamma \leq M-1-\alpha\}, \\
&\qquad \max\{\tau_\gamma \in \mathcal{L}_M | M-1-\alpha < \gamma \leq M-1-\alpha+\beta\}) \\
&= \max\{\tau_\gamma \in \mathcal{L}_M | \gamma \leq M-1-\alpha+\beta\} \\
&= \begin{cases} \tau_{M-1-\alpha+\beta} & \text{if } \tau_\alpha \geq \tau_\beta \\ \tau_{M-1} & \text{otherwise} \end{cases}
\end{aligned}$$

□

Using the sparse representation that leads to a lower triangular matrix, the ABOP configuration of the Łukasiewicz t-norm is thus obtained as:

$$\kappa_L = \begin{bmatrix}
\tau_{M-1} & & & & & \\
\tau_{M-2} & \tau_{M-1} & & & & \\
\tau_{M-3} & \tau_{M-2} & \tau_{M-1} & & & \\
\tau_{M-4} & \tau_{M-3} & \tau_{M-2} & \tau_{M-1} & & \\
\vdots & \vdots & \vdots & & \ddots & \\
\tau_0 & \tau_1 & \tau_2 & \dots & \dots & \tau_{M-1}
\end{bmatrix}$$

Again, this result is, as expected, identical to the one established in [8], but obtained as a special case of Theorem 1 and not arbitrarily proposed.

It also corresponds to a full lower triangular matrix, but differs from the Zadeh case on the cell content: the values are not constant columnwise, column β contains all values from τ_{M-1} to τ_β in decreasing order.

Its graphical representation, given in Figure 3, has the same structure as the Zadeh configuration, given in Figure 2, with different, greater, arrow labels.

C. Nilpotent Minimum

This section considers a more severe t-norm, expected to lead to a sparser matrix. It illustrates the fact that the computation of the coefficients cannot always be simplified.

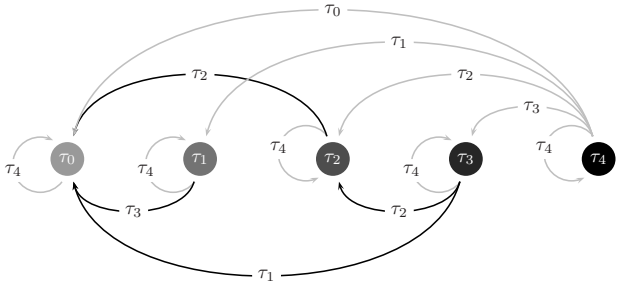

Fig. 4. The nilpotent minimum t-norm, F_{NM}, in $\mathcal{L}_5$

Corollary 3.
The nilpotent minimum t-norm

$$\top(\tau_\alpha, \tau_\beta) = \begin{cases} \min(\tau_\alpha, \tau_\beta) & \textit{if } \tau_{\alpha+\beta} > \tau_{M-1} \\ \tau_0 & \textit{otherwise} \end{cases}$$

is encoded in the ABOP *configuration defined by*

$$\kappa_\alpha^\beta = \begin{cases} \tau_{M-1} & \textit{if } \tau_\alpha \leq \tau_\beta \\ \tau_\beta & \textit{if } \tau_\alpha > \tau_\beta \textit{ and } \tau_{\alpha+\beta} > \tau_{M-1} \\ f & \textit{otherwise} \end{cases}$$

where $f = \max(\tau_{M-1-\alpha}, \max\{\tau_{M-1-\alpha} < \tau_\gamma < \tau_{M-1} | \min(\tau_\alpha, \tau_\gamma) \leq \tau_\beta\})$

Proof. Suppose, first, that $\tau_\alpha \leq \tau_\beta$. Then, the greatest $\tau_\gamma \in \mathcal{L}_M$ such that $\top(\tau_\alpha, \tau_\gamma) \leq \tau_\beta$ is τ_{M-1}, since, for any τ_γ the t-norm will either return τ_α or τ_0, both of which are less than τ_β.

Suppose now that $\tau_\alpha > \tau_\beta$ and $\tau_{\alpha+\beta} > \tau_{M-1}$. Then, the greatest $\tau_\gamma \in \mathcal{L}_M$ such that $\top(\tau_\alpha, \tau_\gamma) \leq \tau_\beta$ is τ_β. Indeed, for candidate $\tau_\gamma = \tau_\beta$, $\top(\tau_\alpha, \tau_\gamma) = \tau_\beta$ which obviously satisfies the desired inequality. For any candidate τ_γ greater than τ_β, $\tau_{\alpha+\gamma} > \tau_{M-1}$, thus $\top(\tau_\alpha, \tau_\gamma) = \min(\tau_\alpha, \tau_\gamma) > \tau_\beta$; thus such a candidate should be discarded.

Finally, in the case $\tau_\alpha > \tau_\beta$ and $\tau_{\alpha+\beta} \leq \tau_{M-1}$, the candidate τ_γ's depend on τ_α, τ_β and how far their sum is to τ_{M-1}, leading to a complex expression that cannot be simplified. □

The matrix is difficult to write, in the general case, because of the comparison between $\tau_{\alpha+\beta}$ and τ_{M-1}. It is given here

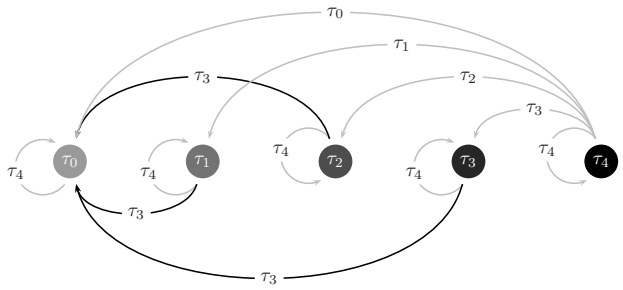

Fig. 5. The drastic t-norm, F_D, in $\mathcal{L}_5$

for the case where $M = 5$, i.e. $\mathcal{L}_5$, using the redundancy induced sparsity:

$$\kappa_{NM} = \begin{bmatrix} \tau_4 & & & & \\ \tau_3 & \tau_4 & & & \\ \tau_2 & & \tau_4 & & \\ \tau_1 & & \tau_2 & \tau_4 & \\ \tau_0 & \tau_1 & \tau_2 & \tau_1 & \tau_4 \end{bmatrix}$$

Note that this matrix has some missing values, specifically in the second column, as the graphical representation, in Figure 4, has 'missing' arrows.

D. Drastic t-norm

This section considers the most severe of all t-norms:

Corollary 4.
The drastic t-norm

$$\top(\tau_\alpha, \tau_\beta) = \begin{cases} \tau_\beta & \text{if } \tau_\alpha = \tau_{M-1} \\ \tau_\alpha & \text{if } \tau_\beta = \tau_{M-1} \\ \tau_0 & \text{otherwise} \end{cases}$$

is encoded in the ABOP *configuration defined by*

$$\kappa_\alpha^\beta = \begin{cases} \tau_\beta & \text{if } \tau_\alpha = \tau_{M-1} \\ \tau_{M-1} & \text{if } \tau_\alpha \leq \tau_\beta \\ \tau_{M-2} & \text{otherwise} \end{cases}$$

Proof. The first case is a direct consequence of the definition of the t-norm and the constraint relative to τ_β. The second also stems from the same constraints, given the choice of outputs by the t-norm, $\tau_0 \leq \tau_\alpha$. The final line considers the case where $\tau_\beta < \tau_\alpha < \tau_{M-1}$. In that case, the greatest value τ_γ can take, while respecting the constraint relative to τ_β, is the second greatest value in $\mathcal{L}_M$, τ_{M-2}. □

Using the sparse representation where only the left most occurrence of each observed value is represented, this definition is equivalent to

$$\kappa_D = \begin{bmatrix} \tau_{M-1} & & & & & \\ \tau_{M-2} & \tau_{M-1} & & & & \\ \tau_{M-2} & & \tau_{M-1} & & & \\ \tau_{M-2} & & & \tau_{M-1} & & \\ \vdots & & & & \ddots & \\ \tau_0 & \tau_1 & \cdots & & \tau_{M-2} & \tau_{M-1} \end{bmatrix}$$

This matrix is sparser than all previous ones, as it only contains $3(M-1)$ values. Equivalently, the graphical representation, shown in Figure 5, contains far less arrows. Indeed, for the drastic t-norm, the only allowed transitions are the ones pointing to τ_0, except for the combination with the neutral element.

V. CONCLUSION

This paper proposed a representation theorem for any t-norm in the ABOP framework, further supporting previous results regarding the expressiveness of this MVL aggregation operator: it can encode any t-norm operator, as well as other, less constrained ones. In addition, its graphical representation increases the interpretability of the considered aggregation operator, offering a visualisation of its parameters. This graphical representation may be enriched, e.g. by using distinct arrow colours or width to give indications about their associated weights that represent the transition costs, κ_α^β.

Directions for future works include the further exploitation of the proposed matrix and graph representation of the t-norms, for instance to define a degree of severity depending on the κ_α^β parameters. The latter may help for instance characterise t-norms depending on their intermediate behaviour between the extreme Zadeh and drastic cases. Other directions for future works include the characterisation of the more general ABOP parameter matrices that guarantee properties such as commutativity, associativity or idempotence.

REFERENCES

[1] T. Calvo, G. Mayor, and R. Mesiar, editors. *Aggregation Operators: New Trends and Applications*, volume 97. Springer, 2002.

[2] P. Capet and T. Delavallade, editors. *Information Evaluation*. Wiley, 2014.

[3] S. Gottwald. *A treatise on many-valued logics*, volume 9 of *Studies in logic and computation*. Research Studies Press, Baldock, 2001.

[4] M. Grabisch, J. Marichal, R. Mesiar, and E. Pap. *Aggregation Functions*. Number 127 in Encyclopedia of Mathematics and its Applications. Cambridge University Press, 2009.

[5] E. Klement, R. Mesiar, and E. Paps. *Triangular Norms*. Springer, 2000.

[6] N. Rescher. *Many-valued Logic*. McGraw Hill, 1969.

[7] A. Revault d'Allonnes and M.-J. Lesot. Formalising information scoring in a multivalued logic framework. In *Proc. of the Int. Conf. on Information Processing and Management of Uncertainty in Knowledge-Based Systems, IPMU'14*, pages 314–323. Springer, 2014.

[8] A. Revault d'Allonnes and M.-J. Lesot. Study of an abating aggregation operator in many-valued logic. In *Proc. of the Int. Conf. on Fuzzy Systems, FuzzIEEE'19*. IEEE, 2019.

[9] H. Seridi and H. Akdag. Approximate Reasoning for Processing Uncertainty. *Journal of Advanced Computational Intelligence and Intelligent Informatics (JACIII)*, 5(2):110–118, 2001.